STAT 3602 Statistical Inference

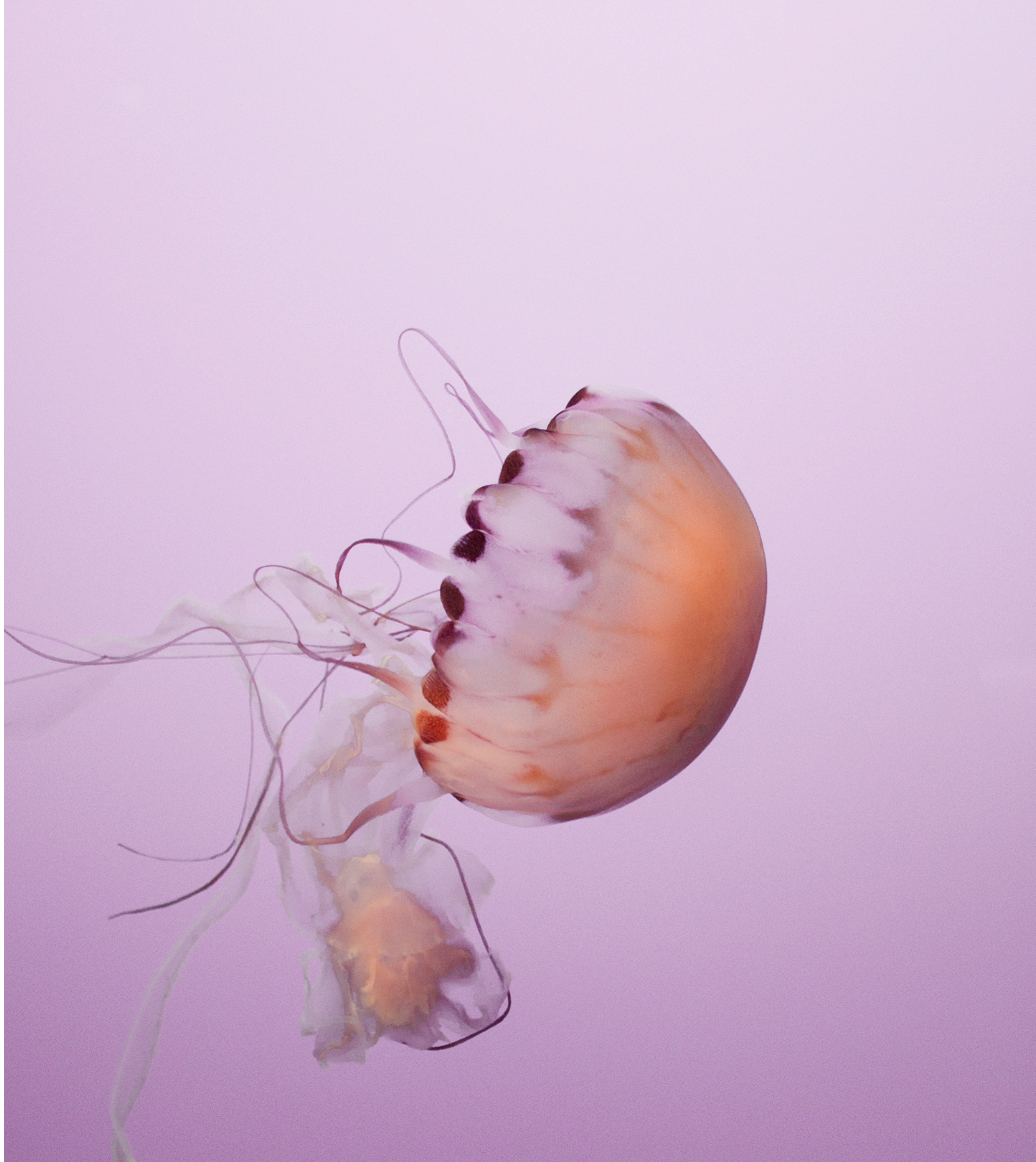
Example Class 8

Chapter 5 Estimation I

LIU Chen
Department of Statistics and Actuarial Science, HKU
10/11/2020

Outline
Example Class 8

- Review: Rao-Blackwell Theorem.
- Q1: Course Selection
- Q2: Uniform Distribution
- Q3: Quality Control



Review

Rao-Blackwell Theorem

$T(X)$ is sufficient for θ, the loss function $L(\theta; a)$ is convex in a and $\rho(X)$ is an estimator of $\phi(\theta)$ with finite expectation and risk : $E_\theta(\rho(X))$ and $E_\theta(L(\theta, \rho(X)))$ is finite.

