STAT3602 Statistical Inference

Example Class 10 (Hypothesis Testing)

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OUTLINE

1. Review: Hypothesis Testing
    1. Formulation
    2. Optimal Test
    3. Likelihood Ratio Test
    4. UMP test Under mlr
    5. Two-sided UMPU test under exponential Family
    6. Conditional test under exponential family
2. Three Exercises

Hypothesis Testing
Definitions

- Test function;
- Critical region; critical value
- Power function of the test
- The size of a test
- The power of a test

Hypothesis Testing

Optimal Test

- The test ψ is unbiased of size α if

$$\sup_{\theta \in \Theta_0} \mathbb{E}_\theta\big[\varphi(\boldsymbol{X})\big] = \alpha \quad \text{and} \quad \mathbb{E}_\theta\big[\varphi(\boldsymbol{X})\big] \geq \alpha \ \ \forall \theta \in \Theta_1 \setminus \Theta_0.$$

- A test ψ_0 is **uniformly most powerful [UMP]** among all tests ψ of $size \leq \alpha$ if

$$\begin{cases} \mathbb{E}_\theta\big[\varphi_0(\boldsymbol{X})\big] \leq \alpha \ \ \forall \theta \in \Theta_0, \quad \text{and} \\ \mathbb{E}_\theta\big[\varphi_0(\boldsymbol{X})\big] \geq \mathbb{E}_\theta\big[\varphi(\boldsymbol{X})\big] \ \ \forall \theta \in \Theta_1 \setminus \Theta_0 \text{ and } \forall \text{ tests } \varphi \text{ of size} \leq \alpha. \end{cases}$$

- A test ψ_0 is **uniformly most powerful unbiased [UMPU]** among all unbiased tests ψ of $size \leq \alpha$ if

$$\begin{cases} \mathbb{E}_\theta\big[\varphi_0(\boldsymbol{X})\big] \leq \alpha \ \ \forall \theta \in \Theta_0, \quad \text{and} \\ \mathbb{E}_\theta\big[\varphi_0(\boldsymbol{X})\big] \geq \mathbb{E}_\theta\big[\varphi(\boldsymbol{X})\big] \ \ \forall \theta \in \Theta_1 \setminus \Theta_0 \text{ and } \forall \boxed{\text{unbiased tests } \varphi} \text{ of size} \leq \alpha. \end{cases}$$

Likelihood Ratio Test

- Definition. The likelihood ratio for the test of $H_0 : \theta \in \Theta_0$ vs $H_1 : \theta \in \Theta_1$, given data X, is defined to be

$$\Lambda_{\boldsymbol{X}}\left(H_0, H_1\right) = \frac{\sup_{\theta \in \Theta_1} \ell_{\boldsymbol{X}}(\theta)}{\sup_{\theta \in \Theta_0} \ell_{\boldsymbol{X}}(\theta)}$$

- where $\ell_{\boldsymbol{X}}(\theta)$ is the likelihood function.
- The likelihood ratio may be viewed as the odds of H_1 against H_0.

Likelihood Ratio Test

Neyman-Pearson

Problem: Test $\underbrace{H_0 : \theta = \theta_0}_{\text{simple hypothesis}}$ vs $\underbrace{H_1 : \theta = \theta_1}_{\text{simple hypothesis}}$

LR test $\varphi_0(\boldsymbol{X}) = \mathbf{1}\{\Lambda_{\boldsymbol{X}}(H_0, H_1) > c\}$ is <u>most powerful</u>

among all tests of size $\leq \mathbb{E}_{\theta_0}[\varphi_0(\boldsymbol{X})]$ $\Big(= \text{size of } \varphi_0 \Big)$

i.e. $\forall \varphi$ with $\mathbb{E}_{\theta_0}[\varphi(\boldsymbol{X})] \leq \mathbb{E}_{\theta_0}[\varphi_0(\boldsymbol{X})]$ $(= \text{size of } \varphi_0)$

we have $\underbrace{\mathbb{E}_{\theta_1}[\varphi(\boldsymbol{X})]}_{\text{power at } \boldsymbol{H_1 : \theta = \theta_1}} \leq \underbrace{\mathbb{E}_{\theta_1}[\varphi_0(\boldsymbol{X})]}_{\text{power at } \boldsymbol{H_1 : \theta = \theta_1}}$

