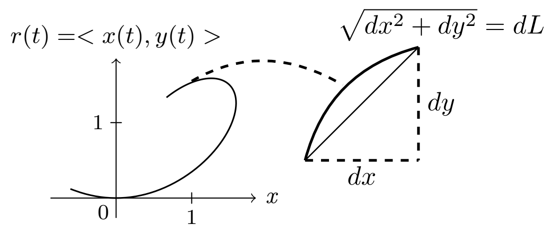
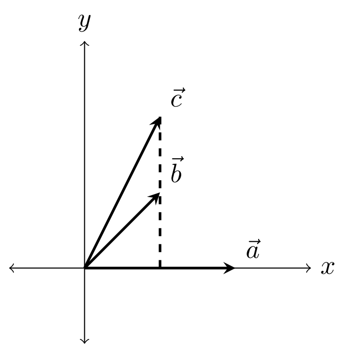
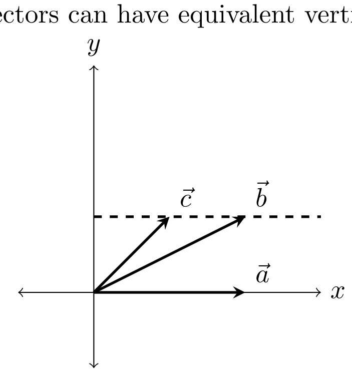


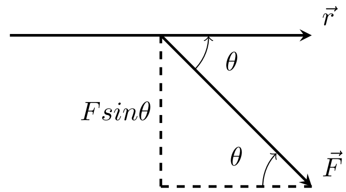

$$\frac{dx}{dt} = f'(t)$$
$$dx = f'(t)dt, dy = g'(t)dt$$
$$dL = \sqrt{dx^2 + dy^2} = \sqrt{(f'(t)dt)^2 + (g'(t)dt)^2} = \sqrt{(\frac{dx}{dt})^2 + \frac{dy}{dt})^2}$$
$$L = \int dL = \int_a^b \sqrt{(\frac{dx}{dt})^2 + \frac{dy}{dt})^2} = \int_a^b |r'(t)|dt$$
$$s(t) = \int_a^b |r'(t)|dt$$

If $\vec{a} \cdot \vec{b} = \vec{a} \cdot \vec{c}$, that does not mean that $\vec{b} = \vec{c}$ because two separate vectors can have the same horizontal component




Also, if $\vec{a} \times \vec{b} = \vec{a} \times \vec{c}$, $\vec{b}$ does not have to be equal to $\vec{c}$ because two separate vectors can have equivalent vertical components





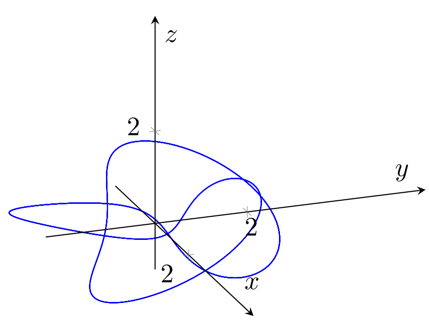

$$\vec{\tau} = (|\vec{r}||\vec{F}|sin\theta)\vec{n}$$
$$\vec{\tau} = (|\vec{r}||\vec{F}|sin\theta)\vec{n}$$
$$\vec{a} \times \vec{b} = (|\vec{a}||\vec{b}|sin\theta)\vec{n}$$
$$\vec{a} =< a_1, a_2, a_3 >, \vec{b} =< b_1, b_2, b_3 >$$
$$\vec{a} \times \vec{b} = \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix} =< (a_2b_3 - a_3b_2), -(a_1b_3 - a_3b_1), (a_1b_2 - a_2b_1) >$$

Directional Derivative
If $g(h) = f(x_o + ha, y_o + hb)$, then by the the definition of a derivative

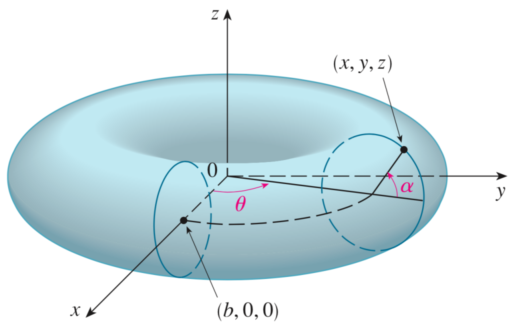
$$g'(0) = \lim_{h \to 0} \frac{f(x_o + ha, y_o + hb) - f(x_o, y_o)}{h} = D_u f(x_o, y_o)$$

