MA3071 Financial Mathematics - DLI, Year 2023-2024

Exercises for Feedback session 2

Exercise 1 *Given that $(1+\rho)^{-t}S_t$ is a martingale, present a general proof of $q-$probabilities such that,*

$$q_u = \frac{1+\rho-d}{u-d}, \quad q_d = 1 - q_u$$

Exercise 2 *Consider a discrete market with one non-risky asset (bond) and one risky asset (share). The interest rate $\rho = \frac{1}{3}$ is fixed over each period and the share price is defined by a $3-$period binomial tree model, such that*

$$S_0 = 2, S_t = S_{t-1}Y_t$$

Y and Y_t, $t = 1, 2, 3$ are iid random variables with

$$P(Y = 2) + P(Y = 1) = 1$$

Write down the 3-step binomial tree and find the no-arbitrage time 0 option price for the option claim $(S_3 - 3)_+$.

Exercise 3 *Consider a 4-period binomial tree model for a non-dividend paying stock, with current price 120. Assume that over each 3 month period the stock can either increase or decrease by 10%. The risk-free rate of interest is 2% p.a.*

i) Construct the binomial tree that has been specified.

ii) Calculate the price of a European call option on the stock with strike price 125 and time to expiry of one year.

Now consider a special European call option with the same strike price $K = 125$ and time to expiry of one year. The owner of such an option has the right to exercise the option at the end of the year only if the stock price goes above the level $L = 145$ during or at the end of the year. This is a type of exotic option known as a barrier option, specifically an "up and in call" since it requires the stock price to increase in order to be valid.

iii) Calculate the price at time 0 of this barrier option.

Solution 1 We want to show

$$\mathbb{E}_Q[(1+\rho)^{-T}S_T|\mathcal{F}_t] = (1+\rho)^{-t}S_t$$

where $\mathcal{F}_t$ is the filtration and $T > t \geqslant 0$. Based on Markov property,

$$\mathbb{E}_Q[(1+\rho)^{-T}S_T|\mathcal{F}_t] = \mathbb{E}_Q[(1+\rho)^{-T}S_T|S_t]$$

As $(1+\rho)^{-T}$ is a constant, we also have

$$\mathbb{E}_Q[(1+\rho)^{-T}S_T|S_t] = (1+\rho)^{-T}\mathbb{E}_Q[S_T|S_t]$$

Given S_t,

$$[S_T|S_t] = S_t Y_{t+1} Y_{t+2} \cdots Y_T$$

Thus,

$$\mathbb{E}_Q[S_T|S_t] = \mathbb{E}_Q[S_t Y_{t+1} Y_{t+2} \cdots Y_T]$$

Since S_t is known, and $Y_{t+1}, Y_{t+2}, \cdots, Y_T$ are independently and identically distributed, we have

$$(1+\rho)^{-T}\mathbb{E}_Q[S_T|S_t] = (1+\rho)^{-T}S_t\left(\mathbb{E}_Q[Y]\right)^{T-t} = (1+\rho)^{-T}S_t(uq_u + dq_d)^{T-t}$$

and only when $E_Q[Y] = uq_u + dq_d = 1+\rho$, we can obtain

$$\mathbb{E}_Q[(1+\rho)^{-T}S_T|\mathcal{F}_t] = (1+\rho)^{-T}S_t(1+\rho)^{T-t} = (1+\rho)^{-t}S_t$$

