LINEAR ALGEBRA II

Ch. VI DETERMINANTS

Ch. 6. Determinants

- $\mathrm{Det}(A) = |A| = D(A) = D(A^1, \ldots, A^n)$. $A \in \mathrm{Mat}_{n\times n}(K)$

  1. $D(A^1, \ldots, C + C', \ldots, A^n) = D(A^1, \ldots, C, \ldots, A^n) + D(A^1, \ldots, C', \ldots, A^n)$;
  2. $D(A^1, \ldots, tC, \ldots, A^n) = tD(A^1, \ldots, C, \ldots, A^n)$; multi-linear
  3. $D(A^1, \ldots, A^j, A^j, \ldots, A^n) = 0$; alternating
  4. $D(A^1, \ldots, A^i, \ldots, A^j, \ldots, A^n) = -D(A^1, \ldots, A^j, \ldots, A^i, \ldots, A^n)$;
  5. $D(A^1, \ldots, A^j, \ldots, A^j, \ldots, A^n) = 0$;
  6. $D(A^1, \ldots, A^i, \ldots, A^j + tA^i, \ldots, A^n) = D(A^1, \ldots, A^i, \ldots, A^j, \ldots, A^n)$;
  7. $D(A^{\mathrm{T}}) = D(A)$;

Ch. 6. Determinants

- Definition of the determinant:
  - By induction, expansion according to the 1st row:

$$D(A) = (-1)^{1+1}a_{11}D(A_{11}) + \cdots + (-1)^{1+n}a_{1n}D(A_{1n}).$$

  - By permutation, the number of inversions, expansion formula.

$$D(A) = \sum_{\sigma} \epsilon(\sigma) a_{\sigma(1),1} \cdots a_{\sigma(n),n}.$$

$\{1\ 2\ 3\ 4\}$

$\sigma = \{2\ 4\ 3\ 1\}$

$N(\sigma) = 1+2+1 = 4$

$\epsilon(\sigma) = (-1)^4 = 1$

  - By Property 1-3 with $D(I) = 1$.

① $D(A) = \begin{vmatrix} a_{11} & * & \cdots & * \\ 0 & & & \\ \vdots & & & \\ 0 & * & \cdots & * \end{vmatrix} + \begin{vmatrix} 0 & * & \cdots & * \\ a_{21} & \vdots & & \vdots \\ 0 & \vdots & & \vdots \\ \vdots & & & \\ 0 & * & & * \end{vmatrix} + \cdots + \begin{vmatrix} 0 & * & \cdots & * \\ \vdots & \vdots & & \vdots \\ 0 & \vdots & & \vdots \\ a_{n1} & * & \cdots & * \end{vmatrix}$ $N(\sigma)$

$\underline{n^n \text{ terms}}$ ③. ② $\epsilon(\sigma) = \begin{vmatrix} a_{\sigma(1),1} & & \\ & a_{\sigma(2),2} & \\ & & \ddots \\ & & & a_{\sigma(n),n} \end{vmatrix}$ $\Rightarrow n!$ terms

$\sigma(1), \cdots, \sigma(n)$

$\{1, \cdots, n\}$

$\begin{vmatrix} a_{\sigma(1),1} & & a_{\sigma(1),i} \\ & a_{\sigma(2),2} & \\ & & \ddots \end{vmatrix}$

$(-1)^{N(\sigma)}$

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- $D(A) = (-1)^{i+1} a_{i1} D(A_{i1}) + \cdots + (-1)^{i+n} a_{in} D(A_{in})$. i-th row
- $D(A) = (-1)^{1+j} a_{1j} D(A_{1j}) + \cdots + (-1)^{n+j} a_{nj} D(A_{nj})$. j-th column
- Laplace Theorem.

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- $D(AB) = D(A)D(B)$;
- $D(A^k) = D(A)^k$;
- $D(A) \neq 0 \Leftrightarrow A$ is invertible (non-singular) $\Leftrightarrow$ rank $A = n \Leftrightarrow$ Columns (rows) of A are L. I.
- $D(A^{-1}) = D(A)^{-1}$.
- Cramer's rule: $x_i = \dfrac{D(A^1, \ldots, \overset{i}{B}, \ldots, A^n)}{D(A)}$.

$AX = B$

$|A| \neq 0$

- $B = (b_{ij})$, where,

$$b_{ij} = \frac{D(A^1, \ldots, \overset{i}{E^j}, \ldots, A^n)}{D(A)}.$$

is the inverse of A.

