LINEAR ALGEBRA II

Ch. I VECTOR SPACES

Module Description

- Mon. 7-8+Wed. 1-2+Fri. 1-2.
  - Lectures+Problem Solving&Discussing;
  - Homework.
- UoL Module Mark:
  - Linear Algebra I: 30% (skills test)+10% (course work)
  - Linear Algebra II: 50% (exam)+10% (course work)
- DUT Module Mark:
  - Linear Algebra II: 70% (exam)+30% (course work)
- Resit: exam resit 100%

Module Description

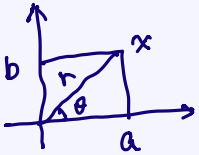
- Linear Algebra (Advanced Algebra)
  - Special set: vector space (linear space), convex set
  - Special mapping: linear mapping, quadratic form, univariate polynomial
  - Special (system of) equation(s): system of linear equations, univariate polynomial equation
  - Analytic Geometry
  - Calculus & Analysis
  - Functional Analysis
  - Abstract Algebra, Commutative Algebra, Algebraic Geometry

Module Description

- Ch. I Vector Spaces
- Ch. II Matrices
- Ch. III 3 Linear Mappings
- Ch. IV Linear Maps and Matrices
- Ch. V Scalar Products and Orthogonality
- Ch. VI Determinants
- Ch. VII Symmetric, Hermitian, and Unitary Operators
- Ch. VIII Eigenvectors and Eigenvalues
- Ch. IX Polynomials and Matrices
- Ch. X Triangulation of Matrices and Linear Maps
- Ch. XI Polynomials and Primary Decomposition
- Ch. XII Convex Sets
- Appendix I Complex Numbers
- Appendix II Iwasawa Decomposition and Others

Appendix I Complex Numbers

- Let $x = a + bi$, $y = c + di$.
  - $x \pm y = (a \pm c) + (b \pm d)i$
  - $x \cdot y = (ac - bd) + (ad + bc)i$
  - $\bar{x} = a - bi$
  - $|x| = \sqrt{a^2 + b^2}$
  - $x\bar{x} = a^2 + b^2 \Rightarrow x^{-1} = \dfrac{\bar{x}}{a^2+b^2} \quad (x \neq 0)$
  - $\dfrac{y}{x} = y \cdot x^{-1}$
  - $x = re^{i\theta} = r(\cos\theta + i\sin\theta)$
  - $x^m = \underset{re^{i\theta}}{\underset{\parallel}{a}} \Rightarrow x = \sqrt[m]{r}\, e^{i\frac{2\pi k}{m}} \quad (k = 0, \cdots, m-1)$



Fundamental Theorem of Algebra

- **Theorem.** The complex numbers are algebraically closed, in other words, every polynomial $f \in C[t]$ of degree $\geqslant 1$ has a root in C.

Proof. Let $f(t) = a_n t^n + a_{n-1} t^{n-1} + \cdots + a_0$, $(n \geqslant 1,\ a_n \neq 0)$. Then

$|f|(t) = |f(t)| \geqslant |a_n||t|^n \left(1 - \frac{|a_{n-1}|}{|a_n| \cdot |t|} - \cdots - \frac{|a_0|}{|a_n||t|^n}\right)$ for $t \neq 0$ and hence $|f|(t) \to \infty$ as $|t| \to \infty$.

For given $B \geqslant |f(0)|$, $\exists\, M > 0$, s.t. $|f(t)| \geqslant B$ for $|t| \geqslant M$.

Because $|f|$ is continuous on the disc $D = \{t \in C \mid |t| \leq M\}$, it has a minimal point $z_0 \in D$, i.e., $|f(t)| \geqslant |f(z_0)|$ for $t \in D$. From $|f(t)| \geqslant B \geqslant |f(0)| \geqslant |f(z_0)|$ for $|t| \geqslant M$, we know that z_0 is a minimal point of $|f|$ on C.

We will prove that $f(z_0) = 0$.

Rewrite f as $f(t) = c_0 + c_1(t - z_0) + \cdots + c_n(t - z_0)^n$. If $c_0 = f(z_0) \neq 0$, let $z = t - z_0$ and let m be the smallest positive integer such that $c_m \neq 0$ (m exists because $\deg(f) \geqslant 1$!)

