Homework Sheet 4, available online November 10th, online delivery November 17th before 4:00 pm UK time.

(Partial Derivatives. Multiple Integrals)

Problem 1. (4 points) By about how much will $h(x, y, z) = \cos(\pi xy) + xz^2$ change if the point $P(x, y, z)$ moves from $P_0(-1, -1, -1)$ a distance of $ds = 0.1$ unit toward the origin?

Solution: $\overrightarrow{P_0O} = (1, 1, 1) \Rightarrow u = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

$$\frac{\partial h}{\partial x} = -\pi y \sin(\pi y x) + z^2, \quad \frac{\partial h}{\partial y} = -\pi y \sin(\pi y x), \quad \frac{\partial h}{\partial z} = 2xz$$

$$\nabla h\big|_{(-1,-1,-1)} = (1, 0, 2)$$

Hence the change of h with respect to u is

$$\nabla h\big|_{P_0} \cdot u = \sqrt{3}$$

$$dh = \nabla h\big|_{P_0} \cdot u \cdot ds = \frac{\sqrt{3}}{10}$$

Problem 2. (4 points) Given a function $f(x, y) = x^2 + kxy + y^2$ of two independent variables x and y with constant parameter k

a) Show that $(0, 0)$ is a critical point of $f(x, y)$ no matter what value the constant k has. (Hint: Consider two cases: $k = 0$ and $k \neq 0$).

b) For what values of the constant k does the Second Derivative Test guarantee that $f(x, y)$ will have a saddle point at $(0, 0)$? A local minimum at $(0, 0)$? For what values of k is the Second Derivative Test inconclusive? Give reasons for your answers.

Solution:

a) i) when $k=0$, $f(x,y) = x^2+y^2$

$$\Rightarrow \left.\frac{\partial f}{\partial x}\right|_{\substack{x=0\\y=0}} = 2x = 0 \quad \left.\frac{\partial f}{\partial y}\right|_{\substack{x=0\\y=0}} = 2y = 0$$

Hence when $k=0$, $(0,0)$ is a critical point of $f(x,y)$

ii) when $k \neq 0$ $f(x,y) = x^2+kxy+y^2$

$$\Rightarrow \left.\frac{\partial f}{\partial x}\right|_{\substack{x=0\\y=0}} = 2x+ky = 0 \quad \left.\frac{\partial f}{\partial y}\right|_{\substack{x=0\\y=0}} = 2y+kx = 0$$

Hence when $k \neq 0$. $(0,0)$ is a critical point of $f(x,y)$

b) When $k=0$ $f(x,y) = x^2+y^2$, $(0,0)$ is not the saddle point

When $k \neq 0$, $f(x,y) = x^2+kxy+y^2$ and $\frac{\partial f}{\partial x} = 2x+ky$. $\frac{\partial f}{\partial y} = 2y+kx$

Hence, $\frac{\partial^2 f}{\partial x^2} = 2$ $\frac{\partial^2 f}{\partial^2 y} = 2$ $\frac{\partial^2 f}{\partial x \partial y} = k$ $\frac{\partial^2}{\partial y \partial x} = k$

If $(0,0)$ is saddle point, $f_{xx}f_{yy} - f_{xy}^2 = 4-k^2 < 0$, $k \in (-\infty,-2) \cup (2,+\infty)$

If $(0,0)$ is local minimum. $f_{xx} = 2 > 0$ $f_{xx}f_{yy} - f_{xy}^2 = 4-k^2 > 0$. $k \in (-2,2)$

For $k = \pm 2$, second derivative test is inconclusive. Because Hession of f is zero when $k = \pm 2$.

Problem 3. (4 points) Maximizing a product: Find the largest product the positive numbers x, y, and z can have if $x + y + z^2 = 16$.

Solution: Let $f(x,y,z) = xyz$, $(x>0, y>0, z>0)$

and $g(x,y,z) = x+y+z^2-16=0$

We first find λ

$\nabla f(x,y,z) = \lambda \nabla g(x,y,z)$

$yz\,i + xz\,j + xy\,k = \lambda i + \lambda j + 2\lambda z\,k$

$$\begin{cases} y = x = 2z^2 \\ x+y+z^2 = 16 \end{cases} \Rightarrow \begin{cases} x = \frac{32}{5} \\ y = \frac{32}{5} \\ z = \frac{4}{\sqrt{5}} \end{cases}$$

Problem 4. (4 points) Find all values of the constant c so that

$$\int_{-1}^{c}\int_{0}^{2}(xy+1)dy\,dx = 4+4c$$

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Solution:

$$\int_{-1}^{c}\int_{0}^{2}(xy+1)dy\,dx$$

$$= \int_{-1}^{c} 2x+2\,dx$$

$$= x^2+2x\Big|_{-1}^{c}$$

$$= c^2+2c-(1-2)$$

$$= c^2+2c+1$$

Since, $c^2+2c+1 = 4+4c$

$$c^2-2c-3=0$$

$$(c+1)(c-3)=0$$

$$c=-1 \text{ or } c=3$$

However, c cannot be -1, we find that $c=3$

Problem 5. (4 points) Integrate $f(s,t) = e^s \ln t$ over the region in the first quadrant of the st-plane that lies above the curve $s = \ln t$ from $t = 1$ to $t = 2$.

Solution:

$u = \ln(x)$

$du = \frac{dx}{x}$

$$\iint_D f(s,t)\,dt = \int_1^2 \int_0^{\ln(t)} e^s \ln t \, ds\, dt$$

$$= \int_1^2 \ln t (t-1)\, dt$$

$$= \int_1^2 \ln t \cdot t\, dt - \int_1^2 \ln t\, dt$$

$$= \left(\frac{1}{2}x^2 \ln x - \frac{x^2}{4}\Big|_1^2\right) - \left(x \ln x - x\Big|_1^2\right)$$

$$= \left((2\ln 2 - 1) + \frac{1}{4}\right) - (2\ln 2 - 2 + 1)$$

$$= \frac{1}{4}$$