# Lecture 1: Introduction. Vectors and the Geometry of Space.

MA2032 Vector Calculus

Lecturer: Larissa Serdukova

School of Computing and Mathematical Science
University of Leicester

September 21, 2022

# Introduction

**Lectures**: Larissa Serdukova, ls563@leicester.ac.uk
**Tutorials**: Fang Chen, fc160@leicester.ac.uk

Some recommended **textbooks**:

1. Thomas, Calculus: Early Transcendentals, fourteenth edition.
2. E. Kreyszig, Advanced Engineering Mathematics.
3. L. Marder, Vector Analysis.
4. P. Moon and D.E. Spencer, Vectors.
5. J. Stewart, Multivariable Calculus (Seventh edition).
6. Schaum's Outlines (book series), Vector Analysis and Advanced Calculus.

Useful free Internet sources for **3D Graphics**:
https://www.geogebra.org/3d
https://www.geogebra.org/m/XUv5mXTmchapter/400289

# Introduction

**Assessment**:

Examination: 70%
Basic skills test: 15%
Continuous Assessment (Homework Problem Sheets): 15%

**Problem sheets**:

- Tutorial Problem Sheets for practice (not for evaluation) will be set up weekly on Tuesdays on Blackboard "Learning Material".
- Homework Problem Sheets for evaluation will be set up biweekly on Tuesdays on Blackboard "Assessment".
- Solutions must be submitted online as pdf-file: Please make sure to label your submission file by "lastname_firstname_sheetnr". In my case, a submission for the first sheet would be called "serdukova_larissa_sheet1.pdf"
- Submission deadline: Tuesday 16:00 (UK time) one week after being made available. Late submission of homework will not be accepted (unless there are clear mitigating circumstances).

# Introduction

**Communication, questions  feedback**

- Three online live Lectures per week at 9:00 am-10:00 am UK time (Monday, Wednesday, Friday). Lectures Recordings will be published in "Learning Material".
- Online live Tutorial on Thursday at 9:00 am-10:00 am (questions, examples, feedback on problem sheets).
- Email communication is welcome but you can also post questions anonymously in the discussion board on blackboard.
- **Office hours** by appointment: please request a meeting by sending email (best times to meet: Monday 8:00 am-9:00 am UK time).

# Introduction

We hope that you will enjoy Vector Calculus!

Please let us know when you have any questions or comments, we are here to support you in your learning.

# Overview

- We begin the study of **multivariable calculus**.
- To apply calculus in many real-world situations, we introduce **three-dimensional coordinate systems** and **vectors**.
- We establish coordinates in space by adding a **third z-axis** that measures distance above and below the $xy$-plane.
- Then we define **vectors**, which provide simple ways to define equations for lines, planes, curves, and surfaces in space.

# Three-Dimensional Coordinate Systems

To locate a point in space, we use three mutually perpendicular coordinate axes, arranged as in Figure 1.

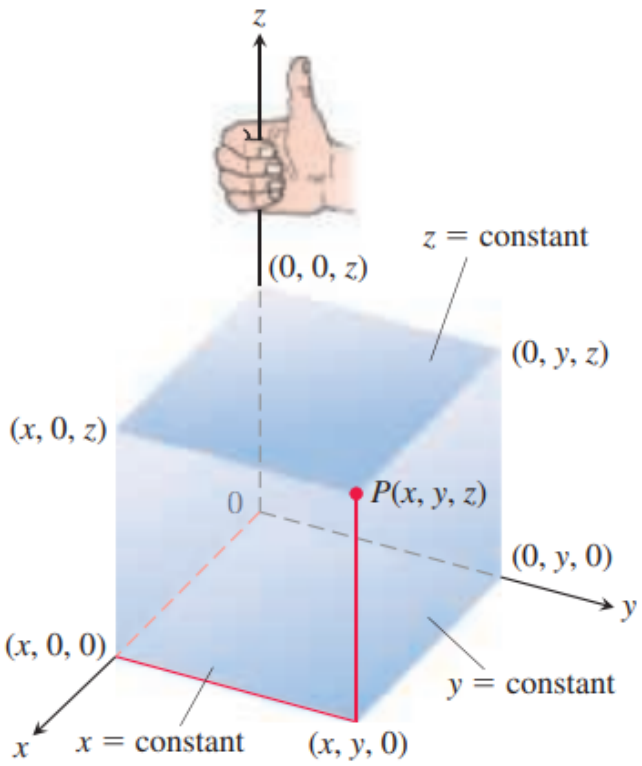



Figure 1: The Cartesian coordinate system is right-handed.

The coordinates $(x, y, z)$ of a point P in space are the values at which the planes through P perpendicular to the axes cut the axes.

The $xy$-plane, $z = 0$.

The $yz$-plane, $x = 0$.

The $xz$-plane, $y = 0$.

They meet at the origin (0, 0, 0).

# Three-Dimensional Coordinate Systems

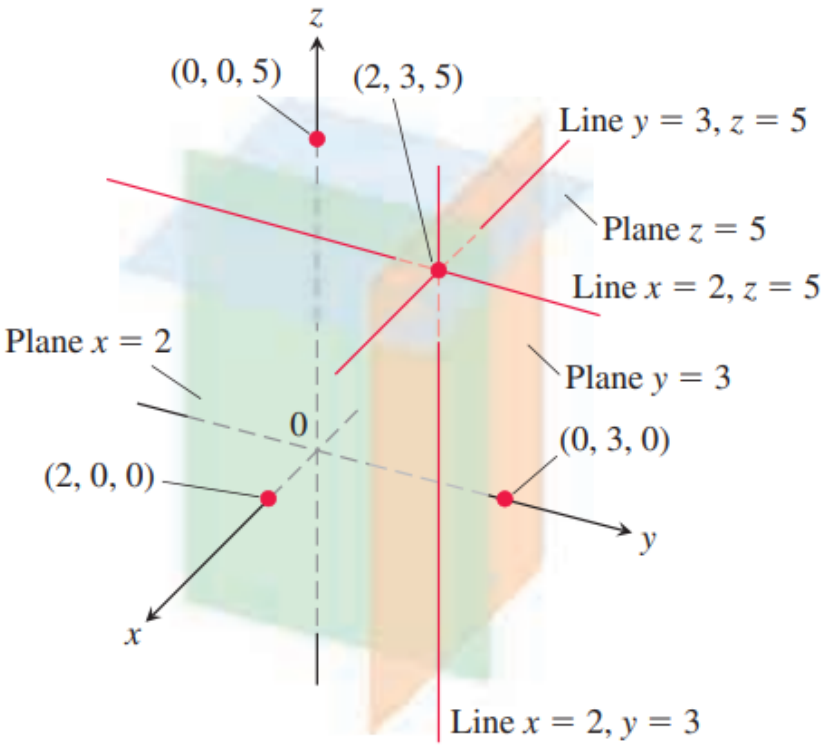



Figure 2: The planes x = 2, y = 3, and z = 5 determine three lines through the point (2, 3, 5).

The plane x = 2 is the plane perpendicular to the x-axis at x = 2.

The plane y = 3 is the plane perpendicular to the y-axis at y = 3.

The plane z = 5 is the plane perpendicular to the z-axis at z = 5.

# Example

We interpret these equations and inequalities geometrically.
(a) $z \geq 0$: The half-space consisting of the points on and above the xy-plane.
(b) x = -3: The plane perpendicular to the x-axis at x = -3.
(c) z = 0, $x \leq 0, y \geq 0$: The second quadrant of the xy-plane.

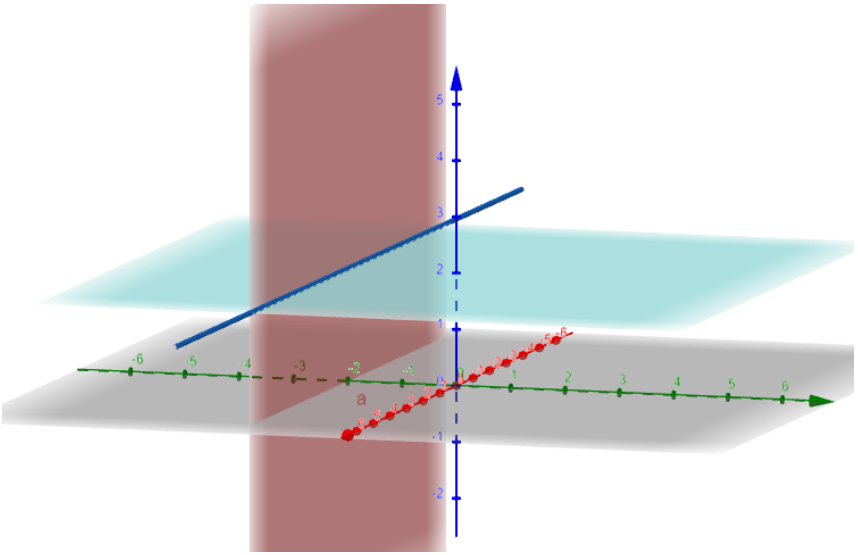


Figure 3: (f) The planes y = -2, z = 2 and their line of intersection.

(d) $x \geq 0, y \geq 0, z \geq 0$: The first octant.
(e) $-1 \leq y \leq 1$: The slab between the planes y = -1 and y = 1 (planes included).
(f) y = -2, z = 2 The line in which the planes y = -2 and z = 2 intersect.

# Distance in Space

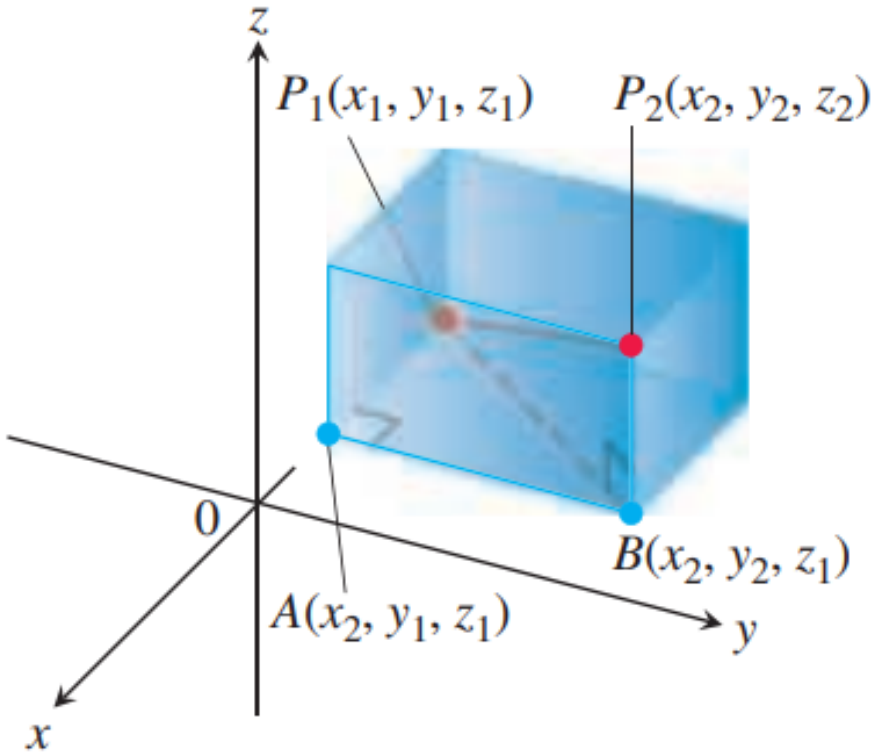
## The Distance Between $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$:

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \tag{1}$$

**Proof**: If $A(x_2, y_1, z_1)$ and $B(x_2, y_2, z_1)$ are the vertices of the box indicated in the figure, then the three box edges $P_1A, AB$, and $BP_2$ have lengths $|P_1A| = |x_2 - x_1|$, $|AB| = |y_2 - y_1|$, $|BP_2| = |z_2 - z_1|$.

Because triangles $P_1BP_2$ and $P_1AB$ are both right-angled, two applications of the Pythagorean theorem give $|P_1P_2|^2 = |P_1B|^2 + |BP_2|^2$ and $|P_1B|^2 = |P_1A|^2 + |AB|^2$.

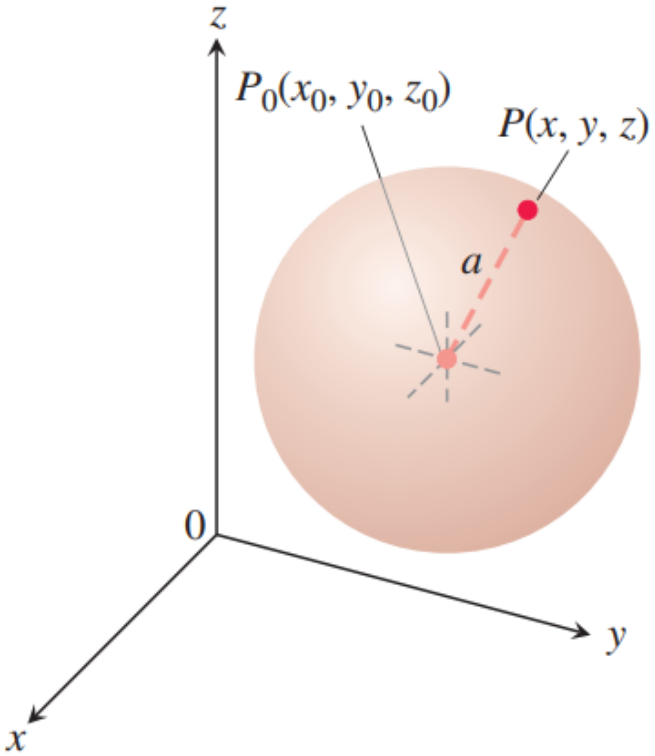
Substitution of $|P_1B|^2$ into $|P_1P_2|^2$ gives $|P_1P_2|^2 = |P_1A|^2 + |AB|^2 + |BP_2|^2 = |x_2 - x_1|^2 + |y_2 - y_1|^2 + |z_2 - z_1|^2 = (x_2 - x_1|)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2$



# Spheres in Space

We can use the distance formula to write equations for spheres in space (Right side Figure).

A point P(x, y, z) lies on the sphere of radius a centered at $P_0(x_0, y_0, z_0)$ precisely when $|P_0P| = a$




**The Standard Equation for the Sphere of Radius $a$ and Center $(x_0, y_0, z_0)$:**

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = a^2 \tag{2}$$

# Lecture 2: Vectors and the Geometry of Space.

MA2032 Vector Calculus

Lecturer: Larissa Serdukova

School of Computing and Mathematical Science
University of Leicester

September 25, 2022

# Vectors

In this section we show how to represent things that have both **magnitude and direction** in the plane or in space.

## Definition:

The **vector** represented by the directed line segment $\overrightarrow{\mathrm{AB}}$ has **initial point A** and **terminal point B** and **its length** is denoted by $|\overrightarrow{\mathrm{AB}}|$. Two **vectors are equal** if they have the same length and direction, regardless of the initial point.

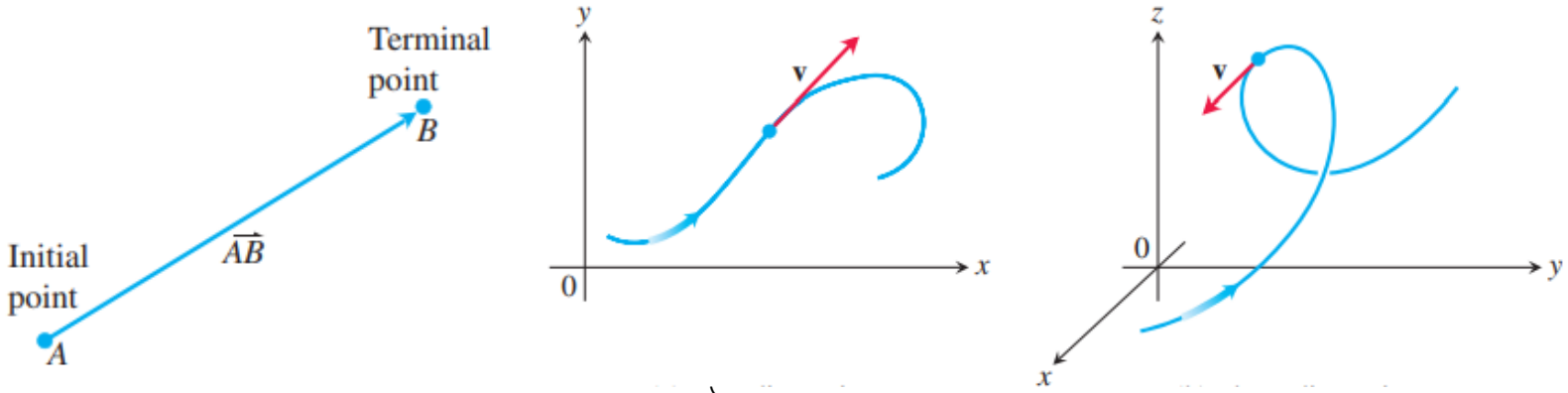



Figure 4: (left) The directed line segment $\overrightarrow{\mathrm{AB}}$ is called a vector; (middle/right) Velocity vector v at a specific location for a particle moving along a path in the plane/space.

# Component Form of Vectors

## Definition:

If v is a **two-dimensional** vector in the plane equal to the vector with initial point at the origin (vector in **standard position**) and terminal point $(v_1, v_2)$, then the **component form** of v is $v = \langle v_1, v_2 \rangle$.

If v is a **three-dimensional** vector equal to the vector with initial point at the origin and terminal point $(v_1, v_2, v_3)$, then the **component form** of v is $v = \langle v_1, v_2, v_3 \rangle$.

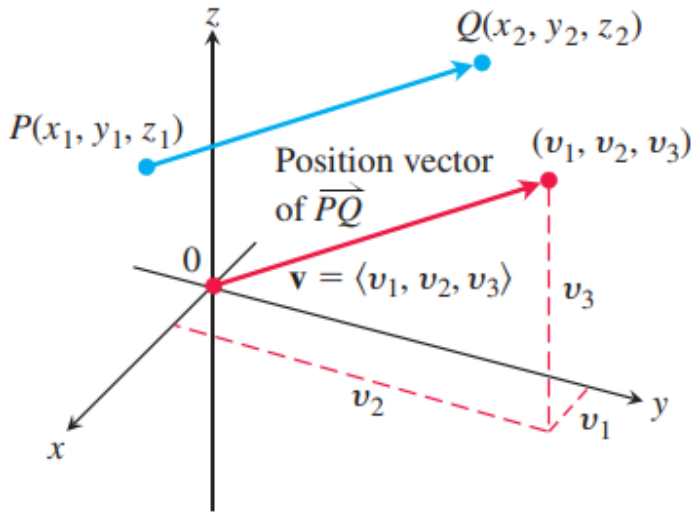



Figure 5: A vector $\overrightarrow{PQ}$ in standard position has its initial point at the origin. The directed line segments $\overrightarrow{PQ}$ and v are parallel and have the same length.

Given the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$, the standard position vector $v = \langle v_1, v_2, v_3 \rangle$ equal to $\overrightarrow{PQ}$ is $v = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$.

# Component Form of Vectors

Two vectors are **equal** if and only if their standard position vectors are identical. Thus $\langle u_1, u_2, u_3 \rangle$ and $\langle v_1, v_2, v_3 \rangle$ are equal if and only if $u_1 = v_1$, $u_2 = v_2$, and $u_3 = v_3$.

## Definition:

The **magnitude** or **length** of the vector $v = \overrightarrow{\mathrm{PQ}}$ is the nonnegative number

$$||v|| = \sqrt{v_1^2 + v_2^2 + v_3^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

The only vector with length 0 is the **zero vector** $0 = \langle 0, 0 \rangle$ or $0 = \langle 0, 0, 0 \rangle$. This vector is also the **only vector with no specific direction**.

# Example

Find the (a) component form and (b) length of the vector with initial point P(-3, 4, 1) and terminal point Q(-5, 2, 2).

**Solution:**

(a) The standard position vector v representing $\overrightarrow{\mathrm{PQ}}$ has components $\upsilon_1 = x_2 - x_1 = -5 - (-3) = -2,\ \ \upsilon_2 = y_2 - y_1 = 2 - 4 = -2,$
and $\upsilon_3 = z_2 - z_1 = 2 - 1 = 1.$
The component form of $\overrightarrow{\mathrm{PQ}}$ is $v = \langle -2, -2, 1 \rangle$.

(b) The length or magnitude of $v = \overrightarrow{\mathrm{PQ}}$ is
$||v|| = \sqrt{(-2)^2 + (-2)^2 + (1)^2} = \sqrt{9} = 3.$

# Vector Algebra Operations

## Definitions:

Let $u = \langle u_1, u_2, u_3 \rangle$ and $v = \langle v_1, v_2, v_3 \rangle$ be vectors with $\kappa$ a scalar.

**Addition**: $u + v = \langle u_1 + v_1, u_2 + v_2, u_3 + v_3 \rangle$
**Scalar multiplication**: $\kappa u = \langle \kappa u_1, \kappa u_2, \kappa u_3 \rangle$
**Difference**: $u - v = \langle u_1 + (-1)v_1, u_2 + (-1)v_2, u_3 + (-1)v_3 \rangle$

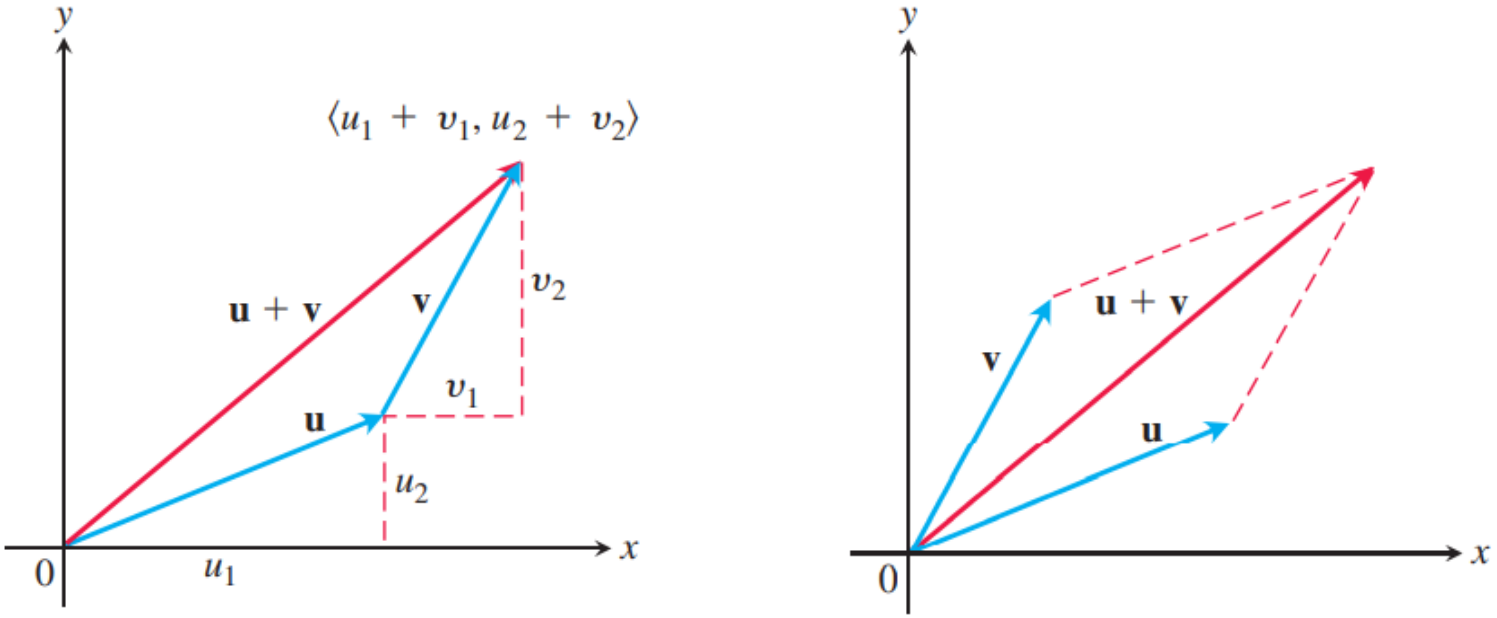



Figure 6: (Left) **Geometric interpretation** of the vector sum. (Right) The **parallelogram law** of vector addition in which both vectors are in standard position.

# Vector Algebra Operations

Vector operations have many of the properties of ordinary arithmetic.

## Properties of Vector Operations

Let u, v, w be vectors and a, b be scalars.

1. $\mathrm{u} + \mathrm{v} = \mathrm{v} + \mathrm{u}$
2. $(\mathrm{u} + \mathrm{v}) + \mathrm{w} = \mathrm{u} + (\mathrm{v} + \mathrm{w})$
3. $\mathrm{u} + 0 = \mathrm{u}$
4. $\mathrm{u} + (\text{-}\mathrm{u}) = 0$
5. $0 \times u = 0$
6. $1 \times u = u$
7. $a \times (bu) = (ab) \times u$
8. $a \times (u + v) = au + av$
9. $(a + b) \times u = au + bu$

When three or more space vectors lie in the same plane, we say they are **coplanar vectors**.
For **example**, the vectors u, v, and u + v are always coplanar.

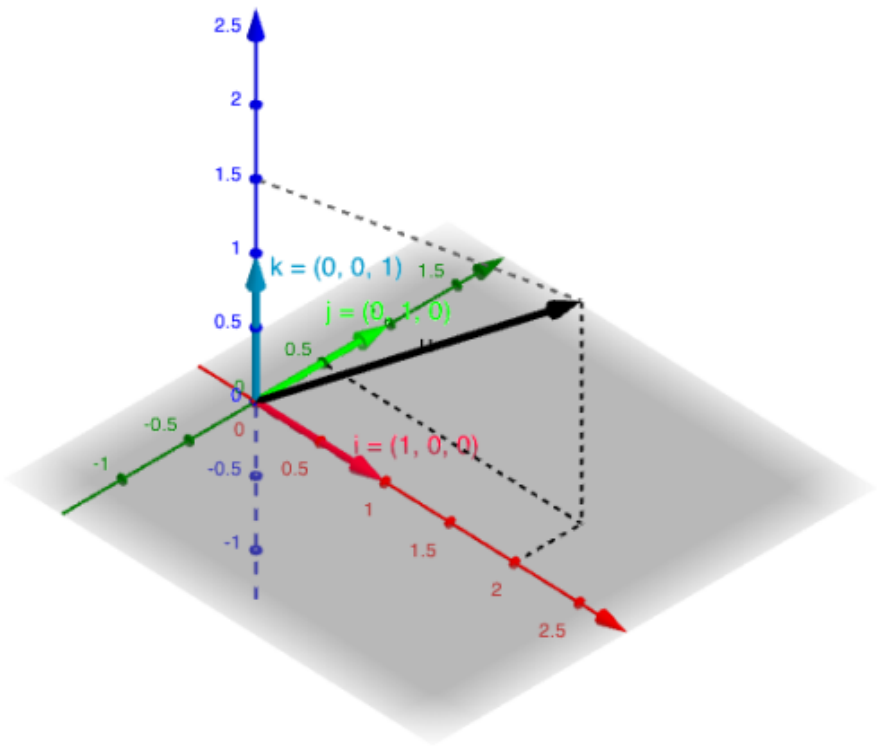
# Unit Vectors

A vector v of length 1 is called a **unit vector**.

The **standard unit vectors** are $i = \langle 1, 0, 0 \rangle$, $j = \langle 0, 1, 0 \rangle$, $k = \langle 0, 0, 1 \rangle$.

Any vector $v = \langle v_1, v_2, v_3 \rangle$ can be written as a **linear combination** of the standard unit vectors as follows:

$$v = \langle v_1, v_2, v_3 \rangle = \langle v_1, 0, 0 \rangle + \langle 0, v_2, 0 \rangle + \langle 0, 0, v_3 \rangle = v_1 \langle 1, 0, 0 \rangle + v_2 \langle 0, 1, 0 \rangle + v_3 \langle 0, 0, 1 \rangle = v_1 i + v_2 j + v_3 k.$$



# Unit Vectors

If $v \neq 0$, then its length $||v||$ is not zero and $\left\| \frac{1}{||v||} v \right\| = \frac{1}{||v||} ||v|| = 1$.

That is, $\frac{v}{||v||}$ is a unit vector in the direction of v, called the **direction** of the nonzero vector v.

We can express any nonzero vector v in terms of its two important features, **length** and **direction**, by writing $v = ||v|| \frac{v}{||v||}$.

**Example**: If $v = 3i - 4j$ is a velocity vector, express v as a product of its speed times its direction of motion.

**Solution**: Speed is the magnitude (length) of v:
$||v|| = \sqrt{(3)^2 + (-4)^2} = \sqrt{9 + 16} = 5$.
The unit vector $v/||v||$ is the direction of v:
$\frac{v}{||v||} = \frac{3}{5} i - \frac{4}{5} j$
So $v = 3i - 4j = 5 \left( \frac{3}{5} i - \frac{4}{5} j \right)$.

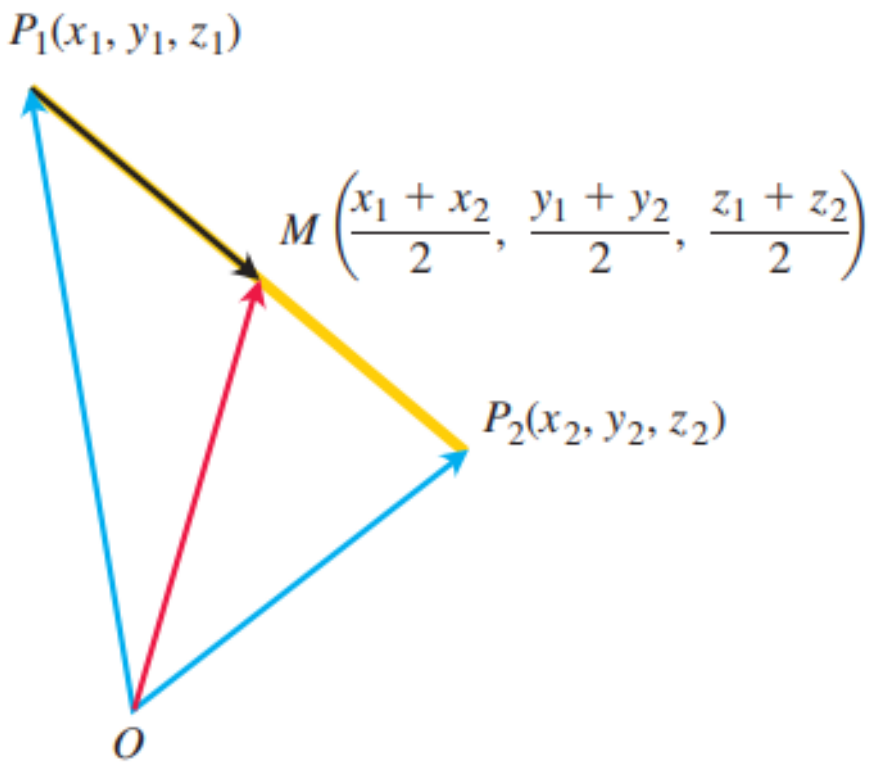
# Midpoint of a Line Segment

**The midpoint M of the line segment**

joining points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is the point

$$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right).$$

To see why, observe Figure at the right that $\overrightarrow{\mathrm{OM}} = \overrightarrow{\mathrm{OP_1}} + \frac{1}{2}(\overrightarrow{\mathrm{P_1P_2}}) = \overrightarrow{\mathrm{OP_1}} + \frac{1}{2}(\overrightarrow{\mathrm{OP_2}} - \overrightarrow{\mathrm{OP_1}}) = \frac{1}{2}(\overrightarrow{\mathrm{OP_1}} + \overrightarrow{\mathrm{OP_2}}) = \frac{x_1+x_2}{2}i + \frac{y_1+y_2}{2}j + \frac{z_1+z_2}{2}k.$



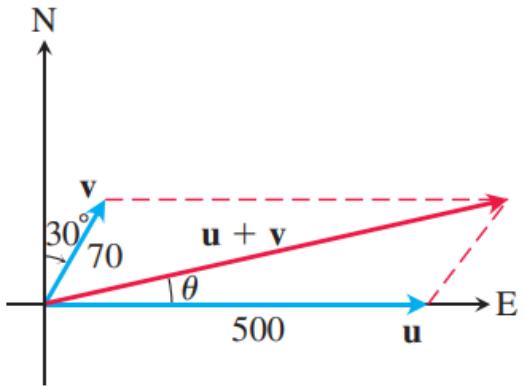
# Applications

An important application of vectors occurs in navigation.

## Example

A jet airliner, flying due east at 500 mph in still air, encounters a 70-mph tailwind blowing in the direction 60° north of east. The airplane holds its compass heading due east but, because of the wind, acquires a new **ground speed** and **direction**. What are they?

**Solution**:
If u is the velocity of the airplane alone and v is the velocity of the tailwind, then $|u| = 500$ and $|v| = 70$. The velocity of the airplane with respect to the ground is given by the magnitude and direction of the resultant vector u + v.



# Solution

If we let the positive x-axis represent east and the positive y-axis represent north, then the component forms of u and v are $u = \langle 500, 0 \rangle$ and $v = \langle 70\cos 60^\circ, 70\sin 60^\circ \rangle = \langle 35, 35\sqrt{3} \rangle$.

Therefore,

$$u + v = \langle 535, 35\sqrt{3} \rangle = 535i + 35\sqrt{3}j$$
$$|u + v| = \sqrt{535^2 + (35\sqrt{3})^2} \approx 538.4$$

and

$\theta = \tan^{-1} \frac{35\sqrt{3}}{535} \approx 6.5^\circ$.
The new ground speed of the airplane is about 538.4 mph, and its new direction is about $6.5^\circ$ north of east.

# Lecture 3: Vectors and the Geometry of Space.

MA2032 Vector Calculus

Lecturer: Larissa Serdukova

School of Computing and Mathematical Science
University of Leicester

September 26, 2022

# The Dot Product

Dot products are also called **inner or scalar products** because the product results in a scalar, not a vector.
The Dot Product is a key part of the calculation of the **angle between two vectors** from their components.

## Theorem: Angle Between Two Vectors

The angle $\theta$ between two nonzero vectors $u = \langle u_1, u_2, u_3 \rangle$ and $v = \langle v_1, v_2, v_3 \rangle$ is given by

$$\theta = \cos^{-1}\left(\frac{u_1 v_1 + u_2 v_2 + u_3 v_3}{||u||\,||v||}\right) = \cos^{-1}\left(\frac{u \cdot v}{||u||\,||v||}\right).$$

## Definition

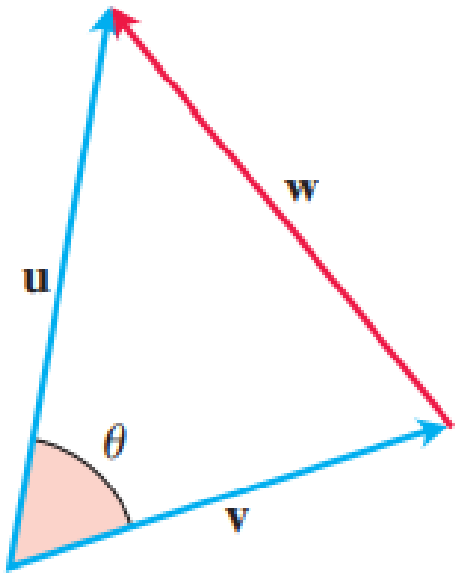
The **dot product** $u \cdot v$ of vectors $u = \langle u_1, u_2, u_3 \rangle$ and $v = \langle v_1, v_2, v_3 \rangle$ is the scalar

$$u \cdot v = u_1 v_1 + u_2 v_2 + u_3 v_3 = ||u||\,||v|| \cos\theta.$$

# Proof of Theorem: Angle Between Two Vectors

Applying the law of cosines to the triangle in Figure, we find that

$$|\mathbf{w}|^2 = |\mathbf{u}|^2 + |\mathbf{v}|^2 - 2|\mathbf{u}||\mathbf{v}|\cos\theta$$
$$2|\mathbf{u}||\mathbf{v}|\cos\theta = |\mathbf{u}|^2 + |\mathbf{v}|^2 - |\mathbf{w}|^2.$$




Because w = u - v, the component form of w is $\langle u_1 - v_1, u_2 - v_2, u_3 - v_3\rangle$. So

$$\begin{aligned}
|\mathbf{u}|^2 &= \left(\sqrt{u_1{}^2 + u_2{}^2 + u_3{}^2}\right)^2 = u_1{}^2 + u_2{}^2 + u_3{}^2 \\
|\mathbf{v}|^2 &= \left(\sqrt{v_1{}^2 + v_2{}^2 + v_3{}^2}\right)^2 = v_1{}^2 + v_2{}^2 + v_3{}^2 \\
|\mathbf{w}|^2 &= \left(\sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + (u_3 - v_3)^2}\right)^2 \\
&= (u_1 - v_1)^2 + (u_2 - v_2)^2 + (u_3 - v_3)^2 \\
&= u_1{}^2 - 2u_1v_1 + v_1{}^2 + u_2{}^2 - 2u_2v_2 + v_2{}^2 + u_3{}^2 - 2u_3v_3 + v_3{}^2
\end{aligned}$$

$\Rightarrow$

$$|\mathbf{u}|^2 + |\mathbf{v}|^2 - |\mathbf{w}|^2 = 2(u_1v_1 + u_2v_2 + u_3v_3).$$

# Proof of Theorem: Angle Between Two Vectors

Therefore,

$$2|\mathbf{u}||\mathbf{v}|\cos\theta = |\mathbf{u}|^2 + |\mathbf{v}|^2 - |\mathbf{w}|^2 = 2(u_1v_1 + u_2v_2 + u_3v_3)$$

$$|\mathbf{u}||\mathbf{v}|\cos\theta = u_1v_1 + u_2v_2 + u_3v_3$$

$$\cos\theta = \frac{u_1v_1 + u_2v_2 + u_3v_3}{|\mathbf{u}||\mathbf{v}|}.$$

Since $0 \leqslant \theta < \pi$, we have

$$\theta = \cos^{-1}\left(\frac{u_1v_1 + u_2v_2 + u_3v_3}{|\mathbf{u}||\mathbf{v}|}\right).$$

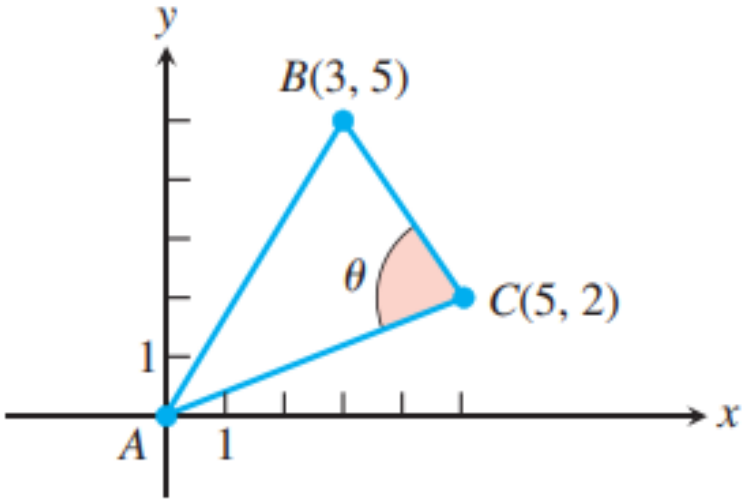
# Example

Find the angle $\theta$ in the triangle $ABC$ determined by the vertices $A = (0, 0)$, $B = (3, 5)$, and $C = (5, 2)$ as showed on Figure.

**Solution:**

The angle $\theta$ is the angle between the vectors $\overrightarrow{\mathrm{CA}}$ and $\overrightarrow{\mathrm{CB}}$. The component forms of these two vectors are

$$\overrightarrow{CA} = \langle -5, -2 \rangle \qquad \text{and} \qquad \overrightarrow{CB} = \langle -2, 3 \rangle.$$




First we calculate the dot product and magnitudes of these two vectors.

$$\overrightarrow{CA} \cdot \overrightarrow{CB} = (-5)(-2) + (-2)(3) = 4$$
$$|\overrightarrow{CA}| = \sqrt{(-5)^2 + (-2)^2} = \sqrt{29}$$
$$|\overrightarrow{CB}| = \sqrt{(-2)^2 + (3)^2} = \sqrt{13}$$

Then applying the angle formula, we have $\approx 78.1^\circ$ or 1.36 radians:

$$\theta = \cos^{-1}\left(\frac{\overrightarrow{CA} \cdot \overrightarrow{CB}}{|\overrightarrow{CA}||\overrightarrow{CB}|}\right) = \cos^{-1}\left(\frac{4}{(\sqrt{29})(\sqrt{13})}\right)$$

# Orthogonal Vectors

Two nonzero vectors u and v are**perpendicular (orthogonal)** if the angle between them is $\pi/2$.

For such vectors, we have $u \cdot v = 0$ because $\cos(\pi/2) = 0$.

The **converse is also true**: If u and v are nonzero vectors with $u \cdot v = ||u||\,||v|| \cos\theta = 0$, then $cos\theta = 0$ and $\theta = \cos^{-1} = \pi/2$.

The following definition also allows for one or both of the vectors to be the zero vector.

## Definition

Vectors u and v are orthogonal if $u \cdot v = 0$.

# Example

To determine if two vectors are orthogonal, calculate their dot product.

**Solution:**

**(a)** $\mathbf{u} = \langle 3, -2 \rangle$ and $\mathbf{v} = \langle 4, 6 \rangle$ are orthogonal because $\mathbf{u} \cdot \mathbf{v} = (3)(4) + (-2)(6) = 0$.

**(b)** $\mathbf{u} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ and $\mathbf{v} = 2\mathbf{j} + 4\mathbf{k}$ are orthogonal because

$$\mathbf{u} \cdot \mathbf{v} = (3)(0) + (-2)(2) + (1)(4) = 0.$$

**(c)** $\mathbf{0}$ is orthogonal to every vector $\mathbf{u}$ since

$$\begin{aligned} \mathbf{0} \cdot \mathbf{u} &= \langle 0, 0, 0 \rangle \cdot \langle u_1, u_2, u_3 \rangle \\ &= (0)(u_1) + (0)(u_2) + (0)(u_3) = 0. \end{aligned}$$

# Dot Product Properties

## Properties of the Dot Product

If $u, v$, and $w$ are any vectors and $c$ is a scalar, then

1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
2. $(c\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (c\mathbf{v}) = c(\mathbf{u} \cdot \mathbf{v})$
3. $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$
4. $\mathbf{u} \cdot \mathbf{u} = |\mathbf{u}|^2$
5. $\mathbf{0} \cdot \mathbf{u} = 0.$

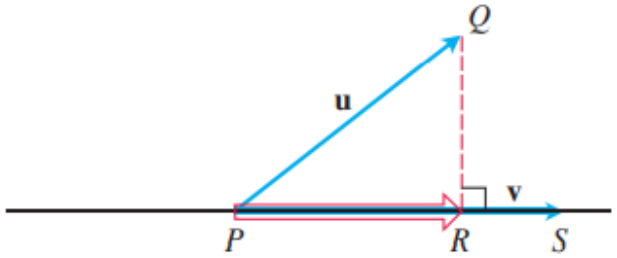
The properties are easy to prove using the definition. For instance, here are the proofs of Properties 1 and 3.

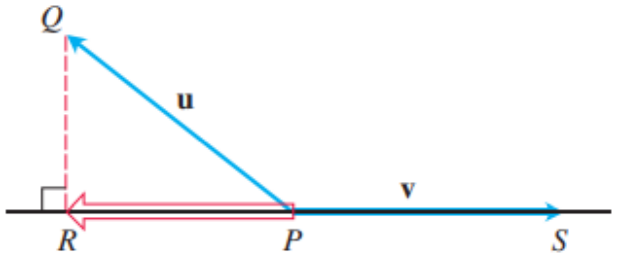
**1.** $\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3 = v_1 u_1 + v_2 u_2 + v_3 u_3 = \mathbf{v} \cdot \mathbf{u}$

**3.**
$$\begin{aligned}
\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) &= \langle u_1, u_2, u_3 \rangle \cdot \langle v_1 + w_1, v_2 + w_2, v_3 + w_3 \rangle \\
&= u_1(v_1 + w_1) + u_2(v_2 + w_2) + u_3(v_3 + w_3) \\
&= u_1 v_1 + u_1 w_1 + u_2 v_2 + u_2 w_2 + u_3 v_3 + u_3 w_3 \\
&= (u_1 v_1 + u_2 v_2 + u_3 v_3) + (u_1 w_1 + u_2 w_2 + u_3 w_3) \\
&= \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}
\end{aligned}$$

# Vector Projections

We now return to the problem of **projecting one vector onto another**. The vector projection of $u = \overrightarrow{PQ}$ onto a nonzero vector $v = \overrightarrow{PS}$, see Figure, is the vector $\overrightarrow{PR} = proj_v u$ determined by **dropping a perpendicular** from $Q$ to the line $PS$.




If the angle $\theta$ between u and v is **acute**, $proj_v u$ has length $|u| \cos\theta$ called **scalar component** of u in the direction of v and direction $v/|v|$.
If u is **obtuse**, cos $< 0$ and $proj_v u$ has length $-|u| \cos\theta$ and direction $-v/|v|$. In both cases

$$\text{proj}_{\mathbf{v}}\,\mathbf{u} = \left(|\mathbf{u}|\cos\theta\right)\frac{\mathbf{v}}{|\mathbf{v}|}$$

$$= \left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|}\right)\frac{\mathbf{v}}{|\mathbf{v}|} \qquad |\mathbf{u}|\cos\theta = \frac{|\mathbf{u}||\mathbf{v}|\cos\theta}{|\mathbf{v}|} = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|}$$

$$= \left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|^2}\right)\mathbf{v}.$$

# Example

Find the vector projection of $u = 6i + 3j + 2k$ onto $v = i - 2j - 2k$ and the scalar component of u in the direction of v.

**Solution:**

Using the previously given Equations we find:

$$\text{proj}_{\mathbf{v}}\,\mathbf{u} = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|^2}\mathbf{v} = \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v} = \frac{6-6-4}{1+4+4}(\mathbf{i}-2\mathbf{j}-2\mathbf{k})$$

$$= -\frac{4}{9}(\mathbf{i}-2\mathbf{j}-2\mathbf{k}) = -\frac{4}{9}\mathbf{i} + \frac{8}{9}\mathbf{j} + \frac{8}{9}\mathbf{k}.$$

$$|\mathbf{u}|\cos\theta = \mathbf{u}\cdot\frac{\mathbf{v}}{|\mathbf{v}|} = (6\mathbf{i}+3\mathbf{j}+2\mathbf{k})\cdot\left(\frac{1}{3}\mathbf{i} - \frac{2}{3}\mathbf{j} - \frac{2}{3}\mathbf{k}\right)$$

$$= 2 - 2 - \frac{4}{3} = -\frac{4}{3}.$$

# Application: Work

## Definition

The work done by a constant force $F$ acting through a displacement $D = \overrightarrow{PQ}$ is

$$W = (\text{scalar component of F in the direction of D})(\text{length of D}) = (|F|\cos\theta)|D| = F \cdot D$$

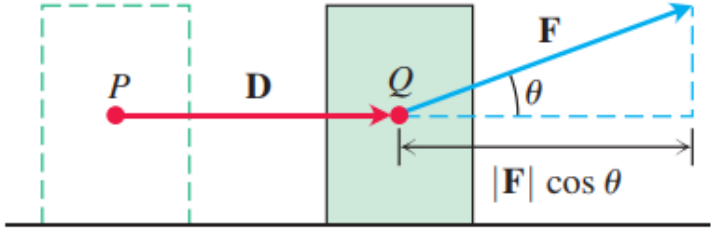



Figure: The work done by a constant force $F$ during a displacement $D$ is the dot product $F \cdot D$.

# Lecture 4: Vectors and the Geometry of Space.

MA2032 Vector Calculus

Lecturer: Larissa Serdukova

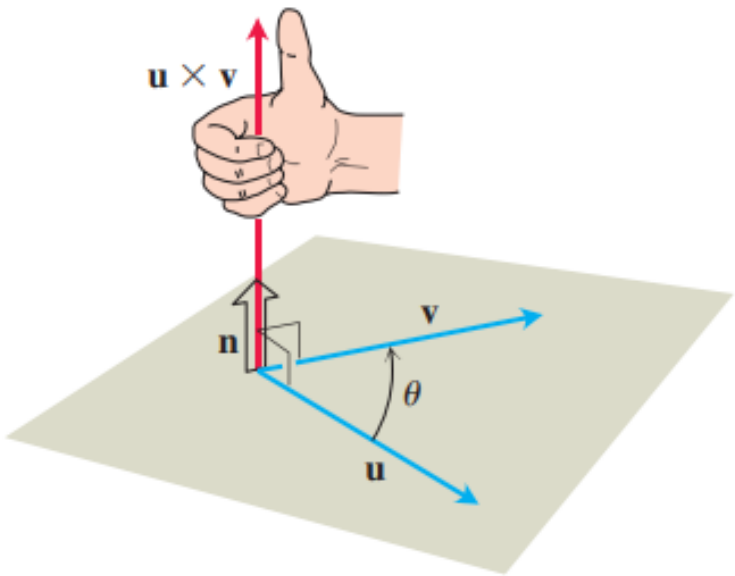
School of Computing and Mathematical Science
University of Leicester

September 28, 2022

# The Cross Product

- In studying lines in the plane, when we needed to describe **how a line was tilting**, we used the notions of slope and angle of inclination.
- In space, we want a way to describe **how a plane is tilting**.
- We accomplish this by **multiplying two vectors** in the plane together to get a third **vector perpendicular to the plane**.
- The **direction of this third vector tells us the "inclination" of the plane**.
- The product we use to multiply the vectors together is the vector or **cross product**, the second of the two vector multiplication methods.
- The cross product gives us a simple **way to find a variety of geometric quantities**, including volumes, areas, and perpendicular vectors.

# The Cross Product of Two Vectors in Space




- Two vectors are **parallel** if one is a nonzero multiple of the other.
- If $u$ and $v$ are not parallel, they **determine a plane**.
- The vectors in this plane are **linear combinations of $u$ and $v$**, so they can be written as a sum $au + bv$.
- We select the **unit vector $n$ perpendicular to the plane** by the right-hand rule, see Figure.
- Then we define a new vector as follows.

**Definition:**

The cross product $u \times v$ ("u cross v") is the vector

$$u \times v = (|u|\,|v| \sin\theta)n.$$

# The Cross Product of Two Vectors in Space

- Unlike the dot product, **the cross product is a vector**.
- For this reason it is also called the **vector product** of u and v, and can be applied only to vectors in space.
- The **vector** $u \times v$ **is orthogonal to both** $u$ **and** $v$ because it is a scalar multiple of $n$.

## Parallel Vectors

Nonzero vectors $u$ and $v$ are **parallel** ($\theta = 0$ or $\theta = \pi \Rightarrow \sin = 0$) if and only if $u \times v = 0$.

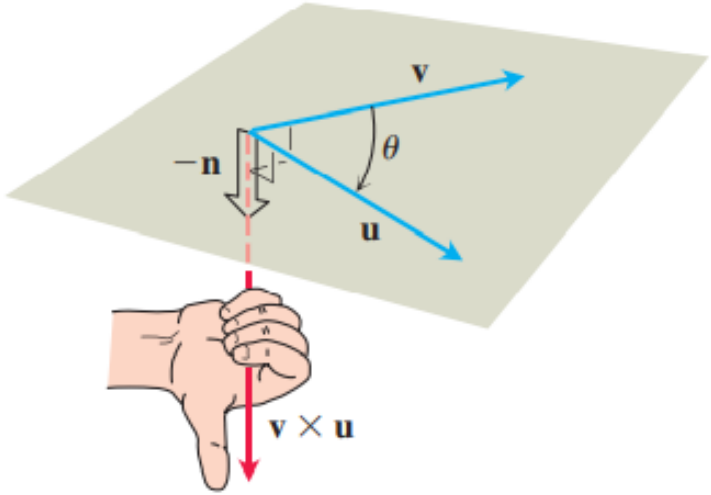
## Properties of the Cross Product

If $u$, $v$, and $w$ are any vectors and $r$, $s$ are scalars, then

1. $(r\mathbf{u}) \times (s\mathbf{v}) = (rs)(\mathbf{u} \times \mathbf{v})$
2. $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$
3. $\mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v})$
4. $(\mathbf{v} + \mathbf{w}) \times \mathbf{u} = \mathbf{v} \times \mathbf{u} + \mathbf{w} \times \mathbf{u}$
5. $\mathbf{0} \times \mathbf{u} = \mathbf{0}$
6. $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}$

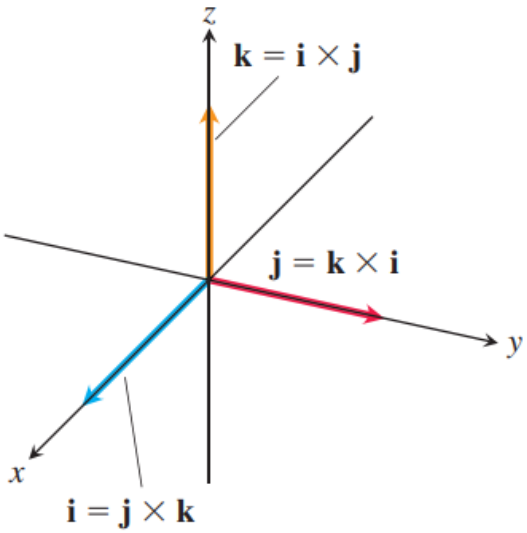
# Property 3

• Visualization of Property 3: when the fingers of your right hand curl through the angle $\theta$ from $v$ to $u$, your thumb points the opposite way.
• The unit vector we choose in forming $v \times u$ is the negative of the one we choose in forming $u \times v$, see Figure.




When we apply the definition and Property 3 to calculate the **pairwise cross products of** $i$**,** $j$**, and** $k$, we find

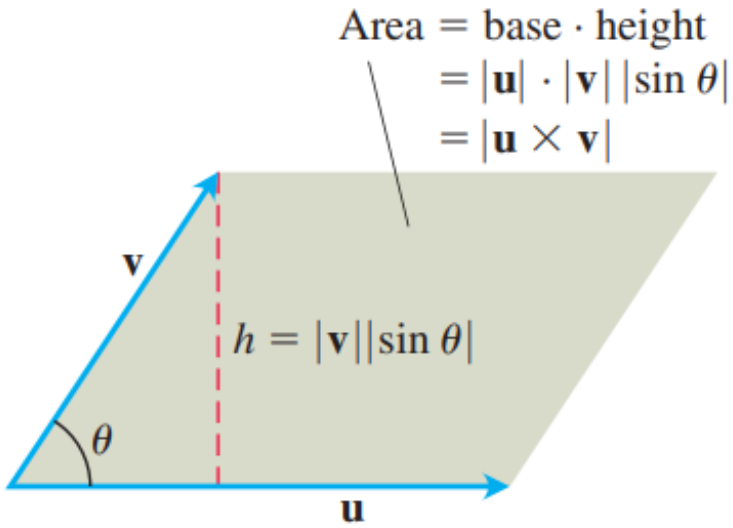
$$
\begin{aligned}
i \times j &= -(j \times i) = k, \\
j \times k &= -(k \times j) = i, \\
k \times i &= -(i \times k) = j, \\
i \times i = j \times j &= k \times k = 0.
\end{aligned}
$$



# Area of a Parallelogram

• Because $n$ is a unit vector, the magnitude of $u \times v$ is
$|u \times v| = |u|\,|v|\,|\sin\theta|\,|n| = |u|\,|v|\sin\theta$.

• This is the **area of the parallelogram** determined by $u$ and $v$, see Figure $|u|$ being the base of the parallelogram and $|v|\,|\sin\theta|$ the height.



# Determinant Formula for $u \times v$

• Our next objective is to calculate $u \times v$ **from the components** of $u$ and $v$ relative to a Cartesian coordinate system.

Suppose that $u = u_1 i + u_2 j + u_3 k$ and $v = \nu_1 i + \nu_2 j + \nu_3 k$.
Then the distributive laws and the rules for multiplying $i$, $j$, and $k$ tell us that

$$\begin{aligned}
\mathbf{u} \times \mathbf{v} &= (u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}) \times (v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}) \\
&= u_1v_1\mathbf{i} \times \mathbf{i} + u_1v_2\mathbf{i} \times \mathbf{j} + u_1v_3\mathbf{i} \times \mathbf{k} \\
&\quad + u_2v_1\mathbf{j} \times \mathbf{i} + u_2v_2\mathbf{j} \times \mathbf{j} + u_2v_3\mathbf{j} \times \mathbf{k} \\
&\quad + u_3v_1\mathbf{k} \times \mathbf{i} + u_3v_2\mathbf{k} \times \mathbf{j} + u_3v_3\mathbf{k} \times \mathbf{k} \\
&= (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}.
\end{aligned}$$

The component terms in the last line are the same as the terms in the expansion of the symbolic **determinant**

$$\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ u_1 & u_2 & u_3 \\ \nu_1 & \nu_2 & \nu_3 \end{vmatrix}$$

# Determinant Formula for $u \times v$

## Calculating the Cross Product as a Determinant

If $u = u_1 i + u_2 j + u_3 k$ and $v = \nu_1 i + \nu_2 j + \nu_3 k$, then

$$u \times v = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ u_1 & u_2 & u_3 \\ \nu_1 & \nu_2 & \nu_3 \end{vmatrix}$$

**Example**: Find a vector perpendicular to the plane of $P(1, -1, 0)$, $Q(2, 1, -1)$, and $R(-1, 1, 2)$, see Figure.

**Solution**:

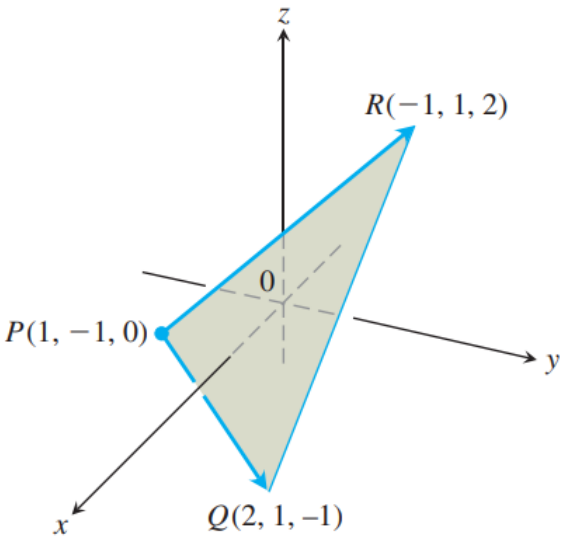
The vector $\overrightarrow{\mathrm{PQ}} \times \overrightarrow{\mathrm{PR}}$ is perpendicular to the plane because it is perpendicular to both vectors. In terms of components,

$$\overrightarrow{PQ} = (2-1)\mathbf{i} + (1+1)\mathbf{j} + (-1-0)\mathbf{k} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$$

$$\overrightarrow{PR} = (-1-1)\mathbf{i} + (1+1)\mathbf{j} + (2-0)\mathbf{k} = -2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$$

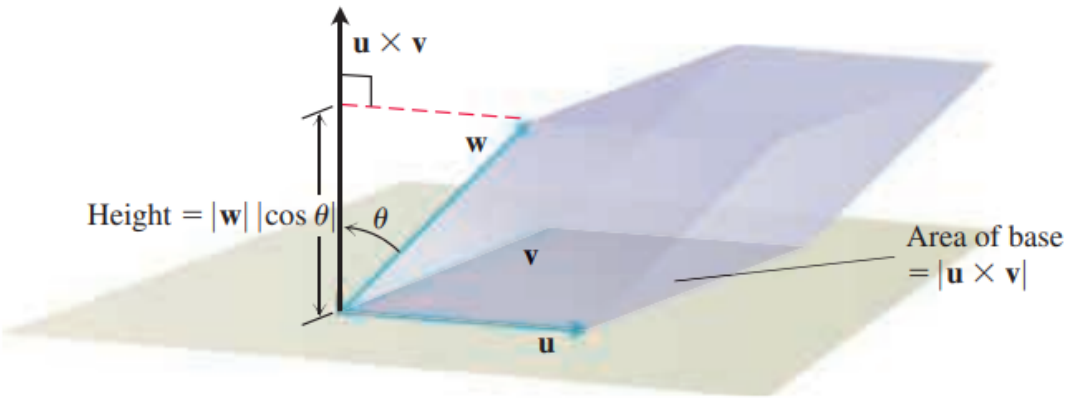
$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & -1 \\ -2 & 2 & 2 \end{vmatrix} = \begin{vmatrix} 2 & -1 \\ 2 & 2 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & -1 \\ -2 & 2 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 2 \\ -2 & 2 \end{vmatrix}\mathbf{k}$$

$$= 6\mathbf{i} + 6\mathbf{k}.$$



# Triple Scalar or Box Product

- The product $(u \times v) \cdot w$ is called the **triple scalar product** of $u$, $v$, and $w$ (in that order).
- As you can see from the formula
$|(u \times v) \cdot w| = |u \times v||w||\cos\theta|$,
the absolute value of this product is the **volume of the parallelepiped** (parallelogram-sided box) determined by u, v, and w, see Figure.
- The number $|u \times v|$ is the **area of the base** parallelogram.
- The number $|w||\cos\theta|$ is the **parallelepiped's height**.
- Because of **this geometry**, $(u \times v) \cdot w$ is also called the box product of $u$, $v$, and $w$.




Volume = area of base · height
$= |\mathbf{u} \times \mathbf{v}|\,|\mathbf{w}|\,|\cos\theta|$

# Triple Scalar Product

The triple scalar product can be evaluated **as a determinant**:

$$
\begin{aligned}
(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} &= \left( \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \mathbf{k} \right) \cdot \mathbf{w} \\
&= w_1 \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} - w_2 \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} + w_3 \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \\
&= \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}.
\end{aligned}
$$

## Calculating the Triple Scalar Product as a Determinant

$$
(u \times v) \cdot w = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}
$$

# Lecture 5: Vectors and the Geometry of Space.

MA2032 Vector Calculus

Lecturer: Larissa Serdukova

School of Computing and Mathematical Science
University of Leicester

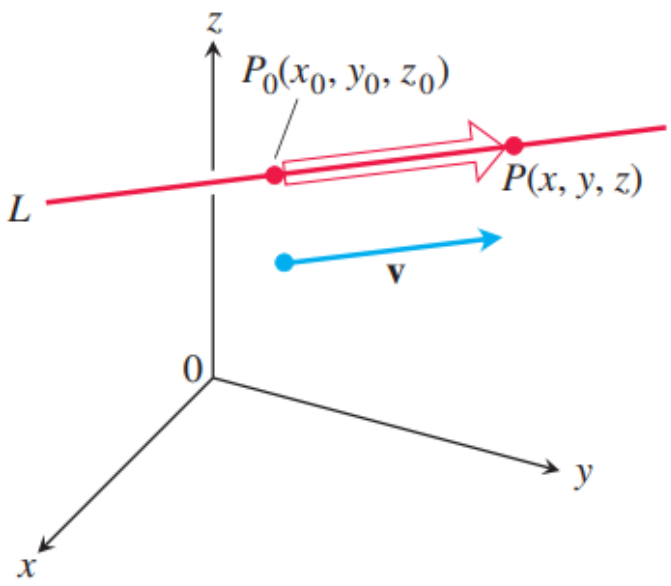
September 30, 2022

# Lines and Planes in Space

- Today I will show how to use scalar and vector products to write **equations for lines, line segments, and planes in space**.
- We will use **these representations** throughout the rest of the course in studying the calculus of curves and surfaces in space.
- **In the plane**, a line is determined by a **point** and a number giving the **slope** of the line.
- **In space** a line is determined by a **point** and a **vector** giving the direction of the line.

# Lines and Line Segments in Space

- Suppose that $L$ is a line in space passing through a point $P_0(x_0, y_0, z_0)$ parallel to a vector $v = \nu_1 i + \nu_2 j + \nu_3 k$.
- Then $L$ is the set of all points $P(x, y, z)$ for which $\overrightarrow{P_0P}$ is parallel to v as shown in Figure.
- Thus, $\overrightarrow{P_0P} = tv$ for some scalar parameter t.
- The value of t depends on the location of the point P along the line, and the domain of t is $(-\infty, \infty)$.




- The expanded form of the equation $\overrightarrow{P_0P} = tv$ is

$$(x - x_0)i + (y - y_0)j + (z - z_0)k = t(\nu_1 i + \nu_2 j + \nu_3 k),$$

which can be rewritten as

$$xi + yj + zk = x_0 i + y_0 j + z_0 k + t(\nu_1 i + \nu_2 j + \nu_3 k).$$

# Lines and Planes in Space

If $r(t)$ is the position vector of a point $P(x, y, z)$ on the line and $r_0$ is the position vector of the point $P_0(x_0, y_0, z_0)$, then Equation above gives the following vector form for the equation of a line in space

## Vector Equation for a Line

A **vector equation for the line** $L$ through $P_0(x_0, y_0, z_0)$ parallel to v is

$$r(t) = r_0 + tv, \quad -\infty < t < \infty$$

where r is the position vector of a point $P(x, y, z)$ on $L$ and $r_0$ is the position vector of $P_0(x_0, y_0, z_0)$.

## Parametric Equations for a Line

The **standard parametrization of the line** through $P_0(x_0, y_0, z_0)$ parallel to $v = \nu_1 i + \nu_2 j + \nu_3 k$ is

$$x = x_0 + t\nu_1, y = y_0 + t\nu_2, z = z_0 + t\nu_3, \quad -\infty < t < \infty$$

## Example

Find parametric equations for the line through $P(-3, 2, -3)$ and $Q(1, -1, 4)$.

**Solution**

• The vector $\overrightarrow{PQ} = (1-(-3))i + (-1-2)j + (4-(-3))k = 4i - 3j + 7k$ is parallel to the line, and the standard parametrization of the line equations with $(x_0, y_0, z_0) = (-3, 2, -3)$ give

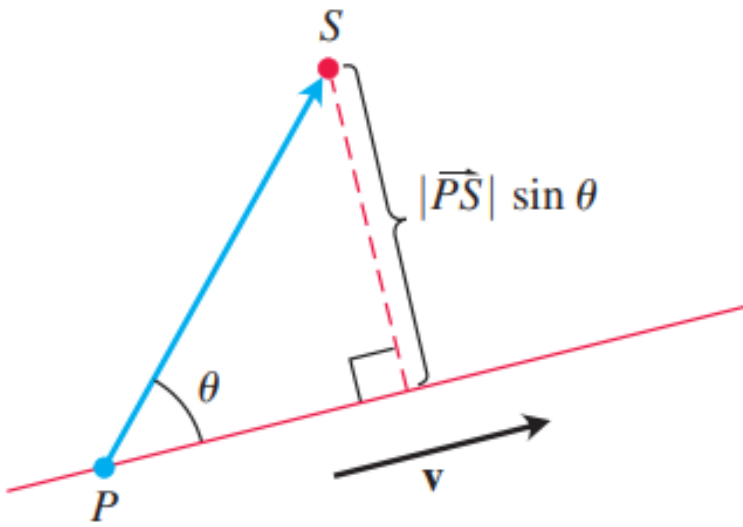
$$x = -3 + 4t,\ y = 2 - 3t,\ z = -3 + 7t.$$

• We could have chosen $Q(1, -1, 4)$ as the "base point" and written $x = 1 + 4t, y = -1 - 3t, z = 4 + 7t$.

• These equations serve as well as the first; they simply place you at a different point on the line for a given value of t.

# The Distance from a Point to a Line in Space

To find the **distance from a point S to a line** that passes through a point P parallel to a vector v, we find the absolute value of the scalar component of $\overrightarrow{PS}$ in the direction of a vector normal to the line, see Figure. In the notation of the figure, the absolute value of the scalar component is $|\overrightarrow{PS}| \sin\theta$, which is




$$\frac{|\overrightarrow{PS}||\mathbf{v}|\sin\theta}{|\mathbf{v}|} = \frac{|\overrightarrow{PS} \times \mathbf{v}|}{|\mathbf{v}|}$$

## Distance from a Point S to a Line Through P Parallel to v

$$d = \frac{|\overrightarrow{PS} \times \mathbf{v}|}{|\mathbf{v}|}$$

# The Distance from a Point to a Line in Space

## Example

Find the distance from the point $S(1,1,5)$ to the line $L:\ x = 1 + t, y = 3 - t, z = 2t$.

**Solution**

We see from the equations for L that L passes through $P(1,3,0)$ parallel to $v = i - j + 2k$. With

$$\overrightarrow{PS} = (1 - 1)\mathbf{i} + (1 - 3)\mathbf{j} + (5 - 0)\mathbf{k} = -2\mathbf{j} + 5\mathbf{k}$$

and

$$\overrightarrow{PS} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -2 & 5 \\ 1 & -1 & 2 \end{vmatrix} = \mathbf{i} + 5\mathbf{j} + 2\mathbf{k},$$

Equation (5) gives

$$d = \frac{|\overrightarrow{PS} \times \mathbf{v}|}{|\mathbf{v}|} = \frac{\sqrt{1 + 25 + 4}}{\sqrt{1 + 1 + 4}} = \frac{\sqrt{30}}{\sqrt{6}} = \sqrt{5}.$$

# An Equation for a Plane in Space

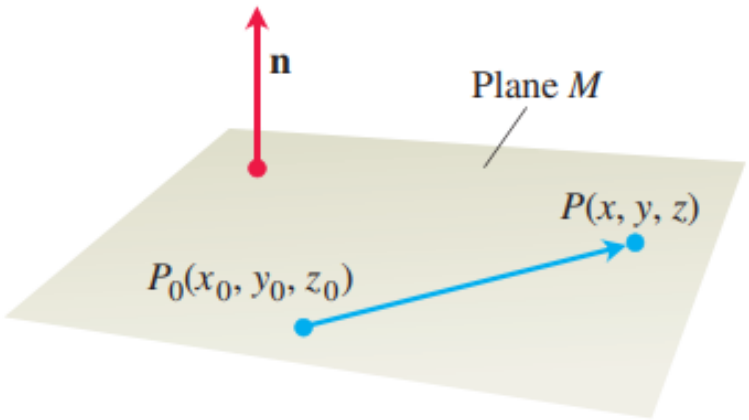
• A **plane in space is determined** by knowing a **point** on the plane and its **"tilt"** or orientation.

• This "tilt" is defined by specifying a vector that is perpendicular or **normal** to the plane.

• Suppose that plane M passes through a point $P_0(x_0, y_0, z_0)$ and is normal to the nonzero vector $n = Ai + Bj + Ck$.

• Vector from $P_0$ to any point $P$ on the plane is orthogonal to $n$.

• Then M is the set of all points $P(x, y, z)$ for which $\overrightarrow{P_0P}$ is orthogonal to n.




• Thus, the dot product $n \cdot \overrightarrow{P_0P} = 0$.

• This equation is equivalent to

$$(Ai + Bj + Ck) \cdot [(x - x_0)i + (y - y_0)j + (z - z_0)k] = 0,$$

so the plane M consists of the points $(x, y, z)$ satisfying

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0.$$

# Equation for a Plane

The plane through $P_0(x_0, y_0, z_0)$ normal to $n = Ai + Bj + Ck$ has

| | |
|---|---|
| **Vector equation:** | $\mathbf{n} \cdot \overrightarrow{P_0P} = 0$ |
| **Component equation:** | $A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$ |
| **Component equation simplified:** | $Ax + By + Cz = D,$ where |
| | $D = Ax_0 + By_0 + Cz_0$ |

## Example

Find an equation for the plane through $A(0, 0, 1)$, $B(2, 0, 0)$, and $C(0, 3, 0)$.

## Solution

We find a vector normal to the plane and use it with one of the points (it does not matter which) to write an equation for the plane. The cross product

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & -1 \\ 0 & 3 & -1 \end{vmatrix} = 3\mathbf{i} + 2\mathbf{j} + 6\mathbf{k}$$

is normal to the plane. We substitute the components of this vector and the coordinates of $A(0, 0, 1)$ into the component form of the equation to obtain

$$\begin{aligned} 3(x - 0) + 2(y - 0) + 6(z - 1) &= 0 \\ 3x + 2y + 6z &= 6. \end{aligned}$$

# Lines of Intersection

- Just as lines are parallel if and only if they have the same direction, two **planes are parallel** if and only if their **normals are parallel**, or $n_1 = kn_2$ for some scalar k.
- Two planes that are not parallel **intersect in a line**.

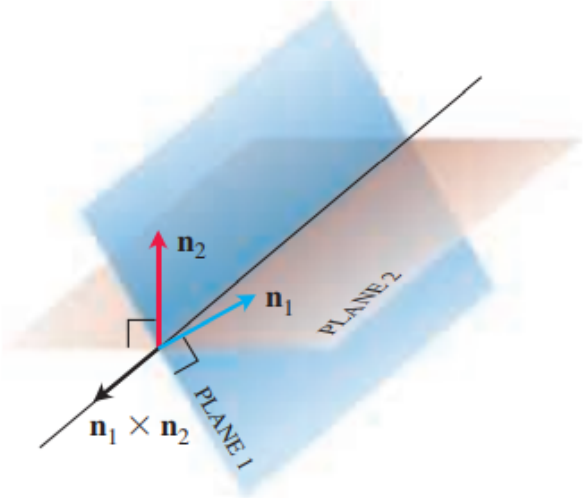
## Example

Find a vector parallel to the line of intersection of the planes $3x - 6y - 2z = 15$ and $2x + y - 2z = 5$.

## Solution

- The line of intersection of two planes is perpendicular to both planes' normal vectors $n_1$ and $n_2$ (see Figure) and therefore parallel to $n_1 \times n_2$.
- Turning this around, $n_1 \times n_2$ is a vector parallel to the planes' line of intersection. In our case,

$$\mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -6 & -2 \\ 2 & 1 & -2 \end{vmatrix} = 14\mathbf{i} + 2\mathbf{j} + 15\mathbf{k}.$$



# A line and a plane intersection

## Example

Find the point where the line $x = \frac{8}{3} + 2t,\ y = -2t,\ z = 1 + t$ intersects the plane $3x + 2y + 6z = 6$.

**Solution**

The point $\left(\frac{8}{3} + 2t, -2t, 1 + t\right)$ lies in the plane if its coordinates satisfy the equation of the plane, that is, if

$$3\left(\frac{8}{3} + 2t\right) + 2(-2t) + 6(1 + t) = 6$$

$$8 + 6t - 4t + 6 + 6t = 6$$

$$8t = -8$$

$$t = -1.$$

The point of intersection is

$$(x, y, z)\big|_{t=-1} = \left(\frac{8}{3} - 2, 2, 1 - 1\right) = \left(\frac{2}{3}, 2, 0\right).$$

# The Distance from a Point to a Plane

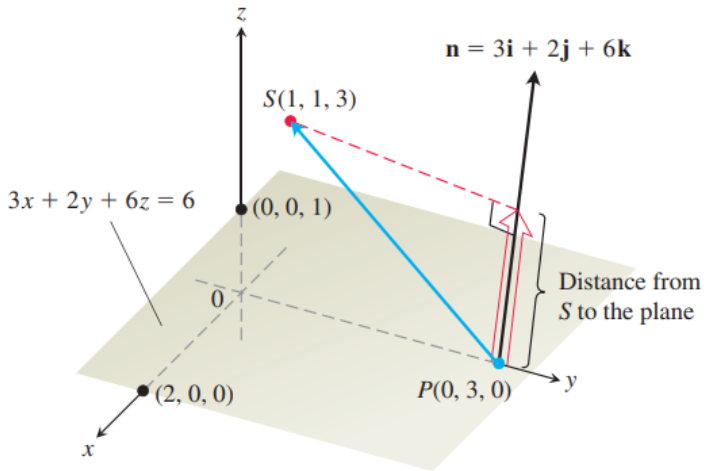
If $P$ is a point on a plane with normal $n$, then the distance from any point $S$ to the plane is the **length of the vector projection** of $\overrightarrow{PS}$ onto $n$, as given in the following formula.

## Distance from a Point S to a Plane with Normal n at Point P

$$d = \left| \overrightarrow{PS} \cdot \frac{n}{|n|} \right|$$

## Example

Find the distance from $S(1, 1, 3)$ to the plane $3x + 2y + 6z = 6$.



# The Distance from a Point to a Plane

**Solution**

• We find a point P in the plane and calculate the length of the vector projection of $\overrightarrow{PS}$ onto a vector n normal to the plane.

• The coefficients in the equation $3x + 2y + 6z = 6$ give $n = 3i + 2j + 6k$.

• The points on the plane easiest to find from the plane's equation are the intercepts.

• If we take P to be the y-intercept $(0, 3, 0)$, then

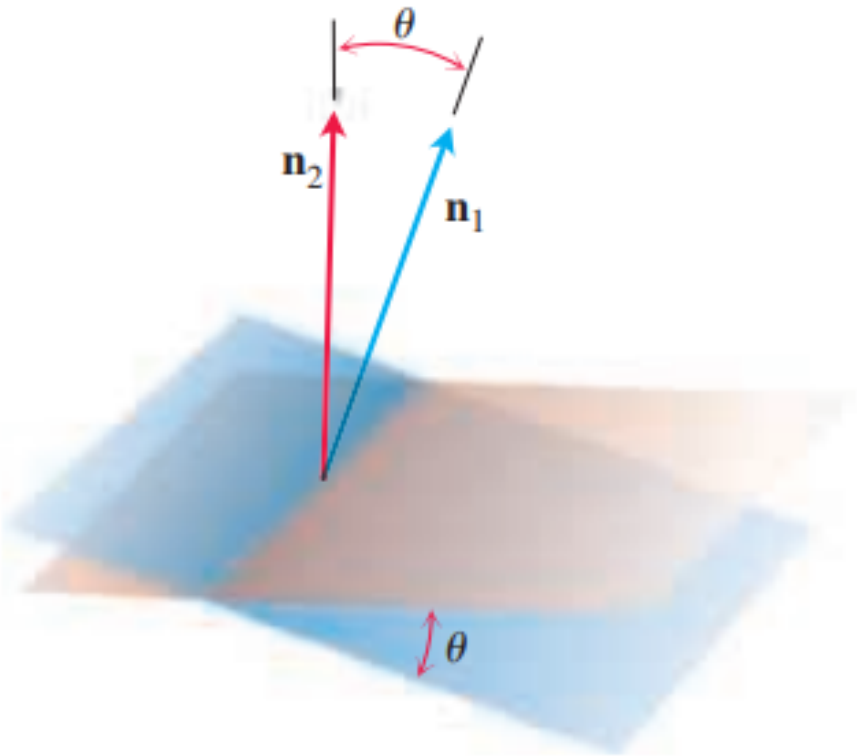
$$\overrightarrow{PS} = (1 - 0)\mathbf{i} + (1 - 3)\mathbf{j} + (3 - 0)\mathbf{k} = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k},$$
$$|\mathbf{n}| = \sqrt{(3)^2 + (2)^2 + (6)^2} = \sqrt{49} = 7.$$

• Therefore, the distance from S to the plane is

$$d = \left|\overrightarrow{PS} \cdot \frac{\mathbf{n}}{|\mathbf{n}|}\right| \qquad \text{Length of } \mathrm{proj}_{\mathbf{n}} \overrightarrow{PS}$$
$$= \left|(\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}) \cdot \left(\frac{3}{7}\mathbf{i} + \frac{2}{7}\mathbf{j} + \frac{6}{7}\mathbf{k}\right)\right|$$
$$= \left|\frac{3}{7} - \frac{4}{7} + \frac{18}{7}\right| = \frac{17}{7}.$$

# Angles Between Planes

• The angle between two intersecting planes is defined to be the **acute angle between their normal vectors**



# Angles Between Planes

## Example

Find the angle between the planes $3x - 6y - 2z = 15$ and $2x + y - 2z = 5$.

**Solution**

The vectors $n_1 = 3i - 6j - 2k$, $n_2 = 2i + j - 2k$ are normals to the planes. The angle between them is

$$\theta = \cos^{-1}\left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|}\right)$$

$$= \cos^{-1}\left(\frac{4}{21}\right) \approx 1.38 \text{ radians.}$$

About 79 degrees

# Lecture 6: Vectors and the Geometry of Space.

MA2032 Vector Calculus

Lecturer: Larissa Serdukova

School of Computing and Mathematical Science
University of Leicester

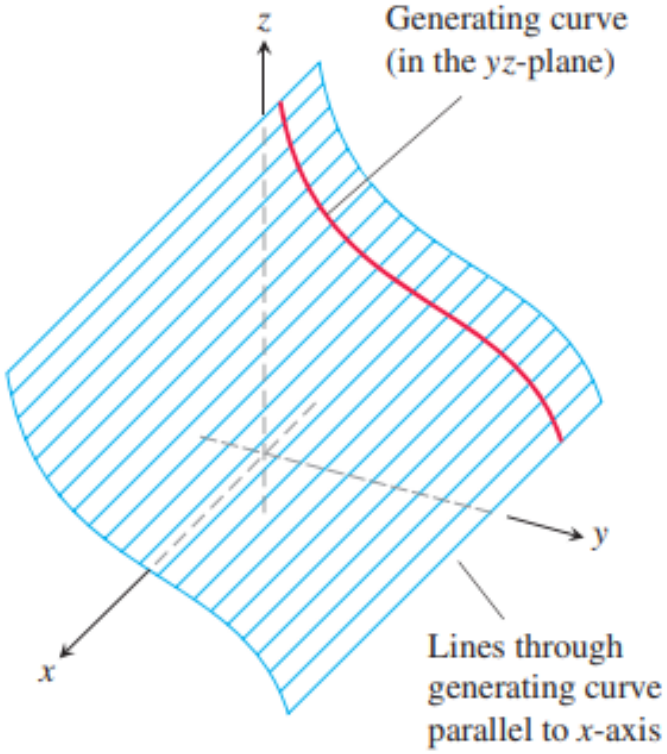
October 3, 2022

# Cylinders and Quadric Surfaces

- Up to now, we have studied two special types of surfaces: **spheres and planes**.
- In today's lecture, we extend our inventory to include a variety of **cylinders and quadric surfaces**.
- Quadric surfaces are surfaces **defined by second-degree equations** in x, y, and z.
- Spheres are quadric surfaces, but there are other of equal interest.

# Cylinders

• A cylinder is a surface that is **generated by moving a straight line along a given planar curve** while holding the line parallel to a given fixed line.
• The curve is called a **generating curve** for the cylinder (see Figure).
• In solid geometry, where cylinder means circular cylinder, the **generating curves are circles**, but now we allow generating curves of **any kind**.




• Any curve $g(x, z) = c$ in the xz-plane defines a cylinder parallel to the y-axis whose space equation is also $g(x, z) = c$.
• Any curve $h(y, z) = c$ defines a cylinder parallel to the x-axis whose space equation is also $h(y, z) = c$.
• The axis of a cylinder need not be parallel to a coordinate axis, however.