$AX = E^j$

$\Uparrow$

$B^j = A^{-1}E^j$

$A^{-1} = B$

Ch. 6. Determinants

- $\tilde{A} = (\tilde{a}_{ij})$, where,
$$\tilde{a}_{ij} = D(A^1, \ldots, E^j, \ldots, A^n).$$
is the called the adjugate matrix of A. It satisfies
$$A\tilde{A} = \tilde{A}A = D(A)I.$$

伴随矩阵

- rank $\tilde{A}$ =?.

Ch. 6. Determinants

- The column rank and row rank of a matrix A.
- The rank of a set of vectors $\{v_1, \ldots, v_n\}$. $v_i \in V$ $L: V \to W$
- The rank of a linear map. L, $\dim \operatorname{Im} L = \operatorname{rank}\{L(v_1), \ldots, L(v_n)\}$
- Rank r of A = the highest order of non-zero minors (subdeterminant, determinant of submatrix) of A. $\Leftrightarrow$ There exists a nonzero minor of order r, and all minors of order $r+1$ are 0.
- rank A =colum rank of A=row rank of A.

Proof. ① If $c.r(A) = r.r(A) = r \Rightarrow \operatorname{rank}(A) = r$

Any $r+1$ rows of A are linearly dependent. $\Rightarrow$

Rows of any submatrix of A of order $r+1$ are L.D.

$\Rightarrow$ Any minor of order $r+1$ is 0

$r.r.(A)=r. \Rightarrow \exists$ r rows of A, say, the first r rows, are L.I.

$c.r.\begin{pmatrix} A_1 \\ \vdots \\ A_r \end{pmatrix} = r.r\begin{pmatrix} A_1 \\ \vdots \\ A_r \end{pmatrix} = r \Rightarrow \exists$ r columns of it, say the first r columns, are L.I.

$\Rightarrow$ The minor of A consists of the first r rows and the first r columns is nonzero. $\Rightarrow$ $\text{rank}(A)=r$

(2) $\text{rank}(A)=r \Rightarrow$ All minors of A of order $r+1$ are 0.

If $c.r.(A)=t$. $\underline{t \geq r}$ $\quad (t<r \Rightarrow \text{rank}(A)<r)$

If $t>r$, then there are $t>r$ rows of A, L.I.

$\Rightarrow \exists$ a Minor of A of order $t>r$ nonzero. contradicts with $\text{rank}(A)=r$.

$\Rightarrow c.r.(A)=r=r.r(A)$

Ch. 6. Determinants

- column (row) operation;
- column (row) equivalent;
- Triangulation by column (row) operations.
- rank $A_{m\times n}$=r $\Leftrightarrow$ There exist non-singular matrices $P_{m\times m}$ and $Q_{n\times n}$, s.t.

$$A = P^{-1}\begin{pmatrix} I_r & O \\ O & O \end{pmatrix} Q^{-1}$$

$$PAQ = \begin{pmatrix} I_r & O \\ O & O \end{pmatrix}.$$

full-rank decomposition

$$A \xrightarrow{c_i + \alpha c_j} A \cdot \begin{array}{c} \begin{array}{cc} i & j \end{array} \\ \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & \alpha & 1 & \\ & & & & \ddots \\ & & & & & 1 \end{pmatrix} \end{array}$$

$$A \xrightarrow{\alpha c_i} A \cdot \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & \alpha & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}$$

$$A \xrightarrow{c_i \leftrightarrow c_j} A \cdot \begin{array}{c} \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 0 & & 1 & & \\ & & & \ddots & & & \\ & & 1 & & 0 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix} \\ \begin{array}{cc} i & j \end{array} \end{array}$$

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- $\begin{vmatrix} A & C \\ O & B \end{vmatrix} = |A| \cdot |B|, \quad \begin{vmatrix} A & O \\ C & B \end{vmatrix} = |A| \cdot |B|, \quad \begin{vmatrix} A & O \\ O & B \end{vmatrix} = |A| \cdot |B|.$

$$|A| \xrightarrow[\text{of type 1}]{\text{row operations}} \begin{vmatrix} s_1 & \cdots & * \\ & \ddots & \vdots \\ 0 & & s_m \end{vmatrix} = s_1 \cdots s_m$$

$$|B| \xrightarrow[\text{of type 1}]{\text{row operation}} \begin{vmatrix} t_1 & * & \cdots & * \\ & \ddots & \ddots & \vdots \\ & & \ddots & * \\ & & & t_n \end{vmatrix} = t_1 \cdots t_n$$

$$\begin{vmatrix} A & C \\ O & B \end{vmatrix} \xrightarrow[\text{of type 1 according to that for } A, B]{\text{row operations}} \begin{vmatrix} s_1 & * & \cdots & * & * & \cdots & * \\ & \ddots & & \vdots & \vdots & & \vdots \\ 0 & & s_m & & * & \cdots & * \\ & & & t_1 & * & \cdots & * \\ & & & & \ddots & & \vdots \\ & & & 0 & & & t_n \end{vmatrix} = s_1 \cdots s_m t_1 \cdots t_n = |A| \cdot |B|$$

