Answer to question 1

(a) i. [Similar to seen] Let the proportion invested in asset i be x_i, with expected return E_i, standard deviation σ_i and covariance σ_{ij}. Let λ and μ be Lagrange multipliers. Then, the Lagrangian function W satisfies:

$$\begin{aligned}\mathcal{L} &= \sum_{i=1}^{3} x_i^2\sigma_i^2 + 2x_2x_3\sigma_{23} - \lambda(x_1E_1 + x_2E_2 + x_3E_3 - 3) - \mu(x_1 + x_2 + x_3 - 1)\\ &= 4x_1^2 + x_2^2 + x_3^2 + 2x_2x_3 - \lambda(x_1 + 2x_2 + 3x_3 - 3) - \mu(x_1 + x_2 + x_3 - 1)\end{aligned}$$

Then, we have

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial x_1} &= 8x_1 - \lambda - \mu = 0\\ \frac{\partial\mathcal{L}}{\partial x_2} &= 2x_2 + 2x_3 - 2\lambda - \mu = 0\\ \frac{\partial\mathcal{L}}{\partial x_3} &= 2x_3 + 2x_2 - 3\lambda - \mu = 0\\ \frac{\partial\mathcal{L}}{\partial \lambda} &= -(x_1 + 2x_2 + 3x_3 - 3) = 0\\ \frac{\partial\mathcal{L}}{\partial \mu} &= -(x_1 + x_2 + x_3 - 1) = 0\end{aligned}$$

Solve these linear equations, we get

$$\begin{aligned}x_1 &= 0.2\\ x_2 &= -0.4\\ x_3 &= 1.2\\ \lambda &= 0\\ \mu &= 1.6\end{aligned}$$

[12 marks]

ii. [Unseen] The minimum variance is

$$\begin{aligned}Var &= \sum_{i=1}^{3} x_i^2\sigma_i^2 + 2x_2x_3\sigma_{23}\\ &= 4x_1^2 + x_2^2 + x_3^2 + 2x_2x_3\\ &= 4\times(0.2)^2 + (-0.4)^2 + (1.2)^2 + 2\times(-0.4)\times 1.2\\ &= 0.8\end{aligned}$$

[3 marks]

(b) i. [Similar to seen] As the market capitalisation of Asset 2 is 40,000 and the market capitalisation of Asset 3 is 60,000, the market portfolio weights invested in Asset 2 and Asset 3 are 40% and 60%, respectively. We then have four paths of the possible returns of market portfolio

Probabilities	R_M
0.1	$0.4\times5+0.6\times6=5.6$
0.3	$0.4\times3+0.6\times3=3$
0.4	$0.4\times4+0.6\times5=4.6$
0.2	$0.4\times6+0.6\times8=7.2$

Thus, we have

$$\begin{aligned}
\mathbb{E}[R_M] &= 0.1\times5.6+0.3\times3+0.4\times4.6+0.2\times7.2=4.74\\
\sigma_M^2 &= \sum_{i=1}^{4} P_i(r_i-\mathbb{E}[R_M])^2\\
&= 0.1(5.6-4.74)^2+0.3(3-4.74)^2\\
&\quad +0.4(4.6-4.74)^2+0.2(7.2-4.74)^2\\
&= 2.2004\\
\sigma_M &\approx 1.483375
\end{aligned}$$

The market price of risk is

$$\frac{\mathbb{E}[R_M]-\rho}{\sigma_M}=\frac{4.74-3}{1.483375}=1.173001$$

[5 marks]

ii. [Similar to seen] Betas are defined as

$$\beta_i=\frac{\mathbb{E}[R_i]-\rho}{\mathbb{E}[R_M]-\rho},\ i=2,3$$

We have

$$\begin{aligned}
\mathbb{E}[R_2] &= 0.1\times5+0.3\times3+0.4\times4+0.2\times6=4.2\\
\mathbb{E}[R_3] &= 0.1\times6+0.3\times3+0.4\times5+0.2\times8=5.1
\end{aligned}$$

