LINEAR ALGEBRA II

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线性代数II（B.YU）

Ch.VIII Eigenvectors and Eigenvalues

§1. EIGENVECTORS AND EIGENVALUES

- **For an** $n \times n$ **matrix** A**,** $0 \neq x \in R^n$ **is called an eigenvector and** λ **the eigenvalue of** A **belonging to** x**, if** $Ax = \lambda x$**.**

- **Let** V **be a vector space over K and** $A : V \to V$ **be a linear map.**

- $0 \neq v \in V$ **is called an eigenvector (characteristic vector) of** A **if there exists a** $\lambda \in$ **K, s.t.,** $Av = \lambda v$

- λ **is uniquely determined and is called the eigenvalue (characteristic value) of** A **belonging to** v**.**

§1. EIGENVECTORS AND EIGENVALUES

Theorem 1.1. *Let V be a vector space and let $A: V \to V$ be a linear map. Let $\lambda \in K$. Let V_λ be the subspace of V generated by all eigenvectors of A having λ as eigenvalue. Then every non-zero element of V_λ is an eigenvector of A having λ as eigenvalue.*

- $V_\lambda = \{ v \in V \mid Av = \lambda v \}$ is a subspace
- $\operatorname{span} V_\lambda = V_\lambda$

Proof: $\forall v_1, v_2 \in V_\lambda$, $c \in K$

$$A(v_1 + v_2) = Av_1 + Av_2 = \lambda v_1 + \lambda v_2 = \lambda(v_1 + v_2)$$

$$\Rightarrow v_1 + v_2 \in V_\lambda$$

$$A(cv_1) = c\,Av_1 = c \cdot \lambda v_1 = \lambda(cv_1) \Rightarrow cv_1 \in V_\lambda$$

$\Rightarrow V_\lambda$ is a subspace.

§1. EIGENVECTORS AND EIGENVALUES

Theorem 1.2. *Let V be a vector space and let $A: V \to V$ be a linear map. Let $v_1, \ldots, v_m$ be eigenvectors of A, with eigenvalues $\lambda_1, \ldots, \lambda_m$ respectively. Assume that these eigenvalues are distinct, i.e.*

$$\lambda_i \neq \lambda_j \qquad \text{if} \quad i \neq j.$$

Then $v_1, \ldots, v_m$ are linearly independent.

Proof. If ① $c_1 v_1 + \cdots + c_m v_m = 0$, $c_i \in K$

$$c_1 A v_1 + \cdots + c_m A v_m = 0$$

② $c_1 \lambda_1 v_1 + \cdots + c_m \lambda_m v_m = 0$

① $\times \lambda_m -$ ② :

③ $c_1 (\lambda_m - \lambda_1) v_1 + \cdots + c_{m-1} (\lambda_m - \lambda_{m-1}) v_{m-1} = 0$

When $m=1$, the result is obvious. Suppose it holds for $m-1$

③ $\Rightarrow c_1 \underbrace{(\lambda_m - \lambda_1)}_{\neq 0} = \cdots = c_{m-1} \underbrace{(\lambda_m - \lambda_{m-1})}_{\neq 0} = 0 \Rightarrow c_1 = \cdots = c_{m-1} = 0 \Rightarrow c_m v_m = 0 \Rightarrow c_m = 0$

$v_1 \ldots v_m$ are L.I.

§1. EIGENVECTORS AND EIGENVALUES

Remark 1. In Theorem 1.2, suppose V is a vector space of dimension n and $A: V \to V$ is a linear map having n eigenvectors $v_1, \ldots, v_n$ whose eigenvalues $\lambda_1, \ldots, \lambda_n$ are distinct. Then $\{v_1, \ldots, v_n\}$ is a basis of V.

$$\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

Example 4. Let V be the vector space consisting of all differentiable functions of a real variable t. Let $\alpha_1, \ldots, \alpha_m$ be distinct numbers. The functions

$$e^{\alpha_1 t}, \ldots, e^{\alpha_m t}$$

are eigenvectors of the derivative, with distinct eigenvalues $\alpha_1, \ldots, \alpha_m$, and hence are linearly independent.

$D(e^{\alpha t}) = \alpha e^{\alpha t}$ $\quad$ $e^{\alpha t}$ is a eigenvector of D

α is the eigenvalue belonging to it

- **Remark 2.**

- $$\frac{dF}{dt} = AF(t).$$

$$A = \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & a_n \end{pmatrix} \quad \text{with } a_i \neq 0 \quad \text{all } i.$$

$$F(t) = \begin{pmatrix} f_1(t) \\ \vdots \\ f_n(t) \end{pmatrix}, \qquad f_i'(t) = a_i f_i(t), \qquad f_i(t) = c_i e^{a_i t}$$

$$\frac{d \ln f_i(t)}{dt} = a_i \qquad \ln f_i(t) = a_i t + b_i$$

$$\underline{f_i(t)} = e^{a_i t + b_i} = e^{b_i} \cdot e^{a_i t} = c_i e^{a_i t}$$

$$F(t) = \begin{pmatrix} c_1 e^{a_1 t} \\ \vdots \\ c_n e^{a_n t} \end{pmatrix} \qquad \begin{pmatrix} e^{a_1 t} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \cdots \begin{pmatrix} 0 \\ \vdots \\ 0 \\ e^{a_n t} \end{pmatrix}$$

§1. EIGENVECTORS AND EIGENVALUES

- **Diagonalizable operator L (matrix).**

- **If $\mathscr{B} = \{v_1, \ldots, v_n\}$ is a basis of V and v_i's are eigenvectors of L, then $\mathscr{B}$ diagonalize L.**

- **A symmetric matrix and the associated operator is diagonalizable.** $\Rightarrow$ §4

§2. THE CHARACTERISTIC POLYNOMIAL

Theorem 2.1. *Let V be a finite dimensional vector space, and let λ be a number. Let $A: V \to V$ be a linear map. Then λ is an eigenvalue of A if and only if $A - \lambda I$ is not invertible.*

