Probabilités et Statistique II Chapitre 6. Lois continues de probabilités

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Uniform Distribution

In[ ]:= PDF[UniformDistribution[{a, b}], x]

Out[ ]=

$$\begin{cases} \frac{1}{-a+b} & a \leq x \leq b \\ 0 & \text{True} \end{cases}$$

In[ ]:= PiecewiseExpand$\left[\text{Piecewise}\left[\left\{\left\{\frac{1}{-a+b}, a \leq x \leq b\right\}\right\}, 0\right]\right]$

Out[ ]=

$$\begin{cases} \frac{1}{-a+b} & a - x \leq 0 \,\&\&\, b - x \geq 0 \\ 0 & \text{True} \end{cases}$$

In[ ]:= Simplify$\left[\text{Piecewise}\left[\left\{\left\{\frac{1}{-a+b}, a \leq x \leq b\right\}\right\}, 0\right]\right]$

Out[ ]=

$$\begin{cases} \frac{1}{-a+b} & a \leq x \leq b \\ 0 & \text{True} \end{cases}$$

In[ ]:=
```
d = UniformDistribution[{1, 11}]
```
Out[ ]=
```
UniformDistribution[{1, 11}]
```
In[ ]:=
```
PDF[d, x]
```
Out[ ]=
$$\begin{cases} \frac{1}{10} & 1 \leq x \leq 11 \\ 0 & \text{True} \end{cases}$$

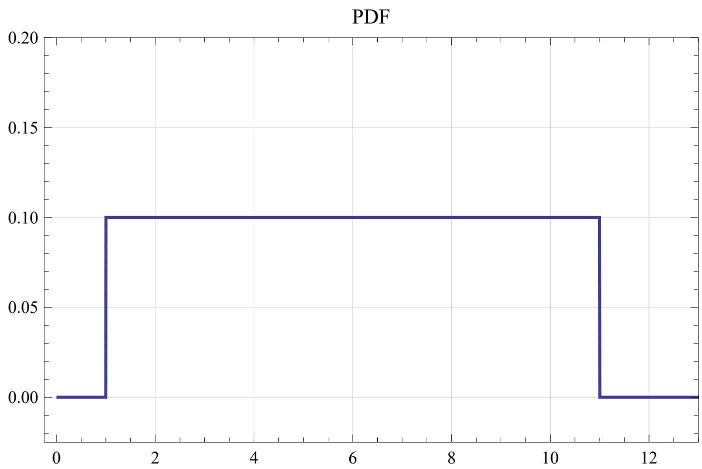
In[ ]:=
```
PiecewiseExpand[Piecewise[{{1/10, 1 ≤ x ≤ 11}}, 0]]
```

Out[ ]=

$$\begin{cases} \frac{1}{10} & 1 \leq x \leq 11 \\ 0 & \text{True} \end{cases}$$

In[ ]:=
```
pdf = Plot[PiecewiseExpand[Piecewise[{{1/10, 1 ≤ x ≤ 11}}, 0]],
  {x, 0, 15}, PlotLabel → "PDF", PlotStyle → ColorData[1, 1],
  GridLines → Automatic, PlotTheme → "Scientific",
  ImageSize → Medium, PlotRange → {{-0.25, 13}, {-0.025, 0.2}}]
```

Out[ ]=



In[∘]:= CDF[d, x]

Out[∘]=

$$\begin{cases} \frac{1}{10}(-1+x) & 1 \le x \le 11 \\ 1 & x > 11 \\ 0 & \text{True} \end{cases}$$

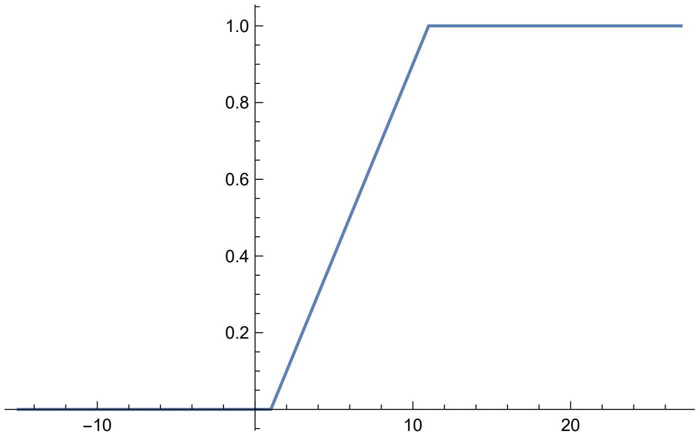
In[∘]:= PiecewiseExpand$\left[\text{Piecewise}\left[\left\{\left\{\frac{1}{10}(-1+x), 1 \le x \le 11\right\}, \{1, x > 11\}\right\}, 0\right]\right]$

Out[∘]=

$$\begin{cases} 1 & x > 11 \\ \frac{1}{10}(-1+x) & 1 \le x \le 11 \\ 0 & \text{True} \end{cases}$$

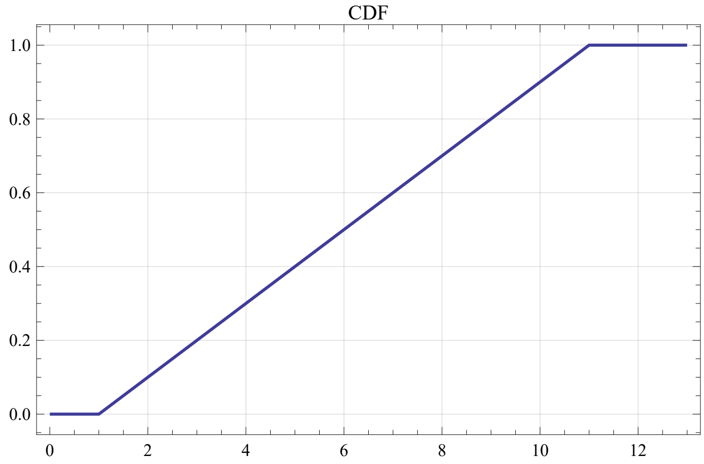
In[∘]:= Plot$\left[\text{Piecewise}\left[\left\{\{1, x > 11\}, \left\{\frac{1}{10}(-1+x), 1 \le x \le 11\right\}\right\}, 0\right], \{x, -15., 27.\}\right]$

Out[∘]=



In[∘]:=
```
cdf = Plot[PiecewiseExpand[
   Piecewise[{{1, x > 11}, {1/10 (-1 + x), 1 ≤ x ≤ 11}}, 0]],
  {x, 0, 13}, PlotLabel → "CDF", PlotStyle → ColorData[1, 1],
  GridLines → Automatic,
  PlotTheme → "Scientific", ImageSize → Medium]
```

