Order Connected Sets are Measurable in $\mathbb{R}^n$

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Bollobas claimed without proof that an order connected set (with respect to the pointwise ordering) in $\mathbb{R}^n$ is measurable. Yael Dillis asked to know the details of the proof and so, I wrote this short note.

Definition (Order connected)**.** Given a partially ordered set $(X, (\leq))$, $S \subseteq X$ is said to be order connected if for all $x, y \in S$,

$$[x, y] := \{z \mid x \leq z \leq y\} \subseteq S.$$

Equipping $\mathbb{R}^n$ with the point-wise ordering (i.e. for $x := (x_i)_{i=1}^n, y := (y_i)_{i=1}^n \in \mathbb{R}^n$, $x \leq y$ if and only if $x_n \leq y_n$ for all $n = 1, \cdots, n$), $\mathbb{R}^n$ form a partially ordered set and we can therefore talk about its order connected subsets.

Theorem. $S \subseteq \mathbb{R}^n$ is Lebesgue measurable if S is order connected with respect to the point-wise ordering on $\mathbb{R}^n$.

Proof. Defining $S^+ := \{x \mid \exists\, y \in S, y \leq x\}$ and $S^- := \{x \mid \exists\, y \in s, x \leq y\}$, we observe $S = S^+ \cap S^-$. Therefore, it is sufficient to show that S^+ and S^- are measurable. We will show measurability for S^+ while the case for S^- is similar.

It is clear that for all $x \in \mathbb{R}^n$, $\{x\}^+$ is measurable. So, defining $Q := S^+ \cap \mathbb{Q}^n$,

$$Q^+ = \bigcup_{q \in Q} \{q\}^+ \subseteq S^+$$

is measurable. Furthermore, $S^+ \setminus Q^+ \subseteq \partial S^+$ since for all $x \in S^\circ$, there exists some open neighborhood $B \subseteq S^\circ$ of x; so, as Q is dense in S^+, there exists an element $q \in Q \cap B$ such that $q \leq x$ and hence, $x \in \{q\}^+ \subseteq Q^+$. Now, as the Lebesgue σ-algebra is complete and $S^+ = Q^+ \cup (S^+ \setminus Q^+)$, it suffices to show that ∂S^+ is a null-set.

To show $\text{Leb}(\partial S^+) = 0$ we will invoke the Lebesgue density theorem. Namely, by showing

$$\partial S^+ \subseteq \{x \in \overline{S^+} \mid d(x) \notin \{0, 1\}\}$$

where

$$d(x) := \lim_{\epsilon \to 0} d_\epsilon(x) = \lim_{\epsilon \to 0} \frac{\text{Leb}(\overline{S^+} \cap B_\epsilon(x))}{\text{Leb}(B_\epsilon(x))}$$

as $\overline{S^+}$ is closed and hence measurable, the Lebesgue density theorem tells us

$$\text{Leb}(\partial S^+) \leq \text{Leb}(\{x \in \overline{S^+} \mid d(x) \notin \{0,1\}\}) = 0.$$

Indeed, for all $x \in \partial S^+, \epsilon > 0$, $\overline{S^+} \cap B_\epsilon(x)$ must contain the with the right-upper quadrant of the ball by the very definition of S^+, implying $\text{Leb}(\overline{S^+} \cap B_\epsilon(x)) \gtrapprox 2^{-n}\text{Leb}(B_\epsilon(x))$ bounding $d(x)$ from below. Similarly, $\overline{S^+}^c \cap B_\epsilon(x)$ must contain the left-bottom quadrant of the ball as otherwise x is contained by some $\{y\}^+$ for some $y \in \overline{S}^+$ contradicting $x \in \partial S^+$. Hence, $\text{Leb}(\overline{S^+} \cap B_\epsilon(x)) \lessapprox (1 - 2^{-n})\text{Leb}(B_\epsilon(x))$ bounding $d(x)$ from above and so, proving the required inclusion. $\square$