MA3071 Financial Mathematics - DLI, Year 2023-2024

Exercises for Feedback session 9

Exercise 1 *The current price of a non-dividend paying stock is* $\pounds 85$ *and its volatility is* 30% *per annum. The continuously compounded risk-free interest rate is* 1% *per annum. Consider a European call option on this share with strike price* $\pounds 70$ *and expiry date in six months' time. Assume that the Black-Scholes model applies.*

i) *Calculate the time 0 price of the call option.*

ii) *Calculate the values of the Greeks for the call option.*

Exercise 2 *Consider a European put option on a non-dividend paying stock. The current stock price is* $\pounds 15$*, the exercise price, K, is* $\pounds 12$*, the continuously compounded risk-free rate of interest is* 2% *per annum, the volatility is* 20% *per annum and the term to maturity is three months.*

i) *Calculate the time 0 price of the put option using the Black-Scholes model.*

ii) *Calculate the values of the Greeks for the put option.*

Exercise 3 *A one-year European call option on a non-dividend paying stock in Company ABC has a strike price of* $\pounds 150$.

The continuously compounded risk-free rate is 2% *p.a. The current stock price is* $\pounds 117.98$*. Assume that the market follows the assumptions of a Black-Scholes model.*

An institutional investor holds a delta-hedged portfolio with $100,000$ *call options, no cash and short* $18,673$ *stocks of Company ABC.*

i) *Calculate the delta of the call option.*

ii) *Calculate the implied volatility for the underlying.*

Exercise 4 *State how to use Monto-Carlo simulation to estimate* $\mathbb{E}[\cos(B_t)]$.

Exercise 5 *State how to use Monto-Carlo simulation to calculate the time* t *option price for the claim* $f(S_T) = \max(\arctan(S_T) - K, 0)$*, where* S_t *is defined by* $dS_t = \mu S_t dt + \sigma S_t dB_t$ *and the interest rate is* ρ*.*

Solution 1 i) We know $S_0 = 85$, $\sigma = 0.3$, $\rho = 0.01$, $K = 70$, $T = 0.5$, the price of the European call option is

$$g(t, S_t) = S_t \Phi(d_1) - Ke^{-\rho(T-t)} \Phi(d_2)$$

$$d_1 = \frac{\log\left(\frac{S_t}{K}\right) + \left(\rho + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}$$
$$d_2 = d_1 - \sigma\sqrt{T-t}$$

Here, at $t = 0$,

$$\begin{aligned} g(0, S_0) &= S_0 \Phi(d_1) - Ke^{-\rho T} \Phi(d_2) \\ &= 85\Phi(d_1) - 70e^{-0.01\times 0.5} \Phi(d_2) \end{aligned}$$

$$d_1 = \frac{\log\left(\frac{S_0}{K}\right) + \left(\rho + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}} = \frac{\log\left(\frac{85}{70}\right) + \left(0.01 + \frac{1}{2}\times 0.3^2\right)\times 0.5}{0.3\sqrt{0.5}} \approx 1.04$$
$$d_2 = d_1 - \sigma\sqrt{T} = 1.04 - 0.3\sqrt{0.5} \approx 0.83$$

Then,

$$\Phi(d_1) = \Phi(1.04) = 0.85083$$
$$\Phi(d_2) = \Phi(0.83) = 0.79673$$

Thus, the price of the European call option is

$$g(0, S_0) = 85 \times 0.85083 - 70e^{-0.01\times 0.5} \times 0.79673 = 16.8276$$

ii)