Hence,

$$\begin{aligned}
\beta_2 &= \frac{\mathbb{E}[R_2]-\rho}{\mathbb{E}[R_M]-\rho}=\frac{4.2-3}{4.74-3}=\frac{20}{29}\approx 0.689655\\
\beta_3 &= \frac{\mathbb{E}[R_3]-\rho}{\mathbb{E}[R_M]-\rho}=\frac{5.1-3}{4.74-3}=\frac{35}{29}\approx 1.206897
\end{aligned}$$

[5 marks]

Answer to question 2

(a) [Challenging] The loss of short position equals the profit of long position,

$$L = (S_T - K)_+ - D = (S_T - 10)_+ - 2$$

Therefore, when $S_T < 10$, the option won't be exercised, $L = -2$.

For any $x < -2$, $P(L \leq x) = 0$.

When $S_T \geq 10$, the option will be exercised, $L = S_T - 12$.

For any $x \geq -2$,

$$\begin{aligned} P(L \leq x) &= P(S_T \leq x + 12) \\ &= \int_{-\infty}^{x+12} f(t)dt \\ &= \int_{10}^{x+12} \frac{200}{t^3} dt \\ &= \left[-\frac{100}{t^2} \right]_{10}^{x+12} \\ &= 1 - \frac{100}{(x+12)^2} \end{aligned}$$

Hence, the cumulative distribution of loss is

$$F_L(x) = \begin{cases} 0 & x < -2 \\ 1 - \dfrac{100}{(x+12)^2} & x \geqslant -2 \end{cases}$$

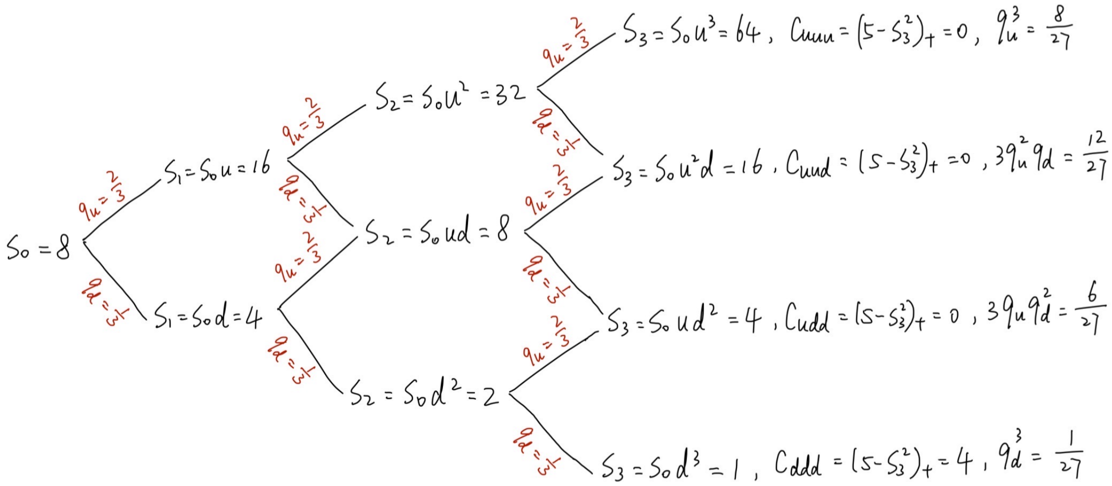
[10 marks]

(b) i. [Similar to seen] As $\mathrm{P}(Y_i = 2) = 0.7, \quad \mathrm{P}(Y_i = 0.5) = 0.3, \quad i = 1, 2, 3$, we have $u = 2$ and $d = 0.5$. Therefore the q-probabilities are

$$q_u = \frac{1 + \rho - d}{u - d} = \frac{1 + 0.5 - 0.5}{2 - 0.5} = \frac{2}{3}, \quad q_d = 1 - q_u = \frac{1}{3}$$

[3 marks]

ii. [Similar to seen]