Proof. λ is an eigenvalue of A

$\iff \exists\, 0 \neq v \in V$, s.t. $Av = \lambda v$

$\iff (A - \lambda I)v = 0$

$\iff \operatorname{Ker}(A - \lambda I) \neq \{0\}$

$\iff A - \lambda I$ is not injective

$\iff$ .. .. invertible

§2. THE CHARACTERISTIC POLYNOMIAL

- **Let $A = (a_{ij})$ be an $n \times n$ matrix (linear map from K^n to K^n). We define the characteristic polynomial P_A to be the determinant**

$$P_A(t) = \mathrm{Det}(tI - A) = \begin{vmatrix} t-a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & \ddots & & -a_{n-1,n} \\ \vdots & & \ddots & \\ -a_{n1} & \cdots & -a_{n,n-1} & t-a_{nn} \end{vmatrix}$$

$$= t^n - \mathrm{tr}(A)\, t^{n-1} + \cdots + (-1)^n \mathrm{Det}(A)$$

§2. THE CHARACTERISTIC POLYNOMIAL

- **Theorem 2.2.** *Let A be an $n \times n$ matrix. A number λ is an eigenvalue of A if and only if λ is a root of the characteristic polynomial of A.*

- **Compute the characteristic polynomial, find eigenvalues and bases for the eigenspaces of a matrix.**

§2. THE CHARACTERISTIC POLYNOMIAL

Example 3. Find the eigenvalues and a basis for the eigenspaces of the matrix

$$\begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 2 & 4 \end{pmatrix}.$$

- Det $\begin{pmatrix} t-2 & -1 & 0 \\ 0 & t-1 & 1 \\ 0 & -2 & t-4 \end{pmatrix} = (t-2)^2(t-3).$

- $$\begin{aligned} (2-\lambda)x + y &= 0, \\ (1-\lambda)y - z &= 0, \\ 2y + (4-\lambda)z &= 0. \end{aligned}$$

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§2. THE CHARACTERISTIC POLYNOMIAL

Example 4, 5. Find the eigenvalues and a basis for the eigenspaces of the matrix

$$\begin{pmatrix} 1 & 1 & 2 \\ 0 & 5 & -1 \\ 0 & 0 & 7 \end{pmatrix}$$

§2. EIGENVECTORS AND EIGENVALUES

- **Theorem 2.3.** *Let A be an $n \times n$ matrix with complex components. Then A has a non-zero eigenvector and an eigenvalue in the complex numbers.*

- **Theorem 2.4.** *Let A, B be two $n \times n$ matrices, and assume that B is invertible. Then the characteristic polynomial of A is equal to the characteristic polynomial of $B^{-1}AB$.*

Proof $|tI - B^{-1}AB| = |B^{-1}(tI - A)B|$

$= |B^{-1}| \cdot |tI - A| \, |B| = |tI - A|$

§2. THE CHARACTERISTIC POLYNOMIAL

- **Let $L : V \to V$ be a linear map of a finite dimensional vector space into itself (operator). Select a basis $\mathscr{B}$ for V and let $A = M_{\mathscr{B}}^{\mathscr{B}}(L)$ be the matrix associated with L w.r.t this basis.**

- **Define the characteristic polynomial of L to be the characteristic polynomial of A.**

- **If we change basis, then A changes to $B^{-1}AB$ where B is invertible.**

- **By Theorem 2.4, the characteristic polynomial does not depend on the choice of basis.**

§2. THE CHARACTERISTIC POLYNOMIAL

- **The eigenvalue and eigenvector of L $\Leftrightarrow$ The eigenvalue and eigenvector of A?**

$$Lv = \lambda v \qquad \mathcal{B} = \{v_1, \cdots, v_n\},$$

$$\Leftrightarrow \quad X_{\mathcal{B}}(Lv) = X_{\mathcal{B}}(\lambda v)$$

$$\Leftrightarrow \quad M_{\mathcal{B}}^{\mathcal{B}}(L) \cdot X_{\mathcal{B}}(v) = \lambda X_{\mathcal{B}}(v)$$

§2. THE CHARACTERISTIC POLYNOMIAL

Example 6. Find the eigenvalues and a basis for the eigenspaces of the matrix

$$A = \begin{pmatrix} 2 & -1 \\ 3 & 1 \end{pmatrix}.$$

- $$\begin{vmatrix} t-2 & 1 \\ -3 & t-1 \end{vmatrix} = (t-2)(t-1) + 3 = t^2 - 3t + 5.$$

- $$\lambda_1 = \frac{3 + \sqrt{-11}}{2} \qquad \text{and} \qquad \lambda_2 = \frac{3 - \sqrt{-11}}{2}.$$

- $$\begin{aligned} (2-\lambda)x - y &= 0, \\ 3x + (1-\lambda)y &= 0. \end{aligned}$$

- $$X(\lambda_1) = \begin{pmatrix} 1 \\ 2-\lambda_1 \end{pmatrix} \qquad \text{and} \qquad X(\lambda_2) = \begin{pmatrix} 1 \\ 2-\lambda_2 \end{pmatrix}.$$

§2. THE CHARACTERISTIC POLYNOMIAL

Example 7. Find the eigenvalues and a basis for the eigenspaces of the matrix

$$A = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

- $P(t) = \begin{vmatrix} t-1 & -1 & 1 \\ 0 & t-1 & 0 \\ -1 & 0 & t-1 \end{vmatrix} = (t-1)(t^2 - 2t + 2).$ $\lambda_1 = 1, \quad \lambda_2 = 1 + \sqrt{-1}, \quad \lambda_3 = 1 - \sqrt{-1}.$

$$\begin{aligned} (1-\lambda)x + y - z &= 0, \\ (1-\lambda)y &= 0, \\ x + (1-\lambda)z &= 0. \end{aligned}$$

§2. THE CHARACTERISTIC POLYNOMIAL

9. Let V be an n-dimensional vector space and assume that the characteristic polynomial of a linear map $A: V \to V$ has n distinct roots. Show that V has a basis consisting of eigenvectors of A.

10. Let A be a square matrix. Show that the eigenvalues of ${}^tA$ are the same as those of A.

11. Let A be an invertible matrix. If λ is an eigenvalue of A show that $\lambda \neq 0$ and that λ^{-1} is an eigenvalue of A^{-1}.

§2. THE CHARACTERISTIC POLYNOMIAL

Ch.VII, §1

12. Let the notation be as in Exercise 10. Let P again be the projection on W. Show that there is a symmetric operator A such that $A^2 = I + P$.

