MA3071 Financial Mathematics - DLI
Year 2023-2024

Coursework 1

INSTRUCTIONS AND DEADLINE:

Please electronically submit one piece of written or typed work per person as a single file via *Blackboard* by **November 8, 2023, at 16:00 (UK) / 23:59 (China)**.

You can use this page as a cover page and write your name, student ID, and signature below.

Name:

Student ID:

Signature:

MARKING CRITERIA:

>> Coursework is marked out of 100 points, with the number of marks for each main question indicated at the beginning of each.

>> Clearly justify and explain your answers. You are expected to use MATLAB for calculations. A printout without full explanations of the formulas and reasoning will result in a deduction of marks.

>> You are required to submit a single PDF file containing justifications, explanations, and codes for each question. Include your MATLAB code in the appendix of your answers, ensuring that it is properly commented. You can copy or screenshot your codes into the PDF file without providing the code files in any other format.

>> You can submit your answers up to 3 attempts when submitting via Blackboard. Only the last attempt of your submission will be assessed. Email submissions won't be accepted.

Please note: *Any numerical results should be rounded to four decimal places.*

Question [100 marks]

Consider a European call option with a strike price of $K = 5.65$ in a discrete market with a fixed interest rate of 5.4% per annum. The price of the underlying asset is modelled by a 20-period (each period is one month) binomial tree model, such that

$$S_0 = 5.35,\ S_t = S_{t-1}Y_t$$

where Y_t, $t = 1, 2, \ldots, 20$ are independent random variables with the following distributions

$$\begin{aligned} P(Y_t = 1.02) + P(Y_t = 0.98) &= 1, \quad t = 1, 5, 9, 13, 17 \\ P(Y_t = 1.006) + P(Y_t = 0.985) &= 1, \quad t = 2, 6, 10, 14, 18 \\ P(Y_t = 1.025) + P(Y_t = 0.975) &= 1, \quad t = 3, 7, 11, 15, 19 \\ P(Y_t = 1.013) + P(Y_t = 0.987) &= 1, \quad t = 4, 8, 12, 16, 20 \end{aligned}$$

For the European call option with a 20-month term to maturity, answer the following questions:

a) **[10 marks]** State the no-arbitrage condition.

b) **[40 marks]** Find the arbitrage-free time 0 option price of the European call option.

c) **[25 marks]** If the European call option can only be exercised when the underlying asset price equals or surpasses the level of $L = 6$ at maturity, calculate the arbitrage-free option price at time 0 for this barrier call option.

Lookback options, in the terminology of finance, are a type of exotic option with path dependency. The payoff depends on the optimal (maximum or minimum) underlying asset's price occurring over the life of the option. The option allows the holder to "look back", or review, the prices of an underlying asset over the lifespan of the option after it has been purchased. The holder may then exercise the option based on the most beneficial price of the underlying asset. The holder can take advantage of the widest differential between the strike price and the price of the underlying asset.

Consider a lookback put option with floating strike. As the name introduces it, the option's strike price is floating and determined at maturity. The floating strike is the highest price of the underlying asset during the option life. The payoff is then the maximum difference between the market asset's price at maturity and the floating strike, such that

$$C = \max(S_{\max} - S_T, 0) = S_{\max} - S_T$$

where $S_{\max}$ denotes the floating strike price, which is equal to the highest price of the underlying asset during the entire option life, such that $S_{\max} = \max_{0 \leqslant t \leqslant T} S_t$.

d) **[25 marks]** Using the above binomial tree model, find the arbitrage-free option price at time 0 for the lookback put option.