UMP test under monotone likelihood ratio (mlr)

Definition

Consider $H_0 : \theta \in \Theta_0$ vs $H_1 : \theta \in \Theta_1$

Definition. Take any $\theta_0 \in \Theta_0$, $\theta_1 \in \Theta_1$:

$$\text{If } \frac{f(\boldsymbol{X}|\theta_1)}{f(\boldsymbol{X}|\theta_0)} \uparrow \text{ as } \underbrace{T(\boldsymbol{X})}_{\textbf{not depending on } \boldsymbol{\theta_0, \theta_1}} \uparrow, \quad \text{then the model has } \underline{\textbf{mlr}} \textbf{ in } \boldsymbol{T(X)}$$

(w.r.t. test of $\boldsymbol{H_0}$ vs $\boldsymbol{H_1}$)

UMP test under monotone likelihood ratio (mlr)

How can we use mlr property to find UMP test?

Theorem. Suppose model mlr in $T(\boldsymbol{X})$ w.r.t.

$$\boxed{\text{test of } \; H_0 : \theta \in \Theta_0 \quad \text{vs} \quad H_1 : \theta \in \Theta_1}$$

Define test function $\varphi_0(\boldsymbol{X}) = \mathbf{1}\{T(\boldsymbol{X}) > t_0\}$

Then: (i) φ_0 is a LR test

(ii) $\sup_{\theta \in \Theta_0} \mathbb{E}_\theta\left[\varphi_0(\boldsymbol{X})\right] \leq \inf_{\theta \in \Theta_1} \mathbb{E}_\theta\left[\varphi_0(\boldsymbol{X})\right]$

(iii) φ_0 is UMP among tests of size $\leq \sup_{\theta \in \Theta_0} \mathbb{E}_\theta\left[\varphi_0(\boldsymbol{X})\right]$

Two-sided UMPU test under exponential family

Data $\boldsymbol{X} \sim f(\boldsymbol{x}|\theta) = c(\theta)h(\boldsymbol{x})e^{\theta t(\boldsymbol{x})}$ $\quad(\theta \in \Pi$, natural parameter space$)$

Test $\to$ $\quad H_0 : \theta \in \underbrace{[\theta_1, \theta_2]}_{\subset \Pi \text{ (assumption)}}$ $\quad$ vs $\quad H_1 : \theta \notin [\theta_1, \theta_2]$

Define test: $\varphi(\boldsymbol{X}) = \mathbf{1}\{t(\boldsymbol{X}) \notin [t_1, t_2]\}$ with $\overbrace{\mathbb{E}_{\theta_1}[\varphi(\boldsymbol{X})] = \mathbb{E}_{\theta_2}[\varphi(\boldsymbol{X})] = \alpha}^{\text{2 equations to solve for } t_1, t_2}$