Or by the chain rule with $x = x_o + ha$ and $y = y_o + hb$

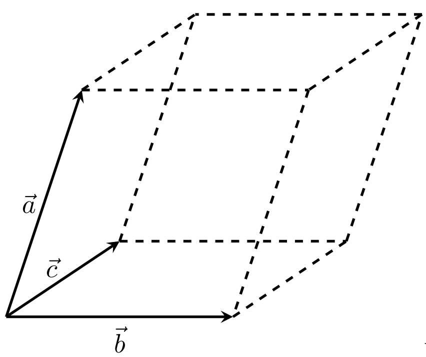
$$g'(h) = \frac{\partial f}{\partial x}\frac{dx}{dh} + \frac{\partial f}{\partial y}\frac{dy}{dh}$$
$$= f_x(x_o, y_o)a + f_y(x_o, y_o)b$$
$$Duf(x_o, y_o) = \nabla f(x_o, y_o) \cdot \vec{u}$$



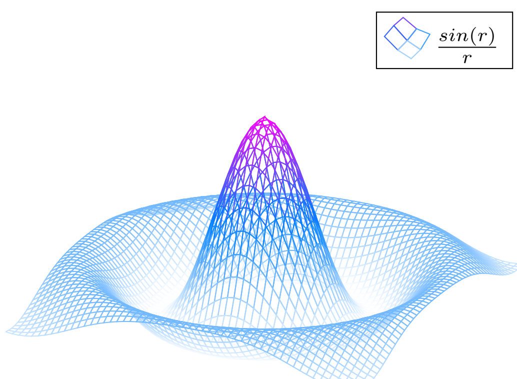

Parametric Torus



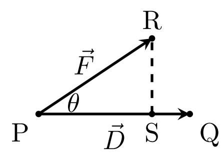

$$x = bcos(\theta) + acos(\alpha)cos(\theta)$$
$$y = bsin(\theta) + acos(\alpha)sin(\theta)$$
$$z = asin(\alpha)$$




$$V = |\vec{a}cos(\theta)|\vec{b} \times \vec{c}| = \vec{a} \cdot (\vec{b} \times \vec{c})$$

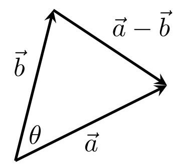


Dot Products




$$|\vec{PS}| = |\vec{F}|cos\theta$$

$$W = |\vec{D}||\vec{PS}| = |\vec{D}||\vec{F}|cos\theta = \vec{D} \cdot \vec{F}$$

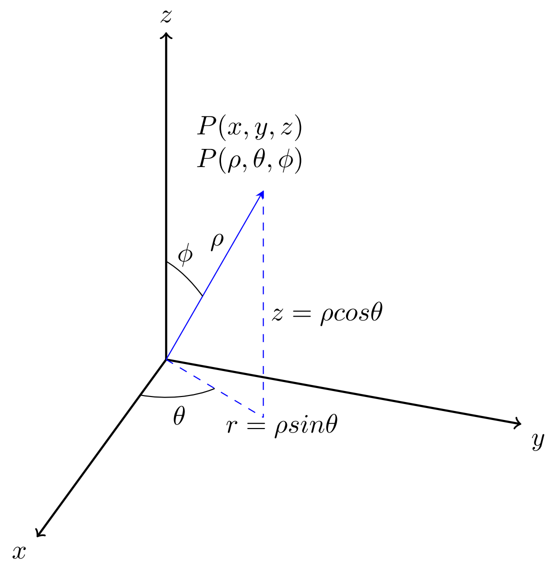



$$|\vec{a} - \vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 - 2|\vec{a}||\vec{b}|cos\theta$$

$$\vec{a} \cdot \vec{b} = \frac{|\vec{a}|^2 + |\vec{b}|^2 - |\vec{a} - \vec{b}|^2}{2}$$

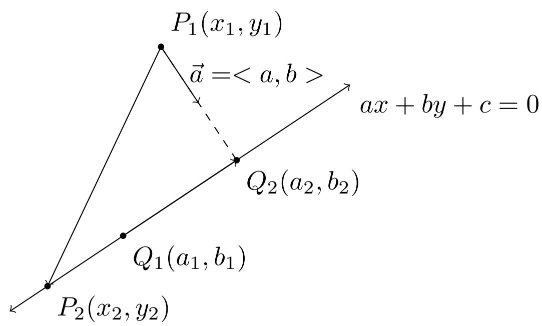
$$\vec{a} \cdot \vec{b} = a_1b_1 + a_2b_2 + a_3b_3$$