Then $f(t) = c_0 + c_m z^m + z^{m+1} g(z) \triangleq h(z)$, where g and h are polynomials.

Let z_1 be a complex number such that $c_0 + c_m z_1^m = 0$, i.e. $z_1^m = -\frac{c_0}{c_m}$, and consider values of z of type $z = \lambda z_1$, where λ is real with $0 \leq \lambda \leq 1$. We have

Fundamental Theorem of Algebra

$$h(z) = h(\lambda z_1) = c_0 - c_0\lambda^m + \lambda^{m+1} z_1^{m+1} g(\lambda z_1) = c_0\left(1 - \lambda^m + \lambda^{m+1} z_1^{m+1} c_0^{-1} g(\lambda z_1)\right).$$

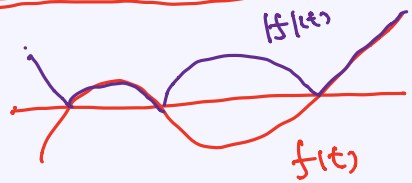
There exists a number $L>0$ such that $|z_1^{m+1} c_0^{-1} g(\lambda z_1)| \le L$ for all $\lambda \in [0,1]$, and hence $|h(\lambda z_1)| \le |c_0|(1-\lambda^m + L\lambda^{m+1})$. It is easy to see that

$$0 < 1 - \lambda^m + L\cdot\lambda^{m+1} < 1$$

for sufficiently small $\lambda \in (0,1)$. Then for such λ,

$$|f(\lambda z_1 - z_0)| = |h(\lambda z_1)| < |c_0| = |f(z_0)|.$$

This contradicts with z_0 is a minimal point of $|f|$. So, $f(z_0)=0$.



$\exists\, z_0 \in \mathbb{C}$, s.t. $f(z_0)=0 \iff \min_{t\in\mathbb{C}} |f|(t) = 0$

$\Rightarrow$ Fermat's Theorem. $f'(x_0)=0$ if $x_0 = \arg\min_{x\in N(x_0)} f(x)$.

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + o(x-x_0)$$

证明思想类似.

Ch. I Vector Spaces

- **Definition （field (of numbers)，数域）.** A subset K of the complex numbers C is said to be a field if it satisfies the following conditions:
  - If x, y are elements of K, then $x + y$ and xy are also elements of K.
  - If $x \in K$, then $-x$ is also an element of K. If furthermore $x \neq 0$, then x^{-1} is an element of K.
  - The elements 0 and 1 are elements of K.

Ch. I Vector Spaces

- **Definition （field，域）.** A set K with two operations "+" and "·" and satisfies the following conditions:
  - If x and y are elements of K, then $x + y$ and $x \cdot y$ are also elements of K.
  - There are zero 0 and identity 1 in K, such that $x + 0 = 0 + x = x$, $x \cdot 1 = 1 \cdot x = x$.
  - For any $x \in K$, there exists $-x \in K$ such that $x + (-x) = (-x) + x = 0$. For any $x \neq 0$, there exists $x^{-1} \in K$ such that $x \cdot x^{-1} = x^{-1} \cdot x = 1$.
  - Associativity: $(x + y) + z = x + (y + z)$, $(x \cdot y) \cdot z = x \cdot (y \cdot z)$.
  - Commutativity: $x + y = y + x$, $x \cdot y = y \cdot x$.
  - Distributivity: $x \cdot (y + z) = x \cdot y + x \cdot z$, $(x + y) \cdot z = x \cdot z + z \cdot z$.

Ch. I Vector Spaces

- **Definition.** A vector space V over the field K is a set of objects which can be added and multiplied by elements of K, in such a way that the sum of two elements of V is again an element of V, the product of an element of V by an element of K is an element of V, and the following properties are satisfied:

  VS 1. Given elements u, v, w of V, we have $(u+v)+w = u+(v+w)$.

  VS 2. There is an element of V, denoted by 0, such that
  $$0 + u = u + 0 = u.$$
  for all elements u of V.

  VS 3. Given an element u of V, there exists an element $-u$ in V such that
  $$u + (-u) = 0.$$

  VS 4. For all elements u, v of V, we have
  $$u + v = v + u.$$

Ch. I Vector Spaces

VS 5. If c is a number, then $c(u+v) = cu + cv$.

