THE UNIVERSITY OF HONG KONG
DEPARTMENT OF STATISTICS AND ACTUARIAL SCIENCE

STAT3602 Statistical Inference

(2020-2021 First Semester)

Example Class 2

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Exercise 1

When Harry met Sally — *a story about fate* (緣份)

The year is 2019. You are Sally, a flashy schoolgirl and a frequent moviegoer. One day you are standing in a long queue, waiting to buy a ticket for a popular film, which is 120 minutes long and will start immediately. Your experience tells you that you need to wait $X \in (0, 120)$ minutes before you can successfully buy a ticket, where X is a random variable with the density function

$$f(x|\theta) = \frac{1}{120\,\theta}\left(1 - \frac{x}{120}\right)^{(1-\theta)/\theta}, \quad x \in (0, 120),$$

for some unknown parameter $\theta \in (0, 1)$. Thus, you will miss the first X minutes of the film if you stay in the queue.

At this juncture, your schoolmate Harry suddenly shows up with a ticket in his hands.

> *"Take my ticket and go watch the film," said Harry. "It's only 80 dollars."*

The normal price of the ticket is 60 dollars.

To help yourself decide whether to buy Harry's ticket, you calculate a loss function as follows. Suppose that each minute of the film is worth 0.5 dollar. If you stay in the queue, you will lose $60 - 0.5(120 - X) = 0.5X$ dollars. If you buy Harry's ticket, you will lose $80 - 0.5(120) = 20$ dollars. Your loss function $L(\theta, a)$ can be defined as $\mathbb{E}_\theta[0.5X]$ if you take the action $a = a_0$, staying in the queue, and the constant 20 if you take the action $a = a_1$, buy Harry's ticket.

Let $X_1, \ldots, X_n$ be independent observations following the common density function $f(x|\theta)$. You may rely on the data $(X_1, \ldots, X_n)$ for your decision.

a. Write down an expression for the risk function $R(\theta, d)$ of a decision rule d. Hence show that a minimax rule is to always buy Harry's ticket.

Exercise 2

Zixia (紫霞) suspects that Joker (至尊寶), a man whom she loves, is having an affair with another woman. She has got so furious that she draws her sword

Figure 1: Zixia and Joker in A Chinese Odyssey Part 1-Pandora's Box(西遊記第壹佰零壹回之月光寶盒).

against Joker's throat at a speed of X cm per second, threatening to kill him.

Zixia's love for Joker is indexed by an unknown parameter $\theta \in (0, \infty)$. The greater the value of θ, the deeper is her love for him. It is known that X is a random variable with the density function

$$f(x|\theta) = \theta (1+x)^{-\theta-1}, x \in (0, \infty). \tag{1}$$

Based on the observed value of X, Joker needs to decide quickly whether to tell Zixia a lie or not. If he keeps his mouth shut, he will be hurt by the sword, so that his body will suffer a loss equal to $1/\theta$, but no harm will be done to his soul. If he tells a lie, he can reduce his bodily loss by one half but his soul will suffer a loss of one unit, so that the total loss becomes $1/(2\theta) + 1$.

a. Define the loss function for Joker's decision problem.

b. Show that the risk function of a decision rule d is given by

$$R(\theta, d) = \frac{1}{\theta} + \left(1 - \frac{1}{2\theta}\right) \mathbb{P}_\theta (d(X) = a), \tag{2}$$

for a particular action a. Specify the action rule a.

c. Deduce from b that $\sup_{\theta>0} R(\theta, d) = \infty$ for any decision rule d.

d. For $\forall c > 0$, define the decision rule d_c by

$$d_c(x) = \begin{cases} \text{“to lie”}, & x > c; \\ \text{“not to lie”}, & x \leq c. \end{cases}$$

(a) let c_1, c_2 be **distinct** positive constants. Show that d_{c_1} does not strictly dominate d_{c_2}.

(b) Is d_c an unbiased decision rule?