Spherical Coordinate System




$$R = \rho sin\theta$$
$$x = Rcos\theta = \rho cos\theta sin\phi$$
$$y = Rsin\theta = \rho sin\theta sin\phi$$
$$z = \rho cos\theta$$
$$\rho^2 = x^2 + y^2 + z^2$$

Distance to a Line




Since the slope of $ax + by + c = 0$ is $-\dfrac{a}{b}$, the slope of a $\perp$ line is $\dfrac{b}{a}$, therefore $< a, b > \perp \vec{Q_1Q_2}$,

$$\text{so} < a, b > \cdot < a_2 - a_1, b_2 - b_1 >= 0$$

Proof:

$$aa_2 - aa_1 + bb_2 - bb_1 = 0$$
$$aa_1 + bb_1 = -c = aa_2 + bb_2$$
$$aa_2 - aa_1 + bb_2 - bb_1 = 0$$

The magnitude of the scalar projection of $\vec{P_1P_2}$ onto $\vec{a}$ is the distance between $P_1$ and the line.

$$|comp_{<a,b>}\vec{P_1P_2}| = \frac{| < x_1 - x_2, y_1 - y_2 > \cdot < a, b > |}{\sqrt{a^2 + b^2}} = \frac{|ax_1 - ax_2 + by_1 - by_2|}{\sqrt{a^2 + b^2}}$$
$$= \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$
$$\text{since } -ax_2 - by_2 = +c$$

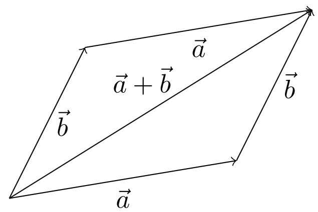
Three Space Distances

$$\sqrt{x^2 + y^2 + z^2} = D$$

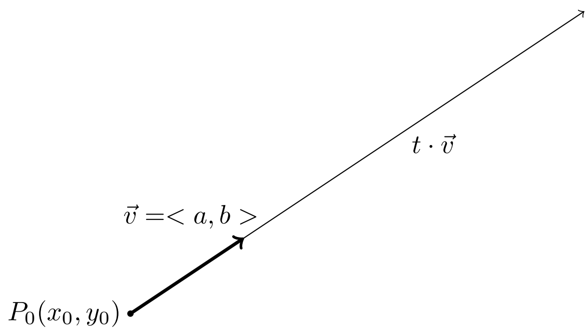
Derivative of a Space Curve

$$\frac{dr}{dt} = \vec{r'} = \lim_{h->0} \frac{\vec{r}(t + h) - \vec{r}(t)}{h}$$

Parallelogram Vector Addition




Equation of a Line




$$\vec{r} = P_0 + t\vec{v} =< x_0, y_0 > +t < a, b >=< x_0 + ta, y_0 + tb >$$
$$x = x_0 + ta$$
$$y = y_0 + tb$$

**Gradient Vectors and Directional Derivative Equations**

$$\nabla f(x, y) =< f_x(x, y), f_y(x, y) >$$
$$D_u f(x, y) = \nabla f(x, y) \cdot u$$

Partial Derivatives

$$g(x) = f(x, b)$$
$$f_x(a, b) = g'(a)$$

Partial Derivatives using limit definition of a derivative

$$f_x(x, y) = lim_{h\rightarrow 0}\frac{f(x + h, y) - f(x, y)}{h}$$
$$f_y(x, y) = lim_{h\rightarrow 0}\frac{f(x, y + h) - f(x, y)}{h}$$

**Unit Vectors**

A unit vector is a vector that has a length of 1.

The unit vector of $\vec{a}$:

$$\vec{u} = \frac{\vec{a}}{|\vec{a}|}$$

Given $s$ is a level surface with equation $F(x, y, z) = k$, a level surface, of the function $F$, and $P(x_o, y_o, z_o)$ is a point on $s$. $C$ is any curve that lies on $s$ and passes through $P$, and $C$ is described by $r(t) =< x(t), y(t), z(t) >$ with $r_o =< x_o, y_o, z_o >$ corresponding to $P$. Any point on $C$ must satisfy the equation of $F(x(t), y(t), z(t)) = k$.

