MA3071 – DLI
Financial Mathematics – Introduction

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My details

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Appointments: E-mail me to book appointments.

Blackboard Site

- Course material and course announcements will be available on Blackboard.
- Lecture slides posted on Bb before the lectures.
- Send me e-mail if I forget to upload something!
- Importance of attending classes.
- Some references are provided on Bb if you are interested in further reading, but it is not necessary to succeed in this course.

Prerequisites

- Essential prerequisites: basic probability (random variables and their distributions, mean, variance, covariance, etc.), basic calculus (derivatives and integrals, Lagrangian approach, etc.), and differential equations.
- We will not cover these in the course, and you are expected to work independently to revise them.
- Desirable skills: Programming (MATLAB, etc.), Excel
- If you have any doubts or questions about prerequisites, please get in touch with me via discussion board or email.

Assessment

- 30% Coursework
- 70% Written examination

Assessment

- *Coursework*: Consisting of 3 computer-based problem sheets, to be done individually. Hand in coursework electronically via Blackboard (one file per person). Deadlines are as follows,
  - Coursework 1: 08/Nov/2023, 16:00 (UK)/23:59 (China)
  - Coursework 2: 08/Dec/2023, 16:00 (UK)/23:59 (China)
  - Coursework 3: 20/Dec/2023, 16:00 (UK)/23:59 (China)
- *Exam*: 4 questions, 25 points each, exam date TBC.

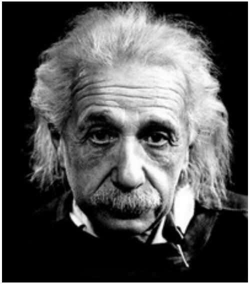
Question

- What do you think are the main topics of financial mathematics?

Answers

- There are many financial games. And in this module, we concentrate on two types of financial games.
    - Option games.
    - Portfolio optimization games.

Answers



"You have to learn the rules of the game. And then you have to play better than anyone else."

– Albert Einstein

Example: Option

- Suppose that a man wants to buy a house in a year from now. The current price is £95, 000 and he believes the price will go up in a year. Therefore, he only wants to pay a guaranteed price of £100, 000 even though the price may be higher than that. Are there any financial tools that would allow him to achieve this goal?

Example, cont.

- European call option contract tells you that it is possible for him to buy the house in a year from now, for a fixed price £$100,000$. However, to do that he needs to pay an extra fee, which is the price to buy the right.
- What is the price for him to buy the right? How much does he pay to have the current homeowner happily grant him this right?
- Obviously, the price should be fair enough for both parties to accept.

Option games

- One of the financial games is option games. Our task is to determine the option prices.

Example: Portfolio optimization

- Suppose that in a financial market, there are only two stocks (Tencent and Alibaba) available for investment. You have a lot of money and would like to invest your money in the two stocks to form a portfolio. Which of the following is a better portfolio?
  - A. 40% of money in Tencent, 60% of money in Alibaba;
  - B. 50% of money in Tencent, 50% of money in Alibaba;
  - C. 60% of money in Tencent, 40% of money in Alibaba.

Example, cont.

- We cannot answer this question intuitively.
- If both stocks have the same expected rates of return,
  - If the Tencent is more risky than the Alibaba, portfolio A would be better.
  - If the Alibaba is more risky than the Tencent, portfolio C would be better.
  - If the Tencent is as risky as the Alibaba, all the above portfolios are equivalent.
- What if both stocks are equivalently risky, but have different expected rates of return?

Example, cont.

- What about "20% of money in Tencent, 80% of money in Alibaba"?
- **Question 1**: Is there a better portfolio than all of the above?
- **Question 2**: What if both stocks have different expected rates of return and different risk levels?

Example, cont.

- Can we determine a pair of numbers (x, y) such that $x + y = 1$ and "x fraction of money in Tencent and y fraction of money in Alibaba" is the best portfolio among all feasible portfolios?

- Please note: x or y may be negative when short selling is allowed in the market.

Portfolio optimization

- Another financial game is portfolio optimization game. Our job is to identify the optimal portfolio.

Syllabus

1. Introduction to options;
2. Binomial tree models;
3. Brownian motion and stochastic differential equations;
4. Black-Scholes pricing models;
5. Monte-Carlo methods for option pricing;
6. Risk measures and Mean-variance portfolio theory;
7. Asset pricing models under equilibrium.

MA3071 – DLI
Financial Mathematics – Section 1
Introduction to options

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Options - definitions

- An **option** gives the buyer of the option (or the person granted the option) the right, but not the obligation, to buy or sell a specified asset (the underlying asset) on a predetermined future date (the expiry/maturity date) for a predetermined price (the strike price, usually denoted by K).
  - A **call option** grants the right, but not the obligation, to **purchase** an underlying asset on a specified maturity date in the future for a predetermined strike price.
  - A **put option** grants the right, but not the obligation, to **sell** an underlying asset on a specified maturity date in the future for a predetermined strike price.
- A **European option** can only be exercised at maturity, while an **American option** can be exercised at any time before its maturity.

Example

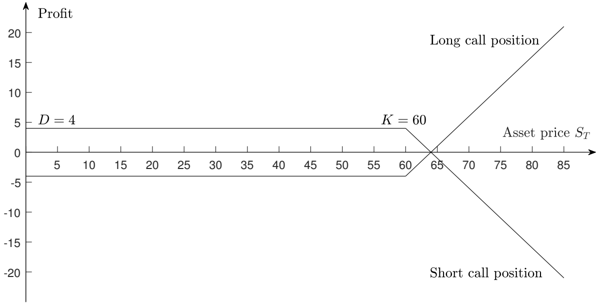
Let S_t denote the price of an underlying asset at time t, T denote the maturity, K be the strike price, and D be the option premium at time T.

- European call option: $T = 2$ years, $K =$£10, $D =$£2, if $S_T =$£20, what is the profit of the option buyer? What if $S_T =$£8?
- European put option: $T = 2$ years, $K =$£10, $D =$£2, if $S_T =$£20, what is the profit of the option buyer? What if $S_T =$£8?

Long and short positions

- A **long position** on an option is when the option has been purchased, while a **short position** is when the option is sold.
- Therefore, a long position on a European option gives the holder (i.e. the buyer, or owner of the option) the right but not the obligation to exercise the option. The holder of the short position (i.e. the seller, or writer of the option) will be obliged to sell, or buy, the underlying asset for the agreed price, if the option is exercised.

European call option profit at maturity

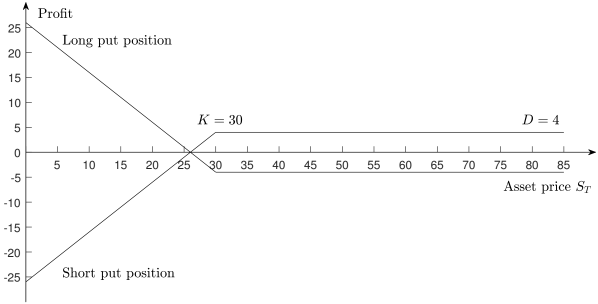


The profit (denoted by P) at maturity T for the holder of a call option is calculated using the formula:

$$P = (S_T - K)_+ - D$$

where $(u)_+ = \max\{u, 0\}$, and $(S_T - K)_+$ is called **call option claim/payoff**.

European put option profit at maturity



The profit P at maturity T for the holder of a put option is calculated using the formula:

$$P = (K - S_T)_+ - D$$

where $(u)_+ = \max\{u, 0\}$, and $(K - S_T)_+$ is called **put option claim/payoff**.

Some discussions

- The loss for a short position and the profit for a long position in a put option are both limited to the difference between the strike price and the premium (i.e. $K - D$).
- In contrast, with a call option, the loss for a short position and the profit for a long position can be positively infinite as the price of the underlying asset theoretically has no upper limit. Therefore, call options are more risky for short positions but can be more profitable for long positions.
- Asset price S_T is a random variable, thus the option profit P is also a random variable.
- The profit for the writer of a European option is always the opposite of the profit for the holder. Thus, only when $\mathbb{E}[P] = 0$ would it be considered fair for both the writer and the holder.

Question

- If you want to buy a European option that expires at T, how much do you need to pay at time 0?

Basic concept of option prices

- Under the assumption that $\mathbb{E}[P] = 0$, the price of a European option at time t (denoted as V_t) is the present value of the expected payoff at time T, that is
  - Call option: $V_t = (1+\rho)^{-(T-t)}\mathbb{E}\left[(S_T - K)_+\right]$
  - Put option: $V_t = (1+\rho)^{-(T-t)}\mathbb{E}\left[(K - S_T)_+\right]$

  where ρ represents the risk-free interest rate for the period $[0, 1]$, and $(1+\rho)^{-(T-t)}$ is the discount factor for the period $[t, T]$.

- In the case of continuous compounding, discount factor is modified as $e^{-\rho(T-t)}$, and
  - Call option: $V_t = e^{-\rho(T-t)}\mathbb{E}\left[(S_T - K)_+\right]$
  - Put option: $V_t = e^{-\rho(T-t)}\mathbb{E}\left[(K - S_T)_+\right]$

- The option premium at T is calculated by $D = V_t(1+\rho)^{T-t}$ or $D = V_t e^{\rho(T-t)}$.

Option pricing models

- Option pricing is one of main tasks of this module.
- We will talk about the following option pricing models in this course:
  - Binomial tree models.
  - Black–Scholes models.
  - Monte-Carlo methods.
- We focus more on the option pricing of European style options.

The concept of arbitrage

- We will encounter the concept of arbitrage when discussing those pricing models. An arbitrage opportunity arises when you can execute a series of transactions that result in a profit without incurring any risk. More formally, arbitrage occurs when:
  - The initial net investment is zero.
  - The probability of making a loss is zero.
  - The probability of making a profit is strictly greater than zero.
- Clearly, if such an opportunity was available then investors would trade as much as they could to take advantage of this "free lunch".

Example

In a financial market, bank A offers a one-year fixed-term bond with an interest rate of ρ_1, and bank B offers another one-year fixed-term bond with an interest rate of ρ_2. Assuming short sales are allowed in this market, are there any arbitrage opportunities when $\rho_1 \neq \rho_2$?

No-arbitrage principle in option pricing

- Since the value of options depends on the value of some other financial variables (price of underlying assets S_t, strike price K, maturity T, etc.), their price will be a function of those variables. If the option price is not consistent with the underlying, then arbitrage would be possible.
- This would result in a loss to one of the parties to the transaction as others traded to take advantage of the arbitrage opportunity. Everyone is therefore keen to avoid this outcome.
- Even if it were to happen, market activity would move prices such that they would fall back in line with the no arbitrage equivalent.
- It ensures that the calculated prices for the relevant options are consistent with the related market variables.

Factors affecting option prices

- The price of the underlying asset (S_t, usually referred to as the spot value) compared to the strike price (K).
- The volatility of the underlying (usually meaning the annualised standard deviation, σ), as a measure of how uncertain the value in future is.
- Interest rates (ρ), since we are estimating present values.
- The term to expiry (T).

Relationship between these key factors and option prices

- A longer term to maturity means a higher option value. The increased uncertainty regarding values further into the future implies a greater potential benefit for the option holder.
- A higher level of volatility also leads to a higher option value (this is the most significant variable because it has the greatest individual impact on the option value). This is due to the greater uncertainty.

Relationship between these key factors and option prices

- Higher interest rates lead to higher call option prices and lower put option prices. As interest rates rise in the market, the expected return demanded by investors in stocks tends to increase. Conversely, the present value of future cash flows generated by option contracts decreases. The overall impact of these two factors is an increase in call option value and a decrease in put option value.
- For fixed levels of the above variables, a higher spot price of the underlying asset increases the option value for call options due to their higher intrinsic value, while it decreases the option value for put options.

MA3071 – DLI
Financial Mathematics – Section 2
Binomial tree models

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Background

- The binomial tree model is a numerical method for estimating option prices in a no-arbitrage framework.
- All such methods are by definition discrete in nature, but with small enough steps (in time and change in modelled variable) the result will converge towards the continuous equivalent (Black-Scholes).
- More steps usually mean a more accurate price, but are more computationally intensive.

Background, cont.

- The model is very flexible and can be used for pricing American options and exotic options where the payoff is path dependent (such as Asian options, barrier options).
- It also provides insight to key concepts in financial economic theory such as hedging portfolios and risk-neutral pricing. These are central to the development of the Black-Scholes formula and option pricing in general.

Financial Assumptions

- The option payoff/claim (C) is a function of the underlying asset price at time T (S_T), i.e. $C = f(S_T)$.
- The risk-free interest rate (ρ) is known and is a constant over a certain time period.
- The volatility/standard deviation (σ) of the return on the underlying asset is a constant.
- There are no transaction costs in buying or selling the underlying asset or the option.
- Short selling is allowed.
- There is no arbitrage opportunity.

Mathematical Assumptions

- **Markov Property**: Behaviour of asset prices satisfies Markov property. Given the present price, the future price does not depend on the past prices.
- **Martingale Property**: To achieve the no-arbitrage condition, we make the assumption that the discounted asset price is a martingale.

Conditional expectation

- If X and Y are two random variables, the conditional expectation of X given $Y = y$ is
  - Discrete: $\mathbb{E}[X|Y = y] = \sum_x xP(X = x|Y = y)$
  - Continuous: $\mathbb{E}[X|Y = y] = \int_{-\infty}^{\infty} xf_{X|Y}(x|y)dx$
- Y as a random variable has different possible values. Therefore, $\mathbb{E}[X|Y]$ **is still a random variable**, since its value depends on the values of the random variable Y.

Properties of conditional expectation

- If X is independent of Y, then

$$\mathbb{E}[X|Y] = \mathbb{E}[X]$$

- Law of total expectation:

$$\mathbb{E}\left[\mathbb{E}[X|Y]\right] = \mathbb{E}[X] \quad \text{and} \quad \mathbb{E}\left[\mathbb{E}[g(X)|Y]\right] = \mathbb{E}[g(X)]$$

- Linearity:

$$\mathbb{E}[aX_1 + bX_2|Y] = a\mathbb{E}[X_1|Y] + b\mathbb{E}[X_2|Y]$$

Markov Property

- **Definition**: Let $(\Omega, \mathcal{F}, P)$ be a probability space with a filtration $\{\mathcal{F}_s, s \in I\}$, for some totally ordered index set I, and let $(S, \mathcal{S})$ be a measurable space. A stochastic process $\{X_t : \Omega \to S\}_{t \in I}$ defined on $(S, \mathcal{S})$ and adapted to the filtration is said to possess the Markov property if, for each $A \in \mathcal{S}$ and each $s, t \in I$ with $t > s \geq 0$,

$$P(X_t \in A | \mathcal{F}_s) = P(X_t \in A | X_s)$$

- Alternatively, the Markov property can be formulated as the following conditional expectation,

$$\mathbb{E}[f(X_t) | \mathcal{F}_s] = \mathbb{E}[f(X_t) | X_s]$$

for all $t > s \geq 0$ and $f : S \to \mathbb{R}$ is a bounded and measurable function.

