Curvature Proof

$$T = \frac{r'}{|r'|} \quad \text{Definition} \quad (1)$$

$$|r'| = \frac{ds}{dt} \quad s(t) = \int_a^t |r'(u)|du, \text{ First Theorem of Calculus} \quad (2)$$

$$r' = |r'|T \quad 1, \text{ Algebra} \quad (3)$$

$$|r'|T = \frac{ds}{dt}T = r' \quad 2, 3, \text{ Algebra} \quad (4)$$

$$r'' = \frac{d^2s}{dt^2}T + \frac{ds}{dt}T' \quad 4, \text{ Product rule} \quad (5)$$

$$T \times T = 0 \quad \text{Property of cross product} \quad (6)$$

$$r' \times r'' = (\frac{ds}{dt})^2(T \times T') \quad 4, 5, 6, c\vec{a} \times c\vec{b} = c^2(\vec{a} \times \vec{b}) \quad (7)$$

$$|r' \times r''| = (\frac{ds}{dt})^2|T||T'|sin(\frac{\pi}{2}) = (\frac{ds}{dt})^2|T'| \quad 7, \text{ definition of cross, } |T| = 1 \text{ so, } T \text{ is orthogonal to } T' \quad (8)$$

$$|T'| = \frac{|r' \times r''|}{(\frac{ds}{dt})^2} \quad 8, \text{ Algebra} \quad (9)$$

$$|T'| = \frac{|r' \times r''|}{|r'|^2} \quad 2, 9, \text{ Algebra} \quad (10)$$

$$\kappa = \frac{|T'|}{|r'|} \quad \kappa = |\frac{dT}{ds}| \text{ chain rule to } |\frac{\frac{dT}{dt}}{\frac{ds}{dt}} = \frac{|T'|}{|r'|} \quad (11)$$

$$\kappa = \frac{|r' \times r''|}{|r'|^3} \quad 10, 11, \text{ Algebra} \quad (12)$$

Tangential and Normal Components of Acceleration

$$v = |\vec{\mathrm{v}}(t)| \quad \text{Definition} \quad (13)$$

$$\vec{T}(t) = \frac{\vec{r'}(t)}{|\vec{r'}(t)|} = \frac{\vec{\mathrm{v}}(t)}{|\vec{\mathrm{v}}(t)} = \frac{\vec{\mathrm{v}}}{v} \quad \text{Definition, } r'(t) = v(t) \quad (14)$$

$$\vec{\mathrm{v}} = v\vec{T} \quad 2, \text{ Algebra} \quad (15)$$

$$\vec{\mathrm{v}}' = \vec{a} = v'\vec{T} + v\vec{T}' \quad 3, \text{ Product rule} \quad (16)$$

$$\kappa = \frac{|\vec{T}'|}{|\vec{r'}|} = \frac{|\vec{T}'|}{v} \quad \text{Definition, chain rule } \kappa = |\frac{dT}{ds}| = \frac{|\frac{dT}{dt}|}{|\frac{ds}{dt}|} = \frac{|T'|}{|r'|} \quad (17)$$

$$|\vec{T}'| = \kappa v \quad 5, \text{ Algebra} \quad (18)$$

$$\vec{N} = \frac{\vec{T}'}{|\vec{T}'|} \quad \text{Definition} \quad (19)$$

$$\vec{T}' = |\vec{T}'|\vec{N} = \kappa v\vec{N} \quad 6, 7, \text{ Algebra} \quad (20)$$

$$\vec{a} = v'\vec{T} + \kappa v^2\vec{N} \quad 4, 8, \text{ Algebra} \quad (21)$$

$$\vec{a} = a_T\vec{T} + a_N\vec{N} \quad \text{Where } a_T = v' \text{ and } a_N = \kappa v^2 \quad (22)$$

Normal Component of Acceleration

$$\vec{a} = v'\vec{T} + \kappa v^2 \vec{N} \quad \text{Definition} \tag{23}$$

$$\vec{\mathrm{v}} = v\vec{T} \quad \text{Definition} \tag{24}$$

$$\vec{\mathrm{v}} \cdot \vec{a} = v\vec{T} \cdot (v'\vec{T} + \kappa v^2 \vec{N}) \quad \text{1, 2, Substitution} \tag{25}$$

$$\vec{\mathrm{v}} \cdot \vec{a} = vv'\vec{T} \cdot \vec{T} + \kappa v^2 \vec{T} \cdot \vec{N} \tag{26}$$

$$\vec{\mathrm{v}} \cdot \vec{a} = vv' \quad 4,\ \vec{T} \cdot \vec{T} = 1 \text{ and } \vec{N} \cdot \vec{T} = 0 \tag{27}$$

$$a_T = v' = \frac{\vec{\mathrm{V}} \cdot \vec{a}}{v} = \frac{r'(t) \cdot r''(t)}{|r'(t)|} \quad \text{5, Definition, Algebra} \tag{28}$$

$$a_n = \kappa v^2 = \frac{|r'(t) \times r''(t)|}{|r'(t)|^3} |r'(t)|^2 \quad \text{Previously proved theorem, definitions} \tag{29}$$

$$a_n = \frac{|r'(t) \times r''(t)|}{|r'(t)|} \quad \text{7, Algebra} \tag{30}$$