Differentiating both sides with the chain rule you get

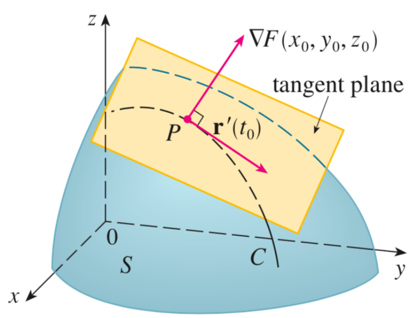
$$\frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt} = 0$$

$$< \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z} > \cdot < \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} >= 0$$

$$\nabla F \cdot r'\vec{(t)} = 0$$

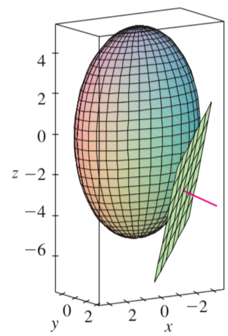
$$t = t_o$$

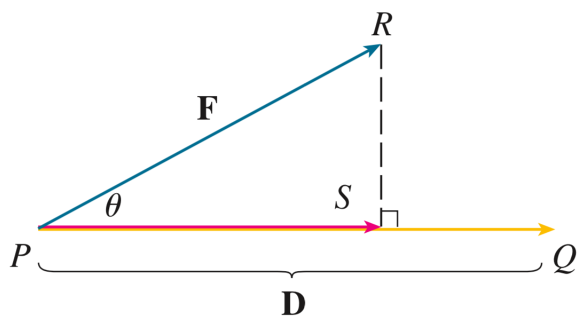
$$\nabla F(x_o, y_o, z_o) \cdot r'\vec{(t_o)} = 0$$




So the gradient vector at $P$ is perpindicular to the tangent vector $r'\vec{(t)}$. Given that $F(x(t), y(t), z(t)) = k$ defines a level curve of the surface $S$, then the gradient vector is always perpindicular to all level surfaces. So the normal vector to a level curve is $\nabla F(x_o, y_o, z_o)$. Therefore, the equation of a tangent plane at point $P$ is given by

$$F_x(x_o, y_o, z_o)(x - x_o) + F_y(x_o, y_o, z_o)(y - y_o) + F_z(x_o, y_o, z_o)(z - z_o) = 0$$



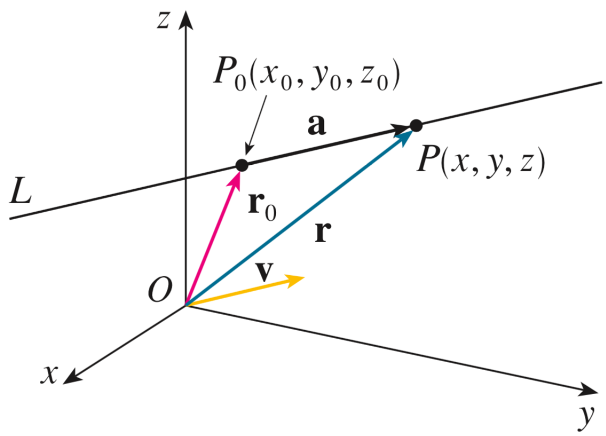




$$\boldsymbol{F} = \vec{PR}$$
$$\boldsymbol{D} = \vec{PQ}$$

The work done by $\mathbf{F}$ is defined as the magnitude of the displacement, $|\mathbf{D}|$, multiplied by the magnitude of the applied force in the direction of the motion

$$|\vec{PS}| = |\mathbf{F}|cos\theta$$

So the work is defined to be $W = |\mathbf{D}|(|\mathbf{F}|cos\theta) = |\mathbf{D}||\mathbf{F}|cos\theta$



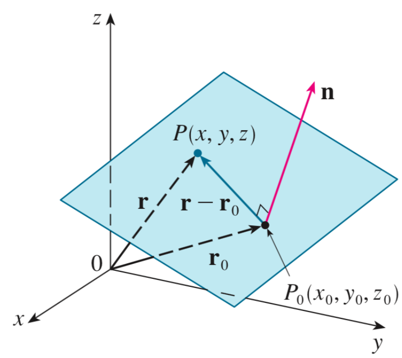

$$\vec{r} = \vec{r_o} + t\vec{v}$$

Let $P$ be a point not on the plane that passes through the points $Q, R$ and $S$. Show that the distance $d$ from $P$ to the plane is

$$d = \frac{|\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|}{|\mathbf{a} \times \mathbf{b}|}$$

where $\boldsymbol{a} = \vec{QR}, \boldsymbol{b} = \vec{QS}$, and $\boldsymbol{c} = \vec{QP}$.