Proof. Let $\{v_1, \cdots, v_r\}$ be an orthonormal basis of W

Expand it to an orthonormal basis of V, $B = \{v_1, \cdots, v_r, v_{r+1}, \cdots, v_n\}$

$\forall v \in V$, $\quad v = c_1 v_1 + \cdots + c_r v_r + c_{r+1} v_{r+1} + \cdots + c_n v_n$

$Pv = c_1 v_1 + \cdots + c_r v_r$

$Pv_i = \begin{cases} v_i, & 1 \le i \le r \\ 0, & i > r \end{cases}$ $\quad M_B(P) = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix}$ $\quad M_B(I+P) = \begin{pmatrix} 2 & & & & & \\ & \ddots & & & & \\ & & 2 & & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix}$

$A \longleftarrow \quad M_B(A) = \begin{pmatrix} \sqrt{2} & & & & & \\ & \ddots & & & & \\ & & \sqrt{2} & & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix}$

$A^2 = I + P \quad M_B(A^2) = (M_B(A))^2 = \begin{pmatrix} 2 & & & & & \\ & \ddots & & & & \\ & & 2 & & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix} = M_B(I+P)$

§2. THE CHARACTERISTIC POLYNOMIAL

- Homework:
  - §1, 4, 5;
  - §2, 14, 15.

§3. EIGENVALUES AND EIGENVECTORS OF SYMMETRIC MATRICES

- **Theorem 3.1.** *Let A be a symmetric $n \times n$ real matrix. Then there exists a non-zero real eigenvector for A.*

- **Two proofs. One is algebraic, the other is analytic.**

- **By Theorem 2.3, we know that A has an eigenvalue A in C, and an eigenvector Z with complex components. It will now suffice to prove:**

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§3. EIGENVALUES AND EIGENVECTORS OF SYMMETRIC MATRICES

Theorem 3.2. *Let A be a real symmetric matrix and let λ be an eigenvalue in $\mathbf{C}$. Then λ is real. If $Z \neq O$ is a complex eigenvector with eigenvalue λ, and $Z = X + iY$ where $X, Y \in \mathbf{R}^n$, then both X, Y are real eigenvectors of A with eigenvalue λ, and X or $Y \neq O$.*

Proof. ① $AZ = \lambda Z \Rightarrow \bar{\lambda} = \lambda$

② ${}^t\bar{Z}A = {}^t\bar{Z}\,{}^t\bar{A} = \bar{\lambda}\,{}^t\bar{Z}$

${}^t\bar{Z} \times$ ①: ${}^t\bar{Z}AZ = \lambda\,{}^t\bar{Z}Z$

② $\times Z$: ${}^t\bar{Z}AZ = \bar{\lambda}\,{}^t\bar{Z}Z$

$\Rightarrow (\lambda - \bar{\lambda})\,{}^t\bar{Z}Z = 0 \Rightarrow \lambda = \bar{\lambda} \Rightarrow \lambda$ is real

$\sum_{i=1}^{n} |z_i|^2 \neq 0$

$A(X + iY) = \lambda(X + iY) \Rightarrow \begin{cases} AX = \lambda X \\ AY = \lambda Y \end{cases}$

$Z \neq 0$, $X \neq 0$ or $Y \neq 0$

X or Y is an eigenvector of A with λ.

§3. EIGENVALUES AND EIGENVECTORS OF SYMMETRIC MATRICES

- **Theorem 3.3.** *Let A be a real symmetric matrix, and let $f(X) = {}^tXAX$ be the associated quadratic form. Let P be a point on the unit sphere such that $f(P)$ is a maximum for f on the sphere. Then P is an eigenvector for A. In other words, there exists a number λ such that $AP = \lambda P$.*

- Proof. $\forall\, w \in P^{\perp}$, s.t. $\|w\|=1$,

consider $C(t) = \cos t \cdot P + \sin t \cdot w$.

$\forall\, t \in [-\pi, \pi]$, $\|C(t)\|^2 = \cos^2 t + \sin^2 t = 1$,

i.e. $C(t)$ is on the unit sphere.

0 is the maximal point of

$g(t) = [C(t)]^T A\, C(t) \Rightarrow g'(0) = 0$

- $g'(t) = [C'(t)]^T A\, C(t) + [C(t)]^T A\, C'(t)$

- $0 = g'(0) = w^T A p + p^T A w = 2 w^T A p$

  $w^T A p = 0$ for all $w \in p^\perp$

  $\Rightarrow \exists\, \lambda \in \mathbb{R}$, s.t. $Ap = \lambda p$

$$\boxed{\begin{aligned} &\max f(x) \\ &\text{s.t. } \|x\|^2 - 1 = 0 \end{aligned}}$$

$\boxed{\text{Lagrange multiplier theorem}}$

- **Corollary 3.4. The maximum (minimum) value of f on the unit sphere is equal to the largest (smallest) eigenvalue of A.**

Proof. $f(P) = \max_{Z \in S} f(Z) = \lambda$ $\quad AP = \lambda P \quad P^T A P = \lambda = f(P)$

$f(P') = \min_{Z \in S} f(Z) = \xi$

$\forall \mu$, an eigenvalue of A, $AY = \mu Y$, $\underline{\|Y\| = 1}$

$f(Y) = Y^T A Y = \mu Y^T Y = \mu$ $\quad Y \in S$

$\lambda = f(P) \geq f(Y) = \mu \geq \xi = f(P')$

$\Rightarrow \lambda = f(P)$ is the largest eigenvalue of A

- **Theorem.** **Let A be a real symmetric matrix. If λ, μ are distinct eigenvalues of A and $\mathbf{x}$, $\mathbf{y}$ are corresponding eigenvectors. Then $\mathbf{x} \perp \mathbf{y}$.**

Proof. $A\underline{x} = \lambda \underline{x}$, $\quad Y^T A \underline{x} = \lambda Y^T \underline{x}$

$AY = \mu Y$, $\quad Y^T A = \mu Y^T$, $\quad Y^T A \underline{x} = \mu Y^T \underline{x}$

$(\lambda - \mu) Y^T \underline{x} = 0$, $\quad \lambda \neq \mu$

$\Rightarrow Y^T \underline{x} = 0 \Rightarrow Y \perp \underline{x}$.

- **Example.** **Let** $f(x_1, x_2) = 2x_1^2 - 3x_1x_2 + x_2^2$. **Let** A **be the symmetric matrix associated with** f**. Find the eigenvectors of** A **on the unit circle, and find the maximum (minimum) of** f **on the unit circle.**

$$A = \begin{pmatrix} 2 & -\frac{3}{2} \\ -\frac{3}{2} & 1 \end{pmatrix}$$

$$|tI - A| = t^2 - 3t - \frac{1}{4}$$

$$\lambda = \frac{3 \pm \sqrt{10}}{2}$$

$$(\lambda I - A)\begin{pmatrix} x \\ y \end{pmatrix} = 0$$

$$\begin{cases} 2x - \frac{3}{2}y = \lambda x \\ -\frac{3}{2}x + y = \lambda y \end{cases}$$

$x = 1$

$$\underline{X}(\lambda) = \begin{pmatrix} 1 \\ \frac{2}{3}(2-\lambda) \end{pmatrix}$$

$$\min f(\underline{X})$$

$$\|\underline{X}\| = 1$$

$$\max$$

§4. DIAGONALIZATION OF A SYMMETRIC LINEAR MAP

Theorem 4.4. Let A be a symmetric real $n \times n$ matrix. Then there exists an $n \times n$ real unitary matrix U such that

$$\Lambda = U^{\mathrm{T}} A U = U^{-1} A U$$

is a diagonal matrix.

