THE UNIVERSITY OF HONG KONG
DEPARTMENT OF STATISTICS AND ACTUARIAL SCIENCE

STAT3602 Statistical Inference

Example Class 8

Review

1. Rao-Blackwell Theorem:

   $T(X)$ is sufficient for θ, the loss function $L(\theta; a)$ is convex in a and $\rho(X)$ is an estimator of $\phi(\theta)$ with finite expectation and risk : $E_\theta(\rho(X))$ and $E_\theta(L(\theta, \rho(X)))$ is finite.

   $$\rho^*(t) = E(\rho(X)|T(X))$$

   (a) $\rho^*(T(X))$ is an estimator of $\phi(\theta)$, such that

   $$E_\theta[L(\theta, \rho^*(T(X)))] \leq E_\theta(L(\theta, \rho(X)))$$

   (b) Specifically, if $\rho(X)$ is unbiased and $T(X)$ is complete,

      i. $\rho^*(T(X))$ is the unique unbiased estimator of $\phi(X)$which is a function of $T(X)$

      ii. $E_\theta[L(\theta, \rho^*(T(X)))] \leq E_\theta[L(\theta, S(X))]$ for any unbiased estimator $S(X)$ of $\phi(\theta)$.

   (c) UMVUE (uniformly minimum variance unbiased estimator)
   if we take $L(\theta; a) = (\theta - a)^2$, and $\rho(X)$ is unbiased , $E_\theta[\rho(X)^2] < \infty$, $T(X)$ is complete.

      i. $Var_\theta(\rho^*(T(X))) \leq Var_\theta(S(X))$ for any other unbiased estimator $S(X)$ of $\phi(X)$, which means $\rho^*(T(X))$ is the unique UMVU eatimator of $\phi(X)$.

2. using Rao-Blackwell Theorem to

   (a) reduce an estimator's risk given a sufficient statistic by taking expectation;

   (b) find the UMVU estimator given a complete sufficient statistic and an unbiased estimator by either finding the expected value of any unbiased estimator conditional on T, or finding an unbiased estimator which is a function of T.

Problem

1. Two students are each required to enroll in one and only one of the following four classes:

   course C1 instructed by lecturer L1, course C1 instructed by lecturer L2,

   course C2 instructed by lecturer L1, course C2 instructed by lecturer L2.

   Assume the two students make their choices independently, each having probabilities α and β to choose course C1 and lecturer L1 respectively. The choice of courses and the choice of lecturers are independent of each other.

   For $i, j = 1, 2$, define $X_{ij} = 1$ if the first student selects course Ci and lecturer Lj, and $X_{ij} = 0$ otherwise. The random variables Y_{ij} are similarly defined for the second student. Define

   $$V = X_{11} + X_{12} + Y_{11} + Y_{12} \textbf{ and } W = X_{11} + X_{21} + Y_{11} + Y_{21}$$

   (a) Show that the likelihood function can be written as

   $$l(\alpha, \beta) = \alpha^V (1 - \alpha)^{2-V} \beta^W (1 - \beta)^{2-W}$$

   (b) Show that (V, W) is complete sufficient for the parameters (α, β).

   (c) Suppose that α, β satisfy $\dfrac{\beta}{1-\beta} = (\dfrac{\alpha}{1-\alpha})^2$,

      i. Show that $V + 2W$ is complete sufficient for the parameters (α, β);

      ii. Calculate the values of the statistic $V + 2W$ for each of the 16 outcomes.

      iii. By considering the probabilities of the 16 cases in (ii), show that

      $$P(X_1 = 1 | V + 2W = 4) = 1/3$$

      iv. If it is observed that $V + 2W = 4$, calculate the uniformly minimum variance unbiased (UMVU) estimate of the parameter $\theta = \alpha\beta$.

2. Suppose $X_1, \ldots, X_n$ is a random sample of size $n(\geq 2)$ from a uniform distribution on the closed interval $[a, b]$ with $a < b \in \mathbb{R}$. Given $T = (\min_i X_i, \max_i X_i)$ is a complete sufficient statistic for $\theta = (a, b)$. Find the UMVU estimate for θ.

3. Independent factory-produced items are packed in boxes each containing k items. The probability that an item is in working order is $\theta, 0 < \theta < 1$. A sample of n boxes are chosen for testing, and X_i, the number of working items in the ith box, is noted. Thus, $X_1, \ldots, Xn$ are sample from

a binomial distribution, $\mathrm{Bin}(k, \theta)$, with index k and parameter θ. It is required to estimate the probability,θ^k, that all items in a box are in working order. Find the UMVU estimator.