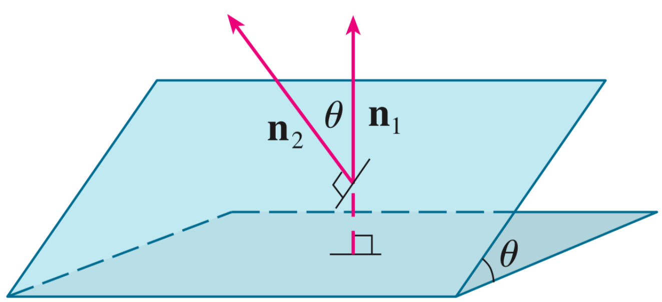
The distance is defined as $|\vec{PS}| = d$. But refering to triangle $PQS$, $d = |\vec{PS}| = |\vec{PS}|sin\theta = |\boldsymbol{b}|sin\theta$. But $\theta$ is the angle between $\vec{QP} = \boldsymbol{b}$ and $\vec{QR} = \boldsymbol{a}$. Thus by definition of cross product, $sin\theta = \frac{|\mathbf{a}\times\mathbf{b}|}{|\mathbf{a}||\mathbf{b}|}$, and so $d = |\boldsymbol{b}|sin\theta = \frac{|\mathbf{b}||\mathbf{a}\times\mathbf{b}|}{|\mathbf{a}||\mathbf{b}|}$



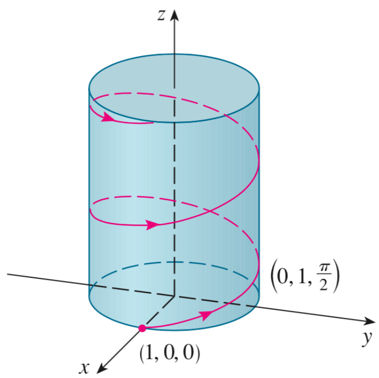

$$\boldsymbol{n} \cdot (\boldsymbol{r} - \boldsymbol{r_o}) = 0$$
$$\boldsymbol{r} = < x, y, z >$$
$$\boldsymbol{r_o} = < x_o, y_o, z_o >$$
$$\boldsymbol{n} = < a, b, c >$$

So the equation of a plane is

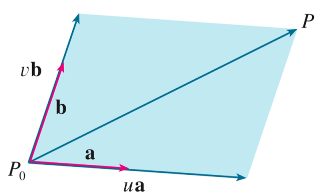
$$a(x - x_o) + b(y - y_o) + c(z - z_o) = 0$$



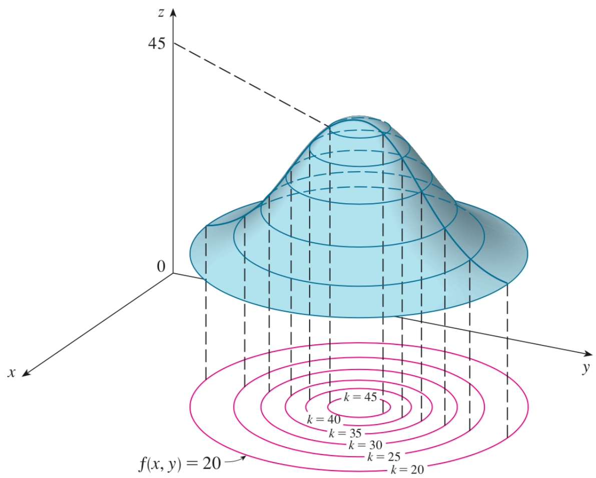

$$cos\theta = \frac{\boldsymbol{n}_1 \cdot \boldsymbol{n}_2}{|\boldsymbol{n}_1||\boldsymbol{n}_2|}$$

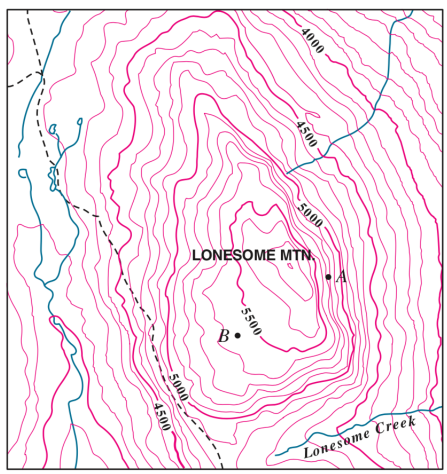


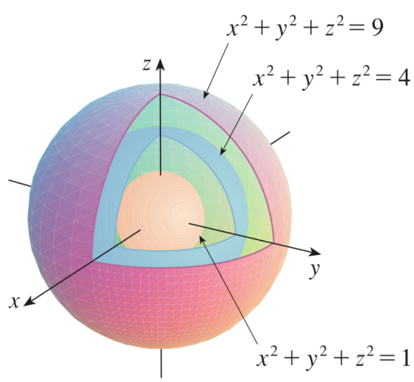

$$r(\vec{t}) = \cos\, t\boldsymbol{i} + \sin\, t\boldsymbol{j} + t\boldsymbol{k}$$