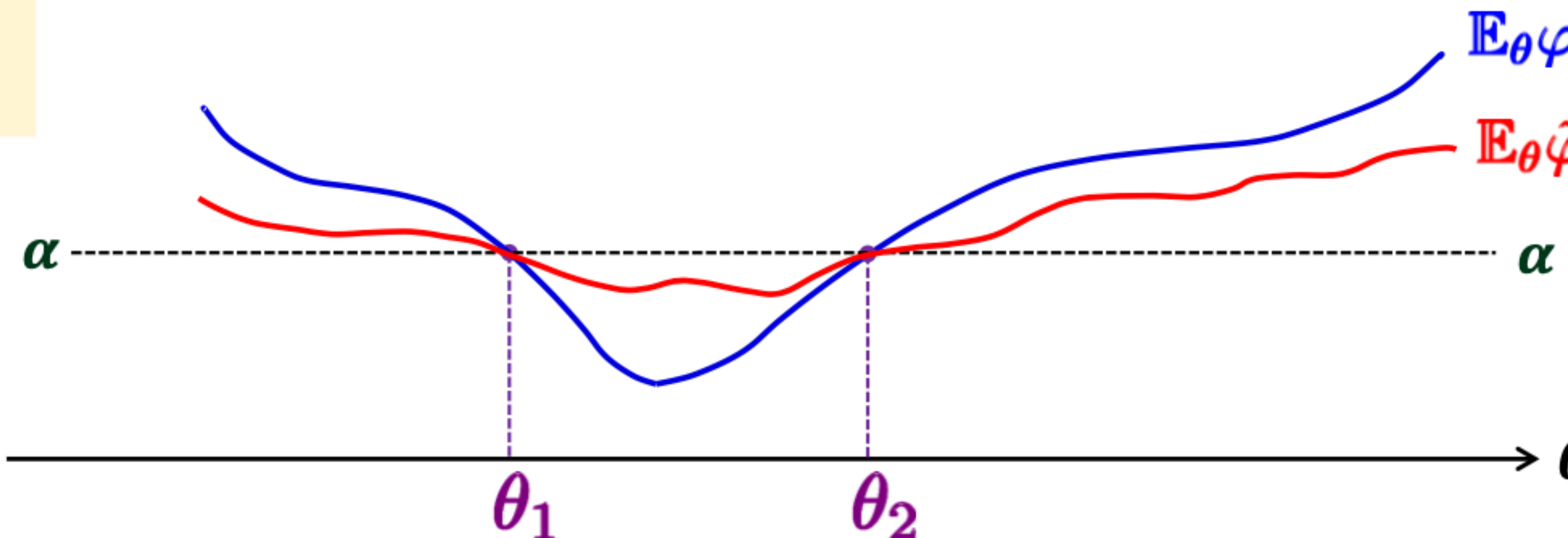
Theorem. $\boxed{\varphi \text{ is UMPU size } \alpha \text{ test of } H_0 \text{ vs } H_1}$

$\to$ **i.e. can achieve optimality (in UMPU sense) by:**

"reject H_0 if natural statistic $t(\boldsymbol{X}) \notin [t_1, t_2]$"

Lemma. For any test $\tilde{\varphi}$ with $\mathbb{E}_{\theta_1}[\tilde{\varphi}(\boldsymbol{X})] = \mathbb{E}_{\theta_2}[\tilde{\varphi}(\boldsymbol{X})] = \alpha$,

$$\begin{cases} \mathbb{E}_\theta[\varphi(\boldsymbol{X})] \geq \mathbb{E}_\theta[\tilde{\varphi}(\boldsymbol{X})] & \forall\, \theta \notin [\theta_1, \theta_2] \\ \mathbb{E}_\theta[\varphi(\boldsymbol{X})] \leq \mathbb{E}_\theta[\tilde{\varphi}(\boldsymbol{X})] & \forall\, \theta \in [\theta_1, \theta_2] \end{cases}$$



Conditional test under exponential family

Definition

$$\text{Data } \boldsymbol{X} \sim f(\boldsymbol{x}|\boldsymbol{\pi}) = C(\boldsymbol{\pi})h(\boldsymbol{x})e^{\sum_{j=1}^{k}\pi_j t_j(\boldsymbol{x})}$$

$$\left[\boldsymbol{\pi} = (\boldsymbol{\pi_1}, \ldots, \pi_k) \in \Pi, \text{ natural parameter space}\right]$$