$$\rho^*(t) = E(\rho(X)|T(X))$$

(a) $\rho^*(T(X))$ is an estimator of $\phi(\theta)$, such that

$$E_\theta[L(\theta, \rho^*(T(X)))] \leq E_\theta(L(\theta, \rho(X)))$$

Review

Rao-Blackwell Theorem

(b) Specifically, if $\rho(X)$ is unbiased and $T(X)$ is complete,

   i. $\rho^*(T(X))$ is the unique unbiased estimator of $\phi(\theta)$which is a function of $T(X)$

   ii. $E_\theta[L(\theta, \rho^*(T(X)))] \leq E_\theta[L(\theta, S(X))]$ for any unbiased estimator $S(X)$ of $\phi(\theta)$.

(c) UMVUE (uniformly minimum variance unbiased estimator)

if we take $L(\theta; a) = (\theta - a)^2$, and $\rho(X)$ is unbiased , $E_\theta[\rho(X)^2] < \infty$, $T(X)$ is complete.

   i. $Var_\theta(\rho^*(T(X))) \leq Var_\theta(S(X))$ for any other unbiased estimator $S(X)$ of $\phi(\theta)$, which means $\rho^*(T(X))$ is the unique UMVU eatimator of $\phi(\theta)$.

Review

We can use Rao-Blackwell Theorem to

- Reduce the risk of an estimator given a sufficient statistic by taking expectation;
- Find the UMVU estimator given a complete sufficient statistic and an unbiased estimator by either finding the expected value of any unbiased estimator conditional on T, or finding an unbiased estimator which is a function of T.

Problems

Exercise 1: Background

Two students are each required to enroll in one and only one of the following four classes:

course C1 instructed by lecturer L1, course C1 instructed by lecturer L2,

course C2 instructed by lecturer L1, course C2 instructed by lecturer L2.

Assume the two students make their choices independently, each having probabilities α and β to choose course C1 and lecturer L1 respectively. The choice of courses and the choice of lecturers are independent of each other.

For $i, j = 1, 2$, define $X_{ij} = 1$ if the first student selects course Ci and lecturer Lj, and $X_{ij} = 0$ otherwise. The random variables Y_{ij} are similarly defined for the second student. Define

$$V = X_{11} + X_{12} + Y_{11} + Y_{12} \textbf{ and } W = X_{11} + X_{21} + Y_{11} + Y_{21}$$

Exercise 1: Problems

(a) Show that the likelihood function can be written as

$$l(\alpha, \beta) = \alpha^V (1-\alpha)^{2-V} \beta^W (1-\beta)^{2-W}$$

(b) Show that (V, W) is complete sufficient for the parameters (α, β).

(c) Suppose that α, β satisfy $\dfrac{\beta}{1-\beta} = (\dfrac{\alpha}{1-\alpha})^2$,

i. Show that $V + 2W$ is complete sufficient for the parameters (α, β);

ii. Calculate the values of the statistic $V + 2W$ for each of the 16 outcomes.

iii. By considering the probabilities of the 16 cases in (ii), show that

$$P(X_1 = 1 | V + 2W = 4) = 1/3$$

iv. If it is observed that $V + 2W = 4$, calculate the uniformly minimum variance unbiased (UMVU) estimate of the parameter $\theta = \alpha\beta$.

Exercise 1 (a) & (b): Solution

(a) **The likelihood function is**

$$
\begin{aligned}
\ell(\alpha,\beta) &= (\alpha\beta)^{X_{11}}(\alpha(1-\beta))^{X_{12}}((1-\alpha)\beta)^{X_{21}}((1-\alpha)(1-\beta))^{X_{22}}\times \\
&\qquad (\alpha\beta)^{Y_{11}}(\alpha(1-\beta))^{Y_{12}}((1-\alpha)\beta)^{Y_{21}}((1-\alpha)(1-\beta))^{Y_{22}} \\
&= \alpha^{V}(1-\alpha)^{2-V}\beta^{W}(1-\beta)^{2-W}.
\end{aligned}
$$

Exercise 1 (a) & (b): Solution

(a) **The likelihood function is**

$$\begin{aligned}\ell(\alpha,\beta) &= (\alpha\beta)^{X_{11}}(\alpha(1-\beta))^{X_{12}}((1-\alpha)\beta)^{X_{21}}((1-\alpha)(1-\beta))^{X_{22}}\times \\ &\quad (\alpha\beta)^{Y_{11}}(\alpha(1-\beta))^{Y_{12}}((1-\alpha)\beta)^{Y_{21}}((1-\alpha)(1-\beta))^{Y_{22}} \\ &= \alpha^{V}(1-\alpha)^{2-V}\beta^{W}(1-\beta)^{2-W}.\end{aligned}$$