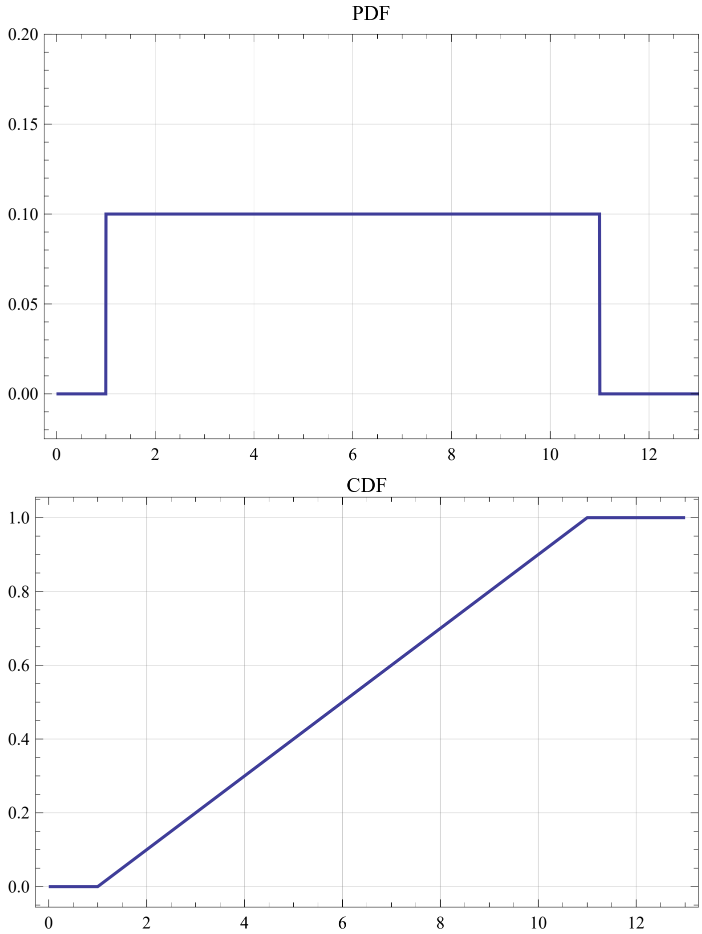
Out[∘]=



In[ ]:=

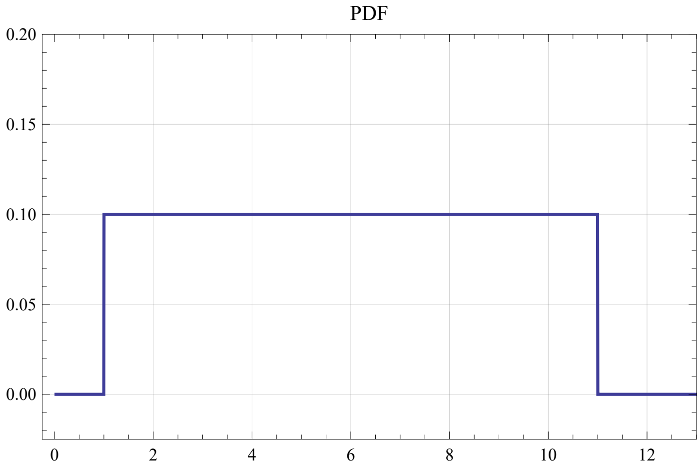
```
Grid[{{pdf}, {cdf}}]
```

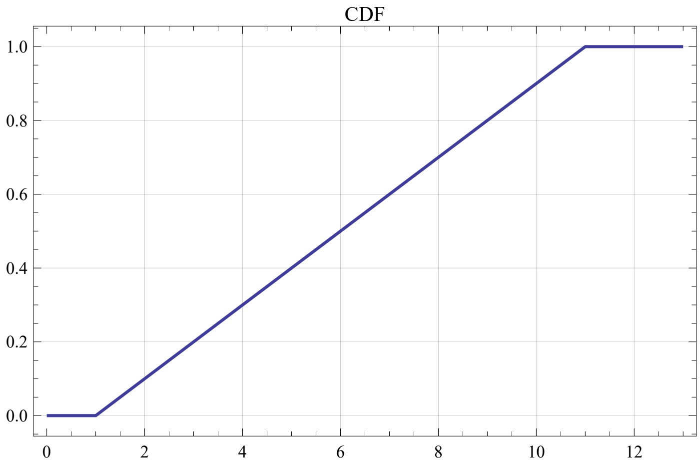
Out[ ]=



In[∘]:=
```
Insert[%40, Alignment → Right, 2]
```

Out[∘]=





Mean Uniform Distribution

The aim is to replicate the demonstration than can be found here :
https://en.wikibooks.org/wiki/Statistics/Distributions/Uniform

In[*]:=
```
Mean[UniformDistribution[{a, b}]]
```

Out[*]=

$$\frac{a+b}{2}$$

We derive the mean as follows.

$$\mathrm{E}[X] = \int_{-\infty}^{\infty} x f(x) dx$$

As the uniform distribution is 0 everywhere but [a, b] we can restrict ourselves that interval

$$\mathrm{E}[X] = \int_a^b \frac{1}{b-a} x dx$$

$$\mathrm{E}[X] = \frac{1}{(b-a)} \frac{1}{2} x^2 \Big|_a^b$$

$$\mathrm{E}[X] = \frac{1}{2(b-a)} \left[b^2 - a^2\right]$$

$$\mathrm{E}[X] = \frac{b+a}{2}$$

Variance Uniform Distribution

The aim is to replicate the demonstration than can be found here :
https://en.wikibooks.org/wiki/Statistics/Distributions/Discrete_Uniform

In[◦]:=

```
Variance[UniformDistribution[{a, b}]]
```

Out[◦]=

$$\frac{1}{12}(-a+b)^2$$

Variance [edit | edit source]

We use the following formula for the variance.

$$\mathrm{Var}(X) = \mathrm{E}[X^2] - (\mathrm{E}[X])^2$$

$$\mathrm{Var}(X) = \left[\int_{-\infty}^{\infty} f(x)\cdot x^2\,dx\right] - \left(\frac{b+a}{2}\right)^2$$

$$\mathrm{Var}(X) = \left[\int_a^b \frac{1}{b-a}x^2\,dx\right] - \frac{(b+a)^2}{4}$$

