MA3071 Financial Mathematics - DLI, Year 2023-2024

Exercises for Feedback session 10

There are four risky assets $A_1,\ A_2,\ A_3,\ A_4$ circulating on a market, with expected returns $\mathbb{E}[R_1] = 0.02$, $\mathbb{E}[R_2] = 0.01$, $\mathbb{E}[R_3] = 0.03$, $\mathbb{E}[R_4] = 0.02$ and the variance-covariance matrix

$$V = \begin{pmatrix} 0.2 & 0.3 & 0 & 0 \\ 0.3 & 0.5 & 0 & 0 \\ 0 & 0 & 0.2 & 0.1 \\ 0 & 0 & 0.1 & 0.1 \end{pmatrix}$$

Assuming that short selling is allowed in this market, answer the following questions:

Exercise 1 *Find the weights, expected return, and variance of the minimum variance portfolio.*

Exercise 2 *Find the minimum variance portfolio with an expected return of* 0.04 *and its variance.*

Exercise 3 *Given that the utility function is* $u = \mathbb{E}_p - \frac{1}{2}\sigma_p^2$, *find the maximum utility portfolio.*

Solution 1 i) **Method 1**:

$$X_p = \frac{V^{-1} \cdot \mathbf{1}}{\mathbf{1}^\top \cdot V^{-1} \cdot \mathbf{1}}$$

$$V = \left(\begin{array}{cc:cc} 0.2 & 0.3 & 0 & 0 \\ 0.3 & 0.5 & 0 & 0 \\ \hdashline 0 & 0 & 0.2 & 0.1 \\ 0 & 0 & 0.1 & 0.1 \end{array}\right) = \begin{pmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{B} \end{pmatrix} \qquad V^{-1} = \begin{pmatrix} \mathbf{A^{-1}} & \mathbf{0} \\ \mathbf{0} & \mathbf{B^{-1}} \end{pmatrix}$$

Then,

$$\left(\begin{array}{cc:cc} 0.2 & 0.3 & 1 & 0 \\ 0.3 & 0.5 & 0 & 1 \end{array}\right) \to \left(\begin{array}{cc:cc} 2 & 3 & 10 & 0 \\ 3 & 5 & 0 & 10 \end{array}\right) \to \left(\begin{array}{cc:cc} 1 & \frac{3}{2} & 5 & 0 \\ 0 & \frac{1}{2} & -15 & 10 \end{array}\right) \to \left(\begin{array}{cc:cc} 1 & 0 & 50 & -30 \\ 0 & 1 & -30 & 20 \end{array}\right)$$

$$\Rightarrow \quad A^{-1} = \begin{pmatrix} 50 & -30 \\ -30 & 20 \end{pmatrix}$$

$$\left(\begin{array}{cc:cc} 0.2 & 0.1 & 1 & 0 \\ 0.1 & 0.1 & 0 & 1 \end{array}\right) \to \left(\begin{array}{cc:cc} 2 & 1 & 10 & 0 \\ 1 & 1 & 0 & 10 \end{array}\right) \to \left(\begin{array}{cc:cc} 1 & 0 & 10 & -10 \\ 1 & 1 & 0 & 10 \end{array}\right) \to \left(\begin{array}{cc:cc} 1 & 0 & 10 & -10 \\ 0 & 1 & -10 & 20 \end{array}\right)$$

$$\Rightarrow \quad B^{-1} = \begin{pmatrix} 10 & -10 \\ -10 & 20 \end{pmatrix}$$

Thus,

$$V^{-1} = \begin{pmatrix} 50 & -30 & 0 & 0 \\ -30 & 20 & 0 & 0 \\ 0 & 0 & 10 & -10 \\ 0 & 0 & -10 & 20 \end{pmatrix} \quad \Rightarrow \quad \text{Minimum variance portfolio } \; X_p = \begin{pmatrix} 1 \\ -0.5 \\ 0 \\ 0.5 \end{pmatrix}$$

Method 2: We have $\sigma_1^2 = 0.2$, $\sigma_2^2 = 0.5$, $\sigma_3^2 = 0.2$, $\sigma_4^2 = 0.1$, $\sigma_{12} = 0.3$, $\sigma_{34} = 0.1$, $\mathbb{E}[R_1] = 0.02$, $\mathbb{E}[R_2] = 0.01$, $\mathbb{E}[R_3] = 0.03$, $\mathbb{E}[R_4] = 0.02$.