# Quadric Surfaces

• A **quadric surface** is the graph in space of a **second-degree equation** in x, y, and z.

• We first focus on quadric surfaces given by the equation

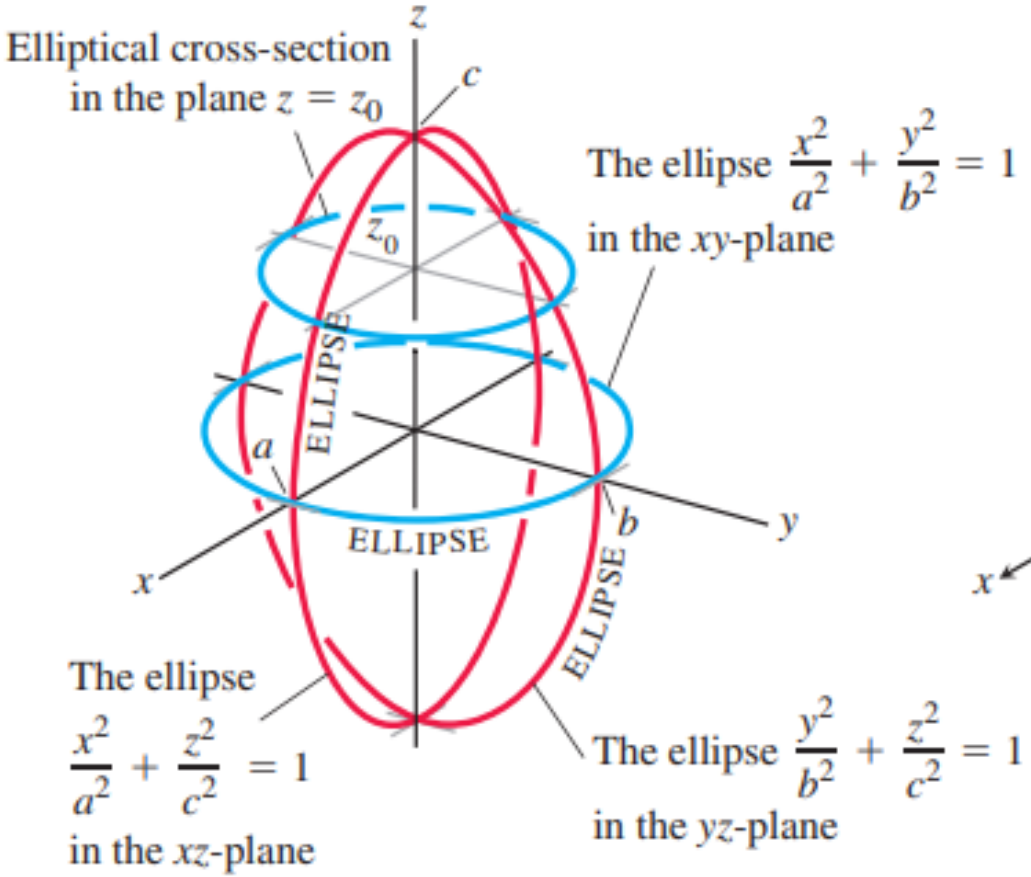
$$Ax^2 + By^2 + Cz^2 + Dz = E,$$

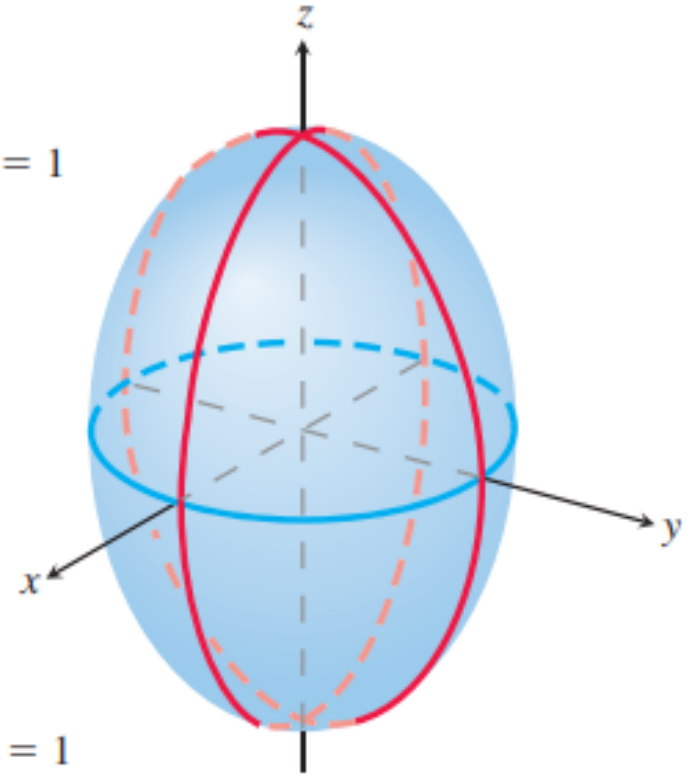
• where A, B, C, D, and E are constants.

• The basic quadric surfaces are **ellipsoids, paraboloids, elliptical cones, and hyperboloids**.

• Spheres are special cases of ellipsoids.

# Ellipsoid

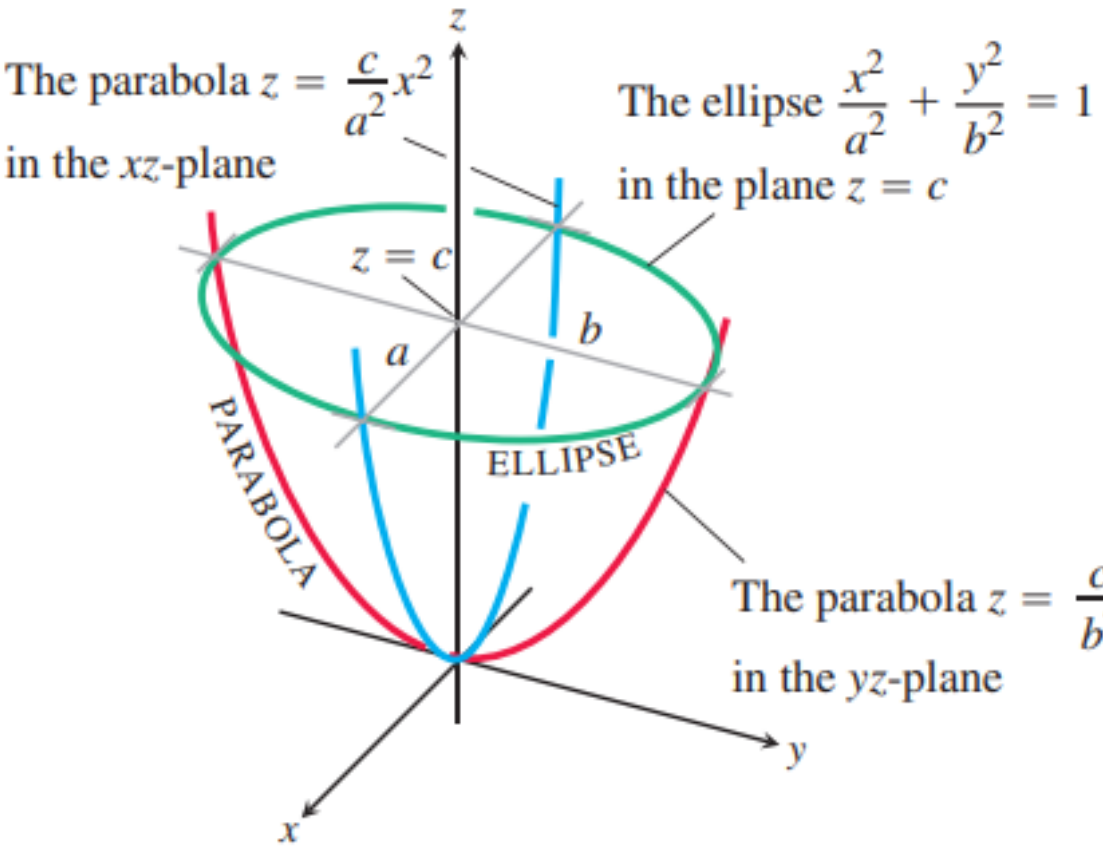


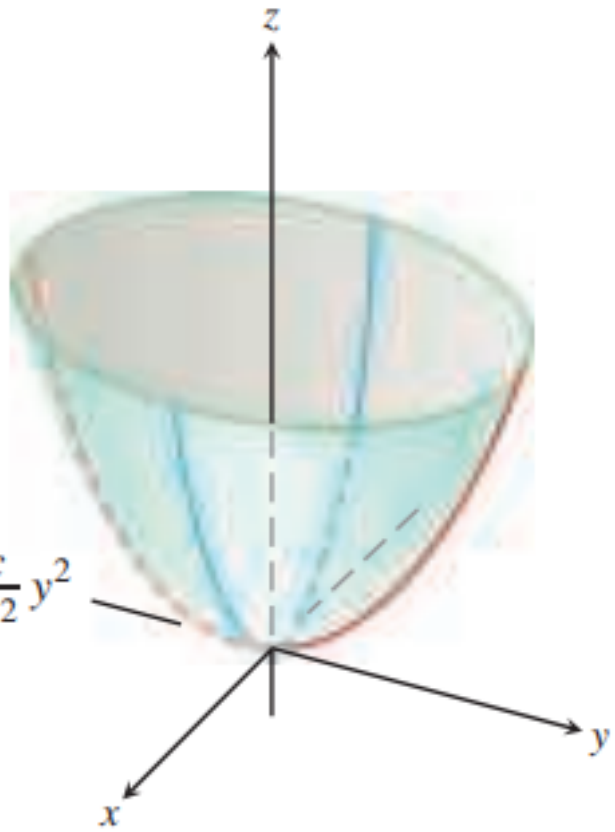




**ELLIPSOID**

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

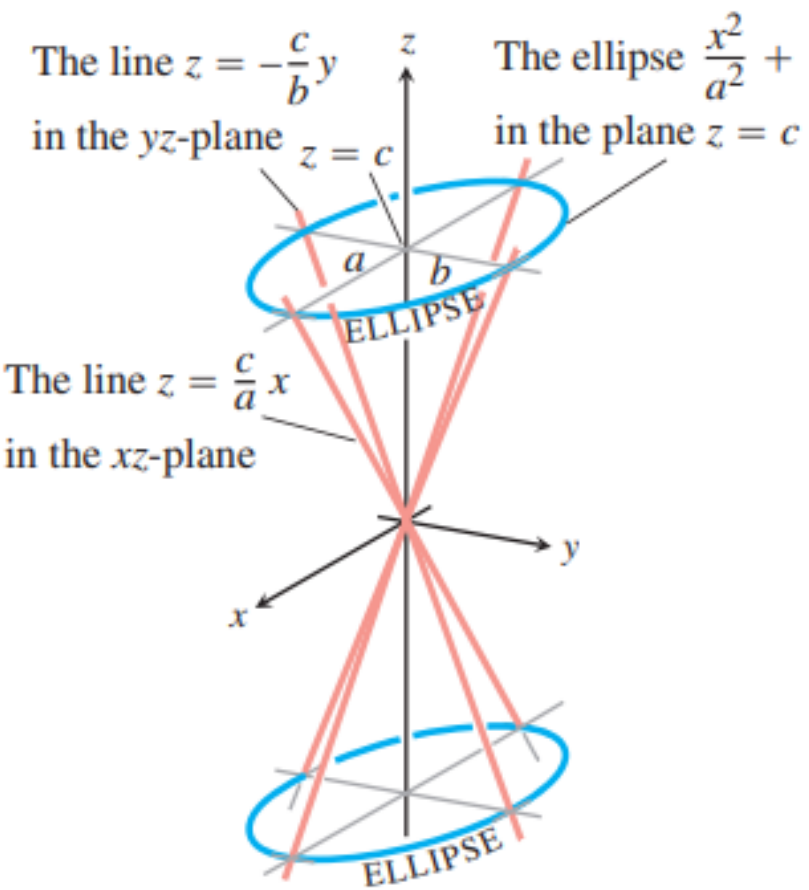
# Elliptical Paraboloid

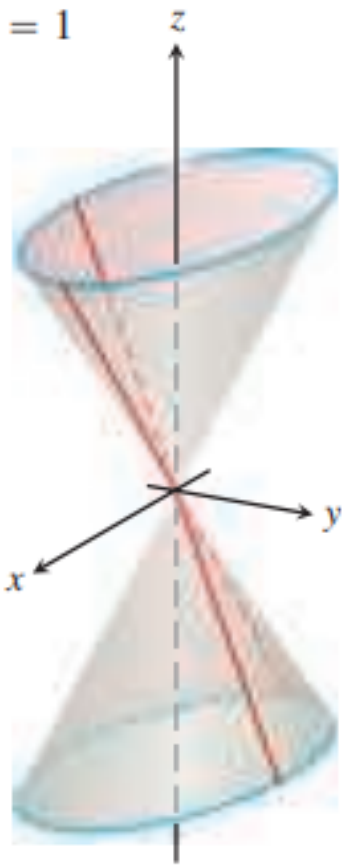



**ELLIPTICAL PARABOLOID** $\quad \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = \dfrac{z}{c}$

# Elliptical Cone

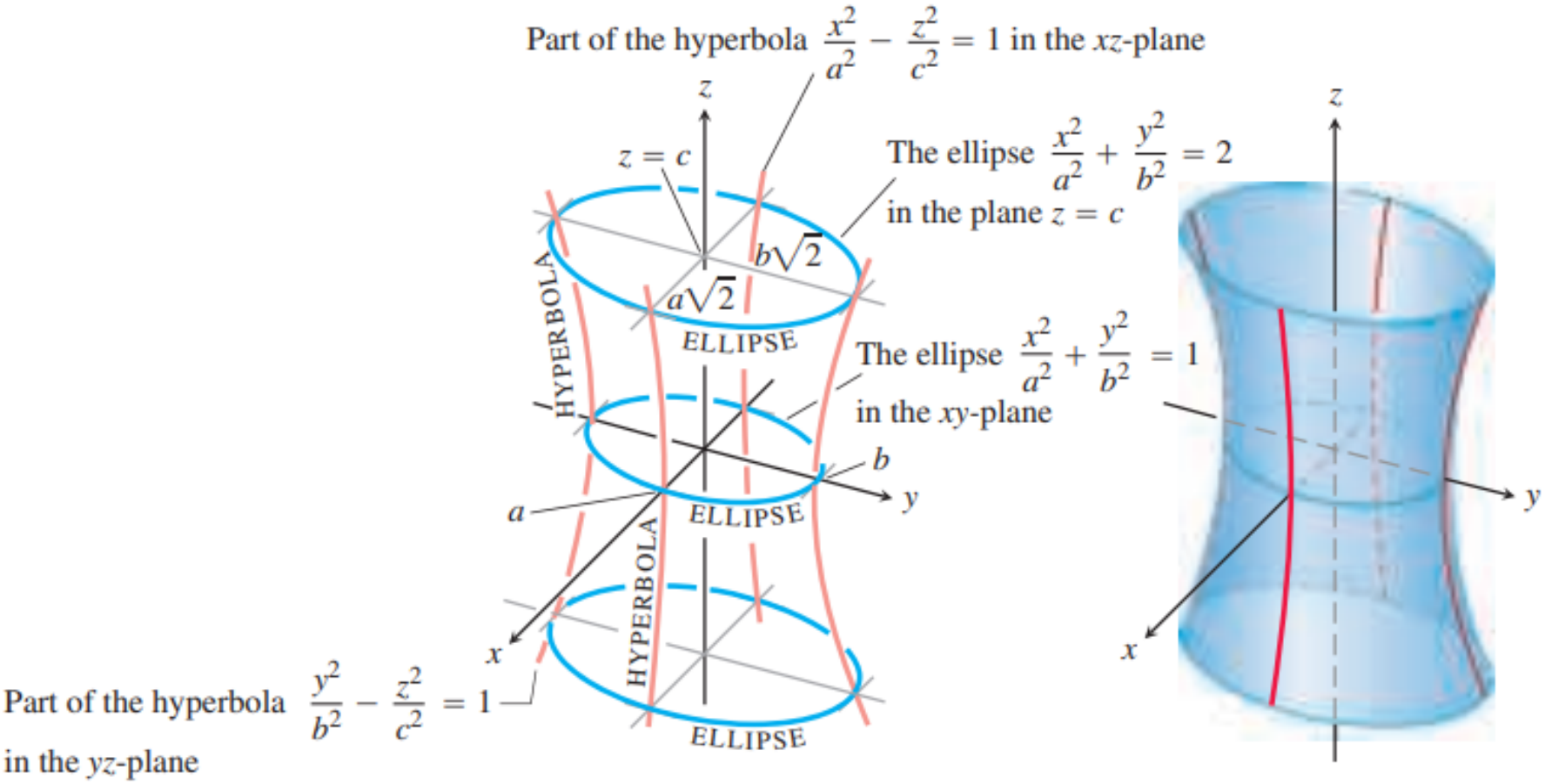



**ELLIPTICAL CONE**

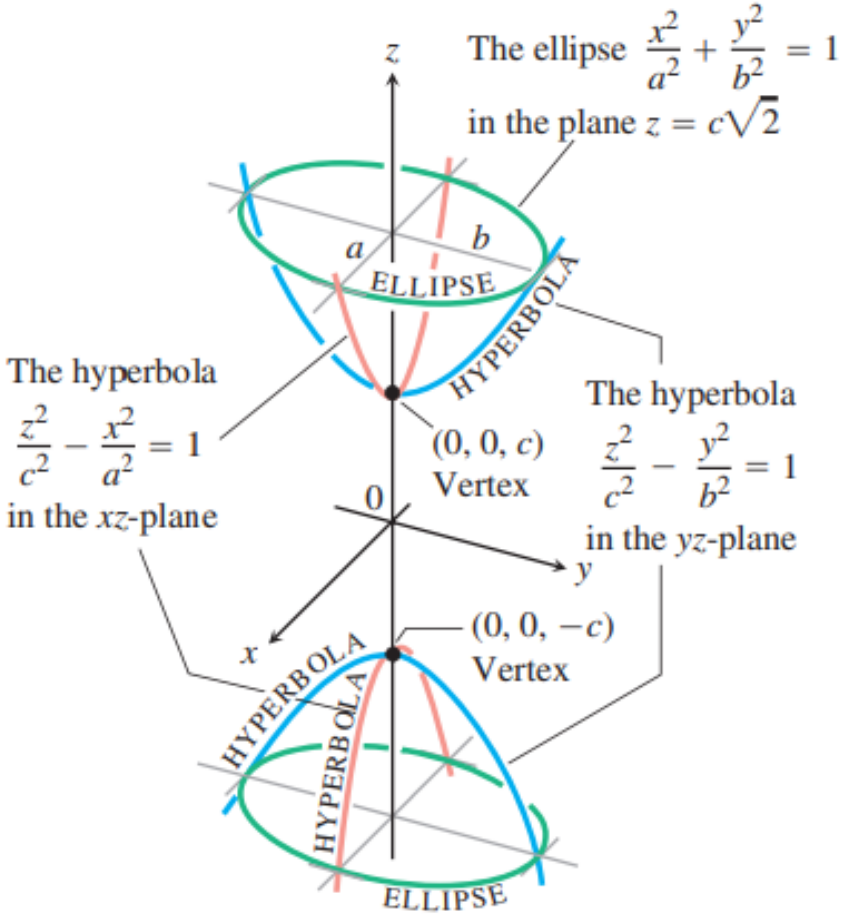
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}$$

# Hyperboloid of one Sheet



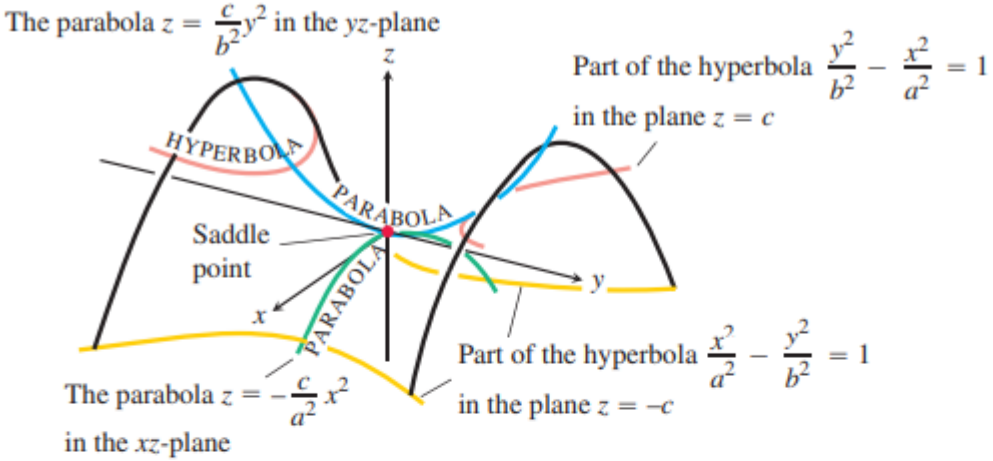

**HYPERBOLOID OF ONE SHEET** $\quad \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 1$

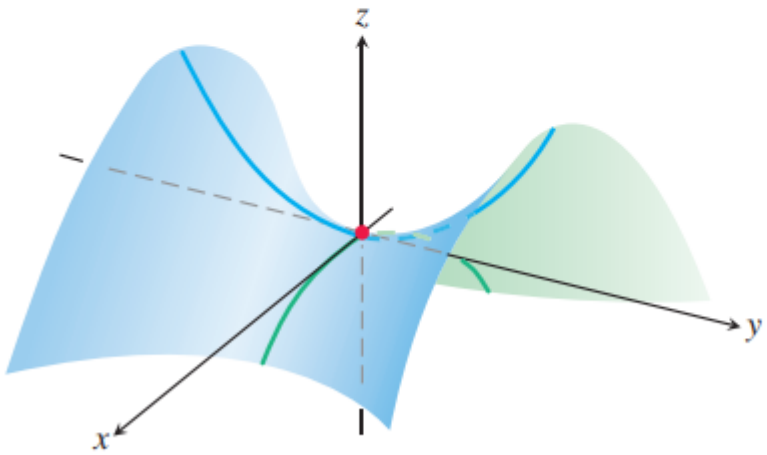
# Hyperboloid of two sheets



**HYPERBOLOID OF TWO SHEETS** $\frac{z^2}{c^2} - \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

# Hyperbolic Paraboloid




**HYPERBOLIC PARABOLOID** $\frac{y^2}{b^2} - \frac{x^2}{a^2} = \frac{z}{c}, c > 0$

# General Quadric Surfaces

• The quadric surfaces we have considered have **symmetries** relative to the x-, y-, or z-axes.

• The general equation of second degree in three variables x, y, z is

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gz + Hy + Iz + J = 0,$$

• where A, B, C, D, E, F, G, H, I, and J are constants.

• This equation leads to surfaces similar to those presented above, but in general these surfaces might be **translated and rotated** relative to the x-, y-, and z-axes.

• Terms of the type Gx, Hy, or Iz in the above formula **lead to translations**, which can be seen by a process of completing the square.

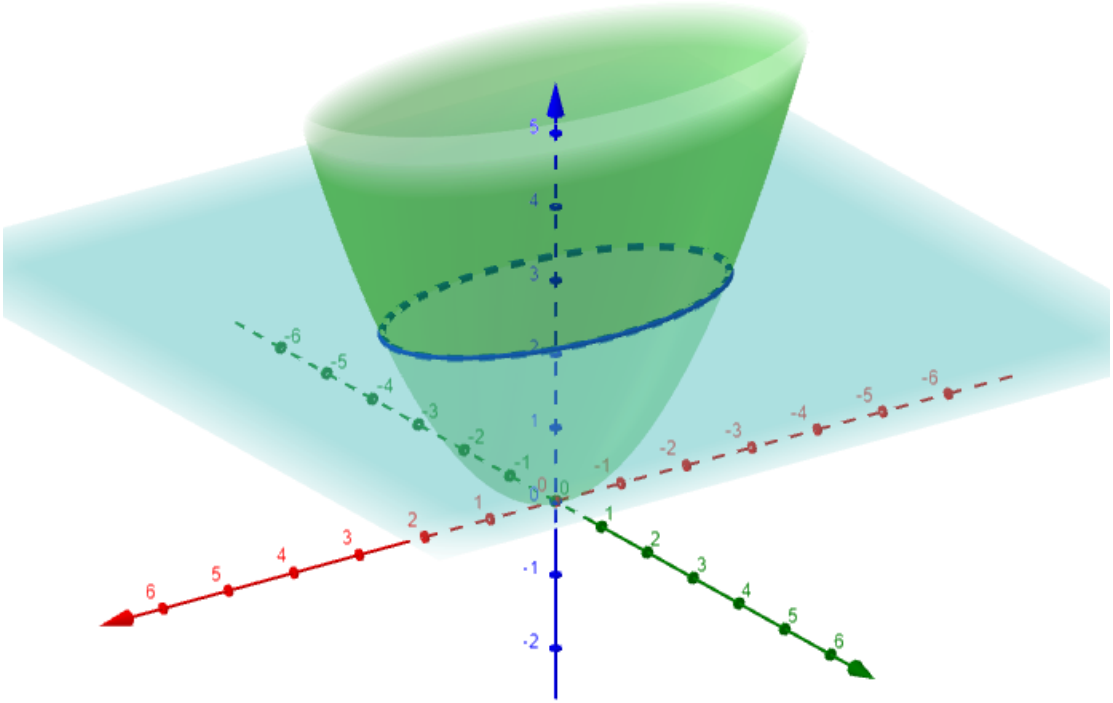
# Examples of Cylinders and Quadric Surfaces

## Example 1

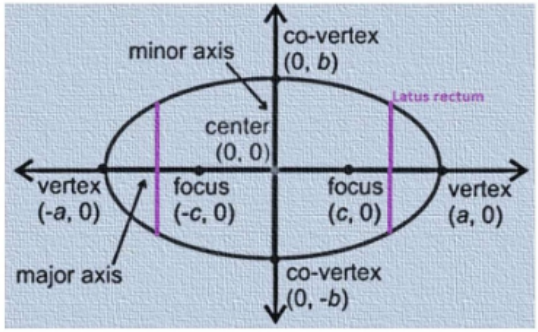
Show that the volume of the segment cut from the paraboloid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}$$

by the plane $z = h$ equals half the segment's base times its altitude.



# Solution for Example 1




The standard form of the equation of an ellipse with center $(0,0)$ is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

where

- $a>b$
- the length of the major axis is $2a$
- the coordinates of the vertices are $(\pm a, 0)$
- the length of the minor axis is $2b$
- the coordinates of the co-vertices are $(0, \pm b)$
- the coordinates of the foci are $(\pm c, 0)$, where $c^2=a^2-b^2$

## Ellipse Equation

Area of ellipse $=\pi ab$

Perimeter of ellipse $=2\pi\sqrt{\frac{(a^2+b^2)}{2}}$

The standard form of the equation of an ellipse with center $(h,\ k)$ is

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$$

where

- $a>b$
- the length of the major axis is $2a$
- the coordinates of the vertices are $(h\pm a, k)$
- the length of the minor axis is $2b$
- the coordinates of the co-vertices are $(h, k\pm b)$
- the coordinates of the foci are $(h\pm c, k)$, where $c^2=a^2-b^2$

We calculate the volume by the slicing method, taking slices parallel to the $xy$-plane. For fixed $z$, $\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{z}{c}$ gives the ellipse $\frac{x^2}{\left(\frac{za^2}{c}\right)}+\frac{y^2}{\left(\frac{zb^2}{c}\right)}=1$. The area of this ellipse is $\pi\left(a\sqrt{\frac{z}{c}}\right)\left(b\sqrt{\frac{z}{c}}\right)=\frac{\pi abz}{c}$ (see Exercise 45a). Hence the volume is given by $V=\int_0^h \frac{\pi abz}{c}\,dz=\left[\frac{\pi abz^2}{2c}\right]_0^h=\frac{\pi abh^2}{2c}$. Now the area of the elliptic base when $z=h$ is $A=\frac{\pi abh}{c}$, as determined previously. Thus, $V=\frac{\pi abh^2}{2c}=\frac{1}{2}\left(\frac{\pi abh}{c}\right)h=\frac{1}{2}$ (base)(altitude), as claimed.

# Examples of Cylinders and Quadric Surfaces

## Example 2

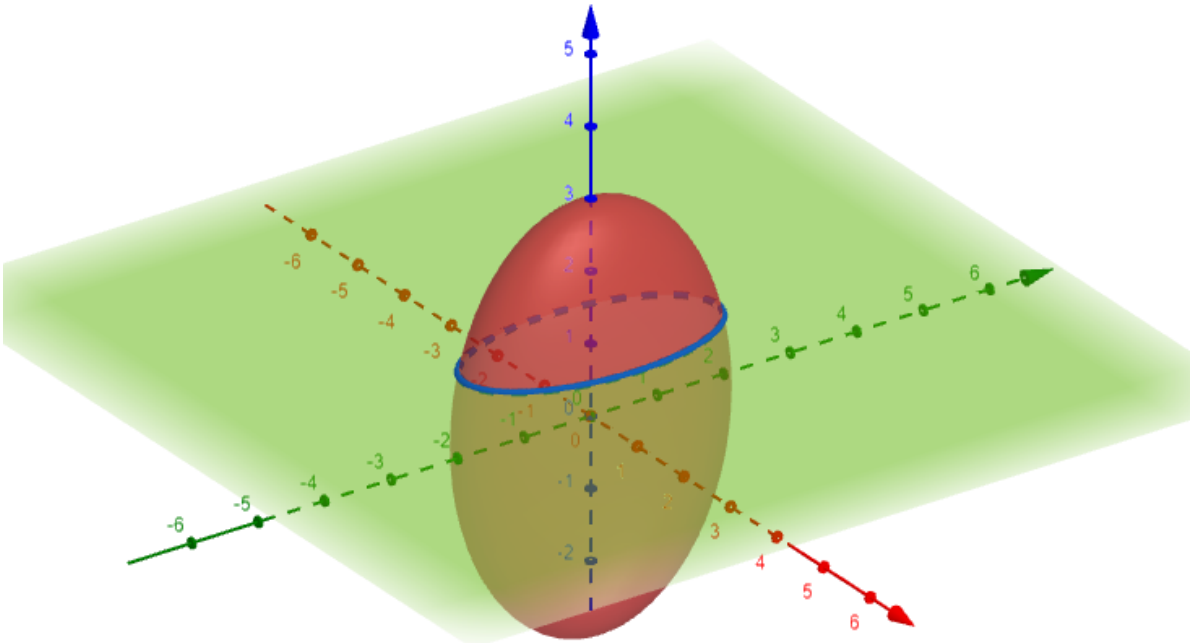
a) Express the area A of the cross-section cut from the ellipsoid

$$x^2 + \frac{y^2}{4} + \frac{z^2}{9} = 1$$

by the plane $z = c$ as a function of c.
b) Use slices perpendicular to the z-axis to find the volume of the ellipsoid in part (a).
c) Now find the volume of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

Does your formula give the volume of a sphere of radius a if a = b = c?



# Solution for Example 2

(a) If $x^2+\frac{y^2}{4}+\frac{z^2}{9}=1$ and $z=c$, then $x^2+\frac{y^2}{4}=\frac{9-c^2}{9} \Rightarrow \frac{x^2}{\left(\frac{9-c^2}{9}\right)}+\frac{y^2}{\left[\frac{4\left(9-c^2\right)}{9}\right]}=1 \Rightarrow A=ab\pi$

$$=\pi\left(\frac{\sqrt{9-c^2}}{3}\right)\left(\frac{2\sqrt{9-c^2}}{3}\right)=\frac{2\pi\left(9-c^2\right)}{9}$$

(b) From part (a), each slice has the area $\frac{2\pi\left(9-z^2\right)}{9}$, where $-3\le z\le 3$. Thus $V=2\int_0^3\frac{2\pi}{9}\left(9-z^2\right)dz$

$$=\frac{4\pi}{9}\int_0^3\left(9-z^2\right)dz=\frac{4\pi}{9}\left[9z-\frac{z^3}{3}\right]_0^3=\frac{4\pi}{9}(27-9)=8\pi$$

(c) $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \Rightarrow \frac{x^2}{\left[\frac{a^2\left(c^2-z^2\right)}{c^2}\right]}+\frac{y^2}{\left[\frac{b^2\left(c^2-z^2\right)}{c^2}\right]}=1 \Rightarrow A=\pi\left(\frac{a\sqrt{c^2-z^2}}{c}\right)\left(\frac{b\sqrt{c^2-z^2}}{c}\right)$

$\Rightarrow V=2\int_0^c\frac{\pi ab}{c^2}\left(c^2-z^2\right)dz=\frac{2\pi ab}{c^2}\left[c^2z-\frac{z^3}{3}\right]_0^c=\frac{2\pi ab}{c^2}\left(\frac{2}{3}c^3\right)=\frac{4\pi abc}{3}$. Note that if $r=a=b=c$, then $V=\frac{4\pi r^3}{3}$, which is the volume of a sphere.

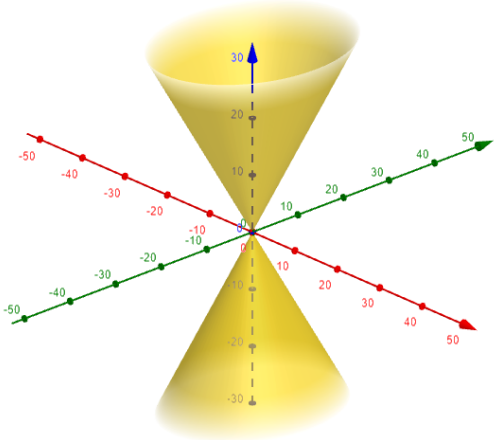
# Examples of Cylinders and Quadric Surfaces

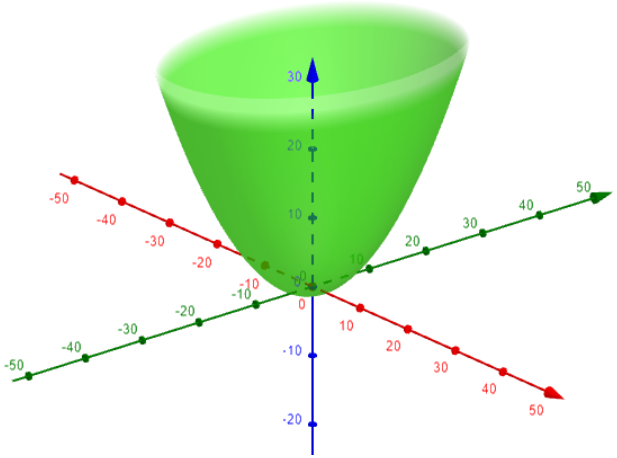
## Example 3

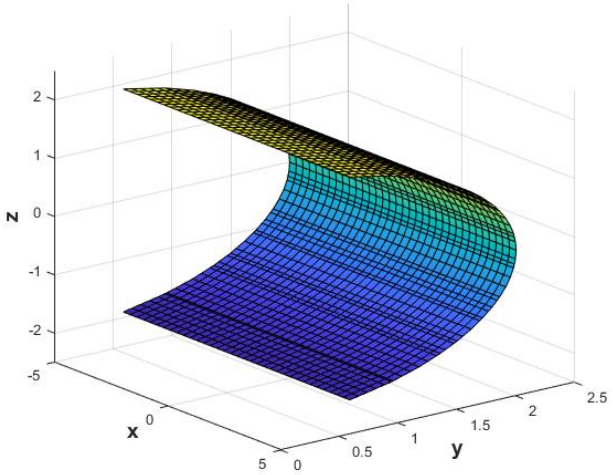
Use a Geogebra to plot the surfaces. Identify the type of quadric surface from your graph.

a) $5x^2 = z^2 - 3y^2$, b) $\frac{y^2}{16} = 1 - \frac{x^2}{9} + z$, c) $y - \sqrt{4 - z^2} = 0$.

Solution








a) Elliptical cone, b) elliptical paraboloid, c) half of cilinder.

# Lecture 7: Vector-Valued Functions and Motion in Space.

MA2032 Vector Calculus

Lecturer: Larissa Serdukova

School of Computing and Mathematical Science
University of Leicester

October 5, 2022

# Vector-Valued Functions and Motion in Space. Overview.

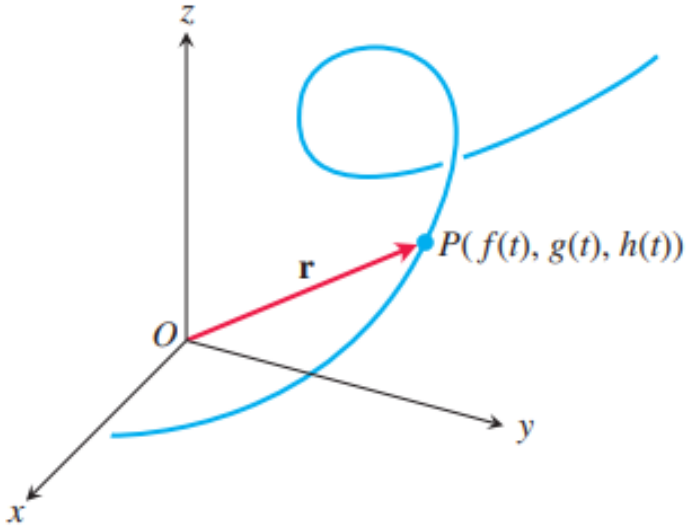
- In the next lectures I will introduce the calculus of **vector-valued functions**.
- The **domains** of these functions are **sets of real numbers**, as before, but their **ranges** consist of **vectors instead of scalars**.
- When a vector-valued function changes, the change can occur in both magnitude and direction, so the **derivative is itself a vector**.
- The **integral** of a vector-valued function is also **a vector**.
- We use the calculus of these functions to **describe the paths and motions of objects** moving in a plane or in space, so their velocities and accelerations are given by vectors.
- I also will introduce new concepts that **quantify** the way that the path of an object moving in space can **twist and turn**.

# Curves in Space and Their Tangents

• When a particle moves through space during a time interval I, we think of the **particle's coordinates** as functions defined on I:

$x = f(t),\ y = g(t),\ z = h(t), t \in I$ (Eq. 1)

• The points $(x, y, z) = ((t), g(t), h(t)),\ t \in I$, make up the curve in space that we call the **particle's path**.




• The equations and interval in Eq. 1 **parametrize the curve**.

• A curve in space can also be represented in **vector form**. The vector

$$r(t) = \overrightarrow{OP} = f(t)i + g(t)j + h(t)k$$

from the origin to the particle's position $P(f(t), g(t), h(t))$ at time $t$ is the **particle's position vector**.

• The functions $f, g$, and $h$ are the **component functions** (or components) of the position vector.

• We think of the particle's **path as the curve traced by r** during the time interval I.

# Limits

• The way we define limits of vector-valued functions is similar to the way we define limits of real-valued functions.

**DEFINITION** Let $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ be a vector function with domain $D$, and let $\mathbf{L}$ be a vector. We say that $\mathbf{r}$ has **limit** $\mathbf{L}$ as $t$ approaches $t_0$ and write

$$\lim_{t \to t_0} \mathbf{r}(t) = \mathbf{L}$$

if, for every number $\varepsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all $t \in D$

$$|\mathbf{r}(t) - \mathbf{L}| < \varepsilon \quad \text{whenever} \quad 0 < |t - t_0| < \delta.$$

• If $L = L_1 i + L_2 j + L_3 k$, then it can be shown that $\lim_{t \to t_0} r(t) = L$ precisely when $\lim_{t \to t_0} f(t) = L_1$, $\lim_{t \to t_0} g(t) = L_2$, and $\lim_{t \to t_0} h(t) = L_3$

• The equation

$$\lim_{t \to t_0} r(t) = \left(\lim_{t \to t_0} f(t)\right) i + \left(\lim_{t \to t_0} g(t)\right) j + \left(\lim_{t \to t_0} h(t)\right) k$$

provides a practical way to calculate limits of vector functions.

# Continuity

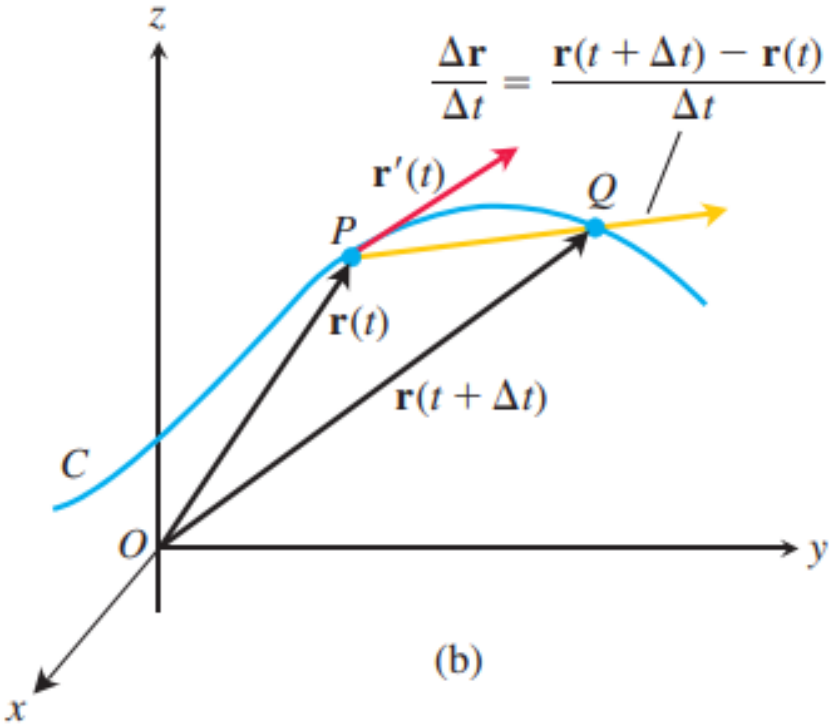
• We define continuity for vector functions the same way we define continuity for scalar functions defined over an interval.

**DEFINITION** A vector function $\mathbf{r}(t)$ is **continuous at a point** $t = t_0$ in its domain if $\lim_{t \to t_0} \mathbf{r}(t) = \mathbf{r}(t_0)$. The function is **continuous** if it is continuous at every point in its domain.

# Derivatives and Motion

**DEFINITION** The vector function $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ has a **derivative (is differentiable) at $t$** if $f$, $g$, and $h$ have derivatives at $t$. The derivative is the vector function

$$\mathbf{r}'(t) = \frac{d\mathbf{r}}{dt} = \lim_{\Delta t \to 0} \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t} = \frac{df}{dt}\mathbf{i} + \frac{dg}{dt}\mathbf{j} + \frac{dh}{dt}\mathbf{k}.$$

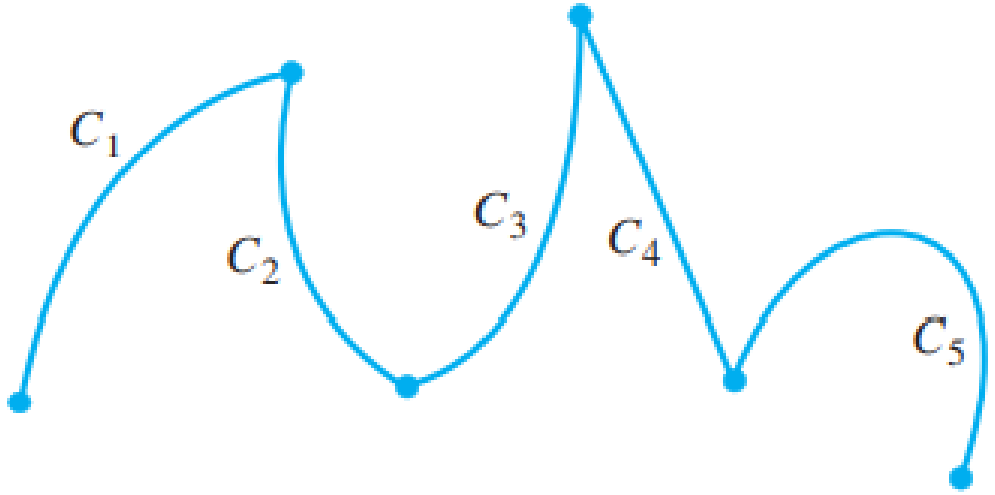


(b)

- As $\Delta t \to 0$, the point Q approaches the point P along the curve C.
- In the limit, the vector $\frac{\overrightarrow{PQ}}{\Delta t}$ becomes the tangent vector $r'(t)$.
- A vector function $r$ is differentiable if it is **differentiable** at every point of its domain.

# Derivatives and Motion

• The curve traced by r is **smooth** if $dr/dt$ is continuous and never 0, that is, if $f, g$, and $h$ have continuous first derivatives that are not simultaneously 0.
• On a smooth curve, there are no sharp, corners or cusps.
• A curve that is made up of a finite number of smooth curves pieced together in a continuous fashion is called **piecewise smooth**.



# Motion

**DEFINITIONS** If $\mathbf{r}$ is the position vector of a particle moving along a smooth curve in space, then

$$\mathbf{v}(t) = \frac{d\mathbf{r}}{dt}$$

is the particle's **velocity vector**, tangent to the curve. At any time $t$, the direction of $\mathbf{v}$ is the **direction of motion**, the magnitude of $\mathbf{v}$ is the particle's **speed**, and the derivative $\mathbf{a} = d\mathbf{v}/dt$, when it exists, is the particle's **acceleration vector**. In summary,

1. Velocity is the derivative of position: $\mathbf{v} = \dfrac{d\mathbf{r}}{dt}$.
2. Speed is the magnitude of velocity: Speed $= |\mathbf{v}|$.
3. Acceleration is the derivative of velocity: $\mathbf{a} = \dfrac{d\mathbf{v}}{dt} = \dfrac{d^2\mathbf{r}}{dt^2}$.
4. The unit vector $\mathbf{v}/|\mathbf{v}|$ is the direction of motion at time $t$.

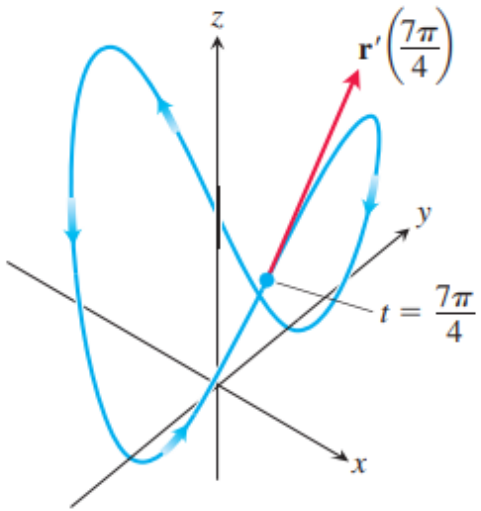
## Example 1

Find the velocity, speed, and acceleration of a particle whose motion in space is given by the position vector $\boldsymbol{r}(t) = 2\cos t\boldsymbol{i} + 2\sin t\boldsymbol{j} + 5\cos^2 t\boldsymbol{k}$. Sketch the velocity vector $\boldsymbol{v}(7\pi/4)$.

**Solution**

The velocity and acceleration vectors at time t are

$$\begin{aligned}\mathbf{v}(t) = \mathbf{r}'(t) &= -2\sin t\,\mathbf{i} + 2\cos t\,\mathbf{j} - 10\cos t\sin t\,\mathbf{k}\\ &= -2\sin t\,\mathbf{i} + 2\cos t\,\mathbf{j} - 5\sin 2t\,\mathbf{k},\\ \mathbf{a}(t) = \mathbf{r}''(t) &= -2\cos t\,\mathbf{i} - 2\sin t\,\mathbf{j} - 10\cos 2t\,\mathbf{k},\end{aligned}$$




and the speed is

$$|\mathbf{v}(t)| = \sqrt{(-2\sin t)^2 + (2\cos t)^2 + (-5\sin 2t)^2} = \sqrt{4 + 25\sin^2 2t}.$$

When $t = 7\pi/4$, we have

$$\mathbf{v}\left(\frac{7\pi}{4}\right) = \sqrt{2}\mathbf{i} + \sqrt{2}\mathbf{j} + 5\mathbf{k}, \qquad \mathbf{a}\left(\frac{7\pi}{4}\right) = -\sqrt{2}\mathbf{i} + \sqrt{2}\mathbf{j}, \qquad \left|\mathbf{v}\left(\frac{7\pi}{4}\right)\right| = \sqrt{29}.$$

# Differentiation Rules

• Because the derivatives of vector functions may be computed component by component, the rules for differentiating vector functions have the **same form** as the rules for differentiating scalar functions.

**Differentiation Rules for Vector Functions**

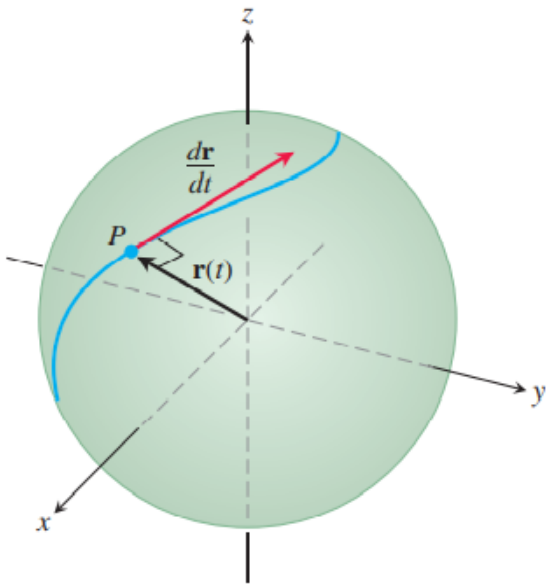
Let $\mathbf{u}$ and $\mathbf{v}$ be differentiable vector functions of $t$, $\mathbf{C}$ a constant vector, $c$ any scalar, and $f$ any differentiable scalar function.

1. *Constant Function Rule:* $\frac{d}{dt}\mathbf{C} = \mathbf{0}$
2. *Scalar Multiple Rules:* $\frac{d}{dt}[c\mathbf{u}(t)] = c\mathbf{u}'(t)$

   $\frac{d}{dt}[f(t)\mathbf{u}(t)] = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t)$
3. *Sum Rule:* $\frac{d}{dt}[\mathbf{u}(t) + \mathbf{v}(t)] = \mathbf{u}'(t) + \mathbf{v}'(t)$
4. *Difference Rule:* $\frac{d}{dt}[\mathbf{u}(t) - \mathbf{v}(t)] = \mathbf{u}'(t) - \mathbf{v}'(t)$
5. *Dot Product Rule:* $\frac{d}{dt}[\mathbf{u}(t)\cdot\mathbf{v}(t)] = \mathbf{u}'(t)\cdot\mathbf{v}(t) + \mathbf{u}(t)\cdot\mathbf{v}'(t)$
6. *Cross Product Rule:* $\frac{d}{dt}[\mathbf{u}(t)\times\mathbf{v}(t)] = \mathbf{u}'(t)\times\mathbf{v}(t) + \mathbf{u}(t)\times\mathbf{v}'(t)$
7. *Chain Rule:* $\frac{d}{dt}[\mathbf{u}(f(t))] = f'(t)\mathbf{u}'(f(t))$

# Vector Functions of Constant Length

• When we track a **particle moving on a sphere** centered at the origin shown on Fig., the **position vector has a constant length** equal to the radius of the sphere.
• The velocity vector $dr/dt$, **tangent to the path of motion**, is tangent to the sphere and hence perpendicular to r.
• This is always the case for a **differentiable vector function of constant length: The vector and its first derivative are orthogonal**.




$$\mathbf{r}(t)\cdot\mathbf{r}(t) = |\mathbf{r}(t)|^2 = c^2 \qquad |\mathbf{r}(t)| = c \text{ is constant.}$$

$$\frac{d}{dt}[\mathbf{r}(t)\cdot\mathbf{r}(t)] = 0 \qquad \text{Differentiate both sides.}$$

$$\mathbf{r}'(t)\cdot\mathbf{r}(t) + \mathbf{r}(t)\cdot\mathbf{r}'(t) = 0 \qquad \text{Rule 5 with } \mathbf{r}(t) = \mathbf{u}(t) = \mathbf{v}(t)$$

$$2\mathbf{r}'(t)\cdot\mathbf{r}(t) = 0.$$

# Motion along a circle

## Example 2

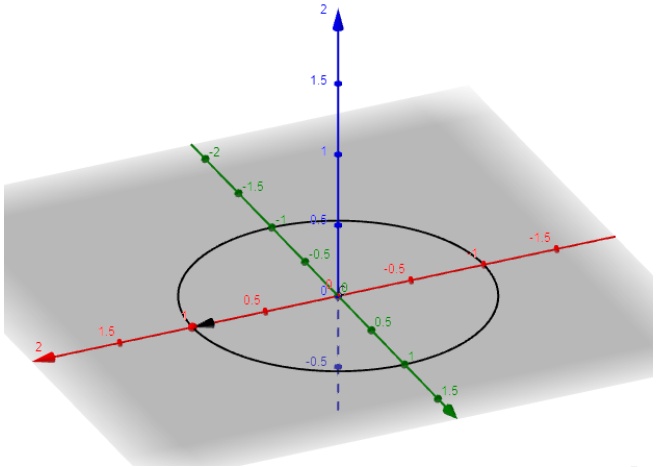
Equation $r(t) = \cos(2t)i + \sin(2t)j, t \geq 0$ describes the motion of a particle having the unit circle $x^2 + y^2 = 1$ path. Answer the following questions about the particle's "dynamics".

i) Does the particle have constant speed? If so, what is its constant speed?

ii) Is the particle's acceleration vector always orthogonal to its velocity vector?

iii) Does the particle move clockwise or counterclockwise around the circle?

iv) Does the particle begin at the point (1, 0)?



# Example 2. Solution.

$\mathbf{v}(t) = -(2\sin 2t)\mathbf{i} + (2\cos 2t)\mathbf{j} \Rightarrow \mathbf{a}(t) = -(4\cos 2t)\mathbf{i} - (4\sin 2t)\mathbf{j};$

(i) $|\mathbf{v}(t)| = \sqrt{4\sin^2 2t + 4\cos^2 2t} = 2 \Rightarrow$ constant speed;

(ii) $\mathbf{v}\cdot\mathbf{a} = 8\sin 2t\cos 2t - 8\cos 2t\sin 2t = 0 \Rightarrow$ yes, orthogonal;

(iii) counterclockwise movement;

(iv) yes, $\mathbf{r}(0) = \mathbf{i} + 0\mathbf{j}$

Direction of motion:

- We will look at $r'(t) \times r''(t)$. This vector is perpendicular to the plane.
- We choose a right framework in 3D, it is parallel to third unit vector k.
- If its third $k$ coordinate is positive, the motion is counterclockwise, otherwise it is clockwise.
- This is because $r(t)$ always points to the side where the center of curvature is located.

$r'(t) \times r''(t) =$

$$\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ -2\sin 2t & 2\cos 2t & 0 \\ -4\cos 2t & -4\sin 2t & 0 \end{vmatrix} = 0i + 0j + (8\sin^2 2t + 8\cos^2 2t) = 8$$

# Example 3

An object is moving with position function $\vec{r}(t) = \langle t^2 - t, t^2 + t \rangle$, $-3 \le t \le 3$, where distances are measured in feet and time is measured in seconds.

(a) Find $\vec{v}(t)$ and $\vec{a}(t)$.
(b) Sketch $\vec{r}(t)$; plot $\vec{v}(-1)$, $\vec{a}(-1)$, $\vec{v}(1)$ and $\vec{a}(1)$, each with their initial point at their corresponding point on the graph of $\vec{r}(t)$.
(c) When is the object's speed minimized?

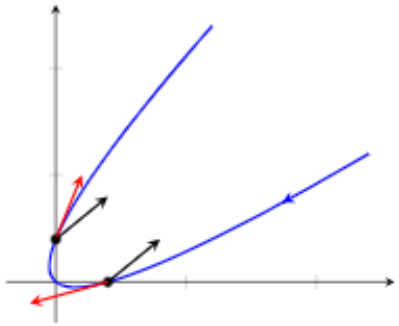
## Solution:

(a) Taking derivatives, we find

$$\vec{v}(t) = \vec{r}'(t) = \langle 2t - 1, 2t + 1 \rangle \quad \text{and} \quad \vec{a}(t) = \vec{r}''(t) = \langle 2, 2 \rangle.$$

Note that acceleration is constant.

(b) $\vec{v}(-1) = \langle -3, -1 \rangle$, $\vec{a}(-1) = \langle 2, 2 \rangle$; $\vec{v}(1) = \langle 1, 3 \rangle$, $\vec{a}(1) = \langle 2, 2 \rangle$. These are plotted with $\vec{r}(t)$



(a)

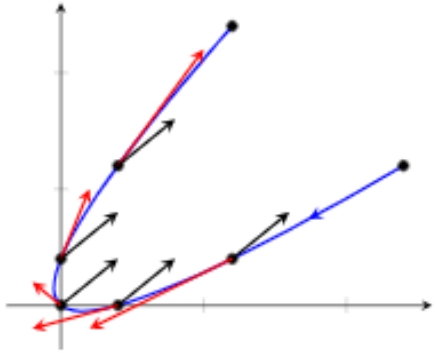
# Example 3. Solution.

We can think of acceleration as "pulling" the velocity vector in a certain direction. At $t = -1$, the velocity vector points down and to the left; at $t = 1$, the velocity vector has been pulled in the $\langle 2, 2\rangle$ direction and is now pointing up and to the right. In Figure 12.3.1(b) we plot more velocity/acceleration vectors, making more clear the effect acceleration has on velocity. Since $\vec{a}(t)$ is constant in this example, as $t$ grows large $\vec{v}(t)$ becomes almost parallel to $\vec{a}(t)$. For instance, when $t = 10$, $\vec{v}(10) = \langle 19, 21\rangle$, which is nearly parallel to $\langle 2, 2\rangle$.

(c) The object's speed is given by

$$\|\vec{v}(t)\| = \sqrt{(2t-1)^2 + (2t+1)^2} = \sqrt{8t^2 + 2}.$$

To find the minimal speed, we could apply calculus techniques (such as set the derivative equal to 0 and solve for $t$, etc.) but we can find it by inspection. Inside the square root we have a quadratic which is minimized when $t = 0$. Thus the speed is minimized at $t = 0$, with a speed of $\sqrt{2}$ ft/s. The graph in Figure 12.3.1(b) also implies speed is minimized here. The filled dots on the graph are located at integer values of $t$ between $-3$ and 3. Dots that are far apart imply the object traveled a far distance in 1 second, indicating high speed; dots that are close together imply the object did not travel far in 1 second, indicating a low speed. The dots are closest together near $t = 0$, implying the speed is minimized near that value.



(b)

# Lecture 8: Vector-Valued Functions and Motion in Space.

MA2032 Vector Calculus

Lecturer: Larissa Serdukova

School of Computing and Mathematical Science
University of Leicester

October 7, 2022

# Integrals of Vector Functions

• Today we investigate **integrals of vector functions** and their **application to motion** along a path in space or in the plane.

• A differentiable vector function $\mathbf{R}(t)$ is an **antiderivative** of a vector function $\mathbf{r}(t)$ on an interval $I$ if $d\mathbf{R}/dt = \mathbf{r}$ at each point of $I$.

• If $\mathbf{R}$ is an antiderivative of $\mathbf{r}$ on $I$, it can be shown, working one component at a time, that every antiderivative of $\mathbf{r}$ on $I$ has the form $\mathbf{R} + \mathbf{C}$ for some constant vector $\mathbf{C}$.

• The set of all antiderivatives of $\mathbf{r}$ on $I$ is the **indefinite integral** of $\mathbf{r}$ on $I$.

DEFINITION The **indefinite integral** of $\mathbf{r}$ with respect to $t$ is the set of all antiderivatives of $\mathbf{r}$, denoted by $\int \mathbf{r}(t)\,dt$. If $\mathbf{R}$ is any antiderivative of $\mathbf{r}$, then

$$\int \mathbf{r}(t)\,dt = \mathbf{R}(t) + \mathbf{C}.$$

# Example 1

• To integrate a vector function, we integrate each of its components.

**Example 1:**

$$\int ((\cos t)\mathbf{i} + \mathbf{j} - 2t\mathbf{k})\,dt = \left(\int \cos t\,dt\right)\mathbf{i} + \left(\int dt\right)\mathbf{j} - \left(\int 2t\,dt\right)\mathbf{k} \qquad (1)$$

$$= (\sin t + C_1)\mathbf{i} + (t + C_2)\mathbf{j} - (t^2 + C_3)\mathbf{k} \qquad (2)$$

$$= (\sin t)\mathbf{i} + t\mathbf{j} - t^2\mathbf{k} + \mathbf{C} \qquad C = C_1\mathbf{i} + C_2\mathbf{j} - C_3\mathbf{k}$$

• **Definite integrals** of vector functions are best defined in terms of **components**.

• The definition is consistent with how we compute limits and derivatives of vector functions.

**DEFINITION** If the components of $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ are integrable over $[a, b]$, then so is $\mathbf{r}$, and the **definite integral** of $\mathbf{r}$ from $a$ to $b$ is

$$\int_a^b \mathbf{r}(t)\,dt = \left(\int_a^b f(t)\,dt\right)\mathbf{i} + \left(\int_a^b g(t)\,dt\right)\mathbf{j} + \left(\int_a^b h(t)\,dt\right)\mathbf{k}.$$

# Integrals of Vector Functions

• The **Fundamental Theorem of Calculus** for continuous vector functions says that

$$\int_a^b \mathbf{r}(t)\,dt = \mathbf{R}(t)\Big]_a^b = \mathbf{R}(b) - \mathbf{R}(a)$$

• where $\mathbf{R}$ is any antiderivative of $\mathbf{r}$, so that $\mathbf{R}'(t) = \mathbf{r}(t)$.

• Notice that an **antiderivative** of a vector function is also a **vector function**, whereas a **definite integral** of a vector function is a **single constant vector**.

**Example 2:**

• As in Example 1, we integrate each component:

$$\begin{aligned}\int_0^\pi ((\cos t)\mathbf{i} + \mathbf{j} - 2t\mathbf{k})\,dt &= \left(\int_0^\pi \cos t\,dt\right)\mathbf{i} + \left(\int_0^\pi dt\right)\mathbf{j} - \left(\int_0^\pi 2t\,dt\right)\mathbf{k}\\ &= \Big[\sin t\Big]_0^\pi \mathbf{i} + \Big[t\Big]_0^\pi \mathbf{j} - \Big[t^2\Big]_0^\pi \mathbf{k}\\ &= [0-0]\mathbf{i} + [\pi - 0]\mathbf{j} - [\pi^2 - 0^2]\mathbf{k}\\ &= \pi\mathbf{j} - \pi^2\mathbf{k}\end{aligned}$$

# Arc Length Along a Space Curve

• We study the mathematical **features of a curve's shape** that describe the **sharpness** of its **turning** and its **twisting**.

• One of the **features** of smooth space and plane curves is that they have a **measurable length**. This enables us to locate points along these curves by giving their **directed distance** $s$ along the curve from some base point.

**DEFINITION** The **length** of a smooth curve $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$, $a \le t \le b$, that is traced exactly once as $t$ increases from $t = a$ to $t = b$, is

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt. \qquad (1)$$

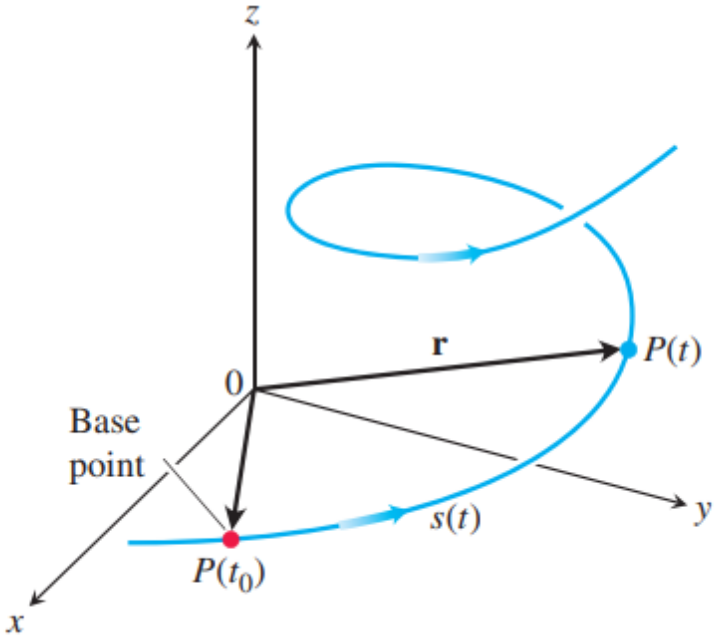
**Arc Length Formula**: $L = \int_b^a |v| dt$.

# Arc Length in Space

• If we choose a **base point** $P(t_0)$ on a smooth curve C parametrized by t, each value of t determines a point $P(t) = (x(t), y(t), z(t))$ on C and a **"directed distance"**

$$s(t) = \int_{t_0}^{t} \sqrt{[x'(\tau)]^2 + [y'(\tau)]^2 + [z'(\tau)]^2}\, d\tau = \int_{t_0}^{t} |\mathbf{v}(\tau)|\, d\tau$$

measured along C from the base point.




- If $t > t_0$, $s(t)$ is the **distance** along the curve from $P(t_0)$ to $P(t)$.
- If $t < t_0$, $s(t)$ is the **negative of the distance**.
- We call $s$ an **arc length parameter** for the curve.
- The parameter's value increases in the direction of increasing t.

# Unit Tangent Vector

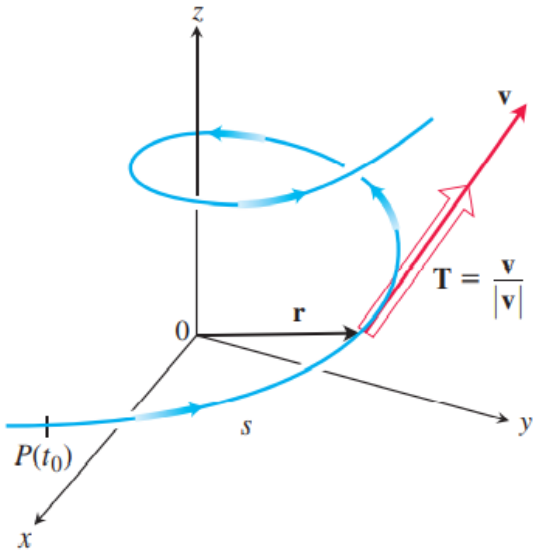
• We already know the velocity vector $\mathbf{v} = d\mathbf{r}/dt$ is tangent to the curve $\mathbf{r}(t)$ and that the vector

$$\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|}$$

is therefore a unit vector tangent to the (smooth) curve, called the **unit tangent vector**.

• The unit tangent vector $\mathbf{T}$ is a differentiable function of t whenever $\mathbf{v}$ is a differentiable function of t.

• $\mathbf{v}$ is one of the vectors in a traveling reference frame that is used to **describe the motion** of objects traveling **in three dimensions**.



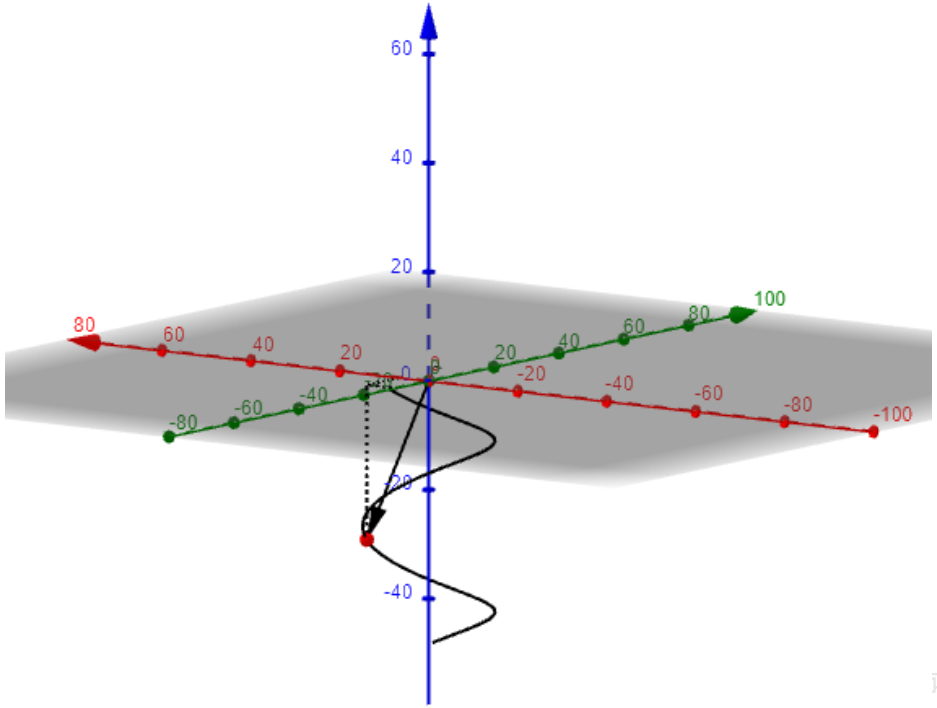
# Arc Length in Space

## Example 2

Find the point on the curve

$$r(t) = (12\sin t)i - (12\cos t)j + 5tk$$

at a distance $13\pi$ units along the curve from the point $(0, -12, 0)$ in the direction opposite to the direction of increasing arc length.



# Solution for Example 2

**Solution:**

Let $P(t_0)$ denote the point. Then $\mathbf{v} = (12\cos t)\mathbf{i} + (12\sin t)\mathbf{j} + 5\mathbf{k}$ and

$-13\pi = \int_0^{t_0} \sqrt{144\cos^2 t + 144\sin^2 t + 25}\, dt = \int_0^{t_0} 13\, dt = 13t_0 \Rightarrow t_0 = -\pi,$ and the point is

$P(-\pi) = (12\sin(-\pi), -12\cos(-\pi), -5\pi) = (0, 12, -5\pi)$

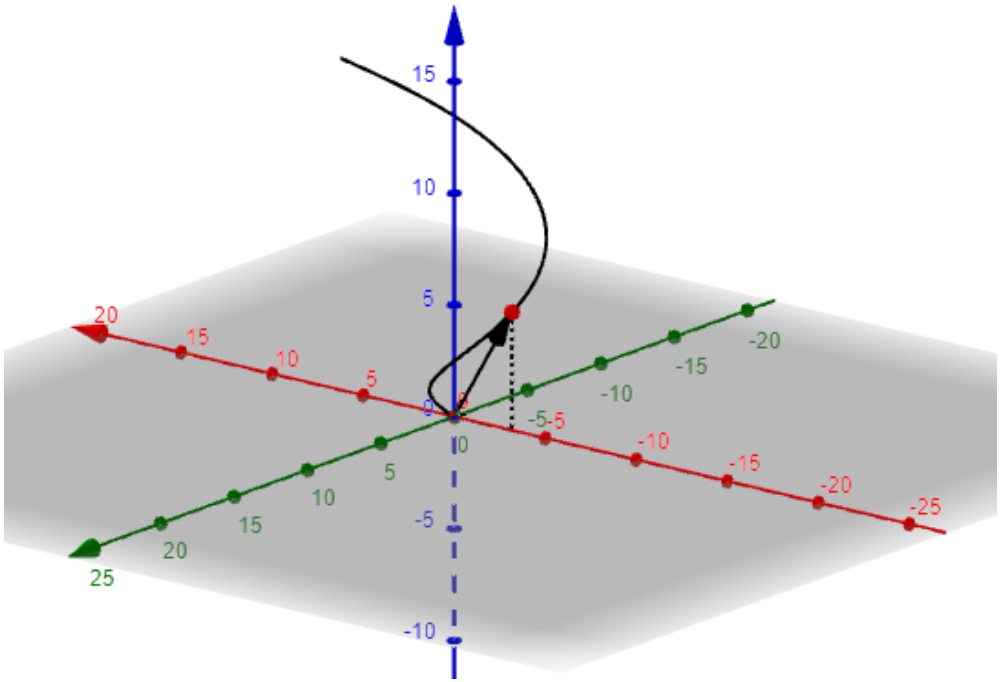
# Arc Length in Space

## Example 3

Find the curve's

$$r(t) = (t\cos t)i + (t\sin t)j + 2\sqrt{2}/3t^{3/2}k, \quad 0 \le t \le \pi$$

unit tangent vector. Also, find the length of the indicated portion of the curve.



# Solution for Example 3

**Solution:**

$$\mathbf{r} = (t\cos t)\mathbf{i} + (t\sin t)\mathbf{j} + \tfrac{2\sqrt{2}}{3}t^{3/2}\mathbf{k} \Rightarrow \mathbf{v} = (\cos t - t\sin t)\mathbf{i} + (\sin t + t\cos t)\mathbf{j} + \left(\sqrt{2}t^{1/2}\right)\mathbf{k}$$

$$\Rightarrow |\mathbf{v}| = \sqrt{(\cos t - t\sin t)^2 + (\sin t + t\cos t)^2 + \left(\sqrt{2}t\right)^2} = \sqrt{1 + t^2 + 2t} = \sqrt{(t+1)^2} = |t+1| = t+1, \text{ if } t \geq 0;$$

$$\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \left(\frac{\cos t - t\sin t}{t+1}\right)\mathbf{i} + \left(\frac{\sin t + t\cos t}{t+1}\right)\mathbf{j} + \left(\frac{\sqrt{2}t^{1/2}}{t+1}\right)\mathbf{k} \text{ and Length } = \int_0^{\pi}(t+1)\,dt = \left[\frac{t^2}{2} + t\right]_0^{\pi} = \frac{\pi^2}{2} + \pi$$

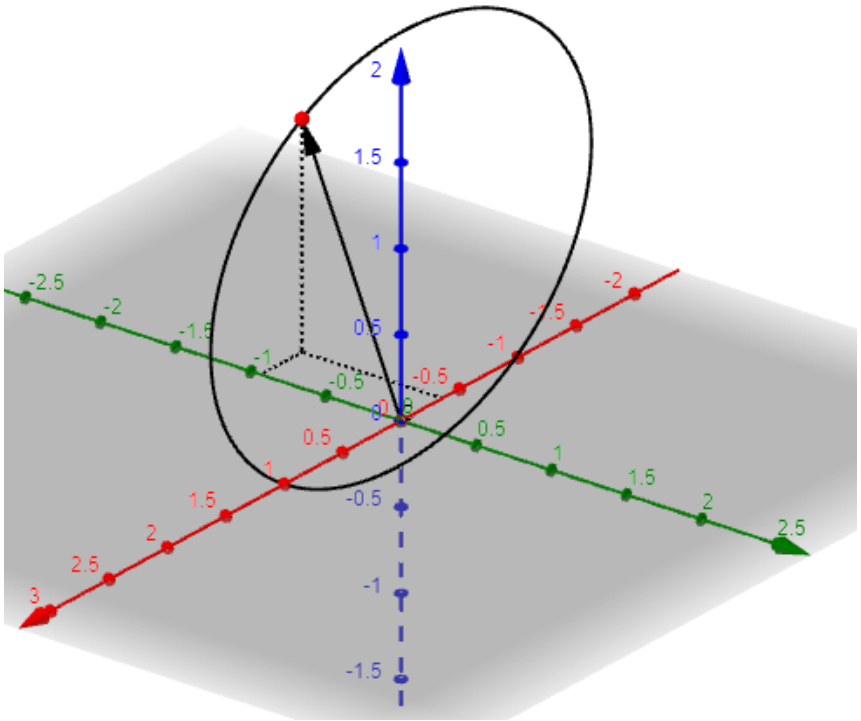
# Arc Length in Space

## Example 4

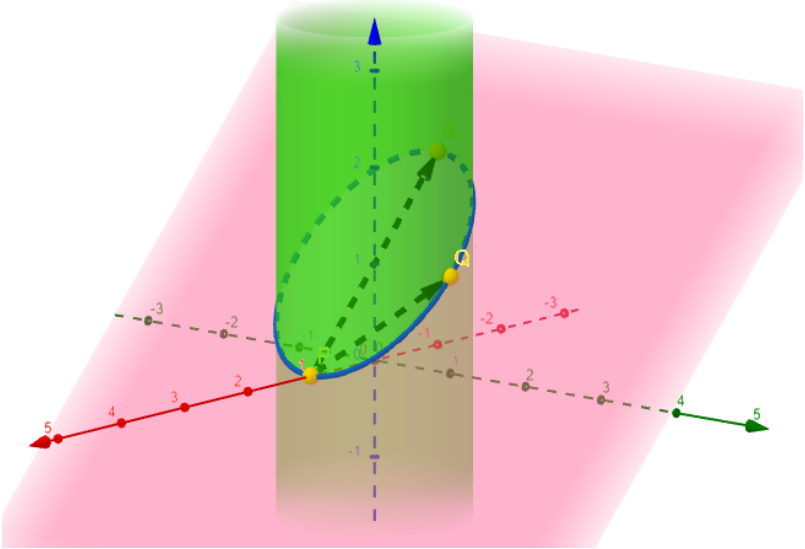
Show that the curve

$$r(t) = (\cos t)i + (\sin t)j + (1 - \cos t)k, \quad 0 \le t \le 2\pi,$$

is an ellipse by showing that it is the intersection of a right circular cylinder and a plane. Find equations for the cylinder and plane.



# Solution for Example 4



$\mathbf{r} = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + (1-\cos t)\mathbf{k},\ 0 \le t \le 2\pi \Rightarrow x = \cos t,\ y = \sin t,\ z = 1-\cos t$

$\Rightarrow x^2 + y^2 = \cos^2 t + \sin^2 t = 1,$ a right circular cylinder with the $z$-axis as the axis and radius $=1$.

Therefore $P(\cos t, \sin t, 1-\cos t)$ lies on the cylinder $x^2 + y^2 = 1$; $t = 0 \Rightarrow P(1, 0, 0)$ is on the curve;

$t = \frac{\pi}{2} \Rightarrow Q(0, 1, 1)$ is on the curve; $t = \pi \Rightarrow R(-1, 0, 2)$ is on the curve. Then $\overrightarrow{PQ} = -\mathbf{i} + \mathbf{j} + \mathbf{k}$ and

$\overrightarrow{PR} = -2\mathbf{i} + 2\mathbf{k} \Rightarrow \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 1 & 1 \\ -2 & 0 & 2 \end{bmatrix} = 2\mathbf{i} + 2\mathbf{k}$ is a vector normal to the plane of $P$, $Q$, and $R$. Then

the plane containing $P$, $Q$, and $R$ has an equation $2x + 2z = 2(1) + 2(0)$ or $x + z = 1$. Any point on the curve will satisfy this equation since $x + z = \cos t + (1 - \cos t) = 1$. Therefore, any point on the curve lies on the intersection of the cylinder $x^2 + y^2 = 1$ and the plane $x + z = 1 \Rightarrow$ the curve is an ellipse.

# Lecture 9: Vector-Valued Functions and Motion in Space.

MA2032 Vector Calculus

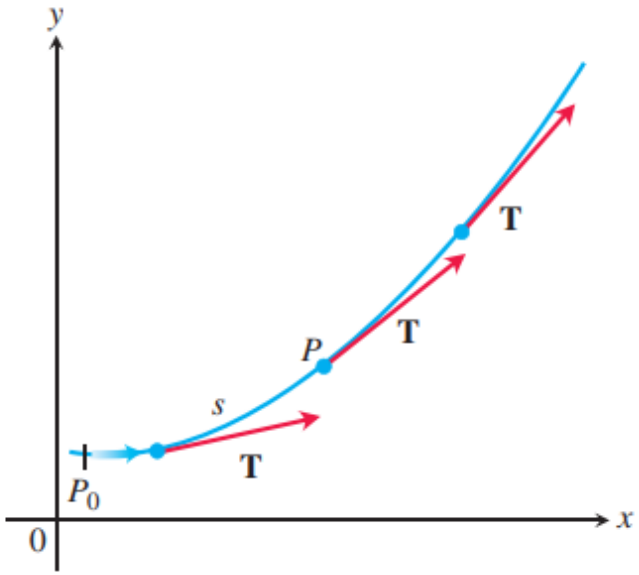
Lecturer: Larissa Serdukova

School of Computing and Mathematical Science
University of Leicester

October 10, 2022

# Curvature and Normal Vectors of a Curve

- We study how a curve **turns or bends**.
- To gain perspective, we look first at curves in the coordinate plane, then we consider curves in space.
- As a particle moves along a smooth curve in the plane, $\mathbf{T} = d\mathbf{r}/ds$ turns as the curve bends.
- Since $\mathbf{T}$ is a unit vector, its length remains constant and **only its direction changes** as the particle moves along the curve.
- The **rate at which T turns per unit of length** along the curve is called the **curvature** $\kappa$.




**DEFINITION** If $\mathbf{T}$ is the unit vector of a smooth curve, the **curvature** function of the curve is

$$\kappa = \left|\frac{d\mathbf{T}}{ds}\right|.$$

# Curvature

• If $|d\mathbf{T}/ds|$ is large, $\mathbf{T}$ turns sharply as the particle passes through P, and the curvature at P is large.
• If $|d\mathbf{T}/ds|$ is close to zero, $\mathbf{T}$ turns more slowly and the curvature at P is smaller.
• We can calculate the curvature as

$$\kappa = \left|\frac{d\mathbf{T}}{ds}\right| = \left|\frac{d\mathbf{T}}{dt}\frac{dt}{ds}\right| \qquad \text{Chain Rule}$$

$$= \frac{1}{|ds/dt|}\left|\frac{d\mathbf{T}}{dt}\right|$$

$$= \frac{1}{|\mathbf{v}|}\left|\frac{d\mathbf{T}}{dt}\right|. \qquad \frac{ds}{dt} = |\mathbf{v}|$$

**Formula for Calculating Curvature**

If $\mathbf{r}(t)$ is a smooth curve, then the curvature is the scalar function

$$\kappa = \frac{1}{|\mathbf{v}|}\left|\frac{d\mathbf{T}}{dt}\right|, \tag{1}$$

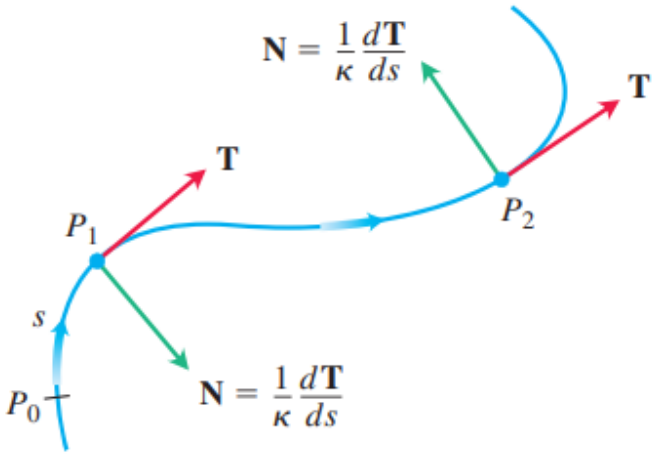
where $\mathbf{T} = \mathbf{v}/|\mathbf{v}|$ is the unit tangent vector.