$$\mathrm{Var}(X) = \frac{1}{b-a}\frac{1}{3}x^3\Big|_a^b - \frac{(b+a)^2}{4}$$

$$\mathrm{Var}(X) = \frac{1}{3(b-a)}[b^3-a^3] - \frac{(b+a)^2}{4}$$

$$\mathrm{Var}(X) = \frac{4(b^3-a^3)-3(b+a)^2(b-a)}{12(b-a)}$$

$$\mathrm{Var}(X) = \frac{(b-a)^3}{12(b-a)}$$

$$\mathrm{Var}(X) = \frac{(b-a)^2}{12}$$

Exponential Distribution

In[∘]:= `exp = ExponentialDistribution[α]`

Out[∘]=

`ExponentialDistribution[α]`

In[∘]:= `PDF[exp, x]`

Out[∘]=

$$\begin{cases} e^{-x\alpha}\alpha & x \geq 0 \\ 0 & \text{True} \end{cases}$$

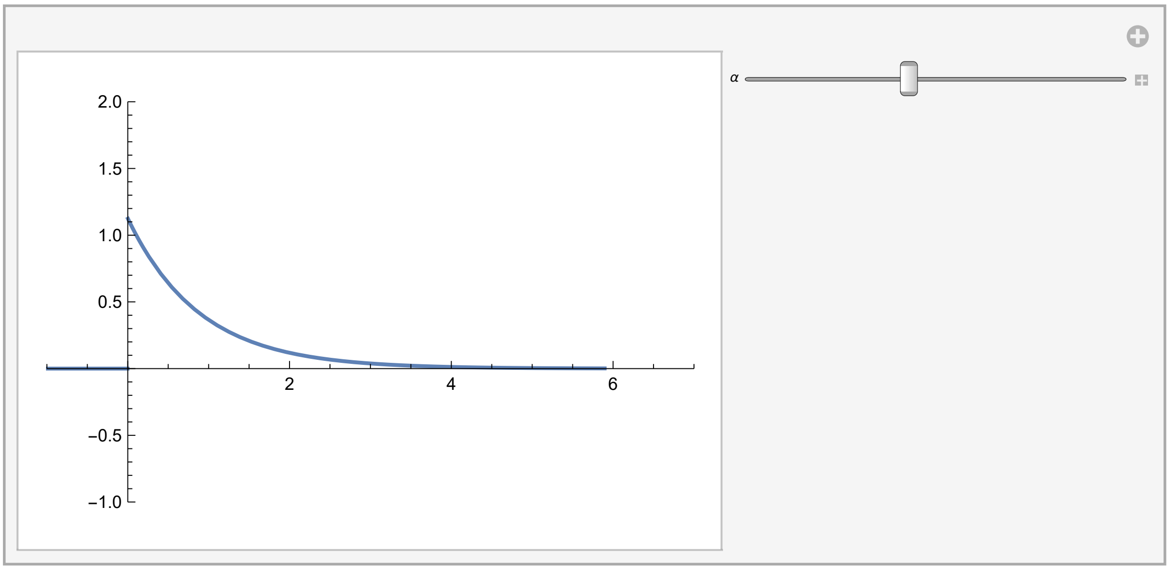
In[∘]:= `PiecewiseExpand[Piecewise[{{`$e^{-x\alpha}$` α, x ≥ 0}}, 0]]`

Out[∘]=

$$\begin{cases} e^{-x\alpha}\alpha & x \geq 0 \\ 0 & \text{True} \end{cases}$$

In[∘]:=
```
Manipulate[Plot[PiecewiseExpand[Piecewise[{{e^(-x α) α, x ≥ 0}}, 0]],
  {x, -1, 5.89588}, PlotRange → {{-1, 7}, {-1, 2}}], {α, 0.5, 2}]
```

Out[∘]=



In[ ]:= CDF[exp, x]

Out[ ]=

$$\begin{cases} 1 - e^{-x\,\alpha} & x \geq 0 \\ 0 & \text{True} \end{cases}$$

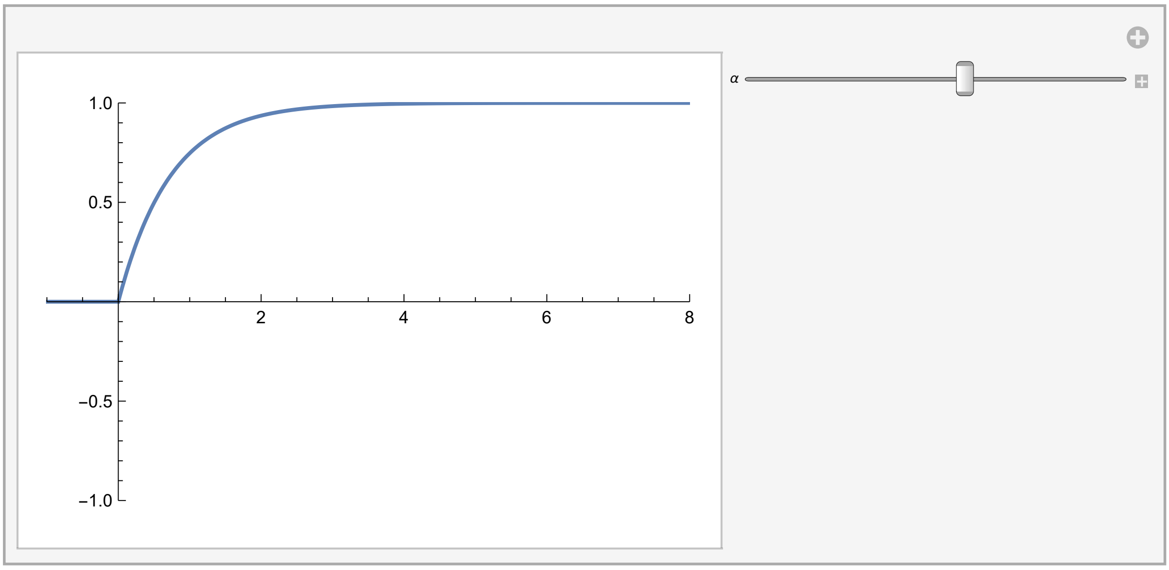
In[ ]:= PiecewiseExpand[Piecewise[{{1 - $e^{-x\,\alpha}$, x ≥ 0}}, 0]]

Out[ ]=

$$\begin{cases} e^{-x\,\alpha}\,(-1 + e^{x\,\alpha}) & x \geq 0 \\ 0 & \text{True} \end{cases}$$

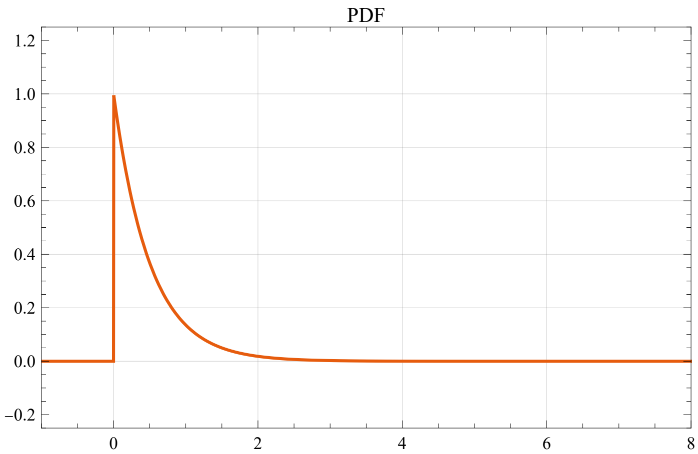
In[ ]:= Manipulate[Plot[PiecewiseExpand[Piecewise[{{1 - $e^{-x\,\alpha}$, x ≥ 0}}, 0]],
{x, -1, 10}, PlotRange → {{-1, 8}, {-1, 1}}], {α, 0.5, 2}]