(I) **One-sided test** $\rightarrow$ $H_0 : \boldsymbol{\pi_1} \le \pi_1^* \quad \text{vs} \quad H_1 : \boldsymbol{\pi_1} > \pi_1^*$

(II) **Two-sided test** $\rightarrow$ $H_0 : \boldsymbol{\pi_1} \in \left[\pi_1^*, \pi_1^{**}\right] \quad \text{vs} \quad H_1 : \boldsymbol{\pi_1} \notin \left[\pi_1^*, \pi_1^{**}\right]$

Conditional test under exponential family

(I) One-sided test $\rightarrow$ $\boxed{H_0 : \boldsymbol{\pi_1} \leq \pi_1^* \quad \text{vs} \quad H_1 : \boldsymbol{\pi_1} > \pi_1^*}$

UMPU size α test: $\boxed{\textbf{reject } \boldsymbol{H_0} \quad \textbf{if} \quad \boldsymbol{t_1(X) > c}}$

where $\boldsymbol{c}$ satisfies $\boxed{\mathbb{P}\Big(\boldsymbol{t_1(X)} > \boldsymbol{c} \,\Big|\, \boldsymbol{\pi_1} = \pi_1^*,\ t_2(\boldsymbol{X}), \ldots, t_k(\boldsymbol{X})\Big) = \alpha}$

[Note: $\boldsymbol{c = c(\alpha, \pi_1^*, t_2(X), \ldots, t_k(X))}$ depends on $\boldsymbol{t_2(X), \ldots, t_k(X)}$ $\rightarrow \therefore$ $\boldsymbol{c}$ is random]

(II) Two-sided test $\rightarrow$ $H_0 : \boldsymbol{\pi_1} \in [\pi_1^*, \pi_1^{**}] \quad \text{vs} \quad H_1 : \boldsymbol{\pi_1} \notin [\pi_1^*, \pi_1^{**}]$

$\boxed{\underline{\textit{or}} \quad \boldsymbol{H_0 : \pi_1 = \pi_1^* \quad \text{vs} \quad H_1 : \pi_1 \neq \pi_1^*}}$

UMPU size α test: $\boxed{\textbf{reject } \boldsymbol{H_0} \quad \textbf{if} \quad \boldsymbol{t_1(X) \notin [c^*, c^{**}]}}$

where $\boldsymbol{c^*, c^{**}}$ satisfy

$$\begin{cases} \mathbb{P}_{\pi_1^{**}}\Big(\boldsymbol{t_1(X)} \notin \boldsymbol{[c^*, c^{**}]} \,\Big|\, t_2(\boldsymbol{X}), \ldots, t_k(\boldsymbol{X})\Big) = \alpha \\ \mathbb{P}_{\pi_1^*}\Big(\boldsymbol{t_1(X)} \notin \boldsymbol{[c^*, c^{**}]} \,\Big|\, t_2(\boldsymbol{X}), \ldots, t_k(\boldsymbol{X})\Big) = \alpha \\ \quad \underline{\textit{or}} \quad \dfrac{\partial}{\partial \boldsymbol{\pi_1}} \mathbb{P}_{\boldsymbol{\pi_1}}\Big(\boldsymbol{t_1(X)} \notin \boldsymbol{[c^*, c^{**}]} \,\Big|\, \boldsymbol{t_2(X), \ldots, t_k(X)}\Big)\Big|_{\boldsymbol{\pi_1 = \pi_1^*}} = 0 \end{cases}$$

[Note: $\boldsymbol{c^*, c^{**}}$ depend on $\boldsymbol{\pi_1^*, \pi_1^{**}, \alpha, t_2(X), \ldots, t_k(X)}$]

Problems

Exercise 1

Given a random sample $X_1, \ldots, X_n$ from the density function

$$f(x \mid \lambda) = \lambda^{-1} e^{-x/\lambda}, x > 0$$

(a) Find the **UMP** size α test of $H_0 : \lambda \leq 1$ against $H_1 : \lambda > 1$.