# Normal Vectors of a Curve

• Among the vectors **orthogonal to the unit tangent vector T**, there is one of particular significance because it points in the **direction in which the curve is turning**.

• Since **T** has constant length (because its length is always 1), the derivative $d\mathbf{T}/ds$ is orthogonal to **T**.

• Therefore, if we divide $d\mathbf{T}/ds$ by its length $\kappa$, we obtain a unit vector **N** orthogonal to **T**.




**DEFINITION** At a point where $\kappa \neq 0$, the **principal unit normal** vector for a smooth curve in the plane is

$$\mathbf{N} = \frac{1}{\kappa}\frac{d\mathbf{T}}{ds}.$$

# Normal Vectors of a Curve

• The principal normal vector **N** will point toward the concave side of the curve.

• Following formula enables us to find **N** without having to find **k** and **s** first.

**Formula for Calculating $N$**

If $\mathbf{r}(t)$ is a smooth curve, then the principal unit normal is

$$\mathbf{N} = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|}, \tag{2}$$

where $\mathbf{T} = \mathbf{v}/|\mathbf{v}|$ is the unit tangent vector.

# Normal Vectors of a Curve

## Example 1

Find **T** and **N** for the circular motion

$$r(t) = (\cos 2t)i + (\sin 2t)j.$$

**Solution:**

We first find **T**, then its derivative:

$$\mathbf{v} = -(2\sin 2t)\mathbf{i} + (2\cos 2t)\mathbf{j} \qquad \frac{d\mathbf{T}}{dt} = -(2\cos 2t)\mathbf{i} - (2\sin 2t)\mathbf{j}$$

$$|\mathbf{v}| = \sqrt{4\sin^2 2t + 4\cos^2 2t} = 2$$

$$\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = -(\sin 2t)\mathbf{i} + (\cos 2t)\mathbf{j}. \qquad \left|\frac{d\mathbf{T}}{dt}\right| = \sqrt{4\cos^2 2t + 4\sin^2 2t} = 2$$

and

$$\mathbf{N} = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|} = -(\cos 2t)\mathbf{i} - (\sin 2t)\mathbf{j}.$$

Notice that $T \cdot N = 0$, verifying that **N** is orthogonal to **T**. Notice too, that for the circular motion here, **N** points from $r(t)$ toward the circle's center at the origin.

# Circle of Curvature for Plane Curves

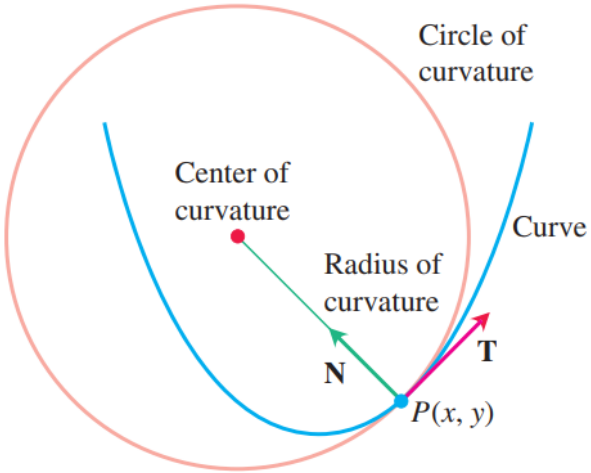
The **circle of curvature** or **osculating circle** at a point P on a plane curve where $\kappa \neq 0$ is the circle in the plane of the curve that

1. is **tangent to the curve** at P (has the same tangent line the curve has)
2. has the **same curvature** the curve has at P
3. has **center that lies toward the concave or inner side** of the curve.

• The **radius of curvature** of the curve at P is the radius of the circle of curvature, which is $\rho = \frac{1}{\kappa}$.

• To find $\rho$, we find $\kappa$ and take the reciprocal.

• The **center of curvature** of the curve at P is the center of the circle of curvature.



## Example 2

Find and graph the osculating circle of the parabola $y = x^2$ at the origin.

**Solution:**

We parametrize the parabola using the parameter $t = x$ $r(t) = ti + t^2 j$.
First we find the curvature of the parabola at the origin

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\mathbf{j}$$

$$|\mathbf{v}| = \sqrt{1 + 4t^2}$$

so that

$$\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = (1 + 4t^2)^{-1/2}\mathbf{i} + 2t(1 + 4t^2)^{-1/2}\mathbf{j}.$$

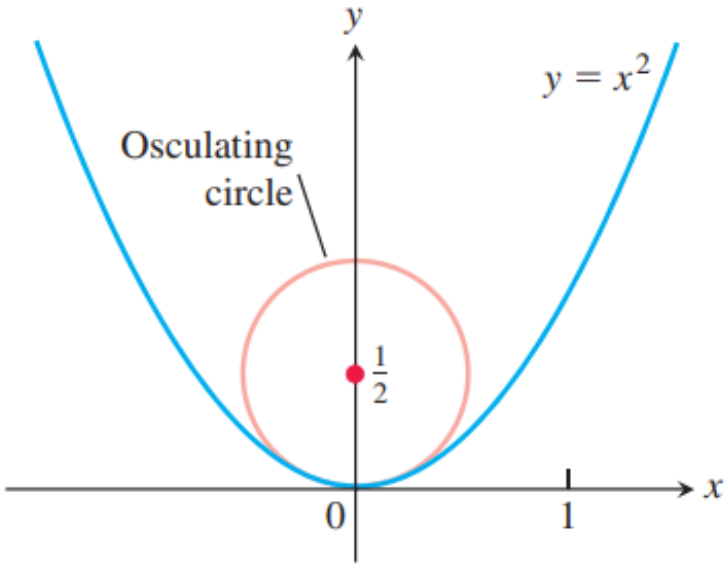
From this we find

$$\frac{d\mathbf{T}}{dt} = -4t(1 + 4t^2)^{-3/2}\mathbf{i} + [2(1 + 4t^2)^{-1/2} - 8t^2(1 + 4t^2)^{-3/2}]\mathbf{j}.$$

**Solution for Example 2 continuation:**

• At the origin, $t = 0$, so the curvature is

$$\begin{aligned}\kappa(0) &= \frac{1}{|\mathbf{v}(0)|}\left|\frac{d\mathbf{T}}{dt}(0)\right| \\ &= \frac{1}{\sqrt{1}}|0\mathbf{i} + 2\mathbf{j}| \\ &= (1)\sqrt{0^2 + 2^2} = 2.\end{aligned}$$




• Therefore, the radius of curvature is $1/k = 1/2$. At the origin we have $t = 0$ and $\mathbf{T} = i$, so $\mathbf{N} = j$. Thus the center of the circle is $(0, 1/2)$. The equation of the osculating circle is $(x - 0)^2 + (y - 1/2)^2 = (1/2)^2$.

• The osculating circle is a better approximation to the parabola at the origin than is the tangent line approximation $y = 0$.

# Curvature and Normal Vectors for Space Curves

If a smooth curve in space is specified by the position vector $r(t)$ as a function of some parameter $t$, and if $s$ is the arc length parameter of the curve, then the unit tangent vector $\mathbf{T}$ is $d\mathbf{r}/ds = \mathbf{v}/|\mathbf{v}|$. The **curvature** in space is then defined to be

$$\kappa = \left|\frac{d\mathbf{T}}{ds}\right| = \frac{1}{|\mathbf{v}|}\left|\frac{d\mathbf{T}}{dt}\right|$$

just as for plane curves. The vector $d\mathbf{T}/ds$ is orthogonal to $\mathbf{T}$, and we define the **principal unit normal** to be

$$\mathbf{N} = \frac{1}{\kappa}\frac{d\mathbf{T}}{ds} = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|}.$$

## Example 3

Find the curvature for the helix

$$r(t) = (a\cos t)\mathbf{i} + (a\sin t)\mathbf{j} + bt\mathbf{k}, \quad a, b \geq 0, \quad a^2 + b^2 \neq 0.$$

Find N for the helix and describe how the vector is pointing.

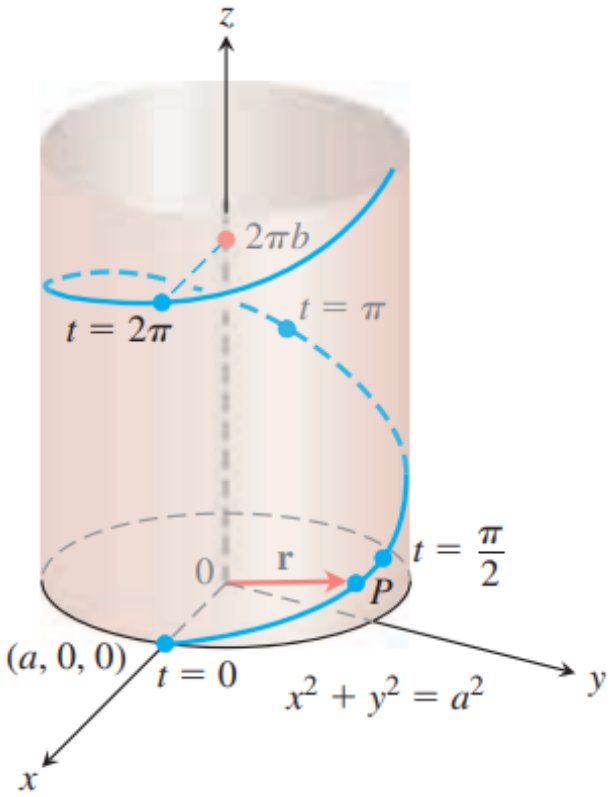
**Solution:**

We calculate T from the velocity vector v then $\kappa$ using definition:

$$\mathbf{v} = -(a\sin t)\mathbf{i} + (a\cos t)\mathbf{j} + b\mathbf{k}$$

$$|\mathbf{v}| = \sqrt{a^2\sin^2 t + a^2\cos^2 t + b^2} = \sqrt{a^2 + b^2}$$

$$\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{1}{\sqrt{a^2 + b^2}}\left[-(a\sin t)\mathbf{i} + (a\cos t)\mathbf{j} + b\mathbf{k}\right].$$



**Solution for Example 3 continuation:**

$$\begin{aligned}
\kappa &= \frac{1}{|\mathbf{v}|}\left|\frac{d\mathbf{T}}{dt}\right| \\
&= \frac{1}{\sqrt{a^2+b^2}}\left|\frac{1}{\sqrt{a^2+b^2}}\left[-(a\cos t)\mathbf{i} - (a\sin t)\mathbf{j}\right]\right| \\
&= \frac{a}{a^2+b^2}\left|-(\cos t)\mathbf{i} - (\sin t)\mathbf{j}\right| \\
&= \frac{a}{a^2+b^2}\sqrt{(\cos t)^2 + (\sin t)^2} = \frac{a}{a^2+b^2}.
\end{aligned}$$

• From this equation, we see that increasing $b$ for a fixed $a$ decreases the curvature.

• Decreasing $a$ for a fixed $b$ eventually decreases the curvature as well.

• If $b = 0$, the helix reduces to a circle of radius $a$ and its curvature reduces to $1/a$, as it should.

• If $a = 0$, the helix becomes the z-axis, and its curvature reduces to 0, again as it should.

**Solution for Example 3 continuation:**

We have

$$\frac{d\mathbf{T}}{dt} = -\frac{1}{\sqrt{a^2+b^2}}\left[(a\cos t)\mathbf{i} + (a\sin t)\mathbf{j}\right]$$

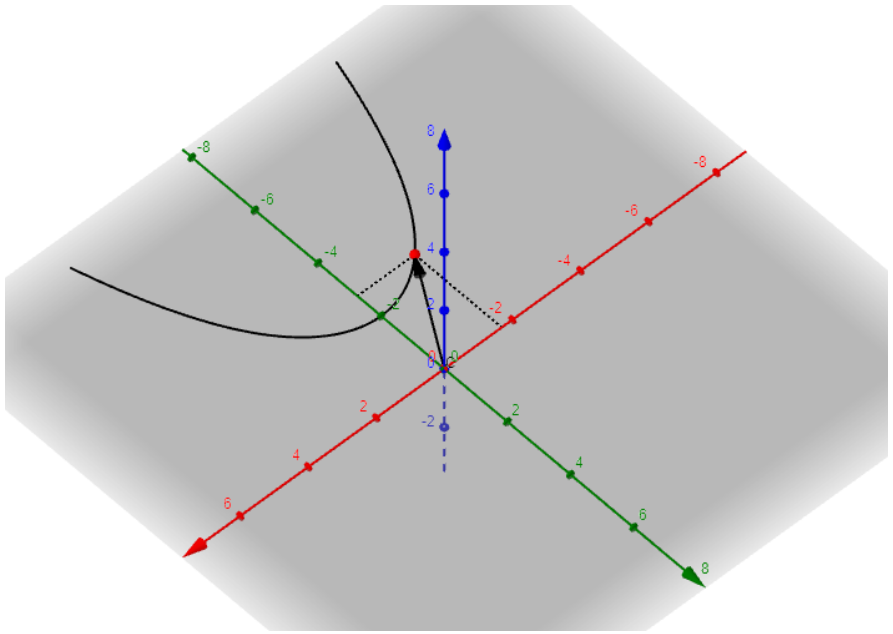
$$\left|\frac{d\mathbf{T}}{dt}\right| = \frac{1}{\sqrt{a^2+b^2}}\sqrt{a^2\cos^2 t + a^2\sin^2 t} = \frac{a}{\sqrt{a^2+b^2}}$$

$$\begin{aligned}\mathbf{N} &= \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|}\\ &= -\frac{\sqrt{a^2+b^2}}{a}\cdot\frac{1}{\sqrt{a^2+b^2}}\left[(a\cos t)\mathbf{i} + (a\sin t)\mathbf{j}\right]\\ &= -(\cos t)\mathbf{i} - (\sin t)\mathbf{j}.\end{aligned}$$

Thus, **N** is parallel to the *xy*-plane and always points toward the z-axis.

**Example 4 :**

Find an equation for the circle of curvature of the curve $\mathbf{r}(t) = (2\ln t)\mathbf{i} - [t + (1/t)]\mathbf{j}$, $e^{-2} \leq t \leq e^{2}$, at the point $(0, -2)$, where $t = 1$.



**Solution for Example 4:**

$$\mathbf{r}=(2\ln t)\mathbf{i}-\left(t+\tfrac{1}{t}\right)\mathbf{j}\Rightarrow \mathbf{v}=\left(\tfrac{2}{t}\right)\mathbf{i}-\left(1-\tfrac{1}{t^2}\right)\mathbf{j}\Rightarrow |\mathbf{v}|=\sqrt{\tfrac{4}{t^2}+\left(1-\tfrac{1}{t^2}\right)^2}=\tfrac{t^2+1}{t^2}\Rightarrow \mathbf{T}=\tfrac{\mathbf{v}}{|\mathbf{v}|}=\tfrac{2t}{t^2+1}\mathbf{i}-\tfrac{t^2-1}{t^2+1}\mathbf{j};$$

$$\frac{d\mathbf{T}}{dt}=\frac{-2\left(t^2-1\right)}{\left(t^2+1\right)^2}\mathbf{i}-\frac{4t}{\left(t^2+1\right)^2}\mathbf{j}\Rightarrow\left|\frac{d\mathbf{T}}{dt}\right|=\sqrt{\frac{4\left(t^2-1\right)^2+16t^2}{\left(t^2+1\right)^4}}=\frac{2}{t^2+1}.\ \text{Thus}\ \kappa=\frac{1}{|\mathbf{v}|}\cdot\left|\frac{d\mathbf{T}}{dt}\right|=\frac{t^2}{t^2+1}\cdot\frac{2}{t^2+1}=\frac{2t^2}{\left(t^2+1\right)^2}$$

$\Rightarrow \kappa(1)=\frac{2}{2^2}=\frac{1}{2}\Rightarrow \rho=\frac{1}{\kappa}=2.$ The circle of curvature is tangent to the curve at $P(0,-2)\Rightarrow$ circle has same tangent as the curve $\Rightarrow \mathbf{v}(1)=2\mathbf{i}$ is tangent to the circle $\Rightarrow$ the center lies on the $y$-axis. If $t\neq 1(t>0)$, then $(t-1)^2>0\Rightarrow t^2-2t+1>0\Rightarrow t^2+1>2t\Rightarrow \frac{t^2+1}{t}>2$ since $t>0\Rightarrow t+\frac{1}{t}>2\Rightarrow -\left(t+\frac{1}{t}\right)<-2\Rightarrow y<-2$ on both sides of $(0,-2)\Rightarrow$ the curve is concave down $\Rightarrow$ center of circle of curvature is $(0,-4)$

$\Rightarrow x^2+(y+4)^2=4$ is an equation of the circle of curvature

# Lecture 10: Vector-Valued Functions and Motion in Space.

MA2032 Vector Calculus

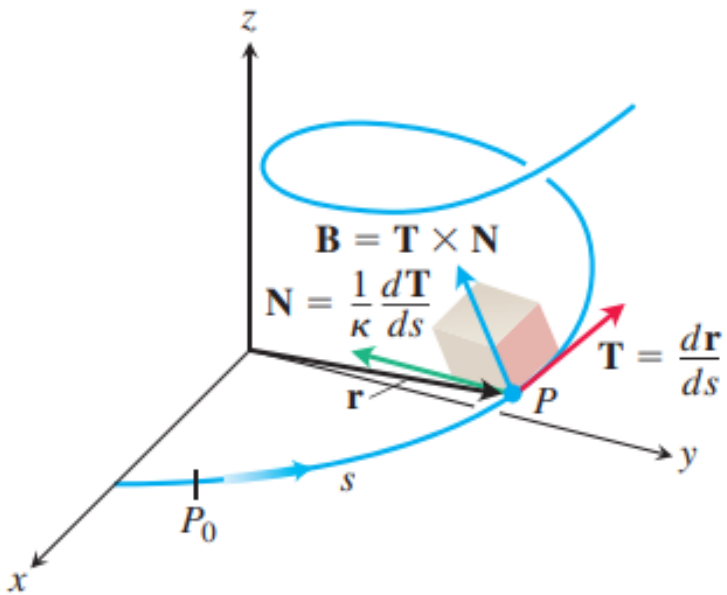
Lecturer: Larissa Serdukova

School of Computing and Mathematical Science
University of Leicester

October 12, 2022

# Tangential and Normal Components of Acceleration

- If you are traveling along a curve in space, the Cartesian i, j, and k coordinate system for representing the vectors describing your motion **may not be very relevant** to you.
- Instead, the vectors that represent your **forward direction** (the unit tangent vector **T**), the direction in which your **path is turning** (the unit normal vector **N**), and the tendency of your motion to **"twist" out of the plane created by these vectors** in the direction perpendicular to this plane (defined by the unit binormal vector $B = T \times N$) are likely to be **more important**.



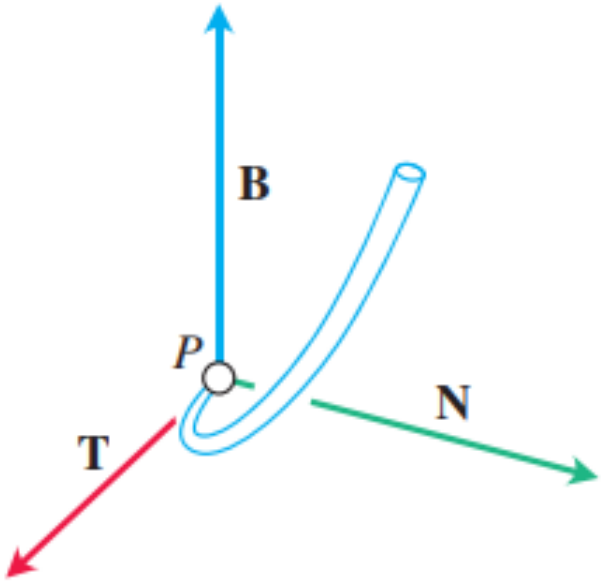

- Expressing the **acceleration vector** along the curve as a **linear combination of this TNB frame** of mutually orthogonal unit vectors traveling with the motion can reveal much about the nature of your path and your motion along it.

# The TNB Frame

•The binormal vector of a curve in space is $B = T \times N$, which is **a unit vector that is orthogonal to both T and N**.

• Together T, N, and B define a moving right-handed vector frame that plays a significant role in calculating the paths of particles moving through space.

• It is called the Frenet ("fre-nay") frame (after Jean-Frédéric Frenet, 1816–1900), or the **TNB frame**.



# Tangential and Normal Components of Acceleration

• We often need to know how much of the **acceleration** acts in the direction of motion, which is the direction of the tangent vector $\mathbf{T}$.

• We can calculate this using the Chain Rule to rewrite $\mathbf{v}$ as $\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{d\mathbf{r}}{ds}\frac{ds}{dt} = \mathbf{T}\frac{ds}{dt}$.

• Then we differentiate both ends of this string of equalities to get

$$\begin{aligned}
\mathbf{a} &= \frac{d\mathbf{v}}{dt} = \frac{d}{dt}\left(\mathbf{T}\frac{ds}{dt}\right) = \frac{d^2s}{dt^2}\mathbf{T} + \frac{ds}{dt}\frac{d\mathbf{T}}{dt} \\
&= \frac{d^2s}{dt^2}\mathbf{T} + \frac{ds}{dt}\left(\frac{d\mathbf{T}}{ds}\frac{ds}{dt}\right) = \frac{d^2s}{dt^2}\mathbf{T} + \frac{ds}{dt}\left(\kappa\mathbf{N}\frac{ds}{dt}\right) \qquad \frac{d\mathbf{T}}{ds} = \kappa\mathbf{N} \\
&= \frac{d^2s}{dt^2}\mathbf{T} + \kappa\left(\frac{ds}{dt}\right)^2\mathbf{N}.
\end{aligned}$$

# Tangential and Normal Components of Acceleration

**DEFINITION** If the acceleration vector is written as

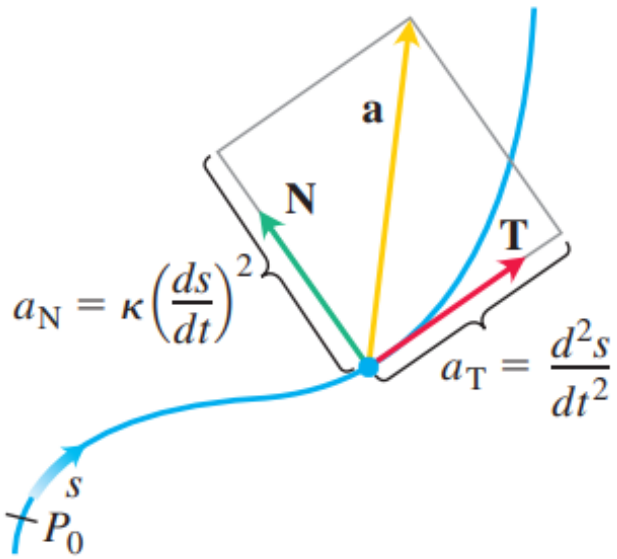
$$\mathbf{a} = a_{\mathrm{T}}\mathbf{T} + a_{\mathrm{N}}\mathbf{N}, \tag{1}$$

then

$$a_{\mathrm{T}} = \frac{d^2s}{dt^2} = \frac{d}{dt}|\mathbf{v}| \qquad \text{and} \qquad a_{\mathrm{N}} = \kappa\left(\frac{ds}{dt}\right)^2 = \kappa|\mathbf{v}|^2 \tag{2}$$

are the **tangential** and **normal** scalar components of acceleration.

- Notice that the binormal vector **B** does not appear in Equation (1).
- No matter how the path of the moving object we are watching may appear to twist and turn in space, the acceleration **a** always lies in the plane of **T** and **N** orthogonal to **B**.
- The equation also tells us exactly how much of the acceleration takes place **tangent** to the motion ($d^2s/dt^2$) and how much takes place **normal** to the motion [$\kappa(ds/dt)^2$]



# Tangential and Normal Components of Acceleration

• What information can we discover from **Equations (2)**?
• By definition, acceleration **a** is the **rate of change of velocity v**, and in general, both the **length** and **direction** of v change as an object moves along its path.
• The tangential component of acceleration $a_T$ measures the **rate of change of the length** of **v** (that is, the change in the speed).
• The normal component of acceleration $a_N$ measures the **rate of change of the direction** of **v**.
• To calculate $a_N$, we usually use the formula

**Formula for Calculating the Normal Component of Acceleration**

$$a_{\mathrm{N}} = \sqrt{|\mathbf{a}|^2 - {a_{\mathrm{T}}}^2} \tag{3}$$

• which comes from solving the equation $|a|^2 = a \cdot a = a_T^2 + a_N^2$ for $a_N$.
• With this formula, we can find $a_N$ without having to calculate $\kappa$ first.

# Tangential and Normal Components of Acceleration

## Example 1

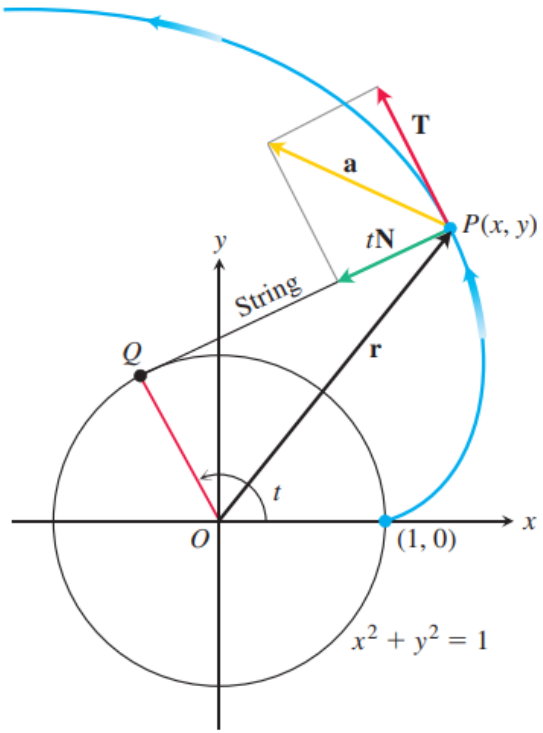
Without finding **T** and **N**, write the acceleration of the motion

$$r(t) = (\cos t + t\sin t)\mathbf{i} + (\sin t - t\cos t)\mathbf{j},\ t > 0.$$

in the form $\mathbf{a} = a_T\mathbf{T} + a_N\mathbf{N}$.

**Solution:**

We use the first of Equations (2) to find $a_T$:




$$\begin{aligned}
\mathbf{v} = \frac{d\mathbf{r}}{dt} &= (-\sin t + \sin t + t\cos t)\mathbf{i} + (\cos t - \cos t + t\sin t)\mathbf{j} \\
&= (t\cos t)\mathbf{i} + (t\sin t)\mathbf{j} \\
|\mathbf{v}| &= \sqrt{t^2\cos^2 t + t^2\sin^2 t} = \sqrt{t^2} = |t| = t && t > 0 \\
a_T &= \frac{d}{dt}|\mathbf{v}| = \frac{d}{dt}(t) = 1. && \text{Eq. (2)}
\end{aligned}$$

Knowing $a_T$, we use Equation (3) to find $a_N$:

$$\begin{aligned}
\mathbf{a} &= (\cos t - t\sin t)\mathbf{i} + (\sin t + t\cos t)\mathbf{j} \\
|\mathbf{a}|^2 &= t^2 + 1 && \text{After some algebra} \\
a_N &= \sqrt{|\mathbf{a}|^2 - {a_T}^2} \\
&= \sqrt{(t^2 + 1) - (1)} = \sqrt{t^2} = t.
\end{aligned}$$

We then use Equation (1) to find **a**:

$$\mathbf{a} = a_T\mathbf{T} + a_N\mathbf{N} = (1)\mathbf{T} + (t)\mathbf{N} = \mathbf{T} + t\mathbf{N}.$$

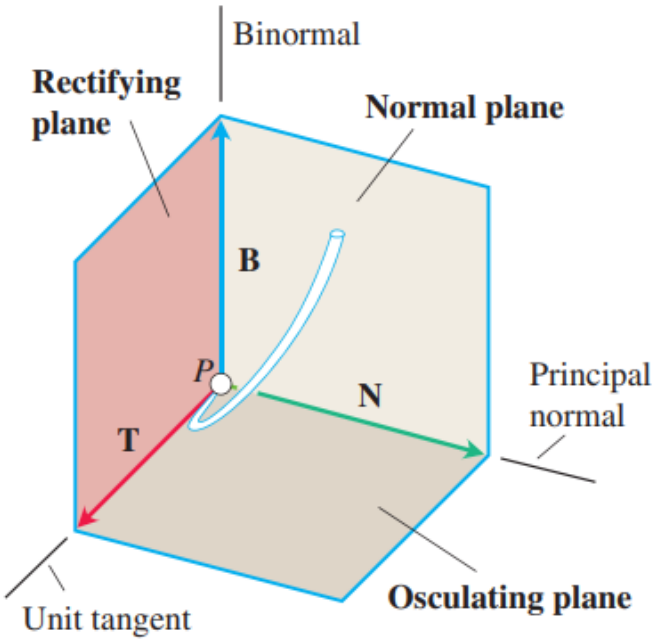
# Torsion

- **Torsion measures how the curve twists**.

**DEFINITION** Let $\mathbf{B} = \mathbf{T} \times \mathbf{N}$. The **torsion** function of a smooth curve is

$$\tau = -\frac{d\mathbf{B}}{ds} \cdot \mathbf{N}. \tag{4}$$

- The three planes determined by **T, N**, and **B** are named and shown in Figure.
- The curvature $\kappa = |dT/ds|$ can be thought of as the rate at which the normal plane turns as the point P moves along its path.
- Similarly, the torsion $\tau = -(dB/ds) \cdot N$ is the rate at which the osculating plane turns about T as P moves along the curve.



# Formulas for Computing Curvature and Torsion

- We now give **easy-to-use formulas for computing the curvature and torsion** of a smooth curve.
- From Equations (1) and (2), we have

$$\mathbf{v} \times \mathbf{a} = \left(\frac{ds}{dt}\mathbf{T}\right) \times \left[\frac{d^2s}{dt^2}\mathbf{T} + \kappa\left(\frac{ds}{dt}\right)^2\mathbf{N}\right] \qquad \mathbf{v} = dr/dt = (ds/dt)\mathbf{T}$$

$$= \left(\frac{ds}{dt}\frac{d^2s}{dt^2}\right)(\mathbf{T} \times \mathbf{T}) + \kappa\left(\frac{ds}{dt}\right)^3(\mathbf{T} \times \mathbf{N})$$

$$= \kappa\left(\frac{ds}{dt}\right)^3\mathbf{B}. \qquad \mathbf{T} \times \mathbf{T} = 0 \text{ and } \mathbf{T} \times \mathbf{N} = \mathbf{B}$$

It follows that

$$|\mathbf{v} \times \mathbf{a}| = \kappa\left|\frac{ds}{dt}\right|^3|\mathbf{B}| = \kappa|\mathbf{v}|^3. \qquad \frac{ds}{dt} = |\mathbf{v}| \quad \text{and} \quad |\mathbf{B}| = 1$$

Solving for $\kappa$ gives the following formula.

**Vector Formula for Curvature**

$$\kappa = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3} \tag{5}$$

# Formulas for Computing Curvature and Torsion

**Formula for Torsion**

$$\tau = \frac{\begin{vmatrix} \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \\ \dddot{x} & \dddot{y} & \dddot{z} \end{vmatrix}}{|\mathbf{v} \times \mathbf{a}|^2} \qquad (\text{if } \mathbf{v} \times \mathbf{a} \neq \mathbf{0}) \tag{6}$$

• This formula calculates the torsion directly from the derivatives of the component functions $x = (t), y = g(t), z = h(t)$ that make up $\mathbf{r}$.

• The determinant's first row comes from $\mathbf{v}$, the second row comes from $\mathbf{a}$, and the third row comes from a $da/dt$.

• This formula for torsion is traditionally written using Newton's dot notation for derivatives.

# Formulas for Computing Curvature and Torsion

## Example 2

Use Equations (5) and (6) to find the curvature $\kappa$ and torsion $\tau$ for the helix

$$r(t) = (a\cos t)\mathbf{i} + (a\sin t)\mathbf{j} + bt\mathbf{k}, \quad a, b \geq 0, \quad a^2 + b^2 \neq 0.$$

**Solution:**

$$\mathbf{v} = -(a\sin t)\mathbf{i} + (a\cos t)\mathbf{j} + b\mathbf{k}$$

$$\mathbf{a} = -(a\cos t)\mathbf{i} - (a\sin t)\mathbf{j}$$

$$\mathbf{v} \times \mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -a\sin t & a\cos t & b \\ -a\cos t & -a\sin t & 0 \end{vmatrix}$$

$$= (ab\sin t)\mathbf{i} - (ab\cos t)\mathbf{j} + a^2\mathbf{k}$$

$$\kappa = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3} = \frac{\sqrt{a^2b^2 + a^4}}{(a^2 + b^2)^{3/2}} = \frac{a\sqrt{a^2 + b^2}}{(a^2 + b^2)^{3/2}} = \frac{a}{a^2 + b^2}.$$

$$\dot{\mathbf{a}} = \frac{d\mathbf{a}}{dt} = (a\sin t)\mathbf{i} - (a\cos t)\mathbf{j}.$$

$$\tau = \frac{\begin{vmatrix} \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \\ \dddot{x} & \dddot{y} & \dddot{z} \end{vmatrix}}{|\mathbf{v} \times \mathbf{a}|^2} = \frac{\begin{vmatrix} -a\sin t & a\cos t & b \\ -a\cos t & -a\sin t & 0 \\ a\sin t & -a\cos t & 0 \end{vmatrix}}{\left(a\sqrt{a^2 + b^2}\right)^2} \quad \text{Value of } |\mathbf{v} \times \mathbf{a}| \text{ from Eq. (7)}$$

$$= \frac{b(a^2\cos^2 t + a^2\sin^2 t)}{a^2(a^2 + b^2)}$$

$$= \frac{b}{a^2 + b^2}.$$

- From this last equation we see that the torsion of a helix about a circular cylinder is constant.
- In fact, constant curvature and constant torsion characterize the helix among all curves in space.

# Formulas for Computing Curvature and Torsion

## Computation Formulas for Curves in Space

*Unit tangent vector:*

$$\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|}$$

*Principal unit normal vector:*

$$\mathbf{N} = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|}$$

*Binormal vector:*

$$\mathbf{B} = \mathbf{T} \times \mathbf{N}$$

*Curvature:*

$$\kappa = \left|\frac{d\mathbf{T}}{ds}\right| = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3}$$

*Torsion:*

$$\tau = -\frac{d\mathbf{B}}{ds} \cdot \mathbf{N} = \frac{\begin{vmatrix} \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \\ \dddot{x} & \dddot{y} & \dddot{z} \end{vmatrix}}{|\mathbf{v} \times \mathbf{a}|^2}$$

*Tangential and normal scalar components of acceleration:*

$$\mathbf{a} = a_{\mathrm{T}}\mathbf{T} + a_{\mathrm{N}}\mathbf{N}$$

$$a_{\mathrm{T}} = \frac{d}{dt}|\mathbf{v}|$$

$$a_{\mathrm{N}} = \kappa|\mathbf{v}|^2 = \sqrt{|\mathbf{a}|^2 - {a_{\mathrm{T}}}^2}$$

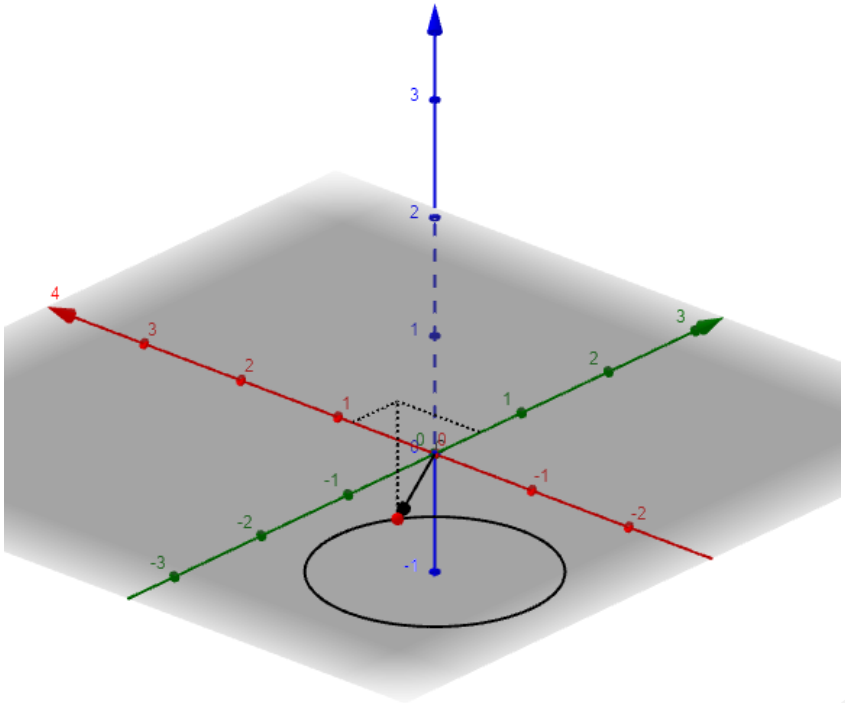
# Formulas for Computing Curvature and Torsion

## Example 3

Given

$$\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} - \mathbf{k},$$

Find **r, T, N**, and **B** at the $t = \pi/4$. Then find equations for the osculating, normal, and rectifying planes at that value of $t = \pi/4$.



# Solution for Example 3

**Solution:**

$$\mathbf{r}=(\cos t)\mathbf{i}+(\sin t)\mathbf{j}-\mathbf{k}\Rightarrow \mathbf{v}=(-\sin t)\mathbf{i}+(\cos t)\mathbf{j}\Rightarrow |\mathbf{v}|=\sqrt{(-\sin t)^2+(\cos t)^2}=1$$

$$\Rightarrow \mathbf{T}=\frac{\mathbf{v}}{|\mathbf{v}|}=(-\sin t)\mathbf{i}+(\cos t)\mathbf{j}\Rightarrow \mathbf{T}\left(\frac{\pi}{4}\right)=-\frac{\sqrt{2}}{2}\mathbf{i}+\frac{\sqrt{2}}{2}\mathbf{j};\ \ \frac{d\mathbf{T}}{dt}=(-\cos t)\mathbf{i}-(\sin t)\mathbf{j}$$

$$\Rightarrow \left|\frac{d\mathbf{T}}{dt}\right|\sqrt{(-\cos t)^2+(-\sin t)^2}=1\Rightarrow \mathbf{N}=\frac{\left(\frac{d\mathbf{T}}{dt}\right)}{\left|\frac{d\mathbf{T}}{dt}\right|}=(-\cos t)\mathbf{i}-(\sin t)\mathbf{j}\Rightarrow \mathbf{N}\left(\frac{\pi}{4}\right)=-\frac{\sqrt{2}}{2}\mathbf{i}-\frac{\sqrt{2}}{2}\mathbf{j};$$

$$\mathbf{B}=\mathbf{T}\times\mathbf{N}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ -\sin t & \cos t & 0\\ -\cos t & -\sin t & 0\end{vmatrix}=\mathbf{k}\Rightarrow \mathbf{B}\left(\frac{\pi}{4}\right)=\mathbf{k},\ \text{the normal to the osculating plane;}\ \mathbf{r}\left(\frac{\pi}{4}\right)=\frac{\sqrt{2}}{2}\mathbf{i}+\frac{\sqrt{2}}{2}\mathbf{j}-\mathbf{k}$$

$\Rightarrow P=\left(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2},-1\right)$ lies on the osculating plane $\Rightarrow 0\left(x-\frac{\sqrt{2}}{2}\right)+0\left(y-\frac{\sqrt{2}}{2}\right)+(z-(-1))=0\Rightarrow z=-1$ is the osculating plane; $\mathbf{T}$ is normal to the normal plane $\Rightarrow \left(-\frac{\sqrt{2}}{2}\right)\left(x-\frac{\sqrt{2}}{2}\right)+\left(\frac{\sqrt{2}}{2}\right)\left(y-\frac{\sqrt{2}}{2}\right)+0(z-(-1))=0$

$\Rightarrow -\frac{\sqrt{2}}{2}x+\frac{\sqrt{2}}{2}y=0\Rightarrow -x+y=0$ is the normal plane; $\mathbf{N}$ is normal to the rectifying plane

$\Rightarrow \left(-\frac{\sqrt{2}}{2}\right)\left(x-\frac{\sqrt{2}}{2}\right)+\left(-\frac{\sqrt{2}}{2}\right)\left(y-\frac{\sqrt{2}}{2}\right)+0(z-(-1))=0\Rightarrow -\frac{\sqrt{2}}{2}x-\frac{\sqrt{2}}{2}y=-1\Rightarrow x+y=\sqrt{2}$ is the rectifying plane.

# Lecture 11: Vector-Valued Functions and Motion in Space.

MA2032 Vector Calculus

Lecturer: Larissa Serdukova

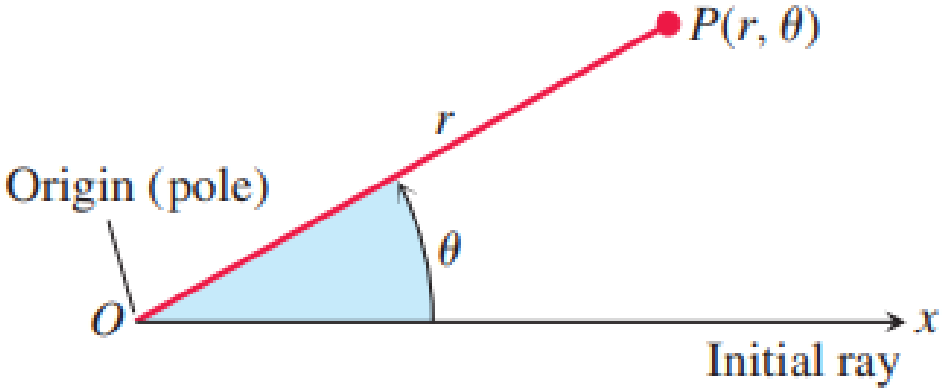
School of Computing and Mathematical Science
University of Leicester

October 14, 2022

# Velocity and Acceleration in Polar and Cylindrical coordinates

- We derive equations for velocity and acceleration in polar and cylindrical coordinates.
- We introduce **polar, cylindrical and spherical coordinates** and their relation to Cartesian coordinates.
- They are useful in **describing the paths of particles** moving in space.

# Definition of Polar Coordinates




• To define polar coordinates, we first **fix an origin** $O$ (called the pole) and an **initial ray** from $O$.

• Usually the **positive x-axis** is chosen as the initial ray.

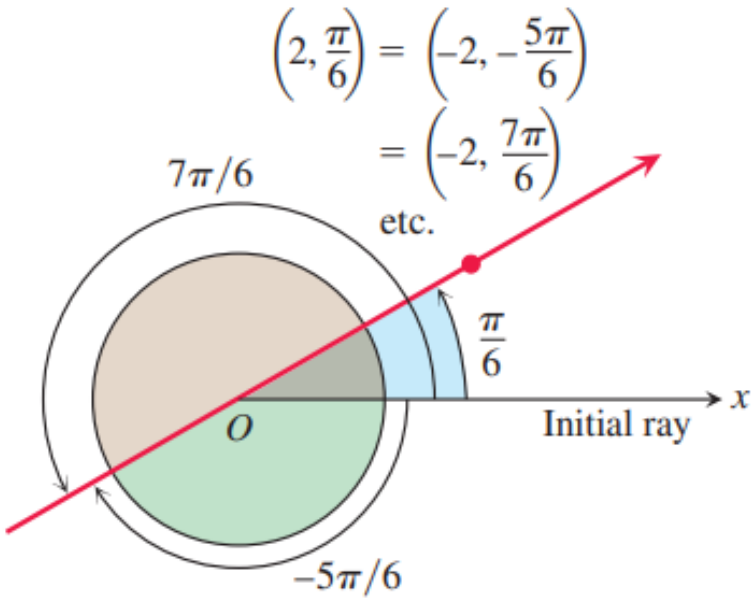
• Then each point P can be located by **assigning to it a polar coordinate pair** $(r, \theta)$ in which $r$ gives the directed distance from $O$ to $P$ and $\theta$ gives the directed angle from the initial ray to ray $OP$. So we label the point $P$ as

$$P(r, \theta)$$

Directed distance from $O$ to $P$ — Directed angle from initial ray to $OP$

# Definition of Polar Coordinates

• As in trigonometry, $\theta$ is **positive** when measured **counterclockwise** and **negative** when measured **clockwise**.
• The **angle** associated with a given point is **not unique**.
• While a point in the plane has just **one pair of Cartesian coordinates**, it has **infinitely many pairs of polar coordinates**.
• For instance, the point 2 units from the origin along the ray $\theta = \pi/6$ has polar coordinates $r = 2, \theta = \pi/6$.
• It also has coordinates shown in Figure.

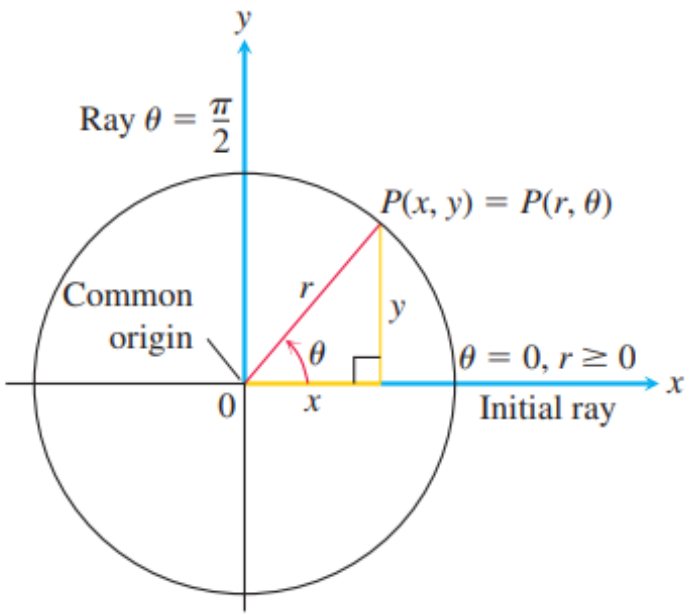


# Relating Polar and Cartesian Coordinates

- When we use both polar and Cartesian coordinates in a plane, we place $\theta = \pi/2, r > 0$, becomes the positive y-axis.
- The two coordinate systems are then related by the following equations

**Equations Relating Polar and Cartesian Coordinates**

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad r^2 = x^2 + y^2, \qquad \tan\theta = \frac{y}{x}$$



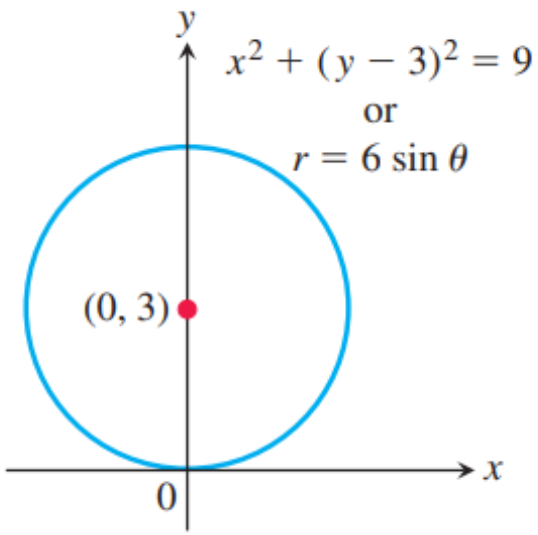
# Relating Polar and Cartesian Coordinates

## Example 1

Find a polar equation for the circle $x^2 + (y-3)^2 = 9$.

**Solution:**

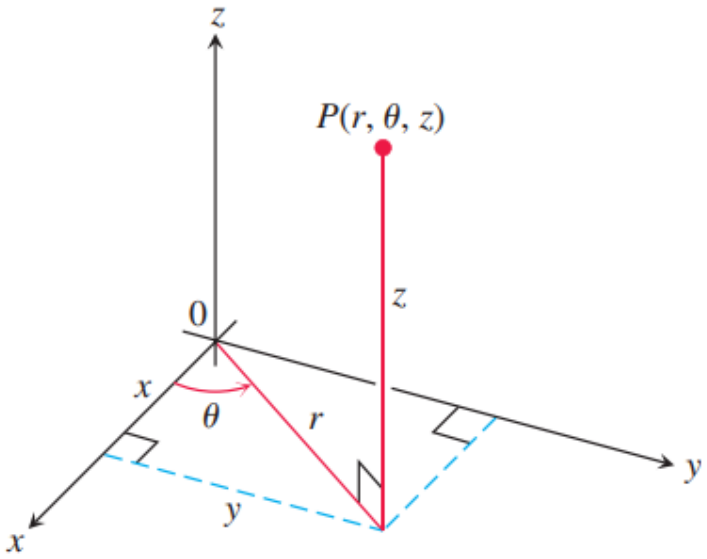
We apply the equations relating polar and Cartesian coordinates:




$$
\begin{aligned}
x^2 + (y-3)^2 &= 9 && \\
x^2 + y^2 - 6y + 9 &= 9 && \text{Expand } (y-3)^2. \\
x^2 + y^2 - 6y &= 0 && \text{Cancelation} \\
r^2 - 6r\sin\theta &= 0 && x^2 + y^2 = r^2,\ y = r\sin\theta \\
r = 0 \quad \text{or} \quad r - 6\sin\theta &= 0 && \\
r &= 6\sin\theta && \text{Includes both possibilities}
\end{aligned}
$$

# Cylindrical Coordinates

• We obtain **cylindrical coordinates** for space by combining **polar coordinates** in the xy-plane with the usual **z-axis**.
• This assigns to every point in space one or more coordinate **triples** of the form $(r, \theta, z)$, as shown in Figure.
• Here we require $r \geq 0$.



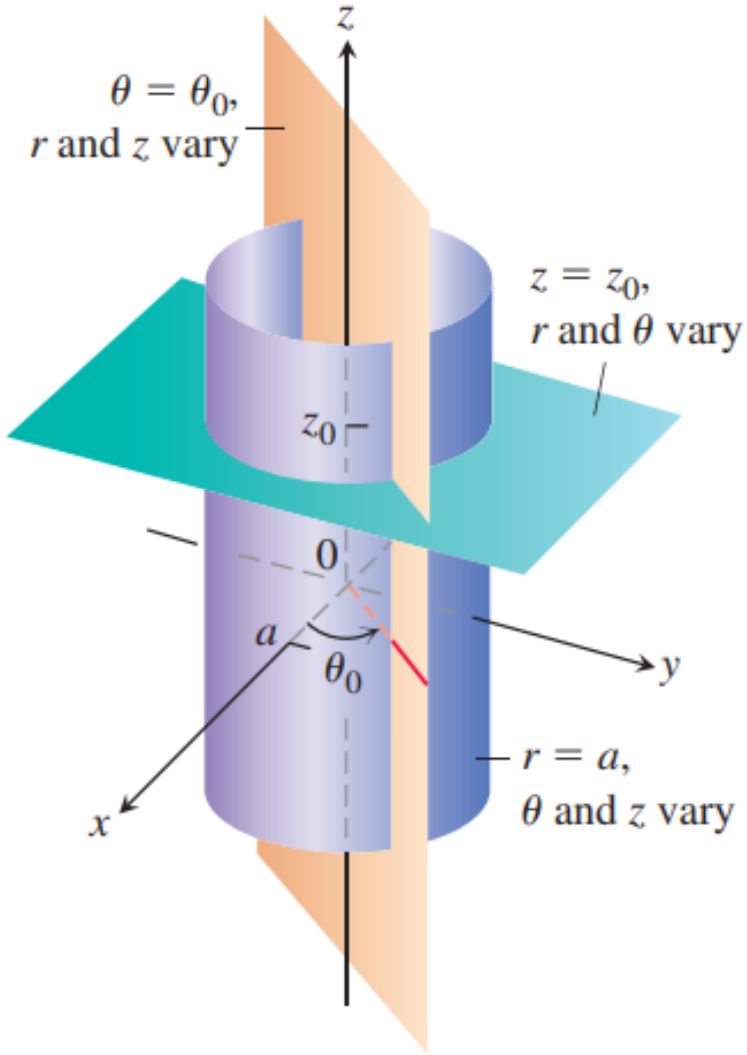

• The values of $x, y, r$, and $\theta$ in rectangular and cylindrical coordinates are related by the usual equations.

**DEFINITION** **Cylindrical coordinates** represent a point $P$ in space by ordered triples $(r, \theta, z)$ in which $r \geq 0$,

1. $r$ and $\theta$ are polar coordinates for the vertical projection of $P$ on the $xy$-plane
2. $z$ is the rectangular vertical coordinate.

# Cylindrical Coordinates

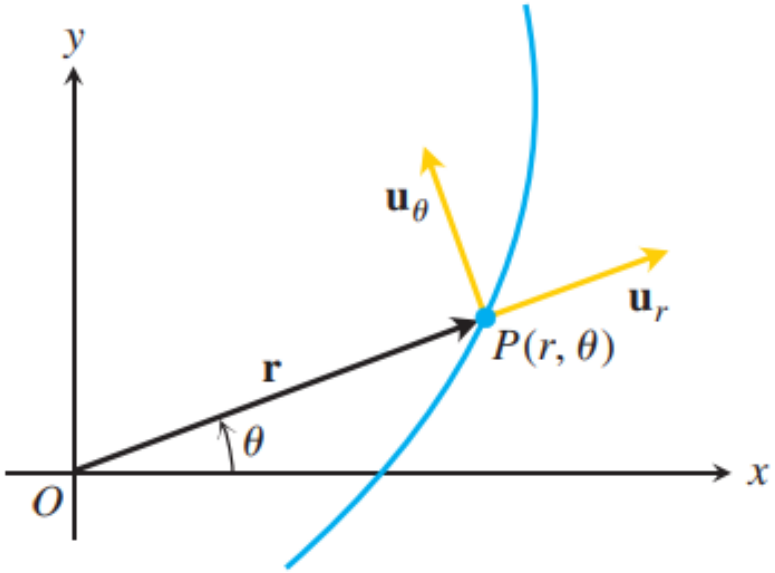
- In cylindrical coordinates, the **equation** $r = a$ describes not just a circle in the xy-plane but an **entire cylinder** about the z-axis.
- The **z-axis** is given by $r = 0$.
- The equation $\theta = \theta_0$ describes the **plane that contains the z-axis** and makes an angle $\theta_0$ with the positive x-axis.
- And, just as in rectangular coordinates, the equation $z = z_0$ describes a **plane perpendicular to the z-axis**.



# Motion in Polar and Cylindrical Coordinates

• When a particle at $P(r, \theta)$ moves along a curve in the polar coordinate plane, we express its position, velocity, and acceleration in terms of the **moving unit vectors**

$$u_r = (\cos\theta)i + (\sin\theta)j, \quad u_\theta = -(\sin\theta)i + (\cos\theta)j,$$




• The length of vector $\mathbf{r}$ is the positive polar coordinate $r$ of the point $P$.
• Thus, $u_r$ is $\mathbf{r}/|\mathbf{r}|$.
• The vector $u_r$ points along the position vector $\overrightarrow{OP}$, so $\mathbf{r} = r\mathbf{u}_r$.
• The vector $u_\theta$ , orthogonal to $u_r$, points in the direction of increasing $u$.

# Motion in Polar and Cylindrical Coordinates

- From equations $u_r$ and $u_\theta$ follow that

$$\frac{d\mathbf{u}_r}{d\theta} = -(\sin\theta)\mathbf{i} + (\cos\theta)\mathbf{j} = \mathbf{u}_\theta$$

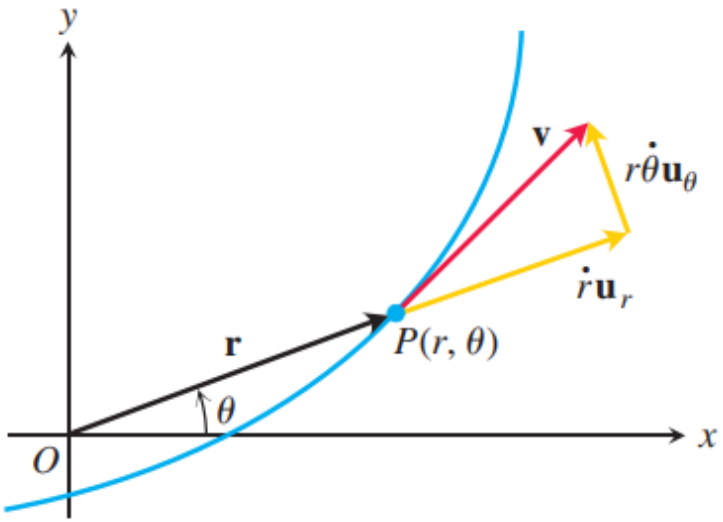
$$\frac{d\mathbf{u}_\theta}{d\theta} = -(\cos\theta)\mathbf{i} - (\sin\theta)\mathbf{j} = -\mathbf{u}_r.$$

- When we differentiate $u_r$ and $u_\theta$ with respect to $t$ to find how they change with time, the Chain Rule gives

$$\dot{\mathbf{u}}_r = \frac{d\mathbf{u}_r}{d\theta}\dot{\theta} = \dot{\theta}\mathbf{u}_\theta, \qquad \dot{\mathbf{u}}_\theta = \frac{d\mathbf{u}_\theta}{d\theta}\dot{\theta} = -\dot{\theta}\mathbf{u}_r.$$

- Hence, we can express the velocity vector in terms of $u_r$ and $u_\theta$ as

$$\mathbf{v} = \dot{\mathbf{r}} = \frac{d}{dt}(r\mathbf{u}_r) = \dot{r}\mathbf{u}_r + r\dot{\mathbf{u}}_r = \dot{r}\mathbf{u}_r + r\dot{\theta}\mathbf{u}_\theta.$$



# Tangential and Normal Components of Acceleration

- The acceleration is

$$\mathbf{a} = \dot{\mathbf{v}} = (\ddot{r}\mathbf{u}_r + \dot{r}\dot{\mathbf{u}}_r) + (\dot{r}\dot{\theta}\mathbf{u}_\theta + r\ddot{\theta}\mathbf{u}_\theta + r\dot{\theta}\dot{\mathbf{u}}_\theta).$$

- Evaluating $\dot{u}_r$ and $\dot{u}_\theta$ and separating components, the equation for acceleration in terms of $\mathbf{u}_r$ and $\mathbf{u}_\theta$ becomes

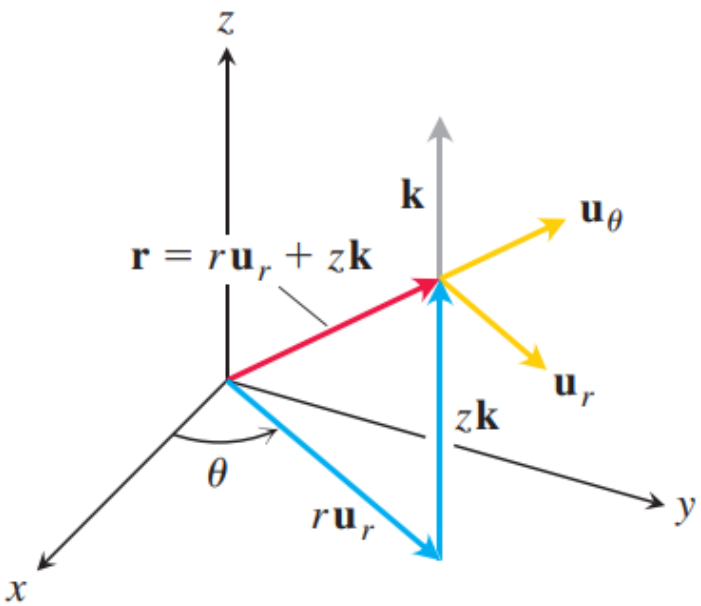
$$\mathbf{a} = (\ddot{r} - r\dot{\theta}^2)\mathbf{u}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\mathbf{u}_\theta.$$

- To extend these equations of motion to space, we add $z\mathbf{k}$ to the right-hand side of the equation $\mathbf{r} = r\mathbf{u}_r$. Then, in these cylindrical coordinates, we have

| | | |
|---|---|---|
| *Position:* | $\mathbf{r} = r\mathbf{u}_r + z\mathbf{k}$ | |
| *Velocity:* | $\mathbf{v} = \dot{r}\mathbf{u}_r + r\dot{\theta}\mathbf{u}_\theta + \dot{z}\mathbf{k}$ | (3) |
| *Acceleration:* | $\mathbf{a} = (\ddot{r} - r\dot{\theta}^2)\mathbf{u}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\mathbf{u}_\theta + \ddot{z}\mathbf{k}$ | |

# Tangential and Normal Components of Acceleration

Position vector and basic unit vectors in cylindrical coordinates.




- The vectors $\mathbf{u}_r$, $\mathbf{u}_\theta$, and $\mathbf{k}$ make a right-handed frame in which

$$\mathbf{u}_r \times \mathbf{u}_\theta = \mathbf{k}, \quad \mathbf{u}_\theta \times \mathbf{k} = \mathbf{u}_r, \quad \mathbf{k} \times \mathbf{u}_r = \mathbf{u}_\theta.$$

## Example 2

Given $r = \frac{1}{\theta}$ and $\frac{d\theta}{dt} = t^2$, find the velocity and acceleration vectors in terms of $u_r$ and $u_\theta$.

**Solution:**

$$\frac{d\theta}{dt} = t^2 = \dot{\theta} \Rightarrow \ddot{\theta} = 2t, \;\; r = \frac{1}{\theta} \Rightarrow \dot{r} = \frac{-\dot{\theta}}{\theta^2} = \frac{-t^2}{\theta^2} \Rightarrow \ddot{r} = \frac{2\theta(\dot{\theta})^2 - \theta^2\ddot{\theta}}{\theta^4} = \frac{2\theta(t^2)^2 - \theta^2(2t)}{\theta^4} = \frac{2t(t^3-\theta)}{\theta^3};$$

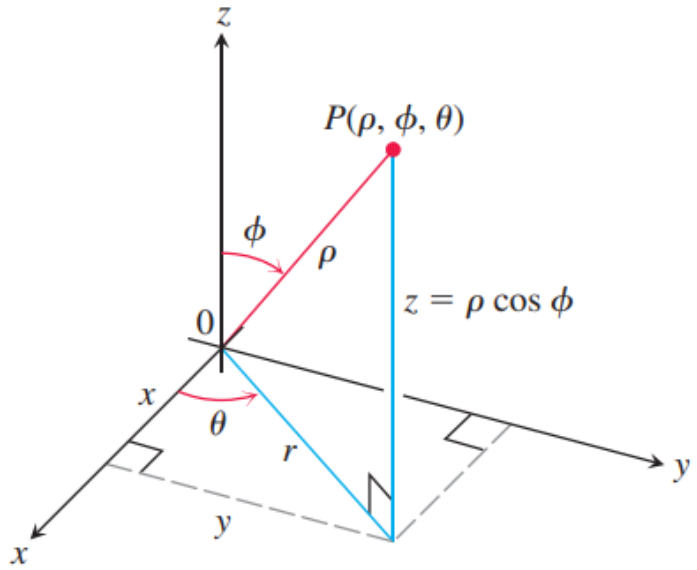
$$\vec{\mathbf{v}}(t) = \left(\frac{-t^2}{\theta^2}\right)\vec{\mathbf{u}}_r + \left(\frac{1}{\theta}\right)\left(t^2\right)\vec{\mathbf{u}}_\theta = \frac{-t^2}{\theta^2}\vec{\mathbf{u}}_r + \frac{t^2}{\theta}\vec{\mathbf{u}}_\theta; \;\; \vec{\mathbf{a}}(t) = \left(\left(\frac{2t(t^3-\theta)}{\theta^3}\right) - \left(\frac{1}{\theta}\right)\left(t^2\right)^2\right)\vec{\mathbf{u}}_r + \left(\left(\frac{1}{\theta}\right)(2t) + 2\left(\frac{-t^2}{\theta^2}\right)\left(t^2\right)\right)\vec{\mathbf{u}}_\theta$$

$$= \frac{t\left(2t^3 - 2\theta - t^3\theta^2\right)}{\theta^3}\vec{\mathbf{u}}_r + \frac{2t\left(\theta - t^3\right)}{\theta^2}\mathbf{u}_\theta$$

# Spherical Coordinates

• **Spherical coordinates** locate points in space with **two angles** and **one distance**, as shown in Figure.

• The first coordinate, $\rho = |\overrightarrow{OP}|$, is the **point's distance from the origin** and is never negative.
• The second coordinate, $\phi$, is the **angle** $\overrightarrow{OP}$ **makes with the positive z-axis**.
• It is required to lie in the interval $[0, \pi]$.
The third coordinate is the **angle** $\theta$ as measured in cylindrical coordinates.



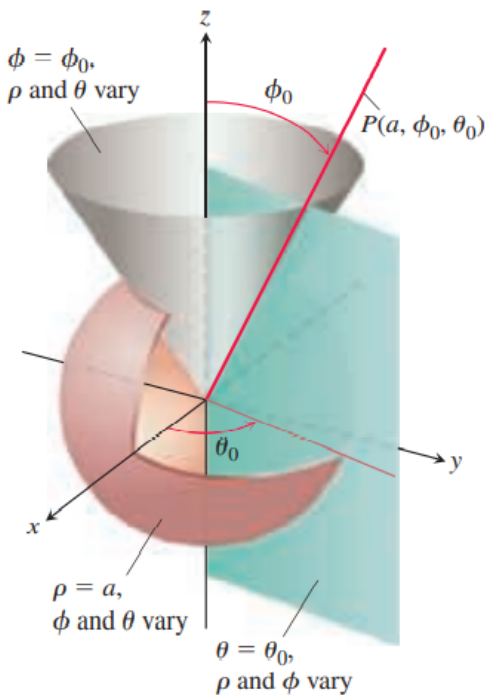

**DEFINITION** **Spherical coordinates** represent a point $P$ in space by ordered triples $(\rho, \phi, \theta)$ in which

1. $\rho$ is the distance from $P$ to the origin ($\rho \geq 0$).
2. $\phi$ is the angle $\overrightarrow{OP}$ makes with the positive $z$-axis ($0 \leq \phi \leq \pi$).
3. $\theta$ is the angle from cylindrical coordinates.

# Spherical Coordinates

• On maps of the earth, $\theta$ is related to the **meridian** of a point on the earth and $\phi$ to its **latitude**, while $r$ is related to **elevation above the earth's surface**.

• The equation $\rho = a$ describes the **sphere** of radius a centered at the origin.
• The equation $\phi = \phi_0$ describes a single **cone** whose vertex lies at the origin and whose axis lies along the z-axis.
• The equation $\theta = \theta_0$ describes the **half-plane** that contains the z-axis and makes an angle $\theta_0$ with the positive x-axis.




**Equations Relating Spherical Coordinates to Cartesian and Cylindrical Coordinates**

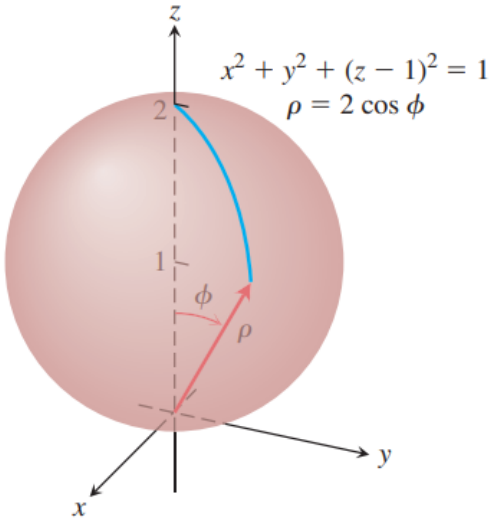
$$r = \rho \sin\phi, \qquad x = r\cos\theta = \rho\sin\phi\cos\theta,$$
$$z = \rho\cos\phi, \qquad y = r\sin\theta = \rho\sin\phi\sin\theta, \qquad (1)$$
$$\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{r^2 + z^2}.$$

## Example 3

Find a spherical coordinate equation for the sphere $x^2 + y^2 + (z-1)^2 = 1$.

**Solution:**

We use Equations (1) to substitute for x, y, and z:




$$
\begin{aligned}
x^2 + y^2 + (z-1)^2 &= 1 \\
\rho^2 \sin^2\phi \cos^2\theta + \rho^2 \sin^2\phi \sin^2\theta + (\rho\cos\phi - 1)^2 &= 1 \qquad \text{Eqs. (1)} \\
\rho^2 \sin^2\phi \underbrace{(\cos^2\theta + \sin^2\theta)}_{1} + \rho^2\cos^2\phi - 2\rho\cos\phi + 1 &= 1 \\
\rho^2 \underbrace{(\sin^2\phi + \cos^2\phi)}_{1} &= 2\rho\cos\phi \\
\rho^2 &= 2\rho\cos\phi \\
\rho &= 2\cos\phi. \qquad \rho > 0
\end{aligned}
$$

- The angle $\phi$ varies from 0 at the north pole of the sphere to $\pi/2$ at the south pole, the angle $\theta$ does not appear in the expression for $\rho$, reflecting the symmetry about the z-axis

# Lecture 12: Partial Derivatives.

MA2032 Vector Calculus

Lecturer: Larissa Serdukova

School of Computing and Mathematical Science
University of Leicester

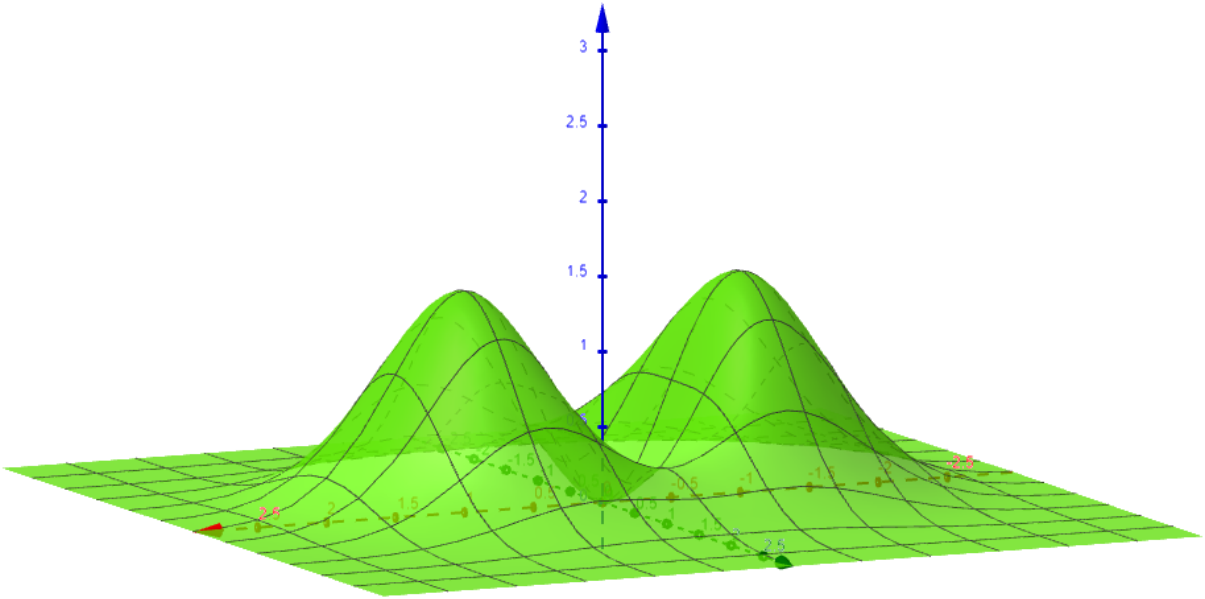
October 17, 2022

# Functions of Several Variables

• We extend the ideas of single-variable differential calculus to **functions of several variables**.

• Their **derivatives are more varied and interesting** because of the different ways the variables can interact.

• The **applications** of these derivatives are also **more varied** than for single-variable calculus, and we will see that the same is true for integrals involving several variables.



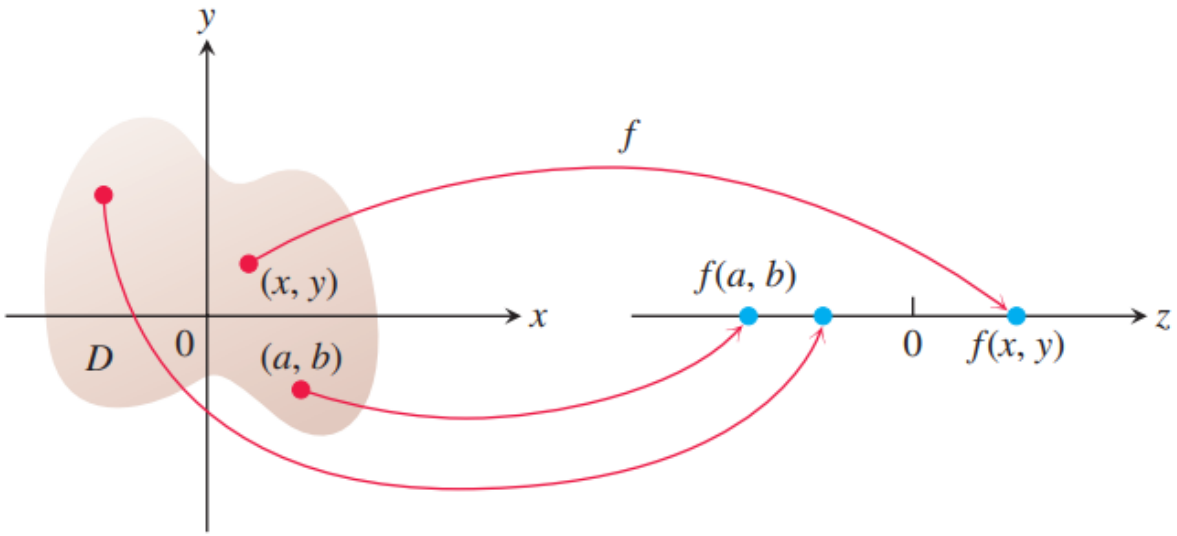
# Functions of Several Variables

**DEFINITIONS** Suppose $D$ is a set of $n$-tuples of real numbers $(x_1, x_2, \ldots, x_n)$. A **real-valued function** $f$ on $D$ is a rule that assigns a unique (single) real number

$$w = f(x_1, x_2, \ldots, x_n)$$

to each element in $D$. The set $D$ is the function's **domain**. The set of $w$-values taken on by $f$ is the function's **range**. The symbol $w$ is the **dependent variable** of $f$, and $f$ is said to be a function of the $n$ **independent variables** $x_1$ to $x_n$. We also call the $x_j$'s the function's **input variables** and call $w$ the function's **output variable**.

- If f is a function of **two independent variables**, we usually call the independent variables **x and y** and the **dependent variable z**, and we picture the **domain of f** as a **region in the xy-plane**.



# Domains and Ranges

• In **defining a function** of more than one variable, we follow the usual practice of **excluding** inputs that lead to **complex numbers or division by zero**.

• The **domain** of a function is assumed to be the largest set for which the defining rule generates real numbers, unless the domain is otherwise specified explicitly.

• The **range** consists of the set of **output values for the dependent variable**.

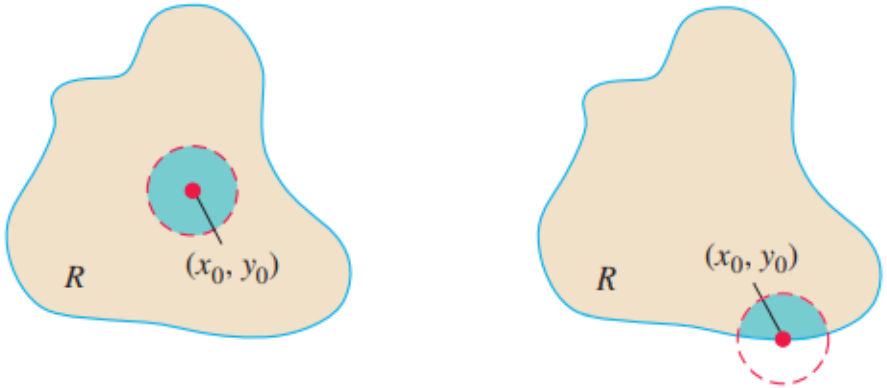
**Example 1:**

| Function | Domain | Range |
|---|---|---|
| $z = \sqrt{y - x^2}$ | $y \geq x^2$ | $[0, \infty)$ |
| $z = \dfrac{1}{xy}$ | $xy \neq 0$ | $(-\infty, 0) \cup (0, \infty)$ |
| $z = \sin xy$ | Entire plane | $[-1, 1]$ |

# Functions of Two Variables

**DEFINITIONS** A point $(x_0, y_0)$ in a region (set) $R$ in the $xy$-plane is an **interior point** of $R$ if it is the center of a disk of positive radius that lies entirely in $R$ (Figure 14.2). A point $(x_0, y_0)$ is a **boundary point** of $R$ if every disk centered at $(x_0, y_0)$ contains points that lie outside of $R$ as well as points that lie in $R$. (The boundary point itself need not belong to $R$.)

The interior points of a region, as a set, make up the **interior** of the region. The region's boundary points make up its **boundary**. A region is **open** if it consists entirely of interior points. A region is **closed** if it contains all its boundary points (Figure 14.3).




(a) Interior point (b) Boundary point

**DEFINITIONS** A region in the plane is **bounded** if it lies inside a disk of finite radius. A region is **unbounded** if it is not bounded.

# Graphs, Level Curves, and Contours

- There are **two standard ways** to picture the values of a function f(x, y).
- One is to draw and **label curves** in the domain on which f has a constant value.
- The other is to **sketch the surface** z = f(x, y) in space.

> **DEFINITIONS** The set of points in the plane where a function $f(x, y)$ has a constant value $f(x, y) = c$ is called a **level curve** of $f$. The set of all points $(x, y, f(x, y))$ in space, for $(x, y)$ in the domain of $f$, is called the **graph** of $f$. The graph of $f$ is also called the **surface** $\boldsymbol{z = f(x, y)}$.

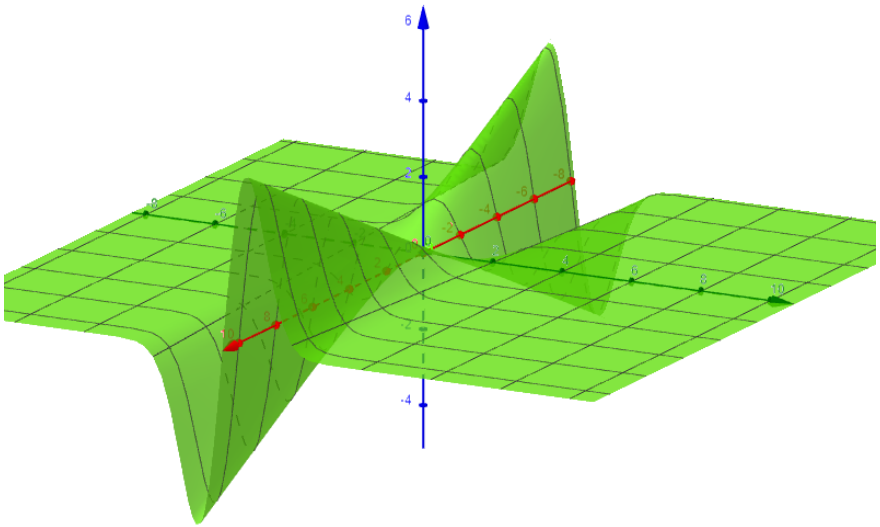
- The curve in space in which the plane $z = c$ cuts a surface $z = f(x, y)$ is made up of the points that represent the function value $f(x, y) = c$.
- It is called the **contour curve** $f(x, y) = c$ to distinguish it from the level curve $f(x, y) = c$ in the domain of $f$.

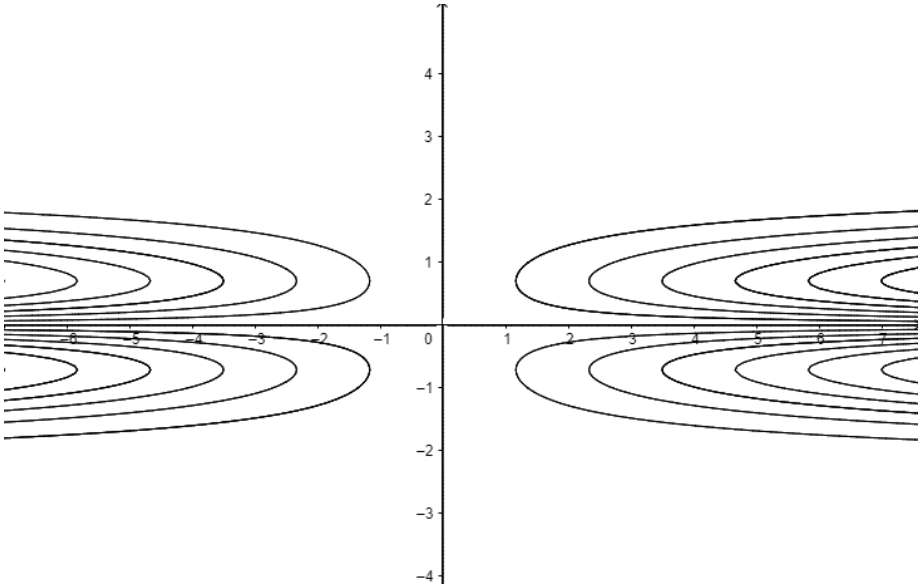
## Example 2

Graph a) $f(x, y) = xye^{-y^2}$ and b) $f(x, y) = \sin x + 2\sin y$ and plot some of the level curves.