Ch. 6. Determinants

- partitioned column (row) operations;

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \xrightarrow{C_2 + C_1 \cdot P} \begin{pmatrix} A & B+AP \\ C & D+CP \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} I & P \\ 0 & I \end{pmatrix}$$

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \xrightarrow{C_2 \leftrightarrow C_1} \begin{pmatrix} B & A \\ D & C \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \xrightarrow[P \text{ nonsi-gular}]{C_1 P} \begin{pmatrix} AP & B \\ CP & D \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} P & 0 \\ 0 & I \end{pmatrix}$$

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$$\xrightarrow{r_i \leftrightarrow r_j}$$

$$\xrightarrow{r_j + r_i}$$

$$\xrightarrow{r_i - r_j}$$

$$r_j + r_i$$

A

B

Ch. 6. Determinants

- $\begin{vmatrix} A & B \\ B & A \end{vmatrix} = |A+B| \cdot |A-B|.$

$A_{n\times n}$ $B_{n\times n}$

$$\begin{vmatrix} A & B \\ A+B & A+B \end{vmatrix} = \begin{vmatrix} A-B & B \\ 0 & A+B \end{vmatrix}$$

- $\begin{vmatrix} I_n & B \\ A & I_m \end{vmatrix} = |I_m - AB| = |I_n - BA|.$

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- compute inverse by (partitioned) row operations;

$$\begin{pmatrix} A & I \end{pmatrix} \xrightarrow{\text{row operations}} \begin{pmatrix} I & A^{-1} \end{pmatrix}$$

$$\begin{pmatrix} \boxed{} & \boxed{I} \end{pmatrix} \xrightarrow{\text{partitioned row operations}} \begin{pmatrix} I & A^{-1} \end{pmatrix}$$

- $\begin{pmatrix} A & O \\ C & B \end{pmatrix}^{-1} = \begin{pmatrix} A^{-1} & O \\ -B^{-1}CA^{-1} & B^{-1} \end{pmatrix}.$

$$\left(\begin{array}{cc|cc} A & O & I & O \\ C & B & O & I \end{array}\right) \xrightarrow{A^{-1}R_1} \left(\begin{array}{cc|cc} I & O & A^{-1} & O \\ C & B & O & I \end{array}\right)$$

$$\xrightarrow{R_2 - CR_1} \left(\begin{array}{cc|cc} I & O & A^{-1} & O \\ O & B & -CA^{-1} & I \end{array}\right)$$

$$\xrightarrow{B^{-1}R_2} \left(\begin{array}{cc|cc} I & O & A^{-1} & O \\ O & I & -B^{-1}CA^{-1} & B^{-1} \end{array}\right)$$

Ch. 6. Determinants

- $\operatorname{rank}\begin{pmatrix} A & C \\ O & B \end{pmatrix} \geq \overset{r}{\operatorname{rank}(A)} + \overset{s}{\operatorname{rank}(B)}$. $\exists$ a submatrix A_1 of A of order r
  a ... B_1 " B ... s

$\begin{pmatrix} A_1 & C_1 \\ O & B_1 \end{pmatrix}$ is the corresponding submatrix of $\begin{pmatrix} A & C \\ O & B \end{pmatrix}$. $\underbrace{\begin{vmatrix} A_1 & C_1 \\ O & B_1 \end{vmatrix}}_{r+s} = |A_1| \cdot |B_1| \neq 0$

A_1, B_1 are square matrices

- $\operatorname{rank}\begin{pmatrix} A & O \\ O & B \end{pmatrix} = \operatorname{rank}(A) + \operatorname{rank}(B)$.