Thus,

$$\begin{cases} uq_u + dq_d = 1+\rho \\ q_u + q_d = 1 \end{cases}$$

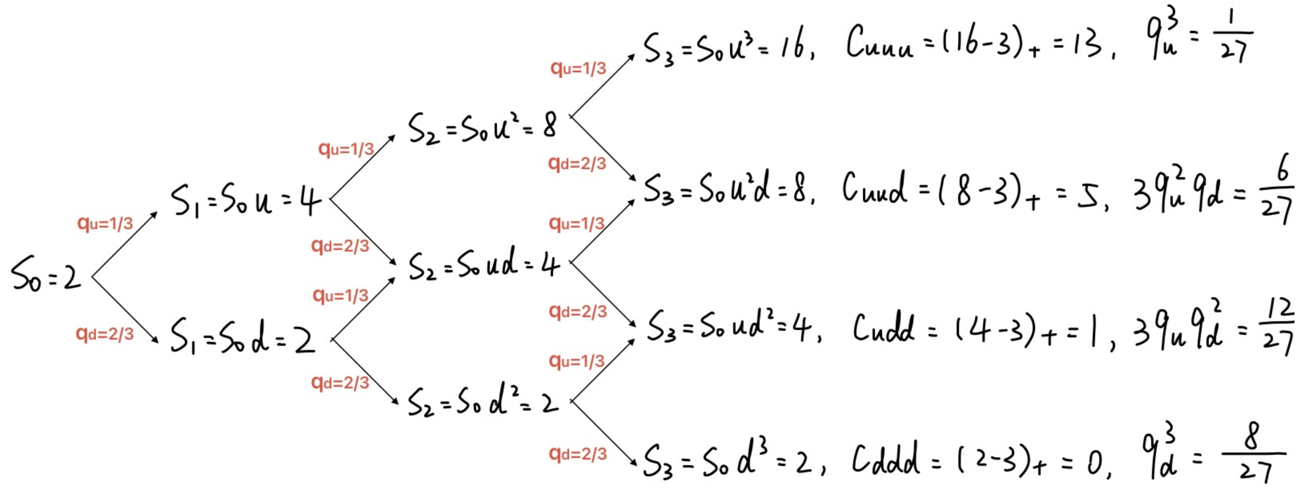
Solving these linear equations, we get

$$q_u = \frac{1+\rho-d}{u-d}, \quad q_d = \frac{u-(1+\rho)}{u-d} = 1 - q_u$$

Solution 2 Here, we have known: $S_0 = 2$, $\rho = \frac{1}{3}$, $u = 2$, $d = 1$, $K = 3$, number of periods $n = 3$. And we calculate the q-probabilities

$$q_u = \frac{1+\rho-d}{u-d} = \frac{1+\frac{1}{3}-1}{2-1} = \frac{1}{3}, \quad q_d = 1 - q_u = \frac{2}{3}$$

Then we write down the 3-period binomial tree as follows,



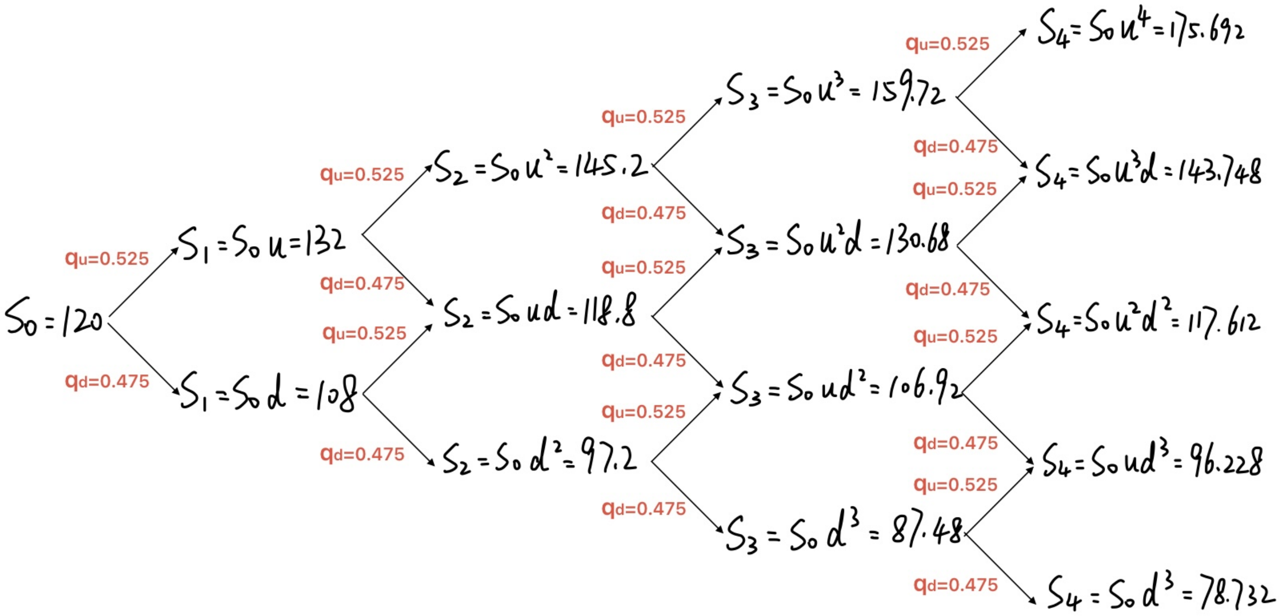
Then, the time 0 option price for the option claim $(S_3 - 3)_+$ is

$$\begin{aligned} V_0 &= \frac{C_{uuu}q_u^3 + 3C_{uud}q_u^2 q_d + 3C_{udd}q_u q_d^2 + C_{ddd}q_d^3}{(1+\rho)^3} \\ &= \frac{13 \times \frac{1}{27} + 5 \times \frac{6}{27} + 1 \times \frac{12}{27} + 0 \times \frac{8}{27}}{\left(1+\frac{1}{3}\right)^3} = \frac{55}{64} \approx 0.859 \end{aligned}$$