Markov Property

- A stochastic process satisfies the Markov property if the conditional probability distribution of future states of the process (conditional on both past and present values) depends only upon the present state.

$$P(X_{t_n} = x_n | X_{t_{n-1}} = x_{n-1}, X_{t_{n-2}} = x_{n-2}, \cdots, X_{t_0} = x_0) \\ = P(X_{t_n} = x_n | X_{t_{n-1}} = x_{n-1})$$

  and

$$\mathbb{E}[f(X_{t_n}) | X_{t_{n-1}}, X_{t_{n-2}}, \cdots, X_{t_0}] = \mathbb{E}[f(X_{t_n}) | X_{t_{n-1}}]$$

  for all $t_n > t_{n-1} > t_{n-2} > \cdots > t_0 \geq 0$.

- An important result is that any process with **independent increments** satisfies the Markov property.

Martingale Property

- **Definition**: A martingale is a stochastic process $\{X_t\}_{t \in I}$ that satisfies:

$$\mathbb{E}[|X_t|] < \infty \text{ for all } t \in I,$$
$$\mathbb{E}[X_t|\mathcal{F}_s] = X_s \text{ for all } s, t \in I \text{ and } t > s \geq 0$$

- For example,

$$\mathbb{E}[X_{t_n}|X_{t_{n-1}}, X_{t_{n-2}}, \cdots, X_{t_0}] = X_{t_{n-1}}$$

for all $t_n > t_{n-1} > t_{n-2} > \cdots > t_0 \geq 0$.

- For our mathematical assumption, the **discounted asset price** is a stochastic process and a martingale.

Assumptions

- Please Note: All of the above financial and mathematical assumptions also apply to the **Black-Scholes model** and the **Monte-Carlo methods** of option pricing.

Single period binomial tree

- In the simplest version of the model, we consider only one time period. Over this time period, the price of the underlying asset (S_t) can only increase or decrease (i.e. is Bernoulli).
- The "increase" or "decrease" is proportional to the current value with factors u and d such that $u > d > 0$.

$$\begin{array}{ccc} & & S_1 = S_0 u \\ & \nearrow & \\ S_0 & & \\ & \searrow & \\ & & S_1 = S_0 d \end{array}$$

Two period binomial tree

- An example of a two period tree is shown here.

$$
\begin{array}{ccccc}
 & & & & S_2 = S_0u^2 \\
 & & & \nearrow & \\
 & & S_1 = S_0u & & \\
 & \nearrow & & \searrow & \\
S_0 & & & & S_2 = S_0ud \\
 & \searrow & & \nearrow & \\
 & & S_1 = S_0d & & \\
 & & & \searrow & \\
 & & & & S_2 = S_0d^2
\end{array}
$$

- The above binomial tree satisfies the **Markov property**.

Example

Consider a single period binomial tree. Assume that $\rho = \frac{1}{3}$ over period $[0, 1]$, and the asset prices at times 0, 1 are defined by

$$S_0 = 6, S_1 = S_0 Y$$

where Y is a random variable with

$$P(Y = 3) + P(Y = 0.5) = 1$$

Write down the single-period binomial tree and calculate the option price V_0 of the option claim $C = (S_1 - 6)_+$.

Questions

- How to make sure the option price is arbitrage free?
- What are the probabilities of the asset price rising and falling?

Martingale

- As binomial tree model is a discrete model, the discounted asset price is $(1+\rho)^{-t}S_t$, where ρ is the risk-free interest rate over each period in the tree.

- According to the mathematical assumptions, $(1+\rho)^{-t}S_t$ is a martingale if

$$\mathbb{E}_Q\left[(1+\rho)^{-T}S_T|\mathcal{F}_t\right] = (1+\rho)^{-t}S_t, \quad T > t \geq 0$$

where $\mathbb{E}_Q[\bullet]$ denotes the expectation based on q-probabilities such that

$$q_u = \frac{1+\rho-d}{u-d}, \quad q_d = 1 - q_u$$

Hedging Portfolio

- Let C_u and C_d denote the option payoff at time 1 when the price of the underlying asset increases and decreases, respectively.
- Suppose we hold a portfolio of stocks and bonds at time 0, with ϕ units of stock and ψ units of bond. At time 1, this portfolio will be worth:
  - $\phi S_0 u + \psi(1+\rho)$ if the stock price increased,
  - $\phi S_0 d + \psi(1+\rho)$ if the stock price decreased.

Hedging Portfolio, cont.

- We now choose ϕ and ψ such that the value of the portfolio at time 1 is equal to the payoff of the option. Therefore,

$$\phi S_0 u + \psi(1+\rho) = C_u$$
$$\phi S_0 d + \psi(1+\rho) = C_d$$

- We then solve the simultaneous equations to get,

$$\phi = \frac{C_u - C_d}{S_0(u-d)} \quad \text{and} \quad \psi = \frac{uC_d - dC_u}{(1+\rho)(u-d)}$$

Hedging Portfolio, cont.

- Since the values of the hedging portfolio and the option are equal at time 1, they must be equal at time 0 to avoid arbitrage. The value of the hedging portfolio is then equal to the price of the option at time 0. Therefore,

$$V_0 = \phi S_0 + \psi = \frac{C_u q_u + C_d q_d}{1+\rho}$$

$$q_u + q_d = 1$$

- We then obtain,

$$q_u = \frac{1+\rho-d}{u-d}, \quad q_d = 1 - q_u$$

Hedging Portfolio, cont.

- The hedging portfolio (ϕ, ψ) is also called a replicating portfolio because it matches the option payoffs with no risk.
- This approach can also be employed for hedging purposes by the option seller/writer: that is an investment strategy which reduces the amount of risk carried by the seller of the option when used in conjunction with the short position in the option.

No-arbitrage condition

- The no-arbitrage condition must hold for the option game

$$d < 1 + \rho < u$$

- Moreover, it is easy to check that
  - $q_u + q_d = 1$
  - $0 < q_u < 1$, and $0 < q_d < 1$ iff the above no arbitrage condition holds.
  - $\mathbb{E}_Q[Y] = uq_u + dq_d = 1 + \rho$
- As long as the above arbitrage-free condition holds, equivalent martingale probabilities ($q-$probabilities) are in effect for arbitrage-free option pricing, and the value of the hedging portfolio equals the value of the option at all times.
- $(1 + \rho)^{-t} S_t$ is a martingale $\Leftrightarrow d < 1 + \rho < u$.

Think about

- If $1 + \rho < d < u$, what's the arbitrage opportunity?
- What if $d < u < 1 + \rho$?

Two period binomial tree

- For the two period binomial model, the discrete time market consists of two assets: one non risky asset (bond) with fixed interest rate ρ and one risky asset such that the asset prices at times 0, 1 and 2 are defined by

$$S_0, S_1 = S_0 Y_1, S_2 = S_1 Y_2$$

where Y, Y_1, Y_2 are iid random variables with

$$P(Y = u) + P(Y = d) = 1$$

- We want to determine the arbitrage free time 0 option price of the option claim $C = f(S_2)$ with expiry date $T = 2$.

Two period binomial tree, cont

- $q-$probabilities are the **same** as found from the single period binomial tree model.
- Then, the general binomial tree is defined by

$$
\begin{tikzcd}[column sep=small]
 & & S_2 = S_0u^2\ , \quad C_{uu} = f(S_0uu) \text{ with } q_u^2 \\
 & S_1 = S_0u \arrow[ur, "q_u"] \arrow[dr, "q_d"'] & \\
S_0 \arrow[ur, "q_u"] \arrow[dr, "q_d"'] & & S_2 = S_0ud\ , \quad C_{ud} = f(S_0ud) \text{ with } 2q_uq_d \\
 & S_1 = S_0d \arrow[ur, "q_u"] \arrow[dr, "q_d"'] & \\
 & & S_2 = S_0d^2\ , \quad C_{dd} = f(S_0dd) \text{ with } q_d^2
\end{tikzcd}
$$

- The option price is

$$V_0 = (1+\rho)^{-2}\left[q_u^2 C_{uu} + 2q_uq_d C_{ud} + q_d^2 C_{dd}\right]$$

Example

Consider a discrete market with one risky asset and one risk free asset. The interest rate $\rho = 0.5$ is fixed over each period, and the asset prices at times 0, 1 and 2 are defined by

$$S_0 = 4, S_1 = S_0 Y_1, S_2 = S_1 Y_2$$

where Y, Y_1, Y_2 are iid random variables with

$$P(Y = 8) + P(Y = 0.5) = 1$$

(i) Determine the equivalent martingale probabilities.

(ii) Write down the two-period binomial tree and find the arbitrage free time 0 option price of the European put option with strike price $K = 5$ and expiry day $T = 2$.

(iii) Determine the hedging portfolio for the two period tree.

Extending to n periods

An n period binomial tree is introduced as follows,

(I) $S_{t_i} = S_{t_{i-1}} Y_{t_i}$ for $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = T$ and $t_i - t_{i-1} = \frac{T}{n}$, $i = 1, 2, \cdots, n$, where S_{t_i} is the time t_i asset price and S_0 is a positive constant;

(II) Y and Y_{t_i}, $i = 1, 2, \cdots, n$ are independent and identically distributed random variables (iid) with

$$P(Y = u) + P(Y = d) = 1$$

where up factor u and down factor d are constants such that $u > d > 0$ and possible outcomes of Y;

(III) Y_{t_i} is independent of $S_{t_1}, \cdots, S_{t_{i-1}}$.

Extending to n periods, cont.

- Distribution of S_{t_n} is defined by

$$S_{t_n} = S_0 u^j d^{n-j}$$
$$P(S_{t_n} = S_0 u^j d^{n-j}) = \binom{n}{j} q_u^j q_d^{n-j}$$

for $j = 0, 1, \cdots, n$.

- The arbitrage free time 0 option price of the option claim $C = f(S_{t_n})$ is calculated by

$$V_0 = (1+\rho)^{-n} \sum_{j=0}^{n} f(S_0 u^j d^{n-j}) \binom{n}{j} q_u^j q_d^{n-j}$$

where ρ is the interest rate over period $[t_{i-1}, t_i]$, $i = 1, 2, \cdots, n$.

Example

Let Y and Y_{t_i}, $i = 1, 2, \cdots, n$ be iid random variables with distribution

$$P(Y = 1) = 0.5, \quad P(Y = 3) = 0.5$$

and the underlying asset price is modelled by $S_{t_i} = S_{t_{i-1}} Y_{t_i}$.

Let $\rho = 0.25$ over each time period. Find the arbitrage free time 0 option price of the option claim $C = S_T^{10}$, when $S_0 = 4$, $n = 5$ and $T = t_n$.

Time varying binomial tree models

- Time varying interest rates ρ_i over period $[t_{i-1}, t_i]$, $i = 1, 2, \cdots, n$.
- Asset price is modelled by $S_{t_i} = S_{t_{i-1}} Y_{t_i}$, $i = 1, 2, \cdots, n$, where Y_{t_i} are independent, but in general not identically distributed
$$P(Y_{t_i} = u_i) + P(Y_{t_i} = d_i) = 1$$
with time varying factors u_i and d_i.
- The time varying $q-$probabilities are
$$q_u^{(i)} = \frac{1 + \rho_i - d_i}{u_i - d_i}, \quad q_d^{(i)} = 1 - q_u^{(i)}, \quad i = 1, 2, \cdots, n$$

Example

- Two period time varying binomial tree model:

$$
\begin{array}{ccccccc}
 & & & \nearrow^{q_u''} & S_2 = S_0 u_1 u_2\,, & C_{u_1 u_2} & \text{with } q_u' q_u'' \\
 & & S_1 = S_0 u_1 & & & & \\
 & \nearrow^{q_u'} & & \searrow_{q_d''} & S_2 = S_0 u_1 d_2\,, & C_{u_1 d_2} & \text{with } q_u' q_d'' \\
S_0 & & & & & & \\
 & \searrow_{q_d'} & & \nearrow^{q_u''} & S_2 = S_0 d_1 u_2\,, & C_{d_1 u_2} & \text{with } q_d' q_u'' \\
 & & S_1 = S_0 d_1 & & & & \\
 & & & \searrow_{q_d''} & S_2 = S_0 d_1 d_2\,, & C_{d_1 d_2} & \text{with } q_d' q_d''
\end{array}
$$

- The option price is

$$V_0 = \frac{C_{u_1 u_2} q_u' q_u'' + C_{u_1 d_2} q_u' q_d'' + C_{d_1 u_2} q_d' q_u'' + C_{d_1 d_2} q_d' q_d''}{(1+\rho_1)(1+\rho_2)}$$

Calibrating binomial trees

- It is convenient to calibrate the tree such that the asset price follows a log-normal distribution,

$$\log\left(\frac{S_T}{S_t}\right) \sim N\left[\left(\rho - \frac{\sigma^2}{2}\right)(T-t), \sigma^2(T-t)\right]$$

- In this case, as the number of time steps increases, the estimated option price will converge towards the Black-Scholes price.

Calibrating binomial trees

- Let Δ_t be the size of the time step, we have
  - $q_u = \frac{e^{\rho \Delta_t} - d}{u - d}$,
  - $u = e^{\sigma \sqrt{\Delta_t}}$,
  - $d = e^{-\sigma \sqrt{\Delta_t}}$.

MA3071 – DLI
Financial Mathematics – Section 3
Brownian motion and stochastic differential equations – Part I

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Background

- The option pricing models that we will encounter in future sections are based on a form of modelling known as stochastic calculus. This is the approach used to model financial variables in **continuous time**.
- This can result, under certain conditions, in closed-form solutions that are therefore very **computationally efficient**.

Brownian motion

- A continuous time process that has proved useful for financial modelling purposes is Brownian motion. This was first developed as an idea in the physical sciences, with botanist Robert Brown noting the erratic motion of pollen particles in water in 1827.
- Norbert Wiener developed a mathematically rigorous construction of the stochastic process in a more abstract, general sense. You will sometimes therefore see it called the Wiener process.
- One way to think about this process is the continuous time equivalent of a binomial tree (i.e. a discrete time process, where at each time point the value can increase or decrease with equal probability).