(b) Find the equation for the **sample size** n required for this test to have power 95% when $\lambda = 2$.

Please illustrating your answer with a sketched graph.

Exercise 1: Solution (a)

$$f(x \mid \lambda) = \lambda^{-1} e^{-x/\lambda}, x > 0$$

For any $0 < \lambda_1 < \lambda_2,$ we note that the likelihood ratio

$$\frac{l(\lambda_2)}{l(\lambda_1)} = \left(\frac{\lambda_2}{\lambda_1}\right)^{-n} e^{-\sum_{i=1}^n X_i\left(\frac{1}{\lambda_2} - \frac{1}{\lambda_1}\right)},$$ **which is increasing in** $\sum_{i=1}^n X_i$.

This monotone likelihood ratio property of the model guarantees the size α UMP test for the one-sided hypotheses is of the form

$$\varphi(X) = 1\left\{\sum_{i=1}^n X_i > c_\alpha\right\}$$

Exercise 1: Solution (a) $f(x \mid \lambda) = \lambda^{-1} e^{-x/\lambda}, x > 0$

For any $0 < \lambda_1 < \lambda_2$, we note that the likelihood ratio

$$\frac{l(\lambda_2)}{l(\lambda_1)} = \left(\frac{\lambda_2}{\lambda_1}\right)^{-n} e^{-\sum_{i=1}^n X_i\left(\frac{1}{\lambda_2} - \frac{1}{\lambda_1}\right)},$$ **which is increasing in** $\sum_{i=1}^{n} X_i$.

This monotone likelihood ratio property of the model guarantees the size α UMP test for the one-sided hypotheses is of the form $\varphi(X) = 1\left\{\sum_{i=1}^{n} X_i > c_\alpha\right\}$, where critical value c_α satisfies

$$\mathbb{P}\left(\sum_{i=1}^{n} X_i > c_\alpha \mid \lambda = 1\right) = \alpha.$$

Since under H_0, the distribution of $\sum_{i=1}^n X_i$ is Gamma with parameters n and 1 (both mean and variance equal n), the critical value is

$$c_\alpha = G_n^{-1}(1 - \alpha)$$

Here and hereafter we use G_n to denote the distribution function of a $\Gamma(n,1)$ distribution and G_n^{-1} denotes its inverse (the lower quantile function).

Exercise 1: Solution (b)

The power of the test when $\lambda = 2$ is given by

$$\mathbb{P}\left(\sum_{i=1}^{n} X_i > c_\alpha \mid \lambda = 2\right) = \mathbb{P}\left(\sum_{i=1}^{n} \frac{X_i}{2} > \frac{c_\alpha}{2} \mid \lambda = 2\right) = 1 - G_n\left(\frac{1}{2}G_n^{-1}(1-\alpha)\right)$$

