THE UNIVERSITY OF HONG KONG
DEPARTMENT OF STATISTICS AND ACTUARIAL SCIENCE

STAT3602 Statistical Inference

(2020-2021 First Semester)

Example Class 6 [Exponential Families]

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1. Show whether the following distributions belong to the exponential family. If so, give its natural parameters and natural parameter space, and describe the natural statistic based on a random sample $x_1, \ldots, x_n$

   (a) Binomial Distribution:
   $$p(x \mid \theta) = \binom{k}{x} \theta^x (1-\theta)^{k-x}, \quad x = 0, \ldots, k; \theta \in (0,1).$$

   (b) Poisson Distribution:
   $$p(x \mid \theta) = \frac{\theta^x e^{-\theta}}{x!}, \quad x = 0, 1, \ldots, \theta > 0.$$

   (c) Gamma Distribution:
   $$f(x \mid \alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} e^{-\beta x} x^{\alpha - 1} \quad x > 0, \beta, \alpha > 0.$$

   (d) Beta Distribution:
   $$f(x \mid \alpha, \beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}, \quad x \in (0,1), \alpha, \beta > 0.$$

   (e) Multinomial Distribution
   $$p(\boldsymbol{x}; \boldsymbol{\theta}) = \binom{N}{x_1, \ldots, x_{k-1}, \left(N - \sum_{i=1}^{k-1} x_i\right)} \theta_1^{x_1} \cdots \theta_{k-1}^{x_{k-1}} \left(1 - \sum_{i=1}^{k-1} \theta_i\right)^{N - \sum_{i=1}^{k-1} x_i},$$
   with $x_1, \ldots, x_{k-1}, \sum_{i=1}^{k-1} x_i \in \{0, 1, \ldots, N\}; \theta_1, \ldots, \theta_{k-1}, \sum_{i=1}^{k-1} \theta_i \in (0,1)$.

   (f) Uniform Distribution:
   $$f(x; \theta) = \theta^{-1}, 0 < x < \theta; \theta \in (0, \infty).$$

   (g) Negative binomial (Pascal distribution):
   $$p(x; r, p) = \binom{r+x-1}{r-1} p^r (1-p)^x, x = 1, 2, \ldots; \quad p \in (0,1), r = \{1, 2, \ldots\}.$$

2. Find the general form of a conjugate prior density for θ in a Bayesian analysis of the one-parameter exponential family density

   If the posterior distribution $p(\theta \mid x)$ and the prior $p(\theta)$ have the same family of distribution, the prior is called a conjugate prior for the likelihood function.
   $$f(x; \theta) = c(\theta) h(x) \exp\{\theta t(x)\}, \quad x \in \mathbb{R}.$$

3. A curved exponential family is a k-parameter exponential family (the dimension of the natural parameter is k) where the dimension of the vector θ is $d < k$. Let $Y_1, \ldots, Y_n$ be independent, identically distributed $N\left(\mu, \mu^2\right)$ variables. Show that this model is an example of a curved exponential family.

4. Let X_1 and X_2 be independent normal random variables with means 0, and variances σ_1^2 and σ_2^2 respectively, so that the density function of X_i is
   $$f\left(x \mid \sigma_i^2\right) = \frac{1}{\sqrt{2\pi\sigma_i^2}} \exp\left\{-\frac{x^2}{2\sigma_i^2}\right\}, \quad x \in R, i = 1, 2.$$

(a) Define $\xi = \frac{1}{2}\left(\frac{1}{\sigma_2^2} - \frac{1}{\sigma_1^2}\right)$ Show that the joint distribution of X_1, X_2 has the form of an exponential family with natural parameter $\left(\xi, -1/\left(2\sigma_2^2\right)\right)$ and natural statistic $\left(X_1^2, X_1^2 + X_2^2\right)$

(b) Show that the conditional distribution of X_1^2 given $X_1^2 + X_2^2$ depends only on the parameter ξ.