Standard Brownian motion

$\{B_t,\ t \geqslant 0\}$ is called a standard Brownian motion (SBM), if

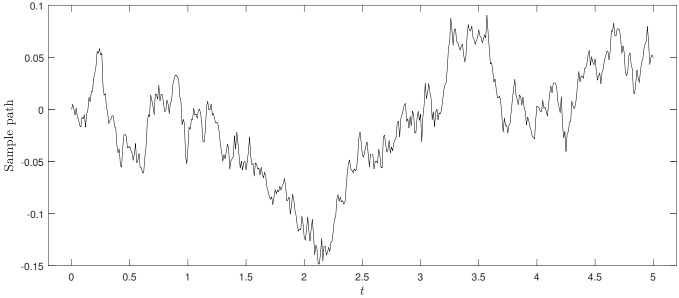
(1) $B_0 = 0$;

(2) $B_t - B_s \backsim N(0, t-s)$ for all $t > s$;

(3) B_t has independent increments such that $B_{t_1},\ B_{t_2} - B_{t_1},\ \cdots,\ B_{t_k} - B_{t_{k-1}}$ are independent for all $t_1 < t_2 < \cdots < t_{k-1} < t_k$;

Sample path of SBM

- An example of a sample path of an SBM is shown in the graph below,



- Notice how jagged the process is. In fact, it is possible to show that the sample path is nowhere differentiable: the limit for a mathematical derivative does not exist because the variable is not bounded, even on a short interval, so no unique tangent can be found.

Properties of SBM

- B_t has stationary increments: the distribution of $(B_t - B_s)$ depends only on $t - s$, where $t > s$.
- $\{B_t,\ t \geqslant 0\}$ is a Markov process.
- $\{B_t,\ t \geqslant 0\}$ returns infinitely often to any level, including 0.
- B_t has continuous sample paths: $t \to B_t$.

Properties of SBM, cont.

- $\mathbb{E}[B_t] = 0$.
- $\mathrm{Var}(B_t) = t$.
- $Cov(aB_t, bB_s) = ab\mathbb{E}(B_t B_s) = ab\min(t, s)$, where a and b are constants.
- $B_t - B_s$ has the same distribution as B_{t-s}, $t > s$. But they are different,
  - $B_t - B_s$ is the difference between two different random variables.
  - B_{t-s} is a single random variable.

Conditional expectation of SBM

The conditional expectation encountered in this course is no longer conditioning on a single random variable, but many (including a filtration, in the abstract sense).

Let $X_t = f(t, B_t)$ be a stochastic process, then we have

$$\mathbb{E}[X_t|\mathcal{F}_s] = \mathbb{E}[f(t, B_t)|\mathcal{F}_s] = \mathbb{E}[f(t, B_s + (B_t - B_s))|\mathcal{F}_s]$$

where $\{\mathcal{F}_s, 0 \leq s < t\}$ is a filtration of X_t.

Specifically, when $s = 0$,

$$\mathbb{E}[X_t|\mathcal{F}_0] = \mathbb{E}[X_t]$$

Useful conclusions

When calculating the conditional expectation, we may need to use the following conclusions,

- $\mathbb{E}[f(s, B_s)|\mathcal{F}_s] = f(s, B_s)$,
- $\mathbb{E}[(B_t - B_s)^{2m+1}|\mathcal{F}_s] = 0,\ m = 0, 1, \ldots$
- $\mathbb{E}[(B_t - B_s)^{2m}|\mathcal{F}_s] = (t-s)^m(2m-1)!!,\ m = 0, 1, \ldots,$ where $(2m-1)!! = 1 \times 3 \times 5 \times (2m-1)$.
- If $f(t, B_t) = g(t)h(B_t)$, $\mathbb{E}[g(t)h(B_t)|\mathcal{F}_s] = g(t)\mathbb{E}[h(B_t)|\mathcal{F}_s]$
- $\mathbb{E}[g(t)|\mathcal{F}_s] = g(t)$ and $\mathrm{Var}[g(t)] = 0$.

for all $t > s \geqslant 0$.

Examples

- Show that B_t is a martingale.
- Show that $B_t^2 - t$ is a martingale.
- Is B_t^3 a martingale?
- Find $\mathbb{E}[t^2 B_t | \mathcal{F}_s]$, $t > s$.

Geometric Brownian Motion (preview)

- One of the shortcomings of standard Brownian motion, from the point of view of financial modelling, is that it allows negative values. It is not helpful when modelling equity prices, since these cannot be worth less than nothing.
- This can be addressed by assuming that the log of a stochastic process follows standard Brownian motion. The process itself will then be log-normally distributed.

Geometric Brownian Motion (preview)

- The asset price at time t can be modelled by

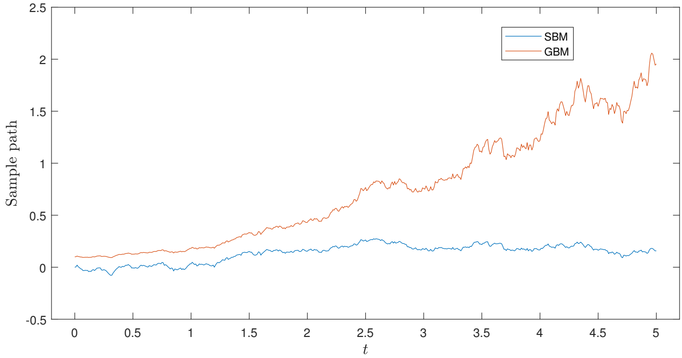
$$S_t = S_0 e^{at + \sigma B_t}$$

  where S_0 is the price at time 0, a is a constant and σ is the standard deviation/volatility of the return.

- This model is referred to as geometric Brownian motion (GBM) and is central to the development of many models in mathematical finance.

Sample path of GBM

- An example sample path, with a comparison to standard Brownian motion is shown below,



Stochastic calculus

- Since the sample paths of Brownian motion are not differentiable, that might suggest that the powerful applications that calculus can bring are not available.
- However, it is in fact possible to develop a definition of integrals of the form $\int_0^t Y_u dB_u$ for suitable random integrands Y_u (i.e. those that are adapted to the filtration of the Brownian motion).
- These stochastic integrals are also referred to as "Ito integrals", after the Japanese mathematician Kiyoshi Ito who first developed the idea in the 1940s.

Stochastic integrals

- The basic principle is that, while Brownian motion is “too erratic” to allow for a sensible definition of a mathematical derivative, there is enough regularity that it is possible to define a sequence of sums that will converge to a finite value under certain conditions. These can be considered an integral, in the same way as a standard integral is the limit of a sequence of Riemann sums.
- These stochastic integrals are themselves random variables, since B_t is random. We will therefore be interested in their distribution, expectation and variance.

Properties of stochastic integrals

(1) $\left(\int_0^t Y_u dB_u, t \geqslant 0\right)$ is a martingale.

(2) $\mathbb{E}[\int_0^t Y_u dB_u] = 0$.

(3) Ito isometry: $\mathbb{E}\left[\left(\int_0^t Y_u dB_u\right)^2\right] = \mathbb{E}\left[\int_0^t Y_u^2 du\right]$.

(4) $\int_0^t Y_u dB_u$ follows a normal distribution with mean 0 and variance $\mathrm{Var}(\int_0^t Y_u dB_u) = \mathbb{E}\left[\int_0^t Y_u^2 du\right]$, that is

$$\int_0^t Y_u dB_u \sim N\left(0, \mathbb{E}\left[\int_0^t Y_u^2 du\right]\right)$$

Example

- State the distribution of the Ito integral $\int_0^t \sqrt{u} dB_u$.

Ito process

- The Ito process X_t is a stochastic process, which can be defined by the stochastic differential equation (SDE)

$$dX_t = A_t dt + Y_t dB_t$$

where A_t and Y_t are two expressions in terms of B_t and t.

- It is common to write the Ito process X_t in integral form

$$X_t = X_0 + \int_0^t A_u du + \int_0^t Y_u dB_u$$

where X_0 is a constant and the initial value of X_t.

Ito process, cont.

- Note that $\int_0^t A_u du$ is deterministic and is called a deterministic integral. Therefore, the Ito process X_t has a deterministic mean and a random component, such that

$$\begin{aligned}\mathbb{E}[X_t] &= \mathbb{E}[X_0] + \mathbb{E}\left[\int_0^t A_u du\right] + \mathbb{E}\left[\int_0^t Y_u dB_u\right] \\ &= X_0 + \mathbb{E}\left[\int_0^t A_u du\right] \\ \mathrm{Var}[X_t] &= \mathrm{Var}[X_0] + \mathrm{Var}\left[\int_0^t A_u du\right] + \mathrm{Var}\left[\int_0^t Y_u dB_u\right] \\ &= \mathbb{E}\left[\int_0^t Y_u^2 du\right]\end{aligned}$$

since the deterministic integral $\int_0^t A_u du$ is independent of the stochastic integral $\int_0^t Y_u dB_u$.

Properties of deterministic integral

(1) Fubini theorem: $\mathbb{E}\left[\int_0^t A_u du\right] = \int_0^t \mathbb{E}[A_u] du$

(2) $\mathrm{Var}\left[\int_0^t A_u du\right] = 0$

(3) If $X_t = X_0 + \int_0^t A_u du + \int_0^t Y_u dB_u$, then

$$X_t \sim N\left(X_0 + \int_0^t \mathbb{E}\left[A_u\right] du, \int_0^t \mathbb{E}\left[Y_u^2\right] du\right)$$

Example

- If $dX_t = -2tdt + 5\sqrt{t}dB_t$, state the distribution of the stochastic integral $\int_0^t \sqrt{u}dX_u$.

MA3071 – DLI
Financial Mathematics – Section 3
Brownian motion and stochastic differential equations – Part II

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Ito's lemma: core concept of stochastic calculus

Classical Ito's Lemma:

Let $f(t,x)$ be a twice partially differentiable function and $X_t = f(t, B_t)$ be a stochastic process. Then the stochastic differential $dX_t = df(t, B_t)$ is defined by the Ito formula

$$df(t, B_t) = f'_t dt + f'_{B_t} dB_t + \frac{1}{2} f''_{B_t B_t} dt$$

Ito rules: $(dB_t)^2 = dt$, $(dt)^2 = 0$, $dB_t dt = 0$

Remark

- Using the Ito's Lemma, we can write the stochastic process $X_t = f(t, B_t)$ in integral form

$$X_t = f(t, B_t) = f(0, B_0) + \int_0^t A_u du + \int_0^t Y_u dB_u$$

where $A_u = f_u' + \frac{1}{2} f''_{B_u B_u}$ and $Y_u = f'_{B_u}$.

- And hence,

$$f(t, B_t) \sim N\left(f(0, B_0) + \int_0^t \mathbb{E}\left[f_u' + \frac{1}{2} f''_{B_u B_u} \right] du, \int_0^t \mathbb{E}\left[\left(f'_{B_u} \right)^2 \right] du \right)$$

Examples

- Find the stochastic differential $df(t, B_t)$ for the following stochastic processes
  - $f(t, B_t) = B_t^2$,
  - $f(t, B_t) = 7e^{tB_t}$.
- Find the integral forms of the stochastic processes B_t^2 and $7e^{tB_t}$.
- Compute $\mathbb{E}[B_t^2]$ using the Ito calculus.

Ito martingale

- The stochastic process $X_t = f(t, B_t)$ is a Ito martingale if

$$df(t, B_t) = Y_t dB_t$$

which has no term with dt (zero drift).

Examples

- Show that B_t is an Ito martingale.
- Show that $B_t^2 - t$ is an Ito martingale.
- Is B_t^3 an Ito martingale?

Martingale versus Ito martingale

- For any stochastic process X_t, if it is a martingale, then zero drift (Ito martingale) is equivalent to the martingale definition $\mathbb{E}[X_t|\mathcal{F}_s] = X_s$ for $0 \leqslant s < t$.

Conditional expectation via Ito calculus

- Conditional Ito integral:

$$\mathbb{E}\left[\int_s^t Y_u dB_u \Big| \mathcal{F}_s\right] = 0$$

- Conditional Fubini theorem:

$$\mathbb{E}\left[\int_s^t A_u du \Big| \mathcal{F}_s\right] = \int_s^t \mathbb{E}[A_u|\mathcal{F}_s] du$$

where $0 \leqslant s < t$.

Examples

- Given $t > s \geqslant 0$, compute the following conditional expectations via the Ito calculus,
  - $\mathbb{E}[t^2 B_t | \mathcal{F}_s]$.
  - $\mathbb{E}[B_t^3 | \mathcal{F}_s]$.

Ito's Lemma, cont.

General Ito's Lemma:

Let $f(t,x)$ be a twice partially differentiable function and $f(t,X_t)$ be a stochastic process with $dX_t = A_t dt + Y_t dB_t$. Then, the general Ito's lemma states that

$$df(t,X_t) = f'_t dt + f'_{X_t} dX_t + \frac{1}{2} f''_{X_t X_t} (dX_t)^2$$

where Ito rules also applied,

$$(dX_t)^2 = Y_t^2 dt \quad \text{since} \quad (dB_t)^2 = dt,\ (dt)^2 = 0,\ dB_t dt = 0$$

Ito's Lemma, cont.

- For the convenience of calculation, one can also express the general Ito's lemma as follows,

$$df(t, X_t) = \left(f'_t + A_t f'_{X_t} + \frac{1}{2} Y_t^2 f''_{X_t X_t} \right) dt + Y_t f'_{X_t} dB_t$$

where A_t and Y_t are given in the SDE of dX_t.

Example

- Let X_t be defined by $dX_t = 2B_t dB_t$, find $d\left[t^3 X_t^5\right]$.

General Ito Isometry

- Given $\int_0^t Y_u dB_u$ and $\int_0^t Q_u dB_u$ are two stochastic integrals,

$$\mathbb{E}\left[\left(\int_0^t Y_u dB_u\right)\left(\int_0^t Q_u dB_u\right)\right] = \mathbb{E}\left[\int_0^t (Y_u Q_u) du\right]$$

Examples

- Let X_t be defined by $dX_t = 2B_t dB_t$ and $X_0 = 0$, find $\mathbb{E}\left[B_t X_t\right]$ and $\mathbb{E}\left[X_t^2\right]$.
- Find $\mathbb{E}[(B_t^3 - 3tB_t + 1)^2]$ by applying the Ito isometry.

Ito's Lemma versus GBM

Let $\{B_t,\ t \geqslant 0\}$ be a standard Brownian motion. Consider a continuous-time market with one risk-free bond offering a fixed interest rate of ρ and one risky asset. The asset price is modelled by a stochastic process $\{S_t,\ t \geqslant 0\}$, which is defined as a solution to the following stochastic differential equation

$$dS_t = \mu S_t dt + \sigma S_t dB_t$$

Ito's Lemma versus GBM, cont.