**Solution:**

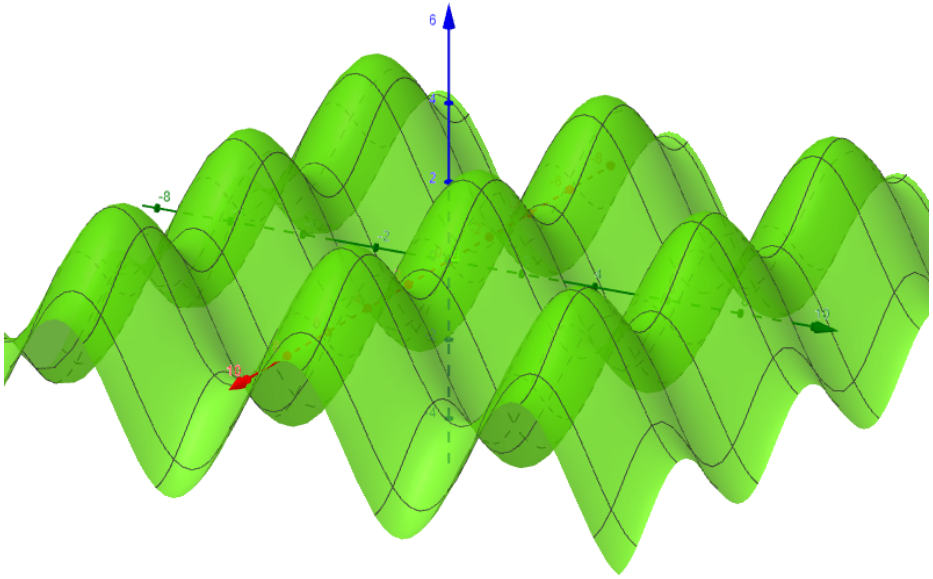
a)

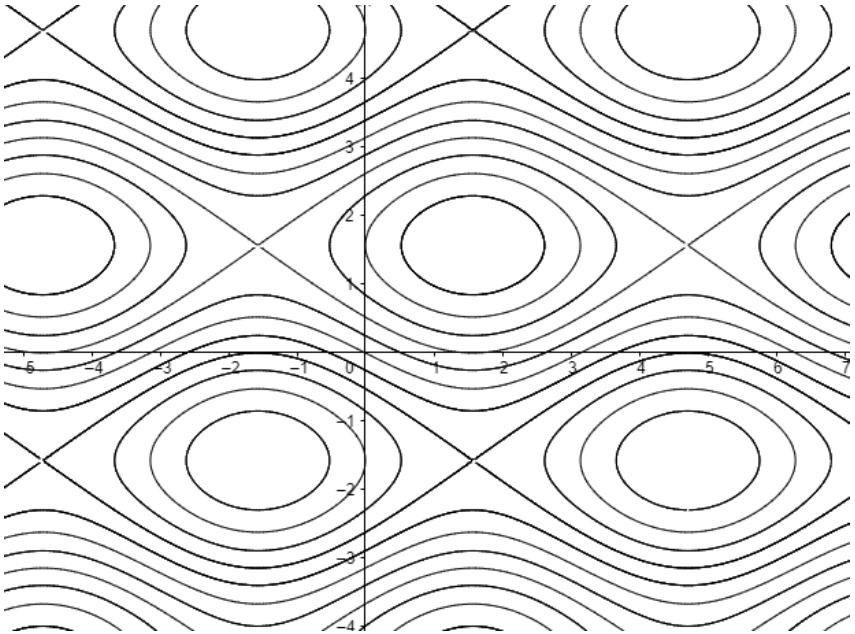




**Solution Example 2:**

b)





# Functions of Three Variables

**DEFINITION** The set of points $(x, y, z)$ in space where a function of three independent variables has a constant value $f(x, y, z) = c$ is called a **level surface** of $f$.

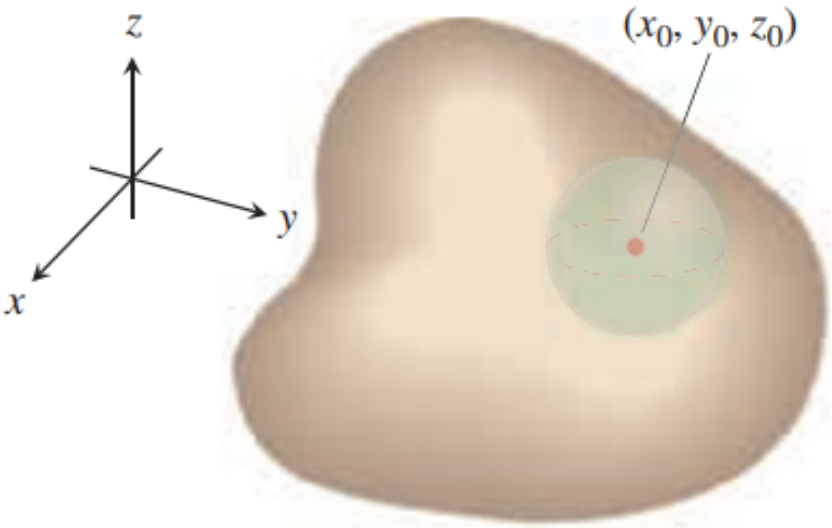
• Since the graphs of functions of three variables consist of points $(x, y, z, f(x, y, z))$ lying in a **four-dimensional space**, we cannot sketch them effectively in our three-dimensional frame of reference.

• We can see how the function behaves, however, by looking at its **three-dimensional level surfaces**.

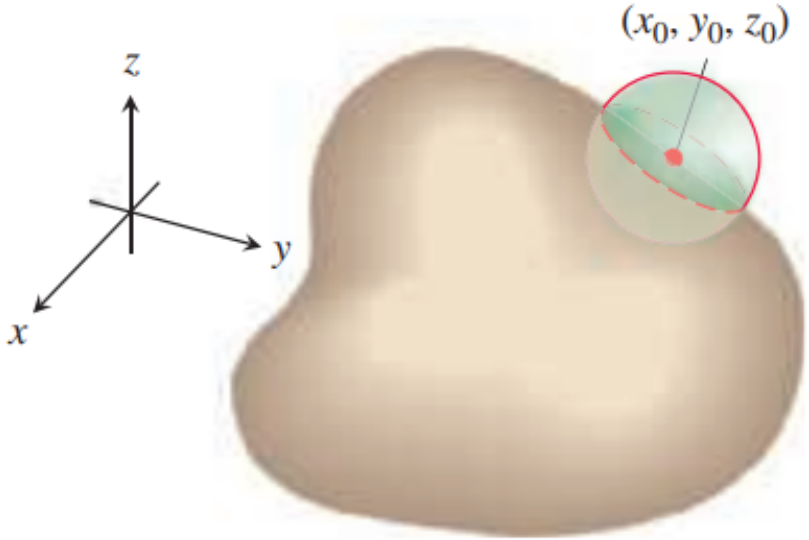
# Functions of Three Variables

**DEFINITIONS** A point $(x_0, y_0, z_0)$ in a region $R$ in space is an **interior point** of $R$ if it is the center of a solid ball that lies entirely in $R$ (Figure 14.9a). A point $(x_0, y_0, z_0)$ is a **boundary point** of $R$ if every solid ball centered at $(x_0, y_0, z_0)$ contains points that lie outside of $R$ as well as points that lie inside $R$ (Figure 14.9b). The **interior** of $R$ is the set of interior points of $R$. The **boundary** of $R$ is the set of boundary points of $R$.

A region is **open** if it consists entirely of interior points. A region is **closed** if it contains its entire boundary.




(a) Interior point




(b) Boundary point

## Example 3

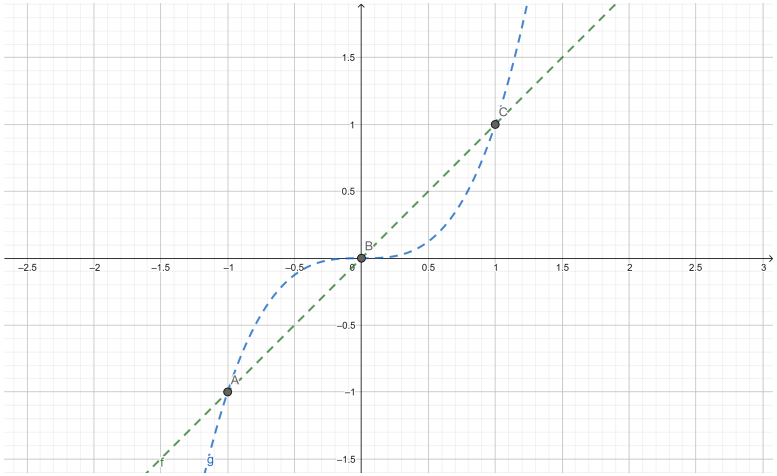
Find and sketch the domain for each function:

a) $f(x,y) = \frac{(x-1)(y+2)}{(y-x)(y-x^3)}$, b) $f(x,y) = cos^{-1}(y - x^2)$,

c) $f(x,y) = \sqrt{(x^2-4)(y^2-9)}$.

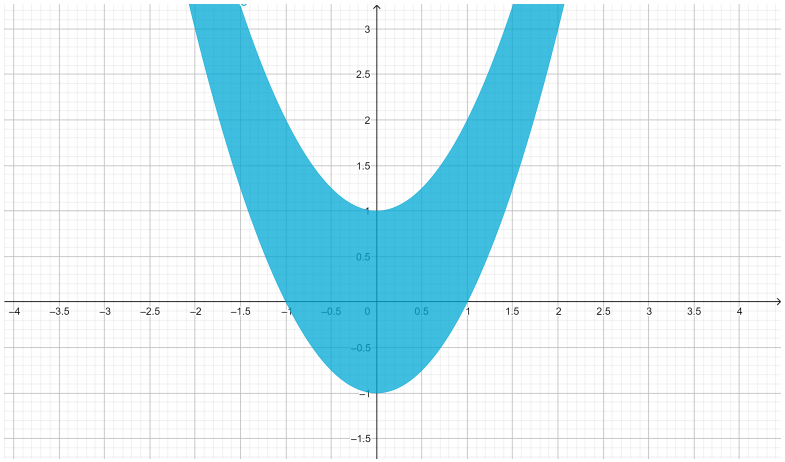
**Solution:**

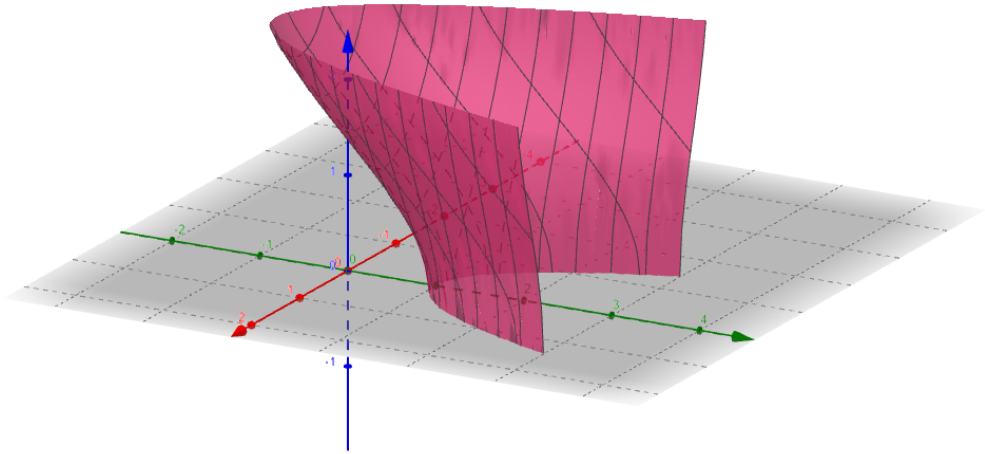
a) Domain: all points $(x,y)$ not lying on the graph of $y = x$, $y = x^3$.



**Solution Example 3:**

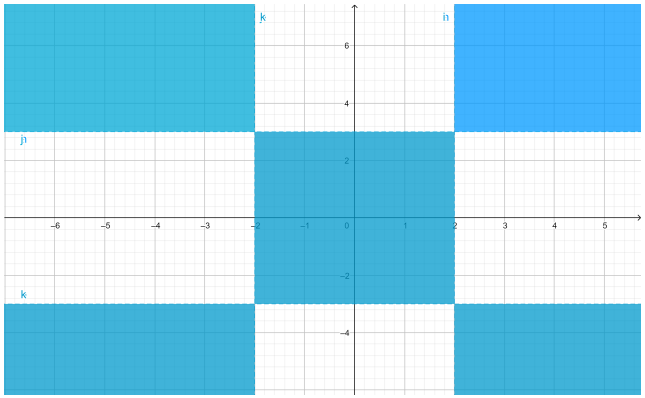
b) Domain: As $\cos^{-1}(y - x^2)$ is defined on the $[-1, 1]$, those all points $(x, y)$ satisfying $x^2 - 1 \leq y \leq x^2 + 1$.

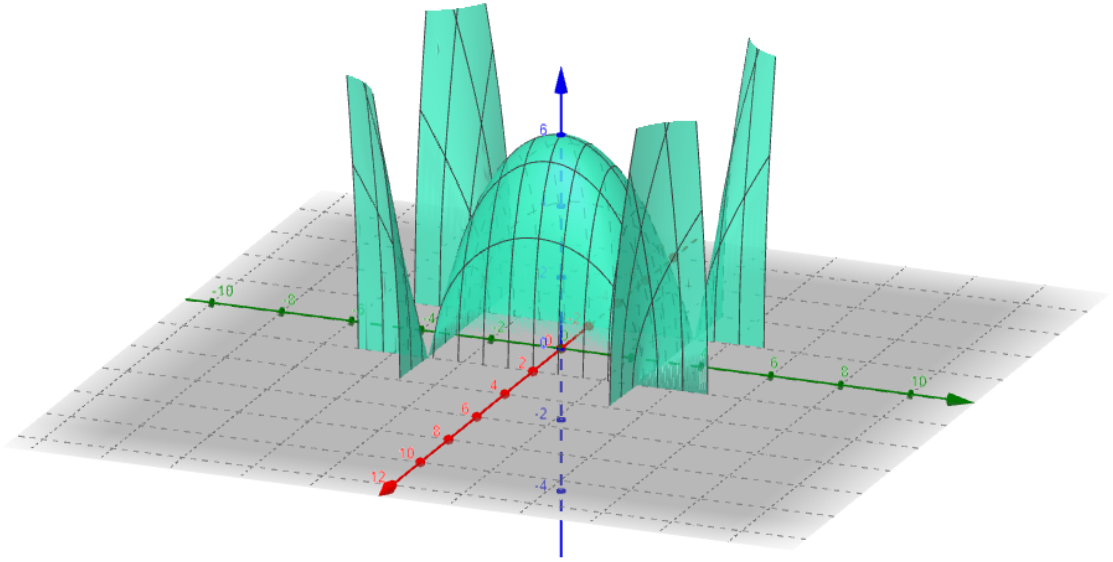




**Solution Example 3:**

c) Domain: all points $(x, y)$ x y satisfying $(x+2)(x-2)(y-3)(y+3) \geq 0$.



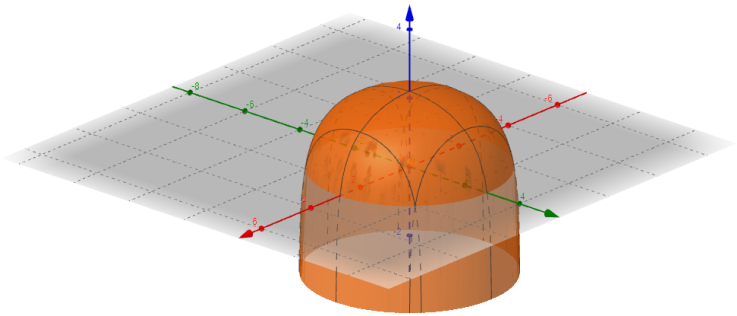


## Example 4

Given function $f(x, y) = \ln(9 - x^2 - y^2)$,
a) find the function's domain, b) find the function's range, c) describe the function's level curves, d) find the boundary of the function's domain, e) determine if the domain is an open region, a closed region, or neither, and (f) decide if the domain is bounded or unbounded.

**Solution:**

(a) Domain: all points $(x, y)$ inside the circle $x^2 + y^2 = 9$
(b) Range: $z < \ln 9$
(c) Circles centered at the origin with radii $r < 9$
(d) Boundary: the circle $x^2 + y^2 = 9$
(e) open
(f) bounded



# Level Surface of the Function

## Example 4

Find an equation for the level surface of the function $f(x, y, z) = \sqrt{x - y} - \ln z$ through the point $(3, -1, 1)$.

**Solution:**

$$f(x, y, z) = \sqrt{x-y} - \ln z \quad \text{at } (3, -1, 1) \Rightarrow w = \sqrt{x-y} - \ln z; \text{ at } (3, -1, 1) \Rightarrow w = \sqrt{3-(-1)} - \ln 1 = 2$$
$$\Rightarrow \sqrt{x-y} - \ln z = 2$$

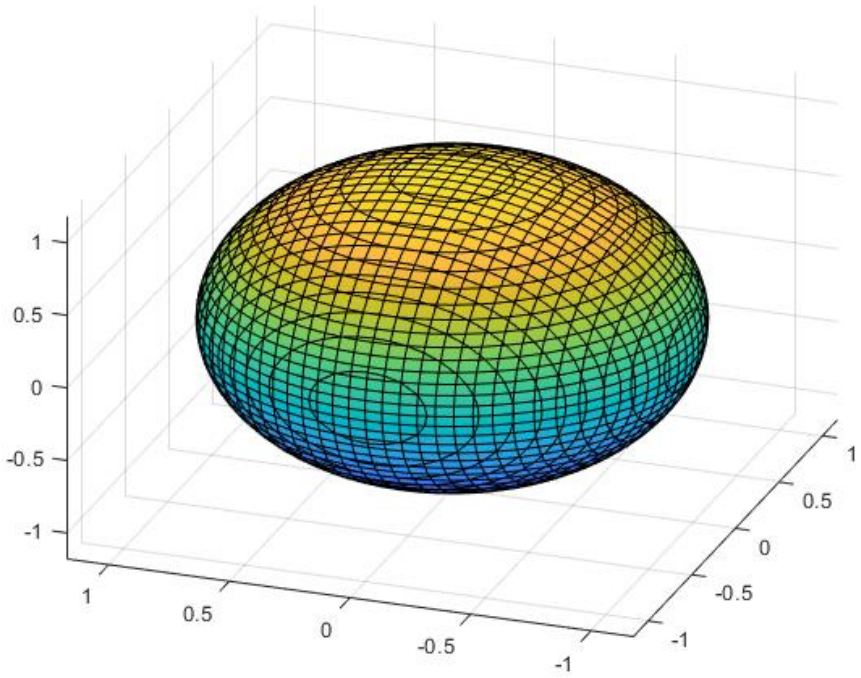
# Level Surface of the Function

## Example 5

Sketch a typical level surface for the function $f(x, y, z) = \ln(x^2 + y^2 + z^2)$.

**Solution:**

For example, $\ln(x^2 + y^2 + z^2) = 0$

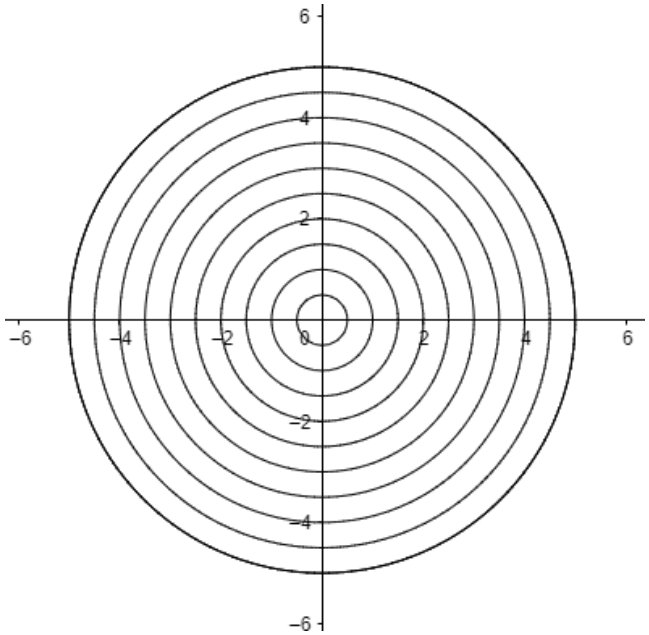


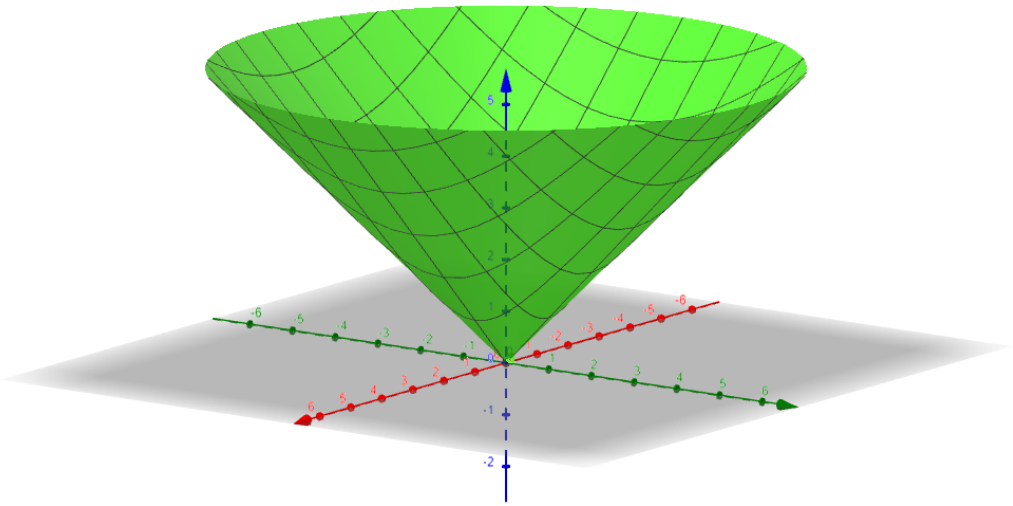
## Example 6

Display the values of the functions in two ways:
(a) by sketching the surface $z = f_n(x, y)$ and (b) by drawing an assortment of level curves in the function's domain.
$f_1(x, y) = \sqrt{x^2 + y^2}, \quad f_2(x, y) = 1 - |x| - |y|, \quad f_3(x, y) = x^2 - y.$

**Solution:**

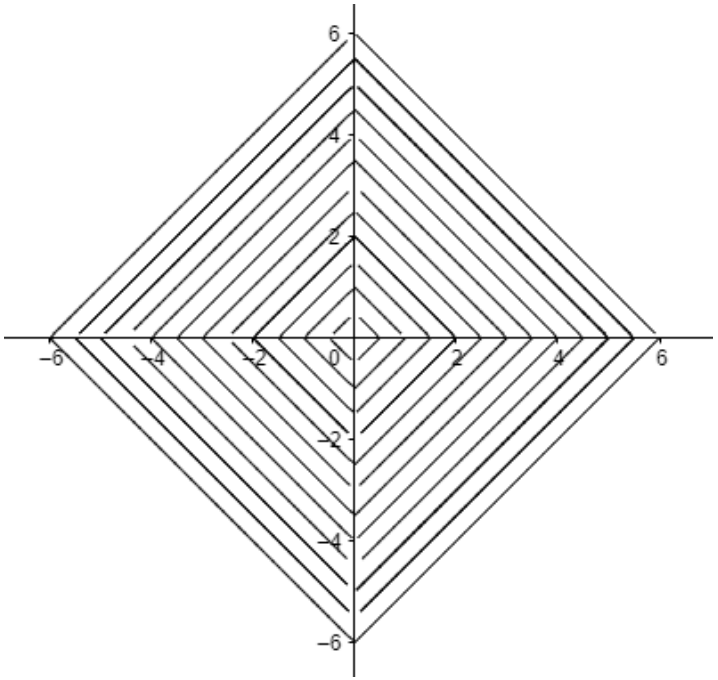
Function $f_1$

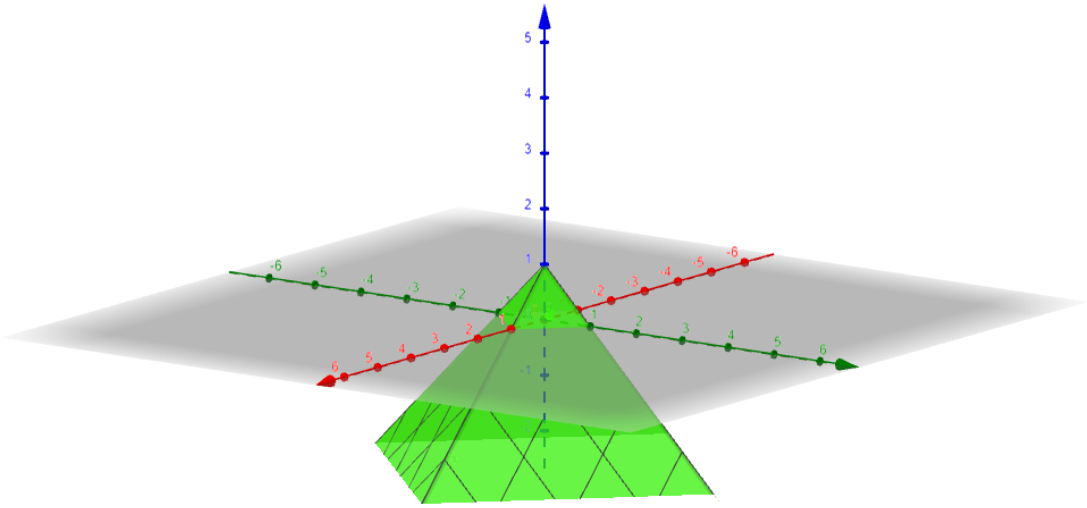


**Solution for Example 6:**

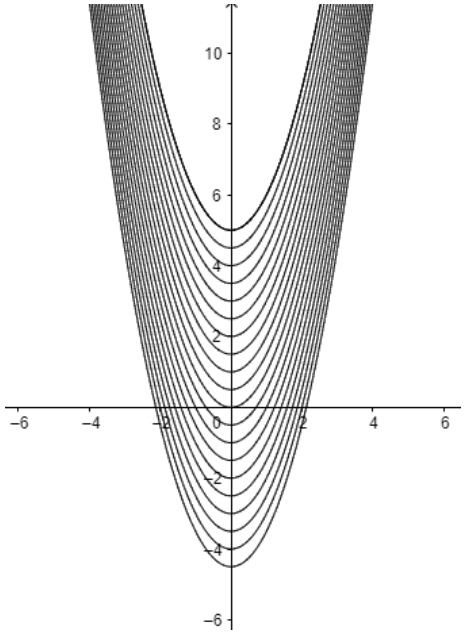
Function $f_2$

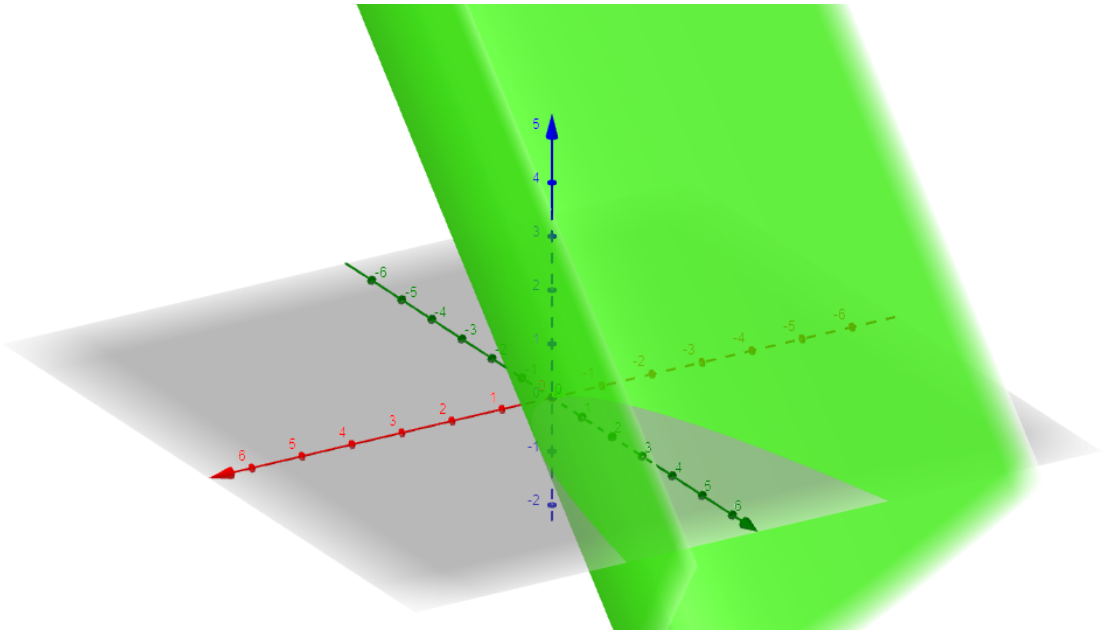


**Solution:**

Function $f_2$



# Lecture 13: Partial Derivatives.

MA2032 Vector Calculus

Lecturer: Larissa Serdukova

School of Computing and Mathematical Science
University of Leicester

October 19, 2022

# Limits and Continuity in Higher Dimensions

• In this section we develop **limits and continuity for multivariable functions**.

• The theory is **similar** to that developed for single-variable functions, but since we now have more than one independent variable, there is **additional complexity** that requires some **new ideas**.

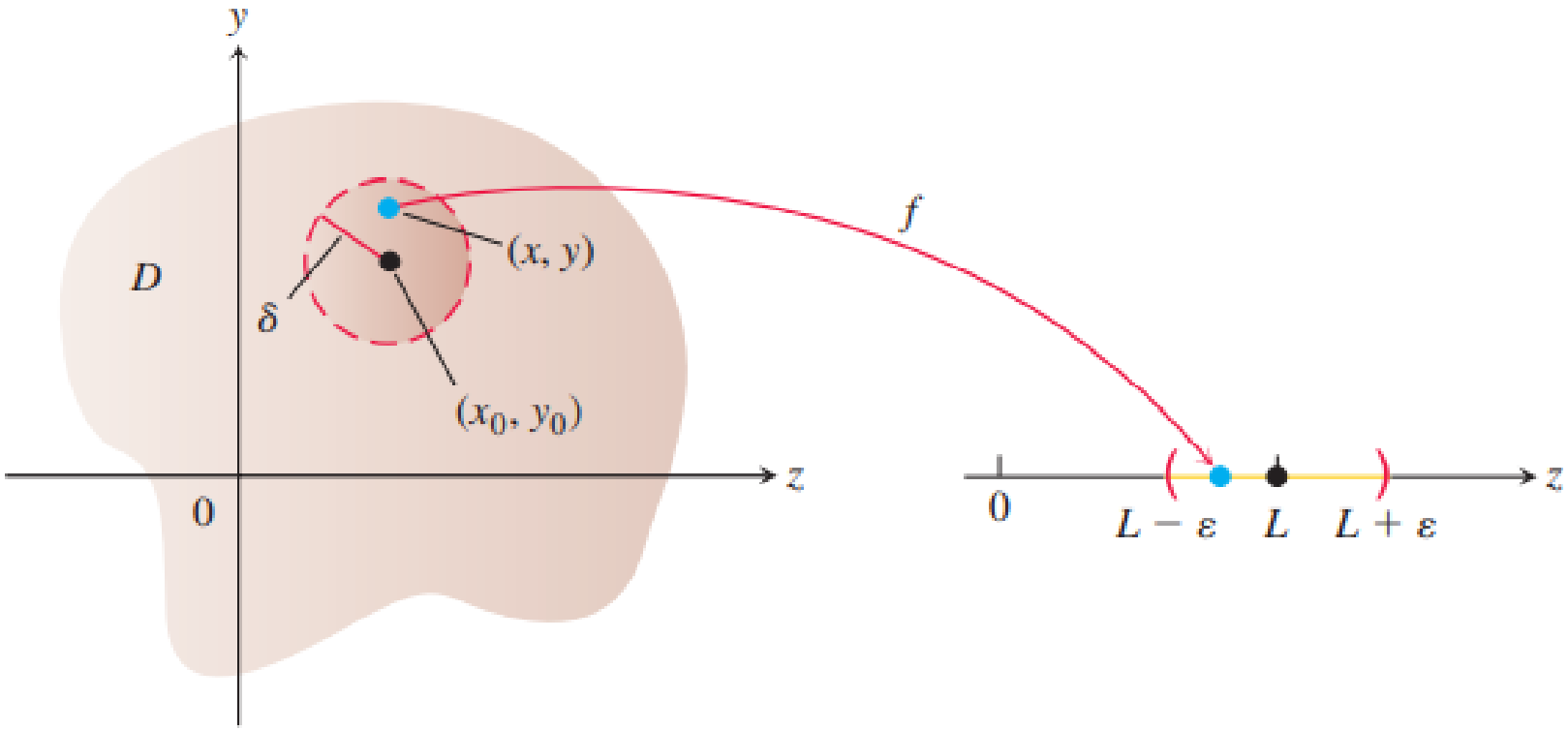
• **Limits for Functions of Two Variables**: If the values of $f(x, y)$ lie arbitrarily close to a fixed real number $L$ for all points $(x, y)$ sufficiently close to a point $(x_0, y_0)$, we say that f approaches the limit $L$ as $(x, y)$ approaches $(x_0, y_0)$.

• This is similar to the **informal definition** for the limit of a function of a single variable.

# Limits for Functions of Two Variables

• Notice, however, that when $(x_0, y_0)$ lies in the interior of f's domain, $(x, y)$ **can approach** $(x_0, y_0)$ **from any direction**, not just from the left or the right.

• **For the limit to exist**, the **same limiting value** must be obtained **whatever direction** of approach is taken.



# Limits for Functions of Two Variables

**DEFINITION** We say that a function $f(x, y)$ approaches the **limit** $L$ as $(x, y)$ approaches $(x_0, y_0)$, and write

$$\lim_{(x,\, y) \to (x_0,\, y_0)} f(x, y) = L$$

if, for every number $\varepsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all $(x, y)$ in the domain of $f$,

$$|f(x, y) - L| < \varepsilon \qquad \text{whenever} \qquad 0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta.$$

• The definition of limit says that the distance between $f(x, y)$ and $L$ **becomes** arbitrarily small whenever the distance from $(x, y)$ to $(x_0, y_0)$ is **made** sufficiently small (but not 0).

• The definition applies to interior points $(x_0, y_0)$ as well as boundary points of the domain of f, although a boundary point need not lie within the domain.

• The points $(x, y)$ that approach $(x_0, y_0)$ are always taken to be in the domain of f.

# Limits for Functions of Two Variables

- As with single-variable functions, the limit of the sum of two functions is the sum of their limits **(when they both exist)**, with similar results for the limits of the differences, constant multiples, products, quotients, powers, and roots.
- These facts are summarized in **Theorem 1**.

**THEOREM 1—Properties of Limits of Functions of Two Variables**
The following rules hold if $L$, $M$, and $k$ are real numbers and

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L \quad \text{and} \quad \lim_{(x,y)\to(x_0,y_0)} g(x,y) = M.$$

1. *Sum Rule:* $\displaystyle\lim_{(x,y)\to(x_0,y_0)} (f(x,y) + g(x,y)) = L + M$
2. *Difference Rule:* $\displaystyle\lim_{(x,y)\to(x_0,y_0)} (f(x,y) - g(x,y)) = L - M$
3. *Constant Multiple Rule:* $\displaystyle\lim_{(x,y)\to(x_0,y_0)} kf(x,y) = kL$ (any number $k$)
4. *Product Rule:* $\displaystyle\lim_{(x,y)\to(x_0,y_0)} (f(x,y)\cdot g(x,y)) = L\cdot M$
5. *Quotient Rule:* $\displaystyle\lim_{(x,y)\to(x_0,y_0)} \frac{f(x,y)}{g(x,y)} = \frac{L}{M}, \quad M \neq 0$
6. *Power Rule:* $\displaystyle\lim_{(x,y)\to(x_0,y_0)} [f(x,y)]^n = L^n$, $n$ a positive integer
7. *Root Rule:* $\displaystyle\lim_{(x,y)\to(x_0,y_0)} \sqrt[n]{f(x,y)} = \sqrt[n]{L} = L^{1/n}$,
   $n$ a positive integer, and if $n$ is even, we assume that $L > 0$.

## Example 1

Find

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}}$$

.

## Solution:

Since the denominator $\sqrt{x} - \sqrt{y}$ approaches 0 as $(x, y) \to (0, 0)$, we cannot use the Quotient Rule from Theorem 1. If we multiply numerator and denominator by $\sqrt{x} + \sqrt{y}$, however, we produce an equivalent fraction whose limit we can find:

$$\begin{aligned}
\lim_{(x,y)\to(0,0)} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}} &= \lim_{(x,y)\to(0,0)} \frac{(x^2 - xy)(\sqrt{x} + \sqrt{y})}{(\sqrt{x} - \sqrt{y})(\sqrt{x} + \sqrt{y})} && \text{Multiply by a form equal to 1.} \\
&= \lim_{(x,y)\to(0,0)} \frac{x(x - y)(\sqrt{x} + \sqrt{y})}{x - y} && \text{Algebra} \\
&= \lim_{(x,y)\to(0,0)} x(\sqrt{x} + \sqrt{y}) && \text{Cancel the nonzero factor } (x - y). \\
&= 0(\sqrt{0} + \sqrt{0}) = 0 && \text{Known limit values}
\end{aligned}$$

We can cancel the factor $(x - y)$ because the path $y = x$ (where we would have $x - y = 0$) is not in the domain of the given function.

# Continuity

• As with functions of a single variable, continuity is defined **in terms of limits**.

**DEFINITION** A function $f(x, y)$ is **continuous at the point $(x_0, y_0)$** if

1. $f$ is defined at $(x_0, y_0)$,
2. $\lim_{(x,y)\to(x_0,y_0)} f(x, y)$ exists,
3. $\lim_{(x,y)\to(x_0,y_0)} f(x, y) = f(x_0, y_0)$.

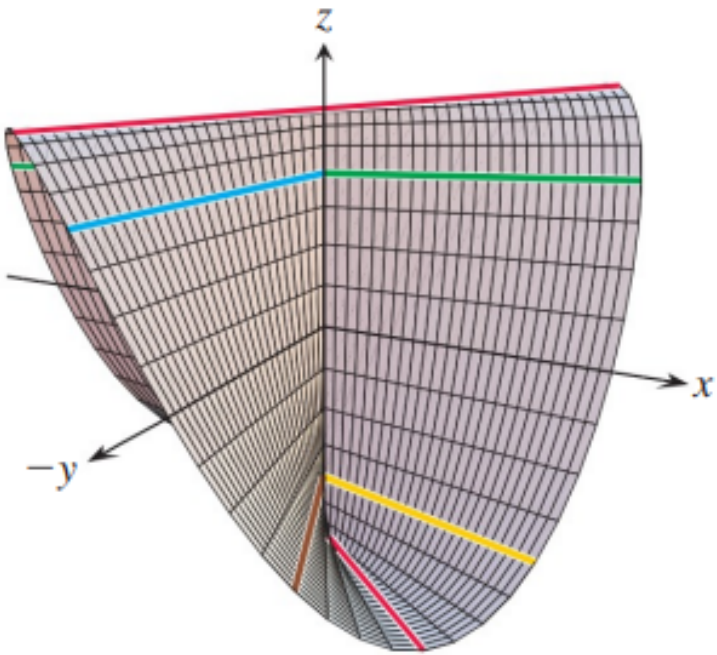
A function is **continuous** if it is continuous at every point of its domain.

# Continuity. Example 2

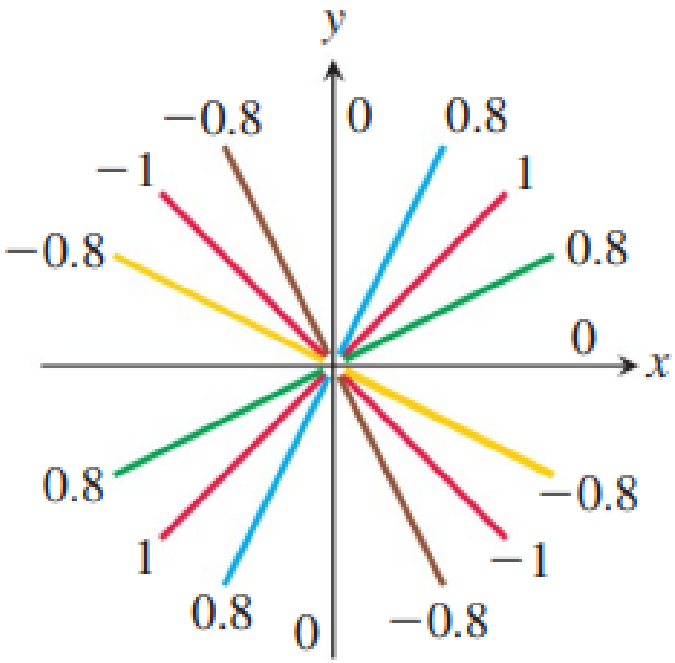
Show that $f(x, y)$ is continuous at every point except the origin.

$$f(x, y) = \begin{cases} \dfrac{2xy}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

**Solution:**

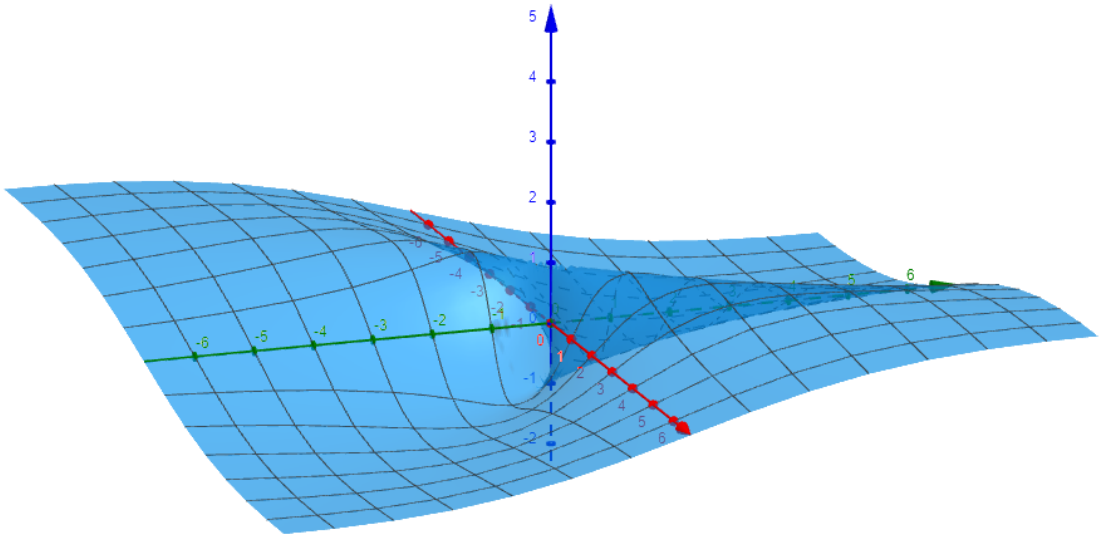


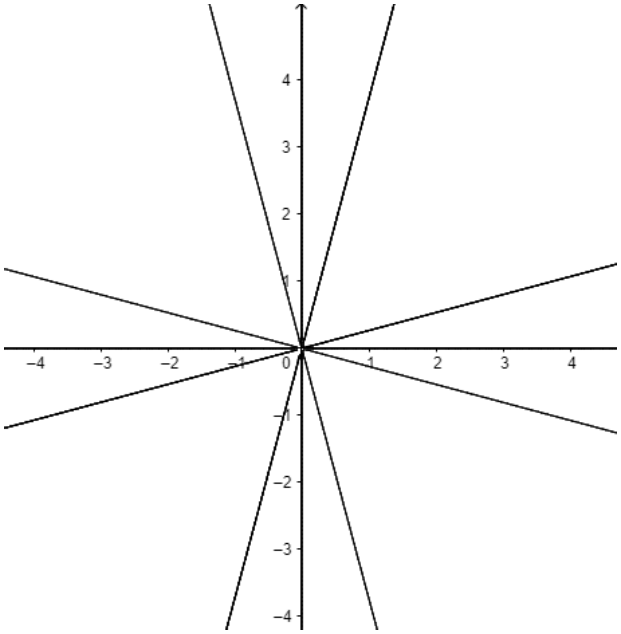

(a)




(b)

# Continuity. Example 2





# Continuity. Example 2

• The function $f$ is **continuous at every point** $(x, y)$ **except** $(0, 0)$ because its values at points other than $(0, 0)$ are given by a rational function of x and y, and therefore at those points the limiting value is simply obtained by **substituting** the values of x and y into that rational expression.

• At $(0, 0)$, the value of f is **defined**, but **f has no limit as** $(x, y) \to (0, 0)$.

• The reason is that **different paths** of approach to the origin can lead to **different results**, as we now see.

# Continuity. Example 2

• For every value of $m$, the function f has a **constant value on the line** $y = mx, x \neq 0$, because

$$f(x, y)\Big|_{y=mx} = \frac{2xy}{x^2 + y^2}\Big|_{y=mx} = \frac{2x(mx)}{x^2 + (mx)^2} = \frac{2mx^2}{x^2 + m^2x^2} = \frac{2m}{1 + m^2}.$$

• Therefore, f has this number as its limit as $(x, y)$ approaches $(0, 0)$ along the line:

$$\lim_{\substack{(x,\,y)\to(0,0)\\ \text{along } y=mx}} f(x, y) = \lim_{(x,\,y)\to(0,0)} \left[ f(x, y)\Big|_{y=mx} \right] = \frac{2m}{1 + m^2}.$$

• This **limit changes** with each value of the slope m.

• There is therefore **no single number** we may call the limit of f as $(x, y)$ approaches the origin.

• The **limit fails to exist**, and the function is **not continuous** at the origin.

# Two-Path Test for Nonexistence of a Limit

• Examples 2 illustrate an **important point** about limits of functions of two or more variables.
• For a limit to exist at a point, the **limit must be the same along every approach path**.
• This result is **analogous to the single-variable case** where both the left- and right-sided limits had to have the same value.
• For functions of two or more variables, if we ever find **paths with different limits**, we know the **function has no limit** at the point they approach.

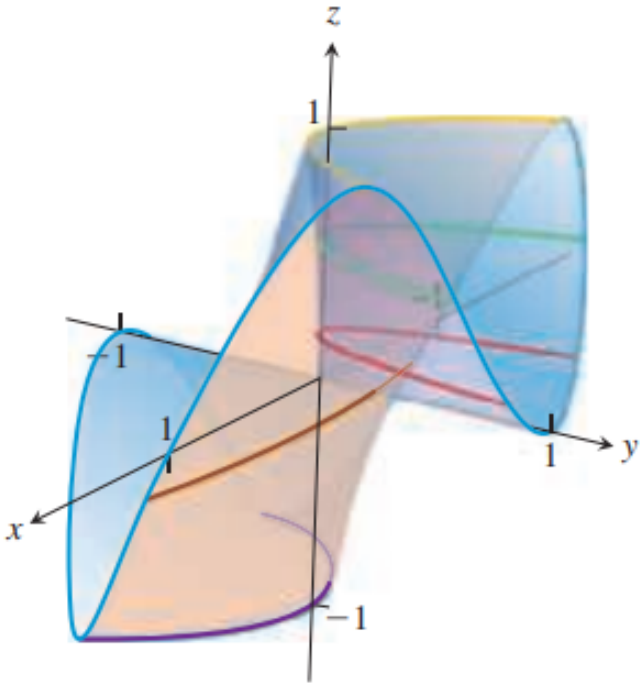
**Two-Path Test for Nonexistence of a Limit**
If a function $f(x, y)$ has different limits along two different paths in the domain of $f$ as $(x, y)$ approaches $(x_0, y_0)$, then $\lim_{(x, y)\to(x_0, y_0)} f(x, y)$ does not exist.

Having the same limit along all straight lines approaching $(x_0, y_0)$ does not imply that a limit exists at $(x_0, y_0)$.

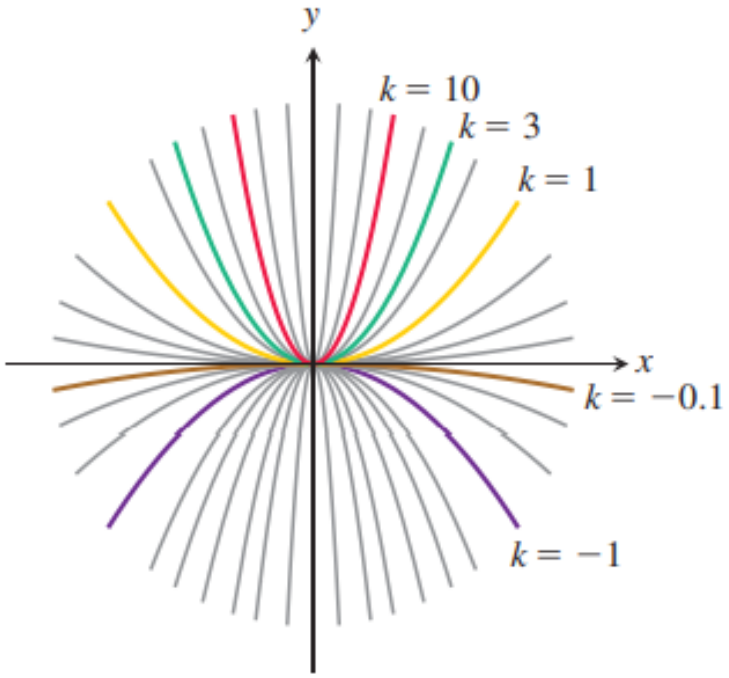
# Continuity. Example 3

Show that the function $f(x, y) = \frac{2x^2y}{x^4+y^2}$ has no limit as $(x, y)$ approaches $(0, 0)$.

**Solution:**

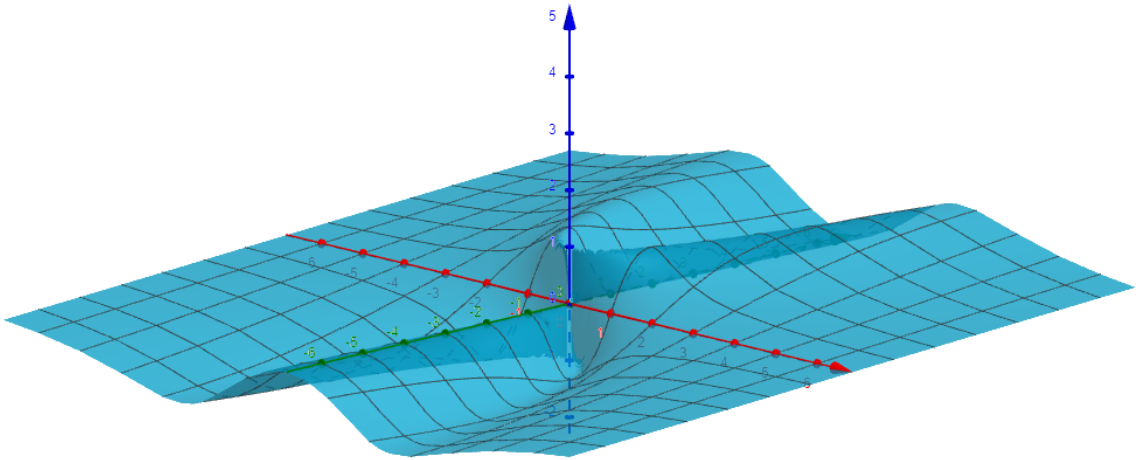


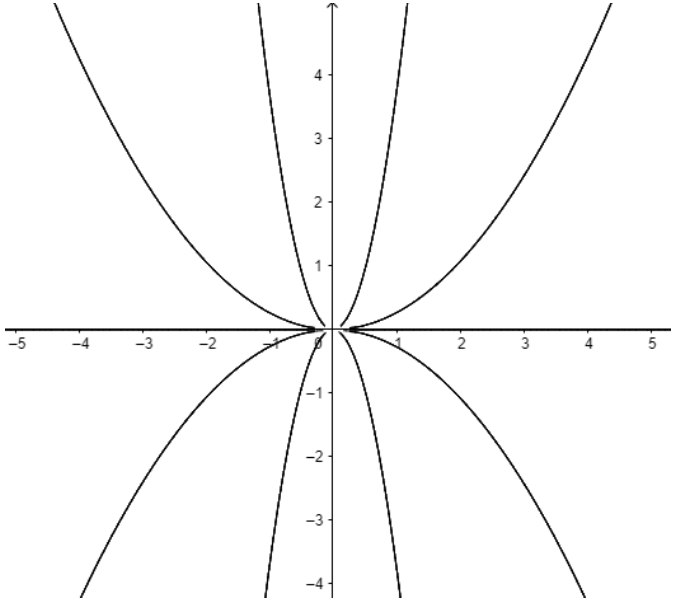

(a)




(b)

# Continuity. Example 3





# Continuity. Example 3

**Solution:**

• The limit cannot be found by direct substitution, which gives the indeterminate form $0/0$.
• We examine the values of f along parabolic curves that end at $(0, 0)$.
• Along the curve $y = kx^2$, $x \neq 0$, the function has the constant value

$$f(x, y)\Big|_{y=kx^2} = \frac{2x^2y}{x^4 + y^2}\Big|_{y=kx^2} = \frac{2x^2(kx^2)}{x^4 + (kx^2)^2} = \frac{2kx^4}{x^4 + k^2x^4} = \frac{2k}{1 + k^2}.$$

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{along } y=kx^2}} f(x, y) = \lim_{(x,y)\to(0,0)} \left[ f(x, y)\Big|_{y=kx^2} \right] = \frac{2k}{1 + k^2}.$$

• This limit varies with the path of approach. If $(x, y)$ approaches $(0, 0)$ along the parabola $y = x^2$, for instance, $k = 1$ and the limit is 1.
• If $(x, y)$ approaches $(0, 0)$ along the x-axis, $k = 0$ and the limit is 0.
• By the two-path test, f has no limit as $(x, y)$ approaches $(0, 0)$.

# Continuity, Extreme Values

**Continuity of Compositions**

If $f$ is continuous at $(x_0, y_0)$ and $g$ is a single-variable function continuous at $f(x_0, y_0)$, then the composition $h = g \circ f$ defined by $h(x, y) = g(f(x, y))$ is continuous at $(x_0, y_0)$.

• **Functions of More Than Two Variables**:
The definitions of limit and continuity for functions of two variables and the conclusions about limits and continuity for sums, products, quotients, powers, and compositions all extend to functions of three or more variables.

• **Extreme Values of Continuous Functions on Closed, Bounded Sets**: A function of a single variable that is continuous at every point of a closed, bounded interval $[a, b]$ takes on an absolute maximum value and an absolute minimum value at least once in $[a, b]$.

# Extreme Values

• The same holds true of a function $z = (x, y)$ that is continuous on a closed, bounded set $R$ in the plane.

• The function takes on **an absolute maximum** value at some point in R and an **absolute minimum value** at some point in R.

• The function may take on a maximum or minimum value **more than once over R**.

• **Similar results hold** for functions of three or more variables.

# Lecture 14: Partial Derivatives.

MA2032 Vector Calculus

Lecturer: Larissa Serdukova

School of Computing and Mathematical Science
University of Leicester

October 21, 2022

# Partial Derivatives

• When we **hold all** but one of the independent variables of a function **constant** and **differentiate with respect to that one** variable, we get a **"partial" derivative**.

• Today, I am going to show how partial derivatives are **defined and interpreted geometrically**, and how to calculate them by applying the rules for differentiating functions of a single variable.

• The idea of **differentiability** for functions of several variables requires more than the existence of the partial derivatives because a point can be **approached from many different directions**.

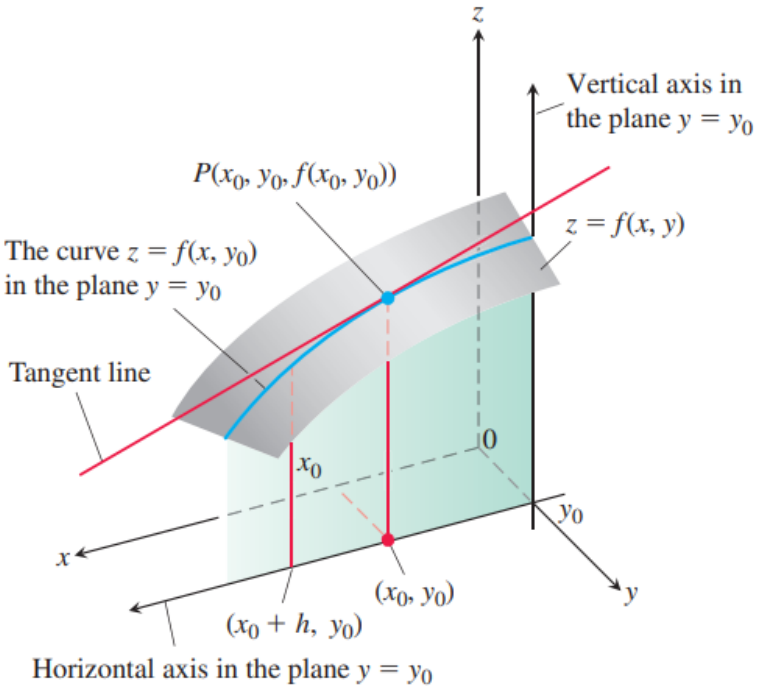
# Partial Derivatives of a Function of Two Variables

• If $(x_0, y_0)$ is a **point in the domain of a function** $f(x, y)$, the **vertical plane** $y = y_0$ will **cut the surface** $z = f(x, y)$ in the curve $z = f(x, y_0)$.

• This **curve is the graph of the function** $z = f(x, y_0)$ in the plane $y = y_0$.

The horizontal coordinate in this plane is x; the vertical coordinate is z.

• The **y-value is held constant at** $y_0$, so y is not a variable.




• We define the **partial derivative of f with respect to x** at the point $(x_0, y_0)$ as the ordinary derivative of $f(x, y_0)$ with respect to x at the point $x = x_0$.

• In the definition, h represents a real number, positive or negative.

# Partial Derivatives of a Function of Two Variables

**DEFINITION** The **partial derivative of $f(x, y)$ with respect to $x$ at the point $(x_0, y_0)$** is

$$\left.\frac{\partial f}{\partial x}\right|_{(x_0,\, y_0)} = \lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h},$$

provided the limit exists.

• The **slope of the curve** $z = f(x, y_0)$ at the point $P(x_0, y_0, f(x_0, y_0))$ in the plane $y = y_0$ is the **value of the partial derivative** of f with respect to x at $(x_0, y_0)$.

• The **tangent line to the curve** at P is the line in the plane $y = y_0$ that passes through P with this slope.

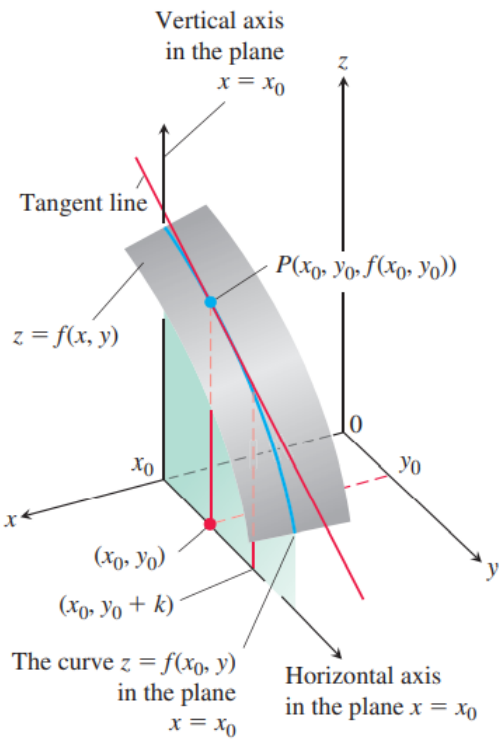
• The partial derivative $\frac{\partial f}{\partial x}$ at $(x_0, y_0)$ gives the **rate of change of f with respect to x** when y is held fixed at the value $y_0$.

# Partial Derivatives of a Function of Two Variables

**DEFINITION** The **partial derivative of $f(x, y)$ with respect to $y$ at the point $(x_0, y_0)$** is

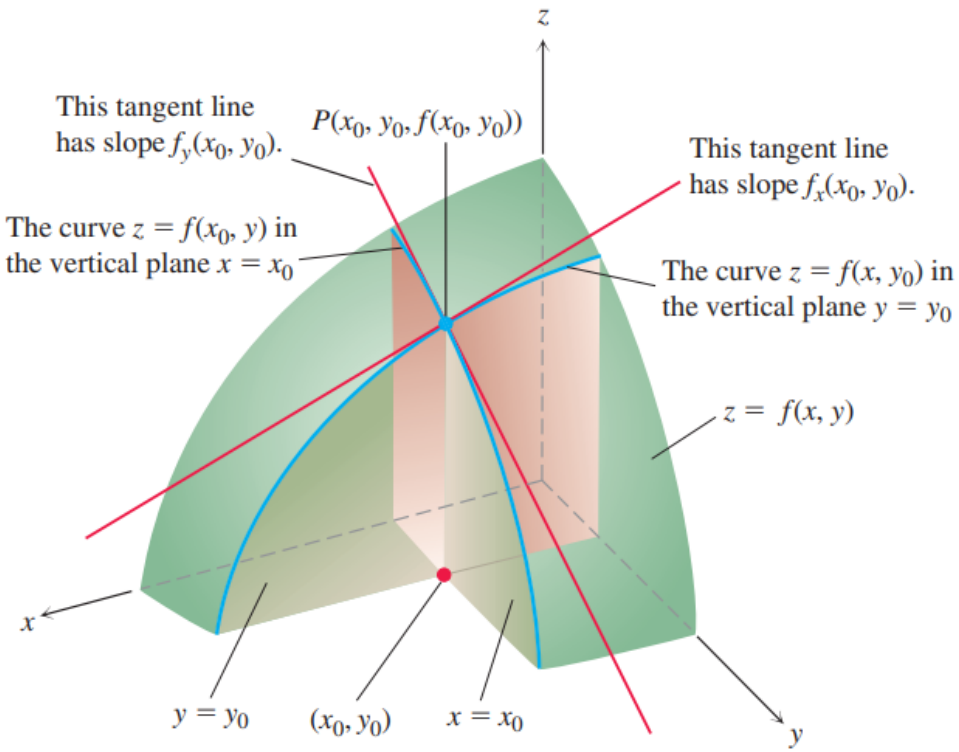
$$\left.\frac{\partial f}{\partial y}\right|_{(x_0,\, y_0)} = \left.\frac{d}{dy} f(x_0, y)\right|_{y=y_0} = \lim_{h \to 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h},$$

provided the limit exists.




- The definition of the partial derivative of $f(x, y)$ with respect to y at a point $(x_0, y_0)$ is similar to the definition of the partial derivative of f with respect to x.

# Partial Derivatives of a Function of Two Variables




- Notice that we now have **two tangent lines** associated with the surface $z = f(x, y)$ at the point $\mathrm{P}(x_0, y_0, f(x_0, y_0))$.
- Is the plane **they determine tangent to the surface at P**?
- We will see that **it is for the differentiable functions** and we will learn how to find the tangent plane.

First we have to better understand partial derivatives.

# Calculations

The definitions of $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ give us **two different ways of differentiating f at a point**:

1) **with respect to x** in the usual way while treating y as a constant and
2) **with respect to y** in the usual way while treating x as a constant.

## Example 1

Find the values of $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ at the point $(4, -5)$ if $f(x, y) = x^2 + 3xy + y - 1$.

**Solution** To find $\partial f/\partial x$, we treat $y$ as a constant and differentiate with respect to $x$:

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^2 + 3xy + y - 1) = 2x + 3 \cdot 1 \cdot y + 0 - 0 = 2x + 3y.$$

The value of $\partial f/\partial x$ at $(4, -5)$ is $2(4) + 3(-5) = -7$.

To find $\partial f/\partial y$, we treat $x$ as a constant and differentiate with respect to $y$:

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^2 + 3xy + y - 1) = 0 + 3 \cdot x \cdot 1 + 1 - 0 = 3x + 1.$$

The value of $\partial f/\partial y$ at $(4, -5)$ is $3(4) + 1 = 13$.

# Calculations

## Example 2

Find $\frac{\partial f}{\partial y}$ as a function if $f(x, y) = y \sin xy$.

**Solution** We treat $x$ as a constant and $f$ as a product of $y$ and $\sin xy$:

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(y \sin xy) = y \frac{\partial}{\partial y} \sin xy + (\sin xy) \frac{\partial}{\partial y}(y)$$

$$= (y \cos xy) \frac{\partial}{\partial y}(xy) + \sin xy = xy \cos xy + \sin xy.$$

# Calculations

## Example 3

Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ as a function if $f(x, y) = \frac{2y}{y+\cos x}$.

**Solution** We treat $f$ as a quotient. With $y$ held constant, we use the quotient rule to get

$$f_x = \frac{\partial}{\partial x}\left(\frac{2y}{y + \cos x}\right) = \frac{(y + \cos x)\frac{\partial}{\partial x}(2y) - 2y\frac{\partial}{\partial x}(y + \cos x)}{(y + \cos x)^2}$$

$$= \frac{(y + \cos x)(0) - 2y(-\sin x)}{(y + \cos x)^2} = \frac{2y \sin x}{(y + \cos x)^2}.$$

With $x$ held constant and again applying the quotient rule, we get

$$f_y = \frac{\partial}{\partial y}\left(\frac{2y}{y + \cos x}\right) = \frac{(y + \cos x)\frac{\partial}{\partial y}(2y) - 2y\frac{\partial}{dy}(y + \cos x)}{(y + \cos x)^2}$$

$$= \frac{(y + \cos x)(2) - 2y(1)}{(y + \cos x)^2} = \frac{2 \cos x}{(y + \cos x)^2}.$$

# Calculations

• **Implicit differentiation** works for partial derivatives the way it works for ordinary derivatives, as the next example illustrates

## Example 4

Find $\frac{\partial z}{\partial x}$ assuming that the equation $yz - \ln z = x + y$ defines z as a function of the two independent variables x and y and the partial derivative exists.

**Solution** We differentiate both sides of the equation with respect to $x$, holding $y$ constant and treating $z$ as a differentiable function of $x$:

$$\frac{\partial}{\partial x}(yz) - \frac{\partial}{\partial x}\ln z = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial x}$$

$$y\frac{\partial z}{\partial x} - \frac{1}{z}\frac{\partial z}{\partial x} = 1 + 0 \qquad \text{With } y \text{ constant, } \frac{\partial}{\partial x}(yz) = y\frac{\partial z}{\partial x}.$$

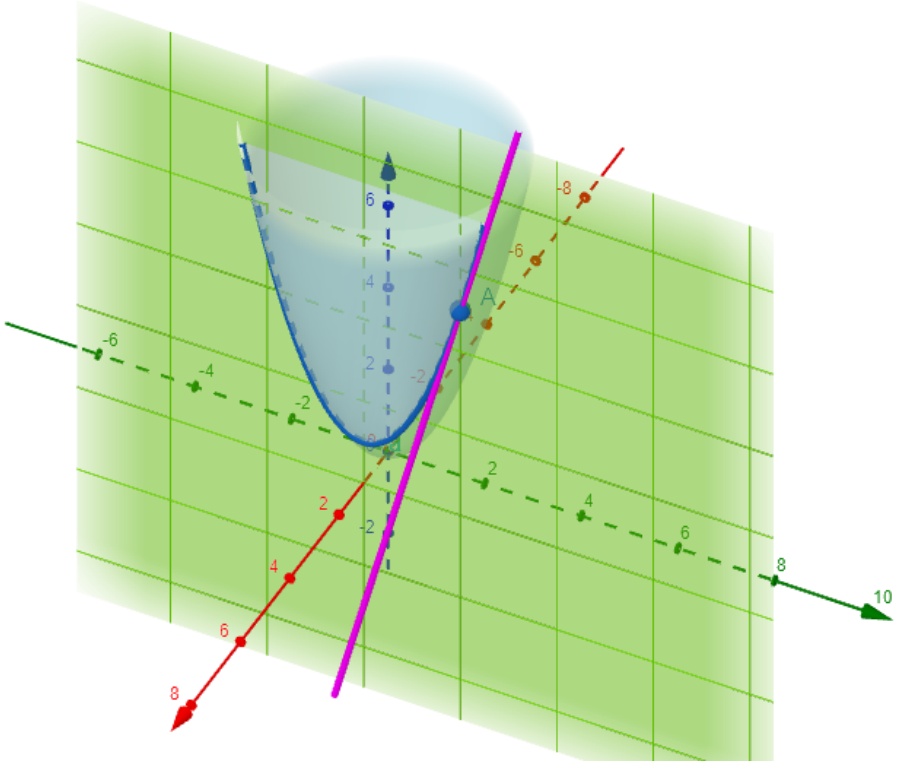
$$\left(y - \frac{1}{z}\right)\frac{\partial z}{\partial x} = 1$$

$$\frac{\partial z}{\partial x} = \frac{z}{yz - 1}.$$

# Calculations

## Example 5

The plane $x = 1$ intersects the paraboloid $z = x^2 + y^2$ in a parabola. Find the slope of the tangent to the parabola at $(1, 2, 5)$.



# Calculations

**Solution** The parabola lies in a plane parallel to the $yz$-plane, and the slope is the value of the partial derivative $\partial z/\partial y$ at $(1, 2)$:

$$\left.\frac{\partial z}{\partial y}\right|_{(1,2)} = \left.\frac{\partial}{\partial y}(x^2 + y^2)\right|_{(1,2)} = \left.2y\right|_{(1,2)} = 2(2) = 4.$$

As a check, we can treat the parabola as the graph of the single-variable function $z = (1)^2 + y^2 = 1 + y^2$ in the plane $x = 1$ and ask for the slope at $y = 2$. The slope, calculated now as an ordinary derivative, is

$$\left.\frac{dz}{dy}\right|_{y=2} = \left.\frac{d}{dy}(1 + y^2)\right|_{y=2} = \left.2y\right|_{y=2} = 4. \qquad \blacksquare$$

# Functions of More Than Two Variables

• The definitions of the partial derivatives of functions of more than two independent variables are **similar to the definitions for functions of two variables**.

• They are ordinary derivatives with respect to one variable, taken while the other independent variables are held constant.

## Example 6

If x, y, and z are independent variables and $f(x, y, z) = x\sin(y + 3z)$, find $\frac{\partial f}{\partial z}$.

$$\begin{aligned}
\frac{\partial f}{\partial z} &= \frac{\partial}{\partial z}\left[x \sin(y + 3z)\right] = x\frac{\partial}{\partial z}\sin(y + 3z) && x \text{ held constant} \\
&= x\cos(y + 3z)\frac{\partial}{\partial z}(y + 3z) && \text{Chain rule} \\
&= 3x\cos(y + 3z). && y \text{ held constant}
\end{aligned}$$

# Second-Order Partial Derivatives

- When we differentiate a function $f(x, y)$ twice, we produce its second-order derivatives.
- These derivatives are usually denoted by $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial y^2}$, $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y \partial x}$

## Example 7

If $f(x, y) = x \cos y + ye^x$, find all second-order derivatives.

**Solution** The first step is to calculate both first partial derivatives.

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x \cos y + ye^x) = \cos y + ye^x$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x \cos y + ye^x) = -x \sin y + e^x$$

Now we find both partial derivatives of each first partial:

$$\frac{\partial^2 f}{\partial y\, \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = -\sin y + e^x$$

$$\frac{\partial^2 f}{\partial x\, \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = -\sin y + e^x$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = ye^x.$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = -x \cos y.$$

# The Mixed Derivative Theorem

**THEOREM 2—The Mixed Derivative Theorem**
If $f(x, y)$ and its partial derivatives $f_x$, $f_y$, $f_{xy}$, and $f_{yx}$ are defined throughout an open region containing a point $(a, b)$ and are all continuous at $(a, b)$, then

$$f_{xy}(a, b) = f_{yx}(a, b).$$

## Example 8

Find $\frac{\partial^2 w}{\partial x \partial y}$ if $w = xy + \frac{e^y}{y^2+1}$.

**Solution** The symbol $\partial^2 w/\partial x\, \partial y$ tells us to differentiate first with respect to $y$ and then with respect to $x$. However, if we interchange the order of differentiation and differentiate first with respect to $x$ we get the answer more quickly. In two steps,

$$\frac{\partial w}{\partial x} = y \qquad \text{and} \qquad \frac{\partial^2 w}{\partial y\, \partial x} = 1.$$

If we differentiate first with respect to $y$, we obtain $\partial^2 w/\partial x\, \partial y = 1$ as well. We can differentiate in either order because the conditions of Theorem 2 hold for $w$ at all points $(x_0, y_0)$.

# Partial Derivatives of Still Higher Order

• Although we will **deal mostly with first- and second-order partial derivatives**, because these appear the most frequently in applications

• There is **no theoretical limit** to how many times we can differentiate a function as long as the **derivatives involved exist**.

• Thus, we get third- and fourth-order derivatives denoted by symbols like $\frac{\partial^3 f}{\partial x \partial y^2} = f_{xyy}$, $\frac{\partial^4 f}{\partial x^2 \partial y^2} = f_{xxyy}$ and so on.

• As with second-order derivatives, the **order of differentiation is immaterial** as long as all the derivatives through the order in question are **continuous**.

# Partial Derivatives of Still Higher Order

## Example 9

Find $f_{yxyz}$ if $f(x, y, z) = 1 - 2xy^2z + x^2y$.

**Solution**: We first differentiate with respect to the variable y, then x, then y again, and finally with respect to z:

$$\begin{aligned} f_y &= -4xyz + x^2 \\ f_{yx} &= -4yz + 2x \\ f_{yxy} &= -4z \\ f_{yxyz} &= -4. \end{aligned}$$

# Partial Derivatives and Continuity. Differentiability

• A function $f(x, y)$ can have partial derivatives with respect to both x and y at a point **without the function being continuous there**.

• This is **different from functions of a single variable**, where the existence of a derivative implies continuity.

• The **concept of differentiability** for functions of several variables is **more complicated** than for single-variable functions because a point in the domain can be approached **along more than one path**.

• For the existence of differentiability, a **property is needed** to ensure that **no abrupt change occurs** in the function resulting from small changes in the independent variables **along any path** approaching $(x_0, y_0)$.

• If the partial derivatives of $f(x, y)$ exist and are continuous throughout a **disk centered at** $(x_0, y_0)$, however, then f is continuous at $(x_0, y_0)$.

# Differentiability

**DEFINITION** A function $z = f(x, y)$ is **differentiable at** $(x_0, y_0)$ if $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ exist and $\Delta z = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0)$ satisfies an equation of the form

$$\Delta z = f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

in which each of $\varepsilon_1, \varepsilon_2 \to 0$ as both $\Delta x, \Delta y \to 0$. We call $f$ **differentiable** if it is differentiable at every point in its domain, and say that its graph is a **smooth surface**.

# How to set the function PATH to a point?

- Two-Path Test for **Nonexistence of a Limit**.
- There is **infinite number of different path** that led to a point $(x_0, y_0) \in \mathbb{R}^2$

**Steps to choose a Path**:

- First, Try two **basic paths** $x = 0$ and $y = 0$.
- If it doesn't work, chose others $x = g(y)$ or $y = h(x)$:

a) **Be certain** that point $(x_0, y_0)$ is actually on your path,

b) Try to substitute so **degrees** of the numerator and denominator **are equal**.

# How to set the function PATH to a point?

$$\lim_{(x,y)\to(1,0)} \frac{x e^y - 1}{x e^y - 1 + y} \qquad \frac{1\cdot e^0 - 1}{1\cdot e^0 - 1 + 0} = \frac{0}{0} \text{ ind.}$$

- $(1,0) \notin x = 0$
- <u>Path</u>: $y = 0$ $\quad \lim_{x\to 1} \frac{x-1}{x-1} = 1$
- <u>Path</u>: $x = 1$ $\quad \lim_{y\to 0} \frac{e^y - 1}{e^y - 1 + y} \overset{\frac{0}{0}\text{ ind}}{=} \lim_{y\to 0} \frac{e^y}{e^y + 1} = \frac{1}{2}$

L'Hopital's Rule

or

- <u>Path</u>: $y = \ln x$ $\quad \lim_{x\to 1} \frac{x\cdot e^{\ln x} - 1}{x\cdot e^{\ln x} - 1 + y} = \lim_{x\to 1} \frac{x^2 - 1}{x^2 - 1 + \ln x} =$

$0 = \ln 1$

$0 = 0$

$$\overset{\frac{0}{0}\text{ ind}}{=} \lim_{x\to 1} \frac{2x}{2x + \frac{1}{x}} = \frac{2}{3}$$

L'H Rule

# Lecture 15: Partial Derivatives.

MA2032 Vector Calculus

Lecturer: Larissa Serdukova

School of Computing and Mathematical Science
University of Leicester

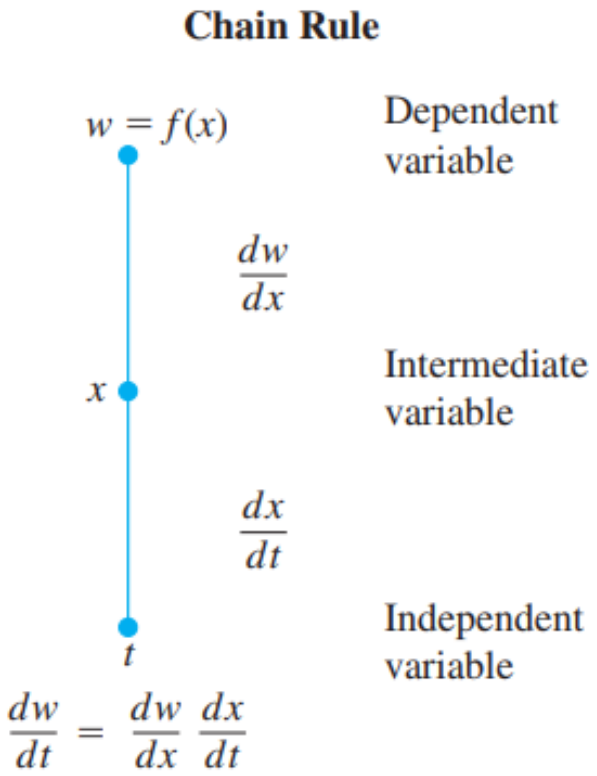
October 24, 2022

# The Chain Rule

• The **Chain Rule** for functions of a **single variable** says that when $w = f(x)$ is a differentiable function of x and $x = g(t)$ is a differentiable function of t, w is a differentiable function of t and $dw/dt$ can be calculated by the formula

$$\frac{dw}{dt} = \frac{dw}{dx}\frac{dx}{dt}.$$

• We display the Chain Rule in a **"dependency diagram"** in the margin. Such diagrams capture which variables depend on which.




• For functions of several variables the **Chain Rule has more than one form**, which depends on how many independent and intermediate variables are involved.

# The Chain Rule. Functions of Two Variables

**THEOREM 5—Chain Rule For Functions of One Independent Variable and Two Intermediate Variables**

If $w = f(x, y)$ is differentiable and if $x = x(t)$, $y = y(t)$ are differentiable functions of $t$, then the composition $w = f(x(t), y(t))$ is a differentiable function of $t$ and

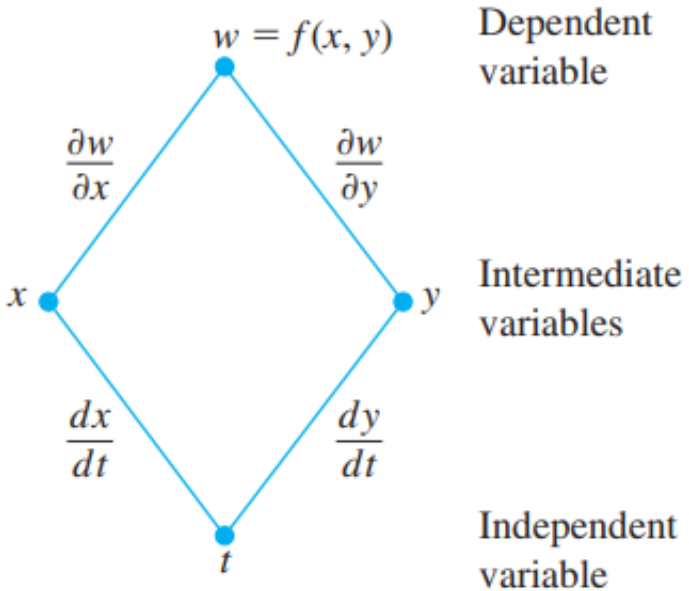
$$\frac{dw}{dt} = f_x(x(t), y(t))\,x'(t) + f_y(x(t), y(t))\,y'(t),$$

or

$$\frac{dw}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$

# The Chain Rule. Functions of Two Variables

**Chain Rule**




$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}$$

## Example 1

Use the Chain Rule to find the derivative of $w = xy$ with respect to t along the path $x = \cos t$, $y = \sin t$. What is the derivative's value at $t = \pi/2$?

**Solution** We apply the Chain Rule to find $dw/dt$ as follows:

$$\begin{aligned}
\frac{dw}{dt} &= \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} \\
&= \frac{\partial(xy)}{\partial x}\frac{d}{dt}(\cos t) + \frac{\partial(xy)}{\partial y}\frac{d}{dt}(\sin t) \\
&= (y)(-\sin t) + (x)(\cos t) \\
&= (\sin t)(-\sin t) + (\cos t)(\cos t) \\
&= -\sin^2 t + \cos^2 t \\
&= \cos 2t.
\end{aligned}$$

In this example, we can check the result with a more direct calculation. As a function of $t$,

$$w = xy = \cos t \sin t = \frac{1}{2}\sin 2t,$$

so

$$\frac{dw}{dt} = \frac{d}{dt}\left(\frac{1}{2}\sin 2t\right) = \frac{1}{2}(2\cos 2t) = \cos 2t.$$

In either case, at the given value of $t$,

$$\left.\frac{dw}{dt}\right|_{t=\pi/2} = \cos\left(2\frac{\pi}{2}\right) = \cos \pi = -1.$$

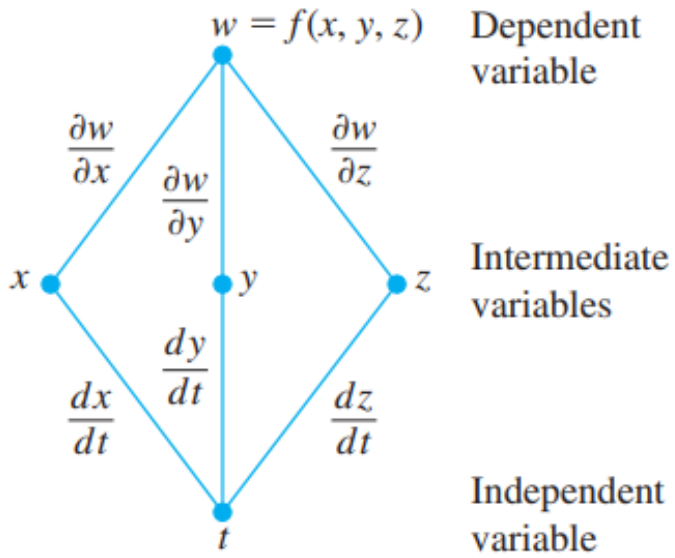
# The Chain Rule. Functions of Three Variable

**THEOREM 6—Chain Rule for Functions of One Independent Variable and Three Intermediate Variables**

If $w = f(x, y, z)$ is differentiable and $x$, $y$, and $z$ are differentiable functions of $t$, then $w$ is a differentiable function of $t$ and

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}.$$

**Chain Rule**




$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

## Example 2

Find $dw/dt$ if $w = xy + z$, $x = \cos t$, $y = \sin t$, $z = t$. In this example the values of $w(t)$ are changing along the path of a helix as t changes. What is the derivative's value at $t = 0$?