[7 marks]

iii. [Similar to seen] The time 0 option price of the option claim $C = \left(5 - S_3^2\right)_+$ is

$$\begin{aligned} V_0 &= \frac{C_{uuu} q_u^3 + 3C_{uud} q_u^2 q_d + 3C_{udd} q_u q_d^2 + C_{ddd} q_d^3}{(1+\rho)^3} \\ &= \frac{0 \times \frac{8}{27} + 0 \times \frac{12}{27} + 0 \times \frac{6}{27} + 4 \times \frac{1}{27}}{(1+0.5)^3} = \frac{32}{729} \approx 0.044 \end{aligned}$$

[5 marks]

Answer to question 3

(a) [Similar to seen] The stochastic integral has a normal distribution

$$\mathbb{E}\left[\int_0^t -2e^{-2u} dB_u\right] = 0$$

$$\begin{aligned} var\left[\int_0^t -2e^{-2u} dB_u\right] &= \mathbb{E}\left[\int_0^t \left(-2e^{-2u}\right)^2 d_u\right] \\ &= \mathbb{E}\left[\int_0^t 4e^{-4u} d_u\right] \\ &= \mathbb{E}\left[-e^{-4u}|_0^t\right] \\ &= \mathbb{E}\left[1-e^{-4t}\right] \\ &= 1-e^{-4t} \end{aligned}$$

Thus, $\int_0^t -2e^{-2u} dB_u \sim N(0, 1-e^{-4t})$. **[5 marks]**

(b) $S_0 = 7$, $\sigma = 0.4$, $\rho = 0.05$, $K = 6.5$, $T = 1$.

i. [Similar to seen] The price of the European call option is

$$\begin{aligned} g(0,S_0) &= S_0\Phi(d_1) - Ke^{-\rho T}\Phi(d_2) = 7\Phi(d_1) - 6.5e^{-0.05}\Phi(d_2) \\ d_1 &= \frac{\log\left(\frac{S_0}{K}\right) + \left(\rho + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} = \frac{\log\left(\frac{7}{6.5}\right) + \left(0.05 + \frac{0.4^2}{2}\right)}{0.4} = 0.5103 \\ d_2 &= d_1 - \sigma\sqrt{T} = 0.5103 - 0.4 = 0.1103 \\ \Phi(d_1) &= \Phi(0.5103) = 0.6951 \\ \Phi(d_2) &= \Phi(0.1103) = 0.5439 \end{aligned}$$

Hence, the price is

$$g(0,S_0) = 7 \times 0.6951 - 6.5e^{-0.05} \times 0.5439 = 1.5034$$

[7 marks]

ii. [Similar to seen] $\Delta = \frac{\partial g}{\partial S_0} = \Phi(d_1) = 0.6951$. **[3 marks]**

iii. [Challenging] As interest rate increases in the market, the expected return required by investors in stock tends to increase. However, the present value of any future cash flow generated by option contracts decreases. The combined impact of these two effects is to increase the value of the call option. **[5 marks]**

(c) [Similar to seen] Let $f(t,S_t) = e^{-\rho t}S_t$

$$\begin{aligned} f_t' &= e^{-\rho t}S_t(-\rho) = -\rho e^{-\rho t}S_t \\ f_{S_t}' &= e^{-\rho t} \\ f_{S_tS_t}'' &= 0 \end{aligned}$$

$$dS_t = \mu S_t dt + \sigma S_t dB_t$$
$$(dS_t)^2 = \sigma^2 S_t^2 dt$$

$$\begin{aligned}
df(t, s_t) &= f_t' dt + f_{S_t}' dS_t + \frac{1}{2} f_{S_t S_t}'' (dS_t)^2 \\
&= -\rho e^{-\rho t} S_t dt + e^{-\rho t} (\mu S_t dt + \sigma S_t dB_t) + \frac{1}{2} \times 0 \\
&= -\rho e^{-\rho t} S_t dt + e^{-\rho t} \mu S_t dt + e^{-\rho t} \sigma S_t dB_t \\
&= e^{-\rho t} S_t (\mu - \rho) dt + e^{-\rho t} \sigma S_t dB_t \\
&= e^{-\rho t} \sigma S_t dB_t
\end{aligned}$$

iff $\mu = \rho$ to make sure $e^{-\rho t} S_t$ martingale. Hence, $\mu = 0.7$. **[5 marks]**