$$\begin{aligned}
\boldsymbol{\Delta} \text{ (Delta)} &= \Phi(d_1) = \Phi(1.04) = 0.85083 \\
\boldsymbol{\Gamma} \text{ (Gamma)} &= \frac{\phi(d_1)}{S_0\sigma\sqrt{T}} = \frac{\frac{1}{\sqrt{2\pi}}e^{-\frac{1.04^2}{2}}}{85 \times 0.3 \times \sqrt{0.5}} = 0.0128 \\
\boldsymbol{\nu} \text{ (Vega)} &= S_0\phi(d_1)\sqrt{T} = 85 \times \frac{1}{\sqrt{2\pi}}e^{-\frac{1.04^2}{2}} \times \sqrt{0.5} = 13.8909 \\
\boldsymbol{\rho} \text{ (Rho)} &= KTe^{-\rho T}\Phi(d_2) = 70 \times 0.5 \times e^{-0.01\times 0.5} \times 0.79673 = 27.7465 \\
\boldsymbol{\Theta} \text{ (Theta)} &= -\frac{S_0\phi(d_1)\sigma}{2\sqrt{T}} - \rho Ke^{-\rho T}\Phi(d_2) \\
&= -\frac{85 \times \frac{1}{\sqrt{2\pi}}e^{-\frac{1.04^2}{2}} \times 0.3}{2\sqrt{0.5}} - 0.01 \times 70e^{-0.01\times 0.5} \times 0.79673 \\
&= -4.7222
\end{aligned}$$

Solution 2 i) We know $S_0 = 15$, $\sigma = 0.2$, $\rho = 0.02$, $K = 12$, $T = 0.25$, the price of the European put option is

$$g(t, S_t) = Ke^{-\rho(T-t)}\Phi(-d_2) - S_t\Phi(-d_1)$$

$$\begin{aligned} d_1 &= \frac{\log\left(\frac{S_t}{K}\right) + \left(\rho + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}} \\ d_2 &= d_1 - \sigma\sqrt{T-t} \end{aligned}$$

Here, at $t = 0$,

$$\begin{aligned} g(0, S_0) &= Ke^{-\rho T}\Phi(-d_2) - S_0\Phi(-d_1) \\ &= 12e^{-0.02\times0.25}\Phi(-d_2) - 15\Phi(-d_1) \end{aligned}$$

$$\begin{aligned} d_1 &= \frac{\log\left(\frac{S_0}{K}\right) + \left(\rho + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}} = \frac{\log\left(\frac{15}{12}\right) + \left(0.02 + \frac{1}{2}\times 0.2^2\right)\times 0.25}{0.2\sqrt{0.25}} \approx 2.3314 \\ d_2 &= d_1 - \sigma\sqrt{T} = 2.3314 - 0.2\sqrt{0.25} \approx 2.2314 \end{aligned}$$

Then,

$$\Phi(-d_1) = \Phi(-2.33) = 1 - \Phi(2.33) = 1 - 0.9901 = 0.0099$$

$$\Phi(-d_2) = \Phi(-2.23) = 1 - \Phi(2.23) = 1 - 0.98713 = 0.01287$$

Thus, the price of the European put option is

$$g(0, S_0) = 12e^{-0.02\times0.25} \times 0.01287 - 15 \times 0.0099 = 0.00517$$

ii)

$$\begin{aligned} \boldsymbol{\Delta}\ \text{(Delta)} &= -\Phi(-d_1) = -0.0099 \\ \boldsymbol{\Gamma}\ \text{(Gamma)} &= \frac{\phi(d_1)}{S_0\sigma\sqrt{T}} = \frac{\frac{1}{\sqrt{2\pi}}e^{-\frac{2.3314^2}{2}}}{15\times 0.2\times\sqrt{0.25}} = 0.0176 \\ \boldsymbol{\nu}\ \text{(Vega)} &= S_0\phi(d_1)\sqrt{T} = 15\times\frac{1}{\sqrt{2\pi}}e^{-\frac{2.3314^2}{2}}\times\sqrt{0.25} = 0.1976 \\ \boldsymbol{\rho}\ \text{(Rho)} &= -KTe^{-\rho T}\Phi(-d_2) = -12\times 0.25\times e^{-0.02\times0.25}\times 0.01287 = -0.0384 \\ \boldsymbol{\Theta}\ \text{(Theta)} &= -\frac{S_0\phi(d_1)\sigma}{2\sqrt{T}} + \rho Ke^{-\rho T}\Phi(-d_2) \\ &= -\frac{15\times\frac{1}{\sqrt{2\pi}}e^{-\frac{2.3314^2}{2}}\times 0.2}{2\sqrt{0.25}} + 0.02\times 12\times e^{-0.02\times0.25}\times 0.01287 \\ &= -0.0759 \end{aligned}$$

