MA3071 Financial Mathematics - DLI, Year 2023-2024

Exercises for Feedback session 11

Exercise 1 *Consider a market where the Capital Asset Pricing Model (CAPM) holds. There are two risky assets with the following attributes:*

		Rate of return (p.a.)	
State	Probability	Asset 1	Asset 2
1	0.2	0.05	0.11
2	0.3	0.1	0.15
3	0.1	0.08	0.12
4	0.4	0.04	0.05

	Variance/Covariance Matrix	
	Asset 1	Asset 2
Asset 1	0.00068	0.00102
Asset 2	0.00102	0.00181

The market capitalisation of Asset 1 is 40,000 and that of Asset 2 is 60,000. The market portfolio is defined by the market capitalisations.

i) *Calculate the betas of Asset 1 and Asset 2.*

ii) *Determine the value of the risk-free rate of interest which is consistent with the results obtained in part (i), under the assumption that the CAPM holds.*

iii) *Determine the market price of risk.*

Solution 1 i) As the market portfolio is the weighted portfolio of the two risky assets in the market, and the given weights are 0.4 and 0.6, then

$$Cov(R_i, R_M) = Cov(R_i, 0.4R_1 + 0.6R_2), \quad i = 1, 2$$

from which it follows that

$$Cov(R_1, R_M) = 0.4\sigma_1^2 + 0.6Cov(R_1, R_2) = 0.00089$$
$$Cov(R_2, R_M) = 0.4Cov(R_1, R_2) + 0.6\sigma_2^2 = 0.0015$$

Also,

$$\sigma_M^2 = 0.4^2 * \sigma_1^2 + 0.6^2 * \sigma_2^2 + 2 * 0.4 * 0.6 * Cov(R_1, R_2) = 0.00125$$

Consequently,

$$\beta_1 = \frac{Cov(R_1, R_M)}{\sigma_M^2} = 0.70915, \quad \beta_2 = \frac{Cov(R_2, R_M)}{\sigma_M^2} = 1.1939$$

ii) Betas can also be calculated as

$$\beta_i = \frac{\mathbb{E}[R_i] - \rho}{\mathbb{E}[R_M] - \rho}, \quad i = 1, 2$$

where ρ is the risk-free rate of interest. We then have

$$\rho = \frac{\mathbb{E}[R_i] - \beta_i \mathbb{E}[R_M]}{1 - \beta_i}$$

Also,

$$\begin{aligned}
\mathbb{E}[R_1] &= 0.2 \times 0.05 + 0.3 \times 0.1 + 0.1 \times 0.08 + 0.4 \times 0.04 = 0.064 \\
\mathbb{E}[R_2] &= 0.2 \times 0.11 + 0.3 \times 0.15 + 0.1 \times 0.12 + 0.4 \times 0.05 = 0.099 \\
\mathbb{E}[R_M] &= 0.4 \times \mathbb{E}[R_1] + 0.6 \times \mathbb{E}[R_2] = 0.085
\end{aligned}$$

Hence,

$$\rho = \frac{\mathbb{E}[R_1] - \beta_1 \mathbb{E}[R_M]}{1 - \beta_1} = 0.012797$$

or

$$\rho = \frac{\mathbb{E}[R_2] - \beta_2 \mathbb{E}[R_M]}{1 - \beta_2} = 0.012797$$

iii) As the market capitalisation of Asset 1 is 40,000 and the market capitalisation of Asset 2 is 60,000, the market portfolio weights invested in Asset 1 and Asset 2 are 40% and 60%, respectively. We then have four paths of the possible returns of market portfolio

Probabilities	R_M
0.2	$0.4 \times 0.05 + 0.6 \times 0.11 = 0.086$
0.3	$0.4 \times 0.1 + 0.6 \times 0.15 = 0.13$
0.1	$0.4 \times 0.08 + 0.6 \times 0.12 = 0.104$
0.4	$0.4 \times 0.04 + 0.6 \times 0.05 = 0.046$

Thus, we have

$$\begin{aligned}
\sigma_M^2 &= \sum_{i=1}^{4} P_i (r_i - \mathbb{E}[R_M])^2 \\
&= 0.2(0.086 - 0.085)^2 + 0.3(0.13 - 0.085)^2 \\
&\qquad + 0.1(0.104 - 0.085)^2 + 0.4(0.046 - 0.085)^2 \\
&= 0.0012522 \\
\sigma_M &\approx 0.035386438
\end{aligned}$$

The market price of risk is

$$\frac{\mathbb{E}[R_M] - \rho}{\sigma_M} = \frac{0.085 - 0.012797}{0.035386438} = 2.0404$$