(b) It can be shown that (V,W) belongs to **the exponential family,**

$$\ell(\alpha,\beta) = (1-\alpha)^2(1-\beta)^2\exp\left\{\log\left(\frac{\alpha}{1-\alpha}\right)V + \log\left(\frac{\beta}{1-\beta}\right)W\right\}$$

with natural parameter $\left(\log\left(\frac{\alpha}{1-\alpha}\right),\log\left(\frac{\beta}{1-\beta}\right)\right)$. As the natural parameter space contains an open rectangle, (V,W) is a complete sufficient statistic.

Exercise 1 (C): Solution

i. With the equation given in the question, the likelihood function can be written as

$$\ell(\alpha, \beta) = (1-\alpha)^2 (1-\beta)^2 \exp\left\{(V+2W)\log\left(\frac{\alpha}{1-\alpha}\right)\right\}.$$

Again, as the natural parameter space contains an open rectangle (an open interval), $V+2W$ is a complete sufficient statistic.

ii. The values of $V+2W$ is listed below

	$Y_{11}=1$	$Y_{12}=1$	$Y_{21}=1$	$Y_{22}=1$
$X_{11}=1$	6	4	5	3
$X_{12}=1$	4	2	3	1
$X_{21}=1$	5	3	4	2
$X_{22}=1$	3	1	2	0

Exercise 1 (C): Solution

ii. The values of $V + 2W$ is listed below

	$Y_{11} = 1$	$Y_{12} = 1$	$Y_{21} = 1$	$Y_{22} = 1$
$X_{11} = 1$	6	4	5	3
$X_{12} = 1$	4	2	3	1
$X_{21} = 1$	5	3	4	2
$X_{22} = 1$	3	1	2	0

iii. The conditional probability can be calculated by

$$
\begin{aligned}
P\left(X_{11} = 1 \mid V + 2W = 4\right) &= \frac{P\left(X_{11} = 1, V + 2W = 4\right)}{P\left(V + 2W = 4\right)} \\
&= P\left(X_{11} = 1, Y_{12} = 1\right) / \{P\left(X_{11} = 1, Y_{12} = 1\right) + \\
&\qquad P\left(X_{12} = 1, Y_{11} = 1\right) + P\left(X_{21} = 1, Y_{21} = 1\right)\} \\
&= \frac{(\alpha\beta)\left(\alpha\left(1 - \beta\right)\right)}{2\left(\alpha\beta\right)\alpha\left(1 - \beta\right) + \left(1 - \alpha\right)^2 \beta^2} \\
&= \frac{1}{2 + 1} = \frac{1}{3}.
\end{aligned}
$$

Exercise 1 (C): Solution

iv. As $\mathbb{E}(X_{11}) = \alpha\beta = \theta$, X_{11} serves as an unbiased estimator of θ. The UMVU estimator is

$$\begin{aligned} \mathbb{E}[X_{11} \mid V + 2W = 4] &= P(X_{11} = 1 \mid V + 2W = 4) \\ &= \frac{1}{3}. \end{aligned}$$

Exercise 2

Suppose $X_1, \ldots, X_n$ is a random sample of size $n(\geq 2)$ from a uniform distribution on the closed interval $[a, b]$ with $a < b \in \mathbb{R}$. Given $T = (\min_i X_i, \max_i X_i)$ is a complete sufficient statistic for $\theta = (a, b)$. Find the UMVU estimate for θ.

Exercise 2: Solution

The pdf of $\min_i \{X_i\}$:

$$p(\min_i \{X_i\} = t) = \frac{d}{dt} P\left(\min_i \{X_i\} \le t\right) = -\frac{1}{(b-a)^n}\frac{d}{dt}(b-t)^n$$

$$\mathbb{E}_\theta\left[\min_i \{X_i\}\right] = -\int_a^b \frac{t}{(b-a)^n}\frac{d}{dt}(b-t)^n\,dt = \frac{an+b}{n+1},$$