Out[ ]=



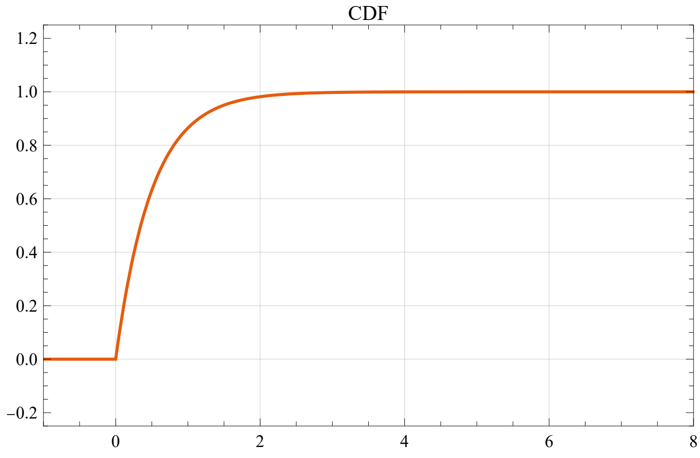
In[ ]:=
```
pdf = Plot[PiecewiseExpand[Piecewise[{{e^(-x 2), x ≥ 0}}, 0]],
  {x, -1, 10}, PlotRange → {{-1, 8}, {-0.25, 1.25}},
  GridLines → Automatic, PlotLabel → "PDF",
  PlotTheme → "Scientific", ImageSize → Medium]
```

Out[ ]=



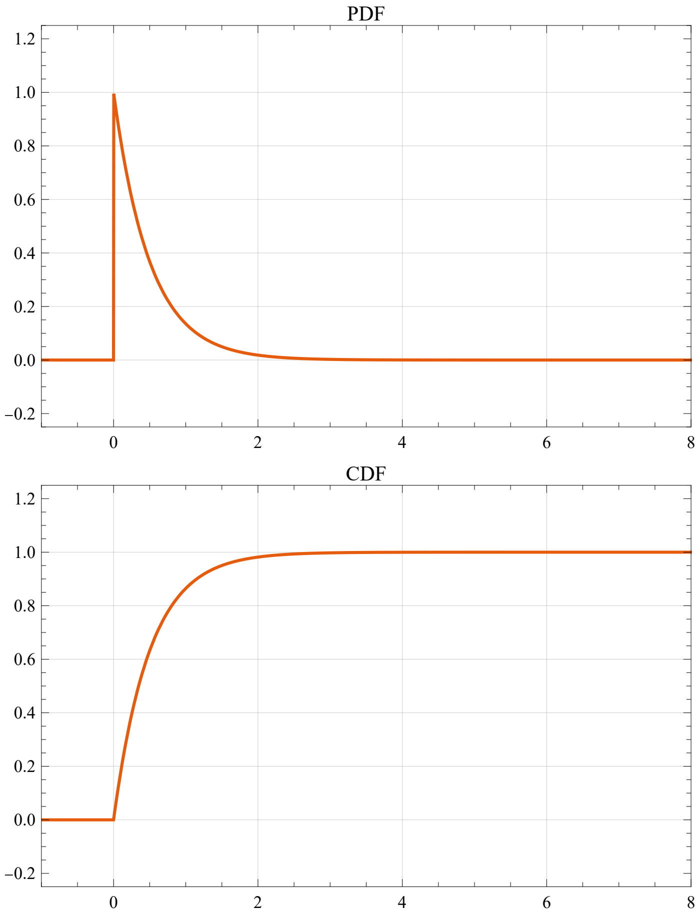
In[ ]:=
```
cdf = Plot[PiecewiseExpand[Piecewise[{{1 - e^(-x 2), x ≥ 0}}, 0]],
  {x, -1, 10}, PlotRange → {{-1, 8}, {-0.25, 1.25}}, GridLines → Automatic,
  PlotLabel → "CDF", PlotTheme → "Scientific", ImageSize → Medium]
```

Out[ ]=



In[ ]:=
```
Grid[{{pdf}, {cdf}}]
```

Out[ ]=



Demonstration (Mean)

https://en.wikibooks.org/wiki/Statistics/Distributions/Exponential

In[∘]:= Mean[ExponentialDistribution[λ]]

Out[∘]=

$$\frac{1}{\lambda}$$

Mean [edit | edit source]

We derive the mean as follows.

$$\mathrm{E}[X] = \int_{-\infty}^{\infty} x \cdot f(x) dx$$

$$\mathrm{E}[X] = \int_{0}^{\infty} x \lambda e^{-\lambda x} dx$$

$$\mathrm{E}[X] = \int_{0}^{\infty} (-x)(-\lambda e^{-\lambda x}) dx$$

We will use integration by parts with $u=-x$ and $v=e^{-\lambda x}$. We see that $du=-1$ and $dv=-\lambda e^{-\lambda x}$.

$$\mathrm{E}[X] = \left[-x \cdot e^{-\lambda x}\right]_0^{\infty} - \int_{0}^{\infty} (e^{-\lambda x})(-1) dx$$

$$\mathrm{E}[X] = [0-0] + \left[\frac{-1}{\lambda}(e^{-\lambda x})\right]_0^{\infty}$$

$$\mathrm{E}[X] = \left[0 - \frac{-1}{\lambda}\right]$$

$$\mathrm{E}[X] = \frac{1}{\lambda}$$

Demonstration (Variance)

https://www.statlect.com/probability-distributions/exponential-distribution

In[ ]:=
```
Variance[ExponentialDistribution[λ]]
```
Out[ ]=

$$\frac{1}{\lambda^2}$$

Variance

The variance of an exponential random variable X is

$$\text{Var}[X] = \frac{1}{\lambda^2}$$

Proof

It can be derived thanks to the usual variance formula ($\text{Var}[X] = \text{E}[X^2] - \text{E}[X]^2$):

$$\begin{aligned}
\text{E}[X^2] &= \int_0^\infty x^2 \lambda \exp(-\lambda x) dx \\
&= \left[-x^2 \exp(-\lambda x)\right]_0^\infty + \int_0^\infty 2x \exp(-\lambda x) dx \qquad \text{(integrating by parts)} \\
&= (0-0) + \left[-\frac{2}{\lambda} x \exp(-\lambda x)\right]_0^\infty + \frac{2}{\lambda}\int_0^\infty \exp(-\lambda x) dx \qquad \text{(integrating by parts again)} \\
&= (0-0) + \frac{2}{\lambda}\left[-\frac{1}{\lambda}\exp(-\lambda x)\right]_0^\infty \\
&= \frac{2}{\lambda^2}
\end{aligned}$$

$$\text{E}[X]^2 = \left(\frac{1}{\lambda}\right)^2 = \frac{1}{\lambda^2}$$

$$\text{Var}[X] = \text{E}[X^2] - \text{E}[X]^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$$

