MA3071 Financial Mathematics - DLI, Year 2023-2024

Exercises for Feedback session 7

Exercise 1 *Consider a stochastic process* $\{M_t,\, t \geqslant 0\}$ *defined by* $dM_t = t^{10}B_t dB_t$ *and* $M_0 = 0$. *Find* $\mathbb{E}[B_t^2 M_t]$ *using bivariate Ito's lemma.*

Exercise 2 *Consider a stochastic process* $\{M_t,\, t \geqslant 0\}$ *defined by* $dM_t = t^{10}B_t dB_t$ *and* $M_0 = 0$. *Find* $\mathbb{E}[B_t^2 M_t]$ *using Ito isometry. [Hint:* $B_t^2 - t$ *is a martingale.]*

Exercise 3 *Let the stochastic process* $\{M_t,\, t \geqslant 0\}$ *be defined by* $dM_t = t^3 B_t dB_t$ *and* $M_0 = 0$. *Does there exists a real number* a *such that* $B_t M_t + aB_t^2$ *is an Ito martingale?*

Solution 1 Let $f(B_t, M_t) = B_t^2 M_t$

$$d\left[B_t^2 M_t\right] = df(B_t, M_t) = f'_{B_t} dB_t + f'_{M_t} dM_t + \frac{1}{2} f''_{B_t B_t}(dB_t)^2 + \frac{1}{2} f''_{M_t M_t}(dM_t)^2 + f''_{B_t M_t} dB_t dM_t$$

We need to calculate

$$f'_{B_t} = 2B_t M_t,\;\; f'_{M_t} = B_t^2,\;\; f''_{B_t B_t} = 2M_t,\;\; f''_{M_t M_t} = 0,\;\; f''_{B_t M_t} = 2B_t$$

$$(dB_t)^2 = dt,\;\; dM_t = t^{10} B_t dB_t,\;\; (dM_t)^2 = t^{20} B_t^2 (dB_t)^2 = t^{20} B_t^2 dt,\;\; dB_t dM_t = t^{10} B_t (dB_t)^2 = t^{10} B_t dt$$

Then, we get

$$\begin{aligned} d\left[B_t^2 M_t\right] &= 2B_t M_t dB_t + t^{10} B_t^3 dB_t + \frac{1}{2} \times 2M_t dt + 0 + 2t^{10} B_t^2 dt \\ &= (2B_t M_t + t^{10} B_t^3) dB_t + (M_t + 2t^{10} B_t^2) dt \end{aligned}$$

Then, we have the integral form

$$B_t^2 M_t = B_0^2 M_0 + \int_0^t (2B_u M_u + u^{10} B_u^3) dB_u + \int_0^t (M_u + 2u^{10} B_u^2) du$$

Then, we have

$$\begin{aligned} \mathbb{E}\left[B_t^2 M_t\right] &= \mathbb{E}\left[B_0^2 M_0\right] + \mathbb{E}\left[Ito\right] + \mathbb{E}\left[\int_0^t (M_u + 2u^{10} B_u^2) du\right] \\ &= 0 + 0 + \mathbb{E}\left[\int_0^t (M_u + 2u^{10} B_u^2) du\right] \quad (\text{since } M_0 = 0,\; \mathbb{E}\left[Ito\right] = 0) \\ &= \int_0^t \left(\mathbb{E}\left[M_u\right] + 2u^{10} \mathbb{E}\left[B_u^2\right]\right) du \\ &= \int_0^t 2u^{10} \mathbb{E}\left[B_u^2\right] du \quad (\text{since } M_u \text{ is a martingale}, \mathbb{E}\left[M_u\right] = \mathbb{E}\left[M_u | \mathcal{F}_0\right] = M_0 = 0) \\ &= \int_0^t 2u^{11} du \quad \left(\text{since } \mathbb{E}\left[B_u^2\right] = u\right) \\ &= \frac{2}{12} u^{12}\Big|_0^t = \frac{1}{6} t^{12} \end{aligned}$$