$AU = U\Lambda$

$A(u_1, \dots, u_n) = (u_1, \dots, u_n)\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$

$Au_i = \lambda_i u_i$

Proof. If $n=1$, it is obviously true. Suppose it is true for $(n-1)\times(n-1)$ symmetric real matrix. For a symmetric real $n\times n$ matrix A. Let λ_1 is a eigenvalue of A, it must be real. Let v_1 is a real eigenvector with eigenvalue λ_1, and $\|v_1\|=1$. Then $Av_1 = \lambda_1 v_1$. Expend v_1 to an orthonormal basis of R^n $\{v_1, v_2, \dots, v_n\}$. Let

$U_1 = (v_1\ v_2 \cdots v_n)$ $\qquad A U_1 = (Av_1\ Av_2 \cdots Av_n) = (\lambda_1 v_1\ x \cdots x)$

$$\begin{pmatrix} v_1^T \\ \vdots \\ v_n^T \end{pmatrix} (\lambda_1 v_1\ x \cdots x) = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & x & \cdots & x \\ \vdots & \vdots & & \vdots \\ 0 & x & \cdots & x \end{pmatrix} = U_1^T A U_1 \rightarrow \text{symmetric}$$

$$\begin{pmatrix} \lambda_1 & 0 \\ 0 & A_1 \end{pmatrix}$$

§4. DIAGONALIZATION OF A SYMMETRIC LINEAR MAP

$$U_1^T A U_1 = \begin{pmatrix} \lambda_1 & 0 \\ 0 & A_1 \end{pmatrix}$$

A_1 is a symmetric real $(n-1)\times(n-1)$ matrix

By the assumption of induction. $\exists$ U_2, an $(n-1)\times(n-1)$ orthogonal matrix, s.t. $U_2^T A_1 U_2 = \begin{pmatrix} \lambda_2 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$

$$\begin{pmatrix} 1 & 0 \\ 0 & U_2^T \end{pmatrix} U_1^T A U_1 \begin{pmatrix} 1 & 0 \\ 0 & U_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & U_2^T \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & A_1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & U_2 \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & U_2^T A_1 U_2 \end{pmatrix}$$

$$\| $$

$$\begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}$$

Let $U = \begin{pmatrix} 1 & 0 \\ 0 & U_2 \end{pmatrix}$. The U is orthogonal, and

$$U^T A U = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix} \triangleq \Lambda$$

§4. DIAGONALIZATION OF A SYMMETRIC LINEAR MAP

- Given a symmetric real matrix A, find an orthogonal matrix U and a diagonal matrix Λ such that $U^T A U = \Lambda$

① Compute all eigenvalues of A;

② For each eigenvalue λ, find an orthonormal basis of the eigenspace V_λ

a) If λ is a simple eigenvalue, then find a nonzero solution of

$$(\lambda I - A)x = 0$$

and then normalize it;

b) If λ is an eigenvalue of multiplicity r,

then find a set of basic solutions of $(\lambda I - A)x = 0$ and then orthonormalize it by G-S procedure

③ Put all orthonormal bases together, we get the orthogonal matrix U

The diagonal matrix, whose i-th diagonal entry is the eigenvalue of A belonging to eigenvector in the i-th column of U, is the diagonal matrix Λ.

- If λ is an eigenvalue of A with multiplicity m, then $V_\lambda(A) = \{\mathbf{v} \in \mathbf{R}^n \mid A\mathbf{v} = \lambda\mathbf{v}\}$ is an m dimensional subspace of $\mathbf{R}^n$.

$$U^T A U = \Lambda = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$$

$$U^T(\lambda I - A)U = \lambda I - \Lambda = \begin{pmatrix} \lambda-\lambda_1 & & \\ & \ddots & \\ & & \lambda-\lambda_n \end{pmatrix}$$

There are m λ_i's which are equal to λ

$$\operatorname{rank}(\lambda I - A) = \operatorname{rank}\begin{pmatrix} \lambda-\lambda_1 & & \\ & \ddots & \\ & & \lambda-\lambda_n \end{pmatrix} = n-m$$

$$\underbrace{\dim \operatorname{Ker}(\lambda I - A)}_{\dim V_\lambda(A)} = m$$

§4. DIAGONALIZATION OF A SYMMETRIC LINEAR MAP

- **A basis $v_1, \ldots, v_n$ such that each v_i is an eigenvector for A is called a spectral basis for A.**

- **This basis diagonalizes A,**

$$U^T A U = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}$$

$$\frac{dY(t)}{dt} = U^T A U Y(t)$$

- $$\frac{dX(t)}{dt} = AX(t)$$

$$\frac{dy_1}{dt} = \lambda_1 y_1, \quad \ldots, \quad \frac{dy_n}{dt} = \lambda_n y_n.$$

- $$y_i(t) = c_1 e^{\lambda_i t}$$

$$Y(t) = U^T X(t) \qquad X(t) = U Y(t)$$

§3,4

- Homework:
  - §2, 8(e)(f), 12, 13;
  - §3, 1;
  - §4, 9(d)(f).

- If $\lambda_1, \dots, \lambda_m$ are distinct eigenvalues of operator A of V.
  $\underline{v_{i1} \cdots v_{in_i}}$ are linearly independent eigenvectors with λ_i,
  then $v_{11}, \dots, v_{1n_1}; \dots; v_{m1}, \dots, v_{mn_m}$ are L.I.
- A is diagonalizable $\Longleftrightarrow$ For each eigenvalue λ of A, $\dim(\lambda I - A)$ is equal to the multiplicity of λ.

