MA3071 Financial Mathematics - DLI, Year 2023-2024

Exercises for Feedback session 3

Exercise 1 *Find* $\mathbb{E}\left[B_2(B_8 - B_6)(B_6 - B_4)\right]$ *and* $Cov\left[3B_5, 2(B_6 - B_2)\right]$.

Exercise 2 *Find* $\mathbb{E}\left[B_T^5 + B_t^6|\mathcal{F}_s\right]$, $T \geqslant t > s \geqslant 0$.

Exercise 3 *State the distribution of the stochastic integral* $\int_0^t e^{-u} dB_u$.

Solution 1 i) Since $B_2 \perp (B_8 - B_6) \perp (B_6 - B_4)$,

$$\mathbb{E}\left[B_2(B_8 - B_6)(B_6 - B_4)\right] = \mathbb{E}\left[B_2\right]\mathbb{E}\left[B_8 - B_6\right]\mathbb{E}\left[B_6 - B_4\right] = 0$$

ii)

$$\begin{aligned} Cov\left[3B_5, 2(B_6 - B_2)\right] &= 3 \times 2 Cov\left[B_5, B_6 - B_2\right] \\ &= 6\left[Cov(B_5, B_6) - Cov(B_5, B_2)\right] \\ &= 6 \times (5 - 2) = 18 \end{aligned}$$

Solution 2

$$\mathbb{E}\left[B_T^5 + B_t^6|\mathcal{F}_s\right] = \mathbb{E}\left[B_T^5|\mathcal{F}_s\right] + \mathbb{E}\left[B_t^6|\mathcal{F}_s\right]$$

$$\begin{aligned} \mathbb{E}\left[B_T^5|\mathcal{F}_s\right] &= \mathbb{E}\left[(B_s + (B_T - B_s))^5|\mathcal{F}_s\right] \\ &= \mathbb{E}\left[\tbinom{5}{0}B_s^5(B_T - B_s)^0|\mathcal{F}_s\right] + \mathbb{E}\left[\tbinom{5}{1}B_s^4(B_T - B_s)^1|\mathcal{F}_s\right] + \mathbb{E}\left[\tbinom{5}{2}B_s^3(B_T - B_s)^2|\mathcal{F}_s\right] \\ &\quad + \mathbb{E}\left[\tbinom{5}{3}B_s^2(B_T - B_s)^3|\mathcal{F}_s\right] + \mathbb{E}\left[\tbinom{5}{4}B_s^1(B_T - B_s)^4|\mathcal{F}_s\right] + \mathbb{E}\left[\tbinom{5}{5}B_s^0(B_T - B_s)^5|\mathcal{F}_s\right] \\ &= B_s^5 + \tbinom{5}{1}\mathbb{E}\left[B_s^4|\mathcal{F}_s\right]\mathbb{E}\left[B_T - B_s|\mathcal{F}_s\right] + \tbinom{5}{2}\mathbb{E}\left[B_s^3|\mathcal{F}_s\right]\mathbb{E}\left[(B_T - B_s)^2|\mathcal{F}_s\right] \\ &\quad + \tbinom{5}{3}\mathbb{E}\left[B_s^2|\mathcal{F}_s\right]\mathbb{E}\left[(B_T - B_s)^3|\mathcal{F}_s\right] + \tbinom{5}{4}\mathbb{E}\left[B_s|\mathcal{F}_s\right]\mathbb{E}\left[(B_T - B_s)^4|\mathcal{F}_s\right] + \mathbb{E}\left[(B_T - B_s)^5|\mathcal{F}_s\right] \\ &= B_s^5 + 0 + \tbinom{5}{2}B_s^3(T - s) + 0 + \tbinom{5}{4}B_s 3(T - s)^2 + 0 \\ &= B_s^5 + \tbinom{5}{2}B_s^3(T - s) + 3\tbinom{5}{4}B_s(T - s)^2 \\ &= B_s^5 + 10B_s^3(T - s) + 15B_s(T - s)^2 \end{aligned}$$