In[ ]:=
```
Integrate[(λ Power[x, 2]) / Power[E, (λ x)], {x, 0, Infinity}]
```
Out[ ]=

$$\frac{2}{\lambda^2} \quad \text{if } \text{Re}[\lambda] > 0$$

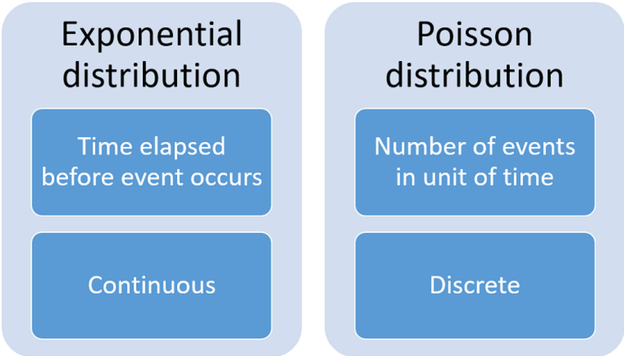
How the distribution is used

The exponential distribution is often used to answer in probabilistic terms questions such as:

- How much time will elapse before an earthquake occurs in a given region?
- How long do we need to wait until a customer enters our shop?
- How long will it take before a call center receives the next phone call?
- How long will a piece of machinery work without breaking down?

All these questions concern the time we need to wait before a given event occurs.

If this waiting time is unknown, it is often appropriate to think of it as a random variable having an exponential distribution.



Normal Distribution

In[ ]:=
```
PDF[NormalDistribution[μ, σ]]
```

Out[ ]=

$$\text{Function}\left[x, \frac{e^{-\frac{(x-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi}\,\sigma}\right]$$

In[ ]:=
```
d = NormalDistribution[μ, σ]
```

Out[ ]=

NormalDistribution[μ, σ]

In[ ]:=
```
PDF[d, x]
```

Out[ ]=

$$\frac{e^{-\frac{(x-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi}\,\sigma}$$

In[∘]:= PDF[d, x]

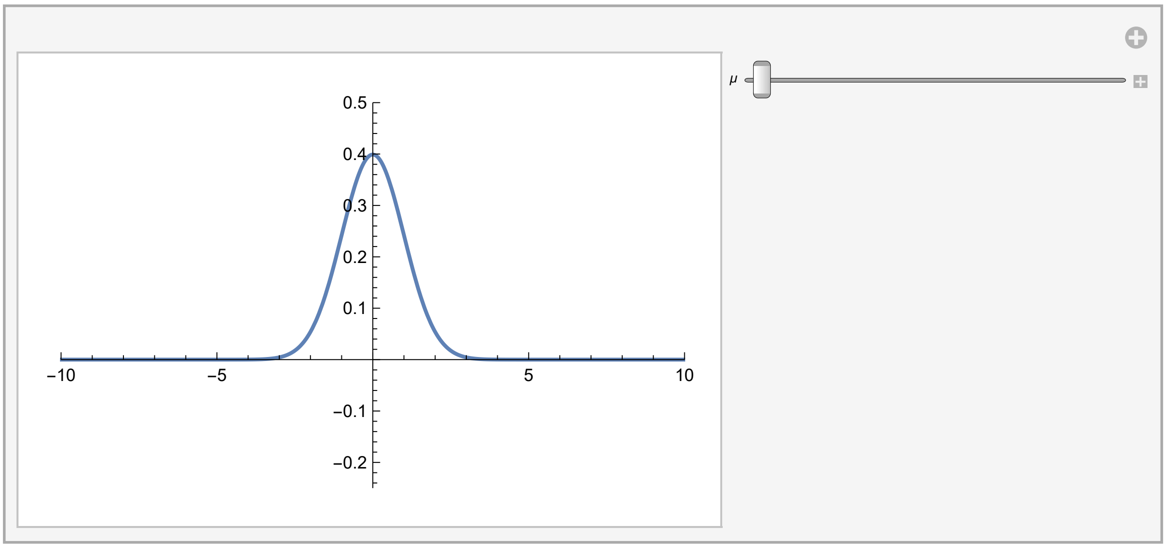
Out[∘]=

$$\frac{e^{-\frac{(x-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi}\,\sigma}$$

In[∘]:= Manipulate[

$$\text{Plot}\left[\frac{e^{-\frac{(x-\mu)^2}{2\times 1^2}}}{\sqrt{2\pi}\,1}, \{x, -10, 10\}, \text{PlotRange} \to \{\{-10, 10\}, \{-0.25, 0.5\}\}\right], \{\mu, 0, 5\}\Big]$$

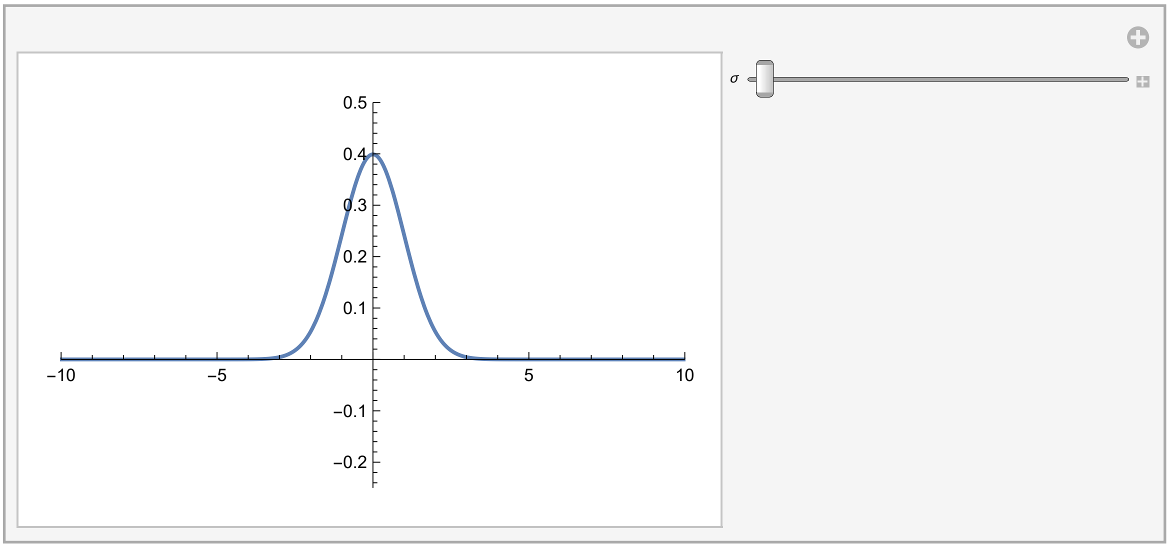
Out[∘]=



In[∘]:= Manipulate[

$$\text{Plot}\left[\frac{e^{-\frac{(x-0)^2}{2\sigma^2}}}{\sqrt{2\pi}\,\sigma}, \{x, -10, 10\}, \text{PlotRange} \to \{\{-10, 10\}, \{-0.25, 0.5\}\}\right], \{\sigma, 1, 5\}\Big]$$

