THE UNIVERSITY OF HONG KONG
DEPARTMENT OF STATISTICS AND ACTUARIAL SCIENCE

STAT3602 Statistical Inference

Example Class 9 (2020-2021 1st Semester)

Review

1. Information Inequality under regularity assumption and also assume $I(\theta)$ is positive definite,

$$Var_\theta(T) \geq \alpha(\theta)^T I(\theta)^{-1}\alpha(\theta)^T$$

where T is a statistic with $E_\theta(T^2) < \infty$ and $\alpha(\theta) = [\frac{\partial}{\partial\theta_1}E_\theta(T), ..., \frac{\partial}{\partial\theta_k}E_\theta(T)]^T$.

   (a) Note: if an unbiased estimator has a variance equal to this lower bound, and the assumptions in the theorem is satisfied, then it must be a UMVU estimator.

   (b) Cramer-Rao Lower Bound: special case when $k = 1$ and $\phi(\theta) = \theta$.

2. MLE

$$\hat{\theta}_n \text{ maximizes } l_n(\theta|x) \quad or \quad S_n(\theta|x)$$

   (a) if $\hat{\theta}_n$ is the MLE of θ_0, then $\hat{\phi}_n = \phi(\hat{\theta}_n)$ is the MLE of $\phi(\theta_0)$.

   (b) property of MLE

   Theorem. Subject to regularity conditions on $p(x|\theta)$, we have

   (i) $\hat{\theta}_n$ *converges in probability* to θ_0,

   (ii) $n^{1/2}(\hat{\theta}_n - \theta_0)$ *converges in distribution* to $N(\mathbf{0}, \mathscr{I}(\theta_0)^{-1})$,

   (iii) $n^{-1/2}\boldsymbol{U}(\theta_0)$ converges in distribution to $N(\mathbf{0}, \mathscr{I}(\theta_0))$,

   where $\mathscr{I}(\theta) = n^{-1}I(\theta) = -\mathbb{E}\left[\frac{\partial^2 \ln p(X_1|\theta)}{\partial\theta\partial\theta^\top}\right]$ has (i,j)th entry $\mathscr{I}_{ij}(\theta) = -\mathbb{E}_\theta\left[\frac{\partial^2 \ln p(X_1|\theta)}{\partial\theta_i\partial\theta_j}\right]$.

   (c) Note: if the regularity conditions fail, $\hat{\theta}_n$ may not be asymptotically normal.

Problem

1. A team of 8 interviewers were recruited to conduct a survey in which each interviewer asked for the opinion (positive, negative, neutral) of 10 respondents about a political issue. The following table summarizes the counts of the three response categories found from the survey:

Interviewer	1	2	3	4	5	6	7	8	Total
Positive	2	3	2	1	0	4	2	2	16
Negative	5	7	6	6	4	6	7	7	48
Neutral	3	0	2	3	6	0	1	1	16

Let X_i and Y_i be the counts of the positive and negative responses obtained by the i-th interviewer respectively, for $i = 1, 2, ..., 8$. Assume that (X_i, Y_i) follows a multinomial distribution with mass function

$$f(x, y|p, q) = \begin{cases} \frac{10!}{x!y!(10-x-y)!} p^x q^y (1-p-q)^{10-x-y}, & x, y \in (0, 1, ..., 10), x + y \leq 10 \\ 0, & otherwise \end{cases}$$

for unknown parameters p and q which can be interpreted as the probabilities that a respondent gives positive and negative responses respectively.

(a) Give the loglikelihood function of p and q based on the observed data.

(b) Find the maximum likelihood estimator of (p, q).

(c) Calculate the expected Fisher information matrix for p, q.

(d) Quoting the large-sample properties of MLE, describe the asymptotic distribution of $(\hat{p}, \hat{q})$

2. Let $X = (X_1, \ldots, X_n)$ be a random sample of size $n \geq 3$ from the exponential distribution with density $f(x) = \theta e^{-\theta x}$ (mean=$1/\theta$ and variance=$1/\theta^2$). Obtain the maximum likelihood estimator $\hat{\theta}_n$ based on the sample of size n for θ.

   Find an unbiased estimator which is a function of $\hat{\theta}_n$.

   Calculate the Cramer-Rao Lower Bound for the variance of the unbiased estimator. Is the bound attained?

   (Hint: Sum of n i.i.d $Exp(\theta)$ distributed random variables follows Gamma(n, θ). If $Y \sim$ Gamma(k, θ), then $Y^{-1} \sim$ InverseGamma(k, θ) with mean $= \theta/(k-1)$ and variance $= \theta^2/[(k-1)^2(k-2)]$)