I) $\{S_t,\ t \geqslant 0\}$ is a geometric Brownian motion with mean parameter $a = \mu - \frac{\sigma^2}{2}$ and volatility parameter σ.

$$S_t = S_0 e^{(\mu - \frac{\sigma^2}{2})t + \sigma B_t}$$

II) Moreover, $e^{-\rho t} S_t$ is a martingale if and only if $\mu = \rho$.

III) The discounted financial derivative $e^{-\rho t} g(t, S_t)$ is a Martingale iff the Black-Scholes equation holds:

$$g_t' + \mu S_t g_{S_t}' + \frac{1}{2}\sigma^2 {S_t}^2 g_{S_t S_t}'' = \rho g$$

GBM, cont.

- S_t follows a log-normal distribution, such that

$$\log\left(\frac{S_t}{S_0}\right) \sim N\left[\left(\mu - \frac{\sigma^2}{2}\right)t, \sigma^2 t\right]$$

- GBM has the following expected value and variance:
  - $\mathbb{E}[S_t] = S_0 e^{\mu t}$
  - $Var[S_t] = S_0^2 e^{2\mu t}\left(e^{\sigma^2 t} - 1\right)$

Conditional expectation of GBM

Let $S_t = S_0 e^{at+\sigma B_t}$ be a GBM, then we have

$$\begin{aligned}\mathbb{E}[f(S_T)|\mathcal{F}_t] &= \mathbb{E}\left[f\left(S_t \cdot \frac{S_T}{S_t}\right)\Big|\mathcal{F}_t\right] \\ &= \mathbb{E}\left[f\left(S_t e^{a(T-t)+\sigma(B_T-B_t)}\right)\Big|\mathcal{F}_t\right]\end{aligned}$$

where $\{\mathcal{F}_t, 0 \leq t < T\}$ is a filtration of S_t.

Specifically, when $t = 0$,

$$\mathbb{E}[f(S_T)|\mathcal{F}_0] = \mathbb{E}[f(S_T)]$$

Useful conclusions

When calculating the conditional expectation, we may need to use the following conclusions,

- $\mathbb{E}[f(S_t)|\mathcal{F}_t] = f(S_t)$,
- $\mathbb{E}[S_T|\mathcal{F}_t] = S_t e^{\mu(T-t)}$
- $\mathbb{E}\left[\left(e^{a(T-t)+\sigma(B_T-B_t)}\right)^k \middle| \mathcal{F}_t\right] = e^{\left(ka+\frac{k^2\sigma^2}{2}\right)(T-t)}$, where k is a constant.
- $S_t \perp e^{a(T-t)+\sigma(B_T-B_t)}$ since $B_t \perp B_T - B_t$.

for all $T > t \geqslant 0$.

Examples

- Find $\mathbb{E}[6S_T^3|\mathcal{F}_t]$ with $S_t = S_0 e^{2t+B_t}$.
- Find $\mathbb{E}[\log(S_T)|\mathcal{F}_t]$ with $S_t = S_0 e^{at+\sigma B_t}$.

Multivariate Ito's Lemma

Multivariate Ito's Lemma:

Let $f(x_1, \cdots, x_k)$ be a twice partially differentiable function and $f\left(X_t^{(1)}, \cdots, X_t^{(k)}\right)$ be a stochastic process with $dX_t^{(i)} = A_t^{(i)} dt + Y_t^{(i)} dB_t,\ i = 1, \cdots, k.$ Then, the multivariate Ito's lemma states that

$$df\left(X_t^{(1)}, \cdots, X_t^{(k)}\right) = \sum_{i=1}^{k} f'_{X_t^{(i)}} dX_t^{(i)} + \frac{1}{2} \sum_{i=1}^{k} f''_{X_t^{(i)} X_t^{(i)}} \left(dX_t^{(i)}\right)^2 + \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} f''_{X_t^{(i)} X_t^{(j)}} dX_t^{(i)} dX_t^{(j)}$$

where $\left(dX_t^{(i)}\right)^2 = \left(Y_t^{(i)}\right)^2 dt$ and $dX_t^{(i)} dX_t^{(j)} = Y_t^{(i)} Y_t^{(j)} dt$.

Bivariate Ito's Lemma

In particular, when $k = 2$, $dX_t = A_t dt + Y_t dB_t$ and $dZ_t = G_t dt + Q_t dB_t$. Then, the bivariate Ito formula is

$$\begin{aligned} df(X_t, Z_t) = f'_{X_t} dX_t + f'_{Z_t} dZ_t + \frac{1}{2} f''_{X_t X_t} (dX_t)^2 \\ + \frac{1}{2} f''_{Z_t Z_t} (dZ_t)^2 + f''_{X_t Z_t} dX_t dZ_t \end{aligned}$$

And in the integral form,

$$\begin{aligned} f(X_t, Z_t) = f(X_0, Z_0) + \int_0^t f'_{X_s} dX_s + \int_0^t f'_{Z_s} dZ_s \\ + \frac{1}{2} \int_0^t f''_{X_s X_s} (dX_s)^2 + \frac{1}{2} \int_0^t f''_{Z_s Z_s} (dZ_s)^2 + \int_0^t f''_{X_s Z_s} dX_s dZ_s \end{aligned}$$

Examples

- Find $d[M_t S_t]$ through the bivariate Ito formula.
- If $dM_t = B_t^4 dB_t$ and $dS_t = B_t^2 dB_t$, $M_0 = S_0 = 0$, find $\mathbb{E}[M_t S_t]$.
- If $dM_t = (1 + t^2 B_t) dB_t$ and $M_0 = 0$, let $Y_t = tB_t M_t$. Is Y_t a martingale?

MA3071 – DLI

Financial Mathematics – Section 4

Black-Scholes pricing models

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Background

- The Black-Scholes model is rightly recognised as one of the great achievements of modern finance. Until it was published in 1973, there was no systematic way to price options. Robert Merton was also a key figure in developing the theory.
- It is related to other developments, most notably the capital asset pricing model (CAPM).
- The options market expanded considerably following the publication of this new approach, also boosted by growing computer power, which aided implementing numerical methods such as the binomial tree models and Monte-Carlo simulations.

Background

- It also provides an important framework for thinking about the risks arising from trading options and how these can be managed. This is central to options trading today. The model also provides a way to consistently quote option prices, based on volatility (known as "implied volatility").
- The model relies on a number of assumptions, each of which can be criticised, however it is still a useful model.

Assumptions of the model

- The price of the underlying asset (S_t) follows a geometric Brownian motion (GBM).
- The option payoff/claim (C) is a function of the underlying asset price at time T (S_T), i.e. $C = f(S_T)$.
- The risk-free interest rate (ρ) is known and a constant, the same for all maturities and the same for borrowing or lending.
- The volatility/standard deviation (σ) of return on the underlying asset is a constant.
- There is no arbitrage opportunity available.
- Unlimited short selling is allowed.
- There are no taxes or transaction costs.

Assumptions of the model

- The underlying asset can be traded continuously and in infinitesimally small units.
- These assumptions imply that every possible contingent payoff has a hedging portfolio (i.e. can be replicated using other assets available in the market). A market that has this property is called a **complete market**. It is further required that the hedging portfolios must also be able to match changes in the payoff as market conditions change, without requiring any external adjustments. This is known as the **self-financing** property.

Mathematical assumptions of the model

- **Markov Property**: Behaviour of asset prices satisfies Markov property. Given the present price, the future price does not depend on the past prices.
- **Martingale property**: To achieve the no-arbitrage condition, we make the assumption that the discounted asset price is a martingale.

The PDE approach

- The idea is to start with the SDE for the underlying asset:

$$dS_t = \mu S_t dt + \sigma S_t dB_t$$

- The option price at time t is a function of this asset value and time such that $V_t = g(t, S_t)$, so we can use Ito's lemma to derive the dynamics of the option value:

$$dg(t, S_t) = g'_{S_t} \sigma S_t dB_t + \left[g'_t + g'_{S_t} \mu S_t + \frac{1}{2} g''_{S_t S_t} \sigma^2 S_t^2 \right] dt$$

Dynamic hedging

- The key insight of Black-Scholes was that by frequently trading in the underlying, you can eliminate all risk.
- Consider a portfolio that is short 1 unit of option and long g'_{S_t} units of the underlying. In other words, the portfolio is held by someone who has sold an option and is trading the underlying asset to manage the risk.
- Remember that $g(t, S_t)$ and its derivatives are functions of time. Therefore, this portfolio will need to be continuously updated to keep the correct value of the underlying asset.
- This approach of continuous updating/rebalancing is known as dynamic hedging.

The PDE approach, cont.

- Let the value of this portfolio be $f(t,S_t) = -g(t,S_t) + g'_{S_t} S_t$. Then the dynamics of the portfolio are then:

$$
\begin{aligned}
df(t,S_t) &= -dg(t,S_t) + g'_{S_t} dS_t \\
&= -\left(g'_{S_t} \sigma S_t dB_t + \left[g'_t + g'_{S_t} \mu S_t + \frac{1}{2} g''_{S_t S_t} \sigma^2 S_t^2 \right] dt \right) \\
&\qquad + g'_{S_t} \left[\mu S_t dt + \sigma S_t dB_t \right] \\
&= -\left(g'_t + \frac{1}{2} g''_{S_t S_t} \sigma^2 S_t^2 \right) dt
\end{aligned}
$$

The PDE approach, cont.

- Notice that the result is completely free of any random terms (i.e. those containing dB_t have been eliminated) and thus of any risk.
- Since g'_{S_t} is a function of time (t) and the price of the asset (S_t), the amount of the asset in the portfolio will need to be continuously updated to maintain this "instantaneously risk-free" position.

The PDE approach, cont.

- Since we have a portfolio that is risk-free, it must earn a return equal to the risk-free rate, since the market is arbitrage-free by assumption. Thus,

$$\begin{aligned} & df(t, S_t) = \rho f(t, S_t) dt \\ \Rightarrow & -\left(g'_t + \frac{1}{2} g''_{S_t S_t} \sigma^2 S_t^2 \right) dt = \rho \left(-g + g'_{S_t} S_t \right) dt \\ \Rightarrow & g'_t + \rho S_t g'_{S_t} + \frac{1}{2} \sigma^2 S_t^2 g''_{S_t S_t} = \rho g \end{aligned}$$

- This is the **Black-Scholes partial differential equation** that tells us how any option value will change over time in a market that satisfies the assumptions of the model.

The PDE approach, cont.

- Recall: The discounted financial derivative $e^{-\rho t}g(t, S_t)$ is a Martingale iff the Black-Scholes equation holds:

$$g'_t + \mu S_t g'_{S_t} + \frac{1}{2}\sigma^2 {S_t}^2 g''_{S_t S_t} = \rho g$$

- To satisfy the no-arbitrage condition in option pricing, the $e^{-\rho t}S_t$ is a martingale iff $\mu = \rho$, which leads to the arbitrage-free version of the Black-Scholes equation on the last page.

- **Note**: In the rest of this course, for any option claim $C = f(S_T)$, we use $g(t, S_t)$ only to denote the arbitrage free time t option price rather than V_t.

Solving the PDE

- Once the Black–Scholes PDE, with boundary and terminal conditions, is derived for an option, the PDE can be solved numerically using standard methods of numerical analysis, such as a type of finite difference method.
- In some specific cases of option claim $C = f(S_T)$, it is possible to solve for a closed-form expression of the option price $g(t, S_t)$.

Solving the PDE, cont.

- The solution for a European call option is:

$$g(t, S_t) = S_t \Phi(d_1) - K e^{-\rho(T-t)} \Phi(d_2)$$

- The solution for a European put option is:

$$g(t, S_t) = K e^{-\rho(T-t)} \Phi(-d_2) - S_t \Phi(-d_1)$$

where

$$d_1 = \frac{\log\left(\frac{S_t}{K}\right) + \left(\rho + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}$$
$$d_2 = d_1 - \sigma\sqrt{T-t}$$

Here $\Phi(x)$ is the cumulative distribution function of a standard normal random variable.

Examples

- The current price of a non-dividend paying share is £14 and its volatility is thought to be 25% per annum. The continuously compounded risk-free interest rate is 4.8% per annum. A European call option on this share has a strike price of £18 and term to maturity of half year. Assume that the Black-Scholes model applies.

  Calculate the price of the call option.

Examples

- Consider a European put option on a non-dividend paying stock when the stock price is £14, the exercise price is £18, the continuously compounded risk-free rate of interest is 4.8% per annum, the volatility is 25% per annum, and the time to maturity is six months.

  Calculate the price of the option using the Black-Scholes model.

Options on assets paying an income

- Many underlying assets such as shares and bonds produce an income in the form of dividends.
- The modified version of the Black-Scholes formula in this case is sometimes called the Garman-Kohlhagen formula, after the researchers who developed this adaptation.
- The basic approach is to assume that the income is continuously payable and reinvested in the underlying asset.

Options on assets paying an income, cont.

- Denote the continuously payable rate of income as q. Then, define a modified version of the underlying asset $\widetilde{S}_t$, where $\widetilde{S}_0 = S_0$ and beyond time 0, $\widetilde{S}_t$ represents the process where dividends are reinvested.
- The asset dynamics then become:

$$d\widetilde{S}_t = \widetilde{S}_t(\mu + q)dt + \sigma\widetilde{S}_t dB_t$$
$$\Rightarrow \widetilde{S}_t = \widetilde{S}_0 e^{(\mu+q-\frac{1}{2}\sigma^2)t+\sigma B_t}$$

Options on assets paying an income, cont.

- We still consider the portfolio that is short 1 unit of the option and long g'_{S_t} units of the underlying. The PDE now becomes

$$g'_t + (\rho - q)S_t g'_{S_t} + \frac{1}{2}\sigma^2 {S_t}^2 g''_{S_t S_t} = \rho g$$

Options on assets paying an income, cont.

- The equation can be solved for a European call option as

$$g(t, S_t) = S_t e^{-q(T-t)}\Phi(d_1) - Ke^{-\rho(T-t)}\Phi(d_2)$$

- For a European put option, the solution is

$$g(t, S_t) = Ke^{-\rho(T-t)}\Phi(-d_2) - S_t e^{-q(T-t)}\Phi(-d_1)$$

where

$$d_1 = \frac{\log\left(\frac{S_t}{K}\right) + \left(\rho - q + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = d_1 - \sigma\sqrt{T-t}$$

Examples

- Consider the examples in page 15 and page 16. The continuously payable rate of income is 2.4% per annum.