VS 6. If a, b are two numbers, then $(a+b)v = av + bv$.

VS 7. If a, b are two numbers, then $(ab)v = a(bv)$.

VS 8. For all elements u of V, we have $1 \cdot u = u$ (1 here is the number one).

- $W \subset V$ is a subspace of V:
  - (i) If $u, v \in W$, then $u + v \in W$.
  - (ii) If $u \in W$ and $c \in K$, then $cu \in W$.
  - (iii) $0 \in W$.

Ch. I Vector Spaces

- $V = K^n$ is a vector space over K.

- $W = \{u \in W | u_n = 0\}$ is a subspace of K^n.

- If V is a vector space over K and $v_1, \ldots, v_n \in V$, then

$$W = \operatorname{span}\{v_1, \ldots, v_n\} \triangleq \{x_1 v_1 + \cdots + x_n v_n | x_1, \ldots, x_n \in K\}$$

  is a subspace of K^n, called the subspace generated (spanned) by $v_1, \ldots, v_n \in V$.

Ch. I Vector Spaces

- Let $A, B \in K^n$, $A = (a_1, \ldots, a_n)$ and $B = (b_1, \ldots, b_n)$. The dot product (scalar product):

$$A \cdot B = a_1 b_1 + \cdots + a_n b_n.$$

SP 1. $A \cdot B = B \cdot A$.
SP 2. $A \cdot (B + C) = A \cdot B + A \cdot C = (B + C) \cdot A$.
SP 3. $(xA) \cdot B = A \cdot (xB) = x(A \cdot B)$.

- $\Rightarrow$ Ch. V
  - Scalar product in general VS over K.
  - Hermitian product in VS over C.

Homework: P281-282, 6-8.

Ch. I Vector Spaces

- $\{v_1, \ldots, v_n\}$ is **a basis of** V, if $\operatorname{span}\{v_1, \ldots, v_n\} = V$ and $v_1, \ldots, v_n$ are linearly independent.
- **Theorem 3.1.** Let V be a vector space over the field K. Let $\{v_1, \ldots, v_m\}$ be a basis of V over K. Let $w_1, \ldots, w_n$ be elements of V, and assume that $n > m$. Then $w_1, \ldots, w_n$ are linearly dependent.

Proof. Assume that $w_1, \cdots, w_n$ are L.I. Because $\{v_1, \cdots, v_m\}$ is a basis of V,

$w_1 = a_1 v_1 + \cdots + a_m v_m$ for some $a_i \in K$. $w_1 \neq 0 \Rightarrow \exists\, i$ (say $i=1$) s.t. $a_i \neq 0 \Rightarrow$

$v_1 = \frac{1}{a_1} w_1 - \frac{a_2}{a_1} v_2 - \cdots - \frac{a_m}{a_1} v_m. \Rightarrow \operatorname{span}\{w_1, v_2, \cdots, v_m\} = V$

If for some $1 \le r < m$, we have $\operatorname{span}\{w_1, \cdots, w_r, v_{r+1}, \cdots, v_m\} = V$, then

$$w_{r+1} = b_1 w_1 + \cdots + b_r w_r + a_{r+1} v_{r+1} + \cdots + a_m v_m \text{ for some } b_i, a_j.$$

Because $w_1, \cdots, w_n$ are L.I., $\exists\, i$ (say $i = r+1$). s.t. $a_i \neq 0$. So,

$$v_{r+1} = -\frac{b_1}{a_{r+1}} w_1 - \cdots - \frac{b_r}{a_{r+1}} w_r + \frac{1}{a_{r+1}} w_{r+1} - \frac{a_{r+2}}{a_{r+1}} v_{r+2} - \cdots - \frac{a_m}{a_{r+1}} v_m$$

$\Rightarrow \operatorname{span}\{w_1, \cdots, w_{r+1}, v_{r+2}, \cdots, v_m\} = V.$

By induction, we have, $\operatorname{span}\{w_1, \cdots, w_m\} = V$. So, $\exists\ a_i \in K$, s.t.

$$w_n = a_1 w_1 + \cdots + a_m w_m.$$

Because $n > m$, this contradicts with that $w_1, \cdots, w_n$ are L.I.

