MA3071 Financial Mathematics - DLI, Year 2023-2024

Exercises for Feedback session 8

Exercise 1 *The current price of a non-dividend paying stock is £85 and its volatility is 30% per annum. The continuously compounded risk-free interest rate is 1% per annum. Consider a European call option on this share with a strike price of £70 and expiry date in six-month time. Assume that the Black-Scholes model applies.*

Calculate the time 0 price of the call option.

Exercise 2 *Consider a European put option on a non-dividend paying stock. The current stock price is £15, the exercise price, K, is £12, the continuously compounded risk-free rate of interest is 2% per annum, the volatility is 20% per annum and the term to maturity is three months.*

Calculate the time 0 price of the put option using the Black-Scholes model.

Exercise 3 *In a continuous time market, a company's share price S_t, is modelled by a stochastic differential equation:*

$$dS_t = \mu S_t dt + \sigma S_t dB_t$$

where $\{B_t; t \geqslant 0\}$ is a standard Brownian motion. Suppose that the risk-free interest rate is ρ.

i) State the no arbitrage condition.

ii) Using the Black-Scholes formula, find the arbitrage free time t option price of the option claim S_T^{2022}.

iii) Show that the option price identified in ii) satisfies the Black-Scholes equation.

Solution 1 We know $S_0 = 85$, $\sigma = 0.3$, $\rho = 0.01$, $K = 70$, $T = 0.5$, the price of the European call option is

$$g(t, S_t) = S_t \Phi(d_1) - K e^{-\rho(T-t)} \Phi(d_2)$$

$$\begin{aligned} d_1 &= \frac{\log\left(\frac{S_t}{K}\right) + \left(\rho + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}} \\ d_2 &= d_1 - \sigma\sqrt{T-t} \end{aligned}$$

Here, at $t = 0$,

$$\begin{aligned} g(0, S_0) &= S_0 \Phi(d_1) - K e^{-\rho T} \Phi(d_2) \\ &= 85\Phi(d_1) - 70 e^{-0.01 \times 0.5} \Phi(d_2) \end{aligned}$$

$$d_1 = \frac{\log\left(\frac{S_0}{K}\right) + \left(\rho + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}} = \frac{\log\left(\frac{85}{70}\right) + \left(0.01 + \frac{1}{2} \times 0.3^2\right) \times 0.5}{0.3\sqrt{0.5}} \approx 1.04$$
$$d_2 = d_1 - \sigma\sqrt{T} = 1.04 - 0.3\sqrt{0.5} \approx 0.83$$

Then,

$$\Phi(d_1) = \Phi(1.04) = 0.85083$$
$$\Phi(d_2) = \Phi(0.83) = 0.79673$$

Thus, the price of the European call option is

$$g(0, S_0) = 85 \times 0.85083 - 70e^{-0.01\times0.5} \times 0.79673 = 16.8276$$

Solution 2 We know $S_0 = 15$, $\sigma = 0.2$, $\rho = 0.02$, $K = 12$, $T = 0.25$, the price of the European put option is

$$g(t, S_t) = Ke^{-\rho(T-t)}\Phi(-d_2) - S_t\Phi(-d_1)$$

$$d_1 = \frac{\log\left(\frac{S_t}{K}\right) + \left(\rho + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}$$
$$d_2 = d_1 - \sigma\sqrt{T-t}$$

Here, at $t = 0$,

$$\begin{aligned} g(0, S_0) &= Ke^{-\rho T}\Phi(-d_2) - S_0\Phi(-d_1) \\ &= 12e^{-0.02\times0.25}\Phi(-d_2) - 15\Phi(-d_1) \end{aligned}$$

$$d_1 = \frac{\log\left(\frac{S_0}{K}\right) + \left(\rho + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}} = \frac{\log\left(\frac{15}{12}\right) + \left(0.02 + \frac{1}{2} \times 0.2^2\right) \times 0.25}{0.2\sqrt{0.25}} \approx 2.3314$$
$$d_2 = d_1 - \sigma\sqrt{T} = 2.3314 - 0.2\sqrt{0.25} \approx 2.2314$$

Then,

$$\Phi(-d_1) = \Phi(-2.33) = 1 - \Phi(2.33) = 1 - 0.9901 = 0.0099$$
$$\Phi(-d_2) = \Phi(-2.23) = 1 - \Phi(2.23) = 1 - 0.98713 = 0.01287$$

Thus, the price of the European put option is

$$g(0, S_0) = 12e^{-0.02\times0.25} \times 0.01287 - 15 \times 0.0099 = 0.00517$$

Solution 3 i) As the share price S_t is modelled by $dS_t = \mu S_t dt + \sigma S_t dB_t$, the no arbitrage condition is $\mu = \rho$.

ii) For the option claim S_T^{2022}

$$g(t, S_t) = e^{-\rho(T-t)}\mathbb{E}\left[\left(S_t e^{a(T-t)+\sigma(B_T-B_t)}\right)^{2022}\middle|\mathcal{F}_t\right]$$

Recall that $\mathbb{E}\left[\left(e^{a(T-t)+\sigma(B_T-B_t)}\right)^k \middle|\mathcal{F}_t\right] = e^{\left(ka+\frac{k^2\sigma^2}{2}\right)(T-t)}$, here $k = 2022$. And the no-arbitrage condition is $a = \rho - \frac{\sigma^2}{2}$.

$$\begin{aligned} g(t, S_t) &= e^{-\rho(T-t)} \cdot S_t^{2022} \cdot e^{\left(2022a+\frac{2022^2\sigma^2}{2}\right)(T-t)} \\ &= e^{-\rho(T-t)} \cdot S_t^{2022} \cdot e^{\left(2022\rho-\frac{2022\sigma^2}{2}+\frac{2022^2\sigma^2}{2}\right)(T-t)} \\ &= e^{\left(2021\rho+1011\times 2021\sigma^2\right)(T-t)} S_t^{2022} \end{aligned}$$

iii) Verify the Black-Scholes equation, which means it should satisfy the following PDE

$$g'_t + \rho S_t g'_{S_t} + \frac{1}{2}\sigma^2 S_t^2 g''_{S_t S_t} = \rho g$$

$$\begin{aligned} g(t, S_t) &= S_t^{2022} \cdot e^{2021\left(\rho+1011\sigma^2\right)(T-t)} \\ g'_t &= S_t^{2022} \cdot e^{2021\left(\rho+1011\sigma^2\right)(T-t)} \cdot 2021\left(-\rho-1011\sigma^2\right) = -2021\left(\rho+1011\sigma^2\right) g \\ g'_{S_t} &= 2022 \cdot S_t^{2021} \cdot e^{2021\left(\rho+1011\sigma^2\right)(T-t)} = \frac{2022g}{S_t} \\ g''_{S_t S_t} &= 2022 \times 2021 \cdot S_t^{2020} \cdot e^{2021\left(\rho+1011\sigma^2\right)(T-t)} = \frac{2022 \times 2021 g}{S_t^2} \end{aligned}$$

Then, verify the results,

$$\begin{aligned} \text{Left hand side} &= -2021\left(\rho+1011\sigma^2\right) g + \rho \cdot S_t \frac{2022g}{S_t} + \frac{1}{2}\sigma^2 S_t^2 \frac{2022 \times 2021 g}{S_t^2} \\ &= -2021\rho g - 2021 \times 1011\sigma^2 g + 2022\rho g + 1011 \times 2021\sigma^2 g \\ &= \rho g = \text{Right hand side} \end{aligned}$$

Thus, $g(t, S_t)$ is a solution of Black-Scholes PDE for the option claim S_T^{2022}.