Solution 2 Notice that

$$\mathbb{E}\left[B_t^2 M_t\right] = \mathbb{E}\left[(B_t^2 - t + t)M_t\right] = \mathbb{E}\left[(B_t^2 - t)M_t\right] + \mathbb{E}\left[tM_t\right]$$

Since $dM_t = t^{10}B_t dB_t$ without dt term, M_t is an Ito martingale, we have

$$\mathbb{E}\left[M_t\right] = M_0 = 0 \quad \Rightarrow \quad \mathbb{E}\left[tM_t\right] = t\mathbb{E}\left[M_t\right] = 0$$

Then, we get

$$\mathbb{E}\left[B_t^2 M_t\right] = \mathbb{E}\left[(B_t^2 - t)M_t\right] \quad \text{here } \ B_t^2 - t \ \text{ is a martingale.}$$

Let $f(t, B_t) = B_t^2 - t$, we have

$$f_t' = -1, \quad f_{B_t}' = 2B_t, \quad f_{B_t B_t}'' = 2$$

$$\begin{aligned} df(t, B_t) &= f_t' dt + f_{B_t}' dB_t + \frac{1}{2} f_{B_t B_t}'' dt \\ d(B_t^2 - t) &= -1dt + 2B_t dB_t + 2 \times \frac{1}{2} dt \\ &= 2B_t dB_t \end{aligned}$$

Thus, the integral form

$$B_t^2 - t = B_0^2 - 0 + \int_0^t 2B_u dB_u = \int_0^t 2B_u dB_u$$

As $dM_t = t^{10}B_t dB_t$, we have

$$M_t = M_0 + \int_0^t u^{10} B_u dB_u = \int_0^t u^{10} B_u dB_u$$

Hence

$$\begin{aligned} \mathbb{E}\left[(B_t^2 - t)M_t\right] &= \mathbb{E}\left[\left(\int_0^t 2B_u dB_u\right)\left(\int_0^t u^{10} B_u dB_u\right)\right] \\ &= \mathbb{E}\left[\int_0^t 2u^{10} B_u^2 du\right] \\ &= \int_0^t 2u^{10}\mathbb{E}\left[B_u^2\right] du \\ &= \int_0^t 2u^{10} u du \\ &= \int_0^t 2u^{11} du \\ &= \frac{2}{12} u^{12}\Big|_0^t \\ &= \frac{1}{6} t^{12} \end{aligned}$$

Therefore, $\mathbb{E}\left[B_t^2 M_t\right] = \mathbb{E}\left[(B_t^2 - t)M_t\right] = \frac{1}{6}t^{12}$.

Solution 3 Let $f(B_t, M_t) = B_t M_t + aB_t^2$

$$df(B_t, M_t) = f'_{B_t} dB_t + f'_{M_t} dM_t + \frac{1}{2} f''_{B_t B_t} (dB_t)^2 + \frac{1}{2} f''_{M_t M_t} (dM_t)^2 + f''_{B_t M_t} dB_t dM_t$$

Then, we need to calculate

$$f'_{B_t} = M_t + 2aB_t, \ \ f'_{M_t} = B_t, \ \ f''_{B_t B_t} = 2a, \ \ f''_{M_t M_t} = 0, \ \ f''_{B_t M_t} = 1$$

$$(dB_t)^2 = dt, \ \ dM_t = t^3 B_t dB_t, \ \ (dM_t)^2 = t^6 B_t^2 (dB_t)^2 = t^6 B_t^2 dt, \ \ dB_t dM_t = t^3 B_t (dB_t)^2 = t^3 B_t dt$$

Hence,

$$\begin{aligned} df(B_t, M_t) &= (M_t + 2aB_t) dB_t + B_t (t^3 B_t dB_t) + \frac{1}{2} \times 2a dt + 0 + t^3 B_t dt \\ &= (M_t + 2aB_t + t^3 B_t^2) dB_t + (a + t^3 B_t) dt \end{aligned}$$

For this to be the Ito martingale, we need $a + t^3 B_t = 0$, which means $a = -t^3 B_t$.

However, by problem formulation, a should be a real number, not a function of (t, B_t). There is no real number a such that $a + t^3 B_t = 0$ for all t. Hence, there is no such a exists.