$$
\begin{aligned}
\mathbb{E}\left[B_t^6|\mathcal{F}_s\right] &= \mathbb{E}\left[(B_s+(B_t-B_s))^6|\mathcal{F}_s\right] \\
&= \mathbb{E}\left[\tbinom{6}{0}B_s^6(B_t-B_s)^0|\mathcal{F}_s\right] + \mathbb{E}\left[\tbinom{6}{1}B_s^5(B_t-B_s)^1|\mathcal{F}_s\right] + \mathbb{E}\left[\tbinom{6}{2}B_s^4(B_t-B_s)^2|\mathcal{F}_s\right] \\
&\quad + \mathbb{E}\left[\tbinom{6}{3}B_s^3(B_t-B_s)^3|\mathcal{F}_s\right] + \mathbb{E}\left[\tbinom{6}{4}B_s^2(B_t-B_s)^4|\mathcal{F}_s\right] + \mathbb{E}\left[\tbinom{6}{5}B_s^1(B_t-B_s)^5|\mathcal{F}_s\right] \\
&\quad + \mathbb{E}\left[\tbinom{6}{6}B_s^0(B_t-B_s)^6|\mathcal{F}_s\right] \\
&= B_s^6 + \tbinom{6}{1}\mathbb{E}\left[B_s^5|\mathcal{F}_s\right]\mathbb{E}\left[B_t-B_s|\mathcal{F}_s\right] + \tbinom{6}{2}\mathbb{E}\left[B_s^4|\mathcal{F}_s\right]\mathbb{E}\left[(B_t-B_s)^2|\mathcal{F}_s\right] \\
&\quad + \tbinom{6}{3}\mathbb{E}\left[B_s^3|\mathcal{F}_s\right]\mathbb{E}\left[(B_t-B_s)^3|\mathcal{F}_s\right] + \tbinom{6}{4}\mathbb{E}\left[B_s^2|\mathcal{F}_s\right]\mathbb{E}\left[(B_t-B_s)^4|\mathcal{F}_s\right] \\
&\quad + \tbinom{6}{5}\mathbb{E}\left[B_s|\mathcal{F}_s\right]\mathbb{E}\left[(B_t-B_s)^5|\mathcal{F}_s\right] + \mathbb{E}\left[(B_t-B_s)^6|\mathcal{F}_s\right] \\
&= B_s^6 + 0 + \tbinom{6}{2}B_s^4(t-s) + 0 + \tbinom{6}{4}B_s^2 3(t-s)^2 + 0 + 15(t-s)^3 \\
&= B_s^6 + 15B_s^4(t-s) + 45B_s^2 3(t-s)^2 + 15(t-s)^3
\end{aligned}
$$

$$
\mathbb{E}\left[B_T^5 + B_t^6|\mathcal{F}_s\right] = B_s^5 + 10B_s^3(T-s) + 15B_s(T-s)^2 + B_s^6 + 15B_s^4(t-s) + 45B_s^2 3(t-s)^2 + 15(t-s)^3
$$

Solution 3 $\int_0^t Y_u dB_u \sim N\left(0, \mathbb{E}\left[\int_0^t Y_u^2 du\right]\right)$, for the given integral $\int_0^t e^{-u} dB_u$, we have

$$
\int_0^t e^{-u} dB_u \sim N\left(0, \mathbb{E}\left[\int_0^t (e^{-u})^2 du\right]\right)
$$

Thus,

$$
\begin{aligned}
\mathbb{E}\left[\int_0^t (e^{-u})^2 du\right] &= \mathbb{E}\left[\int_0^t e^{-2u} du\right] \\
&= \mathbb{E}\left[-\frac{1}{2}e^{-2u}\Big|_0^t\right] \\
&= -\frac{1}{2}e^{-2t} + \frac{1}{2}
\end{aligned}
$$

Thus, $\int_0^t e^{-u} dB_u \sim N\left(0, -\frac{1}{2}e^{-2t} + \frac{1}{2}\right)$.