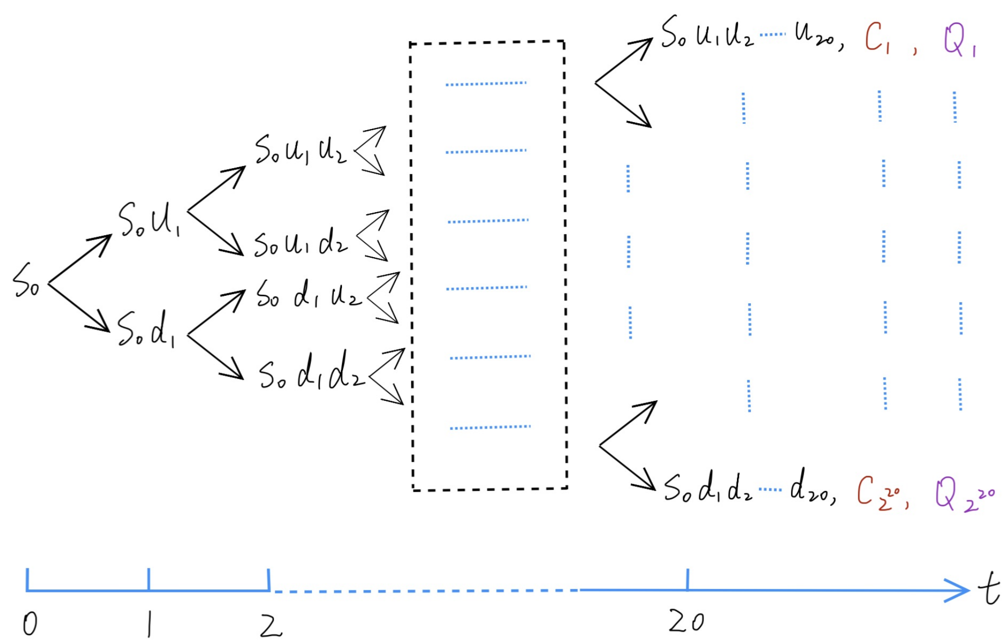
Solutions

a) **[10 marks]** As the yearly interest rate is 5.4%, the monthly interest rate is $\rho = (1+5.4\%)^{\frac{1}{12}} - 1 \approx 0.4392\%$. For each period in the 20−period tree, we have

$$\begin{aligned} u_t &= 1.02 > 1+\rho = 1.004392 > d_t = 0.98, \quad t = 1,5,9,13,17 \\ u_t &= 1.006 > 1+\rho = 1.004392 > d_t = 0.985, \quad t = 2,6,10,14,18 \\ u_t &= 1.025 > 1+\rho = 1.004392 > d_t = 0.975, \quad t = 3,7,11,15,19 \\ u_t &= 1.013 > 1+\rho = 1.004392 > d_t = 0.987, \quad t = 4,8,12,16,20 \end{aligned}$$

Therefore, the no-arbitrage condition is satisfied for all periods in the binomial tree model.

b) **[40 marks]** The binomial tree model is a time varying model with 20 periods. Therefore, there will be 2^{20} paths from S_0 to S_{20}.



Let $S_{20}^{(i)}$ and C_i, $i = 1, \cdots, 2^{20}$ denote the underlying asset price at time 20 and the payoff of the European call option corresponding to path i, respectively, e.g.,

$$\begin{aligned} C_1 &= \max(S_{20}^{(1)} - K, 0) = \max(S_0u_1 \cdots u_{20} - K, 0) \\ &= \max(5.35 \cdot 1.02^5 \cdot 1.006^5 \cdot 1.025^5 \cdot 1.013^5 - 5.65, 0) \approx 1.6953 \\ C_{2^{20}} &= \max(S_{20}^{(2^{20})} - K, 0) = \max(S_0d_1 \cdots d_{20} - K, 0) \\ &= \max(5.35 \cdot 0.98^5 \cdot 0.985^5 \cdot 0.975^5 \cdot 0.987^5 - 5.65, 0) = 0 \end{aligned}$$

Then, the arbitrage-free q-probabilities are calculated by

$$
\begin{aligned}
q_u^{(t)} &= \frac{1+\rho-d_t}{u_t-d_t} = \frac{1.004392-0.98}{1.02-0.98} \approx 0.6098, \quad q_d^{(t)} \approx 0.3902, \quad t=1,5,9,13,17 \\
q_u^{(t)} &= \frac{1+\rho-d_t}{u_t-d_t} = \frac{1.004392-0.985}{1.006-0.985} \approx 0.9234, \quad q_d^{(t)} \approx 0.0766, \quad t=2,6,10,14,18 \\
q_u^{(t)} &= \frac{1+\rho-d_t}{u_t-d_t} = \frac{1.004392-0.975}{1.025-0.975} \approx 0.5878, \quad q_d^{(t)} \approx 0.4122, \quad t=3,7,11,15,19 \\
q_u^{(t)} &= \frac{1+\rho-d_t}{u_t-d_t} = \frac{1.004392-0.987}{1.013-0.987} \approx 0.6689, \quad q_d^{(t)} \approx 0.3311, \quad t=4,8,12,16,20
\end{aligned}
$$