Solution 3 i) When deriving the Black-Scholes equation, we consider a Delta-hedged portfolio, -1 unit option and g'_{S_t} units of the underlying denoted by $\left(-1, g'_{S_t}\right)$. Similarly, in this question, we hold a Delta-hedged portfolio consists of 100000 options and -18673 stocks, that is $(100000, -18673)$.

Both portfolios have the same properties. We can say $(100000, -18673)$ is equivalent to short selling 100000 units of $\left(-1, g'_{S_t}\right)$.

$$\Rightarrow -100000\left(-1, g'_{S_t}\right) \to (100000, -18673)$$
$$\Rightarrow -100000 g'_{S_t} = -18673$$
$$\Rightarrow g'_{S_t} = 0.18673 = \Delta$$

ii) By definition $\Delta = g'_{S_t} = \Phi(d_1)$ for European call options.

$$\Phi(d_1) = 0.18673 \Rightarrow d_1 = -0.89$$

As we know $K = 150$, $\rho = 0.02$, $S_0 = 117.98$, $T = 1$, then

$$\begin{aligned} d_1 &= \frac{\log\left(\frac{S_0}{K}\right) + \left(\rho + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}} \\ &= \frac{\log\left(\frac{117.98}{150}\right) + \left(0.02 + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{1}} \\ &= \frac{-0.24 + 0.02 + \frac{1}{2}\sigma^2}{\sigma} \\ &= -\frac{0.22}{\sigma} + \frac{\sigma}{2} = -0.89 \end{aligned}$$

$$\Rightarrow \quad -0.22 + 0.89\sigma + \frac{1}{2}\sigma^2 = 0$$
$$\Rightarrow \quad \sigma = -0.89 \pm \sqrt{(0.89^2 + 0.44)}$$

Rejecting the negative root, $\sigma \approx 22\%$.

Solution 4 Generate M independent observations $z_i, i = 1, \cdots, M$ from a $N(0,1)$ distribution, such that

$$P(Z \leqslant z_i) = \Phi(z_i) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z_i} e^{-\frac{x^2}{2}} dx$$

where $Z \sim N(0,1)$ is a random variable. Then,

$$\mathbb{E}\left[f(t, B_t) | \mathcal{F}_s\right] \approx \frac{1}{M} \sum_{i=1}^{M} f\left(t, B_s + z_i\sqrt{t-s}\right)$$

Thus, we have

$$\mathbb{E}\left[\cos(B_t)\right] \approx \frac{1}{M} \sum_{i=1}^{M} \cos\left(\sqrt{t} z_i\right)$$

Solution 5 Under the no-arbitrage condition, the geometric Brownian motion is $S_t = S_0 e^{\left(\rho - \frac{1}{2}\sigma^2\right)t + \sigma B_t}$. For the given option claim,

$$f(S_T) = \max(\arctan(S_T) - K, 0) \Longrightarrow f(x) = \max(\arctan(x) - K, 0)$$

Thus, using Monte-Carlo simulation,

$$\begin{aligned} g(t, S_t) &\approx e^{-\rho(T-t)} \frac{1}{M} \sum_{i=1}^{M} f\left(S_t e^{a(T-t) + \sigma z_i \sqrt{T-t}}\right) \\ &= e^{-\rho(T-t)} \frac{1}{M} \sum_{i=1}^{M} \left[\max\left(\arctan\left[S_t e^{a(T-t) + \sigma z_i \sqrt{T-t}}\right] - K, 0\right)\right] \end{aligned}$$

where $a = \rho - \frac{1}{2}\sigma^2$ and z_i, $i = 1, \ldots, M$ are M independent observations from the standard normal distribution $N(0, 1)$.