MA3071 Financial Mathematics - DLI, Year 2023-2024

Exercises for Feedback session 1

Exercise 1 *You are the seller of a European call option with a strike price of $K = 100$ and a premium of $D = 5$. It is expected that, on the expiry date, the price S_T of the underlying asset will follow a probability distribution*

$$\mathbb{P}[S_T = 90 + k] = 0.05, \quad k = 1, 2, \cdots, 20$$

Find the cumulative probability distribution of your loss.

Exercise 2 *A European call option has a strike price of $K = 100$ and a premium of $D = 10$. It is expected that, on the expiry date, the price S_T of the underlying asset will follow a uniform distribution supported on the interval $[0, 120]$. Calculate the cumulative probability distribution of loss for the short position.*

Solution 1 We know that the strike price K and the premium D are constants, while only S_T is a random variable with a discrete uniform distribution. At expiry, S_T will have 20 possible values $(91, \cdots, 110)$, each with a probability of 5%. Then, according to the profit formula,

$$\text{Profit of buyer} = (S_T - K)_+ - D = (S_T - 100)_+ - 5$$

$\boldsymbol{k}$	$\mathbf{1, \cdots, 9}$	$\mathbf{10}$	$\mathbf{11, \cdots, 15}$	$\mathbf{16, \cdots, 20}$
S_T	$91, \cdots, 99$	100	$101, \cdots, 105$	$106, \cdots, 110$
P	$0.05, \cdots, 0.05$	0.05	$0.05, \cdots, 0.05$	$0.05, \cdots, 0.05$
Profit of buyer	$-5, \cdots, -5$	-5	$-4, \cdots, 0$	$1, \cdots, 5$
P	$0.05, \cdots, 0.05$	0.05	$0.05, \cdots, 0.05$	$0.05, \cdots, 0.05$
Loss of seller	$-5, \cdots, -5$	-5	$-4, \cdots, 0$	$1, \cdots, 5$
P	$0.05, \cdots, 0.05$	0.05	$0.05, \cdots, 0.05$	$0.05, \cdots, 0.05$

As we know, Profit of buyer $=$ Loss of seller,

$$\begin{cases} P(L = -5) = 10 \times 0.05 = 0.5 \\ P(L = -4) = 0.05 \\ P(L = -3) = 0.05 \\ \vdots \\ P(L = 4) = 0.05 \\ P(L = 5) = 0.05 \end{cases}$$

Therefore, the cumulative probability distribution of seller's loss:

$$F_L(x) = P(L \leqslant x) = \begin{cases} 0, & x < -5 \\ 0.5, & -5 \leqslant x < -4 \\ 0.55, & -4 \leqslant x < -3 \\ \vdots & \\ 0.95, & 4 \leqslant x < 5 \\ 1, & 5 \leqslant x \end{cases} = \begin{cases} 0, & x < -5 \\ 0.5 + 0.05i, & -5 + i \leqslant x < -5 + i + 1, \quad i = 0, \cdots, 9 \\ 1, & 5 \leqslant x \end{cases}$$

Solution 2 We know that the strike price K and the premium D are constants, while only S_T is a random variable with a uniform distribution on $[0, 120]$. Thus, the density function is

$$f(x) = \begin{cases} 0, & x < 0 \\ \frac{1}{120}, & 0 \leqslant x \leqslant 120 \\ 0, & x > 120 \end{cases}$$

And we know the loss of short position equals the profit of long position,

$$L = (S_T - K)_+ - D = (S_T - 100)_+ - 10$$

Therefore, when $S_T < 100$, the option won't be exercised, and the loss $L = -10$.

For any $x < -10$,

$$P(L \leqslant x) = 0$$

When $100 \leqslant S_T \leqslant 120$, the option will be exercised, $L = S_T - 110$ and $L \in [-10, 10]$.

For any $-10 \leqslant x \leqslant 10$,

$$\begin{aligned} P(L \leqslant x) &= P(S_T \leqslant x + 110) \\ &= \int_{-\infty}^{x+110} f(t) dt \\ &= \int_{0}^{x+110} \frac{1}{120} dt \\ &= \left[\frac{t}{110} \right]_0^{x+110} \\ &= \frac{x + 110}{120} \end{aligned}$$

It is impossible for $S_T > 120$, therefore, $L \not> 10$, and for any $x > 10$,

$$P(L \leqslant x) = 1$$

Hence, the cumulative probability distribution of loss for short position is

$$F_L(x) = \begin{cases} 0, & x < -10 \\ \frac{x+110}{120}, & -10 \leqslant x \leqslant 10 \\ 1, & x > 10 \end{cases}$$