Let Q_i, $i=1,\cdots,2^{20}$ represent the q-probability corresponding to path i, e.g.,

$$
\begin{aligned}
Q_1 &= q_u^{(1)} q_u^{(2)} \cdots q_u^{(20)} \\
&= 0.6098^5 \cdot 0.9234^5 \cdot 0.5878^5 \cdot 0.6689^5 \approx 5.3243 \times 10^{-4} \\
Q_{2^{20}} &= q_d^{(1)} q_d^{(2)} \cdots q_d^{(20)} \\
&= 0.3902^5 \cdot 0.0766^5 \cdot 0.4122^5 \cdot 0.3311^5 \approx 1.1250 \times 10^{-12}
\end{aligned}
$$

Hence, the arbitrage-free option price at time 0 is

$$V_0 = (1+\rho)^{-20} \sum_{i=1}^{2^{20}} C_i Q_i = 0.2641$$

which is calculated by MATLAB.

c) **[25 marks]** The barrier call option has the same setup as mentioned above. However, it can only be exercised when $S_{20} \geq 6$.

Under this barrier, if $S_{20}^{(i)} < 6$, then $C_i = 0$. If $S_{20}^{(i)} \geq 6$, then $C_i = \max(S_{20}^{(i)} - K, 0) = S_{20}^{(i)} - 5.65$.

After the amendment of C_i, the barrier call option price at time 0 is

$$V_0 = (1+\rho)^{-20} \sum_{i=1}^{2^{20}} C_i Q_i = 0.2145$$

which is calculated by MATLAB.

d) **[25 marks]** Since $S_{\max}^{(i)} = \max_{0\leqslant t\leqslant 20} S_t^{(i)} = \max\left(S_0^{(i)}, S_1^{(i)}, \cdots, S_{20}^{(i)}\right)$, $i=1,\cdots,2^{20}$, the payoff of the lookback put option corresponding to path i is

$$C_i = \max\left(S_0^{(i)}, S_1^{(i)}, \cdots, S_{20}^{(i)}\right) - S_{20}^{(i)}, \quad i=1,\cdots,2^{20}$$

For example,

$$
\begin{aligned}
C_1 &= 7.3453 - 7.3453 = 0 \\
C_{2^{20}} &= 5.35 - 3.7006 = 1.6494
\end{aligned}
$$

The lookback put option price at time 0 is

$$V_0 = (1+\rho)^{-20} \sum_{i=1}^{2^{20}} C_i Q_i = 0.1251$$

which is calculated by MATLAB.

MATLAB Codes

```
S0=5.35;
rho=(1+5.4/100)^(1/12)-1; %monthly interest rate, rate over each period
K=5.65;
u=repmat([1.02 1.006 1.025 1.013],1,5);
d=repmat([0.98 0.985 0.975 0.987],1,5);
n=length(u);

%%%%%%%%%%% Compute qu for each period
qu=zeros(1,n);
for i=1:n
    qu(i)=(1+rho-d(i))/(u(i)-d(i));
end
qd=1-qu;

%%%%%%%%%%% Define the tree and q-probability for each path
St=[S0*ones(2^n,1) zeros(2^n,n)];
Q=ones(2^n,1);
for j=2:n+1
    nob=2^(n-j+1); % Number of branchs at time (j-1).
    for k=1:2^(j-1)
        if rem(k,2)==1
            St((k-1)*nob+1:k*nob,j)=St((k-1)*nob+1:k*nob,j-1)*u(j-1);
            Q((k-1)*nob+1:k*nob)=Q((k-1)*nob+1:k*nob)*qu(j-1);
        else
            St((k-1)*nob+1:k*nob,j)=St((k-1)*nob+1:k*nob,j-1)*d(j-1);
            Q((k-1)*nob+1:k*nob)=Q((k-1)*nob+1:k*nob)*qd(j-1);
        end
    end
end

%%%%%%%%%%%% ii) European call
payoff=zeros(2^n,1);
for i=1:2^n
    payoff(i)=max(St(i,n+1)-K,0);
end
V0_European_call=payoff'*Q*(1+rho)^(-n)

%%%%%%%%%%%% iii) Barrier call
L=6;
payoff=zeros(2^n,1);
for i=1:2^n
    if St(i,n+1)≥L
```

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        payoff(i)=max(St(i,n+1)-K,0);
    else
        payoff(i)=0;
    end
end
V0_Barrier_call=payoff'*Q*(1+rho)^(-n)

%%%%%%%%%%%% iv) Lookback put with floating strike
payoff=zeros(2^n,1);
for i=1:2^n
    payoff(i)=max(St(i,:))-St(i,end);
end
V0_Lookback_put=payoff'*Q*(1+rho)^(-n)
```