- **Theorem 3.2.** Let V be a vector space over the field K and $\{v_1, \ldots, v_m\}$, $\{w_1, \ldots, w_n\}$ bases of it. Then $m = n$.$\Rightarrow$ The $\underbrace{\text{dimension of } V}_{\dim V}$ over K!

Ch. I Vector Spaces

- $\{v_1, \ldots, v_n\}$ is **a basis of** V

  $\Leftrightarrow$span$\{v_1, \ldots, v_n\} = V$ and $v_1, \ldots, v_n$ are linearly independent.

  $\Leftrightarrow v_1, \ldots, v_n$ are linearly independent and $n = \dim V$.

  $\Leftrightarrow$ Every element in V can be expressed uniquely as a linear combination of $v_1, \ldots, v_n$.

  $\Leftrightarrow$ $\{v_1, \ldots, v_n\}$ is a maximal set of linearly independent elements of V.

- Let V be a vector space and W a subspace of V. If $\dim W = \dim V$, then $W = V$.

- Let V be a vector space of dimension n. Let r be a positive integer with $r < n$, and let $v_1, \ldots, v_r$ be linearly independent elements of V. Then one can find elements $v_{r+1}, \ldots, v_n$ such that $\{v_1, \ldots, v_n\}$ is a basis of V.

Ch. I Vector Spaces

- Let V be a vector space having a basis consisting of n elements. Let W be a subspace which does not consist of 0 alone. Then W has a basis, and the dimension of W is $\leq n$.
- $\dim K^n = ? = n$ $\quad (1,0,\cdots,0)^T, (0,1,0\cdots,0)^T, \cdots, (0,\cdots,0,1)^T$
- C over R? $\quad a+bi, \{1, i\}, \dim = 2$
- Coordinate vector

Let $\{v_1, \cdots, v_n\}$ be a basis of V. $\forall v \in V$, $\exists\, a_1, \cdots, a_n \in K$, s.t.

$v = a_1 v_1 + \cdots + a_n v_n \iff (a_1, \cdots, a_n)$

$V \iff K^n$

Ch. I Vector Spaces

- $V = \{$all K-valued functions$\}$ is a VS over K. $\dim V = \infty$
- $V = \{$all real valued continuous functions$\}$ is a VS over R. $\dim V = \infty$
- $V = \{$all real valued differentiable functions$\}$ is a VS over R. $\dim V = \infty$
- $V = \left\{ \sum_{k=0}^{n} a_k t^k | a_k \in K \right\}$ is a VS over K. $\dim V = n+1$, $\{1, t, \cdots, t^n\}$ is a basis.
- $V = \left\{ \sum_{k=0}^{n} a_k e^{kt} | a_k \in K \right\}$ is a VS over K. $\dim V = n+1$, $\{1, e^t, e^{2t}, \cdots, e^{nt}\}$ ...
- $V = \left\{ \sum_{k=0}^{n} [a_k \cos(kt) + b_k \sin(kt)] | a_k, b_k \in K \right\}$ is a VS over K. $\dim V = 2n+1$

$\{1, \cos t, \sin t, \cos 2t, \sin 2t, \cdots, \cos nt, \sin nt\}$ is a basis

Ch. I Vector Spaces

- **Sums and Direct Sums**
- Let U and W are subspaces of the vector space V over K.
- $U + W = \{u + w | u \in U, w \in W\}$ is a subspace of V.
- $U \cap W = \{v | v \in U, v \in W\}$ is a subspace of V.
- V is a direct sum of U and W, denotes $V = U \oplus W$, if for any $v \in V$, there exists unique $u \in U$ and $w \in W$ such that $v = u + w$.
- $V = \bigoplus_{i=1}^{n} V_i = V_1 \oplus \cdots \oplus V_n$

- **Theorem 4.1.** Let V be a vector space over the field K, and let U, W be subspaces. If $U + W = V$, then $V = U \oplus W \Leftrightarrow U \cap W = \{O\}$.