§4. DIAGONALIZATION OF A SYMMETRIC LINEAR MAP

- **Theorem 4.1. Let V be a finite dimensional vector space over R with a positive definite scalar product. Let $A : V \to V$ be a symmetric linear map. Then A has a nonzero eigenvector.**

① V over $R \xrightarrow{\text{complexify}} \tilde{V}$ over C, Scalar Product $\to$ Hermitian Product

$A \to \tilde{A} \Rightarrow$ Th. 5.1

② $M_B^B(A^T) \overset{?}{=} [M_B^B(A)]^T$

A is symmetric $\overset{?}{\Leftrightarrow}$ $M_B^B(A)$ is symmetric

$\boxed{\text{If } B \text{ is orthonormal, the answers are "yes."}}$

Let $B = \{v_1, \cdots, v_n\}$ be an orthonormal basis of V

$\forall\ v = x_1 v_1 + \cdots + x_n v_n, \quad w = y_1 v_1 + \cdots + y_n v_n$ in V

$A v_i = a_{1i} v_1 + \cdots + a_{ni} v_n \qquad M = M_B^B(A) = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$

§4. DIAGONALIZATION OF A SYMMETRIC LINEAR MAP

$N = (b_{ij})_{n\times n} = M^{\mathcal{B}}_{\mathcal{B}}(A^T) = M^T$

$\langle Av, w\rangle = \langle x_1 Av_1 + \cdots + x_n Av_n,\ y_1 v_1 + \cdots + y_n v_n\rangle$

$= \langle x_1(a_{11}v_1 + \cdots + a_{n1}v_n) + \cdots + x_n(a_{1n}v_1 + \cdots + a_{nn}v_n),\ y_1 v_1 + \cdots + y_n v_n\rangle$

$= \langle (a_{11}x_1 + \cdots + a_{1n}x_n)v_1 + \cdots + (a_{n1}x_1 + \cdots + a_{nn}x_n)v_n,\ y_1 v_1 + \cdots + y_n v_n\rangle$

$= y_1(a_{11}x_1 + \cdots + a_{1n}x_n) + \cdots + y_n(a_{n1}x_1 + \cdots + a_{nn}x_n) = Y^T M X$

$\langle v, A^T w\rangle = \langle x_1 v_1 + \cdots + x_n v_n,\ y_1 A^T v_1 + \cdots + y_n A^T v_n\rangle$

$= \langle x_1 v_1 + \cdots + x_n v_n,\ (\underbrace{a_{11}}_{b} y_1 + \cdots + \underbrace{a_{1n}}_{b} y_n)v_1 + \cdots + (\underbrace{a_{n1}}_{b} y_1 + \cdots \underbrace{a_{nn}}_{b} y_n)v_n\rangle$

$= X^T \underbrace{M}_{N} Y$

A is symmetric $\Leftrightarrow \forall\, v, w \in V,\ \langle Av, w\rangle = \langle v, Aw\rangle \Leftrightarrow \forall\, X, Y \in \mathbb{R}^n$

$Y^T M X = X^T \underbrace{M}_{N} Y = Y^T \underbrace{M^T}_{N^T} X \Leftrightarrow \forall\, X \in \mathbb{R}^n,\ MX = \underbrace{M^T}_{N^T} X$

$\Leftrightarrow \underline{M = M^T}$

$M = N^T$

If $\mathcal{B}$ is not an orthonormal basis, the answers are "No".

$\boxed{P = M^{\mathcal{B}}_{\mathcal{B}'}(id)\ ?}$

- $A \xrightarrow{\mathcal{B}} M$, If $A^T \xrightarrow{\mathcal{B}} M^T$

$\xrightarrow{\mathcal{B}'} P^{-1}MP$ $\xrightarrow{\mathcal{B}'} P^{-1}M^TP = P^TM^TP^{-T} \Leftarrow \boxed{P^{-1} = P^T}$

§4. DIAGONALIZATION OF A SYMMETRIC LINEAR MAP

- **Let $A : V \to V$ be a symmetric linear map. A subspace W of V is said to be stable (invariant) under A if $A(W) \subset W$.**

稳定的 不变的

- **Theorem 4.2. Let $A : V \to V$ be a symmetric operator. a) Let $v \neq 0$ be an eigenvector of A. If $w \in V$ and $w \perp v$, then $Aw \perp v$ too. b) If W is a subspace of V which is stable under A, then $W^\perp$ is also stable under A.**

Proof. a) $\langle Aw, v\rangle = \langle w, Av\rangle = \langle w, \lambda v\rangle = \lambda\langle w, v\rangle = 0 \Rightarrow Aw \perp v$

b) $\forall v \in W^\perp \Rightarrow Av \in W^\perp$ ✓

$\forall w \in W$, $\langle Av, w\rangle = \langle v, \underbrace{Aw}_{\in W}\rangle = 0$

§4. DIAGONALIZATION OF A SYMMETRIC LINEAR MAP

- **Theorem 4.3 (Spectral theorem). Let V be a n ($n > 0$ finite) dimensional vector space over $\mathbb{R}$ with a positive definite scalar product. Let $A : V \to V$ be a linear map, symmetric with respect to the scalar product. Then V has an orthonormal basis consisting of eigenvectors.**
- **Proof. By induction on n and using Th. 4.2.**

When $n=1$, it is obviously true. Suppose that the conclusion is true for $n-1$ dimensional V.S. For n dimensional case, from Th. 3.1, there exists a unit eigenvector v_1. By Th. 4.2, $V_1^\perp$ is stable under A. A can be viewed as an operator of the $n-1$ dimensional vector space $V_1^\perp$ with the same positive scalar product. By the assumption of induction, $V_1^\perp$ has an orthonormal basis $\{v_2, \dots, v_n\}$ consisting of eigenvectors. Then $\{v_1, v_2, \dots, v_n\}$ is an orthonormal basis of V consisting of eigenvectors.

§4. DIAGONALIZATION OF A SYMMETRIC LINEAR MAP

- **A basis $v_1, \ldots, v_n$ such that each v_i is an eigenvector for A is called a spectral basis for A.**

- **This basis diagonalizes A,**

-

§4. DIAGONALIZATION OF A SYMMETRIC LINEAR MAP

- **Theorem 4.4. Let A be a symmetric real $n \times n$ matrix. Then there exists an $n \times n$ real unitary matrix U such that**

$$U^{\mathrm{T}} A U = U^{-1} A U$$

is a diagonal matrix.

Proof. Let $\mathcal{B} = \{E^1, \cdots, E^n\}$ be the standard basis of $\mathbb{R}^n$ and $F: v \mapsto Av$ be the operator defined by A. Then $A = M_{\mathcal{B}}^{\mathcal{B}}(F)$.