**Solution** Using the Chain Rule for three intermediate variables, we have

$$\begin{aligned}\frac{dw}{dt} &= \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt} \\ &= (y)(-\sin t) + (x)(\cos t) + (1)(1) \\ &= (\sin t)(-\sin t) + (\cos t)(\cos t) + 1 \\ &= -\sin^2 t + \cos^2 t + 1 = 1 + \cos 2t,\end{aligned}$$

Substitute for intermediate variables.

so

$$\left.\frac{dw}{dt}\right|_{t=0} = 1 + \cos(0) = 2.$$

# The Chain Rule. Functions Defined on Surfaces

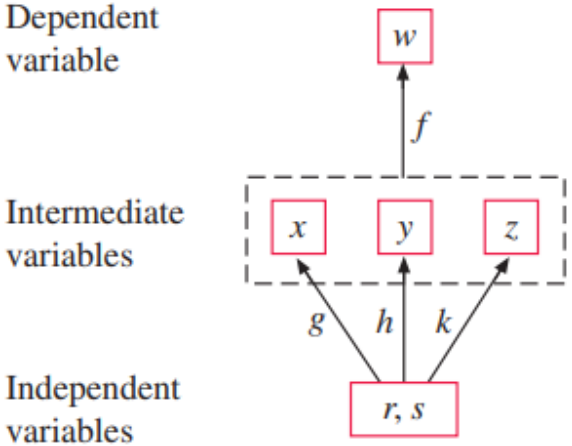
**THEOREM 7—Chain Rule for Two Independent Variables and Three Intermediate Variables**

Suppose that $w = f(x, y, z)$, $x = g(r, s)$, $y = h(r, s)$, and $z = k(r, s)$. If all four functions are differentiable, then $w$ has partial derivatives with respect to $r$ and $s$, given by the formulas

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r}$$

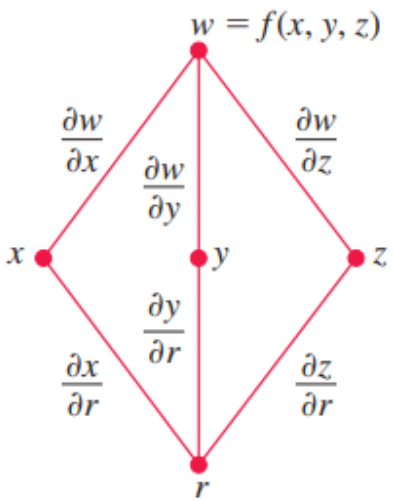
$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}.$$

# The Chain Rule. Functions of Three Variable



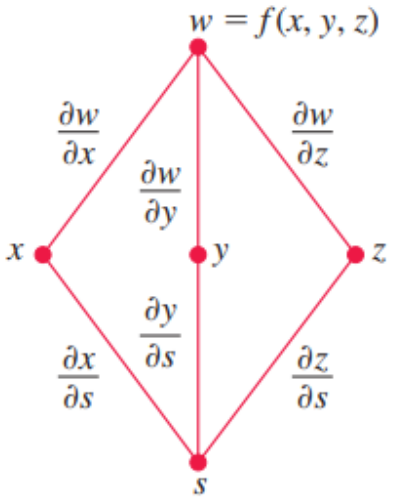

$$w = f(\,g(r, s), h(r, s), k(r, s))$$

(a)




$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\delta y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r}$$

(b)




$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}$$

(c)

## Example 3

Express $\partial w/dr$ and $\partial w/ds$ in terms of r and s if $w = x + 2y + z^2$, $x = \frac{r}{s}$, $y = r^2 + \ln s$, $z = 2r$.

**Solution** Using the formulas in Theorem 7, we find

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r}$$

$$= (1)\left(\frac{1}{s}\right) + (2)(2r) + (2z)(2)$$

$$= \frac{1}{s} + 4r + (4r)(2) = \frac{1}{s} + 12r$$ Substitute for intermediate variable $z$.

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}$$

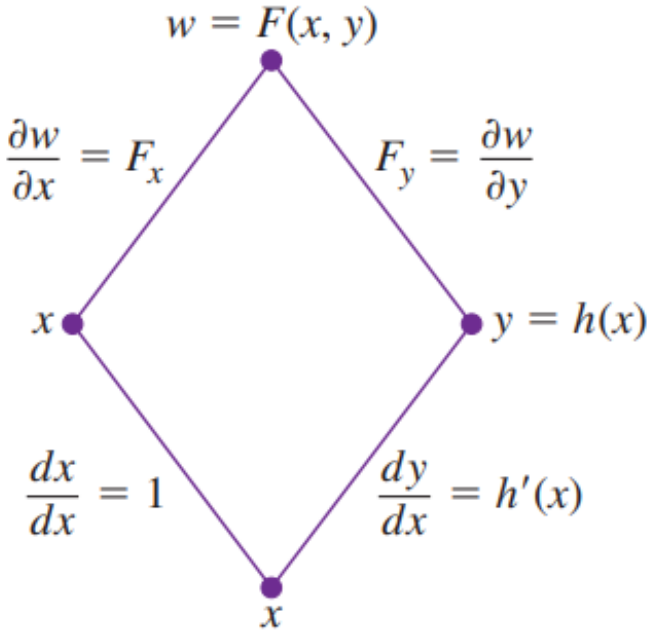
$$= (1)\left(-\frac{r}{s^2}\right) + (2)\left(\frac{1}{s}\right) + (2z)(0) = \frac{2}{s} - \frac{r}{s^2}.$$

# Implicit Differentiation Revisited

• Suppose that
1. The function $F(x, y)$ is differentiable and
2. The equation $F(x, y) = 0$ defines y implicitly as a differentiable function of x, say $y = h(x)$.
• Since $w = F(x, y) = 0$, the derivative $dw/dx$ must be zero.
• Computing the derivative from the Chain Rule, we find

$$0 = \frac{dw}{dx} = F_x \frac{dx}{dx} + F_y \frac{dy}{dx}$$

$$= F_x \cdot 1 + F_y \cdot \frac{dy}{dx}.$$




• If $F_y = dw/dy \neq 0$, we can solve this equation for $dy/dx$ to get

$$\frac{dy}{dx} = -\frac{F_x}{F_y}$$

# Implicit Differentiation Revisited

**THEOREM 8—A Formula for Implicit Differentiation**
Suppose that $F(x, y)$ is differentiable and that the equation $F(x, y) = 0$ defines $y$ as a differentiable function of $x$. Then at any point where $F_y \neq 0$,

$$\frac{dy}{dx} = -\frac{F_x}{F_y}. \tag{1}$$

**EXAMPLE 5** Use Theorem 8 to find $dy/dx$ if $y^2 - x^2 - \sin xy = 0$.

**Solution** Take $F(x, y) = y^2 - x^2 - \sin xy$. Then

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{-2x - y\cos xy}{2y - x\cos xy} = \frac{2x + y\cos xy}{2y - x\cos xy}.$$

This calculation is significantly shorter than a single-variable calculation using implicit differentiation. ■

# Implicit Differentiation Revisited

- The result in Theorem 8 is **easily extended** to three variables.
- Suppose that the equation $F(x, y, z) = 0$ defines the **variable z implicitly** as a function $z = f(x, y)$.
- Then for all $(x, y)$ in the domain of f, we have $F(x, y, f(x, y)) = 0$.
- Assuming that **F and f are differentiable** functions, we can use the Chain Rule to differentiate the equation $F(x, y, z) = 0$ with respect to the independent variable x:

$$
\begin{aligned}
0 &= \frac{\partial F}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} \\
&= F_x \cdot 1 + F_y \cdot 0 + F_z \cdot \frac{\partial z}{\partial x},
\end{aligned}
$$

$y$ is constant when differentiating with respect to $x$.

- A similar calculation for differentiating with respect to the independent variable y is performed.

# Implicit Differentiation Revisited

• If the partial derivatives $F_x, F_y$, and $F_z$ are **continuous** throughout an open region R in space containing the point $(x_0, y_0, z_0)$, and if for some constant c, $F(x_0, y_0, z_0) = c$ and $F_z(x_0, y_0, z_0) \neq 0$, then the equation $F(x, y, z) = c$ **defines z implicitly** as a differentiable function of x and y near $(x_0, y_0, z_0)$, and the **partial derivatives of z** are given by

$$\frac{dz}{dx} = -\frac{F_x}{F_z} \text{ and } \frac{dz}{dy} = -\frac{F_y}{F_z}$$

# Implicit Differentiation Revisited

## Example 4

Find $dz/dx$ and $dz/dy$ at $(0,0,0)$ if $x^3 + z^2 + ye^{xz} + z\cos y = 0$.

**Solution** Let $F(x, y, z) = x^3 + z^2 + ye^{xz} + z\cos y$. Then

$$F_x = 3x^2 + zye^{xz}, \qquad F_y = e^{xz} - z\sin y, \qquad \text{and} \qquad F_z = 2z + xye^{xz} + \cos y.$$

Since $F(0, 0, 0) = 0$, $F_z(0, 0, 0) = 1 \neq 0$, and all first partial derivatives are continuous, the Implicit Function Theorem says that $F(x, y, z) = 0$ defines $z$ as a differentiable function of $x$ and $y$ near the point $(0, 0, 0)$. From Equations (2),

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{3x^2 + zye^{xz}}{2z + xye^{xz} + \cos y} \qquad \text{and} \qquad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{e^{xz} - z\sin y}{2z + xye^{xz} + \cos y}.$$

At $(0, 0, 0)$ we find

$$\frac{\partial z}{\partial x} = -\frac{0}{1} = 0 \qquad \text{and} \qquad \frac{\partial z}{\partial y} = -\frac{1}{1} = -1.$$

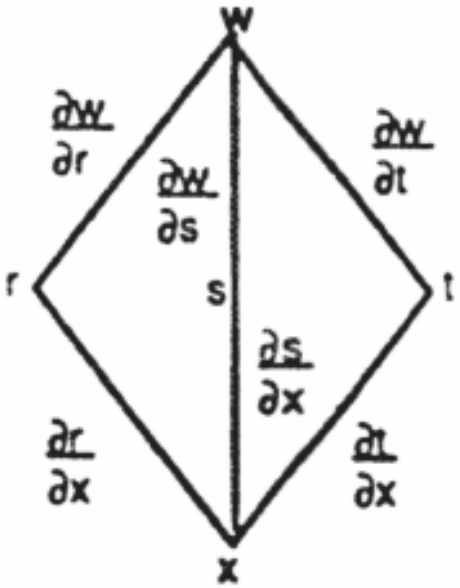
## Example 5

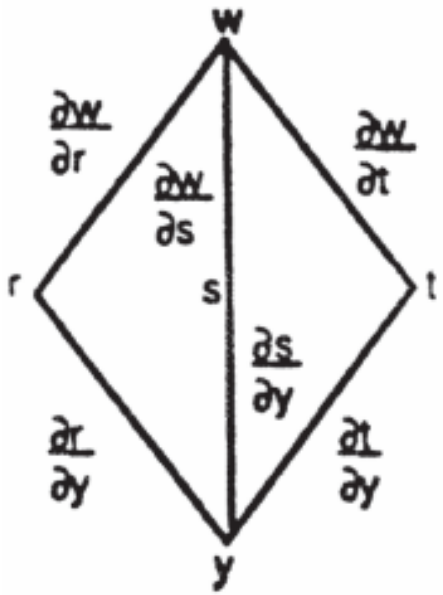
Draw a dependency diagram and write a Chain Rule formula for each derivative
$\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$ for $w = f(r, s, t)$, $r = g(x, y)$, $s = h(x, y)$, $t = k(x, y)$.

**Solution:**

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial w}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial w}{\partial t}\frac{\partial t}{\partial x}$$

$$\frac{\partial w}{\partial y} = \frac{\partial w}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial w}{\partial s}\frac{\partial s}{\partial y} + \frac{\partial w}{\partial t}\frac{\partial t}{\partial y}$$



## Example 6

Assume that $z = f(x, y)^2$, $x = g(t)$, $y = h(t)$, $f_x(1, 0) = -1$, $f_y(1, 0) = 1$, and f(1,0)=2.
If $g(3) = 1$, $h(3) = 0$, $g'(3) = -3$, and $h'(3) = 4$, find $\frac{dz}{dt}\,|_{t=3}$

**Solution:**

$$\frac{dz}{dt} = 2f(x,y)\left[f_x(x,y)g'(t) + f_y(x,y)\cdot h'(t)\right] \Rightarrow \left.\frac{dz}{dt}\right|_{t=3} = 2f(1,0)\left[f_x(1,0)g'(3) + f_y(1,0)h'(3)\right]$$
$$= 2(2)\left[(-1)(-3) + (1)(4)\right] = 28$$

# Lecture 16: Partial Derivatives.

MA2032 Vector Calculus

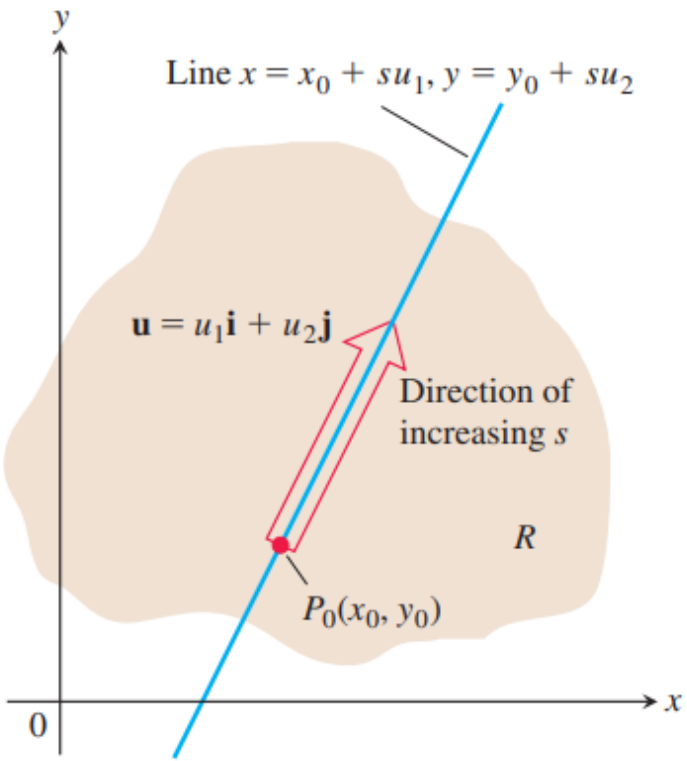
Lecturer: Larissa Serdukova

School of Computing and Mathematical Science
University of Leicester

October 26, 2022

# Directional Derivatives

- Suppose that the function $f(x, y)$ is defined throughout a region **R** in the xy-plane, that $P_0(x_0, y_0)$ is a point in **R**, and that $u = u_1 i + u_2 j$ is a unit vector.
- Then the equations $x = x_0 + su_1$, $y = y_0 + su_2$ parametrize the line through $P_0$ parallel to **u**.
- If the parameter **s** measures arc length from $P_0$ in the direction of **u**, we find the **rate of change of f at $P_0$ in the direction of u by calculating $df/ds$ at $P_0$.**




**DEFINITION** The **derivative of $f$ at $P_0(x_0, y_0)$ in the direction of the unit vector $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$** is the number

$$\left(\frac{df}{ds}\right)_{\mathbf{u}, P_0} = \lim_{s \to 0} \frac{f(x_0 + su_1, y_0 + su_2) - f(x_0, y_0)}{s}, \qquad (1)$$

provided the limit exists.

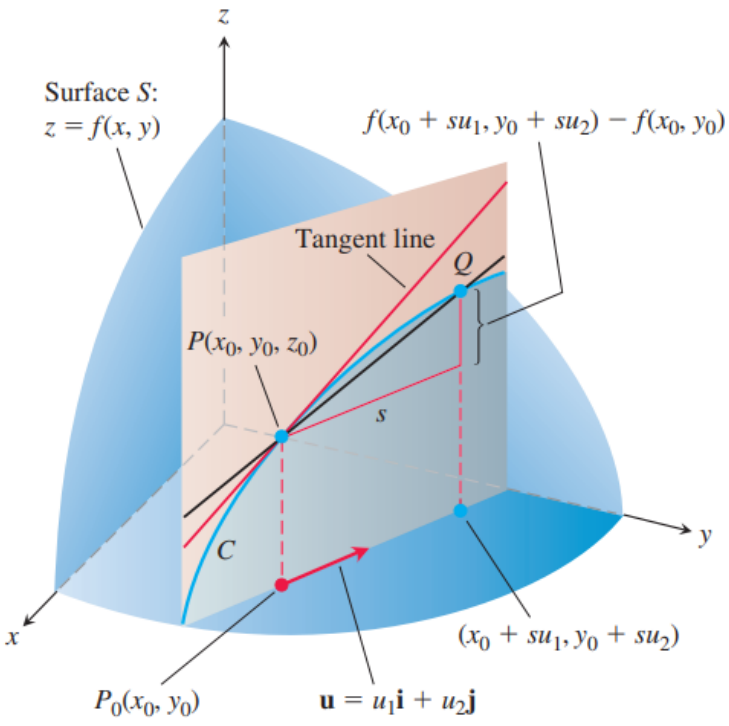
# Interpretation of the Directional Derivative

• The equation $z = f(x, y)$ represents a surface S in space.

• If $z_0 = f(x_0, y_0)$, then the point $P(x_0, y_0, z_0)$ lies on S.

• The vertical plane that passes through P and $P_0(x_0, y_0)$ parallel to u intersects S in a curve C.

• The rate of change of f in the direction of u is the slope of the tangent to C at P in the right-handed system formed by the vectors u and k.




• The partial derivatives $f_x(x_0, y_0)$ and $_y(x_0, y_0)$ are the **directional derivatives** of f at $P_0$ in the **i** and **j** directions.

# Calculation and Gradients

- We now develop an **efficient formula** to calculate the directional derivative for a differentiable function f.
- By the Chain Rule we find

$$\left(\frac{df}{ds}\right)_{\mathbf{u},P_0} = \left.\frac{\partial f}{\partial x}\right|_{P_0}\frac{dx}{ds} + \left.\frac{\partial f}{\partial y}\right|_{P_0}\frac{dy}{ds} \qquad \text{Chain Rule for differentiable } f$$

$$= \left.\frac{\partial f}{\partial x}\right|_{P_0} u_1 + \left.\frac{\partial f}{\partial y}\right|_{P_0} u_2 \qquad \text{From Eqs. (2), } dx/ds = u_1 \text{ and } dy/ds = u_2$$

$$= \underbrace{\left[\left.\frac{\partial f}{\partial x}\right|_{P_0}\mathbf{i} + \left.\frac{\partial f}{\partial y}\right|_{P_0}\mathbf{j}\right]}_{\text{Gradient of } f \text{ at } P_0} \cdot \underbrace{\left[u_1\mathbf{i} + u_2\mathbf{j}\right]}_{\text{Direction } \mathbf{u}}.$$

**DEFINITION** The **gradient vector** (or **gradient**) of $f(x, y)$ is the vector

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}.$$

The value of the gradient vector obtained by evaluating the partial derivatives at a point $P_0(x_0, y_0)$ is written

$$\nabla f|_{P_0} \qquad \text{or} \qquad \nabla f(x_0, y_0).$$

# Directional Derivative

**THEOREM 9—The Directional Derivative Is a Dot Product**

If $f(x, y)$ is differentiable in an open region containing $P_0(x_0, y_0)$, then

$$\left(\frac{df}{ds}\right)_{\mathbf{u},P_0} = \nabla f|_{P_0} \cdot \mathbf{u}, \tag{4}$$

the dot product of the gradient $\nabla f$ at $P_0$ with the vector $\mathbf{u}$. In brief, $D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u}$.

**Properties of the Directional Derivative $D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u} = |\nabla f| \cos\theta$**

1. The function $f$ increases most rapidly when $\cos\theta = 1$, which means that $\theta = 0$ and $\mathbf{u}$ is the direction of $\nabla f$. That is, at each point $P$ in its domain, $f$ increases most rapidly in the direction of the gradient vector $\nabla f$ at $P$. The derivative in this direction is

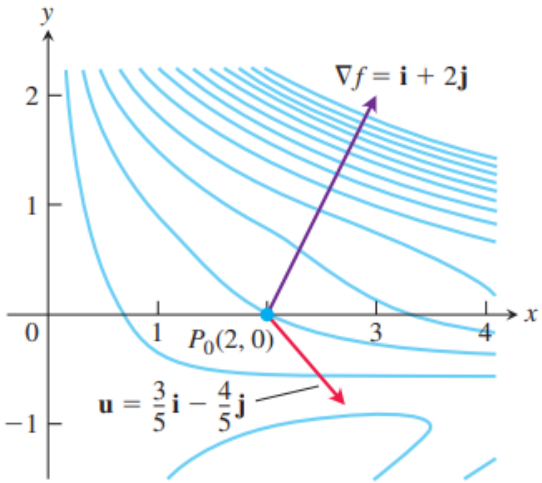
$$D_{\mathbf{u}} f = |\nabla f| \cos(0) = |\nabla f|.$$

2. Similarly, $f$ decreases most rapidly in the direction of $-\nabla f$. The derivative in this direction is $D_{\mathbf{u}} f = |\nabla f| \cos(\pi) = -|\nabla f|$.
3. Any direction $\mathbf{u}$ orthogonal to a gradient $\nabla f \neq 0$ is a direction of zero change in $f$ because $\theta$ then equals $\pi/2$ and

$$D_{\mathbf{u}} f = |\nabla f| \cos(\pi/2) = |\nabla f| \cdot 0 = 0.$$

## Example 1

Find the derivative of $f(x, y) = xe^y + cos(xy)$ at the point $(2, 0)$ in the direction of $v = 3i - 4j$.




**Solution** Recall that the direction of a vector $\mathbf{v}$ is the unit vector obtained by dividing $\mathbf{v}$ by its length:

$$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{\mathbf{v}}{5} = \frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}.$$

The partial derivatives of $f$ are everywhere continuous and at $(2, 0)$ are given by

$$f_x(2, 0) = (e^y - y\sin(xy))\Big|_{(2,0)} = e^0 - 0 = 1$$

$$f_y(2, 0) = (xe^y - x\sin(xy))\Big|_{(2,0)} = 2e^0 - 2\cdot 0 = 2.$$

The gradient of $f$ at $(2, 0)$ is

$$\nabla f|_{(2,0)} = f_x(2, 0)\mathbf{i} + f_y(2, 0)\mathbf{j} = \mathbf{i} + 2\mathbf{j}$$

(Figure 14.29). The derivative of $f$ at $(2, 0)$ in the direction of $\mathbf{v}$ is therefore

$$D_{\mathbf{u}}f|_{(2,0)} = \nabla f|_{(2,0)} \cdot \mathbf{u} \qquad \text{Eq. (4) with the } D_{\mathbf{u}}f|_{P_0} \text{ notation}$$

$$= (\mathbf{i} + 2\mathbf{j}) \cdot \left(\frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}\right) = \frac{3}{5} - \frac{8}{5} = -1.$$

## Example 2

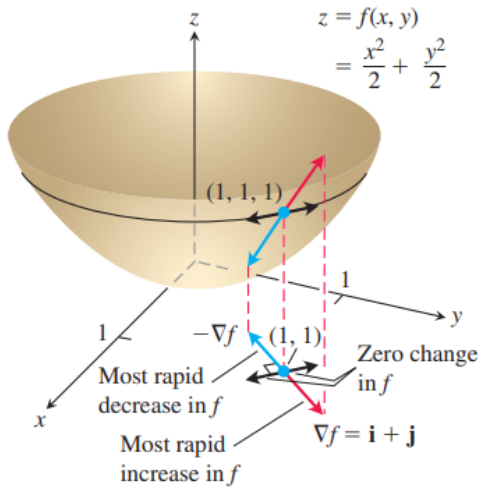
Find the directions in which $f(x, y) = (x^2/2) + (y^2/2)$
(a) increases most rapidly at the point (1, 1), and
(b) decreases most rapidly at (1, 1).

(c) What are the directions of zero change in f at (1, 1)?

### Solution

**(a)** The function increases most rapidly in the direction of $\nabla f$ at (1, 1). The gradient there is

$$\nabla f\big|_{(1,\,1)} = (x\mathbf{i} + y\mathbf{j})\Big|_{(1,\,1)} = \mathbf{i} + \mathbf{j}.$$




Its direction is

$$\mathbf{u} = \frac{\mathbf{i} + \mathbf{j}}{|\mathbf{i} + \mathbf{j}|} = \frac{\mathbf{i} + \mathbf{j}}{\sqrt{(1)^2 + (1)^2}} = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}.$$

**(b)** The function decreases most rapidly in the direction of $-\nabla f$ at (1, 1), which is

$$-\mathbf{u} = -\frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j}.$$

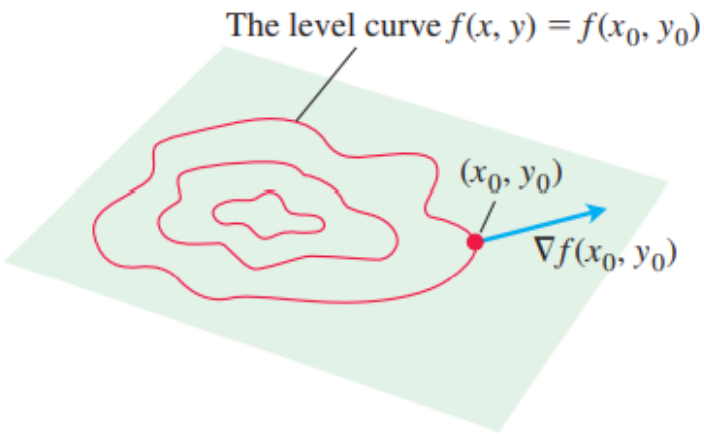
**(c)** The directions of zero change at (1, 1) are the directions orthogonal to $\nabla f$:

$$\mathbf{n} = -\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j} \quad \text{and} \quad -\mathbf{n} = \frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j}.$$

# Gradients and Tangents to Level Curves

• If a differentiable function $f(x, y)$ has a constant value c along a smooth curve $r = g(t)i + h(t)j$ (making the curve part of a level curve of f), then $f(g(t), h(t)) = c$.

• Differentiating both sides of this equation with respect to t leads to the equation




$$\frac{d}{dt} f(g(t), h(t)) = \frac{d}{dt}(c)$$

$$\frac{\partial f}{\partial x}\frac{dg}{dt} + \frac{\partial f}{\partial y}\frac{dh}{dt} = 0 \qquad \text{Chain Rule}$$

$$\underbrace{\left(\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}\right)}_{\nabla f} \cdot \underbrace{\left(\frac{dg}{dt}\mathbf{i} + \frac{dh}{dt}\mathbf{j}\right)}_{\frac{d\mathbf{r}}{dt}} = 0.$$

At every point $(x_0, y_0)$ in the domain of a differentiable function $f(x, y)$, the gradient of $f$ is normal to the level curve through $(x_0, y_0)$ (Figure 14.31).

# Directional Derivative

• This observation also enables us to find **equations for tangent lines to level curves**.

• They are the **lines normal to the gradients**.

• The line through a point $P_0(x_0, y_0)$ normal to a nonzero vector $N = Ai + Bj$ has the equation $A(x - x_0) + B(y - y_0) = 0$

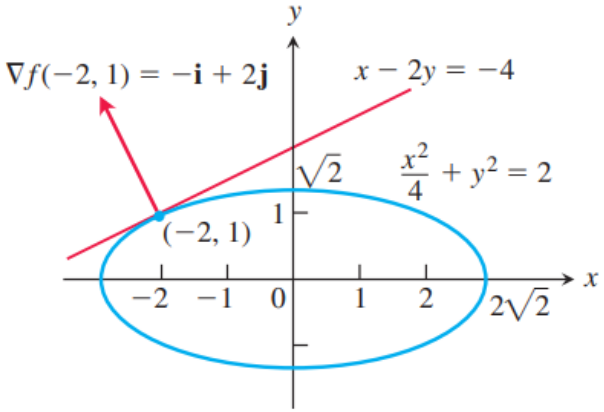
• If N is the gradient $\nabla f|_{(x_0,y_0)} = f_x(x_0, y_0)i + f_y(x_0, y_0)j$, and this gradient is not the zero vector, then this equation gives the following formula.

**Tangent Line to a Level Curve**

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) = 0 \qquad (6)$$

## Example 3

Find an equation for the tangent to the ellipse $x^2/4 + y^2 = 2$ at the point $(-2, 1)$.




**Solution** The ellipse is a level curve of the function

$$f(x, y) = \frac{x^2}{4} + y^2.$$

The gradient of $f$ at $(-2, 1)$ is

$$\nabla f|_{(-2,\,1)} = \left(\frac{x}{2}\mathbf{i} + 2y\mathbf{j}\right)\Big|_{(-2,\,1)} = -\mathbf{i} + 2\mathbf{j}.$$

Because this gradient vector is nonzero, the tangent to the ellipse at $(-2, 1)$ is the line

$$(-1)(x + 2) + (2)(y - 1) = 0 \qquad \text{Eq. (6)}$$

$$x - 2y = -4. \qquad \text{Simplify.}$$

# Directional Derivative

**Algebra Rules for Gradients**

1. *Sum Rule:* $\nabla(f+g) = \nabla f + \nabla g$
2. *Difference Rule:* $\nabla(f-g) = \nabla f - \nabla g$
3. *Constant Multiple Rule:* $\nabla(kf) = k\nabla f$ (any number $k$)
4. *Product Rule:* $\nabla(fg) = f\nabla g + g\nabla f$
5. *Quotient Rule:* $\nabla\left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{g^2}$

Scalar multipliers on left of gradients

# Functions of Three Variables

• For a differentiable function $f(x, y, z)$ and a unit vector $u = u_1 i + u_2 j + u_3 k$ in space, we have

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

$$D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u} = \frac{\partial f}{\partial x} u_1 + \frac{\partial f}{\partial y} u_2 + \frac{\partial f}{\partial z} u_3.$$

•The **directional derivative** can once again be written in the form

$$D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u} = |\nabla f||\mathbf{u}| \cos\theta = |\nabla f| \cos\theta,$$

• so the properties listed earlier for functions of two variables extend to three variables.

• At any given point, f **increases most rapidly** in the direction of $\nabla f$ and **decreases most rapidly** in the direction of $-\nabla f$.

• In any **direction orthogonal** to $\nabla f$, the derivative is zero.

## Example 4

(a) Find the derivative of $f(x, y, z) = x^3 - xy^2 - z$ at $P_0(1, 1, 0)$ in the direction of $v = 2i - 3j + 6k$.
(b) In what directions does f change most rapidly at $P_0$ , and what are the rates of change in these directions?.

**Solution**

**(a)** The direction of **v** is obtained by dividing **v** by its length:

$$|\mathbf{v}| = \sqrt{(2)^2 + (-3)^2 + (6)^2} = \sqrt{49} = 7$$

$$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{2}{7}\mathbf{i} - \frac{3}{7}\mathbf{j} + \frac{6}{7}\mathbf{k}.$$

The partial derivatives of $f$ at $P_0$ are

$$f_x = (3x^2 - y^2)\Big|_{(1,1,0)} = 2, \qquad f_y = -2xy\Big|_{(1,1,0)} = -2, \qquad f_z = -1\Big|_{(1,1,0)} = -1.$$

The gradient of $f$ at $P_0$ is

# Example 4. Solution

$$\nabla f|_{(1,1,0)} = 2\mathbf{i} - 2\mathbf{j} - \mathbf{k}.$$

The derivative of $f$ at $P_0$ in the direction of $\mathbf{v}$ is therefore

$$D_{\mathbf{u}}f|_{(1,1,0)} = \nabla f|_{(1,1,0)} \cdot \mathbf{u} = (2\mathbf{i} - 2\mathbf{j} - \mathbf{k}) \cdot \left(\frac{2}{7}\mathbf{i} - \frac{3}{7}\mathbf{j} + \frac{6}{7}\mathbf{k}\right)$$

$$= \frac{4}{7} + \frac{6}{7} - \frac{6}{7} = \frac{4}{7}.$$

**(b)** The function increases most rapidly in the direction of $\nabla f = 2\mathbf{i} - 2\mathbf{j} - \mathbf{k}$ and decreases most rapidly in the direction of $-\nabla f$. The rates of change in the directions are, respectively,

$$|\nabla f| = \sqrt{(2)^2 + (-2)^2 + (-1)^2} = \sqrt{9} = 3 \qquad \text{and} \qquad -|\nabla f| = -3. \qquad \blacksquare$$

# The Chain Rule for Paths

• If $r(t) = x(t)i + y(t)j + z(t)k$ is a smooth path C, and $w = f(r(t))$ is a scalar function evaluated along C, then according to the Chain Rule

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}.$$

• The partial derivatives on the right-hand side of the above equation are **evaluated along the curve** $r(t)$, and the derivatives of the intermediate variables are **evaluated at t**.

• If we express this equation using **vector notation**, we have

**The Derivative Along a Path**

$$\frac{d}{dt}f(\mathbf{r}(t)) = \nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t). \tag{7}$$

• What Equation (7) says is that the derivative of the **composite function f(r(t))** is the "derivative" (gradient) of the **outside function f** "times" (dot product) the derivative of the **inside function r**.

# Lecture 17: Partial Derivatives.

MA2032 Vector Calculus

Lecturer: Larissa Serdukova

School of Computing and Mathematical Science
University of Leicester

October 28, 2022

# Tangent Planes and Differentials

• In single-variable differential calculus we saw **how the derivative defined the tangent line** to the graph of a differentiable function at a point on the graph.

• The **tangent line then provided for a linearization** of the function at the point.

• In this section, we will see analogously **how the gradient defines the tangent plane** to the level surface of a function $w = (x, y, z)$ at a point on the surface.

• The tangent plane then provides for a **linearization** of f at the point and defines the total differential of the function.

# Tangent Planes and Normal Lines

• If $r(t) = x(t)i + y(t)j + z(t)k$ is a **smooth curve on the level surface** $f(x, y, z) = c$ of a differentiable function f, we found in the last lecture that

$$\frac{d}{dt} f(\mathbf{r}(t)) = \nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t).$$

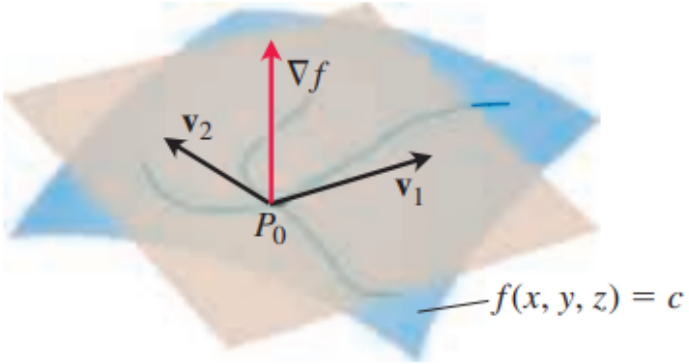
• Since f is constant along the curve **r**, the derivative on the left-hand side of the equation is 0, so the **gradient** $\nabla f$ **is orthogonal to the curve's velocity vector** $r'$.

• Now let us restrict our attention to the curves that **pass through a point** $P_0$.

• All the velocity vectors at $P_0$ are **orthogonal** to $\nabla f$ at $P_0$, so the curves' tangent lines all lie in the plane through $P_0$ normal to $P_0$.

• We now **define this plane**.

# Tangent Planes and Normal Lines




**DEFINITIONS** The **tangent plane** to the level surface $f(x, y, z) = c$ of a differentiable function $f$ at a point $P_0$ where the gradient is not zero is the plane through $P_0$ normal to $\nabla f|_{P_0}$.

The **normal line** of the surface at $P_0$ is the line through $P_0$ parallel to $\nabla f|_{P_0}$.

**Tangent Plane to $f(x, y, z) = c$ at $P_0(x_0, y_0, z_0)$**

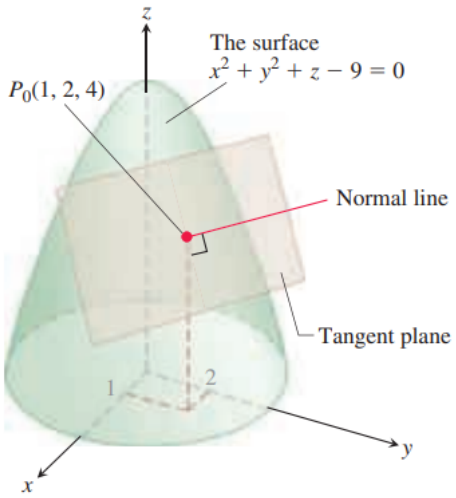
$$f_x(P_0)(x - x_0) + f_y(P_0)(y - y_0) + f_z(P_0)(z - z_0) = 0 \qquad (1)$$

**Normal Line to $f(x, y, z) = c$ at $P_0(x_0, y_0, z_0)$**

$$x = x_0 + f_x(P_0)t, \qquad y = y_0 + f_y(P_0)t, \qquad z = z_0 + f_z(P_0)t \qquad (2)$$

## Example 1

Find the tangent plane and normal line of the level surface $f(x, y, z) = x^2 + y^2 + z - 9 = 0$ at the point $P_0(1, 2, 4)$.




**Solution** The surface is shown in Figure 14.34.

The tangent plane is the plane through $P_0$ perpendicular to the gradient of $f$ at $P_0$. The gradient is

$$\nabla f|_{P_0} = (2x\mathbf{i} + 2y\mathbf{j} + \mathbf{k})\Big|_{(1,2,4)} = 2\mathbf{i} + 4\mathbf{j} + \mathbf{k}.$$

The tangent plane is therefore the plane

$$2(x - 1) + 4(y - 2) + (z - 4) = 0, \qquad \text{or} \qquad 2x + 4y + z = 14.$$

The line normal to the surface at $P_0$ is

$$x = 1 + 2t, \qquad y = 2 + 4t, \qquad z = 4 + t.$$

# Tangent Plane to a smooth surface

• To find an equation for the **plane tangent to a smooth surface** $z = (x, y)$ at a point $P_0(x_0, y_0, z_0)$ where $z_0 = f(x_0, y_0)$, we first observe that the equation $z = f(x, y)$ **is equivalent to** $f(x, y) - z = 0$.

• The surface $z = f(x, y)$ is therefore the **zero level surface** of the function $F(x, y, z) = f(x, y) - z$.

• The **partial derivatives of F** are

$$F_x = \frac{\partial}{\partial x}(f(x, y) - z) = f_x - 0 = f_x$$

$$F_y = \frac{\partial}{\partial y}(f(x, y) - z) = f_y - 0 = f_y$$

$$F_z = \frac{\partial}{\partial z}(f(x, y) - z) = 0 - 1 = -1.$$

• The formula $F_x(P_0)(x - x_0) + F_y(P_0)(y - y_0) + F_z(P_0)(z - z_0) = 0$ for the plane tangent to the level surface at $P_0$ therefore **reduces to**

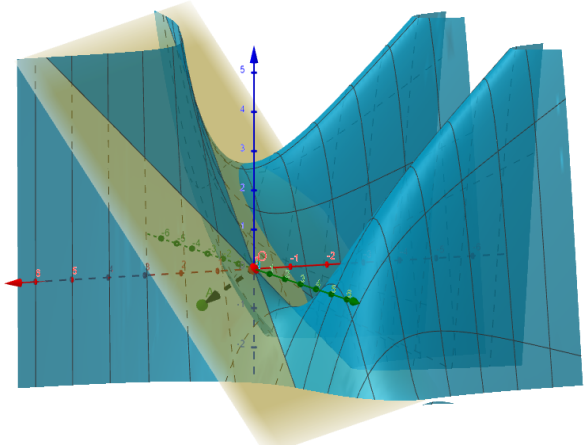
# Tangent Plane to a smooth surface

**Plane Tangent to a Surface $z = f(x, y)$ at $(x_0, y_0, f(x_0, y_0))$**
The plane tangent to the surface $z = f(x, y)$ of a differentiable function $f$ at the point $P_0(x_0, y_0, z_0) = (x_0, y_0, f(x_0, y_0))$ is

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - z_0) = 0. \qquad (3)$$

## Example 2

Find the plane tangent to the surface $z = x\cos y - ye^x$ at $(0, 0, 0)$.



**Solution** We calculate the partial derivatives of $f(x, y) = x\cos y - ye^x$ and use Equation (3):

$$f_x(0, 0) = (\cos y - ye^x)\Big|_{(0,0)} = 1 - 0 \cdot 1 = 1$$

$$f_y(0, 0) = (-x\sin y - e^x)\Big|_{(0,0)} = 0 - 1 = -1.$$

The tangent plane is therefore

$$1 \cdot (x - 0) - 1 \cdot (y - 0) - (z - 0) = 0, \qquad \text{Eq. (3)}$$

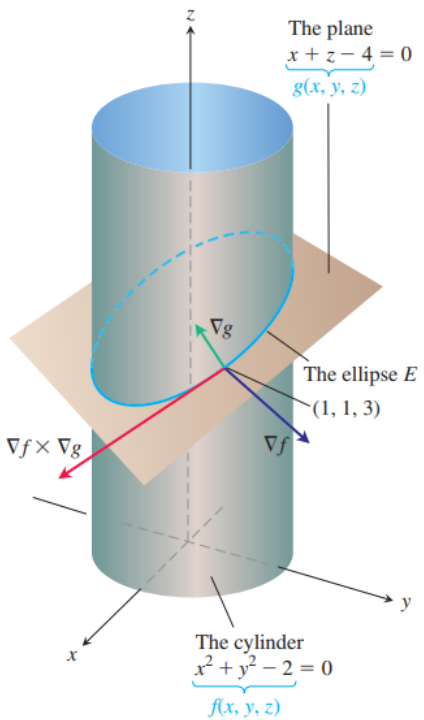
or

$$x - y - z = 0.$$

# Tangent Plane to a smooth surface

## Example 3

The surfaces $f(x, y, z) = x^2 + y^2 - 2 = 0$ a cylinder and $g(x, y, z) = x + z - 4 = 0$ a plane meet in an ellipse E. Find parametric equations for the line tangent to E at the point $P_0(1, 1, 3)$.




**Solution** The tangent line is orthogonal to both $\nabla f$ and $\nabla g$ at $P_0$, and therefore parallel to $\mathbf{v} = \nabla f \times \nabla g$. The components of $\mathbf{v}$ and the coordinates of $P_0$ give us equations for the line. We have

$$\nabla f|_{(1,1,3)} = (2x\mathbf{i} + 2y\mathbf{j})\Big|_{(1,1,3)} = 2\mathbf{i} + 2\mathbf{j}$$

$$\nabla g|_{(1,1,3)} = (\mathbf{i} + \mathbf{k})\Big|_{(1,1,3)} = \mathbf{i} + \mathbf{k}$$

$$\mathbf{v} = (2\mathbf{i} + 2\mathbf{j}) \times (\mathbf{i} + \mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 2 & 0 \\ 1 & 0 & 1 \end{vmatrix} = 2\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}.$$

The tangent line to the ellipse of intersection is

$$x = 1 + 2t, \qquad y = 1 - 2t, \qquad z = 3 - 2t.$$

# Estimating Change in a Specific Direction

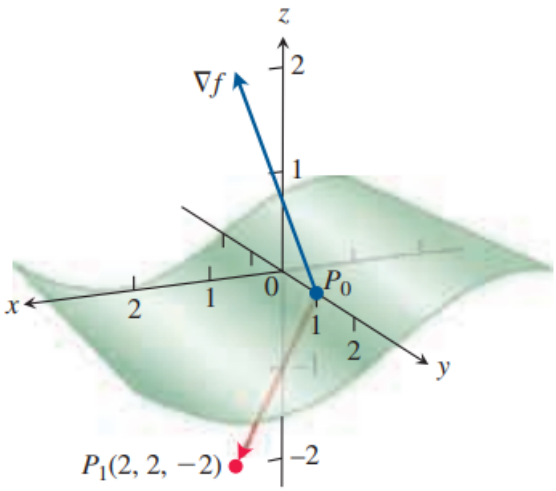
**Estimating the Change in $f$ in a Direction $\mathbf{u}$**

To estimate the change in the value of a differentiable function $f$ when we move a small distance $ds$ from a point $P_0$ in a particular direction $\mathbf{u}$, use the formula

$$df = \underbrace{(\nabla f|_{P_0} \cdot \mathbf{u})}_{\text{Directional derivative}} \underbrace{ds}_{\text{Distance increment}}$$

## Example 4

Estimate how much the value of $f(x, y, z) = y \sin x + 2yz$ will change if the point $P(x, y, z)$ moves 0.1 unit from $P_0(0, 1, 0)$ straight toward $P_1(2, 2, -2)$.




**Solution** We first find the derivative of $f$ at $P_0$ in the direction of the vector $\overrightarrow{P_0P_1} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$. The direction of this vector is

$$\mathbf{u} = \frac{\overrightarrow{P_0P_1}}{|\overrightarrow{P_0P_1}|} = \frac{\overrightarrow{P_0P_1}}{3} = \frac{2}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} - \frac{2}{3}\mathbf{k}.$$

The gradient of $f$ at $P_0$ is

$$\nabla f|_{(0,1,0)} = ((y \cos x)\mathbf{i} + (\sin x + 2z)\mathbf{j} + 2y\mathbf{k})\Big|_{(0,1,0)} = \mathbf{i} + 2\mathbf{k}.$$

Therefore,

$$\nabla f|_{P_0} \cdot \mathbf{u} = (\mathbf{i} + 2\mathbf{k}) \cdot \left(\frac{2}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} - \frac{2}{3}\mathbf{k}\right) = \frac{2}{3} - \frac{4}{3} = -\frac{2}{3}.$$

The change $df$ in $f$ that results from moving $ds = 0.1$ unit away from $P_0$ in the direction of $\mathbf{u}$ is approximately

$$df = (\nabla f|_{P_0} \cdot \mathbf{u})(ds) = \left(-\frac{2}{3}\right)(0.1) \approx -0.067 \text{ unit}.$$

# How to Linearize a Function of Two Variables

• Functions of two variables can be quite **complicated**, and we sometimes **need to approximate** them with simpler ones that give the accuracy required for specific applications without being so difficult to work with.

• We do this in a way that is **similar** to the way we find **linear replacements** for functions of a single variable.

• Suppose the function we wish to approximate is $z = f(x, y)$ near a point $(x_0, y_0)$ at which we know the values of $f$, $f_x$, and $f_y$ and **at which f is differentiable**.

• If we move from $(x_0, y_0)$ to any nearby point $(x, y)$ by increments $\Delta x = x - x_0$ and $\Delta y = y - y_0$, then the definition of differentiability gives the **change**

$$f(x, y) - f(x_0, y_0) = f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y,$$

• where $\varepsilon_1, \varepsilon_2 \to 0$ as $\Delta x, \Delta y \to 0$.

# How to Linearize a Function of Two Variables

• If the increments $\Delta x$ and $\Delta y$ **are small**, the products $\varepsilon_1 \Delta x$ and $\varepsilon_2 \Delta y$ will eventually **be smaller still** and we have the **approximation**

$$f(x, y) \approx \underbrace{f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)}_{L(x,\, y)}.$$

• In other words, as long as $\Delta x$ and $\Delta y$ are small, **f will have approximately the same value as the linear function L**.

**DEFINITIONS** The **linearization** of a function $f(x, y)$ at a point $(x_0, y_0)$ where $f$ is differentiable is the function

$$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

The approximation

$$f(x, y) \approx L(x, y)$$

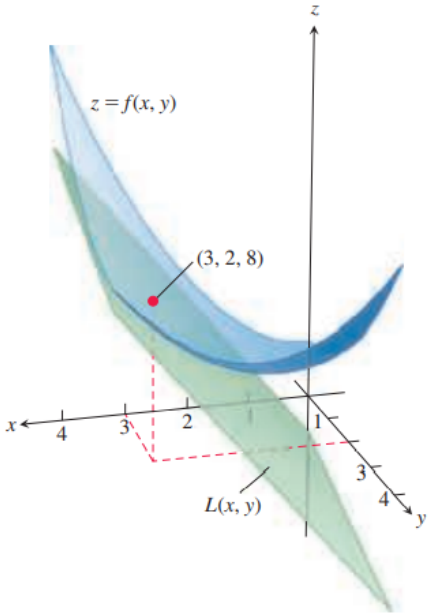
is the **standard linear approximation** of $f$ at $(x_0, y_0)$.

• Plane $z = L(x, y)$ is tangent to the surface $z = f(x, y)$ at the point $(x_0, y_0)$. Thus, the **linearization** of a function of two variables is a **tangent-plane approximation**.

# How to Linearize a Function of Two Variables

## Example 5

Find the linearization of $(x, y) = x^2 - xy + 1/2y^2 + 3$ at the point $(3, 2)$.




**Solution** We first evaluate $f$, $f_x$, and $f_y$ at the point $(x_0, y_0) = (3, 2)$:

$$f(3, 2) = \left(x^2 - xy + \frac{1}{2}y^2 + 3\right)\Big|_{(3,2)} = 8$$

$$f_x(3, 2) = \frac{\partial}{\partial x}\left(x^2 - xy + \frac{1}{2}y^2 + 3\right)\Big|_{(3,2)} = (2x - y)\Big|_{(3,2)} = 4$$

$$f_y(3, 2) = \frac{\partial}{\partial y}\left(x^2 - xy + \frac{1}{2}y^2 + 3\right)\Big|_{(3,2)} = (-x + y)\Big|_{(3,2)} = -1,$$

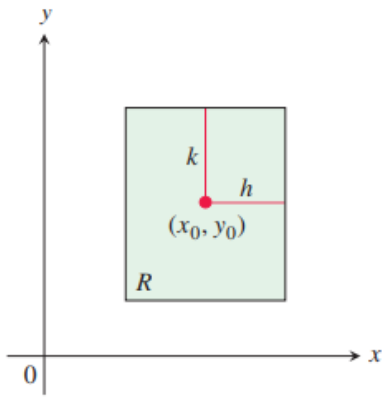
giving

$$\begin{aligned} L(x, y) &= f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) \\ &= 8 + (4)(x - 3) + (-1)(y - 2) = 4x - y - 2. \end{aligned}$$

The linearization of $f$ at $(3, 2)$ is $L(x, y) = 4x - y - 2$ (see Figure 14.38).

# The Error in the Standard Linear Approximation

- The **error** is defined by $E(x, y) = f(x, y) - L(x, y)$.




**The Error in the Standard Linear Approximation**

If $f$ has continuous first and second partial derivatives throughout an open set containing a rectangle $R$ centered at $(x_0, y_0)$ and if $M$ is any upper bound for the values of $|f_{xx}|$, $|f_{yy}|$, and $|f_{xy}|$ on $R$, then the error $E(x, y)$ incurred in replacing $f(x, y)$ on $R$ by its linearization

$$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

satisfies the inequality

$$|E(x, y)| \leq \frac{1}{2} M\big(|x - x_0| + |y - y_0|\big)^2.$$

- To make $|E(x, y)|$ **small** for a given $M$, we just make $|x - x_0|$ and $|y - y_0|$ small.

# Functions of More Than Two Variables

• Analogous results hold for differentiable **functions of more than two variables**.

**1.** The **linearization** of $(x, y, z)$ at a point $P_0(x_0, y_0, z_0)$ is
$L(x, y, z) = f(P_0) + f_x(P_0)(x - x_0) + f_y(P_0)(y - y_0) + f_z(P_0)(z - z_0)$.
**2.** Suppose that R is a closed rectangular solid centered at $P_0$ and lying in an open region on which the second partial derivatives of fare continuous. Suppose also that $|f_{xx}|$, $|f_{yy}|$, $|f_{zz}|$, $|f_{xy}|$, $|f_{xz}|$, and $|f_{yz}|$ are all less than or equal to M throughout R.
• Then the **error** $E(x, y, z) = (x, y, z) - L(x, y, z)$ in the approximation of f by L is bounded throughout R by the inequality

$$|E| \leq \frac{1}{2}M(|x - x_0| + |y - y_0| + |z - z_0|)^2.$$

**3.** If the second partial derivatives of f are continuous and if $x, y$, and $z$ change from $x_0, y_0$, and $z_0$ by small amounts $dx, dy$, and $dz$, the **total differential**

$$df = f_x(P_0)dx + f_y(P_0)dy + f_z(P_0)dz$$

• gives a **good approximation of the resulting change in f**.

## Example 6

Find the linearization $L(x, y, z)$ of $(x, y, z) = x^2 - xy + 3\sin z$ at the point $(x_0, y_0, z_0) = (2, 1, 0)$. Find an upper bound for the error incurred in replacing f by L on the rectangular region.

$$R: \quad |x-2| \le 0.01, \quad |y-1| \le 0.02, \quad |z| \le 0.01$$

**Solution** Routine calculations give

$$f(2,1,0) = 2, \qquad f_x(2,1,0) = 3, \qquad f_y(2,1,0) = -2, \qquad f_z(2,1,0) = 3.$$

Thus,

$$L(x,y,z) = 2 + 3(x-2) + (-2)(y-1) + 3(z-0) = 3x - 2y + 3z - 2.$$

Since

$$f_{xx} = 2, \qquad f_{yy} = 0, \qquad f_{zz} = -3\sin z, \qquad f_{xy} = -1, \qquad f_{xz} = 0, \qquad f_{yz} = 0,$$

and $|-3\sin z| \le 3\sin 0.01 \approx 0.03$, we may take $M = 2$ as a bound on the second partials. Hence, the error incurred by replacing $f$ by $L$ on $R$ satisfies

$$|E| \le \frac{1}{2}(2)(0.01 + 0.02 + 0.01)^2 = 0.0016.$$

# Lecture 18: Partial Derivatives.

MA2032 Vector Calculus

Lecturer: Larissa Serdukova

School of Computing and Mathematical Science
University of Leicester

October 31, 2022

# Extreme Values and Saddle Points

• Continuous functions of two variables assume **extreme values** on closed, bounded domains.

• We see in this section that we can narrow the search for these extreme values by **examining the functions' first partial derivatives**.

• A function of two variables can assume extreme values only at **boundary points** of the domain or at **interior domain points** where both first partial derivatives are zero or where one or both of the first partial derivatives fail to exist.

• However, the vanishing of derivatives at an interior point $(a, b)$ does not always signal the presence of an extreme value.

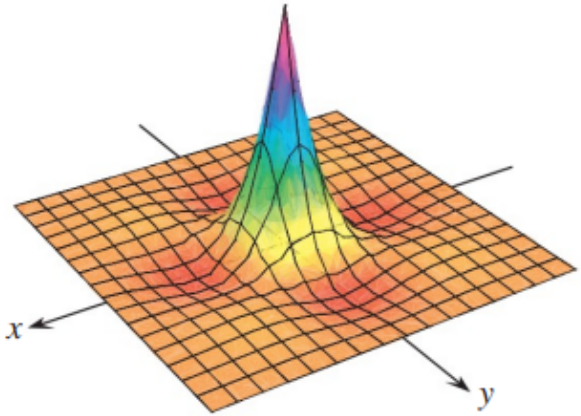



Figure: $z = (\cos x)(\cos y)e - \sqrt{x^2 + y^2}$ with max=1 and a min=-0.067 on $|x| \leq 3\pi/2, |y| \leq 3\pi/2$.

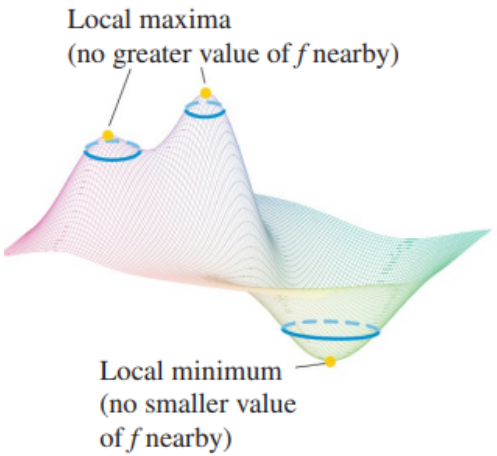
• The surface that is the graph of the function might be shaped like a **saddle** right above $(a, b)$ and cross its tangent plane there.

# Derivative Tests for Local Extreme Values

• For a function $f(x, y)$ of two variables, we look for points where the surface $z = f(x, y)$ has a **horizontal tangent plane**.
• At such points, we then look for **local maxima, local minima, and saddle points**.

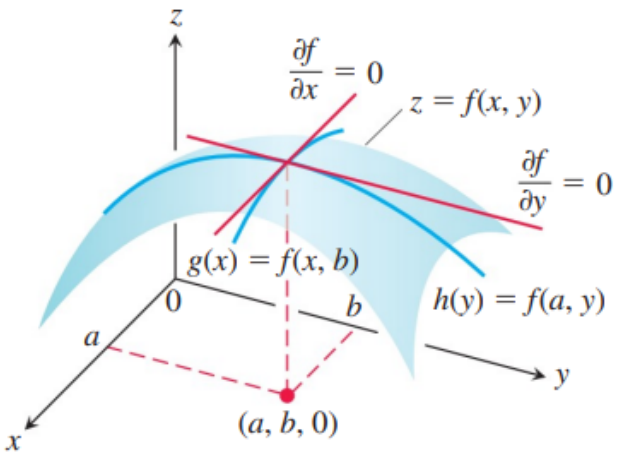
**DEFINITIONS** Let $f(x, y)$ be defined on a region $R$ containing the point $(a, b)$. Then

1. $f(a, b)$ is a **local maximum** value of $f$ if $f(a, b) \geq f(x, y)$ for all domain points $(x, y)$ in an open disk centered at $(a, b)$.
2. $f(a, b)$ is a **local minimum** value of $f$ if $f(a, b) \leq f(x, y)$ for all domain points $(x, y)$ in an open disk centered at $(a, b)$.



# Derivative Tests for Local Extreme Values

**THEOREM 10—First Derivative Test for Local Extreme Values**
If $f(x, y)$ has a local maximum or minimum value at an interior point $(a, b)$ of its domain and if the first partial derivatives exist there, then $f_x(a, b) = 0$ and $f_y(a, b) = 0$.

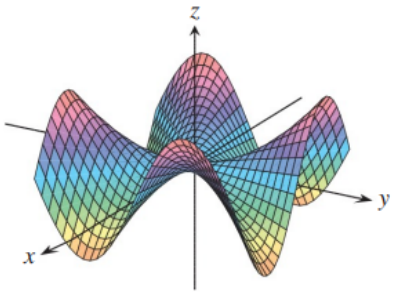



**DEFINITION** An interior point of the domain of a function $f(x, y)$ where both $f_x$ and $f_y$ are zero or where one or both of $f_x$ and $f_y$ do not exist is a **critical point** of $f$.

# Extreme Values and Saddle Points

• As with differentiable functions of a single variable, **not every critical point gives rise to a local extremum**.
• A differentiable function of a single variable might have a **point of inflection**.
• A differentiable function of two variables might have a **saddle point**, with the graph of f **crossing the tangent plane defined there**.

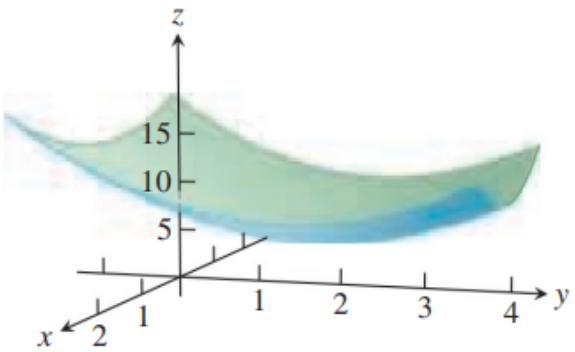
**DEFINITION** A differentiable function $f(x, y)$ has a **saddle point** at a critical point $(a, b)$ if in every open disk centered at $(a, b)$ there are domain points $(x, y)$ where $f(x, y) > f(a, b)$ and domain points $(x, y)$ where $f(x, y) < f(a, b)$. The corresponding point $(a, b, f(a, b))$ on the surface $z = f(x, y)$ is called a saddle point of the surface (Figure 14.45).




$$z = \frac{xy(x^2 - y^2)}{x^2 + y^2}$$

## Example 1

Find the local extreme values of $f(x, y) = x^2 + y^2 - 4y + 9$.




**Solution** The domain of $f$ is the entire plane (so there are no boundary points) and the partial derivatives $f_x = 2x$ and $f_y = 2y - 4$ exist everywhere. Therefore, local extreme values can occur only where

$$f_x = 2x = 0 \qquad \text{and} \qquad f_y = 2y - 4 = 0.$$

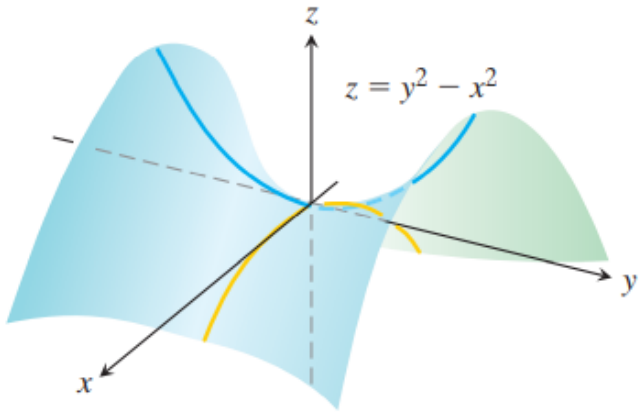
The only possibility is the point (0, 2), where the value of $f$ is 5. Since $f(x, y) = x^2 + (y - 2)^2 + 5$ is never less than 5, we see that the critical point (0, 2) gives a local minimum (Figure 14.46). ■

# Extreme Values and Saddle Points

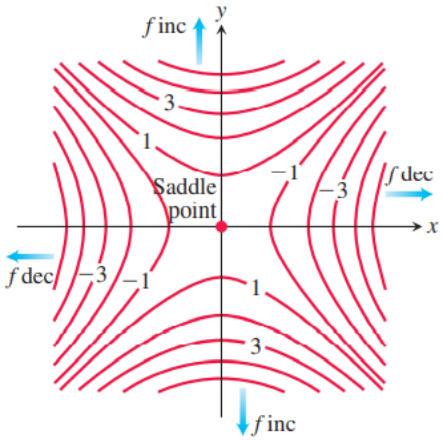
## Example 2

Find the local extreme values (if any) of $f(x, y) = y^2 - x^2$.

**Solution** The domain of $f$ is the entire plane (so there are no boundary points) and the partial derivatives $f_x = -2x$ and $f_y = 2y$ exist everywhere. Therefore, local extrema can occur only at the origin $(0, 0)$ where $f_x = 0$ and $f_y = 0$. Along the positive $x$-axis, however, $f$ has the value $f(x, 0) = -x^2 < 0$; along the positive $y$-axis, $f$ has the value $f(0, y) = y^2 > 0$. Therefore, every open disk in the $xy$-plane centered at $(0, 0)$ contains points where the function is positive and points where it is negative. The function has a saddle point at the origin and no local extreme values (Figure 14.47a). Figure 14.47b displays the level curves (they are hyperbolas) of $f$, and shows the function decreasing and increasing in an alternating fashion among the four groupings of hyperbolas. ■




(a)




(b)

# Second Derivative Test for Local Extreme Values

- That $f_x = f_y = 0$ at an interior point $(a, b)$ of R **does not guarantee** f has a local extreme value there.
- If f and its **first and second partial derivatives are continuous on R**, however, we may be able to learn more from the following theorem

**THEOREM 11—Second Derivative Test for Local Extreme Values**
Suppose that $f(x, y)$ and its first and second partial derivatives are continuous throughout a disk centered at $(a, b)$ and that $f_x(a, b) = f_y(a, b) = 0$. Then

**i)** $f$ has a **local maximum** at $(a, b)$ if $f_{xx} < 0$ and $f_{xx}f_{yy} - {f_{xy}}^2 > 0$ at $(a, b)$.

**ii)** $f$ has a **local minimum** at $(a, b)$ if $f_{xx} > 0$ and $f_{xx}f_{yy} - {f_{xy}}^2 > 0$ at $(a, b)$.

**iii)** $f$ has a **saddle point** at $(a, b)$ if $f_{xx}f_{yy} - {f_{xy}}^2 < 0$ at $(a, b)$.

**iv)** **the test is inconclusive** at $(a, b)$ if $f_{xx}f_{yy} - {f_{xy}}^2 = 0$ at $(a, b)$. In this case, we must find some other way to determine the behavior of $f$ at $(a, b)$.

- The expression $f_{xx}f_{yy} - f_{xy}^2$ is called the discriminant or **Hessian of f**.
- It is sometimes easier to remember it in **determinant form**,

$$f_{xx}f_{yy} - {f_{xy}}^2 = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix}.$$

## Example 3

Find the local extreme values of the function
$f(x, y) = xy - x^2 - y^2 - 2x - 2y + 4.$

**Solution** The function is defined and differentiable for all $x$ and $y$, and its domain has no boundary points. The function therefore has extreme values only at the points where $f_x$ and $f_y$ are simultaneously zero. This leads to

$$f_x = y - 2x - 2 = 0, \qquad f_y = x - 2y - 2 = 0,$$

or

$$x = y = -2.$$

Therefore, the point $(-2, -2)$ is the only point where $f$ may take on an extreme value. To see if it does so, we calculate

$$f_{xx} = -2, \qquad f_{yy} = -2, \qquad f_{xy} = 1.$$

The discriminant of $f$ at $(a, b) = (-2, -2)$ is

$$f_{xx}f_{yy} - {f_{xy}}^2 = (-2)(-2) - (1)^2 = 4 - 1 = 3.$$

The combination

$$f_{xx} < 0 \qquad \text{and} \qquad f_{xx}f_{yy} - {f_{xy}}^2 > 0$$

tells us that $f$ has a local maximum at $(-2, -2)$. The value of $f$ at this point is $f(-2, -2) = 8$.

## Example 4

Find the local extreme values of $f(x, y) = 3y^2 - 2y^3 - 3x^2 + 6xy$.

**Solution** Since $f$ is differentiable everywhere, it can assume extreme values only where

$$f_x = 6y - 6x = 0 \qquad \text{and} \qquad f_y = 6y - 6y^2 + 6x = 0.$$

From the first of these equations we find $x = y$, and substitution for $y$ into the second equation then gives

$$6x - 6x^2 + 6x = 0 \qquad \text{or} \qquad 6x(2 - x) = 0.$$

The two critical points are therefore (0, 0) and (2, 2).

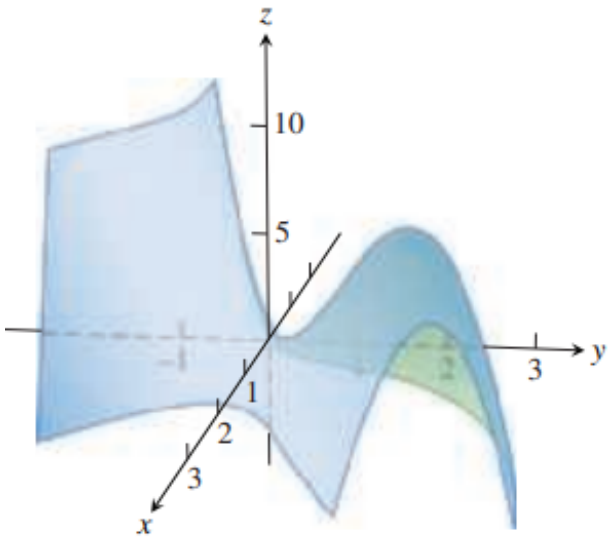
To classify the critical points, we calculate the second derivatives:

$$f_{xx} = -6, \qquad f_{yy} = 6 - 12y, \qquad f_{xy} = 6.$$

The discriminant is given by

$$f_{xx}f_{yy} - f_{xy}{}^2 = (-36 + 72y) - 36 = 72(y - 1).$$

At the critical point (0, 0) we see that the value of the discriminant is the negative number $-72$, so the function has a saddle point at the origin. At the critical point (2, 2) we see that the discriminant has the positive value 72. Combining this result with the negative value of the second partial $f_{xx} = -6$, Theorem 11 says that the critical point (2, 2) gives a local maximum value of $f(2, 2) = 12 - 16 - 12 + 24 = 8$. A graph of the surface is shown in Figure 14.48. ■



# Absolute Maxima and Minima on Closed Bounded Regions

We organize the search for the **absolute extrema** of a continuous function $f(x, y)$ on a closed and bounded region R into three steps.

**1.** List the **interior points of R** where f may have local maxima and minima and evaluate f at these points. These are the critical points of f.

**2.** List the **boundary points of R** where f has local maxima and minima and evaluate f at these points. We show how to do this in the next example.

**3.** Look through the lists for the maximum and minimum values of f. These will be the **absolute maximum and minimum values of f on R**.

## Example 5

Find the absolute maximum and minimum values of $f(x, y) = 2 + 2x + 4y - x^2 - y^2$ on the triangular region in the first quadrant bounded by the lines $x = 0, y = 0$, and $y = 9 - x$.

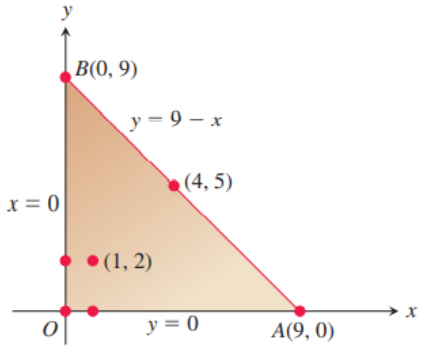
**Solution** Since $f$ is differentiable, the only places where $f$ can assume these values are points inside the triangle where $f_x = f_y = 0$ and points on the boundary (Figure 14.50a).

**(a) Interior points.** For these we have

$$f_x = 2 - 2x = 0, \qquad f_y = 4 - 2y = 0,$$

yielding the single point $(x, y) = (1, 2)$. The value of $f$ there is

$$f(1, 2) = 7.$$




(a)

# Example 5

**(b) Boundary points.** We take the triangle one side at a time:

**i)** On the segment $OA$, $y = 0$. The function

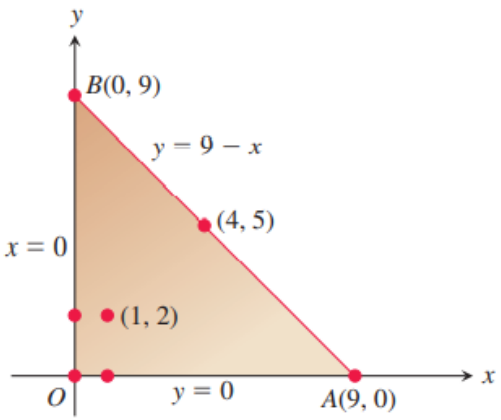
$$f(x, y) = f(x, 0) = 2 + 2x - x^2$$

may now be regarded as a function of $x$ defined on the closed interval $0 \leq x \leq 9$. Its extreme values (as we know from Chapter 4) may occur at the endpoints

$$x = 0 \quad \text{where} \quad f(0, 0) = 2$$
$$x = 9 \quad \text{where} \quad f(9, 0) = 2 + 18 - 81 = -61$$

or at the interior points where $f'(x, 0) = 2 - 2x = 0$. The only interior point where $f'(x, 0) = 0$ is $x = 1$, where

$$f(x, 0) = f(1, 0) = 3.$$




(a)

# Example 5

**ii)** On the segment $OB$, $x = 0$ and

$$f(x, y) = f(0, y) = 2 + 4y - y^2.$$

As in part i), we consider $f(0, y)$ as a function of $y$ defined on the closed interval $[0, 9]$. Its extreme values can occur at the endpoints or at interior points where $f'(0, y) = 0$. Since $f'(0, y) = 4 - 2y$, the only interior point where $f'(0, y) = 0$ occurs at $(0, 2)$, with $f(0, 2) = 6$. So the candidates for this segment are

$$f(0, 0) = 2, \qquad f(0, 9) = -43, \qquad f(0, 2) = 6.$$

**iii)** We have already accounted for the values of $f$ at the endpoints of $AB$, so we need only look at the interior points of the line segment $AB$. With $y = 9 - x$, we have

$$f(x, y) = 2 + 2x + 4(9 - x) - x^2 - (9 - x)^2 = -43 + 16x - 2x^2.$$

Setting $f'(x, 9 - x) = 16 - 4x = 0$ gives

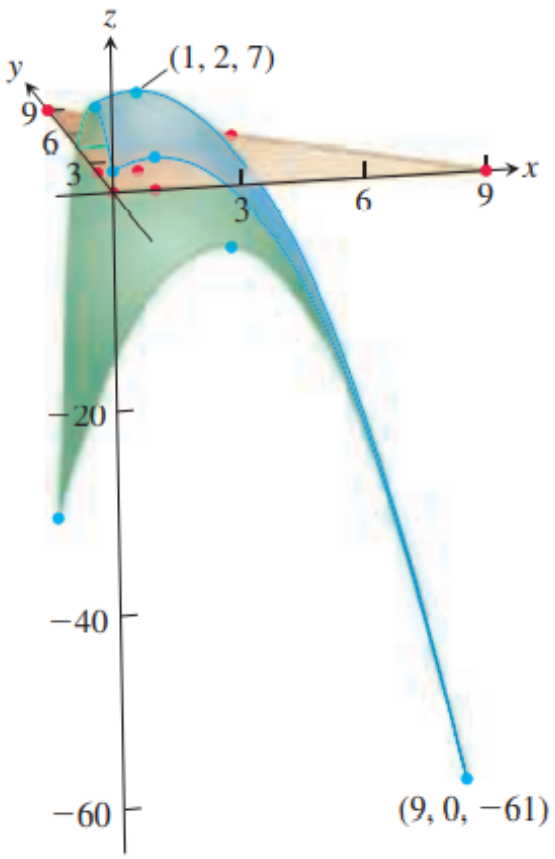
$$x = 4.$$

At this value of $x$,

$$y = 9 - 4 = 5 \qquad \text{and} \qquad f(x, y) = f(4, 5) = -11.$$

# Example 5

**Summary** We list all the function value candidates: 7, 2, −61, 3, −43, 6, −11. The maximum is 7, which $f$ assumes at (1, 2). The minimum is −61, which $f$ assumes at (9, 0). See Figure 14.50b. ■




(b)

# Summary of Max-Min Tests

### Summary of Max-Min Tests

The extreme values of $f(x, y)$ can occur only at

**i)** **boundary points** of the domain of $f$

**ii)** **critical points** (interior points where $f_x = f_y = 0$ or points where $f_x$ or $f_y$ fails to exist)

If the first- and second-order partial derivatives of $f$ are continuous throughout a disk centered at a point $(a, b)$ and $f_x(a, b) = f_y(a, b) = 0$, the nature of $f(a, b)$ can be tested with the **Second Derivative Test**:

**i)** $f_{xx} < 0$ and $f_{xx}f_{yy} - {f_{xy}}^2 > 0$ at $(a, b)$ $\Rightarrow$ **local maximum**

**ii)** $f_{xx} > 0$ and $f_{xx}f_{yy} - {f_{xy}}^2 > 0$ at $(a, b)$ $\Rightarrow$ **local minimum**

**iii)** $f_{xx}f_{yy} - {f_{xy}}^2 < 0$ at $(a, b)$ $\Rightarrow$ **saddle point**

**iv)** $f_{xx}f_{yy} - {f_{xy}}^2 = 0$ at $(a, b)$ $\Rightarrow$ **test is inconclusive**

# Lecture 19: Partial Derivatives.

MA2032 Vector Calculus

Lecturer: Larissa Serdukova

School of Computing and Mathematical Science
University of Leicester

November 2, 2022

# Lagrange Multipliers

• Sometimes we need to find the extreme values of a function whose **domain is constrained** to lie within some particular subset of the plane — for example, a disk, a closed triangular region, or along a curve.

• We saw an instance of this situation in **Example 5 of the previous lecture 18**.

• Today we explore a powerful method for finding **extreme values of constrained functions**: the method of **Lagrange multipliers**.

# The Method of Lagrange Multipliers

• The method says that the **local extreme values** of a function $f(x, y, z)$ whose variables are subject to a **constraint** $g(x, y, z) = 0$ are to be found on the surface $g = 0$ among the points where

$$\nabla f = \lambda \nabla g$$

• for some scalar $\lambda$ (called a **Lagrange multiplier**).

• To explore the method further and see why it works, we first make the following observation, which we state as a **theorem**.

# The Method of Lagrange Multipliers

**THEOREM 12—The Orthogonal Gradient Theorem**
Suppose that $f(x, y, z)$ is differentiable in a region whose interior contains a smooth curve

$$C: \quad \mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}.$$

If $P_0$ is a point on $C$ where $f$ has a local maximum or minimum relative to its values on $C$, then $\nabla f$ is orthogonal to $C$ at $P_0$.

**Proof** We show that $\nabla f$ is orthogonal to the curve's tangent vector $\mathbf{r}'$ at $P_0$. The values of $f$ on $C$ are given by the composition $f(x(t), y(t), z(t))$, whose derivative with respect to $t$ is

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} = \nabla f \cdot \mathbf{r}'.$$

At any point $P_0$ where $f$ has a local maximum or minimum relative to its values on the curve, $df/dt = 0$, so

$$\nabla f \cdot \mathbf{r}' = 0. \qquad \blacksquare$$

# The Method of Lagrange Multipliers

• By dropping the z-terms in Theorem 12, we obtain a similar result for **functions of two variables**.

> **COROLLARY** At the points on a smooth curve $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$ where a differentiable function $f(x, y)$ takes on its local maxima and minima relative to its values on the curve, $\nabla f \cdot \mathbf{r}' = 0$.

• **Theorem 12 is the key** to the method of Lagrange multipliers.

• Suppose that $f(x, y, z)$ and $g(x, y, z)$ are differentiable and that $P_0$ is a **point on the surface** $g(x, y, z) = 0$ **where f has a local maximum or minimum** value relative to its other values on the surface.

• We assume also that $\nabla g \neq 0$ at points on the surface $g(x, y, z) = 0$.

• Then f takes on a local maximum or minimum at $P_0$ relative to its values on every differentiable curve through $P_0$ on the surface $g(x, y, z) = 0$.

# The Method of Lagrange Multipliers

- Therefore, $\nabla f$ is **orthogonal to the tangent vector** of every such differentiable curve through $P_0$.
- So is $\nabla g$, moreover (because $\nabla g$ is **orthogonal to the level surface** $g = 0$).
- Therefore, at $P_0$ , $\nabla f$ is some **scalar multiple** $\lambda$ of $\nabla g$.

**The Method of Lagrange Multipliers**

Suppose that $f(x, y, z)$ and $g(x, y, z)$ are differentiable and $\nabla g \neq \mathbf{0}$ when $g(x, y, z) = 0$. To find the local maximum and minimum values of $f$ subject to the constraint $g(x, y, z) = 0$ (if these exist), find the values of $x$, $y$, $z$, and $\lambda$ that simultaneously satisfy the equations

$$\nabla f = \lambda \nabla g \qquad \text{and} \qquad g(x, y, z) = 0. \tag{1}$$

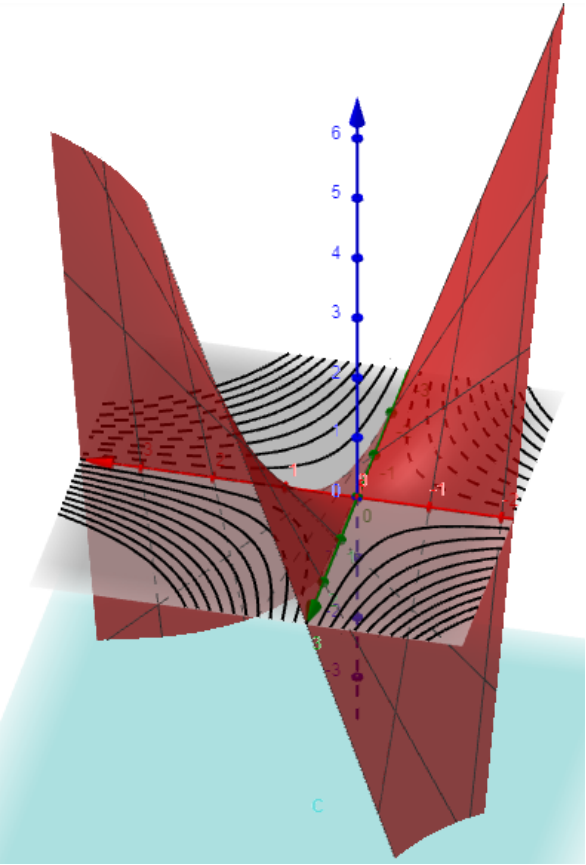
For functions of two independent variables, the condition is similar, but without the variable $z$.

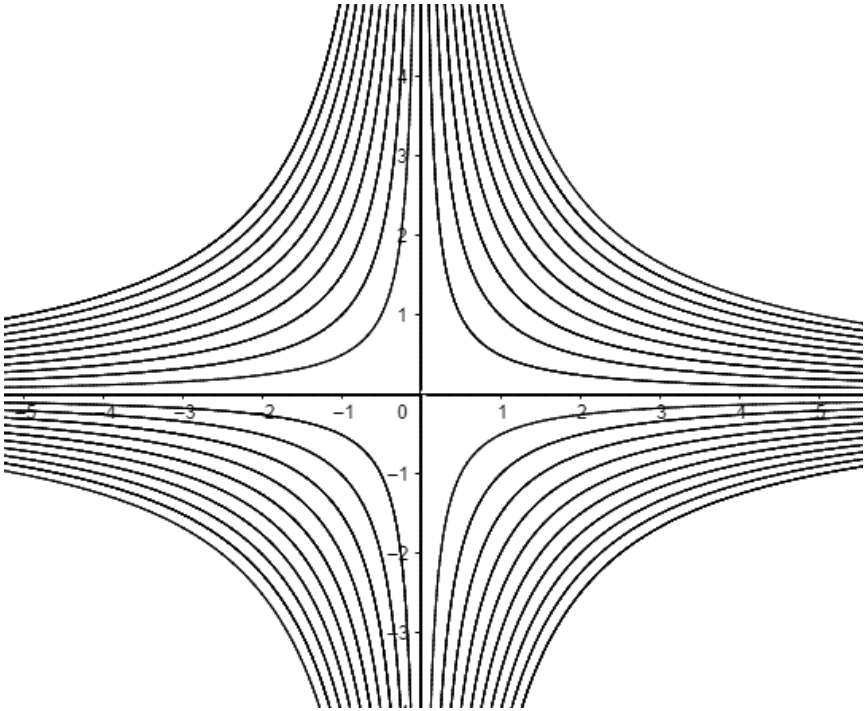
- Some **care must be used** in applying this method. **An extreme value may not actually exist**.

# The Method of Lagrange Multipliers

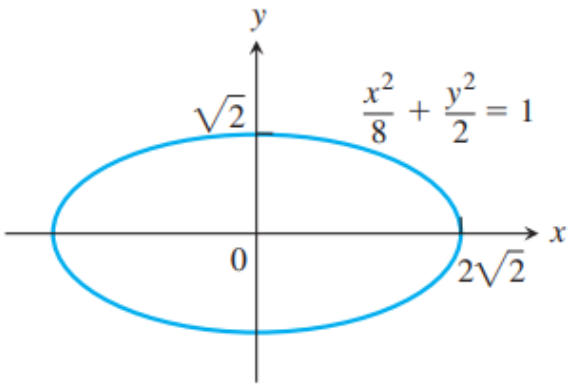
## Example 1

Find the greatest and smallest values that the function $f(x, y) = xy$ takes on the ellipse $\frac{x^2}{8} + \frac{y^2}{2} = 1$.