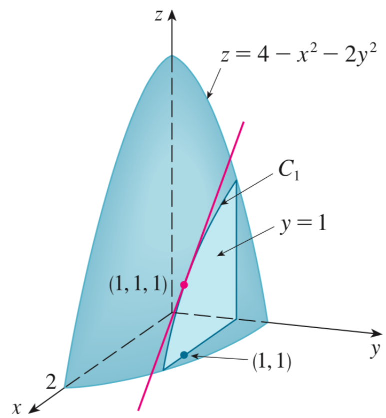

$$\boldsymbol{r} = \boldsymbol{r}_o + u\boldsymbol{a} + v\boldsymbol{b}$$


z
45
0
x
y
k = 45
k = 40
k = 35
k = 30
k = 25
k = 20
f(x, y) = 20

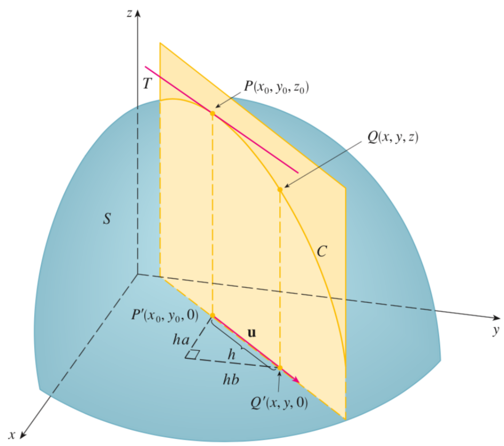

4000
4500
5000
LONESOME MTN.
A
B
5500
5000
4500
Lonesome Creek

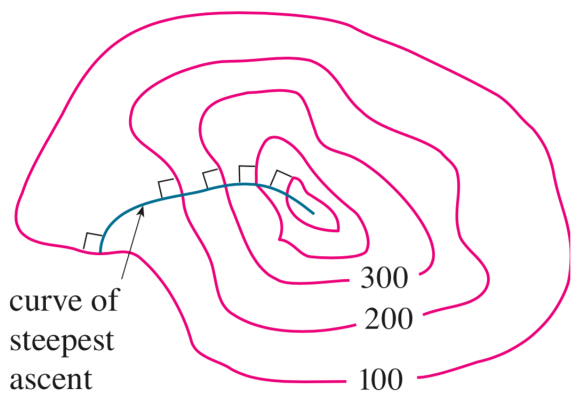

x² + y² + z² = 9
x² + y² + z² = 4
z
y
x
x² + y² + z² = 1

$$f(x, y, z) = x^2 + y^2 + z^2$$

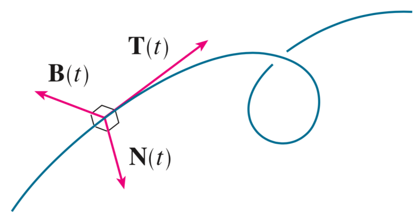



The tangent line on $C$ is $\dfrac{\partial F}{\partial x}$




The direction of the gradient vector, $\nabla f(x, y, z)$, is the direction of steepest ascent because $D_u$ is maximized with $\nabla f \cdot \vec{u}$ when $u$ is in the direction of $\nabla f$




$$\vec{T(t)} = \frac{\vec{r'(t)}}{|\vec{r'(t)}|}$$

$$\vec{N(t)} = \frac{\vec{T'(t)}}{|\vec{T'(t)}|}$$

$$\vec{B(t)} = \vec{T(t)} \times \vec{N(t)}$$

$$\kappa = \frac{|d\vec{T}|}{|ds|}$$

Tangent Plane to a Level Surface

$\nabla f(x_0, y_0, z_0) \cdot < x - x_0, y - y_0, z - z_0 >= 0$

Tangent Line to a Level Curve

$\nabla f(x_0, y_0) \cdot < x - x_0, y - y_0 >= 0$