  Calculate the option prices.

Put-call parity

- The term "put-call" parity refers to a principle that defines the relationship between the price of European put and call options of the same class, that is

$$P_t + S_t = C_t + Ke^{-\rho(T-t)}$$

  where:
  - P_t denotes the European put option price at time t
  - C_t denotes the European call option price at time t

- This relationship always holds for European put and call options on the same underlying, with the same maturity and strike price. It is based on no-arbitrage arguments.

Put-call parity, cont.

- It can be used as a shortcut in pricing. Once you know one price, you automatically know the other.
- It can also be used to test new pricing models: all should satisfy this relationship.
- If the put-call parity is violated, then arbitrage opportunities arise.

General solution of the Black-Scholes equation

- Suppose that there is no dividend payment, the arbitrage free time t option price with claim $f(S_T)$ can be expressed by

$$g(t, S_t) = e^{-\rho(T-t)}\mathbb{E}\left[f\left(S_t e^{a(T-t)+\sigma(B_T-B_t)}\right)\middle|\mathcal{F}_t\right]$$

where the no-arbitrage condition is $a = \rho - \frac{\sigma^2}{2}$.

- This is the general solution of the Black-Scholes equation for arbitrary option claim $f(S_T)$.

Examples

- The underlying asset price S_t, is currently £10 and can be modelled by the stochastic differential equation:

$$dS_t = \mu S_t dt + 0.2 S_t dB_t$$

where $\{B_t; t \geq 0\}$ is a standard Brownian motion. Suppose that the risk-free interest rate is $\rho = 0.5$.

(i) State the no arbitrage condition (you need to know how to prove it).

(ii) Show that $g(t, S_t) = S_t$ for the option claim $C = S_T$.

(iii) Find the arbitrage free time t option price for an exotic option claim $C = S_T^3$ as a function of S_t, T and t.

(iv) Find the arbitrage free time t option price for an exotic option claim $C = \log(S_T)$ as a function of S_t, T and t.

Property of Black-scholes equation

- If $g_i(t, S_t)$ for $i = 1, \cdots, n$ are solutions of the Black-Scholes equation, then any linear combination among them is still a solution of the Black-Scholes equation, that is, $\sum_{i \in I} a_i g_i(t, S_t)$ is also a solution of the PDE where $I \subseteq \{1, 2, \cdots, n\}$ and a_i, $i \in I$ are some constants.
- This property implies, if option claim is linear, then the option price is still linear.
- For example, if $f(S_T) = \sum_{i=1}^{n} a_i f_i(S_T)$ and $g_i(t, S_t)$ is a solution of the PDE corresponding to the claim $f_i(S_T)$, then $\sum_{i=1}^{n} a_i g_i(t, S_t)$ is the option price of claim $f(S_T)$.

Examples

- Compute the arbitrage free time t option price for the option claim $C = (S_T + 9)^3$, where the underlying asset price S_t is defined by the SDE:

$$dS_t = \mu S_t dt + \sigma S_t dB_t$$

and risk free interest rate ρ is a positive constant.

Dynamic hedging and the Greeks

- A central concept of the Black-Scholes model is to consider how the option price changes value as each of the underlying factors affecting the price change.
- This is done considering the partial derivatives of the option pricing formula, which are (mostly) denoted by various Greek letters.
- This shows that the combined risk of an option can be managed in terms of the different sources of that risk. This allows traders to decide which risks to retain and which to hedge.

Dynamic hedging and the Greeks, cont.

- It also allows for effective risk management of more complex structures and exotic options.
- The Greeks change as time passes and the underlying markets move, so hedging is an ongoing activity. Your risk can change even when you haven't consciously added to your position. Options trading is not for the fainthearted!

The Greeks: definitions

- $\mathbf{\Delta} = g'_{S_t}$, Delta measures the rate of change of the theoretical option value with respect to changes in the underlying asset's price. Long position of call options have positive $\mathbf{\Delta}$ and long position of put options have negative $\mathbf{\Delta}$.
- $\mathbf{\Gamma} = g''_{S_t S_t}$, Gamma measures the rate of change in the delta with respect to changes in the underlying price. Gamma changes as the level of the underlying changes, because it is highest when the option is at the money. As the derivative becomes deeply in or out of the money, $\mathbf{\Gamma}$ falls towards 0.

The Greeks: definitions, cont.

- $\nu = g'_\sigma$, Vega (not the name of any Greek letter) measures sensitivity to volatility. Vega is typically expressed as the amount of money per underlying asset that the option's value will gain or lose as volatility rises or falls by 1 percentage point. All options (both calls and puts) will gain value with rising volatility.
- $\rho = g'_\rho$, Rho measures sensitivity to the interest rate. Except under extreme circumstances, the value of an option is less sensitive to changes in the risk free interest rate than to changes in other parameters. Rho is typically expressed as the amount of money, per share of the underlying, that the value of the option will gain or lose as the risk free interest rate rises or falls by 1% per annum.

The Greeks: definitions, cont.

- $\boldsymbol{\Theta} = g'_t$, Theta measures the sensitivity of the value of the derivative to the passage of time. It is not really a variable that you can hedge against other than by reducing exposure altogether.
- $\boldsymbol{\lambda} = g'_q$, Lambda is the percentage change in option value per percentage change in the underlying dividend yield, a measure of the dividend risk. It is not really possible to hedge this risk, other than to reduce the overall exposure by carrying out equal and opposite option trades.

Greeks

- For European options with no dividend, the exact expressions of the Greeks are

	European Call Option	European Put Option
$\boldsymbol{\Delta}$ (Delta)	$\Phi(d_1)$	$-\Phi(-d_1)$
$\boldsymbol{\Gamma}$ (Gamma)	$\frac{\phi(d_1)}{S_t\sigma\sqrt{T-t}}$	$\frac{\phi(d_1)}{S_t\sigma\sqrt{T-t}}$
$\boldsymbol{\nu}$ (Vega)	$S_t\phi(d_1)\sqrt{T-t}$	$S_t\phi(d_1)\sqrt{T-t}$
$\boldsymbol{\rho}$ (Rho)	$K(T-t)e^{-\rho(T-t)}\Phi(d_2)$	$-K(T-t)e^{-\rho(T-t)}\Phi(-d_2)$
$\boldsymbol{\Theta}$ (Theta)	$-\frac{S_t\phi(d_1)\sigma}{2\sqrt{T-t}} - \rho K e^{-\rho(T-t)}\Phi(d_2)$	$-\frac{S_t\phi(d_1)\sigma}{2\sqrt{T-t}} + \rho K e^{-\rho(T-t)}\Phi(-d_2)$

- Here $\phi(x)$ is the density function of a standard normal distribution such that

$$\phi(x) = \Phi'(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

Greeks

- For European options with dividends, the exact expressions of the Greeks are

	European Call Option	European Put Option
$\boldsymbol{\Delta}$ (Delta)	$e^{-q(T-t)}\Phi(d_1)$	$-e^{-q(T-t)}\Phi(-d_1)$
$\boldsymbol{\Gamma}$ (Gamma)	$e^{-q(T-t)}\frac{\phi(d_1)}{S_t\sigma\sqrt{T-t}}$	$e^{-q(T-t)}\frac{\phi(d_1)}{S_t\sigma\sqrt{T-t}}$
$\boldsymbol{\nu}$ (Vega)	$S_te^{-q(T-t)}\phi(d_1)\sqrt{T-t}$	$S_te^{-q(T-t)}\phi(d_1)\sqrt{T-t}$
$\boldsymbol{\rho}$ (Rho)	$K(T-t)e^{-\rho(T-t)}\Phi(d_2)$	$-K(T-t)e^{-\rho(T-t)}\Phi(-d_2)$
$\boldsymbol{\Theta}$ (Theta)	$-e^{-q(T-t)}\frac{S_t\phi(d_1)\sigma}{2\sqrt{T-t}}$ $-\rho Ke^{-\rho(T-t)}\Phi(d_2)$ $+qS_te^{-q(T-t)}\Phi(d_1)$	$-e^{-q(T-t)}\frac{S_t\phi(d_1)\sigma}{2\sqrt{T-t}}$ $+\rho Ke^{-\rho(T-t)}\Phi(-d_2)$ $-qS_te^{-q(T-t)}\Phi(-d_1)$
$\boldsymbol{\lambda}$ (Lambda)	$-S_t(T-t)e^{-q(T-t)}\Phi(d_1)$	$S_t(T-t)e^{-q(T-t)}\Phi(-d_1)$

Examples

- Consider the examples in page 15 and page 16. Calculate the values of Greeks.

Hedging portfolio (Static Hedging)

- Recall that the hedging portfolio (ϕ, ψ) we learned in Section 2. We still want to identify a hedging portfolio at time t to match the option payoffs at the maturity with no risk.

- In a continuous time market, it has one stock and one bond. Let $A_t = \phi_t S_t + \psi_t e^{\rho t}$ be the total amount of capital held at time t, where ϕ_t is the number of stocks, ψ_t is the number of bonds, S_t is the stock price and $e^{\rho t}$ is the bond price at time t. Then, the pair (ϕ_t, ψ_t) is called a hedging portfolio (static hedging).

- If A_t is the total capital to be paid at time t to match the option claim $f(S_T)$ at the maturity date T, then we have $A_t = g(t, S_t)$, and

$$\phi_t = g'_{S_t}, \quad \text{and} \quad \psi_t = e^{-\rho t}\left[g - \phi_t S_t\right]$$

Examples

- Consider the examples (iii) and (iv) in page 25. Find the hedging portfolios.

MA3071 – DLI
Financial Mathematics – Section 5
Monte-Carlo methods for option pricing

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Background

- Option pricing is one of the key problems encountered in financial markets, and the Black–Scholes model is the most widely used model in practice.
- Despite its success in the pricing of many common types of options, the Black-Scholes model suffers computational intractability for many more exotic and complicated products in financial markets.
- Pricing such products is typically not practical using analytical methods, either because the level of mathematics required is far beyond that reasonably expected within a financial institution, or it is simply impossible using any of today's known techniques.

Background

- Hence computer models using Monte Carlo methods become increasingly important in these cases.
- The Monte Carlo method is a class of computational algorithms based on repeated random sampling to compute their results.
- It provides probabilistic algorithms for simulating systems where underlying randomness exists and is particularly powerful for high-dimensional problems that typically arise in finance.

Starter

- How to calculate the following expectations?
  - $\mathbb{E}[\sin(B_t)|\mathcal{F}_s]$.
  - $\mathbb{E}[(B_t - 1)^{B_t}|\mathcal{F}_s]$.
  - $\mathbb{E}[\cos(S_T)|\mathcal{F}_t]$ when S_t is a GBM.
- Do you think our conclusions still apply in the above cases?

Motivation

- In a binomial tree model, we can generate all possible sample paths for asset prices from time 0 to the maturity date.
- For example, consider a 2-period binomial tree. There are 4 possible paths from $S_0 \to S_2$.

$$\begin{array}{ccccccc} & & & & S_2, & C_1, & \text{with } q_1 \\ & & & \nearrow^u & & & \\ & & S_1 & & & & \\ & \nearrow^u & & \searrow_d & S_2, & C_2, & \text{with } q_2 \\ S_0 & & & & & & \\ & \searrow_d & & \nearrow^u & S_2, & C_3, & \text{with } q_3 \\ & & S_1 & & & & \\ & & & \searrow_d & & & \\ & & & & S_2, & C_4, & \text{with } q_4 \end{array}$$

- Let the option payoff be $C = f(S_2)$. Then the expected payoff is

$$\mathbb{E}(C) = q_1 C_1 + q_2 C_2 + q_3 C_3 + q_4 C_4$$

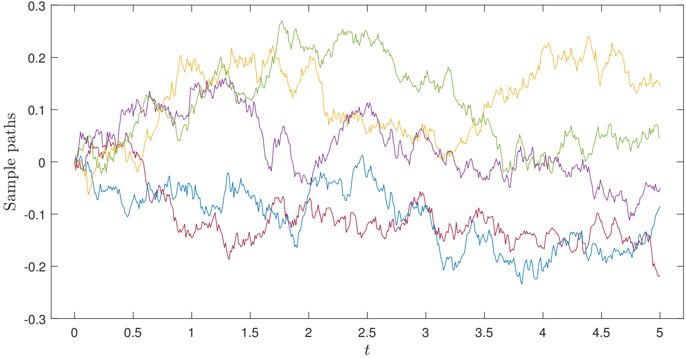
Motivation, cont.

- For an n-period binomial tree, the number of sample paths for $S_0 \to S_T$ is 2^n and therefore, we have 2^n possible values of the payoff.
- Let the option payoff be $C = f(S_T)$. Then the expected payoff is

$$\mathbb{E}(C) = \sum_{i=1}^{2^n} q_i C_i$$

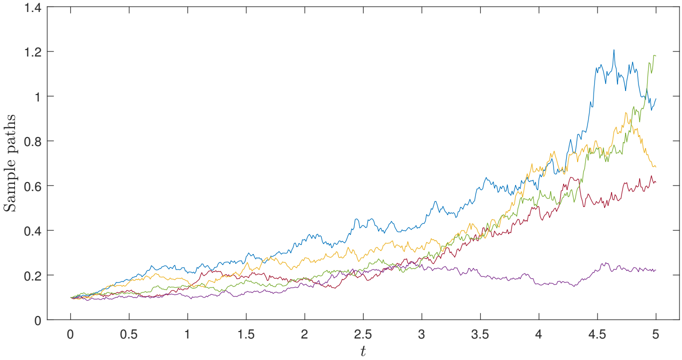
Sample paths for SBM

- We generate 5 sample paths for $B_0 \to B_5$,



Sample paths for GBM

- Let S_t be defined by $dS_t = \mu S_t dt + \sigma S_t dB_t$ with $\mu = 5$ and $\sigma = 3$. When $S_0 = 0.1$, we generate 5 sample paths for $S_0 \to S_5$,



Estimate of expectation

- Obviously, we cannot draw all possible sample paths for SBM or GBM, as B_t has an infinite number of possible values.
- However, as long as we generate enough different sample paths, we can still estimate expectations for SMB or GBM, such that
  - $\mathbb{E}[B_t] \approx \frac{1}{M} \sum_{i=1}^{M} B_t^{(i)}$
  - $\mathbb{E}[S_t] \approx \frac{1}{M} \sum_{i=1}^{M} S_t^{(i)}$

  where M is a large positive integer, $B_t^{(i)}$ and $S_t^{(i)}$ are observations of B_t and S_t, respectively.