The Lagrangian function is:

$$\begin{aligned} \mathcal{L} &= \sum_{i=1}^{4} x_i^2\sigma_i^2 + 2x_1x_2\sigma_{12} + 2x_3x_4\sigma_{34} - \mu\left(\sum_{i=1}^{4} x_i - 1\right) \\ &= 0.2x_1^2 + 0.5x_2^2 + 0.2x_3^2 + 0.1x_4^2 + 2 \times 0.3x_1x_2 + 2 \times 0.1x_3x_4 \\ &\quad - \mu\left(x_1 + x_2 + x_3 + x_4 - 1\right) \end{aligned}$$

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial x_1} &= 0.4x_1 + 0.6x_2 - \mu = 0 \\ \frac{\partial \mathcal{L}}{\partial x_2} &= x_2 + 0.6x_1 - \mu = 0 \\ \frac{\partial \mathcal{L}}{\partial x_3} &= 0.4x_3 + 0.2x_4 - \mu = 0 \\ \frac{\partial \mathcal{L}}{\partial x_4} &= 0.2x_4 + 0.2x_3 - \mu = 0 \\ \frac{\partial \mathcal{L}}{\partial \mu} &= -\left(x_1 + x_2 + x_3 + x_4 - 1\right) = 0 \end{aligned}$$

Hence, we get the minimum variance portfolio

$$X_p = \begin{pmatrix} 1 \\ -0.5 \\ 0 \\ 0.5 \end{pmatrix}, \qquad \mu = 0.1$$

ii) Here,

$$r = \begin{pmatrix} 0.02 \\ 0.01 \\ 0.03 \\ 0.02 \end{pmatrix}$$

Expected return:

$$\mathbb{E}_p = X_p^\top \cdot r = (1, -0.5, 0, 0.5) \begin{pmatrix} 0.02 \\ 0.01 \\ 0.03 \\ 0.02 \end{pmatrix} = 0.02 - 0.005 + 0 + 0.01 = 0.025$$

iii) Variance of the minimum variance portfolio:

$$\begin{aligned}
\sigma_p^2 &= X_p^\top \cdot V \cdot X_p \\
&= (1, -0.5, 0, 0.5) \begin{pmatrix} 0.2 & 0.3 & 0 & 0 \\ 0.3 & 0.5 & 0 & 0 \\ 0 & 0 & 0.2 & 0.1 \\ 0 & 0 & 0.1 & 0.1 \end{pmatrix} \begin{pmatrix} 1 \\ -0.5 \\ 0 \\ 0.5 \end{pmatrix} \\
&= (0.05, 0.05, 0.05, 0.05) \begin{pmatrix} 1 \\ -0.5 \\ 0 \\ 0.5 \end{pmatrix} \\
&= 0.05 - 0.025 + 0 + 0.025 = 0.05
\end{aligned}$$

Solution 2 **Method 1**: We have $\mathbb{E}_0 = 0.04$, and we calculate

$$\begin{aligned}
\alpha &= \mathbf{1}^\top \cdot V^{-1} \cdot \mathbf{1} = 20 \\
\beta &= r^\top \cdot V^{-1} \cdot r = 0.015 \\
\gamma &= \mathbf{1}^\top \cdot V^{-1} \cdot r = 0.5 \\
\Theta &= \alpha\beta - \gamma^2 = 0.05
\end{aligned}$$

Therefore,

$$X_p = \frac{1}{\Theta}\left(\alpha\mathbb{E}_0 - \gamma\right)\left(V^{-1} \cdot r\right) - \frac{1}{\Theta}\left(\gamma\mathbb{E}_0 - \beta\right)\left(V^{-1} \cdot \mathbf{1}\right) = \begin{pmatrix} 2.2 \\ -1.4 \\ 0.6 \\ -0.4 \end{pmatrix}$$

Method 2: We have $\mathbb{E}_0 = 0.04$, $\sigma_1^2 = 0.2$, $\sigma_2^2 = 0.5$, $\sigma_3^2 = 0.2$, $\sigma_4^2 = 0.1$, $\sigma_{12} = 0.3$, $\sigma_{34} = 0.1$, $\mathbb{E}[R_1] = 0.02$, $\mathbb{E}[R_2] = 0.01$, $\mathbb{E}[R_3] = 0.03$, $\mathbb{E}[R_4] = 0.02$

The Lagrangian function is:

$$\begin{aligned}
\mathcal{L} &= \sum_{i=1}^{4} x_i^2\sigma_i^2 + 2x_1x_2\sigma_{12} + 2x_3x_4\sigma_{34} - \lambda\left(\sum_{i=1}^{4} x_i\mathbb{E}[R_i] - \mathbb{E}_0\right) - \mu\left(\sum_{i=1}^{4} x_i - 1\right) \\
&= 0.2x_1^2 + 0.5x_2^2 + 0.2x_3^2 + 0.1x_4^2 + 2 \times 0.3x_1x_2 + 2 \times 0.1x_3x_4 \\
&- \lambda\left(0.02x_1 + 0.01x_2 + 0.03x_3 + 0.02x_4 - 0.04\right) - \mu\left(x_1 + x_2 + x_3 + x_4 - 1\right)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial x_1} &= 0.4x_1 + 0.6x_2 - 0.02\lambda - \mu = 0 \\
\frac{\partial \mathcal{L}}{\partial x_2} &= x_2 + 0.6x_1 - 0.01\lambda - \mu = 0 \\
\frac{\partial \mathcal{L}}{\partial x_3} &= 0.4x_3 + 0.2x_4 - 0.03\lambda - \mu = 0 \\
\frac{\partial \mathcal{L}}{\partial x_4} &= 0.2x_4 + 0.2x_3 - 0.02\lambda - \mu = 0 \\
\frac{\partial \mathcal{L}}{\partial \lambda} &= -\left(0.02x_1 + 0.01x_2 + 0.03x_3 + 0.02x_4 - 0.04\right) = 0 \\
\frac{\partial \mathcal{L}}{\partial \mu} &= -\left(x_1 + x_2 + x_3 + x_4 - 1\right) = 0
\end{aligned}$$