Out[∘]=

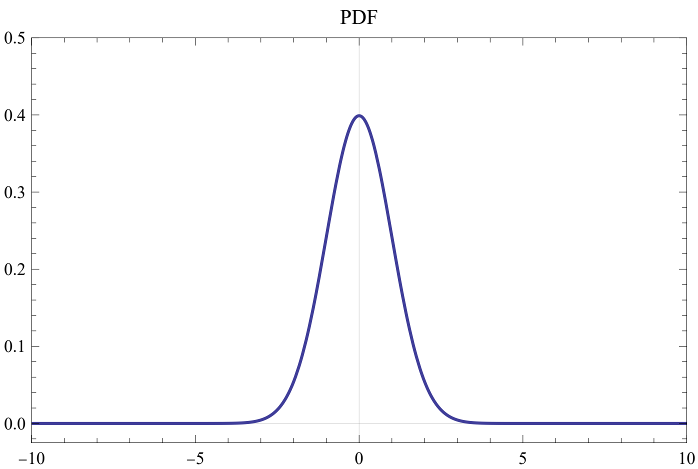


In[ ]:=

```
pdf = Plot[E^(-(x-0)^2/(2×1^2))/(Sqrt[2 π] 1), {x, -10, 10}, PlotLabel → "PDF", PlotStyle → ColorData[1, 1],
  PlotTheme → "Scientific", ImageSize → Medium, PlotRange → {{-10, 10}, {-0.025, 0.5}}]
```

$$\text{pdf} = \text{Plot}\left[\frac{e^{-\frac{(x-0)^2}{2\times 1^2}}}{\sqrt{2\pi}\ 1}, \{x, -10, 10\}, \ldots\right]$$

Out[ ]=



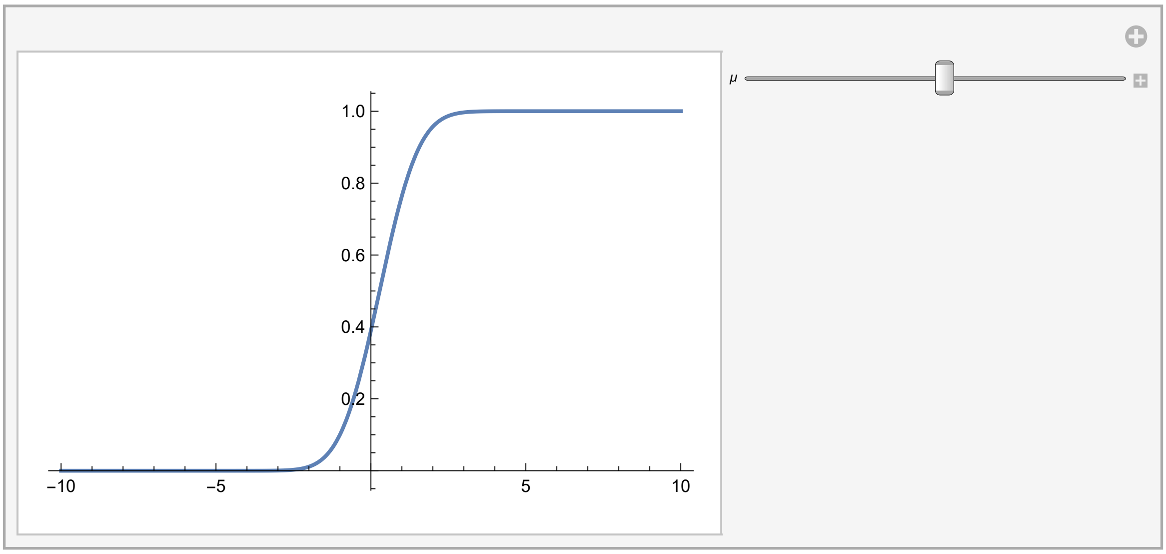
In[∘]:= CDF[d, x]

Out[∘]=

$$\frac{1}{2}\,\mathrm{Erfc}\left[\frac{-x+\mu}{\sqrt{2}\,\sigma}\right]$$

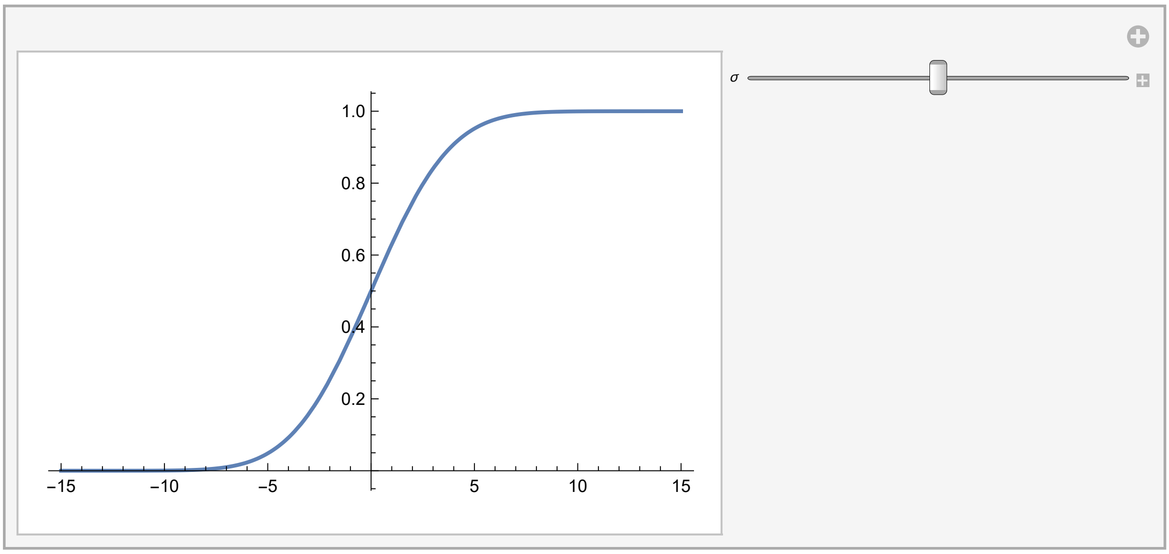
In[∘]:= $\mathrm{Manipulate}\left[\mathrm{Plot}\left[\frac{1}{2}\,\mathrm{Erfc}\left[\frac{-x+\mu}{\sqrt{2}\;1}\right],\ \{x,\ -10,\ 10\}\right],\ \{\mu,\ -5.31623,\ 5.31623\}\right]$