By Th. 4.3, $\mathbb{R}^n$ has an orthonormal basis $\mathcal{B}' = \{w_1, \cdots, w_n\}$, s.t., $\Lambda = M_{\mathcal{B}'}^{\mathcal{B}'}(F)$ is diagonal. Because

$$M_{\mathcal{B}'}^{\mathcal{B}'}(F) = M_{\mathcal{B}'}^{\mathcal{B}}(id)\, M_{\mathcal{B}}^{\mathcal{B}}(F)\, M_{\mathcal{B}}^{\mathcal{B}'}(id),$$

and $U = M_{\mathcal{B}}^{\mathcal{B}'}(id)$ is unitary with $U^{\mathrm{T}} = U^{-1} = M_{\mathcal{B}'}^{\mathcal{B}}(id)$, we have

$$\Lambda = U^{\mathrm{T}} A U = U^{-1} A U.$$

§5. THE HERMITIAN CASE

- Let V be a finite dimensional vector space over **C** with a positive definite hermitian product.

- **Theorem 5.1.** Let $A : V \to V$ be a hermitian operator. Then every eigenvalue of A is real.

Proof. Let λ be an eigenvalue of A and $v \neq 0$ is an eigenvector with λ. Then $Av = \lambda v \Rightarrow \langle Av, v \rangle = \langle \lambda v, v \rangle = \lambda \langle v, v \rangle$

① By Th. 2.5, Ch. VII, $\langle Av, v \rangle$ is real. $\Rightarrow$ λ is real

② $\langle v, Av \rangle = \langle v, \lambda v \rangle = \bar{\lambda} \langle v, v \rangle \Rightarrow (\bar{\lambda} - \lambda) \langle v, v \rangle = 0 \Rightarrow \bar{\lambda} = \lambda$

§5. THE HERMITIAN CASE

Theorem 5.2. *Let $A: V \to V$ be a hermitian operator. Let v be a non-zero eigenvector of A. If w is an element of V perpendicular to v then Aw is also perpendicular to v.*

If W is a subspace of V which is stable under A, then $W^{\perp}$ is also stable under A.

The proof is the same as that of Theorem 4.2.

Theorem 5.3 (Spectral theorem). *Let $A: V \to V$ be a hermitian linear map. Then V has an orthogonal basis consisting of eigenvectors of A.*

Again the proof is the same as that of Theorem 4.3.

§5. THE HERMITIAN CASE

Quadratic forms (revisit)

- $f(X) = X^{\mathrm{T}}AX = \sum_{i,j=1}^{n} a_{ij}x_ix_j,$ **where** $X \in R^n$ **and** A **is a symmetric** $n \times n$ **real matrix.**

Hermitian form

- $f(X) = X^*AX = \sum_{i,j=1}^{n} a_{ij}\bar{x}_ix_j,$ **where** $X \in C^n$ **and** A **is an** $n \times n$ **Hermitian matrix.**

Standard form （标准型）

- $f(X) = d_1x_1^2 + \cdots + d_rx_r^2 - d_{r+1}x_{r+1}^2 - \cdots - d_sx_s^2,$ **where,** $d_i > 0.$

Canonical form (规范型）

- $f(X) = x_1^2 + \cdots + x_r^2 - x_{r+1}^2 - \cdots - x_s^2$
-

§5. THE HERMITIAN CASE

Coordinate transformation

- $X = PY$, **where** P **is an** $n \times n$ **nonsingular real matrix.**
- $g(Y) = f(PY) = Y^{\mathrm{T}}P^{\mathrm{T}}APY = Y^{\mathrm{T}}BY \Rightarrow$ **Standard form, Canonical form**
- $P^{\mathrm{T}}AP = B$, **congruent transformation （合同变换、相合变换）**

Sylvester‘s theorem

- r **(positive inertia index) and** s **do not depend on** P**.**

Find a standard form by unitary coordinate transformation

Find the canonical form

§5. THE HERMITIAN CASE

Positive definite quadratic forms (matrix)

- $f(X) = X^{\mathrm{T}}AX > 0$ **for all** $X \neq O$

Positive semidefinite quadratic forms (matrix)

- $f(X) = X^{\mathrm{T}}AX \geq 0$ **for all** $X \in R^n$

- **If** A **is positive (semi)definite, then for any nonsingular matrix** P, $B = P^{\mathrm{T}}AP$ **is also (semi)definite**

- $\forall\ X \neq 0$, $X \in R^n$ $\quad X^T B X = X^T P^T A \underbrace{PX}_{Y \neq 0} = Y^T A Y > 0$ $\quad \geq 0$

§5. THE HERMITIAN CASE

Let $A_{n\times n}$ be a real symmetric (complex hermitian) matrix. The following statements are equivalent.

① A is positive definite

$\Downarrow$ $\forall \lambda, v \neq 0$. s.t. $Av = \lambda v$, $\langle Av, v\rangle = \lambda \langle v, v\rangle \xrightarrow[\langle Av, v\rangle > 0]{\langle v, v\rangle > 0} \lambda > 0$

② All eigenvalues of A are positive

$\Downarrow$ By Th 4.4 and ②

③ There is exist a unitary matrix U, such that $U^{\mathrm{T}}AU = \Lambda$ $(U^*AU = \Lambda)$ is a diagonal matrix with positive diagonal elements

$\Downarrow$ Let $P = U\,\mathrm{diag}(1/\sqrt{\lambda_1}, \cdots, 1/\sqrt{\lambda_n})$

④ There is exist a nonsingular matrix P, such that $P^{\mathrm{T}}AP = I$ $(P^*AP = I)$

$\Downarrow$ By Sylvester's Theorem

⑤ The positivity index (positive inertia index) is $n \Rightarrow$ ① For any nonsingular P such that $P^TAP = \Lambda$ is diagonal, all diagonal elements of Λ are positive. Therefore Λ and hence A is P.D.

⑥ There exists a nonsingular matrix B, such that $A = B^{\mathrm{T}}B$ $(A = B^*B)$ $\Leftrightarrow$ ④

$P^TAP = I \xleftrightarrow{B = P^{-1}} A = B^TB$

⑦ There exists a nonsingular symmetric (hermitian) matrix B, such that $A = B^2$ and $AB = BA$

③ $\Rightarrow$ ⑦ $A = U\Lambda U^T = U\Lambda^{\frac{1}{2}}\Lambda^{\frac{1}{2}}U^T = U\Lambda^{\frac{1}{2}}U^TU\Lambda^{\frac{1}{2}}U^T = B^2$, $B = U\Lambda^{\frac{1}{2}}U^T$, $AB = U\Lambda U^TU\Lambda^{\frac{1}{2}}U^T = U\Lambda\Lambda^{\frac{1}{2}}U^T$
$= U\Lambda^{\frac{1}{2}}\Lambda U^T = U\Lambda^{\frac{1}{2}}U^TU\Lambda U^T = BA$

⑧ All principal minors of A are positive

⑨ All sequential principal minors of A are positive

⑦ $\Rightarrow$ ⑥ $A = B^2 = B^TB$

§5. THE HERMITIAN CASE

① ⇒ ⑧ For any $1 \le r \le n$ and $1 \le i_1 < \cdots < i_r \le n$, we can prove that the submatrix $A\begin{pmatrix} i_1 \cdots i_r \\ i_1 \cdots i_r \end{pmatrix}$ is positive definite. In fact,

$\forall\ 0 \ne Y \in \mathbb{R}^r$, $Y^T A\begin{pmatrix} i_1, \cdots, i_r \\ i_1, \cdots, i_r \end{pmatrix} Y = X^T A X > 0$, since X with

$$x_i = \begin{cases} y_j, & \text{if } i = i_j \text{ for some } 0 \le j \le r \\ 0, & \text{ortherwise} \end{cases}$$

is nonzero.