For the power to be at least 95%, we need an n such that

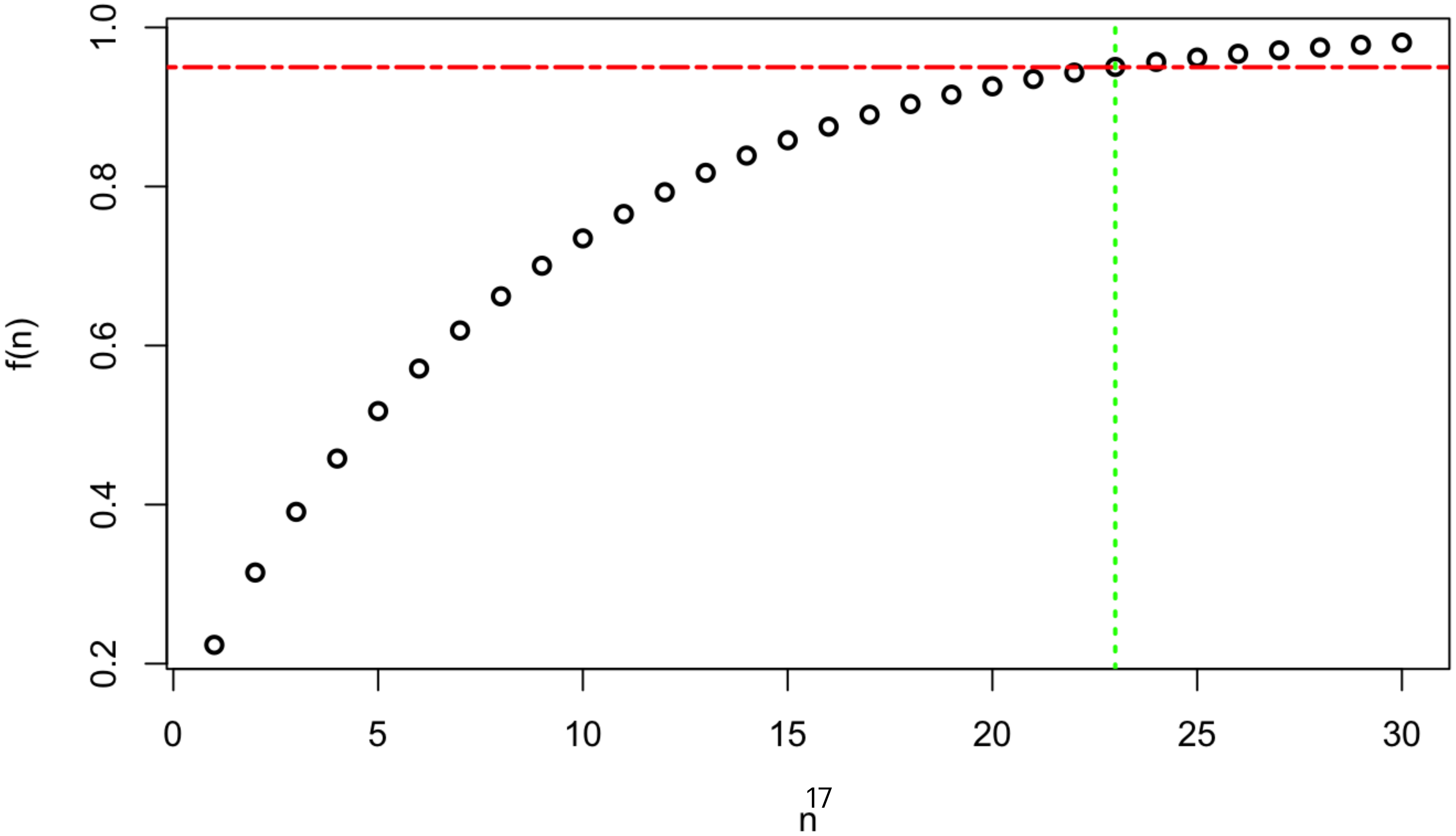
$$1 - G_n\left(\frac{1}{2}G_n^{-1}(1-\alpha)\right) \equiv f(n) \geq 0.95$$

To get n satisfying this requirement, for fixed α we may plot the left hand side as a function of n and locate an n as desired.

Exercise 1: Solution (b)

For example, when $\alpha = 0.05$, from the following sketch, we see a sample size $n = 23$ will suffice.

