MA3071 Financial Mathematics - DLI, Year 2023-2024

Exercises for Feedback session 5

Exercise 1 *Find $\mathbb{E}[B_t^4|\mathcal{F}_s]$ by applying Ito's lemma. [Hint: You may need to use the fact $\mathbb{E}[B_t^2|\mathcal{F}_s] = B_s^2 + (t-s)$.]*

Exercise 2 *Let X_t be defined by the SDE $dX_t = 2X_t dB_t$, is $e^{-t}X_t$ a martingale?*

Exercise 3 *Let X_t be defined by the SDE $dX_t = \log(t)dt + 2X_t dB_t$, is $e^{-t}X_t$ a martingale?*

Solution 1 i) Method one: Classical Ito's lemma.

Let $f(t, B_t) = B_t^4$, we have

$$\begin{aligned} &f_t' = 0 \\ &f_{B_t}' = 4B_t^3 \\ &f_{B_tB_t}'' = 12B_t^2 \end{aligned}$$

By the Ito's lemma:

$$\begin{aligned} df(t, B_t) &= f_{B_t}' dB_t + f_t' dt + \frac{1}{2} f_{B_tB_t}'' dt \\ &= 4B_t^3 dB_t + 0 + \frac{1}{2} \times 12B_t^2 dt \\ &= 4B_t^3 dB_t + 6B_t^2 dt \end{aligned}$$

As we know the integral form: $X_t = X_s + \int_s^t A_u du + \int_s^t Y_u dB_u$, here, $X_s = B_s^4$, $A_u = 6B_u^2$, $Y_u = 4B_u^3$, then we have

$$f(t, B_t) = B_t^4 = B_s^4 + \int_s^t 6B_u^2 du + \int_s^t 4B_u^3 dB_u$$

Thus

$$\begin{aligned}
\mathbb{E}\left[B_t^4|\mathcal{F}_s\right] &= \mathbb{E}\left[B_s^4 + \int_s^t 6B_u^2 du + \int_s^t 4B_u^3 dB_u \Big| \mathcal{F}_s\right] \\
&= \mathbb{E}\left[B_s^4|\mathcal{F}_s\right] + \mathbb{E}\left[\int_s^t 6B_u^2 du \Big| \mathcal{F}_s\right] + \mathbb{E}\left[\int_s^t 4B_u^3 dB_u \Big| \mathcal{F}_s\right] \\
&= B_s^4 + \int_s^t \mathbb{E}\left[6B_u^2|\mathcal{F}_s\right] du + 0 \\
&= B_s^4 + \int_s^t 6\left(B_s^2 + (u-s)\right) du \quad \text{(by hint)} \\
&= B_s^4 + \int_s^t \left(6B_s^2 + 6u - 6s\right) du \\
&= B_s^4 + 6B_s^2[u|_s^t] + 3u^2|_s^t - 6s[u|_s^t] \\
&= B_s^4 + 6B_s^2(t-s) + 3(t^2-s^2) - 6s(t-s) \\
&= B_s^4 + 6B_s^2(t-s) + 3(t-s)^2
\end{aligned}$$

ii) Method two: Conditional expectation.

$$\begin{aligned}
\mathbb{E}\left[B_t^4|\mathcal{F}_s\right] &= \mathbb{E}\left[(B_s + (B_t - B_s))^4|\mathcal{F}_s\right] \\
&= \mathbb{E}\left[\tbinom{4}{0} B_s^4 (B_t - B_s)^0|\mathcal{F}_s\right] + \mathbb{E}\left[\tbinom{4}{1} B_s^3 (B_t - B_s)^1|\mathcal{F}_s\right] + \mathbb{E}\left[\tbinom{4}{2} B_s^2 (B_t - B_s)^2|\mathcal{F}_s\right] \\
&\quad + \mathbb{E}\left[\tbinom{4}{3} B_s^1 (B_t - B_s)^3|\mathcal{F}_s\right] + \mathbb{E}\left[\tbinom{4}{4} B_s^0 (B_t - B_s)^4|\mathcal{F}_s\right] \\
&= \mathbb{E}\left[B_s^4|\mathcal{F}_s\right] + \tbinom{4}{1}\mathbb{E}\left[B_s^3|\mathcal{F}_s\right]\mathbb{E}\left[B_t - B_s|\mathcal{F}_s\right] + \tbinom{4}{2}\mathbb{E}\left[B_s^2|\mathcal{F}_s\right]\mathbb{E}\left[(B_t - B_s)^2|\mathcal{F}_s\right] \\
&\quad + \tbinom{4}{3}\mathbb{E}\left[B_s|\mathcal{F}_s\right]\mathbb{E}\left[(B_t - B_s)^3|\mathcal{F}_s\right] + \mathbb{E}\left[(B_t - B_s)^4|\mathcal{F}_s\right] \\
&= B_s^4 + 0 + \tbinom{4}{2} B_s^2 (t-s) + 0 + 3(t-s)^2 \\
&= B_s^4 + 6B_s^2(t-s) + 3(t-s)^2
\end{aligned}$$

Solution 2 Let $f(t, X_t) = e^{-t}X_t$, we have

$$\begin{aligned}
f_t' &= -e^{-t}X_t \\
f_{X_t}' &= e^{-t} \\
f_{X_tX_t}'' &= 0
\end{aligned}$$

By the General Ito's lemma:

$$\begin{aligned}
df(t, X_t) &= f_{X_t}' dX_t + f_t' dt + \frac{1}{2} f_{X_tX_t}''(dX_t)^2 \\
&= e^{-t}dX_t - e^{-t}X_t dt + 0 \\
&= 2e^{-t}X_t dB_t - e^{-t}X_t dt \;\; (\text{since } \; dX_t = 2X_t dB_t)
\end{aligned}$$

There is a term with dt, so it's not a martingale.

Solution 3 Let $f(t, X_t) = e^{-t}X_t$, we have

$$\begin{aligned}
f'_t &= -e^{-t}X_t \\
f'_{X_t} &= e^{-t} \\
f''_{X_tX_t} &= 0
\end{aligned}$$

By the General Ito's lemma:

$$\begin{aligned}
df(t, X_t) &= f'_{X_t}dX_t + f'_t dt + \frac{1}{2}f''_{X_tX_t}(dX_t)^2 \\
&= e^{-t}dX_t - e^{-t}X_t dt + 0 \\
&= e^{-t}(\log(t)dt + 2X_t dB_t) - e^{-t}X_t dt \ \ (\text{since } \ dX_t = \log(t)dt + 2X_t dB_t) \\
&= e^{-t}(\log(t) - X_t)dt + 2e^{-t}X_t dB_t
\end{aligned}$$

Since $\log(t) - X_t \neq 0$, there is a term with dt, so it's not a martingale.