MA3071 Financial Mathematics - DLI, Year 2023-2024

Exercises for Feedback session 4

Exercise 1 *Using the classical Ito's lemma, find the stochastic differential $df(t, B_t)$ for the stochastic process $f(t, B_t) = t + cos(tB_t)$.*

Exercise 2 *Show that $X_t = B_t^3 - 3tB_t$ is a martingale. [Hint: we have two methods, either via conditional expectation or via classical Ito's lemma.]*

Solution 1 Since $f(t, B_t) = t + cos(tB_t)$, we have

$$
\begin{aligned}
f'_t &= 1 - B_t \sin(tB_t) \\
f'_{B_t} &= -t \sin(tB_t) \\
f''_{B_t B_t} &= -t^2 \cos(tB_t)
\end{aligned}
$$

By the Ito's lemma:

$$
\begin{aligned}
df(t, B_t) &= f'_{B_t} dB_t + f'_t dt + \frac{1}{2} f''_{B_t B_t} dt \\
&= -t \sin(tB_t) dB_t + [1 - B_t \sin(tB_t)]\, dt - \frac{1}{2} t^2 \cos(tB_t) dt \\
&= -t \sin(tB_t) dB_t + \left[1 - B_t \sin(tB_t) - \frac{1}{2} t^2 \cos(tB_t)\right] dt
\end{aligned}
$$

Solution 2 i) Method one: Conditional expectation.

$$
\begin{aligned}
\mathbb{E}\left[X_t | \mathcal{F}_s\right] &= \mathbb{E}\left[B_t^3 - 3tB_t | \mathcal{F}_s\right] \\
&= \mathbb{E}\left[(B_s + (B_t - B_s))^3 - 3t(B_s + (B_t - B_s)) | \mathcal{F}_s\right] \\
&= \mathbb{E}\left[B_s^3 + 3B_s^2(B_t - B_s) + 3B_s(B_t - B_s)^2 + (B_t - B_s)^3 - 3tB_s - 3t(B_t - B_s) | \mathcal{F}_s\right] \\
&= \mathbb{E}\left[B_s^3 | \mathcal{F}_s\right] + 3\mathbb{E}\left[B_s^2 | \mathcal{F}_s\right] \mathbb{E}\left[B_t - B_s | \mathcal{F}_s\right] + 3\mathbb{E}\left[B_s | \mathcal{F}_s\right] \mathbb{E}\left[(B_t - B_s)^2 | \mathcal{F}_s\right] \\
&\quad + \mathbb{E}\left[(B_t - B_s)^3 | \mathcal{F}_s\right] - 3t\mathbb{E}\left[B_s | \mathcal{F}_s\right] - 3t\mathbb{E}\left[B_t - B_s | \mathcal{F}_s\right] \\
&= B_s^3 + 0 + 3B_s(t - s) + 0 - 3tB_s - 0 \\
&= B_s^3 + 3tB_s - 3sB_s - 3tB_s \\
&= B_s^3 - 3sB_s
\end{aligned}
$$

Then, we obtain $\mathbb{E}\left[B_t^3 - 3tB_t|\mathcal{F}_s\right] = B_s^3 - 3sB_s$, thus according to the definition of martingale, X_t is a martingale.

ii) Method two: Classical Ito's lemma.
Let $f(t, B_t) = B_t^3 - 3tB_t$, we have

$$
\begin{aligned}
&f'_t = -3B_t \\
&f'_{B_t} = 3B_t^2 - 3t \\
&f''_{B_tB_t} = 6B_t
\end{aligned}
$$

By the Ito's lemma:

$$
\begin{aligned}
df(t, B_t) &= f'_{B_t}dB_t + f'_t dt + \frac{1}{2}f''_{B_tB_t}dt \\
&= (3B_t^2 - 3t)dB_t - 3B_t dt + \frac{1}{2} \times 6B_t dt \\
&= (3B_t^2 - 3t)dB_t - 3B_t dt + 3B_t dt \\
&= (3B_t^2 - 3t)dB_t
\end{aligned}
$$

Thus, $X_t = B_t^3 - 3tB_t$ is a martingale, as there is no term with dt.