Out[∘]=



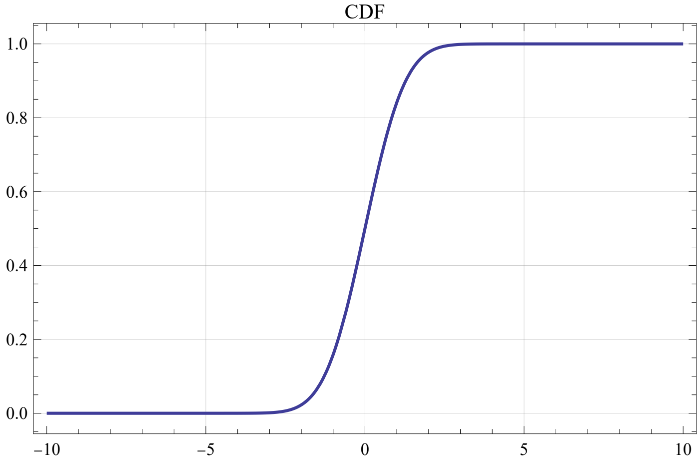
In[∘]:= $\mathrm{Manipulate}\left[\mathrm{Plot}\left[\frac{1}{2}\,\mathrm{Erfc}\left[\frac{-x+0}{\sqrt{2}\;\sigma}\right],\ \{x,\ -15,\ 15\}\right],\ \{\sigma,\ 1,\ 5\}\right]$

Out[∘]=



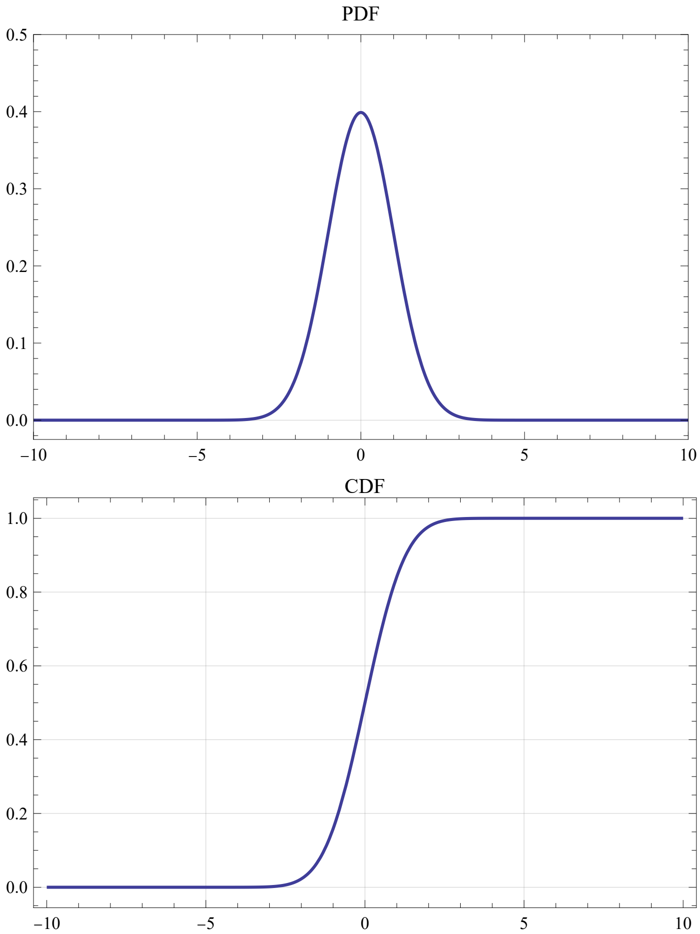
In[ ]:=
```
cdf = Plot[1/2 Erfc[(-x + 0)/(√2 1)], {x, -10, 10}, PlotStyle → ColorData[1, 1], PlotLabel → "CDF",
  GridLines → Automatic, PlotTheme → "Scientific", ImageSize → Medium]
```

Out[ ]=



In[•]:=
```
Grid[{{pdf}, {cdf}}]
```

Out[•]=



Demonstration (Mean)

```
(* (1) Write the expression for the Mean:  ∑_{i=0}^n f(x_i) × (x_i)  *)
```

$$
\begin{aligned}
\mathrm{E}[X] &= \int_{-\infty}^{\infty} x f_X(x)\,dx \\
&= (2\pi)^{-1/2} \int_{-\infty}^{\infty} x \exp\left(-\frac{1}{2}x^2\right)dx \\
&= (2\pi)^{-1/2} \int_{-\infty}^{0} x \exp\left(-\frac{1}{2}x^2\right)dx + (2\pi)^{-1/2} \int_{0}^{\infty} x \exp\left(-\frac{1}{2}x^2\right)dx \\
&= (2\pi)^{-1/2}\left[-\exp\left(-\frac{1}{2}x^2\right)\right]_{-\infty}^{0} + (2\pi)^{-1/2}\left[-\exp\left(-\frac{1}{2}x^2\right)\right]_{0}^{\infty} \\
&= (2\pi)^{-1/2}[-1+0] + (2\pi)^{-1/2}[0+1] \\
&= (2\pi)^{-1/2} - (2\pi)^{-1/2} \\
&= 0
\end{aligned}
$$

https://www.statlect.com/probability-distributions/normal-distribution

Demonstration (Variance)

```
(* (1) Write the expression for the "Squared Mean":  ∑_{i=0}^{n} f(x_i)×(x_i)^2 *)
```

$$
\begin{aligned}
\mathrm{E}\left[X^2\right] &= \int_{-\infty}^{\infty} x^2 f_X(x)dx \\
&= (2\pi)^{-1/2} \int_{-\infty}^{\infty} x^2 \exp\left(-\frac{1}{2}x^2\right)dx \\
&= (2\pi)^{-1/2} \left\{\int_{-\infty}^{0} x\left(x\exp\left(-\frac{1}{2}x^2\right)\right)dx + \int_{0}^{\infty} x\left(x\exp\left(-\frac{1}{2}x^2\right)\right)dx\right\} \\
&= (2\pi)^{-1/2} \left\{\left[-x\exp\left(-\frac{1}{2}x^2\right)\right]_{-\infty}^{0} + \int_{-\infty}^{0} \exp\left(-\frac{1}{2}x^2\right)dx + \left[-x\exp\left(-\frac{1}{2}x^2\right)\right]_{0}^{\infty}\right. \\
&\quad \left. + \int_{0}^{\infty} \exp\left(-\frac{1}{2}x^2\right)dx\right\} \qquad \text{(integrating by parts)} \\
&= (2\pi)^{-1/2} \left\{(0-0) + (0-0) + \int_{-\infty}^{0} \exp\left(-\frac{1}{2}x^2\right)dx + \int_{0}^{\infty} \exp\left(-\frac{1}{2}x^2\right)dx\right\} \\
&= (2\pi)^{-1/2} \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2}x^2\right)dx \\
&= \int_{-\infty}^{\infty} f_X(x)dx = 1 \qquad \text{(the integral of a pdf over its support is equal to 1)}
\end{aligned}
$$