Answer to question 4

(a) [Challenging]

$$
\begin{aligned}
cov(B_4^3, B_5 - B_3) &= \mathbb{E}\left[B_4^3(B_5 - B_3)\right] - \mathbb{E}\left[B_4^3\right]\mathbb{E}\left[B_5 - B_3\right] \\
&= \mathbb{E}\left[B_4^3(B_5 - B_3)\right] \\
&= \mathbb{E}\left[B_4^3 B_5\right] - \mathbb{E}\left[B_4^3 B_3\right] \\
&= \mathbb{E}\left[B_4^3(B_5 - B_4 + B_4)\right] - \mathbb{E}\left[(B_4 - B_3 + B_3)^3 B_3\right] \\
&= \mathbb{E}\left[B_4^3(B_5 - B_4)\right] + \mathbb{E}\left[B_4^4\right] - \mathbb{E}\left[(B_4 - B_3)^3 B_3\right] \\
&\qquad - 3\mathbb{E}\left[(B_4 - B_3)^2 B_3^2\right] - 3\mathbb{E}\left[(B_4 - B_3) B_3^3\right] - \mathbb{E}\left[B_3^4\right] \\
&= 0 + 3 \times 4^2 - 0 - 3 \times (4 - 3) \times 3 - 0 - 3 \times 3^2 \\
&= 12
\end{aligned}
$$

Since $B_4 \perp (B_5 - B_4)$, $B_3 \perp (B_4 - B_3)$. **[7 marks]**

(b) [Similar to seen] As $dX_t = t^2 B_t dB_t$, we have $A_t = 0$, $Y_t = t^2 B_t$,

$$
df(t, X_t) = \left(f_t' + A_t f_{X_t}' + \frac{1}{2} Y_t^2 f_{X_t X_t}''\right) dt + Y_t f_{X_t}' dB_t
$$

since $f(t, X_t) = t\sin(X_t)$, we have

$$
\begin{aligned}
f_t' &= \sin(X_t) \\
f_{X_t}' &= t\cos(X_t) \\
f_{X_t X_t}'' &= -t\sin(X_t)
\end{aligned}
$$

Hence,

$$
\begin{aligned}
df(t, X_t) &= \left(\sin(X_t) + 0 + \frac{1}{2} t^4 B_t^2 (-t\sin(X_t))\right) dt + t^2 B_t t\cos(X_t) dB_t \\
&= \left(\sin(X_t) - \frac{1}{2} t^5 B_t^2 \sin(X_t)\right) dt + t^3 B_t \cos(X_t) dB_t
\end{aligned}
$$

[6 marks]

(c) [Similar to seen] Let $f(t, B_t) = e^{B_t - \frac{a^2 t}{2}}$

$$
\begin{aligned}
f_t' &= -\frac{a^2}{2} e^{B_t - \frac{a^2 t}{2}} = -\frac{a^2}{2} f \\
f_{B_t}' &= e^{B_t - \frac{a^2 t}{2}} = f \\
f_{B_t B_t}'' &= e^{B_t - \frac{a^2 t}{2}} = f
\end{aligned}
$$

$$
\begin{aligned}
df(t, B_t) &= f_t' dt + f_{B_t}' dB_t + \frac{1}{2} f_{B_t B_t}'' dt \\
&= -\frac{a^2}{2} f dt + f dB_t + \frac{1}{2} f dt \\
&= \left(-\frac{a^2}{2} f + \frac{1}{2} f\right) dt + f dB_t
\end{aligned}
$$

For martingale, there should be no dt term. Hence, if and only if $-\frac{a^2}{2}f + \frac{1}{2}f = 0$, which implies $a = \pm 1$. **[7 marks]**

(d) [Similar to seen] Generate M independent observations $z_i, i = 1, \cdots, M$ from a $N(0,1)$ distribution, such that

$$P(Z \leq z_i) = \Phi(z_i) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z_i} e^{-\frac{x^2}{2}} dx$$

where $Z \sim N(0,1)$ is a random variable. Then,

$$\mathbb{E}[\tan(B_t^2)] \approx \frac{1}{M} \sum_{i=1}^{M} \tan(t z_i^2)$$

[5 marks]