In fact, when $n = 23, 1 - G_n\left(\frac{1}{2}G_n^{-1}(1-0.05)\right) = 0.9503$



Exercise 2

Let $X_1, \ldots, X_n$ be an independent sample from a normal distribution with mean 0 and variance σ^2

(a) Find the UMP test of size α to test $H_0 : \sigma^2 = \sigma_0^2$ against $H_1 : \sigma^2 = \sigma_1^2 > \sigma_0^2$

(b) Find the UMP test of size α to test $H_0 : \sigma^2 \leq \sigma_0^2$ against $H_1 : \sigma^2 > \sigma_0^2$

(c) Show that the UMPU test of size α to test $H_0 : \sigma^2 = \sigma_0^2$ against $H_1 : \sigma^2 \neq \sigma_0^2$ is in the form $\varphi(\mathbf{X}) = 1\left\{ \sum_{i=1}^n X_i^2 \notin [t_1, t_2] \right\}$ with t_1, t_2 satisfying

$$\begin{cases} 1 + \mathbb{P}\left(Y < \frac{t_1}{\sigma_0^2}\right) - \mathbb{P}\left(Y < \frac{t_2}{\sigma_0^2}\right) = \alpha, \ \text{where } Y \sim \chi_n^2 \\ \frac{t_1 - t_2}{n(\ln t_1 - \ln t_2)} = \sigma_0^2 \end{cases}$$

Exercise 2: Solution (a)

(a) Find the UMP test of size α to test $H_0 : \sigma^2 = \sigma_0^2$ against $H_1 : \sigma^2 = \sigma_1^2 > \sigma_0^2$

By the **Neyman-Pearson Lemma**, the likelihood ratio test of size α is UMP among all tests of size $\leq \alpha$. The likelihood ratio is

$$\Lambda_{\mathbf{X}}\left(\sigma_0^2, \sigma_1^2\right) = \left(\frac{\sigma_0}{\sigma_1}\right)^n \exp\left\{-\frac{1}{2}\left(\frac{1}{\sigma_1^2} - \frac{1}{\sigma_0^2}\right)\sum X_i^2\right\}$$

which is an increasing function in $\sum X_i^2$. The size \$\alpha\$ likelihood ratio test has critical region equivalent to $\left\{\mathbf{X} : \sum X_i^2 > c_\alpha\right\}$, where c_α satisfies

$$\alpha = P_{\sigma_0^2}\left(\sum X_i^2 > c_\alpha\right) = P\left(Y > c_\alpha/\sigma_0^2\right) \quad \textbf{where } Y \sim \chi_n^2.$$

Thus, $c_\alpha = \sigma_0^2 \chi_n^2(1-\alpha)$, where $\chi_n^2(1-\alpha)$ is the $(1-\alpha)$ th quantile of a chi-square distribution with degree of freedom n, and the UMP size α test is

$$\varphi(\mathbf{X}) = \mathbf{1}\left\{\sum X_i^2 > \sigma_0^2 \chi_n^2(1-\alpha)\right\}$$

Exercise 2: Solution (b)

(b) Find the UMP test of size α to test $H_0 : \sigma^2 \leq \sigma_0^2$ against $H_1 : \sigma^2 > \sigma_0^2$

The likelihood ratio is

$$\Lambda_{\mathbf{X}}\left(\sigma_0^2, \sigma_1^2\right) = \left(\frac{\sigma_0}{\sigma_1}\right)^n \exp\left\{-\frac{1}{2}\left(\frac{1}{\sigma_1^2} - \frac{1}{\sigma_0^2}\right)\sum X_i^2\right\}$$

which is an increasing function in $\sum X_i^2$.

Thus, the model has mlr in $\sum X_i^2$.

The UMP test of size α is exactly the same as (a)

(c) ***Show that the UMPU test of size α to test $H_0 : \sigma^2 = \sigma_0^2$ against $H_1 : \sigma^2 \neq \sigma_0^2$ is in the form*** $\varphi(\mathbf{X}) = 1\left\{ \sum_{i=1}^n X_i^2 \notin \left[t_1, t_2\right] \right\}$ ***with t_1, t_2 satisfying*** $\begin{cases} 1 + \mathbb{P}\left(Y < \frac{t_1}{\sigma_0^2}\right) - \mathbb{P}\left(Y < \frac{t_2}{\sigma_0^2}\right) = \alpha, \\ \frac{t_1 - t_2}{n(\ln t_1 - \ln t_2)} = \sigma_0^2 \end{cases}$