Hence, we get the minimum variance portfolio

$$X_p = \begin{pmatrix} 2.2 \\ -1.4 \\ 0.6 \\ -0.4 \end{pmatrix}, \quad \lambda = 12, \quad \mu = -0.2$$

Variance:

$$\begin{aligned}
\sigma_p^2 &= X_p^\top \cdot V \cdot X_p \\
&= (2.2, -1.4, 0.6, -0.4) \begin{pmatrix} 0.2 & 0.3 & 0 & 0 \\ 0.3 & 0.5 & 0 & 0 \\ 0 & 0 & 0.2 & 0.1 \\ 0 & 0 & 0.1 & 0.1 \end{pmatrix} \begin{pmatrix} 2.2 \\ -1.4 \\ 0.6 \\ -0.4 \end{pmatrix} \\
&= (0.02, -0.04, 0.08, 0.02) \begin{pmatrix} 2.2 \\ -1.4 \\ 0.6 \\ -0.4 \end{pmatrix} \\
&= 0.14
\end{aligned}$$

Solution 3 **Method 1**: As the utility function is $u = \mathbb{E}_p - \frac{1}{2}\sigma_p^2$, we know $\kappa = \frac{1}{2}$. We calculate

$$\begin{aligned}
\alpha &= \mathbf{1}^\top \cdot V^{-1} \cdot \mathbf{1} = 20 \\
\gamma &= \mathbf{1}^\top \cdot V^{-1} \cdot r = 0.5
\end{aligned}$$

Therefore,

$$X_p = \frac{V^{-1} \cdot r}{2\kappa} - \frac{(\gamma - 2\kappa)\left(V^{-1} \cdot \mathbf{1}\right)}{2\kappa\alpha} = \begin{pmatrix} 1.2 \\ -0.65 \\ 0.1 \\ 0.35 \end{pmatrix}$$

Method 2: As the utility function is $u = \mathbb{E}_p - \frac{1}{2}\sigma_p^2$, we know $\kappa = \frac{1}{2}$, $\sigma_1^2 = 0.2$, $\sigma_2^2 = 0.5$, $\sigma_3^2 = 0.2$, $\sigma_4^2 = 0.1$, $\sigma_{12} = 0.3$, $\sigma_{34} = 0.1$

The Lagrangian function is:

$$\begin{aligned}
\mathcal{L} &= \sum_{i=1}^{4} x_i \mathbb{E}[R_i] - \kappa \left(\sum_{i=1}^{4} x_i^2 \sigma_i^2 + 2x_1x_2\sigma_{12} + 2x_3x_4\sigma_{34} \right) - \mu \left(\sum_{i=1}^{4} x_i - 1 \right) \\
&= 0.02x_1 + 0.01x_2 + 0.03x_3 + 0.02x_4 - \frac{1}{2}\left(0.2x_1^2 + 0.5x_2^2 + 0.2x_3^2 + 0.1x_4^2 + 2 \times 0.3x_1x_2 + 2 \times 0.1x_3x_4\right) \\
&- \mu\left(x_1 + x_2 + x_3 + x_4 - 1\right)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial x_1} &= 0.02 - 0.2x_1 - 0.3x_2 - \mu = 0 \\
\frac{\partial \mathcal{L}}{\partial x_2} &= 0.01 - 0.5x_2 - 0.3x_1 - \mu = 0 \\
\frac{\partial \mathcal{L}}{\partial x_3} &= 0.03 - 0.2x_3 - 0.1x_4 - \mu = 0 \\
\frac{\partial \mathcal{L}}{\partial x_4} &= 0.02 - 0.1x_4 - 0.1x_3 - \mu = 0 \\
\frac{\partial \mathcal{L}}{\partial \mu} &= -\left(x_1 + x_2 + x_3 + x_4 - 1\right) = 0
\end{aligned}$$

Thus, we get the maximum utility portfolio

$$X_p = \begin{pmatrix} 1.2 \\ -0.65 \\ 0.1 \\ 0.35 \end{pmatrix}, \quad \mu = -0.025$$