The pdf of $\max_i \{X_i\}$:

$$p(\max_i \{X_i\} = t) = \frac{d}{dt} P\left(\max_i \{X_i\} \le t\right) = \frac{1}{(b-a)^n}\frac{d}{dt}(t-a)^n$$

$$\mathbb{E}_\theta\left[\max_i \{X_i\}\right] = \int_a^b \frac{t}{(b-a)^n}\frac{d}{dt}(t-a)^n\,dt = \frac{bn+a}{n+1},$$

$$\begin{pmatrix} \mathbb{E}_\theta(T_1) \\ \mathbb{E}_\theta(T_2) \end{pmatrix} = \begin{pmatrix} \frac{na+b}{n+1} \\ \frac{a+nb}{n+1} \end{pmatrix}$$

$$= \frac{1}{n+1}\begin{pmatrix} n & 1 \\ 1 & n \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}.$$

Exercise 2: Solution

AFTER REARRANGEMENT, WE HAVE

$$
\begin{aligned}
\begin{pmatrix} a \\ b \end{pmatrix} &= (n+1)\begin{pmatrix} n & 1 \\ 1 & n \end{pmatrix}^{-1}\begin{pmatrix} \mathbb{E}_\theta(T_1) \\ \mathbb{E}_\theta(T_2) \end{pmatrix} \\
&= (n+1)\frac{1}{n^2-1}\begin{pmatrix} n & -1 \\ -1 & n \end{pmatrix}\begin{pmatrix} \mathbb{E}_\theta(T_1) \\ \mathbb{E}_\theta(T_2) \end{pmatrix} \\
&= \frac{1}{n-1}\begin{pmatrix} n & -1 \\ -1 & n \end{pmatrix}\begin{pmatrix} \mathbb{E}_\theta(T_1) \\ \mathbb{E}_\theta(T_2) \end{pmatrix},
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{E}_\theta\left[\frac{nT_1 - T_2}{n-1}\right] &= a, \\
\mathbb{E}_\theta\left[\frac{nT_2 - T_1}{n-1}\right] &= b.
\end{aligned}
$$

So $\left(\frac{1}{n-1}\left(n \min_i X_i - \max_i X_i\right), \frac{1}{n-1}\left(n \max_i X_i - \min_i X_i\right)\right)$ is the unbiased estimator for (a, b) and is a function of complete sufficient statistic T. Therefore, it is the UMVU estimate for (a, b).

Exercise 3

Independent factory-produced items are packed in boxes each containing k items. The probability that an item is in working order is $\theta,\ 0 < \theta < 1.$

A sample of n boxes are chosen for testing, and X_i , the number of working items in the ith box, is noted. Thus, $X_1, \ldots, Xn$ are sample from a binomial distribution, $\mathrm{Bin}(k, \theta)$, with index k and parameter θ.

It is required to estimate the probability, θ^k, that all items in a box are in working order. Find the UMVU estimator.

Exercise 3: Solution I

$\mathcal{X} = (X_1, \ldots, X_n)$ belongs to the exponential family with natural statistic $T = X_1 + \cdots + X_n$, and the natural parameter space contains an open rectangle (verify it yourself if you are not sure). Therefore, T is a complete sufficient statistic. To find the UMVU estimator for θ^k, we start with any unbiased estimator. Since $P(X_i = k) = \theta^k$, we use the estimator

$$\rho(\mathcal{X}) = \mathbf{1}\{X_1 = k\}$$

as

$$\mathbb{E}(\mathbf{1}\{X_1 = k\}) = P(X_1 = k) = \theta^k.$$

Exercise 3: Solution II

By the Rao-Blackwell Theorem, we know that $\rho^*(T)$ is the UMVU estimator with $\rho^*(t) = \mathbb{E}(\rho(\mathcal{X}) \mid T = t)$.

$$
\begin{aligned}
\mathbb{E}(\rho(\mathcal{X}) \mid T = t) &= P\left(X_1 = k \mid \sum X_i = t\right) \\
&= \frac{P(X_1 = k, \sum X_i = t)}{P(\sum X_i = t)} \\
&= \frac{P(X_1 = k)\, P(X_2 + \cdots + X_n = t - k)}{P(\sum X_i = t)} \\
&= \begin{cases} \frac{\theta^k \binom{(n-1)k}{t-k} \theta^{t-k} (1-\theta)^{nk-t}}{\binom{nk}{t} \theta^t (1-\theta)^{nk-t}}, & \text{for } t \geq k, \\ 0, & \text{for } t < k. \end{cases} \\
&= \begin{cases} \frac{[(n-1)k]!}{(nk)!} \frac{t!}{(t-k)!}, & \text{for } t \geq k, \\ 0, & \text{for } t < k. \end{cases}
\end{aligned}
$$

Thus,

$$\rho^*(T) = \frac{[(n-1)k]!}{(nk)!} \frac{T!}{(T-k)!} \mathbf{1}\{T \geq k\}$$

is the UMVU estimator of θ^k.