Proof. "$\Leftarrow$" $\forall v \in V$, $\exists\, u \in U$, $w \in W$, s.t. $v = u + w$

If there exists another pair of vectors $u' \in U$, $w' \in W$, s.t. $u' + w' = v$

$u + w = u' + w'$, $U \ni u - u' = w' - w \in W$, $0 \neq u - u' = w' - w \in U \cap W$

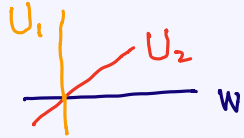
contradicts $U \cap W = \{0\}$.

"$\Rightarrow$" If not, $\exists\, 0 \neq v_0 \in U \cap W$

Let $v \in V$, then $\exists\, u \in U$, $w \in W$, s.t. $v = u + w = \underbrace{(u + v_0)}_{\in U} + \underbrace{(w - v_0)}_{\in W}$

$\qquad\qquad\qquad

- **Theorem 4.2.** Let V be a finite dimensional vector space over the field K. Let W be a subspace. Then there exists a subspace U such that $V = U \oplus W$. $\Rightarrow$ 不唯一！



$V = R^2 = W \oplus U_1 = W \oplus U_2$

Let $\{w_1, \cdots, w_m\}$ be a basis of W. Then, $\exists\, u_1, \cdots, u_n \in V$. s.t.

$\{w_1, \cdots, w_m, u_1, \cdots, u_n\}$ is a basis of V.

Let $U = \text{span}\{u_1, \cdots, u_n\}$, then $V = U \oplus W$.

$\boxed{P_{10}\ 12,\ P_{15}\ 8\ (7),\ P_{22}\ 2}$

- **Theorem 4.2.** $\dim(U \oplus W) = \dim U + \dim W$.

Proof. Let $\{u_1, \cdots, u_m\}$ be a basis of U and $\{w_1, \cdots, w_n\}$ a basis of W.

$\Rightarrow \{u_1, \cdots, u_m, w_1, \cdots, w_n\}$ is a basis of $U \oplus W \Rightarrow \dim(U \oplus W) = \dim U + \dim W$.

$\Uparrow$

$\forall\, v \in U \oplus W$, $\exists!\ u \in U$, $w \in W$, s.t. $v = u + w$

$\exists!\ a_1, \cdots, a_m \in K$, s.t. $u = a_1 u_1 + \cdots + a_m u_m$

$b_1, \cdots, b_n \in K$, s.t. $w = b_1 w_1 + \cdots + b_n w_n$

$v = a_1 u_1 + \cdots + a_m u_m + b_1 w_1 + \cdots + b_n w_n$

Ch. I Vector Spaces

- Let U and W are arbitrary vector spaces over K.
- $U \times W = \{(u, w) | u \in U, w \in W\}$ is a vector space over K, called direct product.

  Cartesian product Descartes Cartesius
- $\prod_{i=1}^{n} W_i = W_1 \times \cdots \times W_n$. $a(w_1, \cdots, w_n) + b(v_1, \cdots, v_n) = (aw_1 + bv_1, \cdots, aw_n + bv_n)$

  $= \{(w_1, \cdots, w_n) \mid \underset{\in W_1}{w_1} \quad \cdots \quad \underset{\in W_n}{w_n}\}$
- $K^{m+n} = K^m \times K^n$. $(x, y) = (x_1, \cdots, x_n, y_1, \cdots, y_m)$
- $\dim(U \times W) = \dim U + \dim W$.

Let $\{u_1, \cdots, u_m\}$ be a basis of U,

$\{w_1, \cdots, w_n\}$ a .. .. W

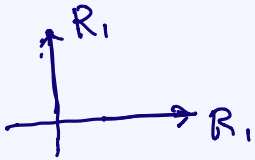
Then $\{\underset{\in W}{(u_1, 0)}, \cdots, (u_m, 0), \underset{\in U}{(0, w_1)}, \cdots, (0, w_n)\}$ is

a basis of $U \times W$.

$\forall (u, w) \in U \times W$, $u = a_1 u_1 + \cdots + a_m u_m$, $w = b_1 w_1 + \cdots + b_n w_n$

$(u, w) = a_1(u_1, 0) + \cdots + a_m(u_m, 0) + b_1(0, w_1) + \cdots + b_n(0, w_n)$

Remark



$R^1 \times R^1 = R^2 \neq R^1 \oplus R^1$

$\{(x, 0) \mid x \in R\} \oplus \{(0, y) \mid y \in R\}$