⑧ ⇒ ⑨ is obvious.

⑨ ⇒ ①: When $n=1$, it is obviously true. Suppose it is true for $(n-1)\times(n-1)$ case. For a symmetric $n \times n$ real matrix A whose sequential principal minors are all positive, let $A = \begin{pmatrix} A_1 & b^T \\ b & a_{nn} \end{pmatrix}$ then A_1 is a symmetric $(n-1)\times(n-1)$ real matrix with positive sequential principal minors. By the assumption of induction, there exists nonsingular matrix P_1. s.t. $P_1' A_1 P_1 = I_{n-1}$.

We have

$$\begin{pmatrix} P_1^T & 0 \\ 0 & 1 \end{pmatrix} A \begin{pmatrix} P_1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} P_1^T A_1 P & P_1^T b^T \\ b P_1 & a_{nn} \end{pmatrix} = \begin{pmatrix} I_{n-1} & b_1^T \\ b_1 & a_{nn} \end{pmatrix}.$$

Let $P = \begin{pmatrix} P_1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} I_{n-1} & -b_1^T \\ 0 & 1 \end{pmatrix}$, we get further

$$P^T A P = \begin{pmatrix} I_{n-1} & 0 \\ 0 & a_{nn} - b_1^T b_1 \end{pmatrix}.$$

From $\det(A) > 0$ and hence $\det(P^T A P) = [\det(P)]^2 \det(A) > 0$, we know $a_{nn} - b_1^T b_1 > 0$. Therefore, $P^T A P$ and hence A is positive definite.

No similar result to ⑨ for the positive semi-definite case!

Counter example: $\begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$ is not positive semidefinite although all sequential principal minors of it are nonnegative!

§5. THE HERMITIAN CASE

Let A be a real symmetric (complex hermitian) matrix. The following statements are equivalent.

① A is positive semidefinite

② All eigenvalues of A are nonnegative

③ There exists a unitary matrix U, such that $U^{\mathrm{T}}AU = \Lambda$ $(U^*AU = \Lambda)$ is a diagonal matrix with nonnegative diagonal elements

④ There exists a nonsingular matrix P, such that $P^{\mathrm{T}}AP = \begin{pmatrix} I & O \\ O & O \end{pmatrix}$ $(P^*AP = \begin{pmatrix} I & O \\ O & O \end{pmatrix})$

⑤ There exists a matrix B, such that $A = B^{\mathrm{T}}B$ $(A = B^*B)$

⑥ There exists a symmetric (hermitian) matrix B, such that $A = B^2$.

⑦ The negativity index (negative inertia index) is 0

⑧ All principal minors of A are nonnegative

§5. THE HERMITIAN CASE

§6. UNITARY OPERATORS

- **I. The complex case**

- Let V be a finite dimensional vector space over C, with a positive definite hermitian product. Let $U : V \to V$ be a unitary operator.

- **Lemma 6.1.** *Let W be a U-invariant subspace of V. Then $\underline{W^{\perp}}$ is also U-invariant.*

Proof. $\forall\, v \in W^{\perp}$, $\langle w, v\rangle = 0$ $\forall\, w \in W$. $U^{*} = U^{-1}$

$U : W \to W$ is invertible since $\ker U = \{0\}$.

$U^{-1} = U^{*} : W \to W$.

$\langle U v, \underline{w}\rangle = \langle v, \underbrace{U^{*} w}_{\in W}\rangle = 0$. $U v \in W^{\perp}$

$W^{\perp}$ is U-invariant.

§6. UNITARY OPERATORS

Theorem 6.2. *Let V be a non-zero finite dimensional vector space over the complex numbers, with a positive definite hermitian product. Let $U: V \to V$ be a unitary operator. Then V has an orthogonal basis consisting of eigenvectors of U.*

Proof. Let v_1 be a non-zero eigenvector, and let V_1 be the 1-dimensional space generated by v_1. Just as in Lemma 6.1, we see that the orthogonal complement $V_1^\perp$ is U-invariant, and by induction, we can find an orthogonal basis $\{v_2, \ldots, v_n\}$ of $V_1^\perp$ consisting of eigenvectors for U. Then $\{v_1, \ldots, v_n\}$ is the desired basis of V.

§6. UNITARY OPERATORS

- **Let $\mathscr{B} = \{v_1, \ldots, v_n\}$ be an orthonormal basis of V. Show that an operator $U : V \to V$ is unitary if and only if $M = M_{\mathscr{B}}^{\mathscr{B}}(U)$ is a unitary matrix.**

Proof. U is unitary $\iff$ $\forall\, v = x_1 v_1 + \cdots + x_n v_n,\ w = y_1 v_1 + \cdots + y_n v_n$

$$\langle Uv, Uw \rangle = \langle v, w \rangle = x_1 \bar{y}_1 + \cdots + x_n \bar{y}_n = Y^* X \qquad (a_{ij})_{n \times n} = M = M_{\mathscr{B}}^{\mathscr{B}}(U)$$

$$\langle Uv, Uw \rangle = \langle x_1 Uv_1 + \cdots + x_n Uv_n,\ y_1 Uv_1 + \cdots + y_n Uv_n \rangle$$

$$= \langle x_1 (a_{11} v_1 + \cdots + a_{n1} v_n) + \cdots + x_n (a_{1n} v_1 + \cdots + a_{nn} v_n),\ y_1 (a_{11} v_1 + \cdots + a_{n1} v_n) + \cdots + y_n (a_{1n} v_1 + \cdots + a_{nn} v_n) \rangle$$