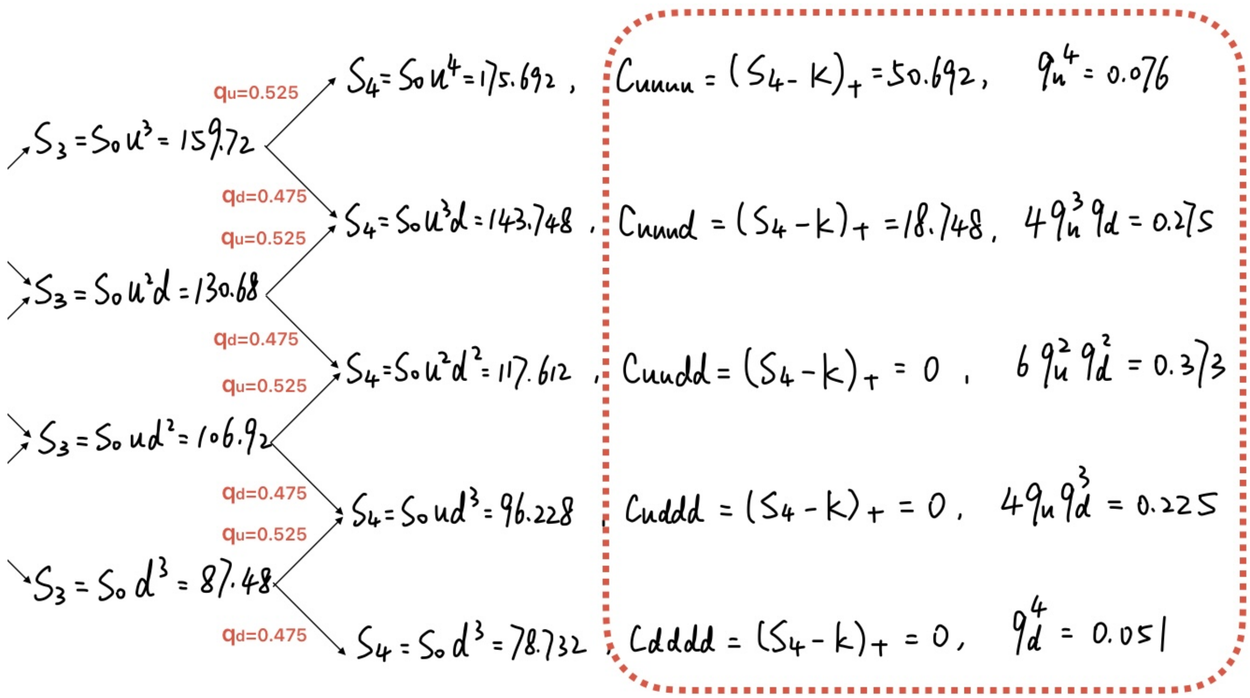
Solution 3 i) We have known: $S_0 = 120$, $u = 1 + 10\% = 1.1$, $d = 1 - 10\% = 0.9$, number of periods $n = 4$, and given the interest rate is 2% per annum, $\rho = (1+2\%)^{\frac{1}{4}} - 1 \approx 0.005$ over each 3-month period. Then, we construct the binomial tree,

$$q_u = \frac{1+\rho-d}{u-d} = \frac{1+0.005-0.9}{1.1-0.9} = \frac{0.105}{0.2} = 0.525, \quad q_d = 1 - q_u = 0.475$$

Then we write down the $4-$period binomial tree as follows,



ii) When $K = 125$ and $T = 1$, we have,



$$
\begin{aligned}
V_0 &= \frac{C_{uuuu}q_u^4 + 4C_{uuud}q_u^3 q_d + 6C_{uudd}q_u^2 q_d^2 + 4C_{uddd}q_u q_d^3 + C_{dddd}q_d^4}{(1+\rho)^4} \\
&= \frac{50.692 \times 0.076 + 18.748 \times 0.275 + 0 + 0 + 0}{(1+0.005)^4} = \frac{9.008292}{1.0201505} \approx 8.83
\end{aligned}
$$

iii) Since the barrier level is $L = 145$, the buyer is eligible to exercise the option only when $S_4 = 175.692$. Thus

$$V_0 = \frac{50.692 \times 0.076 + 0 + 0 + 0 + 0}{(1 + 0.005)^4} \approx 3.78$$