Question

- How to collect observations for B_t and S_t?

Observations

- Recall that $B_t \sim N(0,t)$, than we have

$$\frac{B_t}{\sqrt{t}} \sim N(0,1)$$

- Let z be an observation collected from a standard normal distribution, such that

$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{x^2}{2}} dx$$

- Then, $z\sqrt{t}$ is an observation of B_t.

Observations, cont.

- The GBM is given by $S_t = S_0 e^{(\mu - \frac{\sigma^2}{2})t + \sigma B_t}$, then

$$S_t = S_0 e^{(\mu - \frac{\sigma^2}{2})t + \sigma z \sqrt{t}}$$

  is an observation of S_t.

- Using the following MATLAB commands to collect standard normal observations,

```
z = normrnd(0,1);

z = randn;
```

Estimate of expectation, cont.

- Let z_i, $i = 1, \cdots, M$ be M independent observations collected from a standard normal distribution, such that

$$P(Z \leqslant z_i) = \Phi(z_i) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z_i} e^{-\frac{x^2}{2}} dx$$

  where $Z \sim N(0,1)$ is a random variable.

- Then we have
  - $\mathbb{E}[B_t] \approx \frac{\sqrt{t}}{M} \sum_{i=1}^{M} z_i$
  - $\mathbb{E}[S_t] \approx \frac{S_0}{M} \sum_{i=1}^{M} e^{(\mu - \frac{\sigma^2}{2})t + \sigma z_i \sqrt{t}}$

Error analysis, $\mathbb{E}[B_t]$

- Recall that $\mathbb{E}[B_t] = 0$.
- We can obtain the following estimates of $\mathbb{E}[B_t]$,

t	M	Estimates	CPU time (s)
5	10^4	0.0518	0.091144
	10^5	-0.0101	0.829603
	10^6	0.0013	8.400436
	10^7	0.000994	82.361601
50	10^4	-0.0108	0.089226
	10^5	0.0095	0.863663
	10^6	0.0011	8.340475
	10^7	0.000463	83.029587
500	10^4	0.0893	0.087249
	10^5	-0.0054	0.811274
	10^6	0.0026	8.338143
	10^7	-0.000732	82.605615

Error analysis, $\mathbb{E}[S_t]$

- Recall that $\mathbb{E}[S_t] = S_0 e^{\mu t}$.
- Assume $S_0 = 1$, $\mu = 0.2$ and $\sigma = 0.5$, we can obtain the following estimates of $\mathbb{E}[S_t]$,

t	$\mathbb{E}[S_t]$	M	Estimates	CPU time (s)
5	2.7183	10^4	2.7318	0.092150
		10^5	2.7205	0.910633
		10^6	2.7194	8.555989
		10^7	2.7176	102.599107
6	3.3201	10^4	3.3948	0.095045
		10^5	3.3273	0.859025
		10^6	3.3224	8.284265
		10^7	3.3210	93.565865
7	4.0552	10^4	4.0879	0.086303
		10^5	4.0734	0.820039
		10^6	4.0618	8.478058
		10^7	4.0599	98.005538

Monte-Carlo simulation

- Let $f(t, B_t)$ be a stochastic process, then we have

$$\begin{aligned}\mathbb{E}[f(t, B_t)|\mathcal{F}_s] &= \mathbb{E}[f(t, B_s + (B_t - B_s))|\mathcal{F}_s] \\ &= \mathbb{E}[f(t, B_s + Z\sqrt{t-s})|\mathcal{F}_s]\end{aligned}$$

  where $\{\mathcal{F}_s, 0 \leq s < t\}$ is the filtration and $Z \sim N(0,1)$ is a random variable.

- By Monte-Carlo simulation,

$$\mathbb{E}[f(t, B_t)|\mathcal{F}_s] \approx \frac{1}{M}\sum_{i=1}^{M} f(t, B_s + z_i\sqrt{t-s})$$

Monte-Carlo simulation, cont.

- Let $S_t = S_0 e^{at+\sigma B_t}$ be a GBM, then we have

$$\mathbb{E}[f(S_T)|\mathcal{F}_t] = \mathbb{E}\left[f\left(S_t e^{a(T-t)+\sigma(B_T-B_t)}\right)\middle|\mathcal{F}_t\right]$$
$$= \mathbb{E}\left[f\left(S_t e^{a(T-t)+\sigma Z\sqrt{T-t}}\right)\middle|\mathcal{F}_t\right]$$

  where $\{\mathcal{F}_t, 0 \leq t < T\}$ is the filtration and $Z \sim N(0,1)$ is a random variable.

- By Monte-Carlo simulation,

$$\mathbb{E}[f(S_T)|\mathcal{F}_t] \approx \frac{1}{M}\sum_{i=1}^{M} f\left(S_t e^{a(T-t)+\sigma z_i\sqrt{T-t}}\right)$$

Monte-Carlo simulation in option pricing

- Let $C = f(S_T)$ be the option claim,

$$
\begin{aligned}
g(t, S_t) &= e^{-\rho(T-t)}\mathbb{E}[f(S_T)|\mathcal{F}_t] \\
&\approx e^{-\rho(T-t)}\frac{1}{M}\sum_{i=1}^{M} f\left(S_t e^{a(T-t)+\sigma z_i\sqrt{T-t}}\right)
\end{aligned}
$$

- The no-arbitrage condition is still $a = \rho - \frac{\sigma^2}{2}$.

European options

- European call option:

$$g(t, S_t) = e^{-\rho(T-t)}\mathbb{E}[(S_T - K)_+ | \mathcal{F}_t]$$
$$\approx e^{-\rho(T-t)} \frac{1}{M} \sum_{i=1}^{M} \left(S_t e^{a(T-t)+\sigma z_i \sqrt{T-t}} - K \right)_+$$

- European put option:

$$g(t, S_t) = e^{-\rho(T-t)}\mathbb{E}[(K - S_T)_+ | \mathcal{F}_t]$$
$$\approx e^{-\rho(T-t)} \frac{1}{M} \sum_{i=1}^{M} \left(K - S_t e^{a(T-t)+\sigma z_i \sqrt{T-t}} \right)_+$$

Example

- The current price of a non-dividend paying share is £14 and its volatility is thought to be 25% per annum. The continuously compounded risk-free interest rate is 4.8% per annum. A European option on this share has a strike price of £18 and term to maturity of half year. Assume that the Black-Scholes model applies.
- $S_0 = 14,\ K = 18,\ T = 0.5,\ \sigma = 0.25,\ \rho = 0.048$, we calculate

$$d_1 = \frac{\log\left(\frac{S_0}{K}\right) + \left(\rho + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}} = -1.1975$$
$$d_2 = d_1 - \sigma\sqrt{T} = -1.3743$$

Example, cont.

- If the option is a European call option, then by Black-Scholes model the time 0 option price is:

$$g(0, S_0) = S_0 \Phi(d_1) - K e^{-\rho T} \Phi(d_2) = 0.1297$$

- Then by Monte Carlo,

$$g(0, S_0) \approx e^{-\rho T} \frac{1}{M} \sum_{i=1}^{M} \left(S_0 e^{\left(\rho - \frac{\sigma^2}{2}\right) T + \sigma z_i \sqrt{T}} - K \right)_+$$

Example, cont.

- If the option is a European put option, then by Black-Scholes model the time 0 option price is:

$$g(0, S_0) = Ke^{-\rho T}\Phi(-d_2) - S_0\Phi(-d_1) = 3.7029$$

- Then by Monte Carlo,

$$g(0, S_0) \approx e^{-\rho T}\frac{1}{M}\sum_{i=1}^{M}\left(K - S_0 e^{\left(\rho - \frac{\sigma^2}{2}\right)T + \sigma z_i\sqrt{T}}\right)_+$$

Error analysis

- Results are listed below,

Options	$g(0, S_0)$	M	Estimates	CPU time (s)
Call	0.1297	10^4	0.1312	0.002365
		10^5	0.1304	0.006900
		10^6	0.1302	0.064002
		10^7	0.1298	0.637134
		10^8	0.1297	6.442940
Put	3.7029	10^4	3.7449	0.002414
		10^5	3.7120	0.006709
		10^6	3.7070	0.064126
		10^7	3.7032	0.665505
		10^8	3.7027	6.545800

Examples

- State how to use Monte-Carlo method to calculate the time t option prices for the following exotic options.
  - $f(S_T) = \max\left(|S_T|, sin(S_T^3)\right)$.
  - $f(S_T) = \cos[\log(S_T^3)]$.
  - $f(S_T) = \frac{1}{3+S_T^2}$.

MA3071 - DLI
Financial Mathematics – Section 6
Mean-variance portfolio theory

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Background

- Developed by Harry Markowitz in the 1950s, the theory seeks to mathematically derive optimal portfolios.
- By making some simplifying assumptions, it is possible to develop a rigorous approach to balancing risk and return.
- The output will consist of a set of potential optimal portfolios for an investor to choose from, with the final portfolio selection depending on their preferences.
- This method for portfolio construction focuses on individual security selection, while asset allocation (i.e., the split between asset classes such as equities and bonds) is treated as a separate decision.

Key assumptions

- Investors only consider expected returns and variance of returns over a single time period, making the measure of risk the variance of returns.
- All expected returns, return variances, and return covariances are known for all assets under consideration.
- Investors are never satiated: they always prefer higher returns for a given level of risk.
- Investors are risk averse: for a given level of return, they will always choose a portfolio with a lower level of risk.
- The market is frictionless:
  - there are no taxes or transaction costs.
  - securities are infinitely divisible and can be traded in any (fractional) amount.
  - short selling is permitted in any amount.

Some definitions

- Suppose there are n assets being considered, denoted as $A_1, A_2, \ldots, A_n$, where:
  - R_i is the rate of return on asset A_i.
  - $\mathbb{E}[R_i]$ is the expected rate of return on asset A_i.
  - $\sigma_i^2 = \mathrm{Var}[R_i]$ is the variance of return on asset A_i.
  - $\sigma_{ij} = \mathrm{Cov}[R_i, R_j]$ is the covariance of return between asset A_i and asset A_j. In particular, when $i = j$, $\sigma_{ii} = \sigma_i^2$.
  - $c_{ij} = \frac{\sigma_{ij}}{\sigma_i \sigma_j}$ is the correlation between asset A_i and asset A_j.
  - x_i represents the weight of asset A_i in the portfolio, subject to the budget constraint $\sum_{i=1}^{n} x_i = 1$.
  - R_p is the portfolio return rate such that $R_p = \sum_{i=1}^{n} x_i R_i$.
- ρ represents the risk-free interest rate in the market.

Opportunity set

- The **opportunity set** consists of all possible portfolios, which includes all permutations of assets from the n assets being considered.
- That is a set of portfolios, such that

$$\left\{ R_p \,\middle|\, R_p = \sum_{i=1}^{n} x_i R_i, \quad \sum_{i=1}^{n} x_i = 1, \quad x_i \in \mathbb{R} \right\}$$

Expected return and variance of portfolio

- The expected return and variance of the portfolio are calculated as follows:
  - $\mathbb{E}_p = \mathbb{E}[R_p] = \sum_{i=1}^{n} x_i \mathbb{E}[R_i]$
  - $\sigma_p^2 = \mathrm{Var}[R_p] = \sum_{i=1}^{n} \sum_{j=1}^{n} x_i x_j \sigma_{ij}$
    $= \sum_{i=1}^{n} x_i^2 \sigma_i^2 + 2 \sum_{i=1}^{n} \sum_{j=i+1}^{n} x_i x_j \sigma_{ij}$

Simplest portfolio

- Let us consider the simplest portfolio consisting of only two assets, A_1 and A_2.
- **Question**: What is the opportunity set?

Simplest portfolio, cont.

- The opportunity set is

$$R_p = x_1 R_1 + x_2 R_2, \quad x_1 + x_2 = 1, \quad x_1, x_2 \in \mathbb{R}$$

- Let $x_1 = x$ and $x_2 = 1 - x$. The opportunity set can be rewritten as

$$R_p = xR_1 + (1-x)R_2 = (R_1 - R_2)x + R_2$$

- Then in this case, we have

$$\begin{aligned}
\mathbb{E}_p &= x\mathbb{E}[R_1] + (1-x)\mathbb{E}[R_2] \\
\sigma_p^2 &= x^2\sigma_1^2 + (1-x)^2\sigma_2^2 + 2x(1-x)\sigma_{12} \\
&= x^2\sigma_1^2 + (1-x)^2\sigma_2^2 + 2x(1-x)\sigma_1\sigma_2 c_{12}
\end{aligned}$$

Example: Simplest portfolio

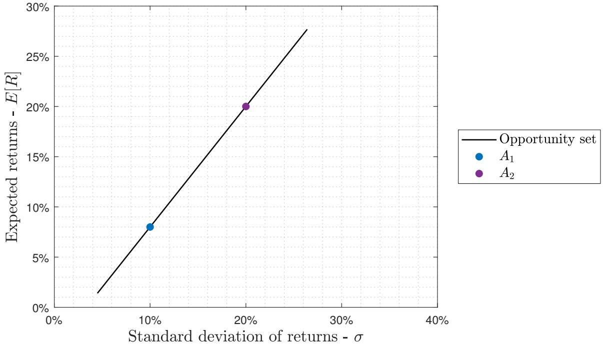
- Suppose that the two assets satisfy the following features:

$$A_1 : \sigma_1 = 10\%, \quad \mathbb{E}[R_1] = 8\%$$
$$A_2 : \sigma_2 = 20\%, \quad \mathbb{E}[R_2] = 20\%$$

- When A_1 and A_2 are perfectly positively correlated, i.e. $c_{12} = 1$, what is the relationship between $\mathbb{E}_p$ and σ_p?
- When A_1 and A_2 are perfectly negatively correlated, i.e. $c_{12} = -1$, what is the relationship between $\mathbb{E}_p$ and σ_p?
- It is typical to plot the portfolios in a $(\mathbb{E}[R], \sigma)$-plane, as is shown below.

Example: Simplest portfolio, cont.

- When A_1 and A_2 are perfectly positively correlated, i.e. $c_{12} = 1$, we have

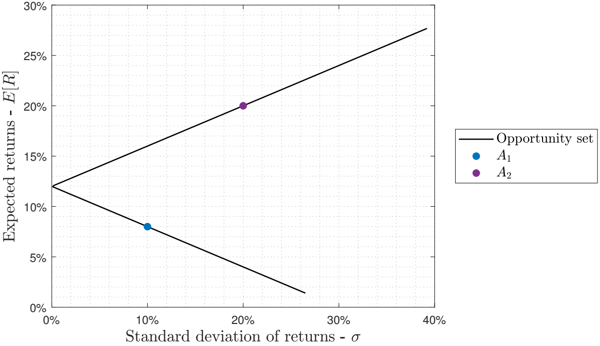


where $\mathbb{E}_p = x\mathbb{E}[R_1] + (1-x)\mathbb{E}[R_2] = 0.2 - 0.12x$ and

$\sigma_p = |x\sigma_1 + (1-x)\sigma_2| = |0.2 - 0.1x|$.