# Example 1




**Solution** We want to find the extreme values of $f(x, y) = xy$ subject to the constraint

$$g(x, y) = \frac{x^2}{8} + \frac{y^2}{2} - 1 = 0.$$

To do so, we first find the values of $x$, $y$, and $\lambda$ for which

$$\nabla f = \lambda \nabla g \qquad \text{and} \qquad g(x, y) = 0.$$

The gradient equation in Equations (1) gives

$$y\mathbf{i} + x\mathbf{j} = \frac{\lambda}{4} x\mathbf{i} + \lambda y\mathbf{j},$$

from which we find

$$y = \frac{\lambda}{4} x, \qquad x = \lambda y, \qquad \text{and} \qquad y = \frac{\lambda}{4}(\lambda y) = \frac{\lambda^2}{4} y,$$

so that $y = 0$ or $\lambda = \pm 2$. We now consider these two cases.

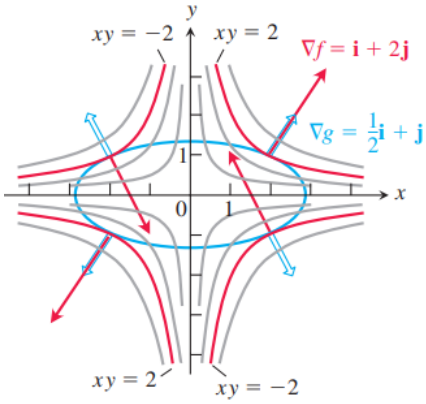
# Example 1

**Case 1:** If $y = 0$, then $x = y = 0$. But $(0, 0)$ is not on the ellipse. Hence, $y \neq 0$.

**Case 2:** If $y \neq 0$, then $\lambda = \pm 2$ and $x = \pm 2y$. Substituting this in the equation $g(x, y) = 0$ gives

$$\frac{(\pm 2y)^2}{8} + \frac{y^2}{2} = 1, \qquad 4y^2 + 4y^2 = 8 \qquad \text{and} \qquad y = \pm 1.$$

The function $f(x, y) = xy$ therefore takes on its extreme values on the ellipse at the four points $(\pm 2, 1)$, $(\pm 2, -1)$. The extreme values are $xy = 2$ and $xy = -2$.




**The Geometry of the Solution** The level curves of the function $f(x, y) = xy$ are the hyperbolas $xy = c$ (Figure 14.56). The farther the hyperbolas lie from the origin, the larger the absolute value of $f$. We want to find the extreme values of $f(x, y)$, given that the point $(x, y)$ also lies on the ellipse $x^2 + 4y^2 = 8$. Which hyperbolas intersecting the ellipse lie farthest from the origin? The hyperbolas that just graze the ellipse, the ones that are tangent to it, are farthest. At these points, any vector normal to the hyperbola is normal to the ellipse, so $\nabla f = y\mathbf{i} + x\mathbf{j}$ is a multiple ($\lambda = \pm 2$) of $\nabla g = (x/4)\mathbf{i} + y\mathbf{j}$. At the point $(2, 1)$, for example,

$$\nabla f = \mathbf{i} + 2\mathbf{j}, \qquad \nabla g = \frac{1}{2}\mathbf{i} + \mathbf{j}, \qquad \text{and} \qquad \nabla f = 2\nabla g.$$

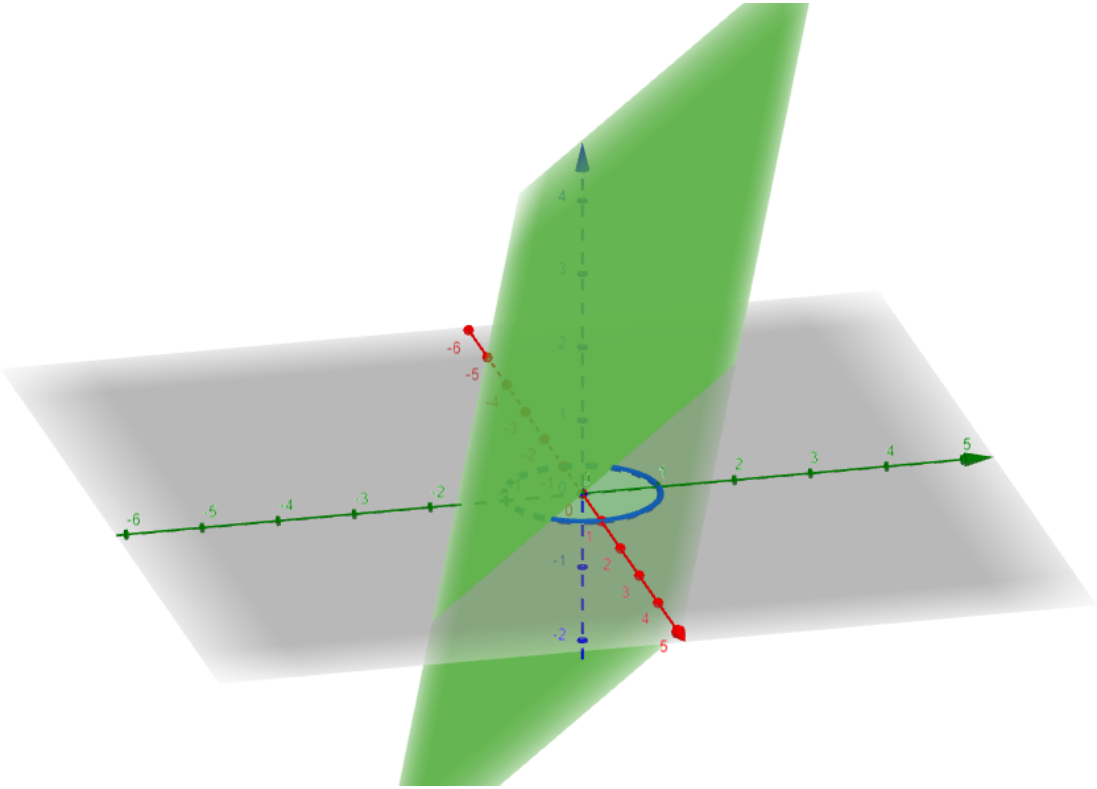
At the point $(-2, 1)$,

$$\nabla f = \mathbf{i} - 2\mathbf{j}, \qquad \nabla g = -\frac{1}{2}\mathbf{i} + \mathbf{j}, \qquad \text{and} \qquad \nabla f = -2\nabla g. \qquad \blacksquare$$

# The Method of Lagrange Multipliers

## Example 2

Find the maximum and minimum values of the function $f(x, y) = 3x + 4y$ on the circle $x^2 + y^2 = 1$.



# Example 2

**Solution** We model this as a Lagrange multiplier problem with

$$f(x, y) = 3x + 4y, \qquad g(x, y) = x^2 + y^2 - 1$$

and look for the values of $x$, $y$, and $\lambda$ that satisfy the equations

$$\nabla f = \lambda \nabla g: \quad 3\mathbf{i} + 4\mathbf{j} = 2x\lambda\mathbf{i} + 2y\lambda\mathbf{j}$$
$$g(x, y) = 0: \quad x^2 + y^2 - 1 = 0.$$

The gradient equation in Equations (1) implies that $\lambda \neq 0$ and gives

$$x = \frac{3}{2\lambda}, \qquad y = \frac{2}{\lambda}.$$

These equations tell us, among other things, that $x$ and $y$ have the same sign. With these values for $x$ and $y$, the equation $g(x, y) = 0$ gives

$$\left(\frac{3}{2\lambda}\right)^2 + \left(\frac{2}{\lambda}\right)^2 - 1 = 0,$$

so

$$\frac{9}{4\lambda^2} + \frac{4}{\lambda^2} = 1, \qquad 9 + 16 = 4\lambda^2, \qquad 4\lambda^2 = 25, \qquad \text{and} \qquad \lambda = \pm\frac{5}{2}.$$

# Example 2

Thus,

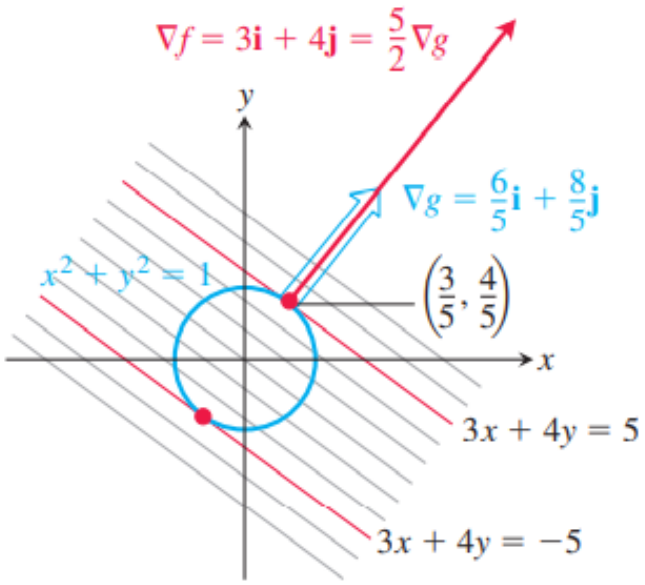
$$x = \frac{3}{2\lambda} = \pm\frac{3}{5}, \qquad y = \frac{2}{\lambda} = \pm\frac{4}{5},$$

and $f(x, y) = 3x + 4y$ has extreme values at $(x, y) = \pm(3/5, 4/5)$.

By calculating the value of $3x + 4y$ at the points $\pm(3/5, 4/5)$, we see that its maximum and minimum values on the circle $x^2 + y^2 = 1$ are

$$3\left(\frac{3}{5}\right) + 4\left(\frac{4}{5}\right) = \frac{25}{5} = 5 \qquad \text{and} \qquad 3\left(-\frac{3}{5}\right) + 4\left(-\frac{4}{5}\right) = -\frac{25}{5} = -5.$$

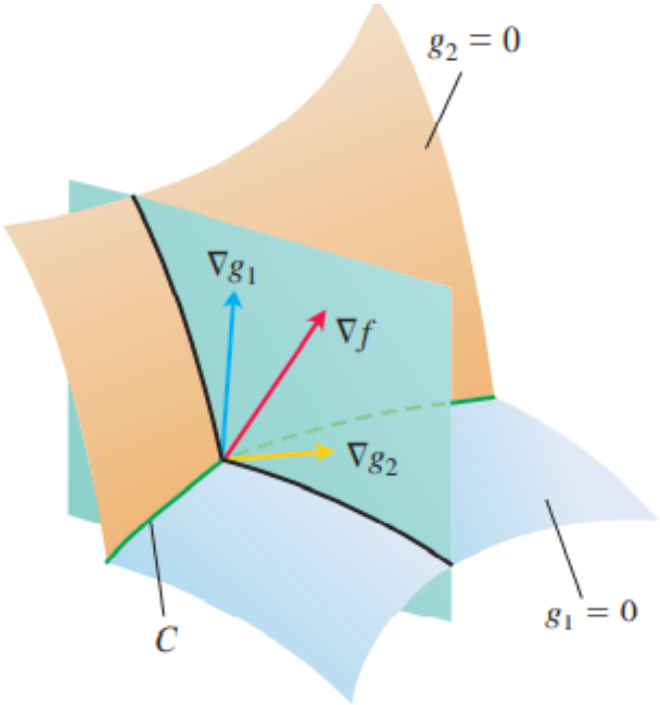
# Example 2




**The Geometry of the Solution** The level curves of $f(x, y) = 3x + 4y$ are the lines $3x + 4y = c$ (Figure 14.57). The farther the lines lie from the origin, the larger the absolute value of $f$. We want to find the extreme values of $f(x, y)$ given that the point $(x, y)$ also lies on the circle $x^2 + y^2 = 1$. Which lines intersecting the circle lie farthest from the origin? The lines tangent to the circle are farthest. At the points of tangency, any vector normal to the line is normal to the circle, so the gradient $\nabla f = 3\mathbf{i} + 4\mathbf{j}$ is a multiple ($\lambda = \pm 5/2$) of the gradient $\nabla g = 2x\mathbf{i} + 2y\mathbf{j}$. At the point $(3/5, 4/5)$, for example,

$$\nabla f = 3\mathbf{i} + 4\mathbf{j}, \qquad \nabla g = \frac{6}{5}\mathbf{i} + \frac{8}{5}\mathbf{j}, \qquad \text{and} \qquad \nabla f = \frac{5}{2}\nabla g.$$

# Lagrange Multipliers with Two Constraints



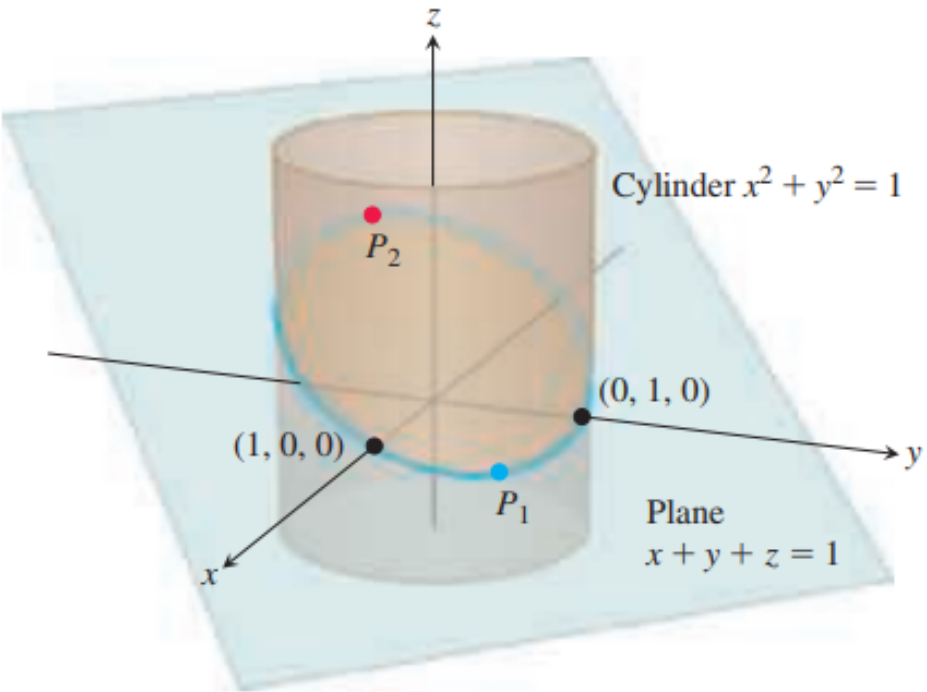

- Many problems require us to find the extreme values of a differentiable function $f(x, y, z)$ whose variables are subject to **two constraints**.
- If the constraints are $g_1(x, y, z) = 0$ and $g_2(x, y, z) = 0$ and $g_1$ and $g_2$ **are differentiable**, with $\nabla g_1$ **not parallel** to $\nabla g_2$, we find the constrained local maxima and minima of f by introducing two Lagrange multipliers $\lambda$ and $\mu$.
- That is, we locate the points $P(x, y, z)$ where f takes on its constrained extreme values by finding the values of x, y, z, $\lambda$, and $\mu$ that **simultaneously satisfy** the three equations

$$\nabla f = \lambda \nabla g_1 + \mu \nabla g_2, \qquad g_1(x, y, z) = 0, \qquad g_2(x, y, z) = 0 \qquad (2)$$

# Lagrange Multipliers with Two Constraints

## Example 3

The plane $x + y + z = 1$ cuts the cylinder $x^2 + y^2 = 1$ in an ellipse. Find the points on the ellipse that lie closest to and farthest from the origin.



# Example 3

**Solution** We find the extreme values of

$$f(x, y, z) = x^2 + y^2 + z^2$$

(the square of the distance from $(x, y, z)$ to the origin) subject to the constraints

$$g_1(x, y, z) = x^2 + y^2 - 1 = 0 \tag{3}$$

$$g_2(x, y, z) = x + y + z - 1 = 0. \tag{4}$$

The gradient equation in Equations (2) then gives

$$\begin{aligned} \nabla f &= \lambda \nabla g_1 + \mu \nabla g_2 \\ 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} &= \lambda(2x\mathbf{i} + 2y\mathbf{j}) + \mu(\mathbf{i} + \mathbf{j} + \mathbf{k}) \\ 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} &= (2\lambda x + \mu)\mathbf{i} + (2\lambda y + \mu)\mathbf{j} + \mu\mathbf{k} \end{aligned}$$

or

$$2x = 2\lambda x + \mu, \qquad 2y = 2\lambda y + \mu, \qquad 2z = \mu. \tag{5}$$

The scalar equations in Equations (5) yield

$$\begin{aligned} 2x &= 2\lambda x + 2z \Rightarrow (1 - \lambda)x = z, \\ 2y &= 2\lambda y + 2z \Rightarrow (1 - \lambda)y = z. \end{aligned} \tag{6}$$

# Example 3

Equations (6) are satisfied simultaneously if either $\lambda = 1$ and $z = 0$ or $\lambda \neq 1$ and $x = y = z/(1 - \lambda)$.

If $z = 0$, then solving Equations (3) and (4) simultaneously to find the corresponding points on the ellipse gives the two points $(1, 0, 0)$ and $(0, 1, 0)$. This makes sense when you look at Figure 14.59.

If $x = y$, then Equations (3) and (4) give

$$\begin{aligned} x^2 + x^2 - 1 &= 0 & \qquad x + x + z - 1 &= 0 \\ 2x^2 &= 1 & z &= 1 - 2x \\ x &= \pm\frac{\sqrt{2}}{2} & z &= 1 \mp \sqrt{2}. \end{aligned}$$

The corresponding points on the ellipse are

$$P_1 = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 1 - \sqrt{2}\right) \quad \text{and} \quad P_2 = \left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, 1 + \sqrt{2}\right).$$

Here we need to be careful, however. Although $P_1$ and $P_2$ both give local maxima of $f$ on the ellipse, $P_2$ is farther from the origin than $P_1$.

The points on the ellipse closest to the origin are $(1, 0, 0)$ and $(0, 1, 0)$. The point on the ellipse farthest from the origin is $P_2$. (See Figure 14.59.) ■

# Lecture 20: Partial Derivatives.

MA2032 Vector Calculus

Lecturer: Larissa Serdukova

School of Computing and Mathematical Science
University of Leicester

November 4, 2022

# Taylor's Formula for Two Variables

• The use of **Taylor's formula** leads to an extension of the formula that provides **polynomial approximations** of all orders for functions of two independent variables.

**Taylor's Formula for $f(x, y)$ at the Point $(a, b)$**

Suppose $f(x, y)$ and its partial derivatives through order $n + 1$ are continuous throughout an open rectangular region $R$ centered at a point $(a, b)$. Then, throughout $R$,

$$\begin{aligned} f(a+h, b+k) = f(a,b) &+ (hf_x + kf_y)\Big|_{(a,b)} + \frac{1}{2!}(h^2 f_{xx} + 2hkf_{xy} + k^2 f_{yy})\Big|_{(a,b)} \\ &+ \frac{1}{3!}(h^3 f_{xxx} + 3h^2 k f_{xxy} + 3hk^2 f_{xyy} + k^3 f_{yyy})\Big|_{(a,b)} + \cdots + \frac{1}{n!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n f\Big|_{(a,b)} \\ &+ \frac{1}{(n+1)!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{n+1} f\Big|_{(a+ch,\, b+ck)}. \end{aligned} \tag{7}$$

• The **first n derivative terms** are evaluated at $(a, b)$. The **last term** is evaluated at some point $(a + ch, b + ck)$ on the line segment joining $(a, b)$ and $(a + h, b + k)$.

# Taylor's Formula for Two Variables

• If $(a, b) = (0, 0)$ and we treat h and k as independent variables (denoting them now by x and y), then Equation (7) assumes the following form.

**Taylor's Formula for $f(x, y)$ at the Origin**

$$
\begin{aligned}
f(x, y) = {} & f(0, 0) + xf_x + yf_y + \frac{1}{2!}(x^2 f_{xx} + 2xyf_{xy} + y^2 f_{yy}) \\
& + \frac{1}{3!}(x^3 f_{xxx} + 3x^2 y f_{xxy} + 3xy^2 f_{xyy} + y^3 f_{yyy}) + \cdots + \frac{1}{n!}\left(x^n \frac{\partial^n f}{\partial x^n} + nx^{n-1}y \frac{\partial^n f}{\partial x^{n-1}\partial y} + \cdots + y^n \frac{\partial^n f}{\partial y^n}\right) \\
& + \frac{1}{(n+1)!}\left(x^{n+1} \frac{\partial^{n+1} f}{\partial x^{n+1}} + (n+1)x^n y \frac{\partial^{n+1} f}{\partial x^n \partial y} + \cdots + y^{n+1} \frac{\partial^{n+1} f}{\partial y^{n+1}}\right)\Bigg|_{(cx,\, cy)} \qquad (8)
\end{aligned}
$$

• The first n derivative terms are evaluated at (0, 0). The last term is evaluated at a point on the line segment joining the origin and $(x, y)$.

• Taylor's formula provides **polynomial approximations** of two-variable functions. The first n derivative terms give the polynomial; the **last term** gives the **approximation error**.

• The first three terms of Taylor's formula give the **function's linearization**. To improve on the linearization, we add higher-power terms.

# Taylor's Formula for Two Variables

## Example 1

Find a quadratic approximation to $f(x, y) = \sin x \sin y$ near the origin. How accurate is the approximation if $|x| \leq 0.1$ and $|y| \leq 0.1$?

**Solution** We take $n = 2$ in Equation (8):

$$f(x, y) = f(0, 0) + (xf_x + yf_y) + \frac{1}{2}(x^2 f_{xx} + 2xyf_{xy} + y^2 f_{yy})$$
$$+ \frac{1}{6}(x^3 f_{xxx} + 3x^2 y f_{xxy} + 3xy^2 f_{xyy} + y^3 f_{yyy})\Big|_{(cx,\, cy)}.$$

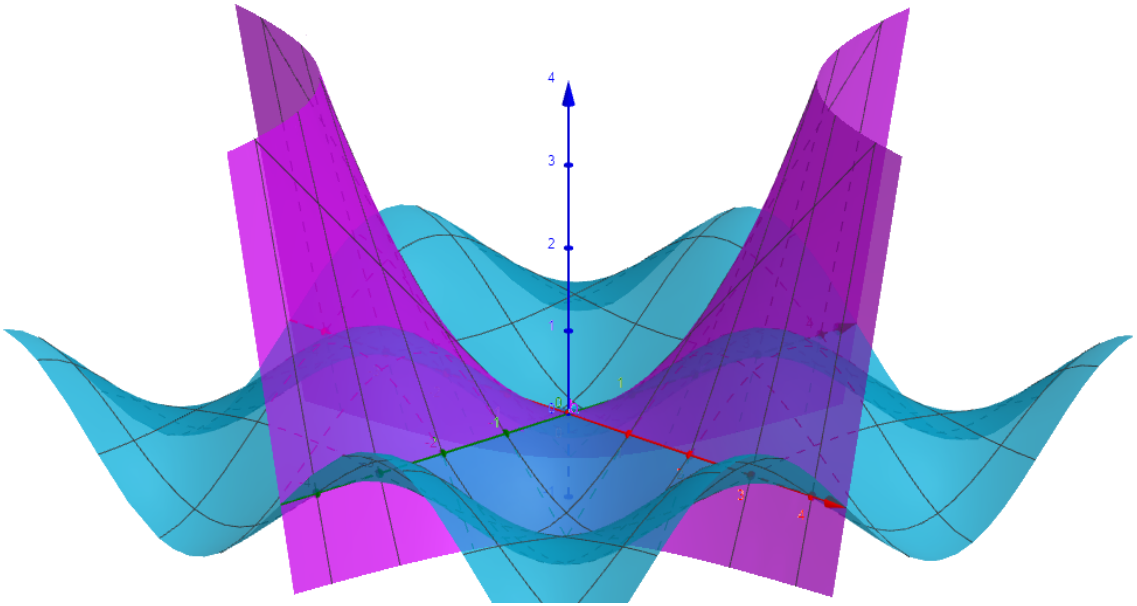
Calculating the values of the partial derivatives,

$$f(0, 0) = \sin x \sin y\Big|_{(0,\,0)} = 0, \qquad f_{xx}(0, 0) = -\sin x \sin y\Big|_{(0,\,0)} = 0,$$
$$f_x(0, 0) = \cos x \sin y\Big|_{(0,\,0)} = 0, \qquad f_{xy}(0, 0) = \cos x \cos y\Big|_{(0,\,0)} = 1,$$
$$f_y(0, 0) = \sin x \cos y\Big|_{(0,\,0)} = 0, \qquad f_{yy}(0, 0) = -\sin x \sin y\Big|_{(0,\,0)} = 0,$$

we have the result

$$\sin x \sin y \approx 0 + 0 + 0 + \frac{1}{2}(x^2(0) + 2xy(1) + y^2(0)), \quad \text{or} \quad \sin x \sin y \approx xy.$$

# Example 1



The error in the approximation is

$$E(x, y) = \frac{1}{6}\left(x^3 f_{xxx} + 3x^2 y f_{xxy} + 3xy^2 f_{xyy} + y^3 f_{yyy}\right)\Big|_{(cx,\, cy)}.$$

The third derivatives never exceed 1 in absolute value because they are products of sines and cosines. Also, $|x| \leq 0.1$ and $|y| \leq 0.1$. Hence

$$|E(x, y)| \leq \frac{1}{6}((0.1)^3 + 3(0.1)^3 + 3(0.1)^3 + (0.1)^3) = \frac{8}{6}(0.1)^3 \leq 0.00134$$

(rounded up). The error will not exceed 0.00134 if $|x| \leq 0.1$ and $|y| \leq 0.1$. ■

# Taylor's Formula for Two Variables

## Example 2

Use Taylor's formula for $f(x, y) = \ln(2x + y + 1)$ at the origin to find quadratic and cubic approximations of f near the origin.

$$f(x, y) = \ln(2x+y+1) \Rightarrow f_x = \frac{2}{2x+y+1},\ f_y = \frac{1}{2x+y+1},\ f_{xx} = \frac{-4}{(2x+y+1)^2},\ f_{xy} = \frac{-2}{(2x+y+1)^2},\ f_{yy} = \frac{-1}{(2x+y+1)^2}$$

$$\Rightarrow f(x, y) \approx f(0,0) + xf_x(0,0) + yf_y(0,0) + \tfrac{1}{2}\left[x^2 f_{xx}(0,0) + 2xyf_{xy}(0,0) + y^2 f_{yy}(0,0)\right]$$

$$= 0 + x\cdot 2 + y\cdot 1 + \tfrac{1}{2}\left[x^2\cdot(-4) + 2xy\cdot(-2) + y^2\cdot(-1)\right] = 2x + y + \tfrac{1}{2}\left(-4x^2 - 4xy - y^2\right)$$

$$= (2x+y) - \tfrac{1}{2}(2x+y)^2, \text{ quadratic approximation;}$$

$$f_{xxx} = \frac{16}{(2x+y+1)^3},\ f_{xxy} = \frac{8}{(2x+y+1)^3},\ f_{xyy} = \frac{4}{(2x+y+1)^3},\ f_{yyy} = \frac{2}{(2x+y+1)^3}$$

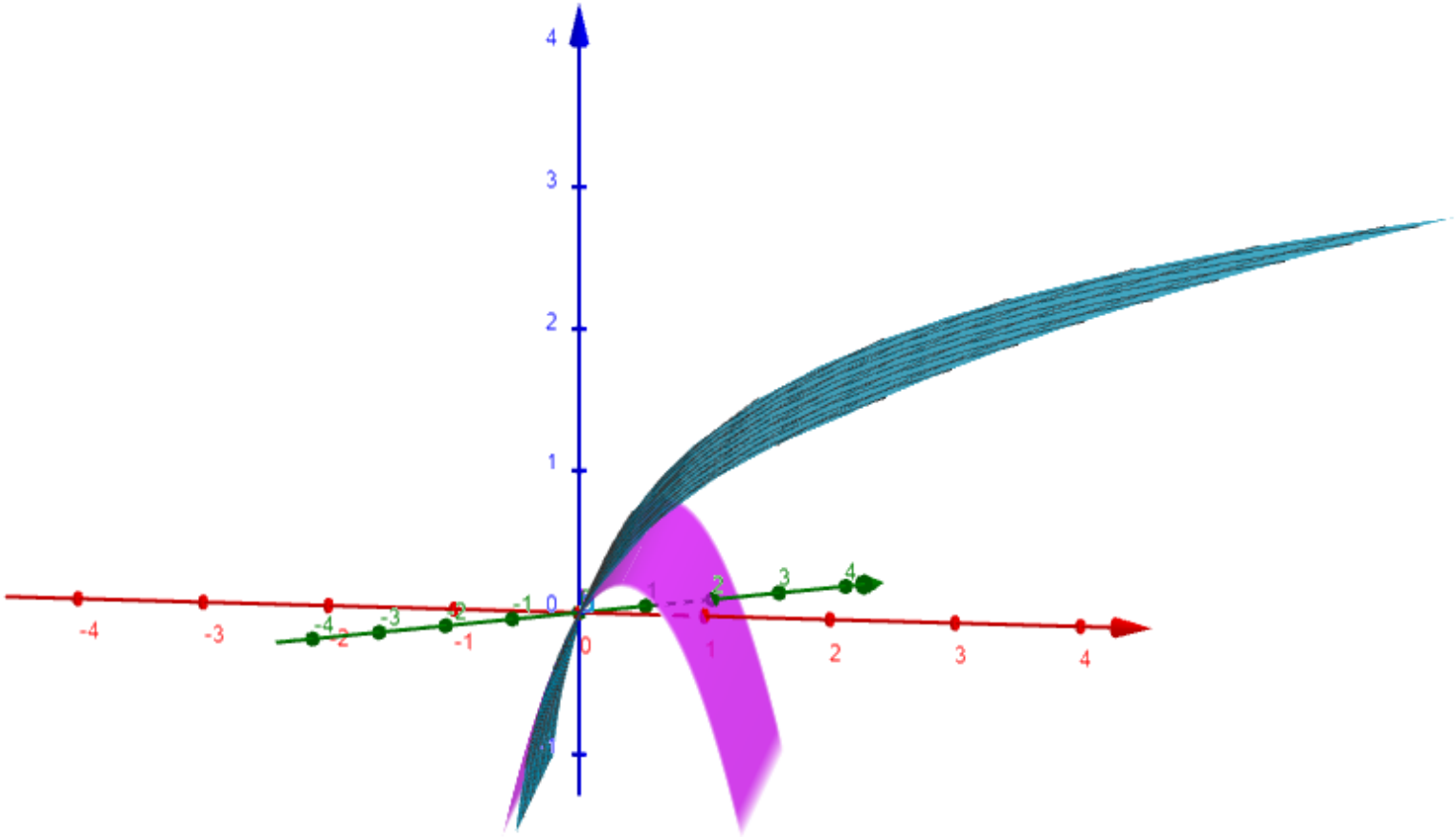
$$\Rightarrow f(x, y) \approx \text{quadratic} + \tfrac{1}{6}\left[x^3 f_{xxx}(0,0) + 3x^2 y f_{xxy}(0,0) + 3xy^2 f_{xyy}(0,0) + y^3 f_{yyy}(0,0)\right]$$

$$= (2x+y) - \tfrac{1}{2}(2x+y)^2 + \tfrac{1}{6}\left(x^3\cdot 16 + 3x^2 y\cdot 8 + 3xy^2\cdot 4 + y^3\cdot 2\right)$$

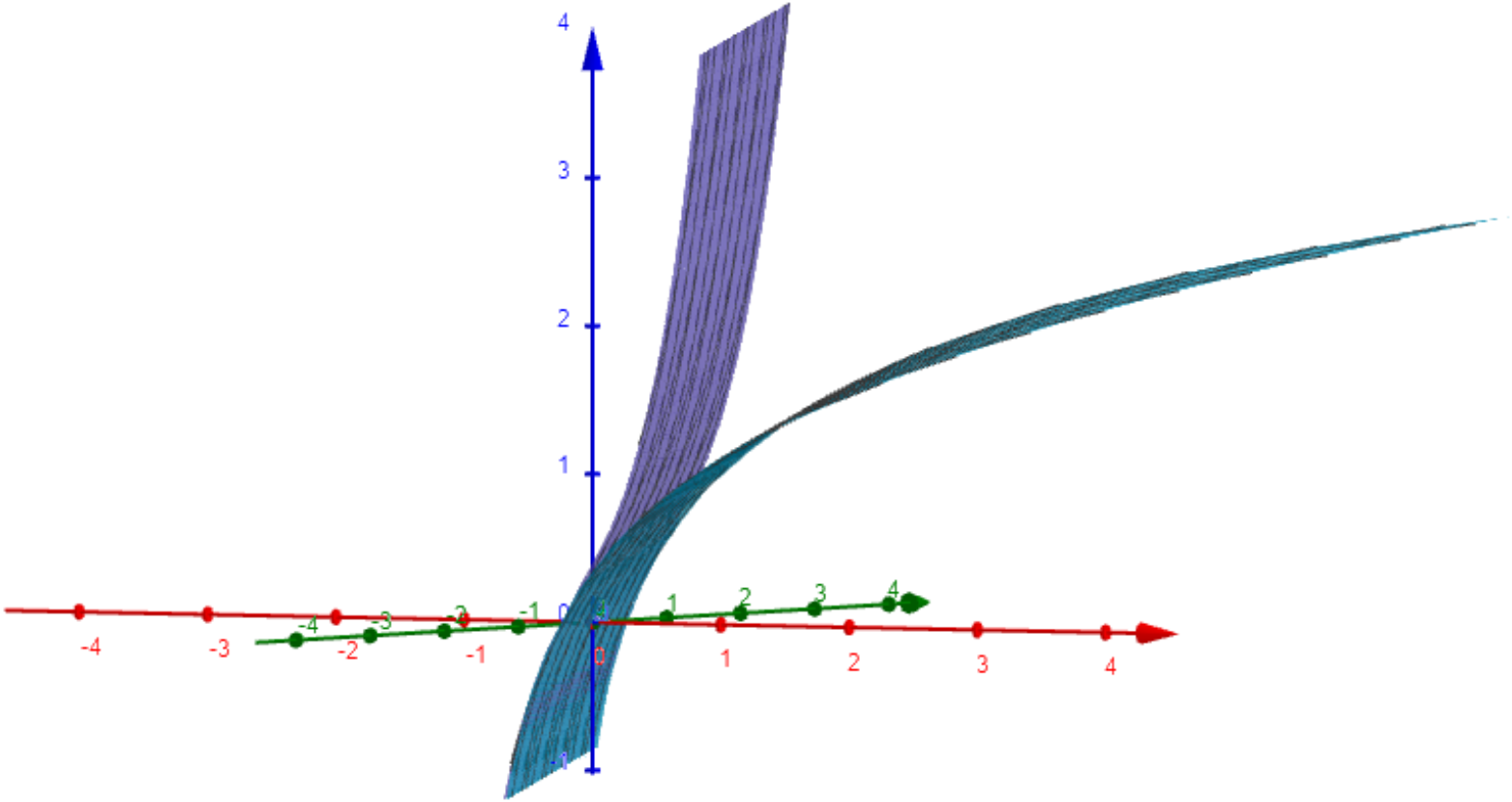
$$= (2x+y) - \tfrac{1}{2}(2x+y)^2 - \tfrac{1}{3}\left(8x^3 + 12x^2 y + 6xy^2 + y^2\right)$$

$$= (2x+y) - \tfrac{1}{2}(2x+y)^2 + \tfrac{1}{3}(2x+y)^3, \text{ cubic approximation}$$

# Example 2



# Example 2



# Taylor's Formula for Two Variables

## Example 3

Use Taylor's formula for $f(x,y) = \frac{1}{1-x-y+xy}$ at the origin to find quadratic and cubic approximations of f near the origin.

$$f(x,y) = \frac{1}{1-x-y+xy} \Rightarrow f_x = \frac{1-y}{(1-x-y+xy)^2},\ f_y = \frac{1-x}{(1-x-y+xy)^2},\ f_{xx} = \frac{2(1-y)^2}{(1-x-y+xy)^3},\ f_{xy} = \frac{1}{(1-x-y+xy)^2},$$

$$f_{yy} = \frac{2(1-x)^2}{(1-x-y+xy)^3} \Rightarrow f(x,y) \approx f(0,0) + xf_x(0,0) + yf_y(0,0) + \tfrac{1}{2}\left[x^2 f_{xx}(0,0) + 2xyf_{xy}(0,0) + y^2 f_{yy}(0,0)\right]$$

$$= 1\cdot x\cdot 1 + y\cdot 1 + \tfrac{1}{2}\left(x^2\cdot 2 + 2xy\cdot 1 + y^2\cdot 2\right) = 1 + x + y + x^2 + xy + y^2,\ \text{quadratic approximation;}$$

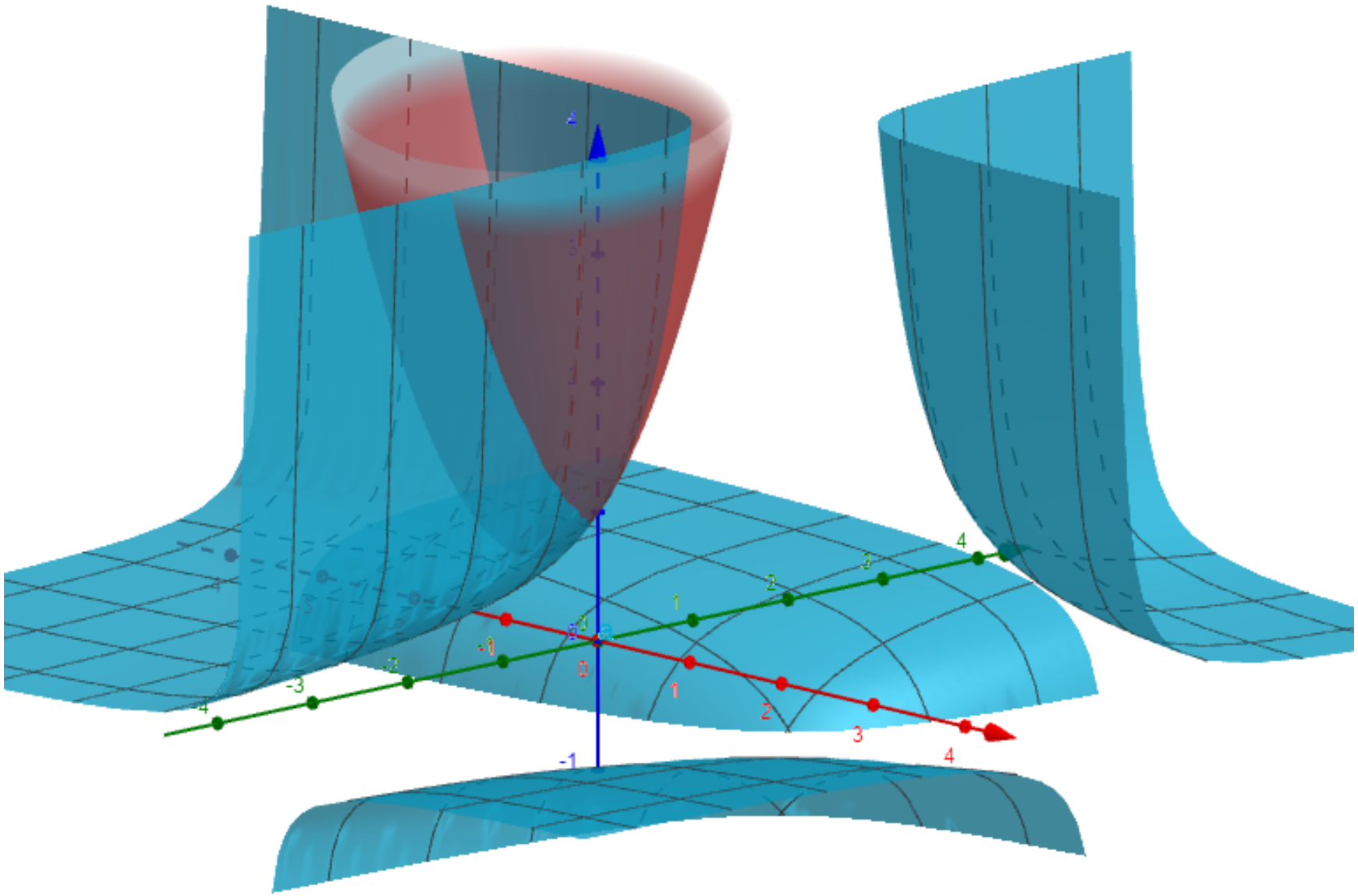
$$f_{xxx} = \frac{6(1-y)^3}{(1-x-y+xy)^4},\ f_{xxy} = \frac{[-4(1-x-y+xy)+6(1-y)(1-x)](1-y)}{(1-x-y+xy)^4},\ f_{xyy} = \frac{[-4(1-x-y+xy)+6(1-x)(1-y)](1-x)}{(1-x-y+xy)^4},$$

$$f_{yyy} = \frac{6(1-x)^3}{(1-x-y+xy)^4} \Rightarrow f(x,y) \approx \text{quadratic} + \tfrac{1}{6}\left[x^3 f_{xxx}(0,0) + 3x^2 y f_{xxy}(0,0) + 3xy^2 f_{xyy}(0,0) + y^3 f_{yyy}(0,0)\right]$$

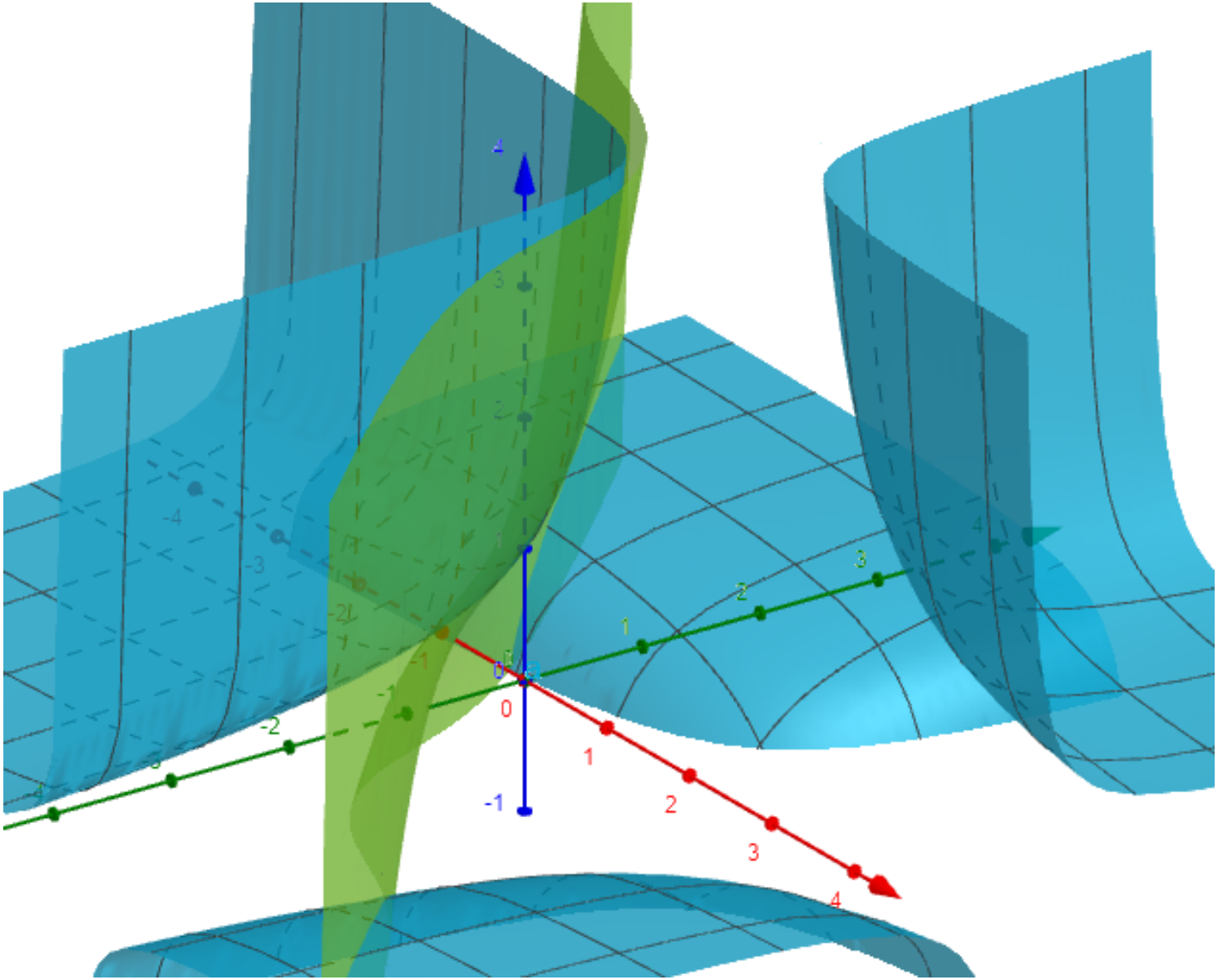
$$= 1 + x + y + x^2 + xy + y^2 + \tfrac{1}{6}\left[x^3\cdot 6 + 3x^2 y\cdot 2 + 3xy^2\cdot 2 + y^3\cdot 6\right]$$

$$= 1 + x + y + x^2 + xy + y^2 + x^3 + x^2 y + xy^2 + y^3,\ \text{cubic approximation}$$

# Example 3



# Example 3



# Taylor's Formula for Two Variables

## Example 4

Use Taylor's formula to find a quadratic approximation of $f(x, y) = e^x \sin y$ at the origin. Estimate the error in the approximation if $|x| \leq 0.1$ and $|y| \leq 0.1$.

$$f(x, y) = e^x \sin y \Rightarrow f_x = e^x \sin y,\ f_y = e^x \cos y,\ f_{xx} = e^x \sin y,\ f_{xy} = e^x \cos y,\ f_{yy} = -e^x \sin y$$

$$\Rightarrow f(x, y) \approx f(0, 0) + x f_x(0, 0) + y f_y(0, 0) + \tfrac{1}{2}\left[x^2 f_{xx}(0, 0) + 2xy f_{xy}(0, 0) + y^2 f_{yy}(0, 0)\right]$$

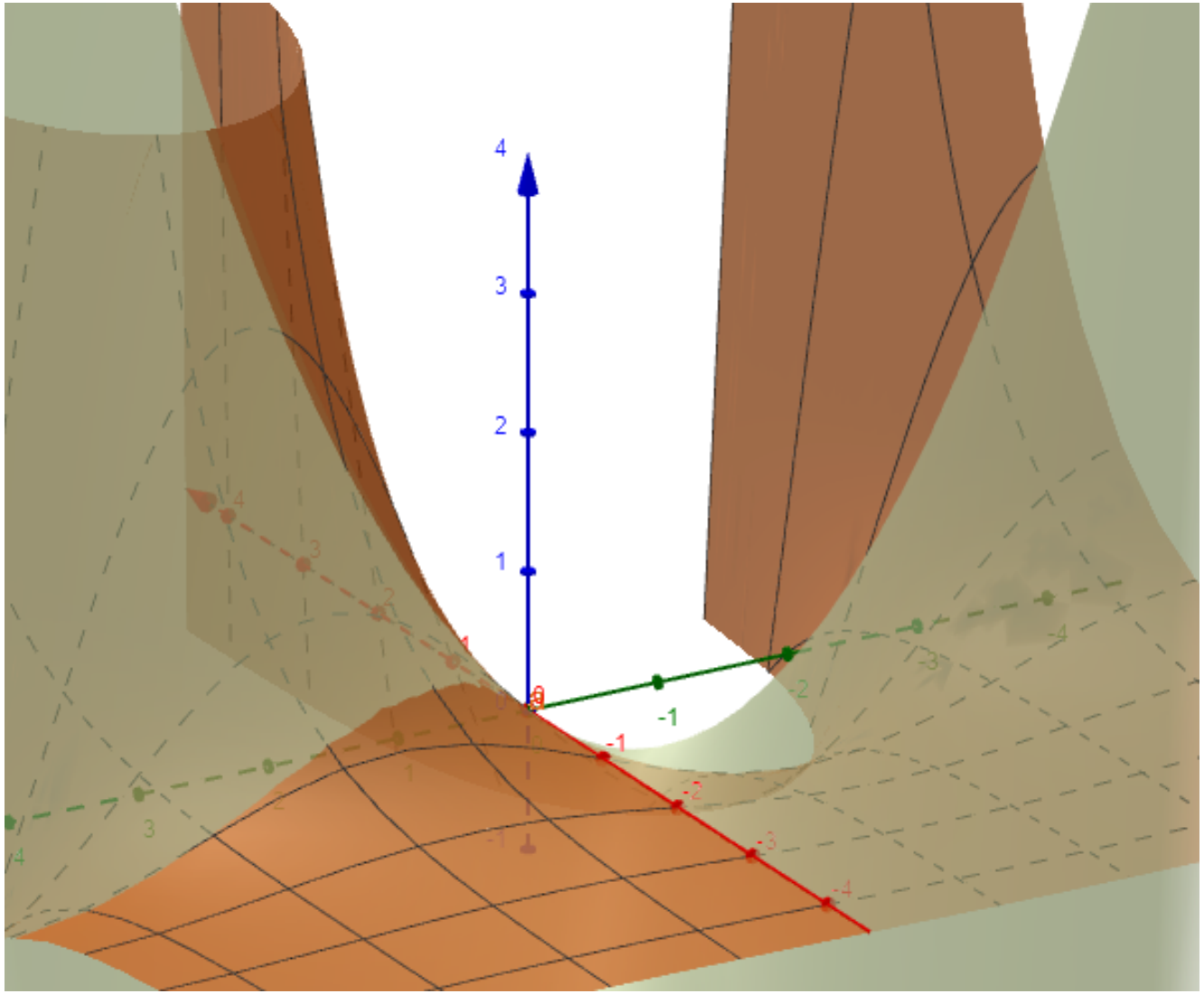
$$= 0 + x \cdot 0 + y \cdot 1 + \tfrac{1}{2}\left(x^2 \cdot 0 + 2xy \cdot 1 + y^2 \cdot 0\right) = y + xy, \text{ quadratic approximation.}$$

Now, $f_{xxx} = e^x \sin y$, $f_{xxy} = e^x \cos y$, $f_{xyy} = -e^x \sin y$, and $f_{yyy} = -e^x \cos y$.

Since $|x| \leq 0.1$, $|e^x \sin y| \leq e^{0.1} \sin 0.1| \approx 0.11$ and $|e^x \cos y| \leq |e^{0.1} \cos 0.1| \approx 1.11$.

Therefore, $E(x, y) \leq \frac{1}{6}\left[(0.11)(0.1)^3 + 3(1.11)(0.1)^3 + 3(0.11)(0.1)^3 + (1.11)(0.1)^3\right] \leq 0.000814.$

# Example 4



# Lecture 21: Multiple Integrals.

MA2032 Vector Calculus

Lecturer: Larissa Serdukova

School of Computing and Mathematical Science
University of Leicester

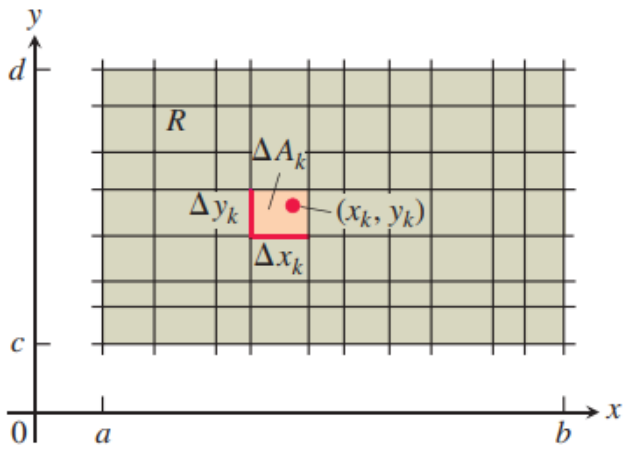
November 7, 2022

# Double and Iterated Integrals over Rectangles

• In integral calculus, we defined the **Definite Integral** of a continuous function $f(x)$ over an interval $[a, b]$ as a **limit of Riemann sums**.

• In this section, we **extend this idea to define the double integral** of a continuous function of two variables $f(x, y)$ over a bounded rectangle R in the plane.

• We consider a function $f(x, y)$ defined on a **rectangular region R**,
$R: \ a \leq x \leq b, \ c \leq y \leq d.$
• We **subdivide R** into $n$ small rectangles which form a **partition of R**.




• **A Riemann sum over R** is defined as

$$S_n = \sum_{k=1}^{n} f(x_k, y_k)\Delta A_k.$$

# Double and Iterated Integrals over Rectangles

• Sometimes the **Riemann sums converge** as the rectangle widths and heights both go to zero and whose number **n goes to infinity**.

• The resulting limit is then

$$\lim_{n\to\infty} S_n = \sum_{k=1}^{n} f(x_k, y_k)\Delta A_k.$$

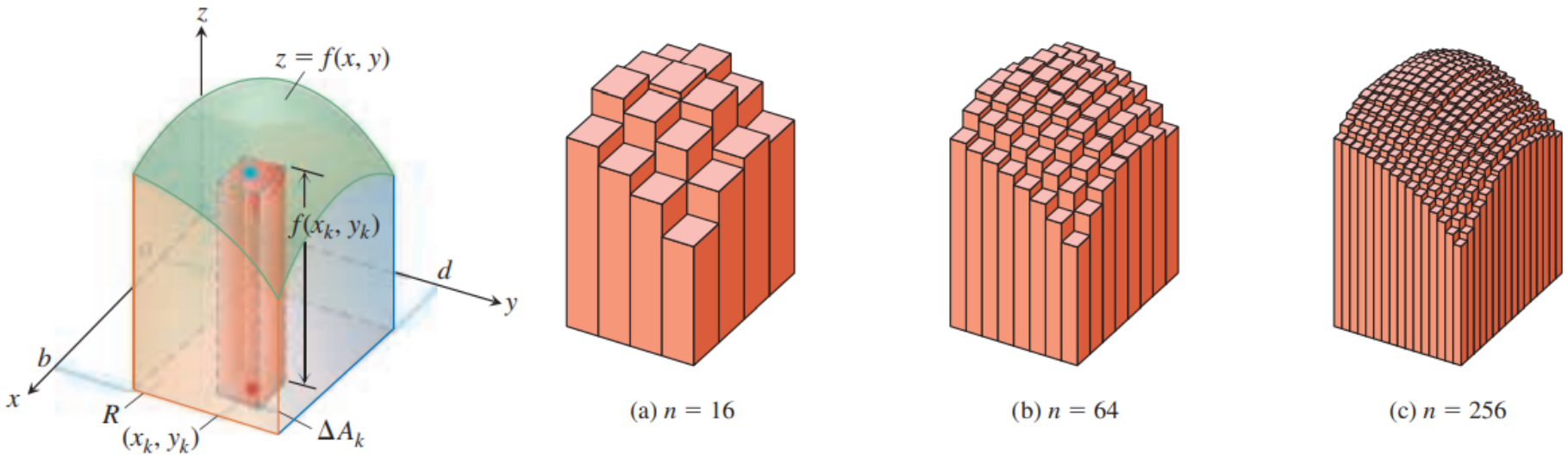
• When **a limit of the sums** $S_n$ **exists**, giving the same limiting value **no matter what choices of an arbitrary point** $(x_k, y_k)$ are made, then the function f is said to be **integrable** and the limit is called the **double integral** of f over **R**, written as

$$\iint_R f(x,y)dA \quad \text{or} \quad \iint_R f(x,y)dxdy$$

# Double Integrals as Volumes

• When $f(x, y)$ is a **positive function** over a rectangular region $R$ in the xy-plane, we may interpret the double integral of f over $R$ as the **volume of the 3-dimensional solid** region over the xy-plane bounded below by $R$ and above by the surface $z = f(x, y)$

$$\text{Volume} = \lim_{n\to\infty} S_n = \iint_R f(x, y)dA$$

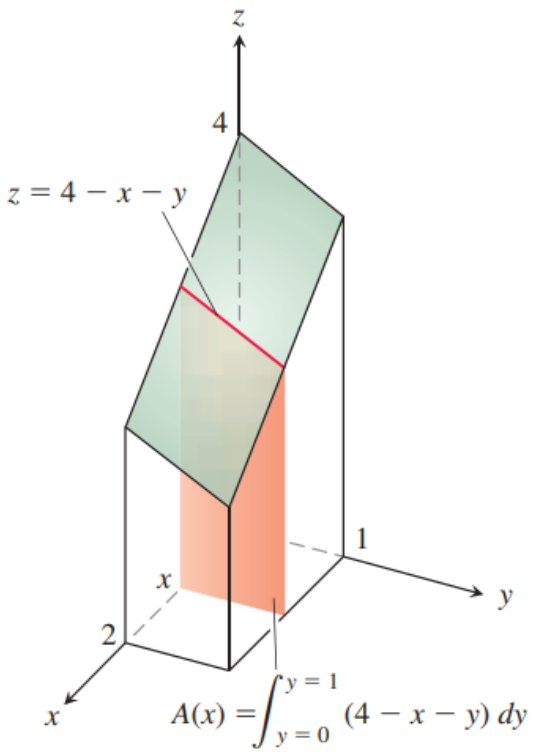


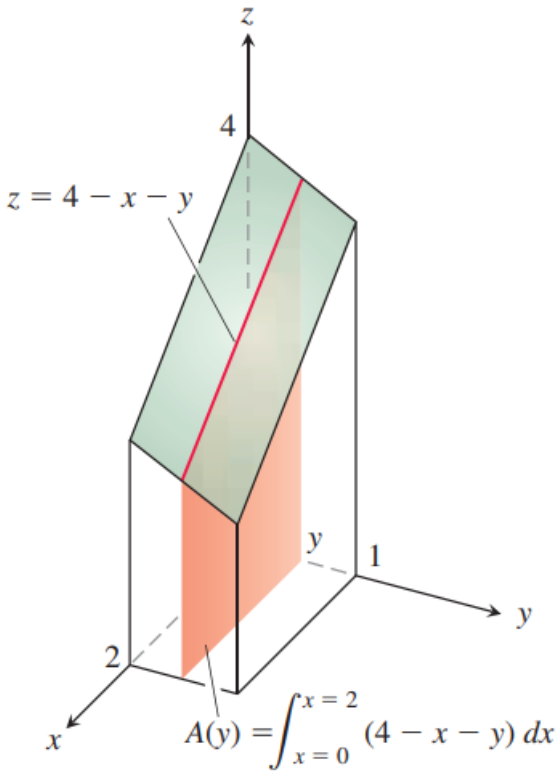

(a) $n = 16$ (b) $n = 64$ (c) $n = 256$

# Fubini's Theorem for Calculating Double Integrals

• If we apply the **method of slicing**, with **slices perpendicular to the x-axis**, then the **volume** is $\int_{x=0}^{x=2} A(x)dx$ where $A(x)$ is the **cross-sectional area at x**.

• If we apply the method of slicing, with **slices perpendicular to the y-axis**, then the volume is $\int_{y=0}^{y=1} A(y)dy$ where $A(y)$ is the **cross-sectional area at y**.



# Double and Iterated Integrals over Rectangles

• A **theorem** published in 1907 by **Guido Fubini** says that the double integral of any continuous function **over a rectangle** can be calculated as an iterated integral **in either order of integration**.

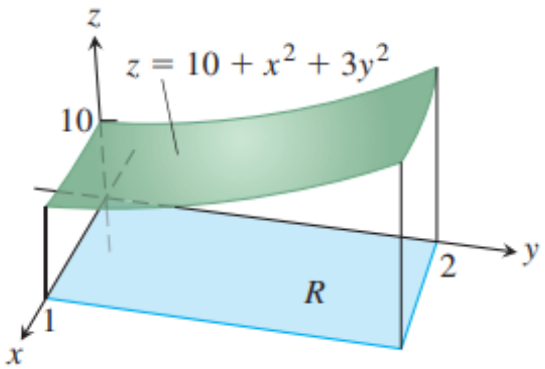
**THEOREM 1—Fubini's Theorem (First Form)**
If $f(x, y)$ is continuous throughout the rectangular region $R: a \leq x \leq b$, $c \leq y \leq d$, then

$$\iint_R f(x, y)\, dA = \int_c^d \int_a^b f(x, y)\, dx\, dy = \int_a^b \int_c^d f(x, y)\, dy\, dx.$$

# Double and Iterated Integrals over Rectangles

## Example 1

Find the volume of the region bounded above by the elliptical paraboloid $z = 10 + x^2 + 3y^2$ and below by the rectangle $R: \ 0 \le x \le 1, \ 0 \le y \le 2$.




**Solution** The surface and volume are shown in Figure 15.7. The volume is given by the double integral

$$V = \iint_R (10 + x^2 + 3y^2)\, dA = \int_0^1 \int_0^2 (10 + x^2 + 3y^2)\, dy\, dx$$

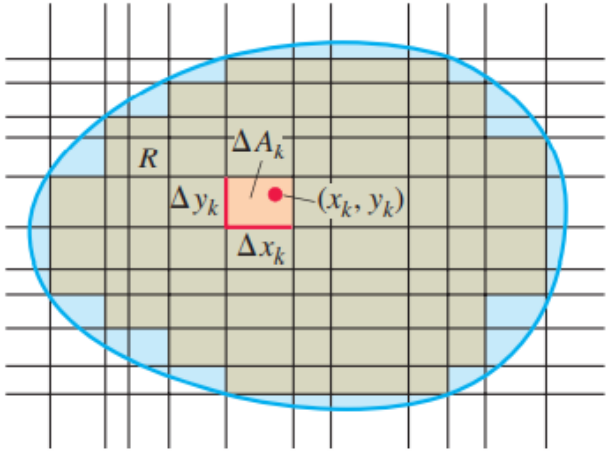
$$= \int_0^1 \Big[ 10y + x^2 y + y^3 \Big]_{y=0}^{y=2} dx$$

$$= \int_0^1 (20 + 2x^2 + 8)\, dx = \Big[ 20x + \frac{2}{3}x^3 + 8x \Big]_0^1 = \frac{86}{3}.$$

# Double Integrals over General Regions

• Now we define and evaluate double integrals over bounded regions in the plane which are **more general** than rectangles.
• These double integrals are also evaluated as **iterated integrals**, with the main practical problem being that of **determining the limits of integration**.

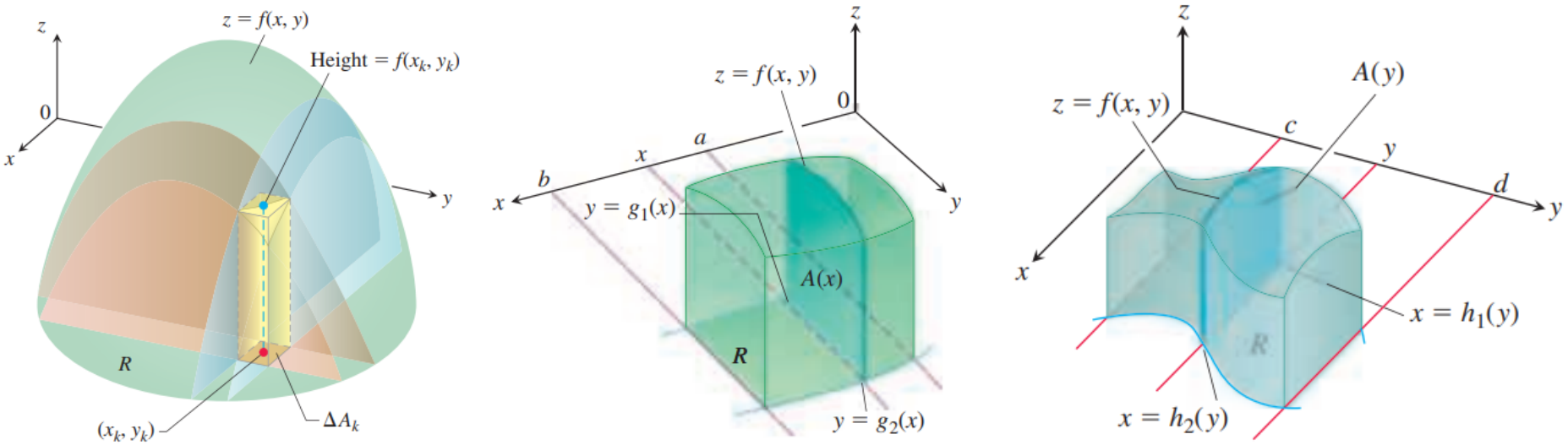
• Since the region of integration may have boundaries other than line segments parallel to the coordinate axes, the **limits of integration often involve variables**, not just constants.
• When **a limit of the Riemann sums** $S_n$ **exists**, then the function f is said to be **integrable** and the limit is called the **double integral** of f over **R**,
written as




$$\lim_{n\to\infty} S_n = \sum_{k=1}^{n} f(x_k, y_k)\Delta A_k = \iint_R f(x,y)dA = \iint_R f(x,y)dxdy$$

# Volumes

• If $f(x, y)$ is **positive and continuous over R**, we define the volume of the solid region between R and the surface $z = f(x, y)$ to be $\iint_R f(x, y)dA$.




• We may again calculate the volume by the method of slicing:

1) If R is a region bounded by the curves $y = g_2(x)$ and $y = g_1(x)$ and on the sides by the lines $x = a, x = b$, then $V = \int_a^b A(x)dx = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)dydx$

2) If R is a region bounded by the curves $x = h_2(y)$ and $x = h_1(y)$ and on the sides by the lines $y = c, y = d$, then $V = \int_c^d A(y)dy = \int_a^b \int_{h_1(y)}^{h_2(y)} f(x, y)dxdy$

# Fubini's Theorem (Stronger Form)

**THEOREM 2—Fubini's Theorem (Stronger Form)**
Let $f(x, y)$ be continuous on a region $R$.

**1.** If $R$ is defined by $a \le x \le b$, $g_1(x) \le y \le g_2(x)$, with $g_1$ and $g_2$ continuous on $[a, b]$, then

$$\iint_R f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx.$$

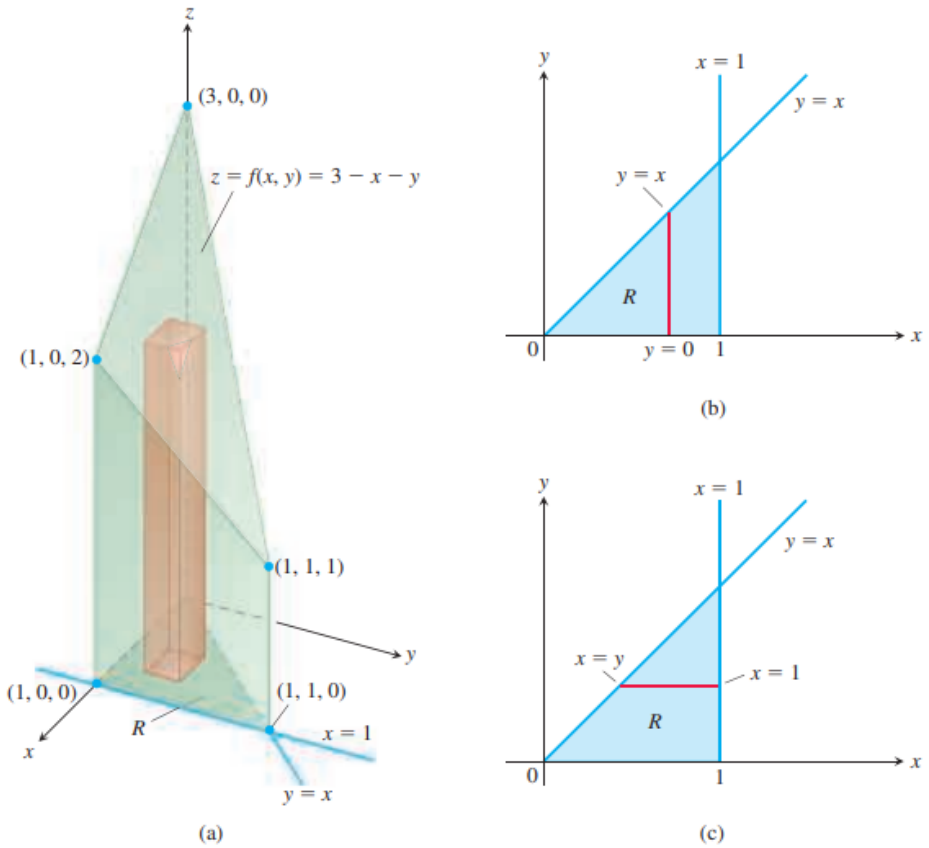
**2.** If $R$ is defined by $c \le y \le d$, $h_1(y) \le x \le h_2(y)$, with $h_1$ and $h_2$ continuous on $[c, d]$, then

$$\iint_R f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy.$$

# Double Integrals over General Regions

## Example 2

Find the volume of the prism whose base is the triangle in the xy-plane bounded by the x-axis and the lines $y = x$ and $x = 1$ and whose top lies in the plane $z = f(x, y) = 3 - x - y$.



# Example 2

**Solution** See Figure 15.12. For any $x$ between 0 and 1, $y$ may vary from $y = 0$ to $y = x$ (Figure 15.12b). Hence,

$$V = \int_0^1 \int_0^x (3 - x - y)\, dy\, dx = \int_0^1 \left[ 3y - xy - \frac{y^2}{2} \right]_{y=0}^{y=x} dx$$

$$= \int_0^1 \left( 3x - \frac{3x^2}{2} \right) dx = \left[ \frac{3x^2}{2} - \frac{x^3}{2} \right]_{x=0}^{x=1} = 1.$$

When the order of integration is reversed (Figure 15.12c), the integral for the volume is

$$V = \int_0^1 \int_y^1 (3 - x - y)\, dx\, dy = \int_0^1 \left[ 3x - \frac{x^2}{2} - xy \right]_{x=y}^{x=1} dy$$

$$= \int_0^1 \left( 3 - \frac{1}{2} - y - 3y + \frac{y^2}{2} + y^2 \right) dy$$

$$= \int_0^1 \left( \frac{5}{2} - 4y + \frac{3}{2}y^2 \right) dy = \left[ \frac{5}{2}y - 2y^2 + \frac{y^3}{2} \right]_{y=0}^{y=1} = 1.$$

The two integrals are equal, as they should be. ■

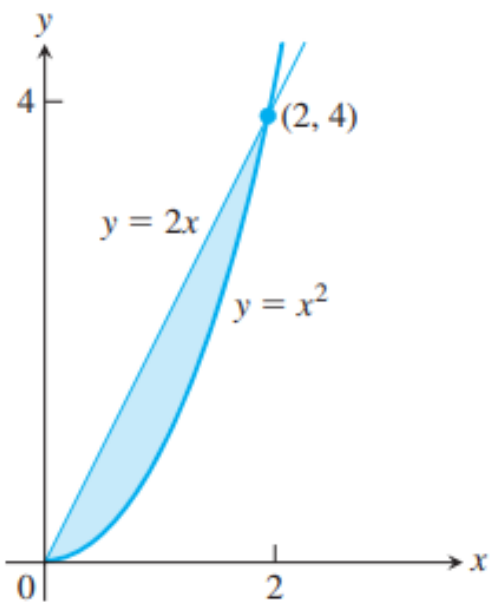
# Double Integrals over General Regions

## Example 3

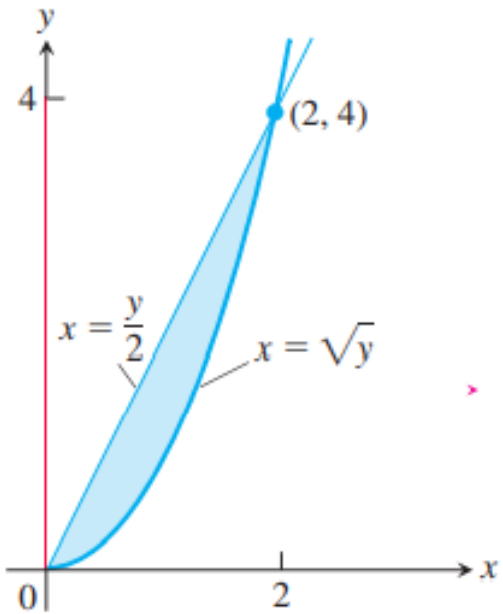
Sketch the region of integration for the integral

$$\int_0^2 \int_{x^2}^{2x} (4x+2)\,dy\,dx$$

and write an equivalent integral with the order of integration reversed.




(a)




(b)

# Example 3

**Solution** The region of integration is given by the inequalities $x^2 \leq y \leq 2x$ and $0 \leq x \leq 2$. It is therefore the region bounded by the curves $y = x^2$ and $y = 2x$ between $x = 0$ and $x = 2$ (Figure 15.16a).

To find limits for integrating in the reverse order, we imagine a horizontal line passing from left to right through the region. It enters at $x = y/2$ and leaves at $x = \sqrt{y}$. To include all such lines, we let $y$ run from $y = 0$ to $y = 4$ (Figure 15.16b). The integral is

$$\int_0^4 \int_{y/2}^{\sqrt{y}} (4x + 2)\, dx\, dy.$$

The common value of these integrals is 8. ■

# Properties of Double Integrals

If $f(x, y)$ and $g(x, y)$ are continuous on the bounded region $R$, then the following properties hold.

**1.** *Constant Multiple:* $\displaystyle\iint_R cf(x, y)\, dA = c \iint_R f(x, y)\, dA$ (any number $c$)

**2.** *Sum and Difference:*

$$\iint_R \big(f(x, y) \pm g(x, y)\big)\, dA = \iint_R f(x, y)\, dA \pm \iint_R g(x, y)\, dA$$

**3.** Domination:

**(a)** $\displaystyle\iint_R f(x, y)\, dA \geq 0$ if $f(x, y) \geq 0$ on $R$

**(b)** $\displaystyle\iint_R f(x, y)\, dA \geq \iint_R g(x, y)\, dA$ if $f(x, y) \geq g(x, y)$ on $R$

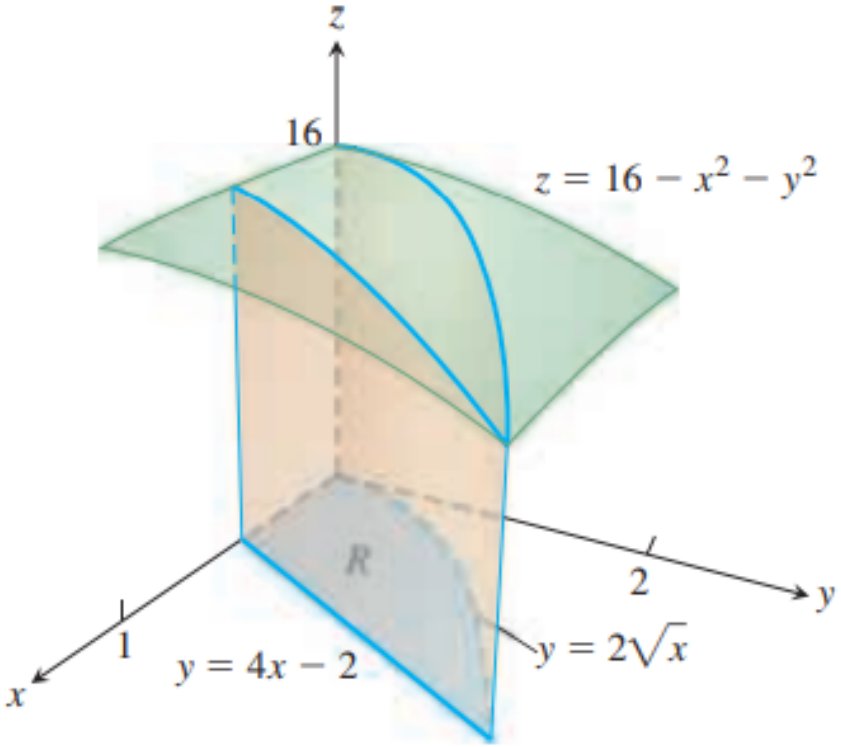
**4.** *Additivity:* If $R$ is the union of two nonoverlapping regions $R_1$ and $R_2$, then

$$\iint_R f(x, y)\, dA = \iint_{R_1} f(x, y)\, dA + \iint_{R_2} f(x, y)\, dA$$

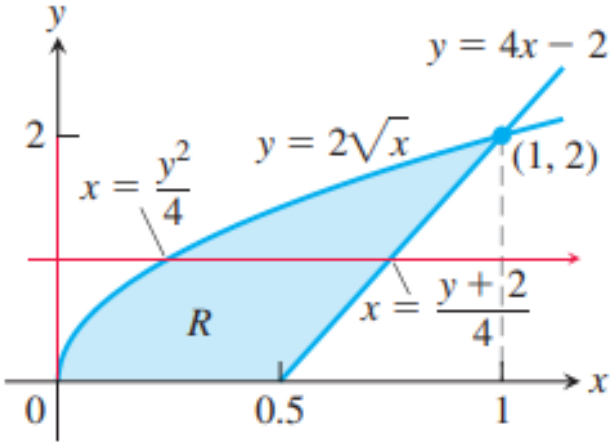
# Double Integrals over General Regions

## Example 4

Find the volume of the wedgelike solid that lies beneath the surface $z = 16 - x^2 - y^2$ and above the region R bounded by the curve $y = 2\sqrt{x}$, the line $y = 4x - 2$, and the x-axis.




(a)




(b)

# Example 4

**Solution** Figure 15.18a shows the surface and the "wedgelike" solid whose volume we want to calculate. Figure 15.18b shows the region of integration in the $xy$-plane. If we integrate in the order $dy\,dx$ (first with respect to $y$ and then with respect to $x$), two integrations will be required because $y$ varies from $y = 0$ to $y = 2\sqrt{x}$ for $0 \le x \le 0.5$, and then varies from $y = 4x - 2$ to $y = 2\sqrt{x}$ for $0.5 \le x \le 1$. So we choose to integrate in the order $dx\,dy$, which requires only one double integral whose limits of integration are indicated in Figure 15.18b. The volume is then calculated as the iterated integral:

$$\iint_R (16 - x^2 - y^2)\,dA$$

$$= \int_0^2 \int_{y^2/4}^{(y+2)/4} (16 - x^2 - y^2)\,dx\,dy$$

$$= \int_0^2 \left[16x - \frac{x^3}{3} - xy^2\right]_{x=y^2/4}^{x=(y+2)/4} dx$$

$$= \int_0^2 \left[4(y+2) - \frac{(y+2)^3}{3\cdot 64} - \frac{(y+2)y^2}{4} - 4y^2 + \frac{y^6}{3\cdot 64} + \frac{y^4}{4}\right] dy$$

$$= \left[\frac{191y}{24} + \frac{63y^2}{32} - \frac{145y^3}{96} - \frac{49y^4}{768} + \frac{y^5}{20} + \frac{y^7}{1344}\right]_0^2 = \frac{20803}{1680} \approx 12.4.$$

# Lecture 22: Multiple Integrals.

MA2032 Vector Calculus

Lecturer: Larissa Serdukova

School of Computing and Mathematical Science
University of Leicester

November 9, 2022

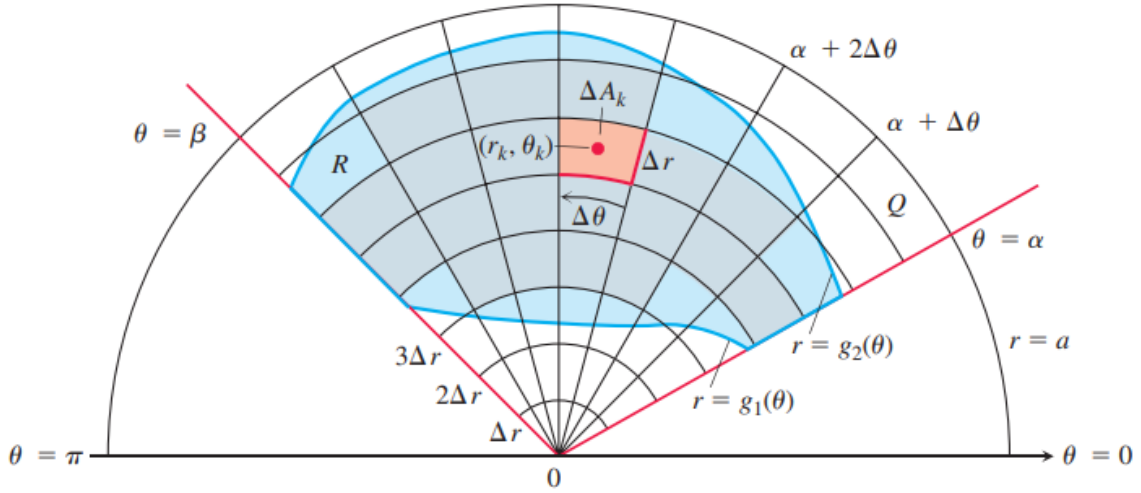
# Double Integrals in Polar Form

• Double integrals are sometimes **easier to evaluate** if we change to polar coordinates.

• Today you will learn **how to accomplish the change** and how to evaluate double integrals over regions whose **boundaries are given by polar equations**.

• We consider a function $f(x, y)$ defined on region **R** that is bounded by the **rays** $\theta = \alpha$ and $\theta = \beta$ and by the continuous **curves** $r = g_1(\theta)$ and $r = g_2(\theta)$.

• Then R lies in a fan-shaped **region** $Q$ defined by the inequalities $0 \leq r \leq a$ and $\alpha \leq \theta \leq \beta$, where $0 \leq \beta - \alpha \leq 2\pi$.



# Double Integrals in Polar Form

• We cover **Q** by a grid of circular arcs and rays, where $\Delta r = a/m$ and $\Delta\theta = (\beta - \alpha)/m'$
• Partition **Q** into small patches called "polar rectangles".
• **A Riemann sum over R** is defined as

$$S_n = \sum_{k=1}^{n} f(r_k, \theta_k)\Delta A_k.$$

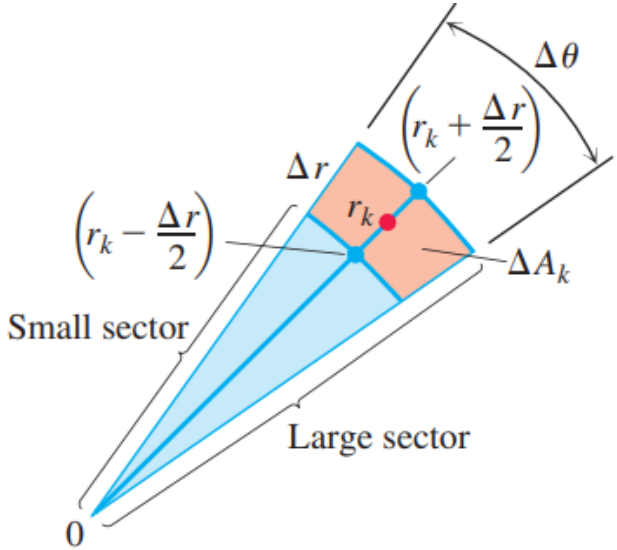
• When **a limit of the sums $S_n$ exists**, is called the **double integral** of f over **R**, written as

$$\lim_{n\to\infty} S_n = \iint_R f(r,\theta)dA$$

• We first have to write the sum $S_n$ in a way that **expresses** $\Delta A_k$ in terms of $\Delta r$ and $\Delta\theta$ .

