UNIVERSITY OF LEICESTER

MA3071
All candidates

Semester 1 Examinations 2023

DO NOT OPEN THE QUESTION PAPER UNTIL INSTRUCTED TO DO SO BY THE CHIEF INVIGILATOR	
School	COMPUTING AND MATHEMATICAL SCIENCES
Module Code	MA3071
Module Title	FINANCIAL MATHEMATICS
Exam Duration	Two Hours
CHECK YOU HAVE THE CORRECT QUESTION PAPER	
Number of Pages	4
Number of Questions	4
Instructions to Candidates	Answer all questions. All marks gained will be counted. This paper consists of four questions. All questions carry equal weight. Marks allocated to parts of questions are indicated in square brackets.
FOR THIS EXAM YOU ARE ALLOWED TO USE THE FOLLOWING:	
Calculators	Approved calculators may be used.
Books/Statutes provided by the University	Formula sheet and standard normal distribution table are provided.
Are students permitted to bring their own Books/Statutes/Notes?	No
Additional Stationery	Not required

1. (a) Consider a market where there are only three assets available to invest in, and where short selling is allowed:

Asset	Expected return	Standard deviation of return
A	1	2
B	2	1
C	3	1

The covariance of returns between assets B and C is 1. The returns on asset A are uncorrelated with the returns of the other assets.

i. Show that the minimum variance portfolio with an expected return of 3 (assuming it will be fully invested in this market) is given by: **[12 marks]**

Asset	Weights
A	0.2
B	-0.4
C	1.2

ii. Calculate the variance of the portfolio you determined in part i). **[3 marks]**

(b) In a market where the assumptions of the CAPM hold, there is one risk-free asset (Asset 1) with an annual rate of return of 3 and two risky assets with the following properties:

		Rate of return (p.a.)	
State	Probability	Asset 2	Asset 3
1	0.1	5	6
2	0.3	3	3
3	0.4	4	5
4	0.2	6	8

The market capitalisation of Asset 2 is 40,000 and the market capitalisation of Asset 3 is 60,000. The market portfolio is defined by the market capitalisations.

i. Determine the market price of risk. **[5 marks]**

ii. Calculate the beta of each risky asset. **[5 marks]**

Total: 25 marks

2. (a) Consider the short position in a European call option with strike price $K = 10$ and premium $D = 2$. On the expiry date, the underlying asset price S_T is anticipated to follow a distribution with the density function

$$f(x) = \begin{cases} 0 & x < 10 \\ \dfrac{200}{x^3} & x \geqslant 10 \end{cases}$$

Calculate for the short position the cumulative distribution function of loss L.

[10 marks]

(b) Consider a discrete market with one non-risky asset (bond) and one risky asset (share). The underlying asset price is modelled by a three-period binomial tree model such that, at times 0, 1, 2, and 3,

$$S_0 = 8, \quad S_1 = S_0 Y_1, \quad S_2 = S_1 Y_2, \quad S_3 = S_2 Y_3$$

Suppose that the interest rate $\rho = 0.5$ is fixed over each period and assume that Y_1, Y_2 and Y_3 are i.i.d. random variables with

$$\mathrm{P}(Y_i = 2) = 0.7, \quad \mathrm{P}(Y_i = 0.5) = 0.3, \quad i = 1, 2, 3$$

For an exotic option $C = \left(5 - S_3^2\right)_+$, answer the following questions:

i. Determine the equivalent martingale probabilities. **[3 marks]**

ii. Write down the binomial tree. **[7 marks]**

iii. Find the arbitrage-free time 0 option price. **[5 marks]**

Total: 25 marks

3. (a) State the distribution of the stochastic integral $\int_0^t -2e^{-2u} dB_u$. **[5 marks]**

(b) The current price of a non-dividend paying share is £7 and its volatility is thought to be 40% per annum. The continuously compounded risk-free interest rate is 5% per annum. A European call option on this share has a strike price of £6.5 and term to maturity of one year. Assume that the Black-Scholes model applies.

i. Calculate the time 0 price of the call option. **[7 marks]**

ii. Calculate the value of the delta of the call option. **[3 marks]**

iii. Discuss how the price of the option in part (b) i) would change if the rate of interest increases. **[5 marks]**

(c) A company's share price S_t can be modelled by the stochastic differential equation:

$$dS_t = \mu S_t dt + 0.85 S_t dB_t$$

where $\{B_t; t \geq 0\}$ is a standard Brownian motion. Suppose that the risk-free interest rate is $\rho = 0.7$. Determine the value of μ such that $e^{-\rho t} S_t$ is a martingale.

[5 marks]

Total: 25 marks

4. Let $\{B_t; t \geq 0\}$ be a standard Brownian motion.

   (a) Evaluate $cov(B_4^3, B_5 - B_3)$. [You may use $cov(X,Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)$.] **[7 marks]**

   (b) Use Ito's lemma to find the stochastic differential $df(t, X_t)$ of the function $f(t, X_t) = t\sin(X_t)$, where X_t is defined by $dX_t = t^2 B_t dB_t$. **[6 marks]**

   (c) Find the value of constant a such that $\exp\left(B_t - \frac{a^2 t}{2}\right)$ is an Ito martingale. **[7 marks]**

   (d) Explain how to use Monto-Carlo simulation to estimate $\mathbb{E}[\tan(B_t^2)]$. **[5 marks]**

Total: 25 marks