$$= \langle (a_{11} x_1 + \cdots + a_{1n} x_n) v_1 + \cdots + (a_{n1} x_1 + \cdots + a_{nn} x_n) v_n,\ (a_{11} y_1 + \cdots + a_{1n} y_n) v_1 + \cdots + (a_{n1} y_1 + \cdots + a_{nn} y_n) v_n \rangle$$

$$= (a_{11} x_1 + \cdots + a_{1n} x_n) \overline{(a_{11} y_1 + \cdots + a_{1n} y_n)} + \cdots + (a_{n1} x_1 + \cdots + a_{nn} x_n) \overline{(a_{n1} y_1 + \cdots + a_{nn} y_n)}$$

$$= (MY)^* (MX) = Y^* M^* M X$$

$\iff M^* M = I \iff M$ is a unitary matrix

§6. UNITARY OPERATORS

- **If matrix A is unitary, then there exists a unitary matrix U, such that $\Lambda = U^*AU$ is diagonal.**

$$Uv = \lambda v$$

$$\|v\| = \|Uv\| = |\lambda| \, \|v\| \Rightarrow |\lambda| = 1$$

§6. UNITARY OPERATORS

- **II. The real case**

- **Theorem 6.3.** *Let V be a finite dimensional vector space over the reals, of dimension > 0, and with a positive definite scalar product. Let T be a real unitary operator on V. Then V can be expressed as a direct sum*

$$V = V_1 \oplus \cdots \oplus V_r$$

of T-invariant subspaces, which are mutually orthogonal (i.e. V_i is orthogonal to V_j if $i \neq j$) and $\dim V_i$ *is* 1 *or* 2, *for each i.*

Proof. Let $\{v_1, \cdots, v_n\}$ be an orthonormal basis of V. Define a vector space over $\mathbb{C}$ by $\tilde{V} = \{c_1 v_1 + \cdots + c_n v_n \mid c_i \in \mathbb{C}\}$ and corresponding Hermitian product by: $\forall\, v = x_1 v_1 + \cdots + x_n v_n$, $w = y_1 v_1 + \cdots + y_n v_n$, $\langle v, w \rangle = x_1 \bar{y}_1 + \cdots + x_n \bar{y}_n$. Then T is a unitary operator of $\tilde{V}$.

$= Y^* X = (X, Y)$

$\forall\, v \in \tilde{V}$. let $Z \in \mathbb{C}^n$ be the coordinate vector of v. then there exist $X, Y \in \mathbb{R}^n$, s.t. $Z = X + iY \Rightarrow v = u + iw$ $(X(u) = X,\ X(w) = Y)$

§6. UNITARY OPERATORS

$T: \tilde{V} \to \tilde{V}$ Let $v \neq 0$ is an eigenvector. λ is the eigenvalue belonging to it. $|\lambda| = 1$ ① $\lambda = \pm 1$ ② $\lambda = \cos\theta + i\sin\theta = e^{i\theta}$ not real

$v = u + iw$

① $Tv = \lambda v$, $Tu + iTw = \lambda u + i\lambda w$

$Tu = \lambda u$, $Tw = \lambda w$, u or w is not zero

Let v_1 be one of u and w which is nonzero. the $Tv_1 = \lambda v_1$

$V_1 = \text{span}\{v_1\}$, $V_1^{\perp}$ is also T-invariant. $V = V_1 \oplus V_1^{\perp}$

By induction, we can prove the result.

② $Tu + iTw = (\cos\theta + i\sin\theta)(u + iw) = (\cos\theta \cdot u - \sin\theta\, w) + i(\sin\theta \cdot u + \cos\theta\, w)$

$$\begin{cases} Tu = \cos\theta \cdot u - \sin\theta \cdot w \\ Tw = \sin\theta \cdot u + \cos\theta \cdot w \end{cases}$$

u, w are linearly independent. $T\bar{v} = \bar{\lambda}\bar{v}$ $\quad u+iw \quad u-iw$ $\quad v, \bar{v}$ are L.I.

$\bar{\lambda} = \lambda$

$T(u - iw) = \bar{\lambda}(u - iw)$

§6. UNITARY OPERATORS

Let $V_1 = \text{span}\{u, w\}$. It is a subspace of dimension 2.

V_1 is T-invariant

$$V = V_1 \oplus \underline{V_1^{\perp}}$$

$T: V_1^{\perp} \to V_1^{\perp}$ is unitary

By induction, we can prove the result.

§6. UNITARY OPERATORS

Theorem 6.4. *Let V be a finite dimensional vector space over the reals, of dimension >0 and with a positive definite scalar product. Let T be a unitary operator on V. Then there exists a basis of V such that the matrix of T with respect to this basis consists of blocks*

$$\begin{pmatrix} M_1 & O & \cdots & O \\ O & M_2 & \cdots & O \\ \vdots & \vdots & & \vdots \\ O & O & \cdots & M_r \end{pmatrix}$$

such that each M_i is a 1×1 matrix or a 2×2 matrix, of the following types:

$$(1), \qquad (-1), \qquad \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

We observe that on each component space V_i in the decomposition

$$V = V_1 \oplus \cdots \oplus V_r$$

the linear map T is either the identity I, or the reflection $-I$, or a rotation. This is the geometric content of Theorem 6.3 and Theorem 6.4.

§6. UNITARY OPERATORS

- **If real matrix A is orthogonal, then there exists a orthogonal matrix U, such that $\Lambda = U^{\mathrm{T}} A U$ is block-wise diagonal with diagonal blocks of types** $(1), \quad (-1), \quad \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$

$\Rightarrow$ Homework.

§6. UNITARY OPERATORS

- **An operator $A : V \to V$ is said to be normal if AA* = A* A.**
- **Let A, B be normal operators such that AB = BA. Show that AB is normal.**
- **Let $\mathscr{B} = \{v_1, \ldots, v_n\}$ be an orthonormal basis of V. Show that an operator $A : V \to V$ is normal if and only if $M = M_{\mathscr{B}}^{\mathscr{B}}(U)$ is a normal matrix.**

§6. UNITARY OPERATORS

- **An operator A: V $\rightarrow$ V is normal iff V has an orthogonal basis consisting of eigenvectors of A.**

§6. UNITARY OPERATORS

- **An matrix A is normal iff there exists a unitary matrix U, such that $\Lambda = U^*AU$ is diagonal.**

Homework

§6. UNITARY OPERATORS

- **Let $A : V \to V$ be a hermitian operator. Show that there exist semipositive operators $P_1,\ P_2$ such that $A = P_1 - P_2$.**

- **Homework:**
  - §4: 3, 5, 23
  - §5: 2, 7
- Problem solving and discussing:
  - §4: 21, 25
  - §5, 9