# Double Integrals in Polar Form

- The area of a wedge-shaped sector of a circle having radius r and angle $\theta$ is $A = 1/2\theta r^2$.




$$\text{Inner radius:} \quad \frac{1}{2}\left(r_k - \frac{\Delta r}{2}\right)^2 \Delta\theta$$

$$\text{Outer radius:} \quad \frac{1}{2}\left(r_k + \frac{\Delta r}{2}\right)^2 \Delta\theta.$$

$$\begin{aligned}\Delta A_k &= \text{area of large sector} - \text{area of small sector} \\ &= \frac{\Delta\theta}{2}\left[\left(r_k + \frac{\Delta r}{2}\right)^2 - \left(r_k - \frac{\Delta r}{2}\right)^2\right] = \frac{\Delta\theta}{2}(2r_k\,\Delta r) = r_k\,\Delta r\,\Delta\theta.\end{aligned}$$

$$S_n = \sum_{k=1}^{n} f(r_k, \theta_k)\, r_k\,\Delta r\,\Delta\theta. \qquad \lim_{n\to\infty} S_n = \iint_R f(r,\theta)\, r\, dr\, d\theta.$$

$$\iint_R f(r,\theta)\, dA = \int_{\theta=\alpha}^{\theta=\beta}\int_{r=g_1(\theta)}^{r=g_2(\theta)} f(r,\theta)\, r\, dr\, d\theta.$$

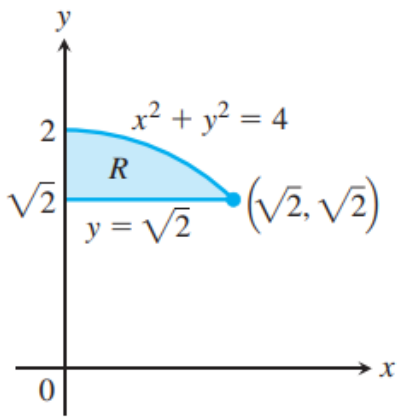
- Fubini's Theorem:

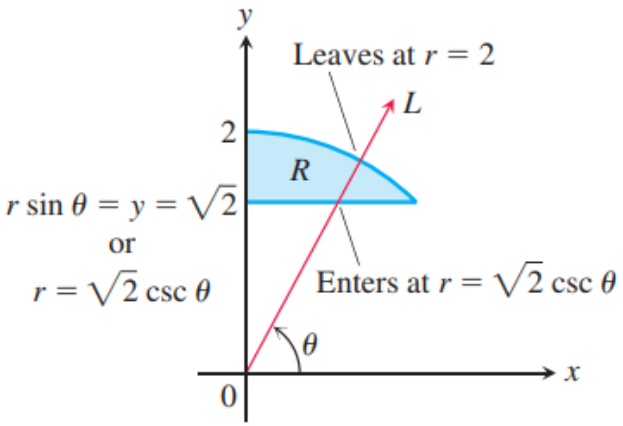
# Finding Limits of Integration

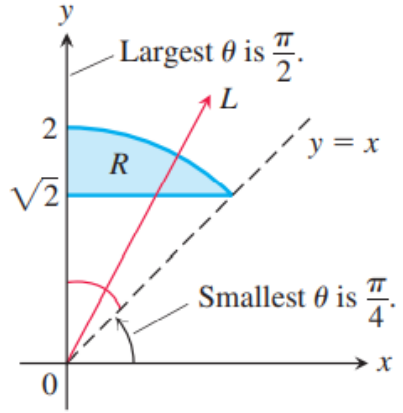
- **Step 1: Sketch** the region and label the bounding curves .

- **Step 2: Find the r-limits of integration**. Imagine a ray L from the origin cutting through R in the direction of increasing r. Mark the r-values where L enters and leaves R. These are the r-limits of integration. They usually depend on the angle $\theta$ that L makes with the positive x-axis

- **Step 3: Find the $\theta$-limits of integration**. Find the smallest and largest $\theta$-values that bound R.



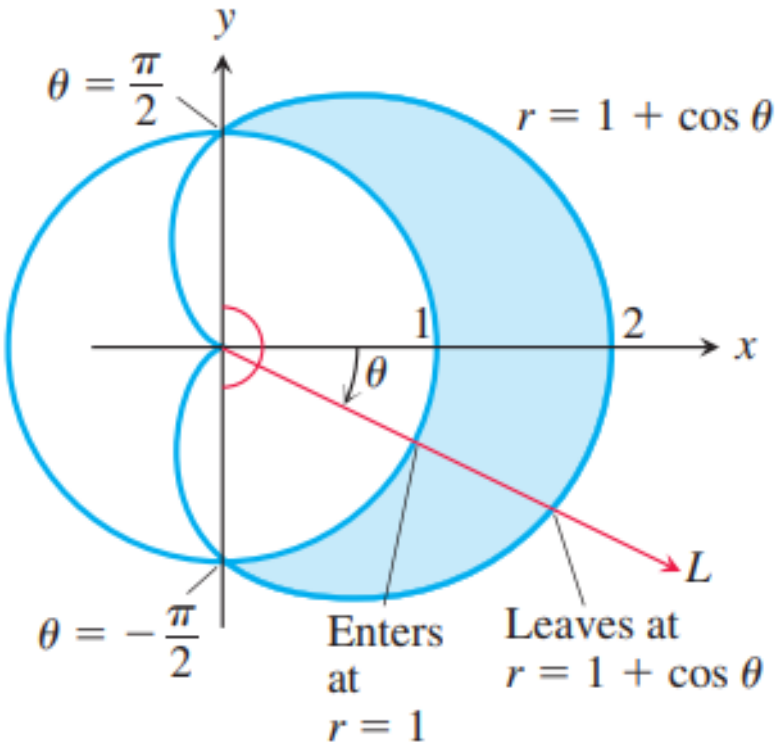



# Double Integrals in Polar Form

## Example 1

Find the limits of integration for integrating $f(r, \theta)$ over the region R that lies inside the cardioid $r = 1 + \cos\theta$ and outside the circle $r = 1$.



# Example 1

**Solution**

1. We first sketch the region and label the bounding curves (Figure 15.25).
2. Next we find the *r-limits of integration*. A typical ray from the origin enters $R$ where $r = 1$ and leaves where $r = 1 + \cos\theta$.
3. Finally we find the *θ-limits of integration*. The rays from the origin that intersect $R$ run from $\theta = -\pi/2$ to $\theta = \pi/2$. The integral is

$$\int_{-\pi/2}^{\pi/2}\int_{1}^{1+\cos\theta} f(r,\theta)\, r\, dr\, d\theta.$$

# Changing Cartesian Integrals into Polar Integrals

• The procedure for changing a Cartesian integral into a polar integral has **two steps**:

• **First substitute** $x = r\cos\theta$ and $y = r\sin\theta$, and replace $dx\ dy$ by $r\ dr\ d\theta$ in the Cartesian integral.

• **Then supply** polar limits of integration for the boundary of R.

• The **Cartesian integral** then becomes

$$\iint_R f(x, y)\, dx\, dy = \iint_G f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta,$$

• where **G** denotes the same region of integration, but now **described in polar coordinates**.

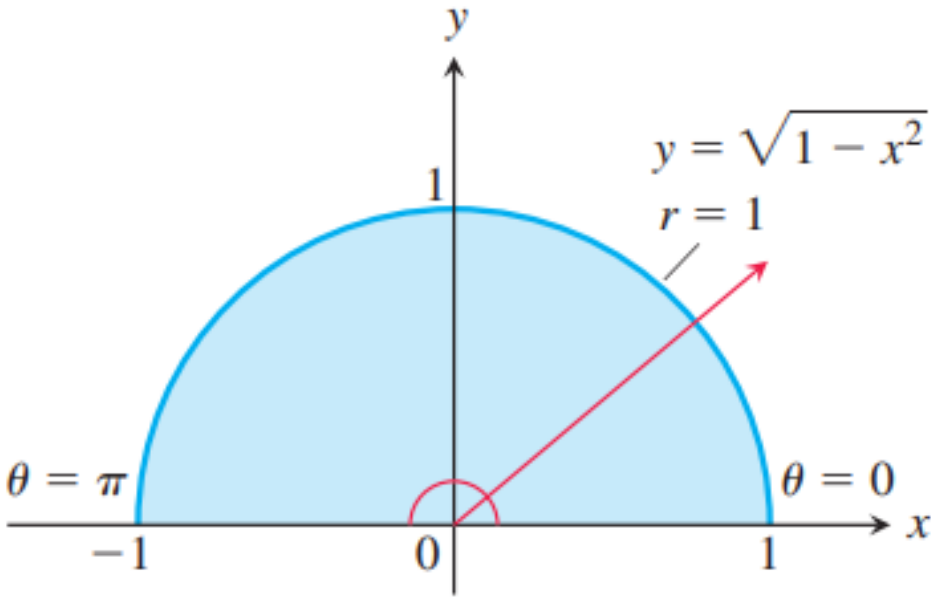
# Changing Cartesian Integrals into Polar Integrals

## Example 2

Evaluate

$$\iint_R e^{x^2+y^2}\,dy\,dx$$

where R is the semicircular region bounded by the x-axis and the curve $y = \sqrt{1-x^2}$.



# Example 2

**Solution** In Cartesian coordinates, the integral in question is a nonelementary integral and there is no direct way to integrate $e^{x^2+y^2}$ with respect to either $x$ or $y$. Yet this integral and others like it are important in mathematics—in statistics, for example—and we need to evaluate it. Polar coordinates make this possible. Substituting $x = r\cos\theta$ and $y = r\sin\theta$ and replacing $dy\,dx$ by $r\,dr\,d\theta$ give

$$\iint_R e^{x^2+y^2}\,dy\,dx = \int_0^{\pi}\int_0^1 e^{r^2}\,r\,dr\,d\theta = \int_0^{\pi}\left[\frac{1}{2}e^{r^2}\right]_0^1 d\theta$$

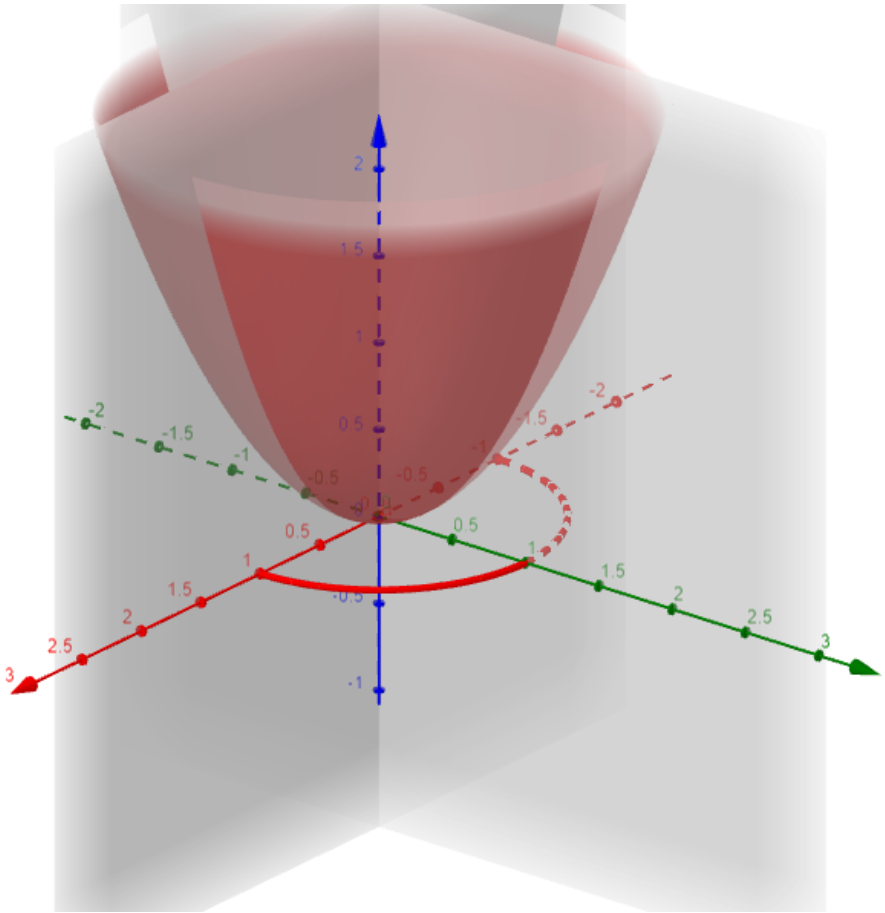
$$= \int_0^{\pi}\frac{1}{2}(e-1)\,d\theta = \frac{\pi}{2}(e-1).$$

The $r$ in the $r\,dr\,d\theta$ is what allowed us to integrate $e^{r^2}$. Without it, we would have been unable to find an antiderivative for the first (innermost) iterated integral. ■

# Changing Cartesian Integrals into Polar Integrals

## Example 3

Evaluate the integral $\int_0^1 \int_0^{\sqrt{1-x^2}} (x^2 + y^2) dy\ dx$.



# Example 3

**Solution** Integration with respect to $y$ gives

$$\int_0^1 \left( x^2 \sqrt{1 - x^2} + \frac{(1 - x^2)^{3/2}}{3} \right) dx,$$

which is difficult to evaluate without tables. Things go better if we change the original integral to polar coordinates. The region of integration in Cartesian coordinates is given by the inequalities $0 \le y \le \sqrt{1 - x^2}$ and $0 \le x \le 1$, which correspond to the interior of the unit quarter circle $x^2 + y^2 = 1$ in the first quadrant. (See Figure 15.27, first quadrant.) Substituting the polar coordinates $x = r \cos \theta$, $y = r \sin \theta$, $0 \le \theta \le \pi/2$, and $0 \le r \le 1$, and replacing $dy\, dx$ by $r\, dr\, d\theta$ in the double integral, we get

$$\int_0^1 \int_0^{\sqrt{1-x^2}} (x^2 + y^2)\, dy\, dx = \int_0^{\pi/2} \int_0^1 (r^2)\, r\, dr\, d\theta$$

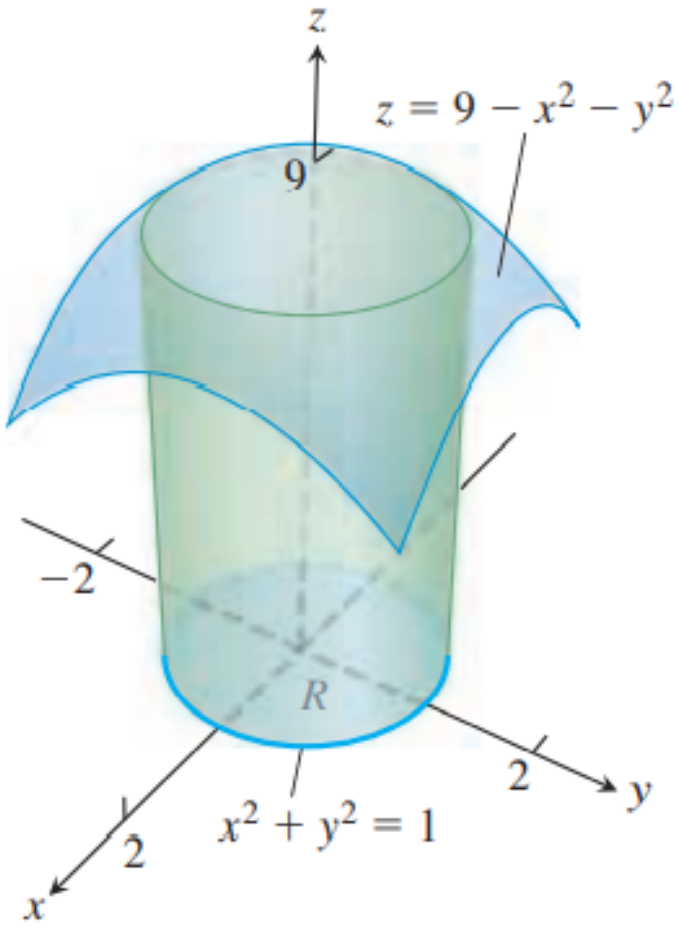
$$= \int_0^{\pi/2} \left[ \frac{r^4}{4} \right]_{r=0}^{r=1} d\theta = \int_0^{\pi/2} \frac{1}{4}\, d\theta = \frac{\pi}{8}.$$

The polar coordinate transformation is effective here because $x^2 + y^2$ simplifies to $r^2$ and the limits of integration become constants. ■

# Changing Cartesian Integrals into Polar Integrals

## Example 4

Find the volume of the solid region bounded above by the paraboloid $z = 9 - x^2 - y^2$ and below by the unit circle in the xy-plane.



# Example 4

**Solution** The region of integration $R$ is bounded by the unit circle $x^2 + y^2 = 1$, which is described in polar coordinates by $r = 1, 0 \leq \theta \leq 2\pi$. The solid region is shown in Figure 15.28. The volume is given by the double integral

$$\iint_R (9 - x^2 - y^2)\, dA = \int_0^{2\pi}\int_0^1 (9 - r^2)\, r\, dr\, d\theta \qquad r^2 = x^2 + y^2, \quad dA = r\, dr\, d\theta.$$

$$= \int_0^{2\pi}\int_0^1 (9r - r^3)\, dr\, d\theta$$

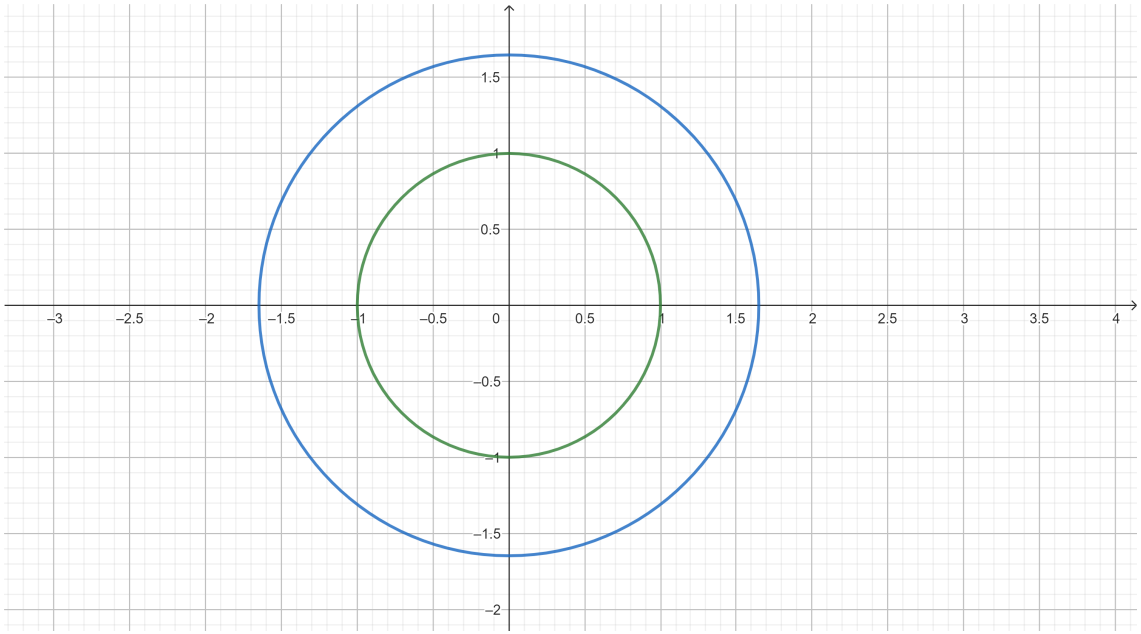
$$= \int_0^{2\pi} \left[\frac{9}{2}r^2 - \frac{1}{4}r^4\right]_{r=0}^{r=1} d\theta$$

$$= \frac{17}{4}\int_0^{2\pi} d\theta = \frac{17\pi}{2}.$$

# Changing Cartesian Integrals into Polar Integrals

## Example 5

Converting to a polar integral then integrate $f(x,y) = \frac{\ln(x^2+y^2)}{\sqrt{x^2+y^2}}$ over the region $1 \leq x^2 + y^2 \leq e$.




$$\int_0^{2\pi}\int_1^{\sqrt{e}}\left(\frac{\ln r^2}{r}\right) r\,dr\,d\theta = \int_0^{2\pi}\int_1^{\sqrt{e}} 2\ln r\,dr\,d\theta = 2\int_0^{2\pi}\left[r\ln r - r\right]_1^{e^{1/2}} d\theta = 2\int_0^{2\pi}\sqrt{e}\left[\left(\tfrac{1}{2}-1\right)+1\right] d\theta = 2\pi\left(2-\sqrt{e}\right)$$

# Lecture 23: Multiple Integrals.

MA2032 Vector Calculus

Lecturer: Larissa Serdukova

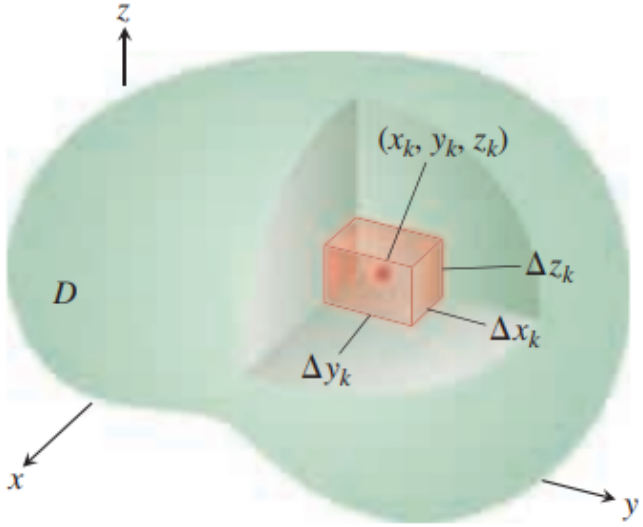
School of Computing and Mathematical Science
University of Leicester

November 11, 2022

# Triple Integrals in Rectangular Coordinates

• We use **triple integrals** to calculate the **volumes** of three-dimensional shapes and the **average value of a function** over a three-dimensional region.

• If $F(x, y, z)$ is a function defined on a **closed bounded region** $D$ **in space** we partition a rectangular boxlike region containing $D$ into **n rectangular cells** by planes parallel to the coordinate axes.




• The kth cell having **dimensions** $\Delta x_k$ by $\Delta y_k$ by $\Delta z_k$ and **volume** $\Delta V_k = \Delta x_k \Delta y_k \Delta z_k$.

• We **choose a point** $(x_k, y_k, z_k)$ in each cell and form the **sum**

$$S_n = \sum_{k=1}^{n} F(x_k, y_k, z_k)\Delta V_k.$$

# Triple Integrals in Rectangular Coordinates

• It can be shown that **when $F$ is continuous** and the bounding surface of $D$ is formed from finitely many **smooth surfaces** joined together along finitely many **smooth curves**, then $F$ is **integrable**.

• As $\Delta x_k \to 0, \Delta y_k \to 0, \Delta z_k \to 0$ and the number of cells $n$ goes to $\infty$, the **sums $S_n$ approach a limit**.

• We call this limit the **triple integral** of $F$ over $D$ and write

$$\lim_{n\to\infty} S_n = \iiint_D F(x,y,z)dV = \iiint_D F(x,y,z)\ dx\ dy\ dz.$$

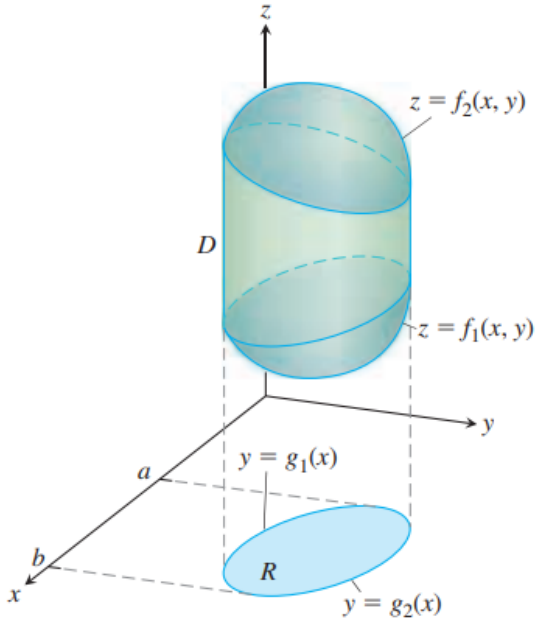
• If $F$ is the **constant function whose value is 1**, then we therefore define the **volume of $D$ to be the triple integral**

**DEFINITION** The **volume** of a closed, bounded region $D$ in space is

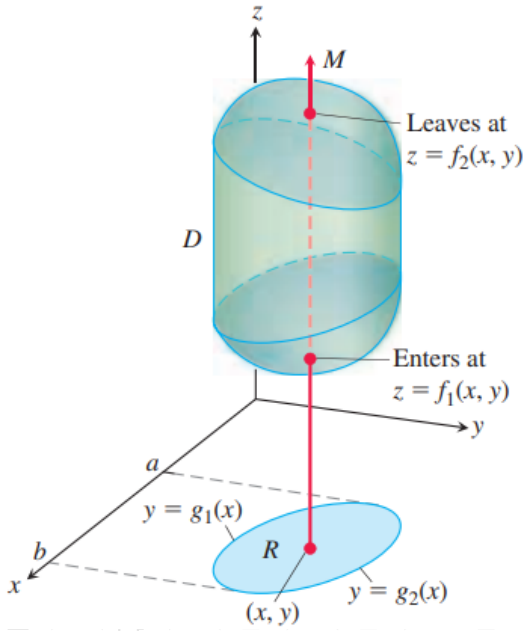
$$V = \iiint_D dV.$$

# Finding Limits of Integration in the Order *dz dy dx*

• **Step 1: Sketch** the region $D$ along with its "shadow" R (vertical projection) in the xy-plane. Label the upper and lower bounding surfaces of $D$ and the upper and lower bounding curves of $R$.



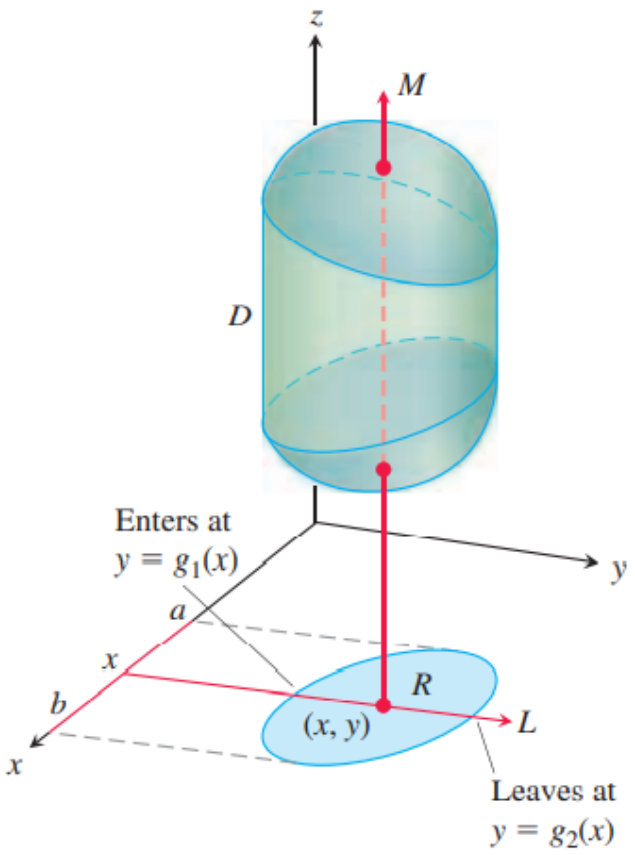

• **Step 2: Find the z-limits of integration**. Draw a line $M$ passing through a typical point $(x, y)$ in $R$ parallel to the $z$-axis. As $z$ increases, $M$ enters $D$ at $z = f_1(x, y)$ and leaves at $z = f_2(x, y)$. These are the z-limits of integration.



# Finding Limits of Integration in the Order *dz dy dx*

• **Step 3: Find the y-limits of integration**. Draw a line $L$ through $(x, y)$ parallel to the $y$-axis. As $y$ increases, $L$ enters $R$ at $y = g_1(x)$ and leaves at $y = g_2(x)$. These are the y-limits of integration.

• **Step 4: Find the x-limits of integration**. Choose $x$-limits that include all lines through $R$ parallel to the $y$-axis ($x = a$ and $x = b$ in the figure). These are the $x$-limits of integration.




• The integral is

$$\int_{x=a}^{x=b} \int_{y=g_1(x)}^{y=g_2(x)} \int_{z=f_1(x,y)}^{z=f_2(x,y)} F(x, y, z)\, dz\, dy\, dx.$$

# Change the order of integration

• Follow similar procedures if you **change the order of integration**.

• The "shadow" of region $D$ **lies in the plane of the last two variables** with respect to which the iterated integration takes place • The preceding

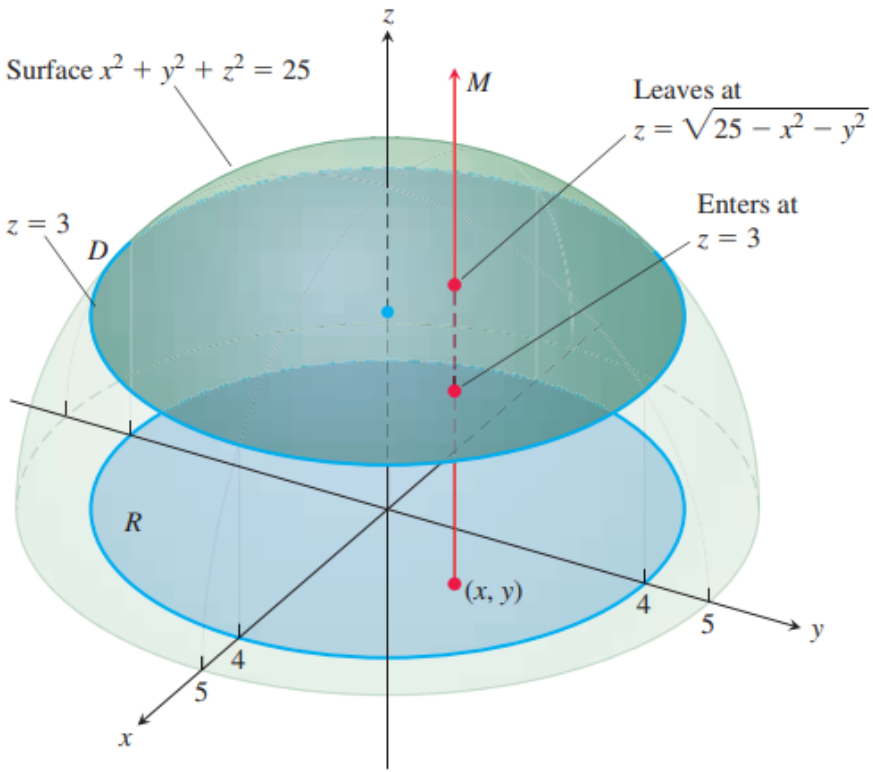
procedure applies whenever a solid region $D$ **is bounded above and below by a surface**, and when the "shadow" region $R$ **is bounded by a lower and upper curve**.

• It **does not apply** to regions with **complicated holes** through them, although sometimes such regions **can be subdivided** into simpler regions for which the procedure does apply.

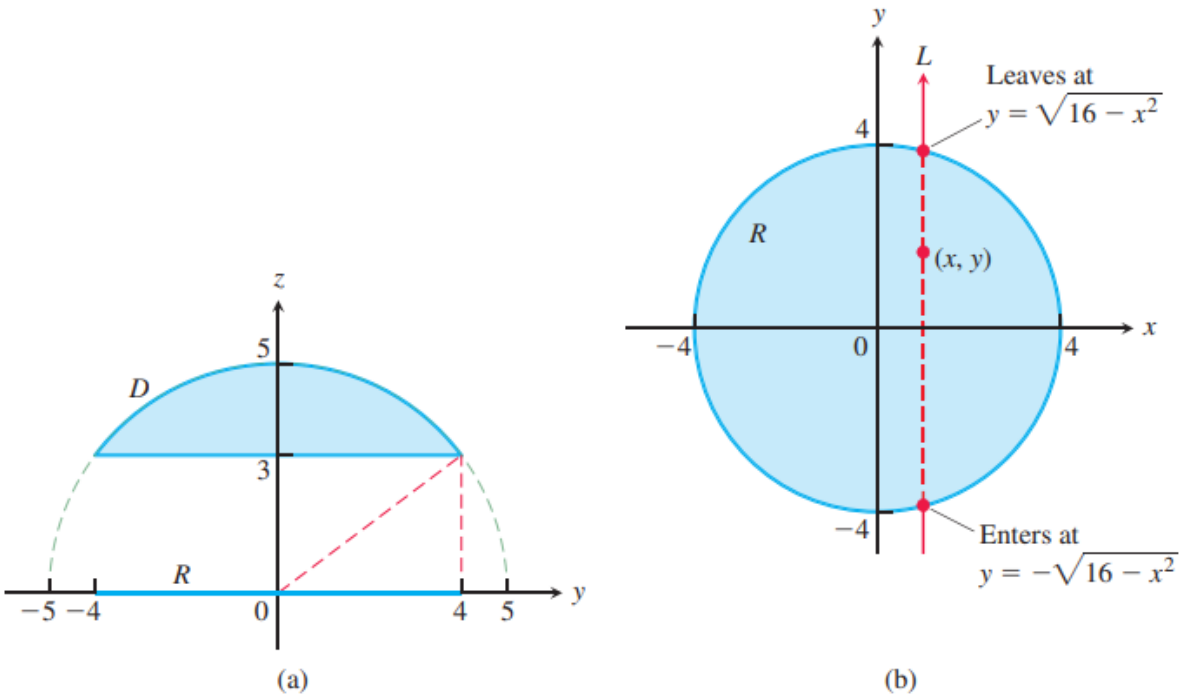
# Triple Integrals in Rectangular Coordinates

## Example 1

Let $S$ be the sphere of radius 5 centered at the origin, and let $D$ be the region under the sphere that lies above the plane $z = 3$. Set up the limits of integration for evaluating the triple integral of a function $F(x, y, z)$ over the region $D$.



# Example 1




**Solution** The region under the sphere that lies above the plane $z = 3$ is enclosed by the surfaces $x^2 + y^2 + z^2 = 25$ and $z = 3$.

To find the limits of integration, we first sketch the region, as shown in Figure 15.31. The "shadow region" $R$ in the $xy$-plane is a circle of some radius centered at the origin. By considering a side view of the region $D$, we can determine that the radius of this circle is 4; see Figure 15.32a.

If we fix a point $(x, y)$ in $R$ and draw a vertical line $M$ above $(x, y)$, then we see that this line enters the region $D$ at the height $z = 3$ and leaves the region at the height $z = \sqrt{25 - x^2 - y^2}$; see Figure 15.31. This gives us the $z$-limits of integration.

# Example 1

To find the $y$-limits of integration, we consider a line $L$ that lies in the region $R$, passes through the point $(x, y)$, and is parallel to the $y$-axis. For clarity we have separately pictured the region $R$ and the line $L$ in Figure 15.32b. The line $L$ enters $R$ when $y = -\sqrt{16 - x^2}$ and exits when $y = \sqrt{16 - x^2}$. This gives us the $y$-limits of integration.

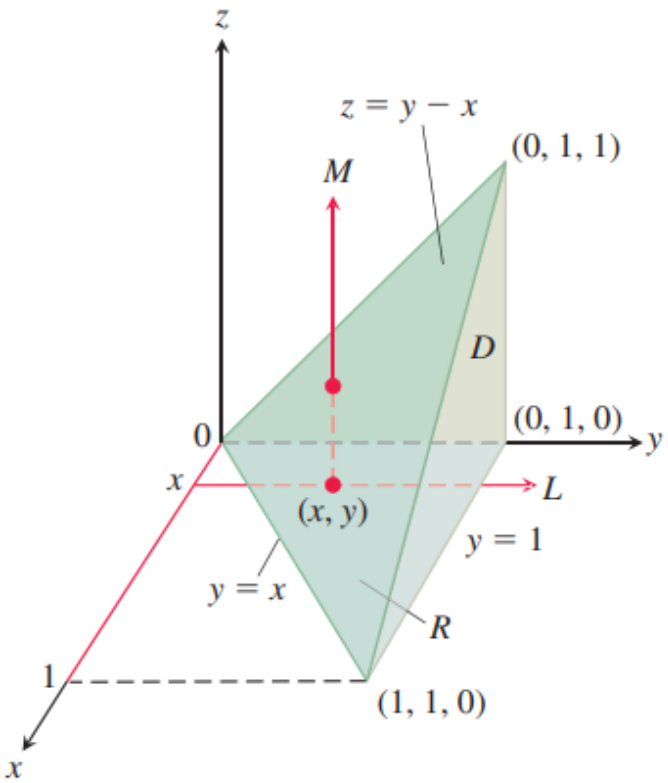
Finally, as $L$ sweeps across $R$ from left to right, the value of $x$ varies from $x = -4$ to $x = 4$. This gives us the $x$-limits of integration. Therefore, the triple integral of $F$ over the region $D$ is given by

$$\iiint_D F(x, y, z)\, dz\, dy\, dx = \int_{-4}^{4} \int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} \int_{3}^{\sqrt{25-x^2-y^2}} F(x, y, z)\, dz\, dy\, dx.$$

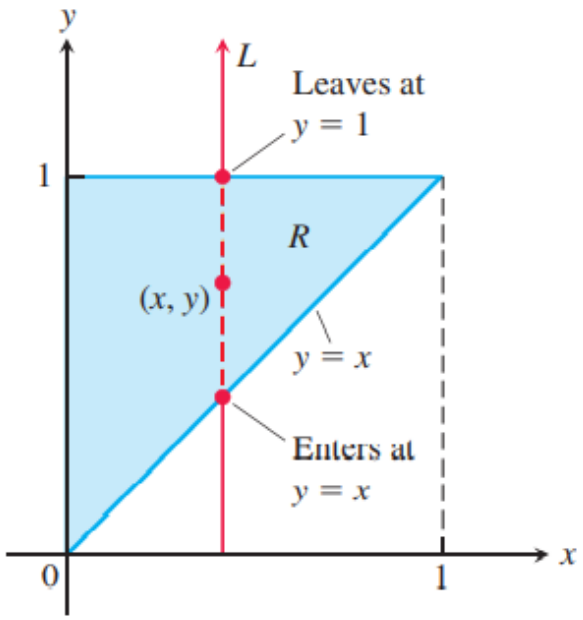
# Triple Integrals in Rectangular Coordinates

## Example 2

Set up the limits of integration for evaluating the triple integral of a function $F(x, y, z)$ over the tetrahedron $D$ whose vertices are $(0, 0, 0)$, $(1, 1, 0)$, $(0, 1, 0)$, and $(0, 1, 1)$. Use the order of integration $dz\ dy\ dx$.




(a)




(b)

# Example 2

**Solution** The region $D$ and its "shadow" $R$ in the $xy$-plane are shown in Figure 15.33a. The "side" face of $D$ is parallel to the $xz$-plane, the "back" face lies in the $yz$-plane, and the "top" face is contained in the plane $z = y - x$.

To find the $z$-limits of integration, fix a point $(x, y)$ in the shadow region $R$, and consider the vertical line $M$ that passes through $(x, y)$ and is parallel to the $z$-axis. This line enters $D$ at the height $z = 0$, and it exits at height $z = y - x$.

To find the $y$-limits of integration we again fix a point $(x, y)$ in $R$, but now we consider a line $L$ that lies in $R$, passes through $(x, y)$, and is parallel to the $y$-axis. This line is shown in Figure 15.33a and also in the face-on view of $R$ that is pictured in Figure 15.33b. The line $L$ enters $R$ when $y = x$ and exits when $y = 1$.

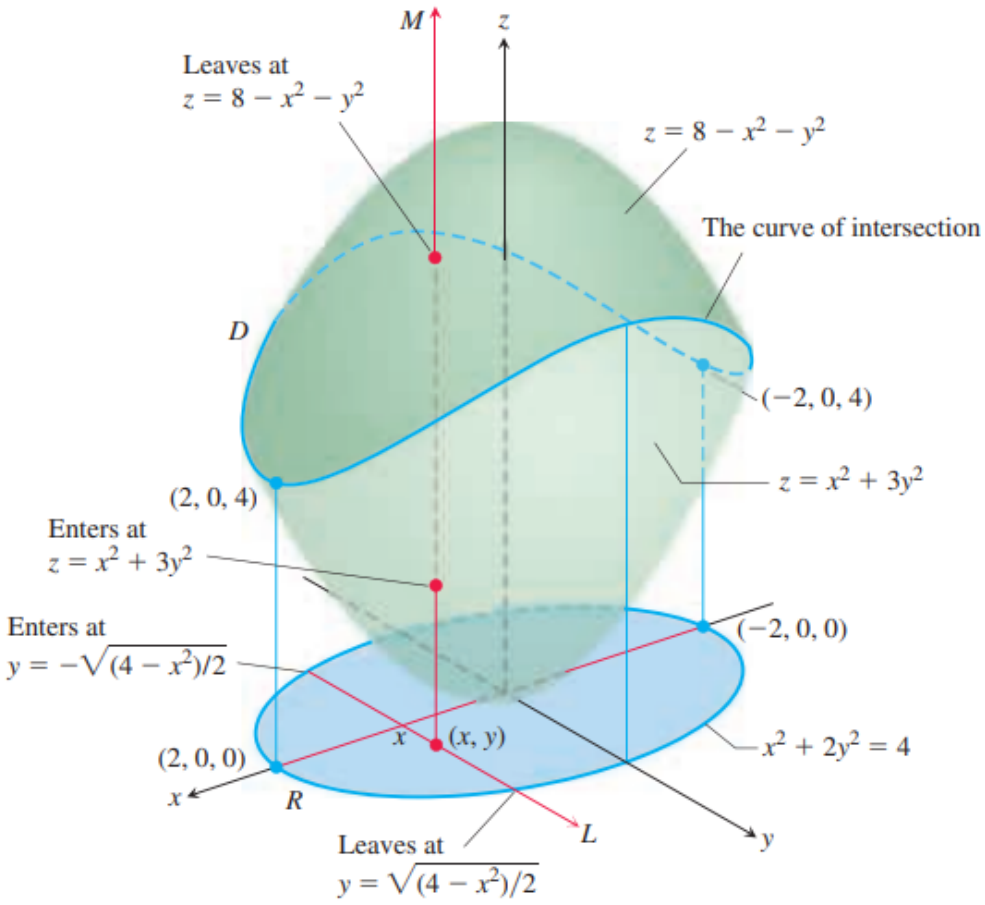
Finally, as $L$ sweeps across $R$, the value of $x$ varies from $x = 0$ to $x = 1$. Therefore, the triple integral of $F$ over the region $D$ is given by

$$\iiint_D F(x, y, z)\, dz\, dy\, dx = \int_0^1 \int_x^1 \int_0^{y-x} F(x, y, z)\, dz\, dy\, dx. \qquad \blacksquare$$

# Triple Integrals in Rectangular Coordinates

## Example 3

Find the volume of the region $D$ enclosed by the surfaces $z = x^2 + 3y^2$ and $z = 8 - x^2 - y^2$.



# Example 3

**Solution** The volume is

$$V = \iiint_D dz\, dy\, dx,$$

the integral of $F(x, y, z) = 1$ over $D$. To find the limits of integration for evaluating the integral, we first sketch the region. The surfaces (Figure 15.35) intersect on the elliptical cylinder $x^2 + 3y^2 = 8 - x^2 - y^2$ or $x^2 + 2y^2 = 4$, $z > 0$. The boundary of the region $R$, the projection of $D$ onto the $xy$-plane, is an ellipse with the same equation: $x^2 + 2y^2 = 4$. The "upper" boundary of $R$ is the curve $y = \sqrt{(4 - x^2)/2}$. The lower boundary is the curve $y = -\sqrt{(4 - x^2)/2}$.

Now we find the $z$-limits of integration. The line $M$ passing through a typical point $(x, y)$ in $R$ parallel to the $z$-axis enters $D$ at $z = x^2 + 3y^2$ and leaves at $z = 8 - x^2 - y^2$.

# Example 3

Next we find the $y$-limits of integration. The line $L$ through $(x, y)$ that lies parallel to the $y$-axis enters the region $R$ when $y = -\sqrt{(4 - x^2)/2}$ and leaves when $y = \sqrt{(4 - x^2)/2}$.

Finally we find the $x$-limits of integration. As $L$ sweeps across $R$, the value of $x$ varies from $x = -2$ at $(-2, 0, 0)$ to $x = 2$ at $(2, 0, 0)$. The volume of $D$ is

$$V = \iiint_D dz\, dy\, dx \qquad \text{Integrand is 1 when computing volume.}$$

$$= \int_{-2}^{2} \int_{-\sqrt{(4-x^2)/2}}^{\sqrt{(4-x^2)/2}} \int_{x^2+3y^2}^{8-x^2-y^2} dz\, dy\, dx \qquad \text{Substitute limits of integration.}$$

$$= \int_{-2}^{2} \int_{-\sqrt{(4-x^2)/2}}^{\sqrt{(4-x^2)/2}} (8 - 2x^2 - 4y^2)\, dy\, dx \qquad \text{Integrate over } z \text{ and evaluate.}$$

$$= \int_{-2}^{2} \left[ (8 - 2x^2)y - \frac{4}{3}y^3 \right]_{y=-\sqrt{(4-x^2)/2}}^{y=\sqrt{(4-x^2)/2}} dx \qquad \text{Integrate over } y.$$

$$= \int_{-2}^{2} \left( 2(8 - 2x^2)\sqrt{\frac{4 - x^2}{2}} - \frac{8}{3}\left(\frac{4 - x^2}{2}\right)^{3/2} \right) dx \qquad \text{Evaluate.}$$

$$= \int_{-2}^{2} \left[ 8\left(\frac{4 - x^2}{2}\right)^{3/2} - \frac{8}{3}\left(\frac{4 - x^2}{2}\right)^{3/2} \right] dx = \frac{4\sqrt{2}}{3} \int_{-2}^{2} (4 - x^2)^{3/2}\, dx$$

$$= 8\pi\sqrt{2}. \qquad \text{After integration with the substitution } x = 2 \sin u$$

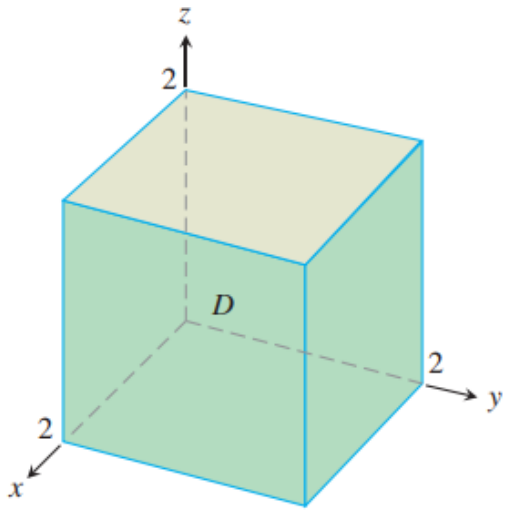
# Average Value of a Function in Space

• The **average value of a function** $F$ **over a region** $D$ in space is defined by the formula

$$\textbf{Average value of } F \text{ over } D = \frac{1}{\text{volume of } D} \iiint_D F\, dV.$$

## Example 4

Find the average value of $F(x, y, z) = xyz$ throughout the cubical region $D$ bounded by the coordinate planes and the planes $x = 2$, $y = 2$, and $z = 2$ in the first octant.



# Example 4

**Solution** We sketch the cube with enough detail to show the limits of integration (Figure 15.36). We then use Equation (2) to calculate the average value of $F$ over the cube.

The volume of the region $D$ is $(2)(2)(2) = 8$. The value of the integral of $F$ over the cube is

$$\int_0^2\int_0^2\int_0^2 xyz\,dx\,dy\,dz = \int_0^2\int_0^2 \left[\frac{x^2}{2}yz\right]_{x=0}^{x=2} dy\,dz = \int_0^2\int_0^2 2yz\,dy\,dz$$

$$= \int_0^2 \left[y^2 z\right]_{y=0}^{y=2} dz = \int_0^2 4z\,dz = \left[2z^2\right]_0^2 = 8.$$

With these values, Equation (2) gives

$$\begin{array}{l}\text{Average value of}\\ xyz \text{ over the cube}\end{array} = \frac{1}{\text{volume}}\iiint_{\text{cube}} xyz\,dV = \left(\frac{1}{8}\right)(8) = 1.$$

In evaluating the integral, we chose the order $dx\,dy\,dz$, but any of the other five possible orders would have done as well. ■

# Lecture 24: Multiple Integrals.

MA2032 Vector Calculus

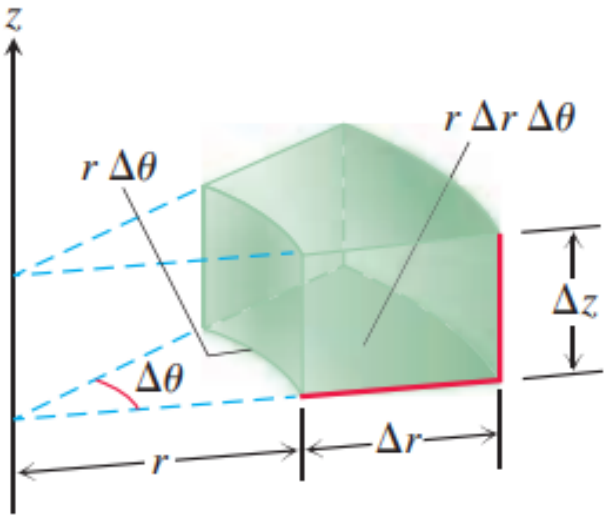
Lecturer: Larissa Serdukova

School of Computing and Mathematical Science
University of Leicester

November 14, 2022

# Integration in Cylindrical Coordinates

• We partition the region of integration $D$ into n small **cylindrical wedges**.
• The kth cylindrical wedge having **dimensions** $\Delta r_k$ by $\Delta\theta_k$ by $\Delta z_k$.
• We calculated in polar coordinates that **area of wedges' base** is $\Delta A_k = r_k \Delta r_k \Delta\theta_k$.
• So the volume of wedge is $\Delta V_k = \Delta z_k r_k \Delta r_k \Delta\theta_k$




• We define the **triple integral as a limit of Riemann sums** using these wedges as

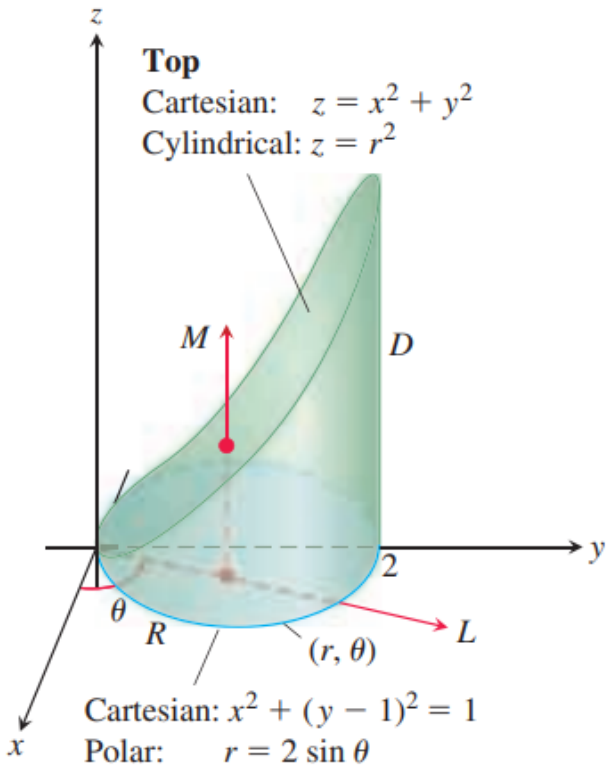
$$\lim_{n\to\infty} \sum_{k=1}^{n} f(r_k, \theta_k, z_k)\Delta V_k = \iiint_D f\, dV = \iiint_D f\, dz\, r\, dr\, d\theta.$$

• In most situations we will need to **restrict** $\theta$ to $\alpha \le \theta \le \beta$, where $0 \le \beta - \alpha \le 2\pi$.

# Integration in Cylindrical Coordinates

## Example 1

Find the limits of integration in cylindrical coordinates for integrating a function $f(r, \theta, z)$ over the region $D$ bounded below by the plane $z = 0$, laterally by the circular cylinder $x^2 + (y-1)^2 = 1$, and above by the paraboloid $z = x^2 + y^2$.



# Example 1

**Solution** The base of $D$ is also the region's projection $R$ on the $xy$-plane. The boundary of $R$ is the circle $x^2 + (y - 1)^2 = 1$. Its polar coordinate equation is

$$\begin{aligned} x^2 + (y - 1)^2 &= 1 \\ x^2 + y^2 - 2y + 1 &= 1 \\ r^2 - 2r \sin\theta &= 0 \\ r &= 2 \sin\theta. \end{aligned}$$

The region is sketched in Figure 15.49.

We find the limits of integration, starting with the $z$-limits. A line $M$ through a typical point $(r, \theta)$ in $R$ parallel to the $z$-axis enters $D$ at $z = 0$ and leaves at $z = x^2 + y^2 = r^2$.

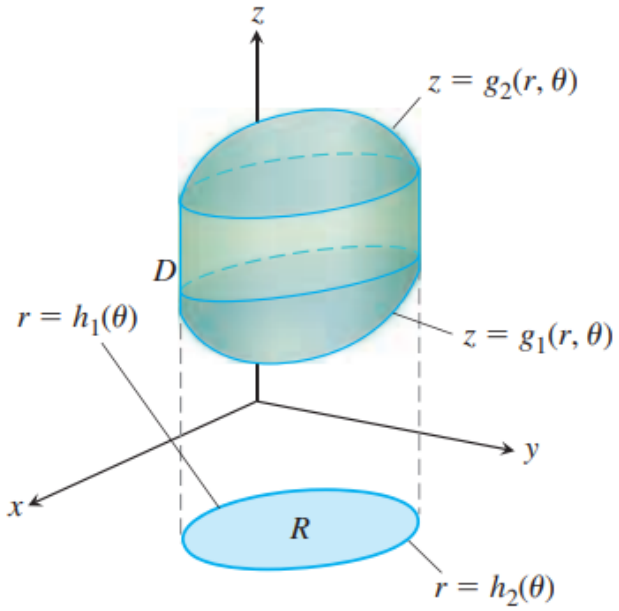
Next we find the $r$-limits of integration. A ray $L$ through $(r, \theta)$ from the origin enters $R$ at $r = 0$ and leaves at $r = 2 \sin\theta$.

Finally we find the $\theta$-limits of integration. As $L$ sweeps across $R$, the angle $\theta$ it makes with the positive $x$-axis runs from $\theta = 0$ to $\theta = \pi$. The integral is

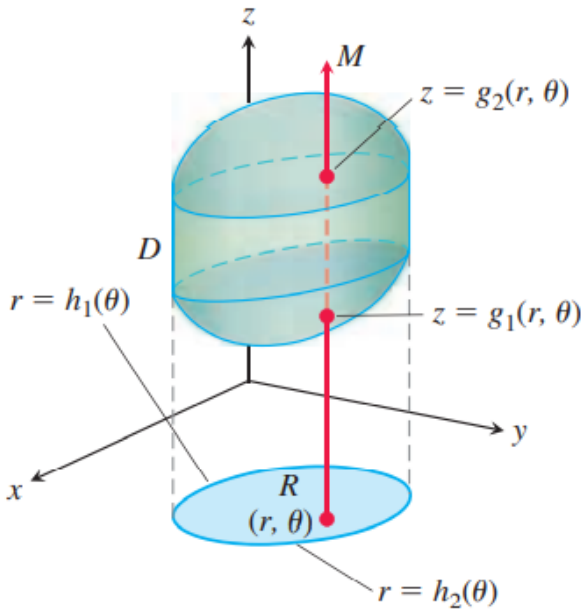
$$\iiint_D f(r, \theta, z)\, dV = \int_0^{\pi} \int_0^{2\sin\theta} \int_0^{r^2} f(r, \theta, z)\, dz\, r\, dr\, d\theta. \quad \blacksquare$$

# Finding Limits of Integration in the Order $dz\ dr\ d\theta$

• **Step 1: Sketch** Sketch the region $D$ along with its projection $R$ on the $xy$-plane. Label the surfaces and curves that bound $D$ and $R$.

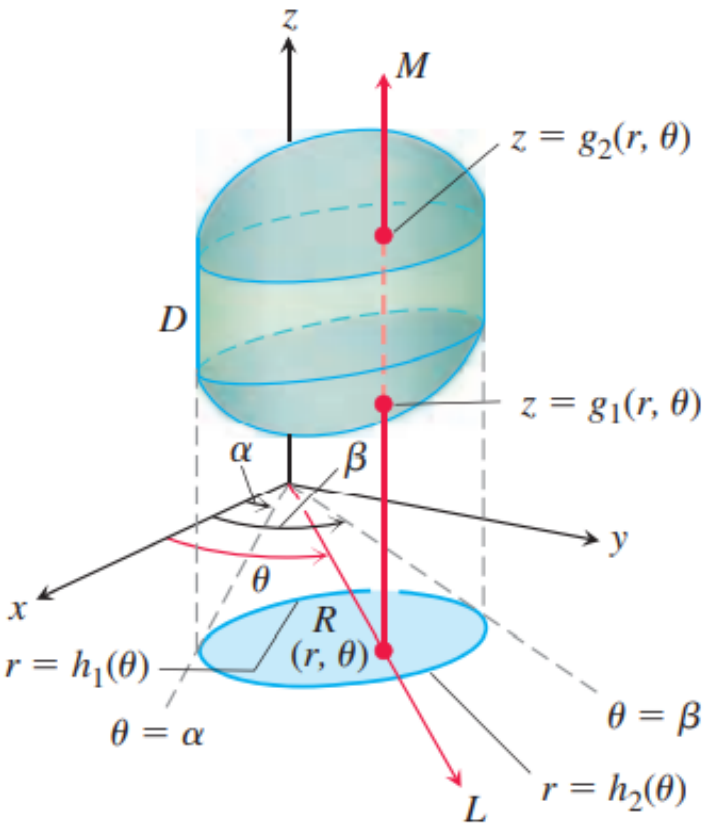



• **Step 2: Find the z-limits of integration**. Draw a line $M$ through a typical point $(r, \theta)$ of $R$ parallel to the $z$-axis. As $z$ increases, $M$ enters $D$ at $z = g_1(r, \theta)$ and leaves at $z = g_2(r, \theta)$. These are the z-limits of integration.



# Whiteboard

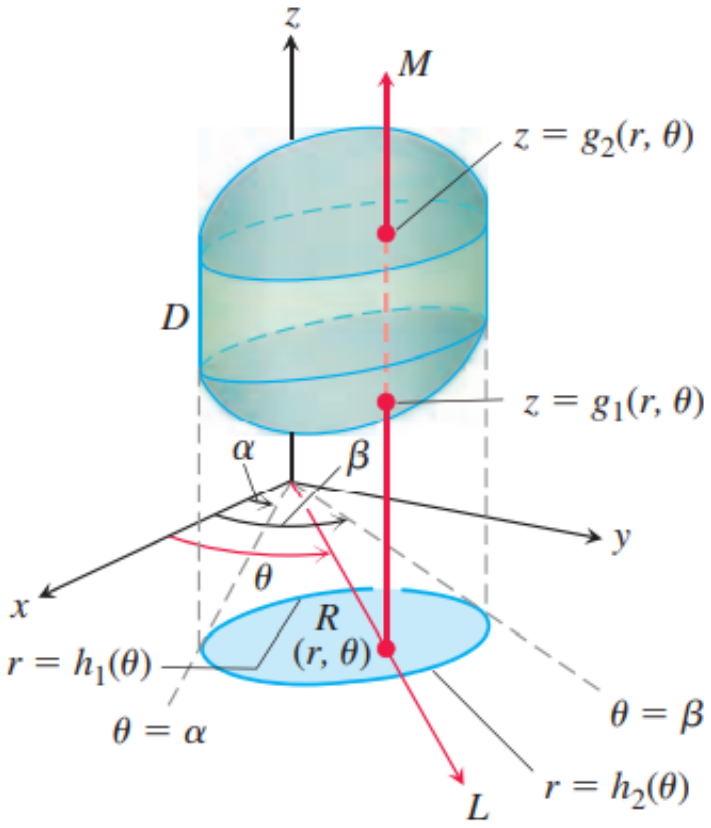
$$\iiint_D f(r,\theta,z)\,dV = \int_{\theta=\alpha}^{\theta=\beta}\int_{r=h_1(\theta)}^{r=h_2(\theta)}\int_{z=g_1(r,\theta)}^{z=g_2(r,\theta)} f(r,\theta,z)\,dz\,r\,dr\,d\theta.$$



# Finding Limits of Integration in the Order *dz dy dx*

- **Step 3: Find the r-limits of integration**. Draw a ray $L$ through $(r, \theta)$ from the origin. The ray enters $R$ at $r = h_1(\theta)$ and leaves at $r = h_2(\theta)$. These are the r-limits of integration.

- **Step 4: Find the $\theta$-limits of integration**. As $L$ sweeps across $R$, the angle $\theta$ it makes with the positive $x$-axis runs from $\theta = \alpha$ to $\theta = \beta$. These are the $\theta$-limits of integration.



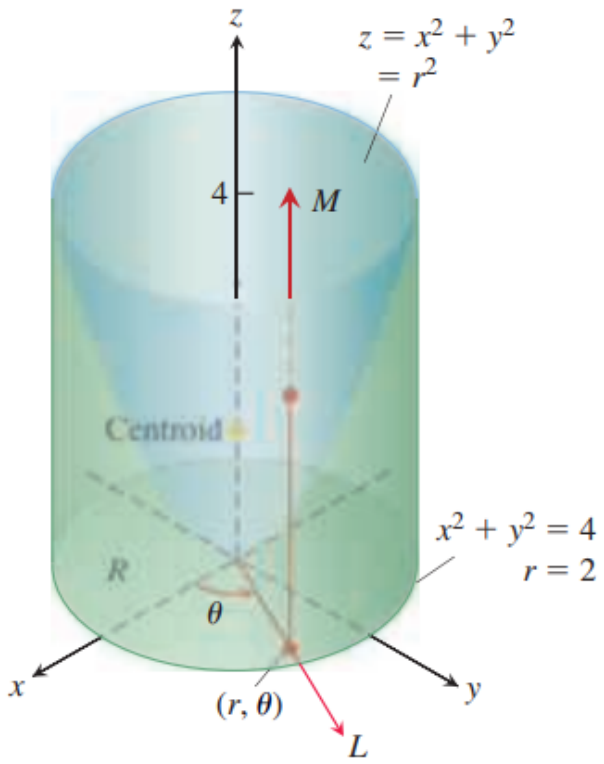

- The integral is

$$\iiint_D f(r, \theta, z)\, dV = \int_{\theta=\alpha}^{\theta=\beta} \int_{r=h_1(\theta)}^{r=h_2(\theta)} \int_{z=g_1(r,\theta)}^{z=g_2(r,\theta)} f(r, \theta, z)\, dz\, r\, dr\, d\theta.$$

# Integration in Cylindrical Coordinates

## Example 2

Find the centroid ($\sigma = 1$) of the solid enclosed by the cylinder $x^2 + y^2 = 4$, bounded above by the paraboloid $z = x^2 + y^2$, and bounded below by the $xy$-plane.



# Example 2

**Solution** We sketch the solid, bounded above by the paraboloid $z = r^2$ and below by the plane $z = 0$ (Figure 15.50). Its base $R$ is the disk $0 \leq r \leq 2$ in the $xy$-plane.

The solid's centroid $(\bar{x}, \bar{y}, \bar{z})$ lies on its axis of symmetry, here the $z$-axis. This makes $\bar{x} = \bar{y} = 0$. To find $\bar{z}$, we divide the first moment $M_{xy}$ by the mass $M$.

To find the limits of integration for the mass and moment integrals, we continue with the four basic steps. We completed our initial sketch. The remaining steps give the limits of integration.

*The z-limits*. A line $M$ through a typical point $(r, \theta)$ in the base parallel to the $z$-axis enters the solid at $z = 0$ and leaves at $z = r^2$.

*The r-limits*. A ray $L$ through $(r, \theta)$ from the origin enters $R$ at $r = 0$ and leaves at $r = 2$.

*The θ-limits*. As $L$ sweeps over the base like a clock hand, the angle $\theta$ it makes with the positive $x$-axis runs from $\theta = 0$ to $\theta = 2\pi$. The value of $M_{xy}$ is

# Example 2

$$M_{xy} = \int_0^{2\pi}\int_0^2\int_0^{r^2} z\,dz\,r\,dr\,d\theta = \int_0^{2\pi}\int_0^2 \left[\frac{z^2}{2}\right]_0^{r^2} r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^2 \frac{r^5}{2}dr\,d\theta = \int_0^{2\pi}\left[\frac{r^6}{12}\right]_0^2 d\theta = \int_0^{2\pi}\frac{16}{3}d\theta = \frac{32\pi}{3}.$$

The value of $M$ is

$$M = \int_0^{2\pi}\int_0^2\int_0^{r^2} dz\,r\,dr\,d\theta = \int_0^{2\pi}\int_0^2 \left[z\right]_0^{r^2} r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^2 r^3\,dr\,d\theta = \int_0^{2\pi}\left[\frac{r^4}{4}\right]_0^2 d\theta = \int_0^{2\pi} 4\,d\theta = 8\pi.$$

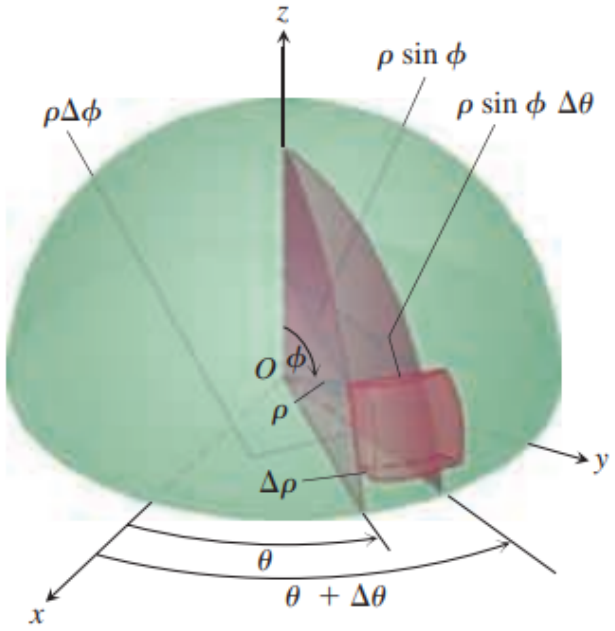
Therefore,

$$\bar{z} = \frac{M_{xy}}{M} = \frac{32\pi}{3}\frac{1}{8\pi} = \frac{4}{3},$$

and the centroid is $(0, 0, 4/3)$. Notice that the centroid lies on the $z$-axis, outside the solid.

# Triple Integrals in Spherical Coordinates

• We partition the region of integration $D$ into n small **spherical wedges**.
• The kth spherical wedge having **dimensions** $\Delta\rho_k$ by $\Delta\phi_k$ by $\Delta\theta_k$.
• It can be shown that the volume of this spherical wedge $\Delta V_k$ is
$\Delta V_k = \rho_k^2 \sin\phi_k \Delta\rho_k \Delta\phi_k \Delta\theta_k$ for $(\rho_k, \phi_k, \theta_k)$, a point chosen inside the wedge.




• We define the **triple integral as a limit of Riemann sums** using these wedges as

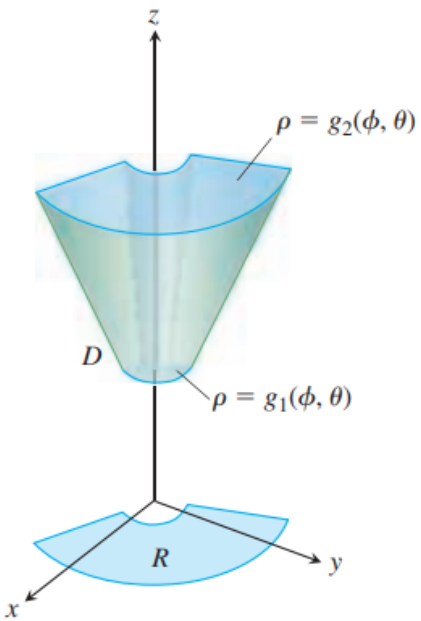
$$\lim_{n\to\infty} \sum_{k=1}^{n} f(\rho_k, \phi_k, \theta_k)\Delta V_k = \iiint_D f(\rho, \phi, \theta)\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta.$$

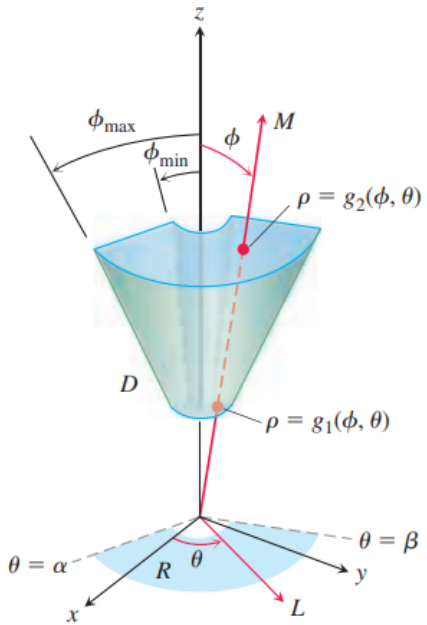
• In most situations we will need to **restrict** $\theta$ to $\alpha \le \theta \le \beta$, where $0 \le \beta - \alpha \le 2\pi$.

# Finding Limits of Integration in the Order $d\rho\ d\phi\ d\theta$

• **Step 1: Sketch** Sketch the region $D$ along with its projection $R$ on the $xy$-plane. Label the surfaces that bound $D$.

• **Step 2: Find the $\rho$-limits of integration**. Draw a ray $M$ from the origin through $D$, making an angle $\phi$ with the positive $z$-axis. Also draw the projection of $M$ on the $xy$-plane (call the projection $L$). The ray $L$ makes an angle $\theta$ with the positive $x$-axis. As $r$ increases, $M$ enters $D$ at $r = g_1(\phi, \theta)$ and leaves at $\rho = g_2(\phi, \theta)$. These are the -limits of integration.



# Finding Limits of Integration in the Order $d\rho\ d\phi\ d\theta$

• **Step 3: Find the $\phi$-limits of integration**. For any given $\theta$, the angle $\phi$ that $M$ makes with the z-axis runs from $f = f_{min}$ to $f = f_{max}$. These are the $\phi$-limits of integration.

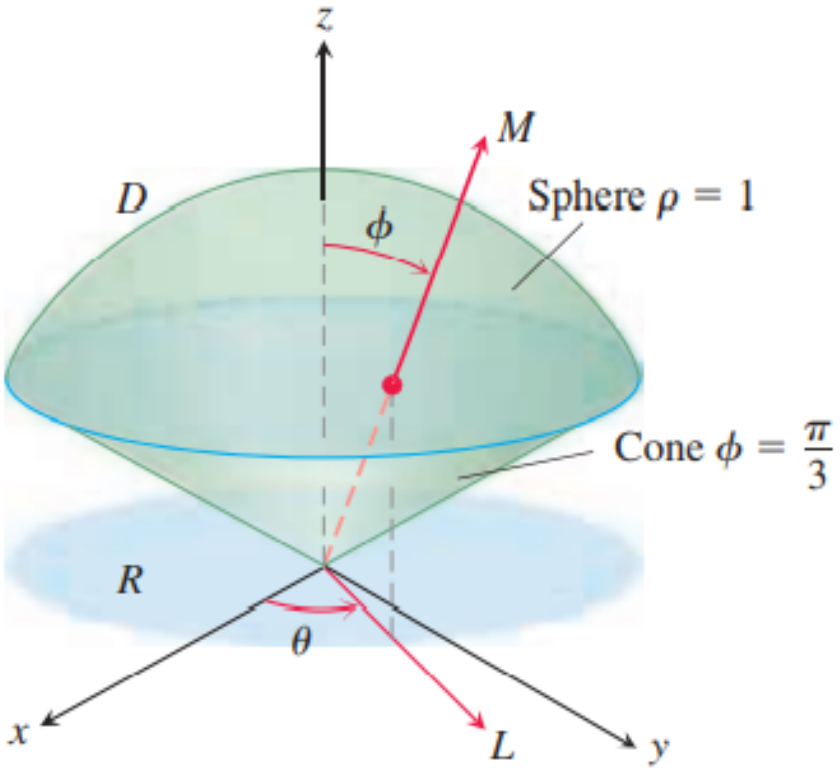
• **Step 4: Find the $\theta$-limits of integration**. The ray $L$ sweeps over $R$ as $\theta$ runs from $\alpha$ to $\beta$. These are the $\theta$-limits of integration.

• The integral is

$$\iiint_D f(\rho, \phi, \theta)\, dV = \int_{\theta=\alpha}^{\theta=\beta} \int_{\phi=\phi_{\min}}^{\phi=\phi_{\max}} \int_{\rho=g_1(\phi,\theta)}^{\rho=g_2(\phi,\theta)} f(\rho, \phi, \theta)\, \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta.$$

## Example 3

Find the volume of the "ice cream cone" $D$ cut from the solid sphere $\rho \leq 1$ by the cone $\phi = \pi/3$.



# Example 3

**Solution** The volume is $V = \iiint_D \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$, the integral of $f(\rho, \phi, \theta) = 1$ over $D$.

To find the limits of integration for evaluating the integral, we begin by sketching $D$ and its projection $R$ on the $xy$-plane (Figure 15.56).

*The $\rho$-limits of integration.* We draw a ray $M$ from the origin through $D$, making an angle $\phi$ with the positive $z$-axis. We also draw $L$, the projection of $M$ on the $xy$-plane, along with the angle $\theta$ that $L$ makes with the positive $x$-axis. Ray $M$ enters $D$ at $\rho = 0$ and leaves at $\rho = 1$.

*The $\phi$-limits of integration.* The cone $\phi = \pi/3$ makes an angle of $\pi/3$ with the positive $z$-axis. For any given $\theta$, the angle $\phi$ can run from $\phi = 0$ to $\phi = \pi/3$.

*The $\theta$-limits of integration.* The ray $L$ sweeps over $R$ as $\theta$ runs from 0 to $2\pi$. The volume is

$$V = \iiint_D \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = \int_0^{2\pi} \int_0^{\pi/3} \int_0^1 \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{2\pi} \int_0^{\pi/3} \left[\frac{\rho^3}{3}\right]_0^1 \sin\phi \, d\phi \, d\theta = \int_0^{2\pi} \int_0^{\pi/3} \frac{1}{3} \sin\phi \, d\phi \, d\theta$$

$$= \int_0^{2\pi} \left[-\frac{1}{3}\cos\phi\right]_0^{\pi/3} d\theta = \int_0^{2\pi} \left(-\frac{1}{6} + \frac{1}{3}\right) d\theta = \frac{1}{6}(2\pi) = \frac{\pi}{3}.$$

## Example 4

A solid of constant density $\sigma = 1$ occupies the region $D$ in Example 3. Find the solid's moment of inertia about the $z$-axis.

**Solution** In rectangular coordinates, the moment is

$$I_z = \iiint_D (x^2 + y^2)\, dV.$$

In spherical coordinates, $x^2 + y^2 = (\rho \sin\phi \cos\theta)^2 + (\rho \sin\phi \sin\theta)^2 = \rho^2 \sin^2\phi$. Hence,

$$I_z = \iiint_D (\rho^2 \sin^2\phi)\, \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta = \iiint_D \rho^4 \sin^3\phi \, d\rho\, d\phi\, d\theta.$$

For the region $D$ in Example 5, this becomes

$$\begin{aligned} I_z &= \int_0^{2\pi}\int_0^{\pi/3}\int_0^1 \rho^4 \sin^3\phi \, d\rho\, d\phi\, d\theta = \int_0^{2\pi}\int_0^{\pi/3} \left[\frac{\rho^5}{5}\right]_0^1 \sin^3\phi \, d\phi\, d\theta \\ &= \frac{1}{5}\int_0^{2\pi}\int_0^{\pi/3} (1 - \cos^2\phi) \sin\phi \, d\phi\, d\theta = \frac{1}{5}\int_0^{2\pi} \left[-\cos\phi + \frac{\cos^3\phi}{3}\right]_0^{\pi/3} d\theta \\ &= \frac{1}{5}\int_0^{2\pi} \left(-\frac{1}{2} + 1 + \frac{1}{24} - \frac{1}{3}\right) d\theta = \frac{1}{5}\int_0^{2\pi} \frac{5}{24}\, d\theta = \frac{1}{24}(2\pi) = \frac{\pi}{12}. \end{aligned}$$

# Triple Integrals in Spherical Coordinates

## Coordinate Conversion Formulas

| Cylindrical to Rectangular | Spherical to Rectangular | Spherical to Cylindrical |
|---|---|---|
| $x = r\cos\theta$ | $x = \rho\sin\phi\cos\theta$ | $r = \rho\sin\phi$ |
| $y = r\sin\theta$ | $y = \rho\sin\phi\sin\theta$ | $z = \rho\cos\phi$ |
| $z = z$ | $z = \rho\cos\phi$ | $\theta = \theta$ |

Corresponding formulas for $dV$ in triple integrals:

$$\begin{aligned} dV &= dx\,dy\,dz \\ &= dz\,r\,dr\,d\theta \\ &= \rho^2\sin\phi\,d\rho\,d\phi\,d\theta \end{aligned}$$

# Lecture 25: Multiple Integrals.

MA2032 Vector Calculus

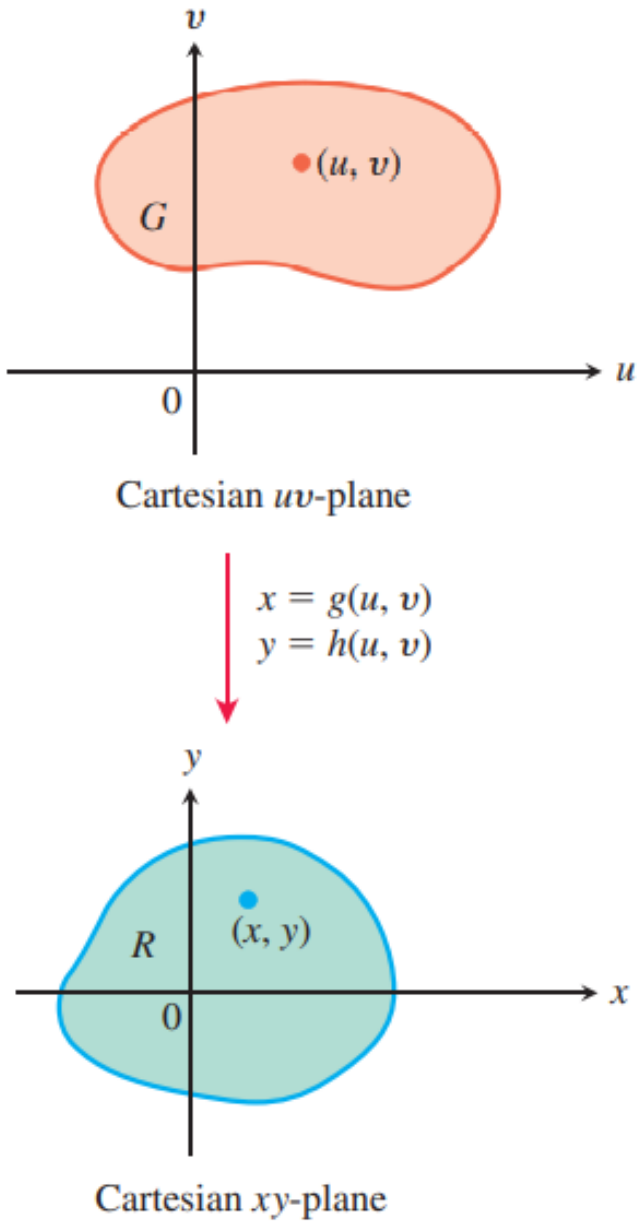
Lecturer: Larissa Serdukova

School of Computing and Mathematical Science
University of Leicester

November 16, 2022

# Substitutions in Multiple Integrals

- Today we introduce the ideas involved in **coordinate transformations** to evaluate multiple integrals by **substitution**.
- The method **replaces complicated integrals** by ones that are **easier to evaluate**.
- Substitutions accomplish this by **simplifying the integrand, the limits of integration, or both**.
- The **polar coordinate substitution is a special case** of a more general substitution method for double integrals.
- Suppose that a region $G$ in the $u\nu$-plane is **transformed** into the region $R$ in the $xy$-plane by equations of the form $x = g(u, \nu), y = h(u, \nu)$



# Substitutions in Double Integrals

• We assume the transformation is **one-to-one** on the interior of $G$.
• We call $R$ the **image** of $G$ under the transformation, and $G$ the **preimage** of $R$.
• Any function $f(x, y)$ defined on R **can be thought** of as a function $f(g(u, \nu), h(u, \nu))$ defined on G as well.
• **How is the integral** of $f(x, y)$ over $R$ **related** to the integral of $f(g(u, y), h(u, y))$ over $G$?
• To gain some **insight into the question**, we look at the **single variable case**: substitution method for single integrals.

$$\int_{g(a)}^{g(b)} f(x)\,dx = \int_a^b f(g(u))\,g'(u)\,du. \qquad x = g(u),\ dx = g'(u)\,du$$

• To propose an analogue for substitution in a double integral $\iint_R f(x, y)dx\ dy$, we need a **derivative factor** like $g'(u)$ as a **multiplier that transforms the area element** $du\ d\nu$ in the region G to its corresponding area element $dx\ dy$ in the region R.

# Substitutions in Double Integrals

- We denote this factor by J:

**DEFINITION** The **Jacobian determinant** or **Jacobian** of the coordinate transformation $x = g(u, v)$, $y = h(u, v)$ is

$$J(u, v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}. \tag{1}$$

- The Jacobian measures **how much the transformation is expanding or contracting the area** around the point (u, $\nu$).