According to the test statistics, t_1, t_2 satisfy

$$\mathbb{E}_{\sigma_0^2}[\varphi(\mathbf{X})] = \alpha \quad \text{, and } \frac{\mathrm{d}}{\mathrm{d}\sigma^2}\mathbb{E}_{\sigma^2}[\varphi(\mathbf{X})]\bigg|_{\sigma_0^2} = 0$$

With $Y \sim \chi_n^2,$ we have

$$\mathbb{E}_{\sigma_0^2}[\varphi(\mathbf{X})] = P\left(Y < \frac{t_1}{\sigma_0^2}\right) + 1 - P\left(Y < \frac{t_2}{\sigma_0^2}\right)$$

$$\frac{\mathrm{d}}{\mathrm{d}\sigma^2}\mathbb{E}_{\sigma^2}[\varphi(\mathbf{X})]\bigg|_{\sigma_0^2} = \frac{\left(t_1/\sigma_0^2\right)^{\frac{n}{2}-1} e^{-t_1/2\sigma_0^2}}{\Gamma(n/2)2^{n/2}}\left(-\frac{t_1}{\sigma_0^4}\right) - \frac{\left(t_2/\sigma_0^2\right)^{\frac{n}{2}-1} e^{-t_2/2\sigma_0^2}}{\Gamma(n/2)2^{n/2}}\left(-\frac{t_2}{\sigma_0^4}\right)$$

Set them equal to α and 0 respectively, we obtain what as required.

Exercise 3

A local councillor suspects that traffic conditions in his village A have become more hazardous than those of a neighbouring village B. He therefore records the **numbers of traffic accidents** N_A, N_B which occur in A and B over a fixed period of time respectively.

Assuming N_A, N_B are **independent Poisson** random variables with parameters $\lambda, \beta\lambda$ respectively, the councillor wishes to test

$H_0 : \beta \geq 1$ **against** $H_1 : \beta < 1$

(a) Derive a form of a UMPU test, of size α, for testing the above hypotheses.

(b) What is the outcome of the test when $\alpha = 0.1, N_A = 7, N_B = 2$?

Exercise 3: Solution

Assuming N_A, N_B are independent Poisson random variables with parameters $\lambda, \beta\lambda$ respectively, the councillor wishes to test

$$H_0 : \beta \geq 1 \text{ against } H_1 : \beta < 1$$

(a) Derive a form of a UMPU test, of size α, for testing the above hypotheses.

The pdf of the independent Poisson random variables can be written as

$$f\left(N_A, N_B\right) = c(\lambda, \beta) h\left(N_A, N_B\right) \exp\left\{\left(N_A + N_B\right) \ln \lambda + N_B \ln \beta\right\}$$

By Theorem, the UMPU test is of the form $1\left\{N_B < c\right\}$, where c satisfy

$\mathbb{P}\left(N_B < c \mid N_A + N_B = n, \beta = 1\right) = \alpha$, which is equivalent to

$$\mathbb{P}(\text{ Binomial } (n,1/2) < c) = \alpha.$$

Exercise 3: Solution

$H_0 : \beta \geq 1$ ***against*** $H_1 : \beta < 1$

(b) What is the outcome of the test when $\alpha = 0.1, N_A = 7, N_B = 2$ ***? The pdf of the independent Poisson random variables can be written as***

Consider

$$\mathbb{P}(\text{Binomial}(9,1/2) < 3) = \frac{46}{2^9} < 0.1$$

Since $N_B = 2 < c = 3$, reject H_0 at 0.1 significance level.

"When I hear you give your reasons, " I remarked, "The thing always appears to me to be so ridiculously simple that I could easily do it myself, though at each successive instance of your reasoning I am baffled until you explain your process. "

Dr. Watson to Sherlock Holmes
A Scandal in Bohemia