$$\mathrm{E}[X]^2 = 0^2 = 0$$

$$\mathrm{Var}[X] = \mathrm{E}\left[X^2\right] - \mathrm{E}[X]^2 = 1 - 0 = 1$$

https://www.statlect.com/probability-distributions/normal-distribution

Properties

https://www.ztable.net/

Properties

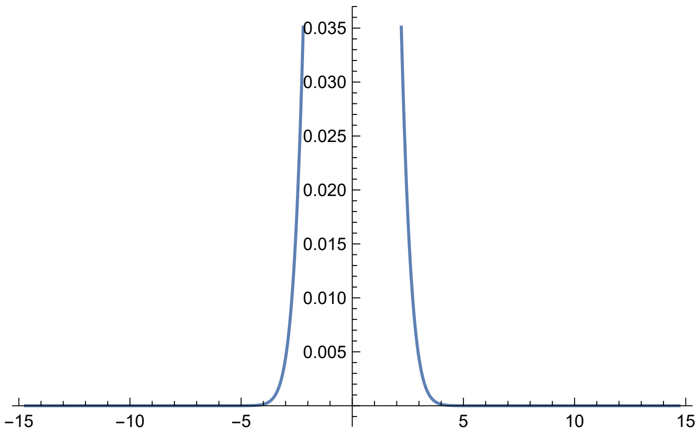
In[∘]:= PDF[NormalDistribution[0, 1], x]

Out[∘]=

$$\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}$$

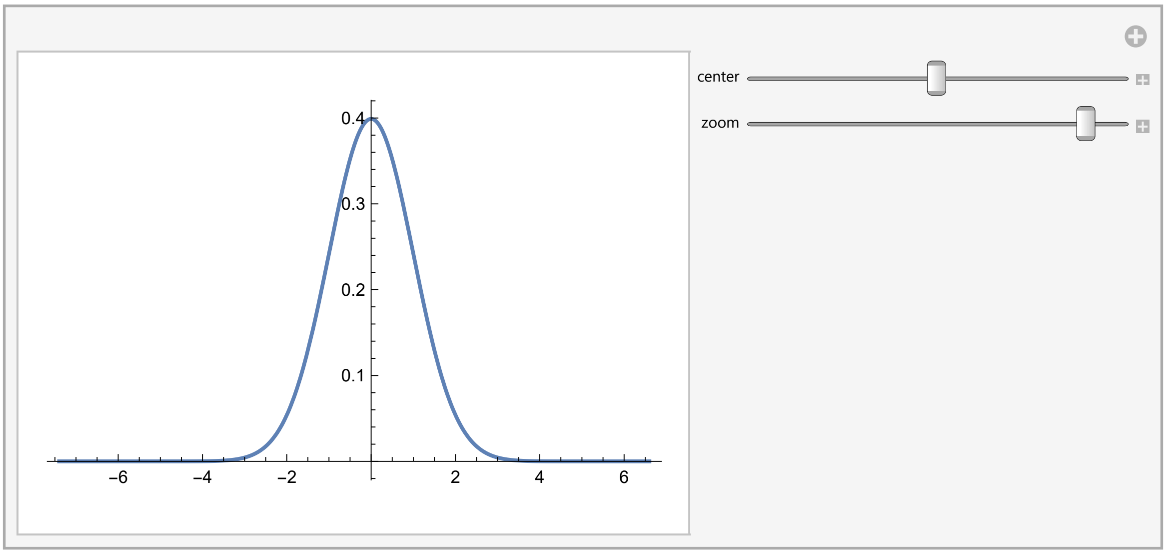
In[∘]:= Plot$\left[\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}, \{x, -14.6969, 14.6969\}\right]$

Out[∘]=



In[∘]:= ▶ interactive plot %12

Out[∘]=



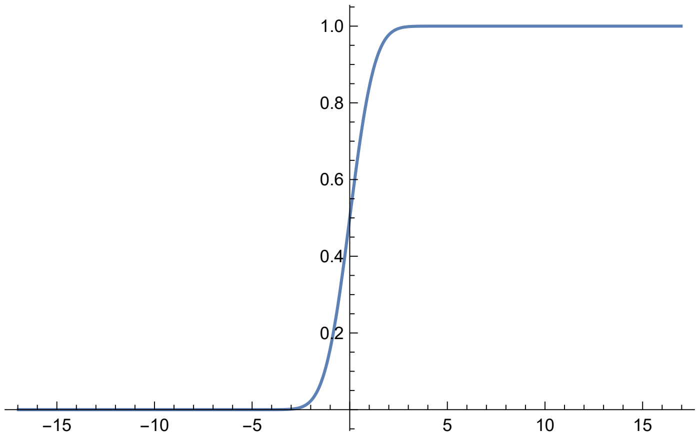
In[∘]:= CDF[NormalDistribution[0, 1], x]

Out[∘]=

$$\frac{1}{2}\,\mathrm{Erfc}\left[-\frac{x}{\sqrt{2}}\right]$$

In[∘]:=
$$\text{Plot}\left[\frac{1}{2}\,\text{Erfc}\left[-\frac{x}{\sqrt{2}}\right],\ \{x,\ -16.9706,\ 16.9706\}\right]$$

Out[∘]=



In[ ]:= **CDF[NormalDistribution[0, 1], 1]**

Out[ ]=

$$\frac{1}{2}\,\mathrm{Erfc}\left[-\frac{1}{\sqrt{2}}\right]$$

In[ ]:= **N[$\frac{1}{2}$ Erfc[$-\frac{1}{\sqrt{2}}$]]**

Out[ ]=

0.841345

In[ ]:= **CDF[NormalDistribution[0, 1], 1.5]**

Out[ ]=

0.933193

In[ ]:= **CDF[NormalDistribution[0, 1], 1.54]**

Out[ ]=

0.93822

In[ ]:= **CDF[NormalDistribution[0, 1], -1.92]**

Out[ ]=

0.0274289

In[ ]:= **1 - CDF[NormalDistribution[0, 1], 1.92]**

Out[ ]=

0.0274289

In[ ]:= **CDF[NormalDistribution[0, 1], 2] - CDF[NormalDistribution[0, 1], -1]**

Out[ ]=

$$-\frac{1}{2}\,\mathrm{Erfc}\left[\frac{1}{\sqrt{2}}\right]+\frac{1}{2}\,\mathrm{Erfc}\left[-\sqrt{2}\right]$$

In[ ]:= **N[$-\frac{1}{2}$ Erfc[$\frac{1}{\sqrt{2}}$] + $\frac{1}{2}$ Erfc[$-\sqrt{2}$]]**

https://www.wolframalpha.com/input?i=normal+distribution+calculator&assumption=%7B%22F%22%2C+%22NormalProbabilities%22%2C+%22mu%22%7D+-%3E%220%22&assumption=%7B%22F%22%2C+%22NormalProbabilities%22%2C+%22sigma%22%7D+-%3E%221%22&assumption=%7B%22F%22%2C+%22NormalProbabilities%22%2C+%22pr%22%7D+-%3E%220.95%22