Example: Simplest portfolio, cont.

- When A_1 and A_2 are perfectly negatively correlated, i.e. $c_{12} = -1$, we have

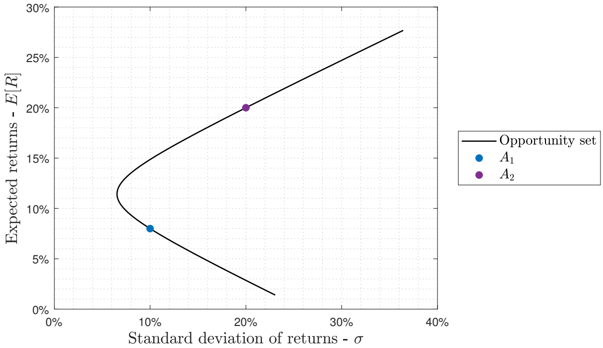


where $\mathbb{E}_p = x\mathbb{E}[R_1] + (1-x)\mathbb{E}[R_2] = 0.2 - 0.12x$ and

$$\sigma_p = |x\sigma_1 - (1-x)\sigma_2| = |0.3x - 0.2|.$$

Example: Simplest portfolio, cont.

- When the correlation is $c_{12} = -0.5$, we have

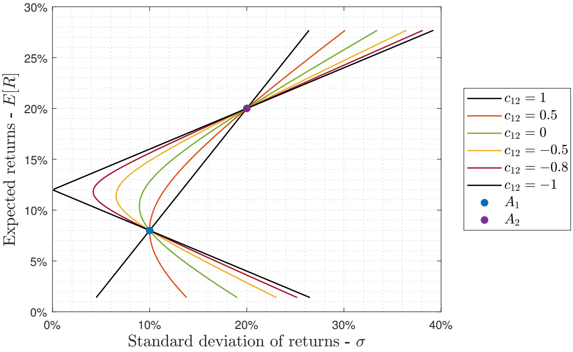


where $\mathbb{E}_p = x\mathbb{E}[R_1] + (1-x)\mathbb{E}[R_2] = 0.2 - 0.12x$ and

$$\sigma_p = \sqrt{0.01x^2 + 0.04(1-x)^2 - 0.02x(1-x)}$$
$$= \sqrt{0.07x^2 - 0.1x + 0.04}.$$

Example: Simplest portfolio, cont.

- When the correlation is $c_{12} \in (-1, 1)$, we have

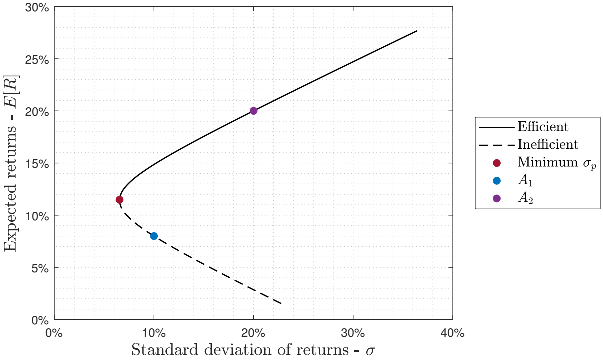


Efficient frontier

- The **efficient frontier** is a portfolio in the opportunity set such that it has the highest possible expected return for a given level of risk, or equivalently has the lowest level of risk for a given level of expected return.
- Investors will only invest in efficient portfolios, given their preferences.

Example: Efficient frontier

- When the correlation is $c_{12} = -0.5$, the solid curve corresponds to a set of efficient frontiers while the dashed curve is a set of inefficient portfolios.



where the minimum variance portfolio is $(x_1, x_2) = (5/7, 2/7)$ with $\sigma_p^2 = 3/700$ and $\mathbb{E}[R_p] = 4/35$.

Finding the minimum variance portfolio

- In the case of two assets, the weights in the minimum variance portfolio can be found using the following formula,

$$x_1 = \frac{\sigma_2^2 - \sigma_{12}}{\sigma_1^2 + \sigma_2^2 - 2\sigma_{12}}, \qquad x_2 = 1 - x_1$$

- Then, the minimum variance and expected return are determined by

$$\sigma_p^2 = x_1^2\sigma_1^2 + x_2^2\sigma_2^2 + 2x_1x_2\sigma_{12}$$
$$\mathbb{E}_p = x_1\mathbb{E}[R_1] + x_2\mathbb{E}[R_2]$$

Some important points to consider

- The shape of the opportunity set, and thus the efficient portfolios, depends critically on the correlations between the assets. This is a result of a wider point, which is that the joint behaviour of the assets is now what matters, not just their individual behaviour.
- The lower the value of correlation, the smaller the variance of return of the minimum variance portfolio.
- The curve will also depend on whether short selling is allowed or not: if short selling is not permitted, this restricts the possible set of efficient portfolios.

Question

- How to determine the minimum variance portfolio consisting of n assets $A_1, \cdots, A_n$?

Minimum variance portfolio consisting of n assets

- We aim to find a portfolio within the opportunity set that minimises σ_p^2, while adhering to the budget constraint $\sum_{i=1}^n x_i = 1$, where x_i represents the weight invested in asset A_i.

- x_i, $i = 1, \cdots, n$ are the variables we are trying to determine to satisfy our conditions.

$$\min_{x_1, \cdots, x_n} \quad \sigma_p^2$$

$$\text{Subject to} \quad \sum_{i=1}^{n} x_i = 1$$

- To solve this constrained optimisation problem, we use the method of Lagrange multipliers.

Minimum variance portfolio consisting of n assets, cont.

- The Lagrangian function is:

$$\begin{aligned}\mathcal{L}(x_1, \cdots, x_n, \mu) &= \sigma_p^2 - \mu\left(\sum_{i=1}^{n} x_i - 1\right)\\ &= \sum_{i=1}^{n} x_i^2\sigma_i^2 + 2\sum_{i=1}^{n}\sum_{j=i+1}^{n} x_i x_j \sigma_{ij} - \mu\left(\sum_{i=1}^{n} x_i - 1\right)\end{aligned}$$

with partial derivatives

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial x_i} &= 2\sum_{j=1}^{n} \sigma_{ij}x_j - \mu, \quad i = 1, \cdots, n\\ \frac{\partial\mathcal{L}}{\partial\mu} &= -\left(\sum_{i=1}^{n} x_i - 1\right)\end{aligned}$$

Minimum variance portfolio consisting of n assets, cont.

- Setting each of these derivatives equal to zero results in:

$$2\sum_{j=1}^{n}\sigma_{ij}x_j - \mu = 0, \quad i = 1, \cdots, n$$

$$\sum_{i=1}^{n} x_i = 1$$

- These equations can be solved simultaneously to find the required portfolio weights. Linear algebra can help to write these equations in a neat way.

Portfolio consisting of n assets, matrix form

- In the case of n assets, let us define the following vectors,

$$r = \begin{pmatrix} \mathbb{E}[R_1] \\ \vdots \\ \mathbb{E}[R_n] \end{pmatrix}, \quad X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \quad \mathbf{1} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

  and define the variance-covariance matrix,

$$V = \begin{pmatrix} \sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1n} \\ \sigma_{21} & \sigma_2^2 & \cdots & \sigma_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{n1} & \sigma_{n2} & \cdots & \sigma_n^2 \end{pmatrix}$$

  The matrix V is symmetric and positive definite ($X^\top \cdot V \cdot X > 0$) for all $X \neq 0$.

Portfolio consisting of n assets, matrix form, cont.

- Then the expected return and variance of the portfolio are given by

$$\mathbb{E}_p = X^\top \cdot r, \qquad \sigma_p^2 = X^\top \cdot V \cdot X$$

- Both expressions are equivalent to the formulas in page 6.

Finding the minimum variance portfolio, matrix form

- The weights in the minimum variance portfolio can be determined by
$$X_p = \frac{V^{-1} \cdot \mathbf{1}}{\mathbf{1}^\top \cdot V^{-1} \cdot \mathbf{1}}$$

- Then, the expected return and variance of the minimum variance portfolio are calculated by
$$\mathbb{E}_p = X_p^\top \cdot r, \qquad \sigma_p^2 = X_p^\top \cdot V \cdot X_p$$

Example

- Three assets have mean return $\mathbb{E}[R_1] = 0.03$, $\mathbb{E}[R_2] = 0.04$, $\mathbb{E}[R_3] = 0.05$, and the variance-covariance matrix

$$V = \begin{pmatrix} 0.1 & -0.1 & 0 \\ -0.1 & 0.2 & 0 \\ 0 & 0 & 0.1 \end{pmatrix}$$

Assume that short selling is allowed in this market.

i) Find the minimum variance portfolio.

ii) Find the variance of the minimum variance portfolio.

iii) Find the expected return of the minimum variance portfolio.

Question

- For the above examples, no constraints on the expected return were taken into account.
- When investors require the expected return to match a specific target level, how can we calculate the minimum variance portfolio that corresponds to this particular expected return?

Minimum variance portfolio with a specific expected return

- We want to find a portfolio within the opportunity set that minimises σ_p^2 for a given expected return such that $\mathbb{E}_p = \mathbb{E}_0$, while adhering to the budget constraint $\sum_{i=1}^{n} x_i = 1$, where x_i represents the weight invested in asset A_i.
- x_i, $i = 1, \cdots, n$ are still the variables we are trying to determine to satisfy our conditions.

$$
\begin{aligned}
\min_{x_1,\cdots,x_n} \quad & \sigma_p^2 \\
\text{Subject to} \quad & \mathbb{E}_p = \mathbb{E}_0 \\
& \sum_{i=1}^{n} x_i = 1
\end{aligned}
$$

Minimum variance portfolio with a specific expected return

- The Lagrangian function is:

$$\mathcal{L}(x_1, \cdots, x_n, \lambda, \mu) = \sigma_p^2 - \lambda(\mathbb{E}_p - \mathbb{E}_0) - \mu\left(\sum_{i=1}^{n} x_i - 1\right)$$
$$= \sum_{i=1}^{n} x_i^2\sigma_i^2 + 2\sum_{i=1}^{n}\sum_{j=i+1}^{n} x_i x_j \sigma_{ij} - \lambda\left(\sum_{i=1}^{n} x_i\mathbb{E}[R_i] - \mathbb{E}_0\right) - \mu\left(\sum_{i=1}^{n} x_i - 1\right)$$

with partial derivatives

$$\frac{\partial\mathcal{L}}{\partial x_i} = 2\sum_{j=1}^{n}\sigma_{ij}x_j - \lambda\mathbb{E}[R_i] - \mu, \quad i = 1, \cdots, n$$
$$\frac{\partial\mathcal{L}}{\partial\lambda} = -\left(\sum_{i=1}^{n} x_i\mathbb{E}[R_i] - \mathbb{E}_0\right)$$
$$\frac{\partial\mathcal{L}}{\partial\mu} = -\left(\sum_{i=1}^{n} x_i - 1\right)$$

Minimum variance portfolio with a specific expected return

- Setting each of these derivatives equal to zero results in:

$$2\sum_{j=1}^{n} \sigma_{ij} x_j - \lambda \mathbb{E}[R_i] - \mu = 0, \quad i = 1, \cdots, n$$

$$\sum_{i=1}^{n} x_i \mathbb{E}[R_i] = \mathbb{E}_0$$

$$\sum_{i=1}^{n} x_i = 1$$

- Solve these equations to find the required portfolio weights.

Matrix form

- Let $\alpha = \mathbf{1}^\top \cdot V^{-1} \cdot \mathbf{1}$, $\beta = r^\top \cdot V^{-1} \cdot r$ and $\gamma = \mathbf{1}^\top \cdot V^{-1} \cdot r$.
- The weights in the minimum variance portfolio with a specific expected return can be determined by

$$X_p = \frac{1}{\Theta}\left(\alpha\mathbb{E}_0 - \gamma\right)\left(V^{-1} \cdot r\right) - \frac{1}{\Theta}\left(\gamma\mathbb{E}_0 - \beta\right)\left(V^{-1} \cdot \mathbf{1}\right)$$

where $\Theta = \alpha\beta - \gamma^2$.

Example

- Three assets have mean return $\mathbb{E}[R_1] = 0.03$, $\mathbb{E}[R_2] = 0.04$, $\mathbb{E}[R_3] = 0.05$, and the variance-covariance matrix

$$V = \begin{pmatrix} 0.1 & -0.1 & 0 \\ -0.1 & 0.2 & 0 \\ 0 & 0 & 0.1 \end{pmatrix}$$

Assume that short selling is allowed in this market.

i) Find the minimum variance portfolio with an expected return of 0.045.

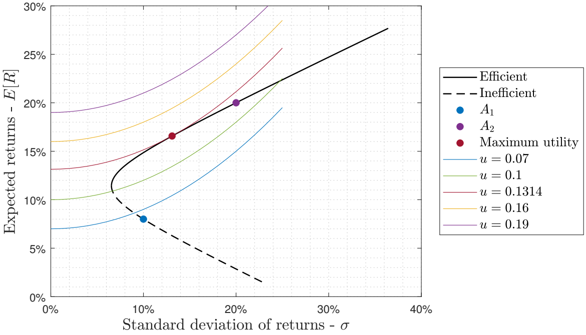
ii) Find the variance of the minimum variance portfolio.

Efficient portfolios with specific preferences

- The analysis that has been presented so far only illustrates how to find efficient portfolios that an investor could choose to invest in. However, the actual selected portfolio will usually depend on the specific preferences of the investor.
- These can also be represented in the $(\mathbb{E}[R], \sigma)$-plane using indifference curves. All points on one indifference curve have the same expected utility value to the investor.
- Note that the indifference curves are upward sloping due to the risk aversion of the investor. In fact, investors want more return as the risk increases, hence upward sloping. The gradient of the upward slope will increase in steepness the more risk averse an investor is.

Efficient portfolios with specific preferences, cont.

- Consider the two assets in page 9 and $c_{12} = -0.5$.
- The efficient portfolio selected will be the one tangent to the highest indifference curve (which will maximise expected utility). In the example below, this is the red curve,



where the utility function is $u = \mathbb{E}_p - 2\sigma_p^2$, and the maximum utility portfolio is $(x_1, x_2) = (2/7, 5/7)$ corresponding to the maximum utility $u = 23/175 \approx 0.1314$.