$\underline{A} \to \begin{pmatrix} I_r & O \\ O & O \end{pmatrix}$, $B \to \begin{pmatrix} I_s & O \\ O & O \end{pmatrix}$

$$\begin{pmatrix} A & O \\ O & B \end{pmatrix} \to \begin{pmatrix} I_r & O & O & \\ O & O & & \\ & & I_s & O \\ O & & O & O \end{pmatrix} \to \begin{pmatrix} I_{r+s} & O \\ O & O \end{pmatrix} \to r+s$$

- $\operatorname{rank}(A_{m\times k})+\operatorname{rank}(B_{k\times n}) - k \overset{②}{\le} \operatorname{rank}(AB) \overset{①}{\le} \min\{\operatorname{rank}(A), \operatorname{rank}(B)\}$.

Proof.

① $(A, O)\begin{pmatrix} I_k & B \\ O & I_n \end{pmatrix} = (A, AB)$

$\operatorname{rank}(A) = \operatorname{rank}(A, O) = \operatorname{rank}(A, AB) \ge \operatorname{rank}(AB)$

$\operatorname{rank}(AB) = \operatorname{rank}((AB)^T) = \operatorname{rank}(\underline{B^T A^T}) \le \operatorname{rank}(B^T) = \operatorname{rank}(B)$

② $\begin{pmatrix} A & O \\ I_k & B \end{pmatrix} \xrightarrow{R_1 - AR_2} \begin{pmatrix} O & -AB \\ I_k & B \end{pmatrix} \xrightarrow{C_2 - C_1 \cdot B} \begin{pmatrix} O & -AB \\ I_k & O \end{pmatrix}$

$\xrightarrow{C_1 \leftrightarrow C_2} \begin{pmatrix} -AB & O \\ O & I_k \end{pmatrix}$

$\operatorname{rank}(A) + \operatorname{rank}(B) \le \operatorname{rank}\begin{pmatrix} A & O \\ I_k & B \end{pmatrix} = \operatorname{rank}\begin{pmatrix} -AB & O \\ O & I_k \end{pmatrix}$

$= \operatorname{rank}(AB) + k$

Ch. 6. Determinants

- $\operatorname{rank}(A, B) \leq \operatorname{rank}(A) + \operatorname{rank}(B)$.

$\geq$

$\max\{\operatorname{rank}(A), \operatorname{rank}(B)\}$

$$(A, B) = (I, I)\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

$$\operatorname{rank}(A, B) \leq \operatorname{rank}\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \operatorname{rank}(A) + \operatorname{rank}(B)$$

- $\operatorname{rank}(A \underline{+} B) \leq \operatorname{rank}(A) + \operatorname{rank}(B)$.

$\geq$

?

$$A + B = (A, B)\begin{pmatrix} I \\ I \end{pmatrix}$$

$$\operatorname{rank}(A+B) \leq \operatorname{rank}(A, B) \leq \operatorname{rank}(A) + \operatorname{rank}(B)$$

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- $$\begin{vmatrix} a & b & 0 & \cdots & 0 & 0 \\ c & a & b & \cdots & 0 & 0 \\ 0 & c & a & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a & b \\ 0 & 0 & 0 & \cdots & c & a \end{vmatrix} = D_n$$

$$D_n = a D_{n-1} - bc D_{n-2}$$

$$x^2 - ax + bc = 0 \Rightarrow x_1, x_2$$

$$x_1 + x_2 = a, \quad x_1 x_2 = bc$$

$$D_n - x_1 D_{n-1} = x_2 (D_{n-1} - x_1 D_{n-2})$$

$$D_n = \begin{cases} \dfrac{x_1^{n+1} - x_2^{n+1}}{x_1 - x_2}, & x_1 \neq x_2 \\ (n+1) x_1^n, & x_1 = x_2 \end{cases}$$

$(a=5, \ b=3, \ c=2)$

$$D_n = 5 D_{n-1} - 6 D_{n-2}$$

$$\underbrace{D_n - 3D_{n-1}}_{H_n} = 2 (\underbrace{D_{n-1} - 3D_{n-2}}_{H_{n-1}})$$

$$= 2^{n-2} H_1 = 2^{n-2} (D_2 - 3D_1) = 2^n$$

$$D_n - 3D_{n-1} = 2^n$$

$$D_n = 3D_{n-1} + 2^n = 3^2 D_{n-2} + 3 \cdot 2^n + 2^n$$

$$D_n = 3^n + 3^{n-1} \cdot 2 + \cdots + 3 \cdot 2^{n-1} + 2^n$$

$$= \frac{3^{n+1} - 2^{n+1}}{3-2} = 3^{n+1} - 2^{n+1}$$