Utility function

- The utility function is usually expressed as

$$u = \mathbb{E}_p - \kappa\sigma_p^2$$

where κ is a positive constant.

- Combining the expressions of $\mathbb{E}_p$ and σ_p^2, it can be rewritten as

$$u = \sum_{i=1}^{n} x_i \mathbb{E}[R_i] - \kappa \left(\sum_{i=1}^{n} x_i^2 \sigma_i^2 + 2 \sum_{i=1}^{n} \sum_{j=i+1}^{n} x_i x_j \sigma_{ij} \right)$$

where $\sum_{i=1}^{n} x_i = 1$.

Maximizing the utility

- Therefore, we want to find a portfolio within the opportunity set that maximises u, while adhering to the budget constraint $\sum_{i=1}^{n} x_i = 1$, where x_i represents the weight invested in asset A_i.
- x_i, $i = 1, \cdots, n$ are still the variables we are trying to determine to satisfy our conditions.

$$\begin{aligned} \max_{x_1,\cdots,x_n} \quad & \mathbb{E}_p - \kappa\sigma_p^2 \\ \text{Subject to} \quad & \sum_{i=1}^{n} x_i = 1 \end{aligned}$$

Maximizing the utility, cont.

- To maximize the utility, the Lagrangian function is:

$$\begin{aligned}\mathcal{L}(x_1, \cdots, x_n, \mu) &= \mathbb{E}_p - \kappa\sigma_p^2 - \mu\left(\sum_{i=1}^{n} x_i - 1\right)\\ &= \sum_{i=1}^{n} x_i \mathbb{E}[R_i] - \kappa\left(\sum_{i=1}^{n} x_i^2\sigma_i^2 + 2\sum_{i=1}^{n}\sum_{j=i+1}^{n} x_i x_j \sigma_{ij}\right) - \mu\left(\sum_{i=1}^{n} x_i - 1\right)\end{aligned}$$

with partial derivatives

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial x_i} &= \mathbb{E}[R_i] - 2\kappa\sum_{j=1}^{n}\sigma_{ij}x_j - \mu, \quad i = 1, \cdots, n\\ \frac{\partial\mathcal{L}}{\partial\mu} &= -\left(\sum_{i=1}^{n} x_i - 1\right)\end{aligned}$$

Maximizing the utility, cont.

- Setting each of these derivatives equal to zero results in:

$$2\kappa \sum_{j=1}^{n} \sigma_{ij} x_j + \mu = \mathbb{E}[R_i], \quad i = 1, \cdots, n$$

$$\sum_{i=1}^{n} x_i = 1$$

- Solve these linear equations to find the weights of the maximum utility portfolio.

Maximizing the utility, matrix form

- The weights in the maximum utility portfolio can be determined by

$$X_p = \frac{V^{-1} \cdot r}{2\kappa} - \frac{(\gamma - 2\kappa)\left(V^{-1} \cdot \mathbf{1}\right)}{2\kappa\alpha}$$

where $\alpha = \mathbf{1}^\top \cdot V^{-1} \cdot \mathbf{1}$ and $\gamma = \mathbf{1}^\top \cdot V^{-1} \cdot r$.

Example

- Three assets have mean return $\mathbb{E}[R_1] = 0.03$, $\mathbb{E}[R_2] = 0.04$, $\mathbb{E}[R_3] = 0.05$, and the variance-covariance matrix

$$V = \begin{pmatrix} 0.1 & -0.1 & 0 \\ -0.1 & 0.2 & 0 \\ 0 & 0 & 0.1 \end{pmatrix}$$

Given the utility function $u = \mathbb{E}_p - \frac{1}{3}\sigma_p^2$ and assuming that short selling is allowed in this market, find the maximum utility portfolio.

MA3071 – DLI
Financial Mathematics – Section 7
Asset pricing models under equilibrium (Capital asset pricing model)

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Aims of the CAPM

- It considers all investors in the market and extends the mean-variance portfolio theory from Section 6.
- Aim to develop a theory of market prices, driven by the actions of individual investors optimising their portfolios using the Markowitz approach.
- This will be an equilibrium theory: i.e. will derive prices under conditions when supply and demand are balanced. This shows the level that prices (and therefore returns) should shift towards over time.
- The capital asset pricing model (CAPM) was developed in the 1960s by William Sharpe, Jon Lintner and Jan Mossin who won the Nobel prize in Economics for this work.

Assumptions

- All assets are marketable (i.e. can be freely bought and sold, meaning that there will always be buyers for assets that are up for sale and there are no other restrictions).
- Capital markets are perfect:
  - All assets are infinitely divisible, that is, any proportion of assets can be traded;
  - All investors are price takers, i.e. no one investor can influence the market price by buying or selling actions;
  - Taxes and transaction costs do not exist or alternatively do not affect the investment decision;
  - Unlimited borrowing and short selling are allowed, without margin/collateral requirements (i.e. there is no credit risk associated with any borrower);
  - Information is available to every investor at no cost, and all investors possess the same information.

Assumptions, cont.

- A risk-free interest rate (ρ) exists at which all investors can undertake unlimited borrowing or lending. This can be thought of as the rate of interest on bank accounts or the yield on Treasury Bills (both of which would be the same in this idealised market).
- All investors are risk-averse and seek to maximize expected utility over single period time horizons.

Assumptions, cont.

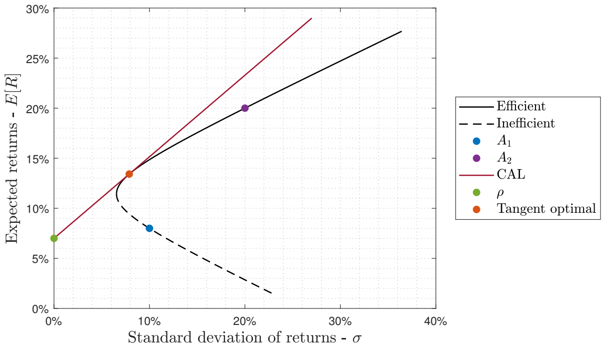
- Investors have homogeneous expectations (and therefore all make the same assumptions):
  - They possess the same investment time horizons and their estimates of the expected returns, variances (or standard deviations), and covariances of returns on risky assets are identical;
  - They all base their portfolio selection decisions on mean-variance optimisation. In other words, all investors perceive investment alternatives and arrive at portfolio decisions in exactly the same manner.

The capital allocation line

- We want to add the risk free asset to the portfolios that form the efficient frontiers of Markowitz.
- The risk free asset always has zero variance of returns, since it is riskless. It also has zero correlation, and therefore zero covariance with any of the risky assets.
- Notice that adding the possibility of investing in a risk free asset extends the choice of portfolio to a mix of risky assets and the risk free asset.

The capital allocation line, cont.

- Since all investors can borrow and lend at the risk-free rate, the efficient frontier simplifies to a straight line passing through the risk free rate and a unique point on the efficient frontier of risky assets, which is called the **capital allocation line** (CAL).



The capital allocation line, cont.

- The orange point at which the straight line is tangent to the frontier comprised of a portfolio containing only risky assets is called the **tangent portfolio**.
- In the picture above, the tangent portfolio is $(x_1, x_2) = \left(\frac{17}{31}, \frac{14}{31}\right)$ with $\sigma_t = \frac{\sqrt{597}}{310} \approx 0.0788$ and $\mathbb{E}[R_t] = \frac{104}{775} \approx 0.1342$.
- All other portfolios (i.e. those made up of the risky assets only) are inefficient compared to the portfolios including the risk-free asset.

The capital allocation line, cont.

- All investors who agree on the expected returns, risk and covariance of risky assets, regardless of their risk preferences, will choose the same portfolio of risky assets (in this simple example it is the **orange dot**).
- This is called a separation result, because the construction of the risky portfolio and the choice of how much risk to take (asset allocation) are separated.
- Firstly, calculate the optimal risky portfolio, which is a mathematical optimisation problem without reference to the investor's preferences with respect to risk.
- Secondly, choose how to split the amount to be invested between the optimal risky portfolio and the risk free asset (i.e. an asset allocation decision). This depends primarily on the investor's preferences. For this reason, the efficient frontier including the risk-free asset is called the **capital allocation line**.

The capital allocation line, cont.

- The slope of the CAL is:

$$\frac{\mathbb{E}[R_t] - \rho}{\sigma_t}$$

where $\mathbb{E}[R_t]$ and σ_t are the expected return and standard deviation of return for the tangent portfolio.

- The equation of CAL is:

$$\mathbb{E}[R_p] - \rho = \left(\frac{\mathbb{E}[R_t] - \rho}{\sigma_t} \right) \sigma_p$$

- The slope can be thought of as a risk/reward ratio that shows the additional return required to take on more risk.

The capital allocation line, cont.

- The **higher** this ratio, the **better** for the investor, since they are achieving a better level of additional return per unit of risk.
- In other words, it is the additional return, in excess of the risk-free rate, that investors require before they will accept an additional unit of risk.
- In portfolio analysis, this is also known as the **Sharpe measure**, which is widely used for fund performance assessment purposes.

Find the tangent portfolio

- To determine the tangent portfolio of risky assets, it needs to maximise this ratio, subject to the constraint that the weights invested across all risky assets must sum to 1.

$$\max_{x_1,\cdots,x_n} \quad \frac{\mathbb{E}[R_t]-\rho}{\sigma_t}$$

$$\text{Subject to} \quad \sum_{i=1}^{n} x_i = 1$$

Find the tangent portfolio, cont

- The Lagrangian function is:

$$\begin{aligned}\mathcal{L}(x_1, \cdots, x_n, \mu) &= \frac{\mathbb{E}[R_t] - \rho}{\sigma_t} - \mu\left(\sum_{i=1}^{n} x_i - 1\right) \\ &= \frac{\sum_{i=1}^{n} x_i \mathbb{E}[R_i] - \rho}{\sqrt{\sum_{i=1}^{n} x_i^2 \sigma_i^2 + 2\sum_{i=1}^{n}\sum_{j=i+1}^{n} x_i x_j \sigma_{ij}}} - \mu\left(\sum_{i=1}^{n} x_i - 1\right)\end{aligned}$$

- We then need to solve the following equations:

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial x_i} &= 0, \quad i = 1, \cdots, n \\ \frac{\partial \mathcal{L}}{\partial \mu} &= 0\end{aligned}$$

Obviously, these equations are non-linear, we can use programming to calculate the weights.

The capital market line and the market portfolio

- If all investors have the same expectations (means, variances and covariances), then the set of efficient portfolios of risky assets and the CAL they use to generate portfolios would be the same. This is because all investors would be using the same expected returns, variances and covariances of returns, which is the situation of **market equilibrium**.
- In equilibrium, the tangent portfolio of risky assets would therefore be the same for all investors. Then the tangent portfolio is called the **market portfolio** and consists of all possible risky assets, weighted by market value (or "**market capitalisation**").
- In practice, the market portfolio is often approximated by broad market indices such as the FTSE100 index in the UK or the S&P500 in the US.

The capital market line and the market portfolio, cont.

- Then, the CAL becomes the capital market line (CML), with equation:

$$\mathbb{E}[R_p] - \rho = \left(\frac{\mathbb{E}[R_M] - \rho}{\sigma_M} \right) \sigma_p$$

where $\mathbb{E}[R_M]$ and σ_M are the expected return and standard deviation of return for the market portfolio.

- The slope of CML is referred to as the '**market price of risk**', being the 'price' that investors as a whole will charge for every unit of risk that they take on.

- **Remark**: The CML is a special case of the CAL. In equilibrium, all investors will hold the same tangent portfolio called the market portfolio. Then everyone's CAL will be called the CML - effectively the market's consensus of the risk frontier.

Capital asset pricing model

- The CML gives an expression that can be used to relate the return on any efficient portfolio to the market portfolio.
- This can be developed to relate the return on any individual asset to the market portfolio:

$$\mathbb{E}[R_i] - \rho = \frac{Cov(R_i, R_M)}{\sigma_M^2} \left(\mathbb{E}[R_M] - \rho \right)$$

  where $\frac{Cov(R_i, R_M)}{\sigma_M^2}$ represents the sensitivity of the expected excess return (over and above the risk-free rate) of asset i to the expected excess market return.
- This equation takes into account the asset's sensitivity to non-diversifiable risk (also known as systematic risk or market risk).

Capital asset pricing model, cont.

- It is common to write the coefficient $\frac{Cov(R_i, R_M)}{\sigma_M^2}$ as β_i, since this is the slope of a linear regression of the expected excess asset return on the expected excess market return. The slope coefficients in regression models are often denoted as β.
- Hence,

$$\mathbb{E}[R_i] = \rho + \beta_i \left(\mathbb{E}[R_M] - \rho\right)$$

  This is the classic form of the CAPM that is most commonly encountered.
- **Remark**: In this model, the appropriate measure of risk is no longer the variance of returns, but rather the degree to which the asset contributes to the risk of the market portfolio, as measured by β. In other words, β can be considered a measure of the systematic risk of the asset in question.

Example

In a market where the assumptions of the CAPM hold, there is one risk-free asset (Asset 1) with an annual rate of return of 3 and two risky assets with the following properties:

		Rate of return (p.a.)	
State	Probability	Asset 2	Asset 3
1	0.1	5	6
2	0.3	3	3
3	0.4	4	5
4	0.2	6	8

The market capitalisation of Asset 2 is 40,000 and the market capitalisation of Asset 3 is 60,000. The market portfolio is defined by the market capitalisations.

i) Determine the market price of risk.

ii) Calculate the beta of each risky asset.

Shortcomings of CAPM

- The main challenge with the model is twofold. Firstly, there is the issue that the market portfolio that it specifies is not observable since it should contain all possible assets including property, fine art, etc.
- The second problem relates to the many assumptions that the model requires. Taken altogether, these are not realistic. The most important assumption that will not be met is that all investors can borrow and lend at the risk-free rate.

Shortcomings of CAPM, cont.

- Academic studies testing the model have therefore tended to find very little evidence in favour of it. However, they use financial indices as a proxy for the market portfolio so are not applying a fully appropriate test.
- There is a model called the "zero beta" model that seeks to overcome the limitation regarding the risk-free rate. It has this name because it relies on constructing an artificial riskless portfolio using portfolios that are uncorrelated with each other (or have zero β with respect to each other). The form of the model is then very similar to that of the classic model, but the zero beta portfolio replaces the risk-free asset.