# Substitutions in Double Integrals

**THEOREM 3—Substitution for Double Integrals**

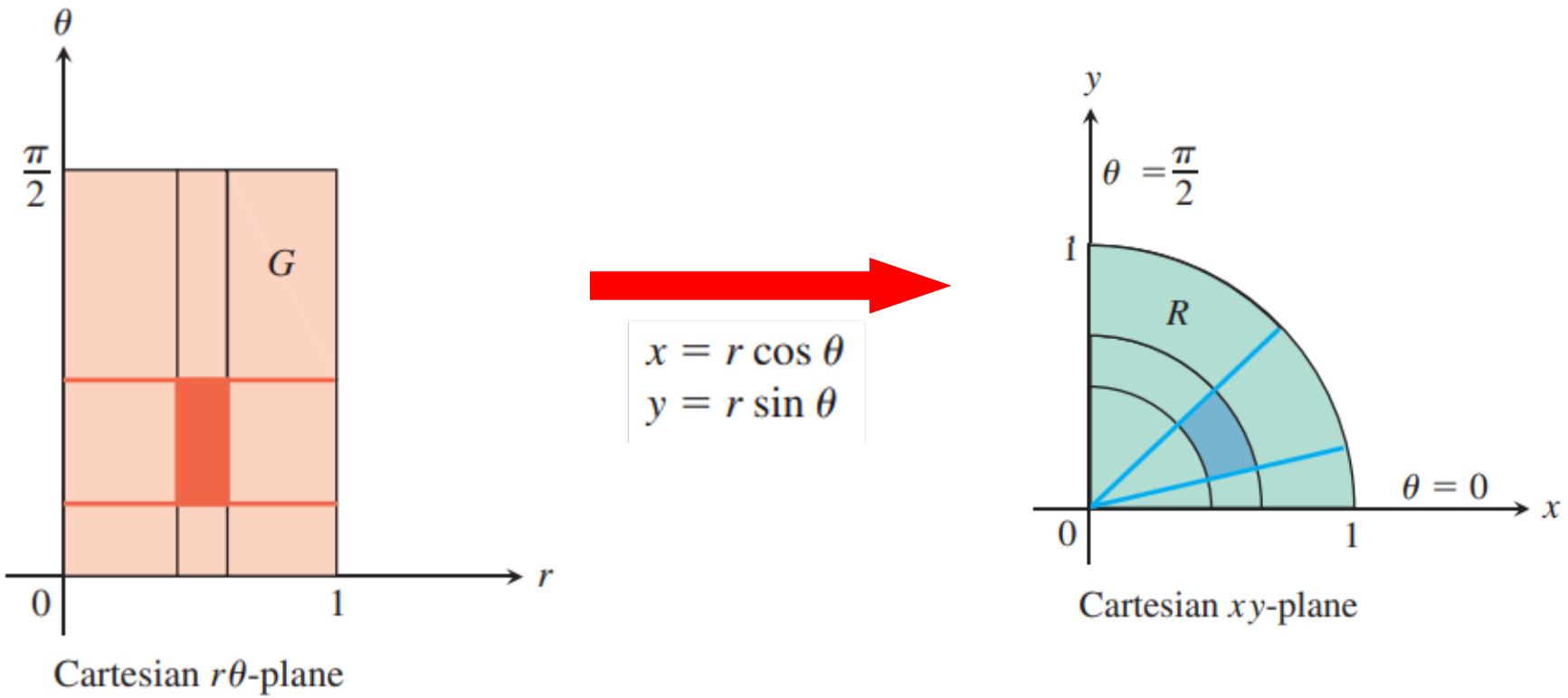
Suppose that $f(x, y)$ is continuous over the region $R$. Let $G$ be the preimage of $R$ under the transformation $x = g(u, v)$, $y = h(u, v)$, which is assumed to be one-to-one on the interior of $G$. If the functions $g$ and $h$ have continuous first partial derivatives within the interior of $G$, then

$$\iint_R f(x, y)\, dx\, dy = \iint_G f(g(u, v), h(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du\, dv. \tag{2}$$

# Substitutions in Double Integrals

## Example 1

Find the Jacobian for the polar coordinate transformation $x = r\cos\theta,\ y = r\sin\theta$, and write the Cartesian integral $\iint_R f(x,y)dx\ dy$ as a polar integral.



# Example 1

**Solution** Figure 15.58 shows how the equations $x = r\cos\theta$, $y = r\sin\theta$ transform the rectangle $G$: $0 \le r \le 1$, $0 \le \theta \le \pi/2$, into the quarter circle $R$ bounded by $x^2 + y^2 = 1$ in the first quadrant of the $xy$-plane.

For polar coordinates, we have $r$ and $\theta$ in place of $u$ and $v$. With $x = r\cos\theta$ and $y = r\sin\theta$, the Jacobian is

$$J(r,\theta) = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r(\cos^2\theta + \sin^2\theta) = r.$$

Since we assume $r \ge 0$ when integrating in polar coordinates, $|J(r,\theta)| = |r| = r$, so that Equation (2) gives

$$\iint_R f(x,y)\,dx\,dy = \iint_G f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta. \tag{3}$$

This is the same formula we derived independently using a geometric argument for polar area in Section 15.4. ■

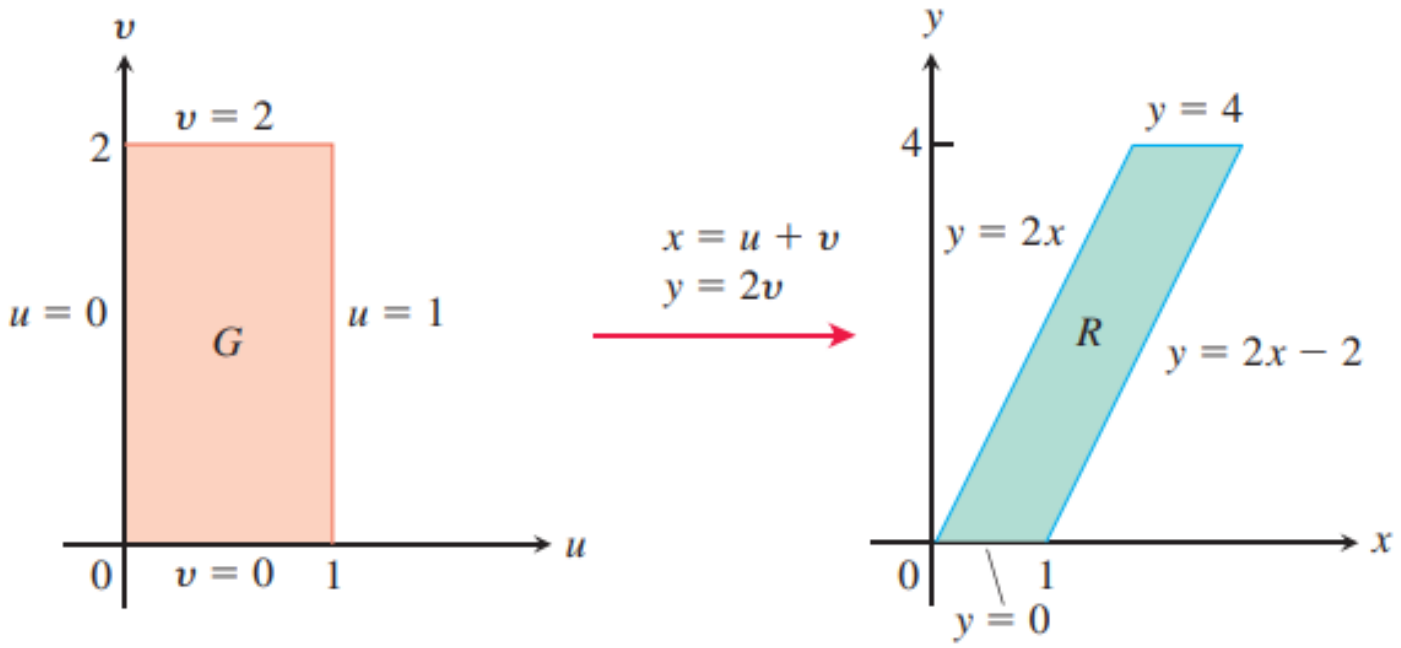
# Substitutions in Double Integrals

## Example 2

Evaluate

$$\int_0^4 \int_{x=y/2}^{x=(y/2)+1} \frac{2x-y}{2}\,dx\,dy$$

by applying the transformation $u = \frac{2x-y}{2}$, $\nu = \frac{y}{2}$ and integrating over an appropriate region in the $u\nu$-plane.



# Example 2

**Solution** We sketch the region $R$ of integration in the $xy$-plane and identify its boundaries (Figure 15.59).

To apply Equation (2), we need to find the corresponding $uv$-region $G$ and the Jacobian of the transformation. To find them, we first solve Equations (4) for $x$ and $y$ in terms of $u$ and $v$. From those equations it is easy to find algebraically that

$$x = u + v, \qquad y = 2v. \tag{5}$$

We then find the boundaries of $G$ by substituting these expressions into the equations for the boundaries of $R$ (Figure 15.59)

| $xy$-equations for the boundary of $R$ | Corresponding $uv$-equations for the boundary of $G$ | Simplified $uv$-equations |
|---|---|---|
| $x = y/2$ | $u + v = 2v/2 = v$ | $u = 0$ |
| $x = (y/2) + 1$ | $u + v = (2v/2) + 1 = v + 1$ | $u = 1$ |
| $y = 0$ | $2v = 0$ | $v = 0$ |
| $y = 4$ | $2v = 4$ | $v = 2$ |

# Example 2

From Equations (5) the Jacobian of the transformation is

$$J(u, v) = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \dfrac{\partial}{\partial u}(u + v) & \dfrac{\partial}{\partial v}(u + v) \\ \dfrac{\partial}{\partial u}(2v) & \dfrac{\partial}{\partial v}(2v) \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} = 2.$$

We now have everything we need to apply Equation (2):

$$\int_0^4 \int_{x=y/2}^{x=(y/2)+1} \frac{2x - y}{2} dx\, dy = \int_{v=0}^{v=2} \int_{u=0}^{u=1} u \left| J(u, v) \right| du\, dv$$

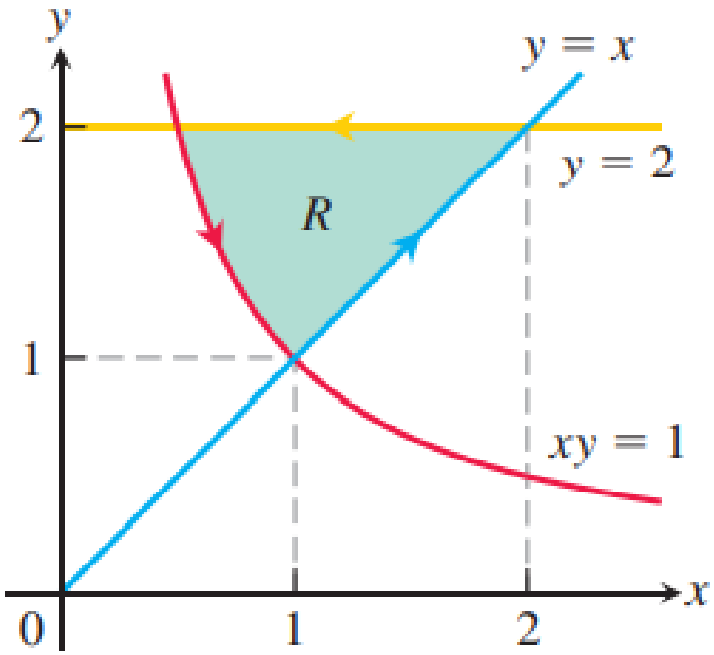
$$= \int_0^2 \int_0^1 (u)(2)\, du\, dv = \int_0^2 \Big[ u^2 \Big]_0^1 dv = \int_0^2 dv = 2. \quad \blacksquare$$

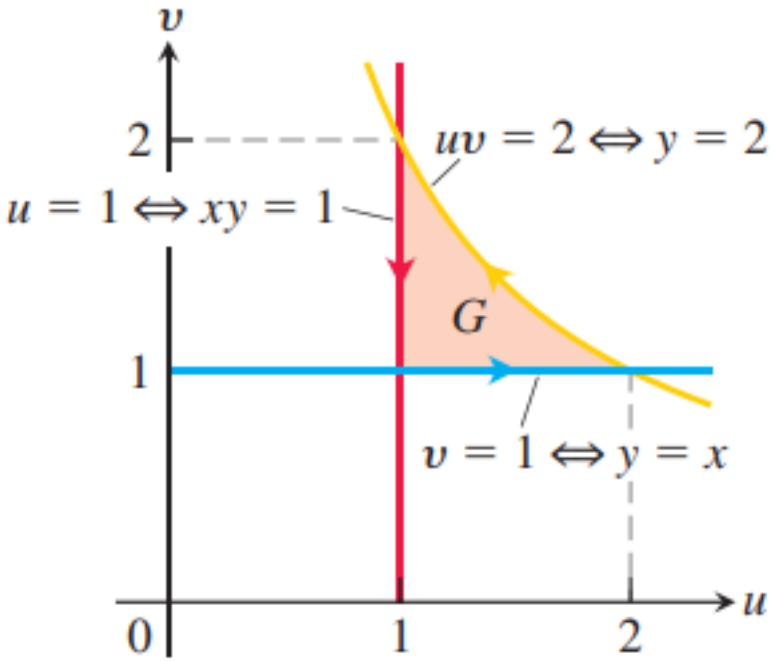
# Substitutions in Double Integrals

## Example 3

Evaluate the integral

$$\int_1^2 \int_{x=1/y}^{x=y} \sqrt{\frac{y}{x}} e^{\sqrt{xy}} \, dx \, dy.$$



# Example 3. Whiteboard

# Example 3

**Solution** The square root terms in the integrand suggest that we might simplify the integration by substituting $u = \sqrt{xy}$ and $v = \sqrt{y/x}$. Squaring these equations gives $u^2 = xy$ and $v^2 = y/x$, which imply that $u^2v^2 = y^2$ and $u^2/v^2 = x^2$. So we obtain the transformation (in the same ordering of the variables as discussed before)

$$x = \frac{u}{v} \qquad \text{and} \qquad y = uv,$$

with $u > 0$ and $v > 0$. Let's first see what happens to the integrand itself under this transformation. The Jacobian of the transformation is not constant:

$$J(u, v) = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \dfrac{1}{v} & \dfrac{-u}{v^2} \\ v & u \end{vmatrix} = \frac{2u}{v}.$$

# Example 3

If $G$ is the region of integration in the $uv$-plane, then by Equation (2) the transformed double integral under the substitution is

$$\iint_R \sqrt{\frac{y}{x}}\, e^{\sqrt{xy}}\, dx\, dy = \iint_G v e^u \frac{2u}{v}\, du\, dv = \iint_G 2ue^u\, du\, dv.$$

The transformed integrand function is easier to integrate than the original one, so we proceed to determine the limits of integration for the transformed integral.

The region of integration $R$ of the original integral in the $xy$-plane is shown in Figure 15.61. From the substitution equations $u = \sqrt{xy}$ and $v = \sqrt{y/x}$, we see that the image of the left-hand boundary $xy = 1$ for $R$ is the vertical line segment $u = 1, 2 \geq v \geq 1$, in $G$ (see Figure 15.62). Likewise, the right-hand boundary $y = x$ of $R$ maps to the horizontal line segment $v = 1, 1 \leq u \leq 2$, in $G$. Finally, the horizontal top boundary $y = 2$ of $R$

# Example 3

maps to $uv = 2$, $1 \le v \le 2$, in $G$. As we move counterclockwise around the boundary of the region $R$, we also move counterclockwise around the boundary of $G$, as shown in Figure 15.62. Knowing the region of integration $G$ in the $uv$-plane, we can now write equivalent iterated integrals:

$$\int_1^2 \int_{1/y}^{y} \sqrt{\frac{y}{x}}\, e^{\sqrt{xy}}\, dx\, dy = \int_1^2 \int_1^{2/u} 2ue^u\, dv\, du.$$

Note the order of integration.

We now evaluate the transformed integral on the right-hand side,

$$\begin{aligned}
\int_1^2 \int_1^{2/u} 2ue^u\, dv\, du &= 2\int_1^2 \Big[ vue^u \Big]_{v=1}^{v=2/u} du \\
&= 2\int_1^2 (2e^u - ue^u)\, du \\
&= 2\int_1^2 (2-u)e^u\, du \\
&= 2\Big[ (2-u)e^u + e^u \Big]_{u=1}^{u=2} \qquad \text{Integrate by parts.} \\
&= 2(e^2 - (e+e)) = 2e(e-2).
\end{aligned}$$

# Substitutions in Triple Integrals

• The **cylindrical and spherical coordinate substitutions** are special cases of a substitution method that pictures changes of variables in **triple integrals** as transformations of three-dimensional regions.

• The method is like the method for double integrals except that now **we work in three dimensions** instead of two.

• Suppose that a region G in $uvw$-space **is transformed** one-to-one into the region D in $xyz$-space by differentiable equations of the form

$$x = g(u, v, w), \qquad y = h(u, v, w), \qquad z = k(u, v, w),$$

• Then any function $F(x, y, z)$ defined on D can be thought of as a function

$$F(g(u, v, w), h(u, v, w), k(u, v, w)) = H(u, v, w)$$

• defined on G.

# Substitutions in Triple Integrals

• If g, h, and k have **continuous first partial derivatives**, then the integral of $F(x, y, z)$ over D is related to the integral of $H(u, y, w)$ over G by the equation

$$\iiint_D F(x, y, z)\, dx\, dy\, dz = \iiint_G H(u, v, w)\, |J(u, v, w)|\, du\, dv\, dw.$$

• The factor $J(u, \nu, w)$, whose absolute value appears in this equation, is

$$J(u, v, w) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \frac{\partial(x, y, z)}{\partial(u, v, w)}.$$

the **Jacobian determinant**

• This determinant measures how much the **volume** near a point in G is being **expanded or contracted** by the transformation from $(u, \nu, w)$ to $(x, y, z)$ coordinates.

# Substitutions in Triple Integrals. Cylindrical coordinates

For cylindrical coordinates, $r$, $\theta$, and $z$ take the place of $u$, $v$, and $w$. The transformation from Cartesian $r\theta z$-space to Cartesian $xyz$-space is given by the equations

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad z = z$$

(Figure 15.64). The Jacobian of the transformation is

$$J(r,\theta,z) = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

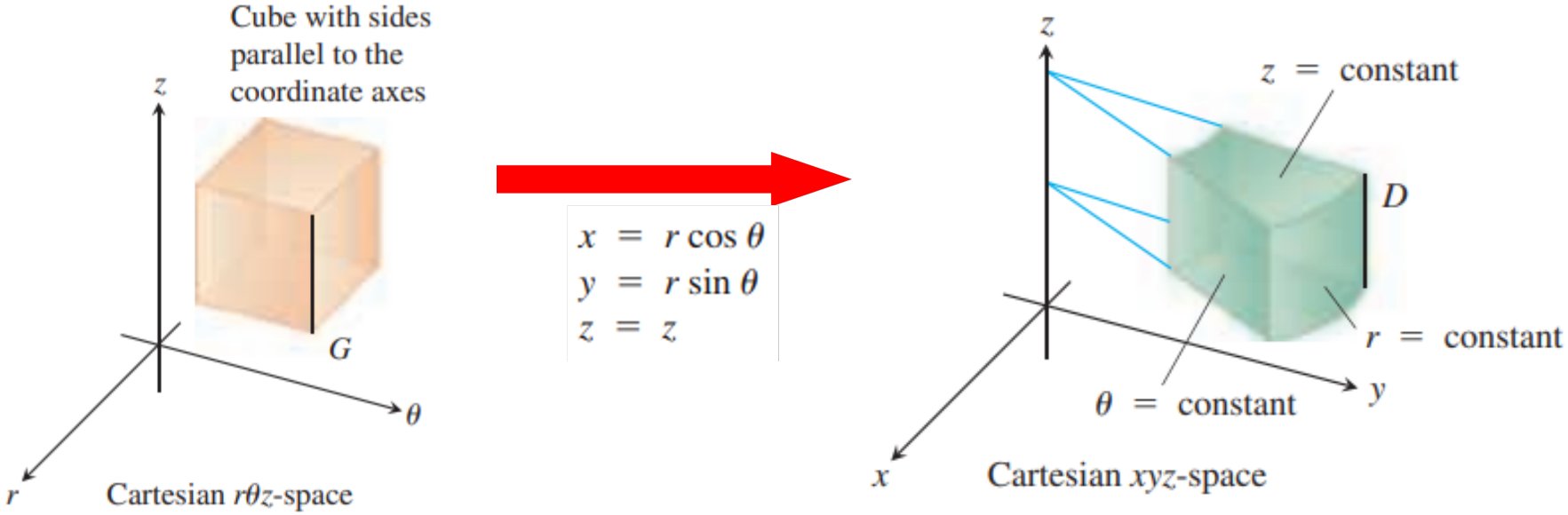
$$= r\cos^2\theta + r\sin^2\theta = r.$$

The corresponding version of Equation (7) is

$$\iiint_D F(x,y,z)\,dx\,dy\,dz = \iiint_G H(r,\theta,z)|r|\,dr\,d\theta\,dz.$$

We can drop the absolute value signs because $r \geq 0$.

# Substitutions in Triple Integrals. Cylindrical coordinates



# Substitutions in Triple Integrals. Spherical coordinates

For spherical coordinates, $\rho$, $\phi$, and $\theta$ take the place of $u$, $v$, and $w$. The transformation from Cartesian $\rho\phi\theta$-space to Cartesian $xyz$-space is given by

$$x = \rho \sin \phi \cos \theta, \qquad y = \rho \sin \phi \sin \theta, \qquad z = \rho \cos \phi$$

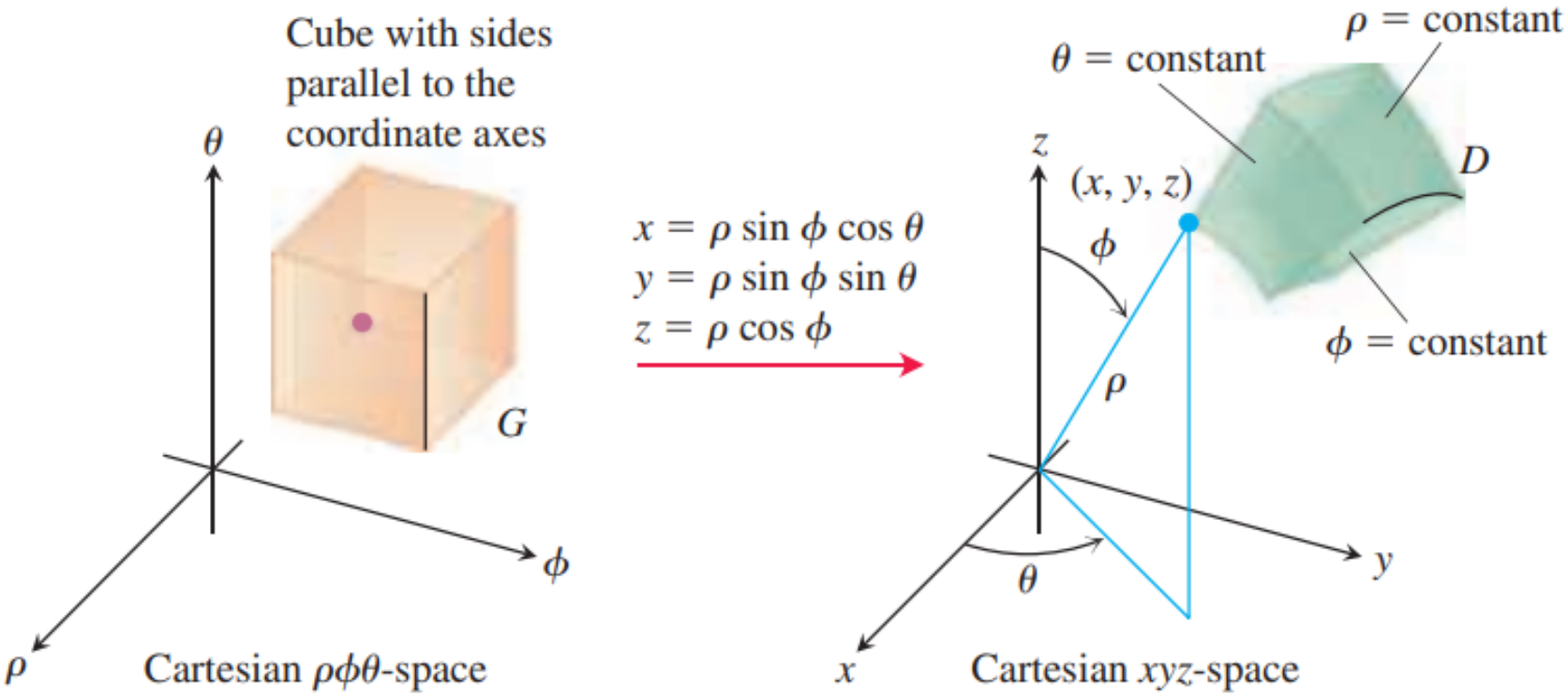
(Figure 15.65). The Jacobian of the transformation (see Exercise 23) is

$$J(\rho, \phi, \theta) = \begin{vmatrix} \dfrac{\partial x}{\partial \rho} & \dfrac{\partial x}{\partial \phi} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial \rho} & \dfrac{\partial y}{\partial \phi} & \dfrac{\partial y}{\partial \theta} \\ \dfrac{\partial z}{\partial \rho} & \dfrac{\partial z}{\partial \phi} & \dfrac{\partial z}{\partial \theta} \end{vmatrix} = \rho^2 \sin \phi.$$

The corresponding version of Equation (7) is

$$\iiint_D F(x, y, z)\, dx\, dy\, dz = \iiint_G H(\rho, \phi, \theta)\, \left|\rho^2 \sin \phi\right|\, d\rho\, d\phi\, d\theta.$$

# Substitutions in Triple Integrals. Spherical coordinates



# Lecture 26: Integrals and Vector Fields.

MA2032 Vector Calculus

Lecturer: Larissa Serdukova

School of Computing and Mathematical Science
University of Leicester

November 18, 2022

# Integrals and Vector Fields. Overview

• **We extend the theory of integration** to general curves and surfaces in space.

• The resulting **line and surface integrals** give powerful mathematical tools for science and engineering.

• **Line integrals are used** to find the **work done by a force** in moving an object along a path, and to find the **mass of a curved wire** with variable density.

• **Surface integrals** are used to find the **rate of flow** of a fluid across a surface and to describe the **interactions of electric and magnetic forces**.

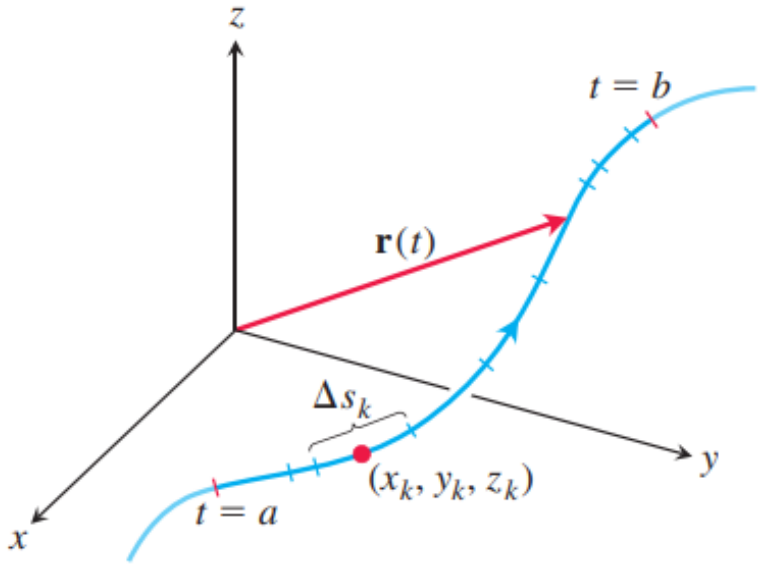
• We present the **fundamental theorems of vector integral calculus**, and discuss their mathematical consequences and physical applications.

• The theorems of vector calculus are then shown to be **generalized versions of the Fundamental Theorem of Calculus**.

# Line Integrals of Scalar Functions

• We need a **more general notion of integral** than was defined in Integral Calculus.
• We need **to integrate over a curve C rather than over an interval** $[a, b]$.
• These more general integrals are called **line integrals**.
• Suppose that $f(x, y, z)$ is a **real-valued function** we wish to integrate over the curve $C \in D_f$ **parametrized by** $r(t) = g(t)i + h(t)j + k(t)k$, $a \leq t \leq b$.

• The **values of** $f$ **along** $C$ **are given by the composite function** $f(g(t), h(t), k(t))$.
• We are going to **integrate this composition** with respect **to arc length** $s(t)$ **from** $t = a$ **to** $t = b$.
• We first **partition** $C$ into a finite number $n$ of **subarcs** with length $\Delta s_k$.



# Line Integrals of Scalar Functions

- In each subarc we choose a point $(x_k, y_k, z_k)$ and form the **sum**

$$S_n = \sum_{k=1}^{n} \underbrace{f(x_k, y_k, z_k)}_{\text{value of } f \text{ at a point on the subarc}} \underbrace{\Delta s_k}_{\text{length of a small subarc of the curve}},$$

- which is similar to a **Riemann sum**.
- If $f$ is **continuous** and the functions $g$, $h$, and $k$ **have continuous first derivatives**, then **these sums approach a limit as $n$ increases** and the lengths $\Delta s_k$ approach zero.
- This leads to the following **definition**

**DEFINITION** If $f$ is defined on a curve C given parametrically by $\mathbf{r}(t) = g(t)\mathbf{i} + h(t)\mathbf{j} + k(t)\mathbf{k}$, $a \le t \le b$, then the **line integral of $f$ over $C$** is

$$\int_C f(x, y, z)\, ds = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k, y_k, z_k)\, \Delta s_k, \qquad (1)$$

provided this limit exists.

# Line Integrals of Scalar Functions

• If the **curve** $C$ **is smooth** for $a \leq t \leq b$ (so $v = dr/dt$ is continuous and never 0) and the function $f$ is **continuous on** $C$, then the limit in Equation (1) can be shown to **exist**.
• We can then apply the Fundamental Theorem of Calculus **to differentiate the arc length equation**,

$$s(t) = \int_a^t |\mathbf{v}(\tau)| \, d\tau,$$

• to express $ds$ in Equation (1) as $ds = |v(t)|dt$ and evaluate the integral of $f$ over $C$ as

$$\int_C f(x, y, z) \, ds = \int_a^b f(g(t), h(t), k(t)) \, |\mathbf{v}(t)| \, dt.$$

• The formula evaluates the line integral correctly **no matter what smooth parametrization for** $C$ is used.
• Note that the parameter **t defines a direction** along the path.

# How to Evaluate a Line Integral

**How to Evaluate a Line Integral**

To integrate a continuous function $f(x, y, z)$ over a curve $C$:

**1.** Find a smooth parametrization of $C$,

$$\mathbf{r}(t) = g(t)\mathbf{i} + h(t)\mathbf{j} + k(t)\mathbf{k}, \qquad a \le t \le b.$$

**2.** Evaluate the integral as

$$\int_C f(x, y, z)\, ds = \int_a^b f(g(t), h(t), k(t))\, |\mathbf{v}(t)|\, dt.$$

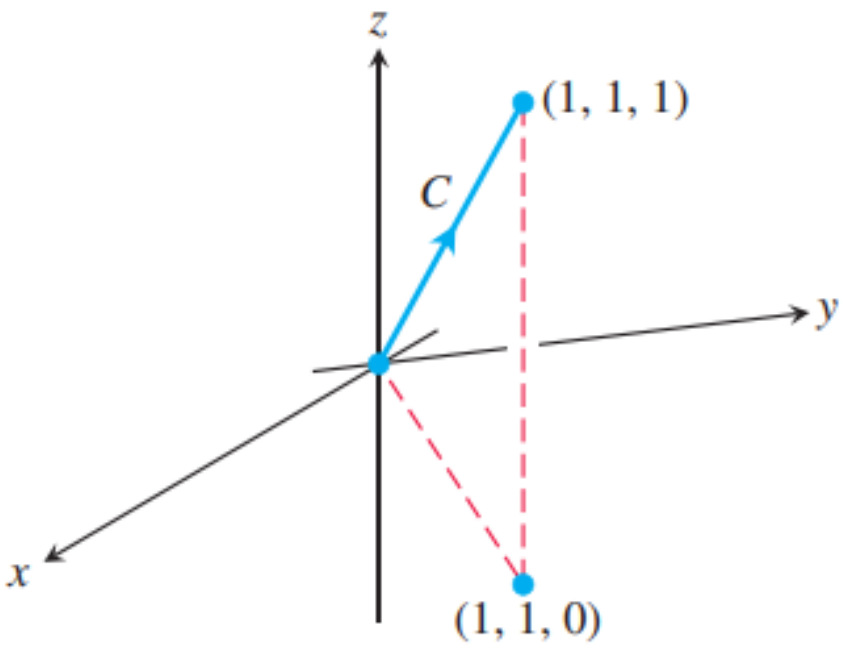
• If $f$ has the constant value 1, then the integral of $f$ over $C$ gives the length of $C$ from $t = a$ to $t = b$.

• We also write $f(\mathbf{r}(t))$ for the evaluation $f(g(t), h(t), k(t))$ along the curve $\mathbf{r}$.

# Line Integrals of Scalar Functions

## Example 1

Integrate $f(x, y, z) = x - 3y^2 + z$ over the line segment $C$ joining the origin to the point $(1, 1, 1)$.



# Example 1

**Solution** Since any choice of parametrization will give the same answer, we choose the simplest parametrization we can think of:

$$\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j} + t\mathbf{k}, \qquad 0 \le t \le 1.$$

The components have continuous first derivatives and $|\mathbf{v}(t)| = |\mathbf{i} + \mathbf{j} + \mathbf{k}| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$ is never 0, so the parametrization is smooth. The integral of $f$ over $C$ is

$$\int_C f(x, y, z)\, ds = \int_0^1 f(t, t, t)\sqrt{3}\, dt \qquad \text{Eq. (2), } ds = |\mathbf{v}(t)|\, dt = \sqrt{3}\, dt$$

$$= \int_0^1 (t - 3t^2 + t)\sqrt{3}\, dt$$

$$= \sqrt{3}\int_0^1 (2t - 3t^2)\, dt = \sqrt{3}\left[t^2 - t^3\right]_0^1 = 0.$$

# Additivity

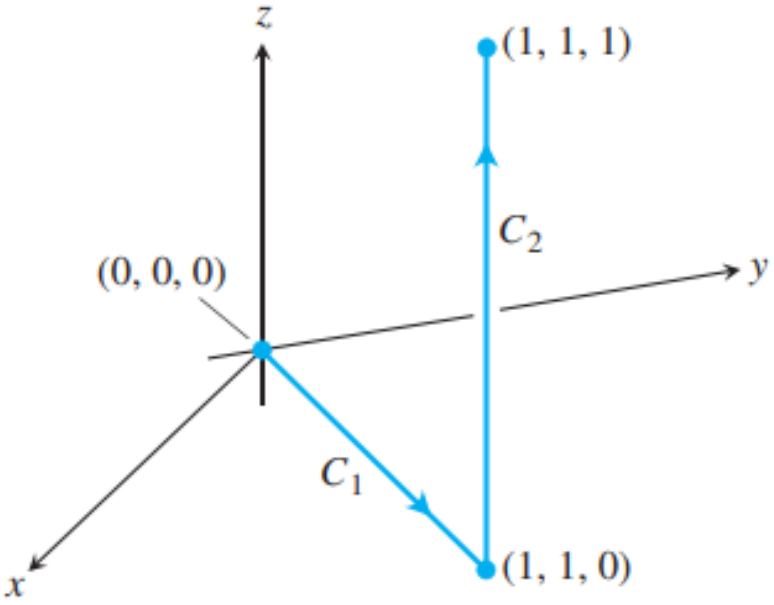
• Line integrals have the **useful property** that if a piecewise smooth curve $C$ is **made by joining a finite number of smooth curves** $C_1, C_2, ..., C_n$ end to end, then the integral of a function over $C$ is the sum of the integrals over the curves that make it up:

$$\int_C f\,ds = \int_{C_1} f\,ds + \int_{C_2} f\,ds + \cdots + \int_{C_n} f\,ds.$$

• The **value of the line integral along a path** joining two points **can change if you change the path** between them.

## Example 2

Figure shows another path from the origin to $(1, 1, 1)$, formed from two line segments $C_1$ and $C_2$. Integrate $f(x, y, z) = x - 3y^2 + z$ over $C_1 \cup C_2$.



# Example 2

**Solution** We choose the simplest parametrizations for $C_1$ and $C_2$ we can find, calculating the lengths of the velocity vectors as we go along:

$$C_1: \quad \mathbf{r}(t) = t\mathbf{i} + t\mathbf{j}, \quad 0 \le t \le 1; \quad |\mathbf{v}| = \sqrt{1^2 + 1^2} = \sqrt{2}$$

$$C_2: \quad \mathbf{r}(t) = \mathbf{i} + \mathbf{j} + t\mathbf{k}, \quad 0 \le t \le 1; \quad |\mathbf{v}| = \sqrt{0^2 + 0^2 + 1^2} = 1.$$

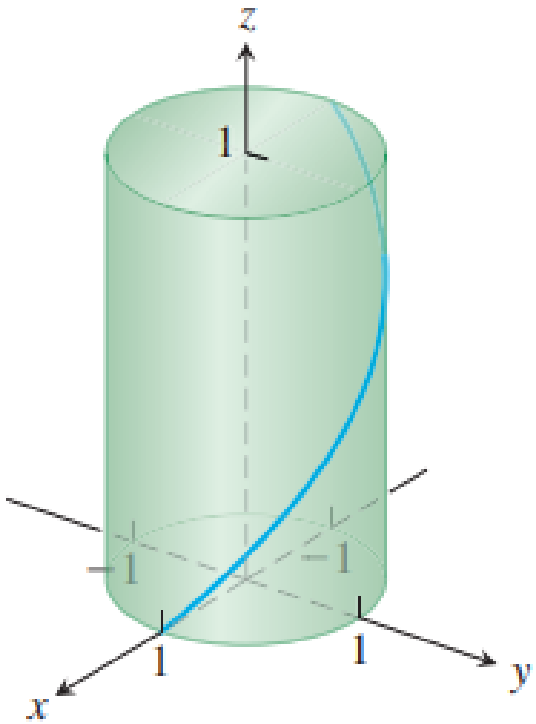
With these parametrizations we find that

$$\begin{aligned}
\int_{C_1 \cup C_2} f(x, y, z)\, ds &= \int_{C_1} f(x, y, z)\, ds + \int_{C_2} f(x, y, z)\, ds && \text{Eq. (3)} \\
&= \int_0^1 f(t, t, 0)\sqrt{2}\, dt + \int_0^1 f(1, 1, t)(1)\, dt && \text{Eq. (2)} \\
&= \int_0^1 (t - 3t^2 + 0)\sqrt{2}\, dt + \int_0^1 (1 - 3 + t)(1)\, dt \\
&= \sqrt{2}\left[\frac{t^2}{2} - t^3\right]_0^1 + \left[\frac{t^2}{2} - 2t\right]_0^1 = -\frac{\sqrt{2}}{2} - \frac{3}{2}.
\end{aligned}$$

# Line Integrals of Scalar Functions

## Example 3

Find the line integral of $f(x, y, z) = 2xy + \sqrt{z}$ over the helix $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}$, $0 \leq t \leq \pi$.



# Example 3

**Solution** For the helix (Figure 16.4) we find $\mathbf{v}(t) = \mathbf{r}'(t) = -\sin t\mathbf{i} + \cos t\mathbf{j} + \mathbf{k}$ and $|\mathbf{v}(t)| = \sqrt{(-\sin t)^2 + (\cos t)^2 + 1} = \sqrt{2}$. Evaluating the function $f$ at the point $\mathbf{r}(t)$, we obtain

$$f(\mathbf{r}(t)) = f(\cos t, \sin t, t) = 2\cos t \sin t + \sqrt{t} = \sin 2t + \sqrt{t}.$$

The line integral is given by

$$\begin{aligned}\int_C f(x, y, z)\, ds &= \int_0^{\pi} \left(\sin 2t + \sqrt{t}\right)\sqrt{2}\, dt \\ &= \sqrt{2}\left[-\frac{1}{2}\cos 2t + \frac{2}{3}t^{3/2}\right]_0^{\pi} \\ &= \frac{2\sqrt{2}}{3}\pi^{3/2} \approx 5.25.\end{aligned}$$

# Mass and Moment Calculations

• We treat **coil springs** (Left Figure) and **wires** (Right Figure) as **masses distributed along smooth curves in space**.

# Mass and Moment Calculations

• The distribution is described by a **continuous density function** $\delta(x, y, z)$ representing **mass per unit length**.

• When a curve $C$ is parametrized by $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$, $a \leq t \leq b$ then $x$, $y$, and $z$ are functions of the parameter $t$, the **density is the function** $\delta(x(t), y(t), z(t))$, and the **arc length differential** is given by

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt.$$

• The spring's or wire's **mass, center of mass, and moments** are then calculated using the formulas in Table on next slide, with the integrations in terms of the parameter $t$ over the interval $[a, b]$ .

• For example, the formula for mass becomes

$$M = \int_a^b \delta(x(t), y(t), z(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt.$$

# Mass and Moment Calculations

**TABLE 16.1** Mass and moment formulas for coil springs, wires, and thin rods lying along a smooth curve $C$ in space

**Mass:** $M = \int_C \delta \, ds$ $\quad$ $\delta = \delta(x, y, z)$ is the density at $(x, y, z)$

**First moments about the coordinate planes:**

$$M_{yz} = \int_C x \, \delta \, ds, \qquad M_{xz} = \int_C y \, \delta \, ds, \qquad M_{xy} = \int_C z \, \delta \, ds$$

**Coordinates of the center of mass:**

$$\bar{x} = M_{yz}/M, \qquad \bar{y} = M_{xz}/M, \qquad \bar{z} = M_{xy}/M$$

**Moments of inertia about axes and other lines:**

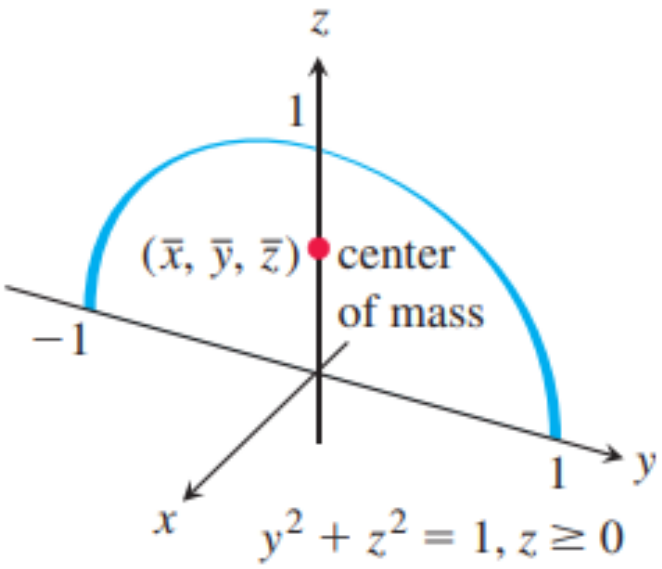
$$I_x = \int_C (y^2 + z^2)\delta \, ds, \qquad I_y = \int_C (x^2 + z^2)\delta \, ds, \qquad I_z = \int_C (x^2 + y^2)\delta \, ds,$$

$$I_L = \int_C r^2 \delta \, ds$$ $\quad$ $r(x, y, z)$ = distance from the point $(x, y, z)$ to line $L$

# Mass and Moment Calculations

## Example 4

A slender metal arch, denser at the bottom than top, lies along the semicircle $y^2 + z^2 = 1,\ z \geq 0$, in the $yz$-plane. Find the center of the arch's mass if the density at the point $(x, y, z)$ on the arch is $\delta(x, y, z) = 2 - z$.



# Example 4

**Solution** We know that $\bar{x} = 0$ and $\bar{y} = 0$ because the arch lies in the $yz$-plane with its mass distributed symmetrically about the $z$-axis. To find $\bar{z}$, we parametrize the circle as

$$\mathbf{r}(t) = (\cos t)\mathbf{j} + (\sin t)\mathbf{k}, \qquad 0 \le t \le \pi.$$

For this parametrization,

$$|\mathbf{v}(t)| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} = \sqrt{(0)^2 + (-\sin t)^2 + (\cos t)^2} = 1,$$

so $ds = |\mathbf{v}|\, dt = dt$.

The formulas in Table 16.1 then give

$$M = \int_C \delta\, ds = \int_C (2 - z)\, ds = \int_0^\pi (2 - \sin t)\, dt = 2\pi - 2$$

$$M_{xy} = \int_C z\delta\, ds = \int_C z(2 - z)\, ds = \int_0^\pi (\sin t)(2 - \sin t)\, dt$$

$$= \int_0^\pi (2 \sin t - \sin^2 t)\, dt = \frac{8 - \pi}{2} \qquad \text{Routine integration}$$

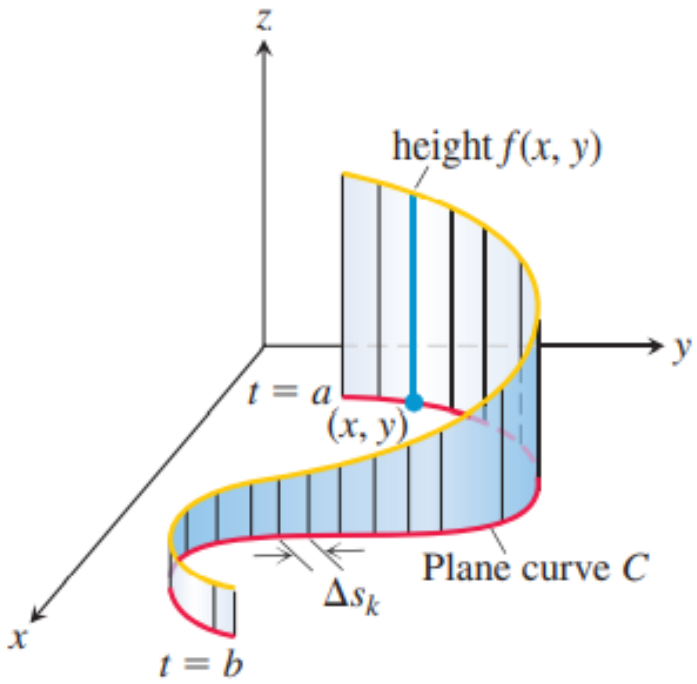
$$\bar{z} = \frac{M_{xy}}{M} = \frac{8 - \pi}{2} \cdot \frac{1}{2\pi - 2} = \frac{8 - \pi}{4\pi - 4} \approx 0.57.$$

With $\bar{z}$ to the nearest hundredth, the center of mass is (0, 0, 0.57).

# Line Integrals in the Plane

- Line integrals **for curves in the plane** have a natural geometric interpretation.

- If $C$ is a smooth curve in the $xy$-plane parametrized by $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}, a \leq t \leq b$, we generate a **cylindrical surface** by moving a straight line along $C$ orthogonal to the plane, holding the line parallel to the $z$-axis.
- The cylinder cuts through the surface $z = (x, y)$, forming a **"curved wall"**.




- At any point $(x, y)$ along the curve, the height of the wall is $f(x, y)$.

$$\int_C f\, ds = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k, y_k)\, \Delta s_k,$$

- The line integral $\int_C f\ ds$ is the area of the wall shown in the figure.

# Lecture 27: Integrals and Vector Fields.

MA2032 Vector Calculus

Lecturer: Larissa Serdukova

School of Computing and Mathematical Science
University of Leicester

November 21, 2022

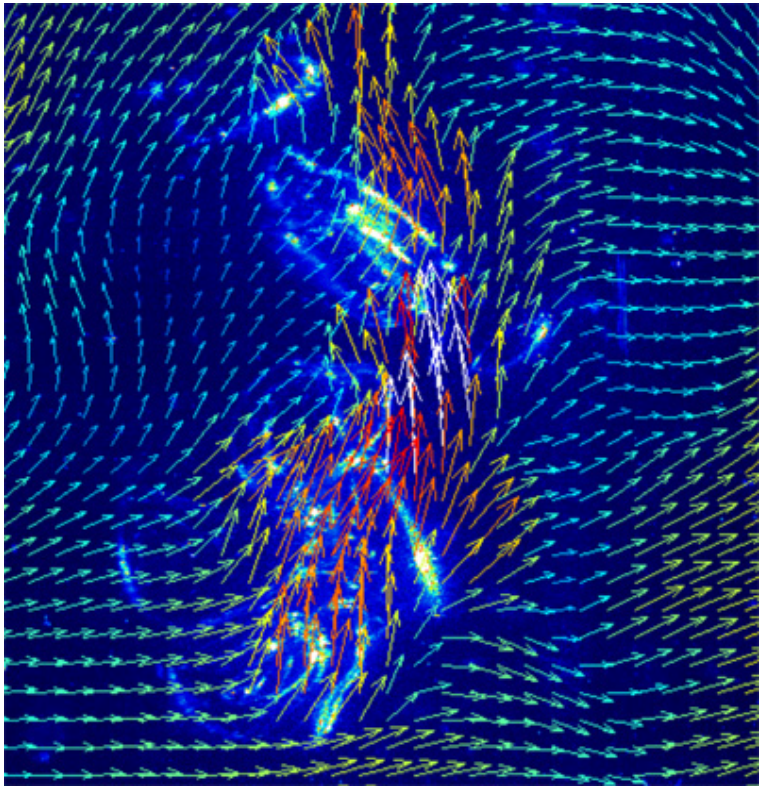
# Vector Fields and Line Integrals: Work, Circulation, and Flux

• **Gravitational and electric forces** have both a **direction** and a **magnitude**.

They are represented by a **vector at each point in their domain**, producing a **vector field**.

• We show **how to compute the work done in moving an object through such a field** by using a **line integral** involving the vector field.

• We also discuss **velocity fields**, such as the vector field representing the velocity of a flowing fluid in its domain.



• A **line integral** can be used to find the **rate at which the fluid flows** along or across a curve within the domain.

# Vector Fields

• Suppose a **region** in the plane or in space **is occupied by a moving fluid**, such as air or water.

• The fluid is made up of a **large number of particles**, and at any instant of time, a **particle has a velocity** $v$.

• Such a fluid flow is an **example of a vector field**.

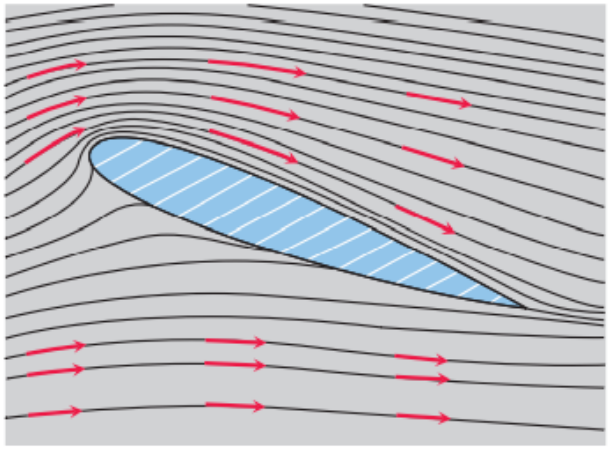


Figure - Velocity vectors of a flow around an airfoil in a wind tunnel.

• Generally, a **vector field is a function** that **assigns a vector to each point in its domain**.

• A **vector field** on a three-dimensional domain in space **might have a formula** like

$$\mathbf{F}(x, y, z) = M(x, y, z)\mathbf{i} + N(x, y, z)\mathbf{j} + P(x, y, z)\mathbf{k}.$$

• The vector field is **continuous** if the component functions M, N, and P are continuous;

• It is **differentiable** if each of the component functions is differentiable.

# Gradient Fields

• The **gradient vector** of a differentiable scalar-valued function at a point **gives the direction of greatest increase of the function**.

• An **important type of vector field** is formed by all the gradient vectors of the function.

• We define the gradient field of a differentiable function $f(x, y, z)$ to be the field of gradient vectors

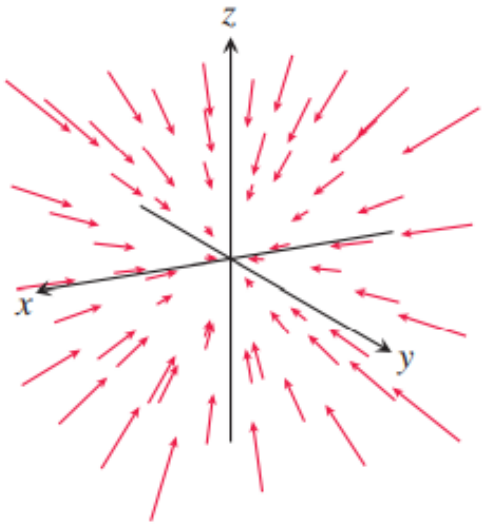
$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}.$$

• At each point $(x, y, z)$, the **gradient field gives a vector pointing in the direction of greatest increase** of $f$, with **magnitude being the value of the directional derivative** in that direction.

# Gradient Fields

## Example 1

Suppose that a material is heated, that the resulting temperature $T$ at each point $(x, y, z)$ in a region of space is given by $T = 100 - x^2 - y^2 - z^2$, and that $\mathbf{F}(x, y, z)$ is defined to be the gradient of $T$. Find the vector field $\mathbf{F}$.




**Solution** The gradient field $\mathbf{F}$ is the field $\mathbf{F} = \nabla T = -2x\mathbf{i} - 2y\mathbf{j} - 2z\mathbf{k}$. At each point in the region, the vector field $\mathbf{F}$ gives the direction for which the increase in temperature is greatest. The vectors point toward the origin, where the temperature is greatest. See Figure 16.17. ■

# Line Integrals of Vector Fields

• In previous lecture we defined the **line integral of a scalar function** $f(x, y, z)$ **over a path C • We turn our attention now to the idea of a line integral of a vector field** $F$ **along the curve** $C$.

• Assume that the vector field

$$\mathbf{F}(x, y, z) = M(x, y, z)\mathbf{i} + N(x, y, z)\mathbf{j} + P(x, y, z)\mathbf{k}.$$

• has **continuous components**, and that the curve $C$ has a **smooth parametrization (forward direction)**

$$\mathbf{r}(t) = g(t)\mathbf{i} + h(t)\mathbf{j} + k(t)\mathbf{k}, \quad a \leq t \leq b.$$

• At each point along the path $C$, the **tangent vector** $\mathbf{T} = d\mathbf{r}/ds = \mathbf{v}/|\mathbf{v}|$ is a unit vector tangent to the path and pointing in this forward direction.

# Line Integrals of Vector Fields

• The line integral of the vector field is the line integral of the **scalar tangential component** of **F** along $C$

• This tangential component **is given by the dot product** $\mathbf{F} \cdot \mathbf{T} = \mathbf{F} \cdot d\mathbf{r}/ds$, so

**DEFINITION** Let $\mathbf{F}$ be a vector field with continuous components defined along a smooth curve $C$ parametrized by $\mathbf{r}(t)$, $a \leq t \leq b$. Then the **line integral of F along $C$** is

$$\int_C \mathbf{F} \cdot \mathbf{T}\, ds = \int_C \left( \mathbf{F} \cdot \frac{d\mathbf{r}}{ds} \right) ds = \int_C \mathbf{F} \cdot d\mathbf{r}. \tag{1}$$

# Line Integrals of Vector Fields

We evaluate line integrals of vector fields in a following way

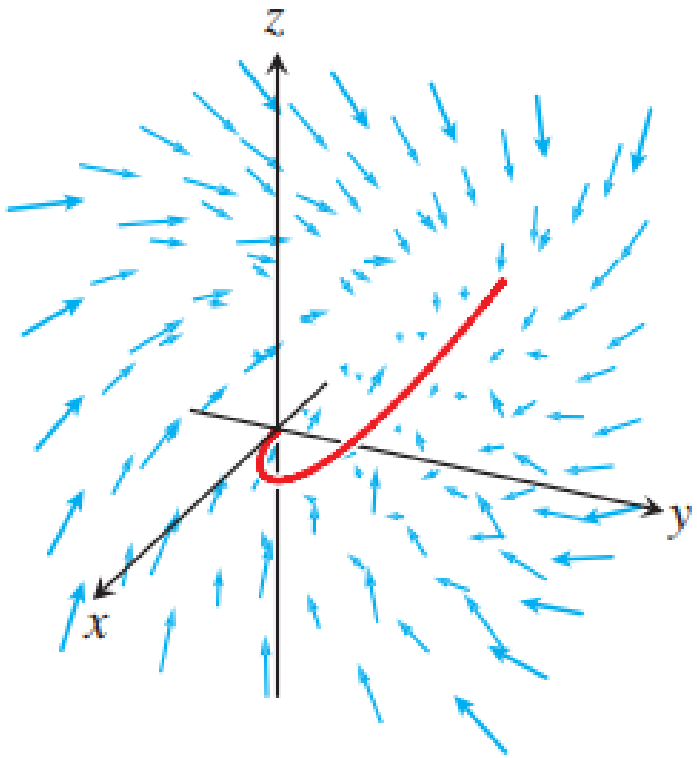
**Evaluating the Line Integral of $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ Along $C$: $\mathbf{r}(t) = g(t)\mathbf{i} + h(t)\mathbf{j} + k(t)\mathbf{k}$**

1. Express the vector field $\mathbf{F}$ along the parametrized curve $C$ as $\mathbf{F}(\mathbf{r}(t))$ by substituting the components $x = g(t)$, $y = h(t)$, $z = k(t)$ of $\mathbf{r}$ into the scalar components $M(x, y, z)$, $N(x, y, z)$, $P(x, y, z)$ of $\mathbf{F}$.
2. Find the derivative (velocity) vector $d\mathbf{r}/dt$.
3. Evaluate the line integral with respect to the parameter $t$, $a \leq t \leq b$, to obtain

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \frac{d\mathbf{r}}{dt}\, dt. \tag{2}$$

## Example 2

Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $F(x, y, z) = z\mathbf{i} + xy\mathbf{j} - y^2\mathbf{k}$ along the curve $C$ given by $\mathbf{r}(t) = t^2\mathbf{i} + t\mathbf{j} + \sqrt{t}\mathbf{k}, 0 \leq t \leq 1$ and shown in Figure.



# Example 2

**Solution** We have

$$\mathbf{F}(\mathbf{r}(t)) = \sqrt{t}\,\mathbf{i} + t^3\mathbf{j} - t^2\mathbf{k} \qquad z = \sqrt{t},\ xy = t^3, -y^2 = -t^2$$

and

$$\frac{d\mathbf{r}}{dt} = 2t\mathbf{i} + \mathbf{j} + \frac{1}{2\sqrt{t}}\mathbf{k}.$$

Thus,

$$\begin{aligned}
\int_C \mathbf{F}\cdot d\mathbf{r} &= \int_0^1 \mathbf{F}(\mathbf{r}(t))\cdot\frac{d\mathbf{r}}{dt}\,dt && \text{Eq. (2)}\\
&= \int_0^1 \left(2t^{3/2} + t^3 - \frac{1}{2}t^{3/2}\right)dt\\
&= \left[\left(\frac{3}{2}\right)\left(\frac{2}{5}t^{5/2}\right) + \frac{1}{4}t^4\right]_0^1 = \frac{17}{20}.
\end{aligned}$$

# Line Integrals with Respect to $dx$, $dy$, or $dz$

- It is often useful to consider **each component direction separately**.
- So we want to **evaluate a line integral** of a scalar function **with respect to only one of the coordinates**, such as $\int_C M\,dx$.
- This type of integral **is not the same** as the arc length line integral $\int_C M\,ds$ we defined, since it picks out displacement in the direction of only one coordinate.
- To define the integral $\int_C M\,dx$ for the scalar function $M(x, y, z)$, we **specify a vector field** $F = M(x, y, z)i$ having a component **only in the** $x$-**direction**, and none in the $y$- or $z$-direction.
- Then, over the curve $C$ parametrized by $\mathbf{r}(t) = g(t)\mathbf{i} + h(t)\mathbf{j} + k(t)\mathbf{k}, \quad a \le t \le b$, we have $x = g(t),\ dx = g'(t)\,dt$, and

$$\mathbf{F}\cdot d\mathbf{r} = \mathbf{F}\cdot\frac{d\mathbf{r}}{dt}\,dt = M(x, y, z)\mathbf{i}\cdot\left(g'(t)\mathbf{i} + h'(t)\mathbf{j} + k'(t)\mathbf{k}\right)dt$$

$$= M(x, y, z)\,g'(t)\,dt = M(x, y, z)\,dx.$$

# Line Integrals with Respect to $dx$, $dy$, or $dz$

Expressing everything in terms of the parameter $t$ along the curve $C$, we have the **following formulas** for three integrals with respect to $dx$, $dy$, and $dz$:

$$\int_C M(x, y, z)\, dx = \int_a^b M(g(t), h(t), k(t))\, g'(t)\, dt$$

$$\int_C N(x, y, z)\, dy = \int_a^b N(g(t), h(t), k(t))\, h'(t)\, dt$$

$$\int_C P(x, y, z)\, dz = \int_a^b P(g(t), h(t), k(t))\, k'(t)\, dt$$

# Line Integrals with Respect to $dx$, $dy$, or $dz$

## Example 3

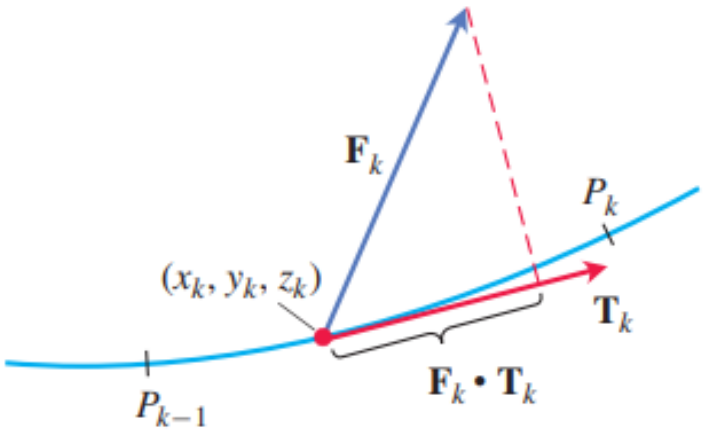
Evaluate the line integral $\int_C (-ydx + zdy + 2x)\, dz$, where $C$ is the helix $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k},\ 0 \le t \le 2\pi$.

**Solution** We express everything in terms of the parameter $t$, so $x = \cos t$, $y = \sin t$, $z = t$, and $dx = -\sin t\, dt$, $dy = \cos t\, dt$, $dz = dt$. Then,

$$\begin{aligned}
\int_C -y\,dx + z\,dy + 2x\,dz &= \int_0^{2\pi} [(-\sin t)(-\sin t) + t\cos t + 2\cos t]\, dt \\
&= \int_0^{2\pi} [2\cos t + t\cos t + \sin^2 t]\, dt \\
&= \left[2\sin t + (t\sin t + \cos t) + \left(\frac{t}{2} - \frac{\sin 2t}{4}\right)\right]_0^{2\pi} \\
&= [0 + (0+1) + (\pi - 0)] - [0 + (0+1) + (0-0)] \\
&= \pi.
\end{aligned}$$

# Work Done by a Force over a Curve in Space

- Suppose that the vector field $F$ represents a **force** throughout a region in space.
- For a curve $C$ in space, **we define the work** done by a continuous force field $F$ **to move an object** along $C$ from a point $A$ to another point $B$ as follows.




$$W \approx \sum_{k=1}^{n} W_k \approx \sum_{k=1}^{n} \mathbf{F}(x_k, y_k, z_k) \cdot \mathbf{T}(x_k, y_k, z_k)\, \Delta s_k.$$

- As $n \to \infty$ and $\Delta s_k \to 0$, these sums approach the line integral

**DEFINITION** Let $C$ be a smooth curve parametrized by $\mathbf{r}(t)$, $a \le t \le b$, and let $\mathbf{F}$ be a continuous force field over a region containing $C$. Then the **work** done in moving an object from the point $A = \mathbf{r}(a)$ to the point $B = \mathbf{r}(b)$ along $C$ is

$$W = \int_C \mathbf{F} \cdot \mathbf{T}\, ds = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \frac{d\mathbf{r}}{dt}\, dt. \qquad (6)$$

# Work Done by a Force over a Curve in Space

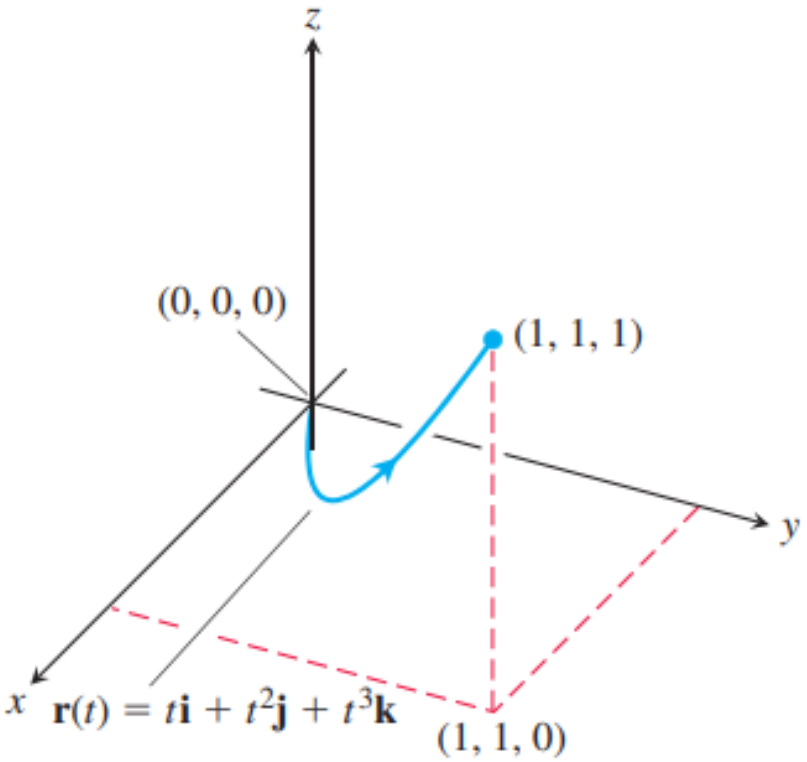
**TABLE 16.2** Different ways to write the work integral for $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ over the curve $C: \mathbf{r}(t) = g(t)\mathbf{i} + h(t)\mathbf{j} + k(t)\mathbf{k}, a \le t \le b$

| | |
|---|---|
| $W = \int_C \mathbf{F} \cdot \mathbf{T}\, ds$ | The definition |
| $= \int_C \mathbf{F} \cdot d\mathbf{r}$ | Vector differential form |
| $= \int_a^b \mathbf{F} \cdot \frac{d\mathbf{r}}{dt}\, dt$ | Parametric vector evaluation |
| $= \int_a^b \big(M g'(t) + N h'(t) + P k'(t)\big)\, dt$ | Parametric scalar evaluation |
| $= \int_C M\, dx + N\, dy + P\, dz$ | Scalar differential form |

# Work Done by a Force over a Curve in Space

## Example 4

Find the work done by the force field
$\mathbf{F} = (y - x^2)\mathbf{i} + (z - y^2)\mathbf{j} + (x - z^2)\mathbf{k}$ in moving an object along the curve $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}, 0 \leq t \leq 1$, from $(0, 0, 0)$ to $(1, 1, 1)$.



# Example 4

**Solution** First we evaluate $\mathbf{F}$ on the curve $\mathbf{r}(t)$:

$$\begin{aligned}\mathbf{F} &= (y - x^2)\mathbf{i} + (z - y^2)\mathbf{j} + (x - z^2)\mathbf{k} \\ &= \underbrace{(t^2 - t^2)}_{0}\mathbf{i} + (t^3 - t^4)\mathbf{j} + (t - t^6)\mathbf{k}. \qquad \text{Substitute } x = t, y = t^2, z = t^3.\end{aligned}$$

Then we find $d\mathbf{r}/dt$,

$$\frac{d\mathbf{r}}{dt} = \frac{d}{dt}(t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}) = \mathbf{i} + 2t\mathbf{j} + 3t^2\mathbf{k}.$$

Finally, we find $\mathbf{F} \cdot d\mathbf{r}/dt$ and integrate from $t = 0$ to $t = 1$:

$$\begin{aligned}\mathbf{F} \cdot \frac{d\mathbf{r}}{dt} &= [\,(t^3 - t^4)\mathbf{j} + (t - t^6)\mathbf{k}\,] \cdot (\mathbf{i} + 2t\mathbf{j} + 3t^2\mathbf{k}) \\ &= (t^3 - t^4)(2t) + (t - t^6)(3t^2) = 2t^4 - 2t^5 + 3t^3 - 3t^8. \qquad \text{Evaluate dot product.}\end{aligned}$$

So,

$$\begin{aligned}\text{Work} &= \int_a^b \mathbf{F} \cdot \frac{d\mathbf{r}}{dt}\,dt = \int_0^1 (2t^4 - 2t^5 + 3t^3 - 3t^8)\,dt \\ &= \left[\frac{2}{5}t^5 - \frac{2}{6}t^6 + \frac{3}{4}t^4 - \frac{3}{9}t^9\right]_0^1 = \frac{29}{60}.\end{aligned}$$

# Flow Integrals and Circulation for Velocity Fields

• Suppose that $F$ represents the **velocity field of a fluid** flowing through a region in space

• Under these circumstances, the integral of $\mathbf{F} \cdot \mathbf{T}$ along a curve in the region gives the **fluid's flow** along, or circulation around, the curve.

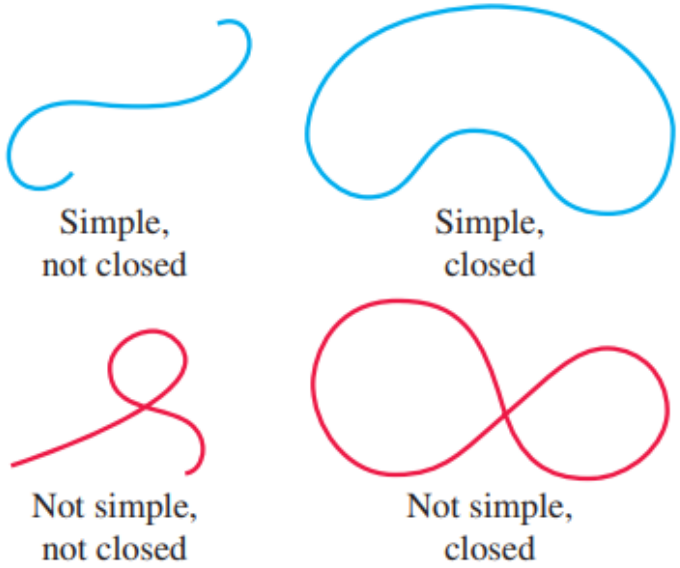
**DEFINITION** If $\mathbf{r}(t)$ parametrizes a smooth curve $C$ in the domain of a continuous velocity field $\mathbf{F}$, the **flow** along the curve from $A = \mathbf{r}(a)$ to $B = \mathbf{r}(b)$ is

$$\text{Flow} = \int_C \mathbf{F} \cdot \mathbf{T}\, ds. \tag{7}$$

The integral is called a **flow integral**. If the curve starts and ends at the same point, so that $A = B$, the flow is called the **circulation** around the curve.

# Flux Across a Simple Closed Plane Curve

- A curve in the $xy$-plane is **simple** if it does not cross itself.
- When a curve starts and ends at the same point, it is a **closed curve** or loop.
- To find the **rate** at which a **fluid is entering or leaving a region** enclosed by a smooth simple closed curve C in the $xy$-plane,




- we calculate the line integral over $C$ of $\mathbf{F} \cdot \mathbf{n}$, the scalar component of the fluid's velocity field in the direction of the curve's outward-pointing normal vector.

**DEFINITION** If $C$ is a smooth simple closed curve in the domain of a continuous vector field $\mathbf{F} = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$ in the plane, and if $\mathbf{n}$ is the outward-pointing unit normal vector on $C$, the **flux** of $\mathbf{F}$ across $C$ is

$$\text{Flux of } \mathbf{F} \text{ across C} = \int_C \mathbf{F} \cdot \mathbf{n}\, ds. \tag{8}$$

# Flux Across a Simple Closed Plane Curve

**Calculating Flux Across a Smooth Closed Plane Curve**

$$\text{Flux of } \mathbf{F} = M\mathbf{i} + N\mathbf{j} \text{ across } C = \oint_C M\,dy - N\,dx \tag{9}$$

The integral can be evaluated from any smooth parametrization $x = g(t)$, $y = h(t)$, $a \le t \le b$, that traces $C$ counterclockwise exactly once.

# Lecture 28: Integrals and Vector Fields.

MA2032 Vector Calculus

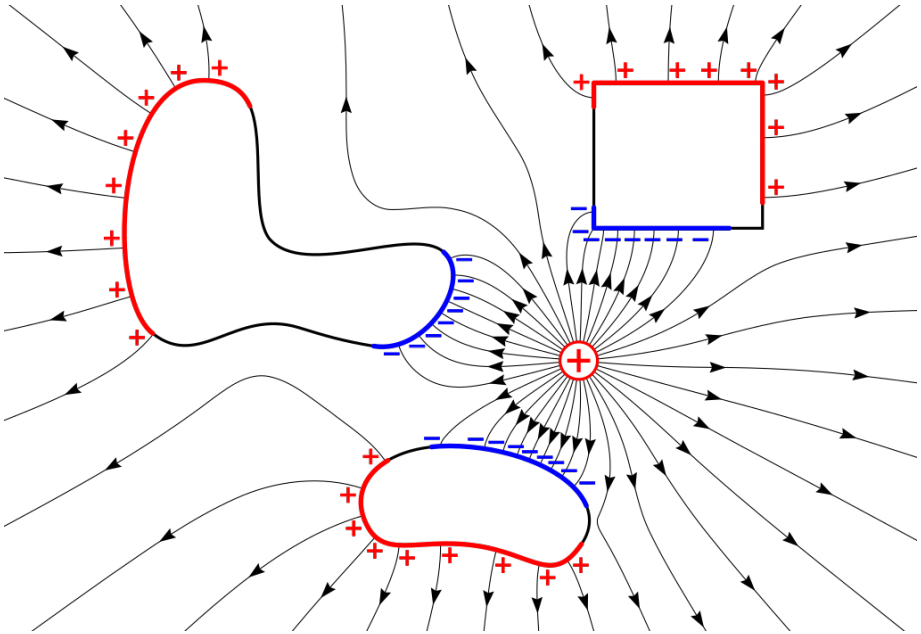
Lecturer: Larissa Serdukova

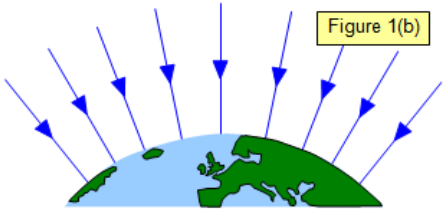
School of Computing and Mathematical Science
University of Leicester

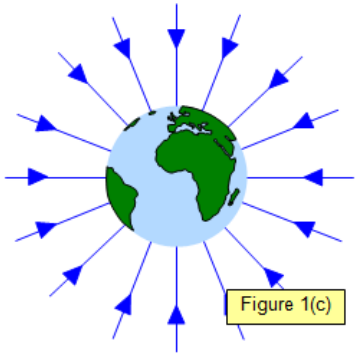
November 23, 2022

# Gravitational Field. Electric Field

• A **gravitational field** $G$ is a vector field that represents the **effect of gravity** at a point in space due to the **presence of a massive object**. The **gravitational force** on a body of mass m placed in the field is given by $F = mG$.






• Similarly, an **electric field** $E$ is a vector field in space that represents the **effect of electric forces** on a **charged particle** placed within it.

• The **force on a body of charge** q placed in the field is given by $F = qE$.

# Gravitational Field. Electric Field. Path Independence

• In gravitational and electric fields, the amount of **work it takes to move a mass or charge** from one point to another **depends on the initial and final positions** of the object — **not on which path is taken** between these positions.

• We study vector fields with this **independence-of-path property** and the calculation of **work integrals** associated with them.

• If A and B are two points in an open region D in space, the **line integral** of F along C from A to B for a field **F** defined on D **usually depends on the path C** taken, as we saw in the previous lectures.

• For some **special fields**, however, the **integral's value is the same for all paths from A to B**.

# Path Independence

**DEFINITIONS** Let $\mathbf{F}$ be a vector field defined on an open region $D$ in space, and suppose that for any two points $A$ and $B$ in $D$ the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ along a path $C$ from $A$ to $B$ in $D$ is the same over all paths from $A$ to $B$. Then the integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ is **path independent in $D$** and the field $\mathbf{F}$ is **conservative on $D$**.

- The **word conservative** comes **from physics**, where it refers to fields in which the **principle of conservation of energy holds**.

- Under reasonable differentiability conditions that we will specify, we will show that a field **F is conservative if and only if it is the gradient field of a scalar function** $f$ — that is, if and only if $F = \nabla f$ for some $f$.

- The function $f$ then has a **special name**.

**DEFINITION** If $\mathbf{F}$ is a vector field defined on $D$ and $\mathbf{F} = \nabla f$ for some scalar function $f$ on $D$, then $f$ is called a **potential function for $\mathbf{F}$**.

# Assumptions on Curves, Vector Fields, and Domains

• Once we have found a **potential function** $f$ for a field **F**, we **can evaluate all the line integrals** in the domain of **F over any path between** $A$ **and** $B$ by

$$\int_A^B \mathbf{F} \cdot d\mathbf{r} = \int_A^B \nabla f \cdot d\mathbf{r} = f(B) - f(A).$$

• The curves we consider are **piecewise smooth**.

• We consider vector fields **F** whose components have **continuous first partial derivatives**.

• The domains D we consider are **connected**: this means that any two points in D can be joined by a smooth curve that lies in the region. Being in **"one piece"**

• Some results require D to be **simply connected**, which means that every loop in D can be contracted to a point in D without ever leaving D. **Not having any "loop-catching holes."**

# Line Integrals in Conservative Fields

• Like the Fundamental Theorem of Calculus, Theorem 1 gives a **direct way to evaluate line integrals**

**THEOREM 1—Fundamental Theorem of Line Integrals**
Let $C$ be a smooth curve joining the point $A$ to the point $B$ in the plane or in space and parametrized by $\mathbf{r}(t)$. Let $f$ be a differentiable function with a continuous gradient vector $\mathbf{F} = \nabla f$ on a domain $D$ containing $C$. Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(B) - f(A).$$

# Proff of Theorem 1

**Proof of Theorem 1** Suppose that $A$ and $B$ are two points in the region $D$ and that $C$: $\mathbf{r}(t) = g(t)\mathbf{i} + h(t)\mathbf{j} + k(t)\mathbf{k}$, $a \le t \le b$, is a smooth curve in $D$ joining $A$ to $B$. In Section 14.5 we found that the derivative of a scalar function $f$ along a path $C$ is the dot product $\nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t)$, so we have

$$\begin{aligned}
\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_A^B \nabla f \cdot d\mathbf{r} && \mathbf{F} = \nabla f \\
&= \int_{t=a}^{t=b} \nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt && \text{Eq. (2) of Section 16.2 for computing } d\mathbf{r} \\
&= \int_a^b \frac{d}{dt} f(\mathbf{r}(t))\, dt && \text{Eq. (7) of Section 14.5 giving derivative along a path} \\
&= f(\mathbf{r}(b)) - f(\mathbf{r}(a)) && \text{Fundamental Theorem of Calculus} \\
&= f(B) - f(A). && \mathbf{r}(a) = A, \mathbf{r}(b) = B
\end{aligned}$$

■

# Conservative Fields

**THEOREM 2—Conservative Fields are Gradient Fields**
Let $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ be a vector field whose components are continuous throughout an open connected region $D$ in space. Then $\mathbf{F}$ is conservative if and only if $\mathbf{F}$ is a gradient field $\nabla f$ for a differentiable function $f$.

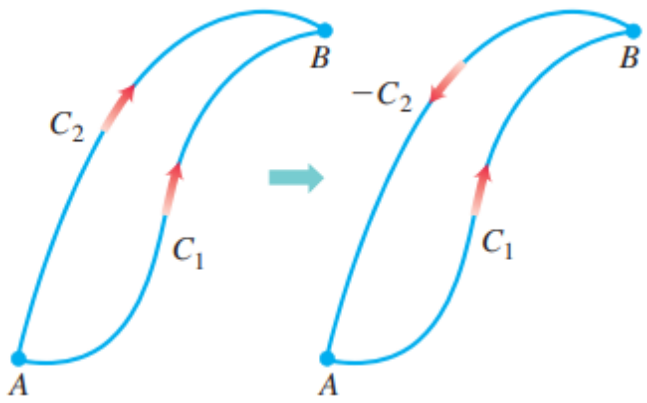
**THEOREM 3—Loop Property of Conservative Fields**
The following statements are equivalent.

1. $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ around every loop (that is, closed curve $C$) in $D$.
2. The field $\mathbf{F}$ is conservative on $D$.

- The following diagram **summarizes the results of Theorems 2 and 3**.

$$\mathbf{F} = \nabla f \text{ on D} \quad \overset{\text{Theorem 2}}{\Longleftrightarrow} \quad \mathbf{F} \text{ conservative on D} \quad \overset{\text{Theorem 3}}{\Longleftrightarrow} \quad \oint_C \mathbf{F} \cdot d\mathbf{r} = 0 \text{ over any loop in D}$$

# Proof of Theorem 3

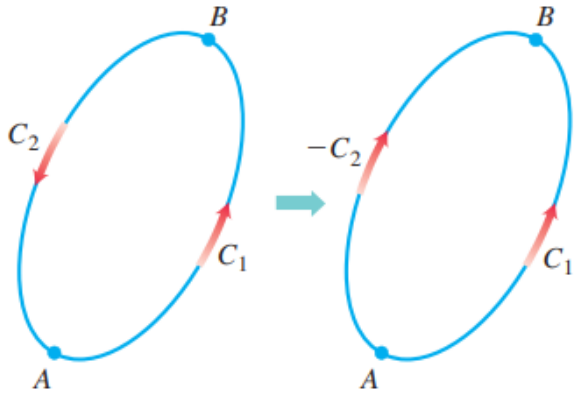



**Proof that Part 1 ⇒ Part 2** We want to show that for any two points $A$ and $B$ in $D$, the integral of $\mathbf{F} \cdot d\mathbf{r}$ has the same value over any two paths $C_1$ and $C_2$ from $A$ to $B$. We reverse the direction on $C_2$ to make a path $-C_2$ from $B$ to $A$ (Figure 16.27). Together, $C_1$ and $-C_2$ make a closed loop $C$, and by assumption,

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} - \int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{-C_2} \mathbf{F} \cdot d\mathbf{r} = \int_{C} \mathbf{F} \cdot d\mathbf{r} = 0.$$

Thus, the integrals over $C_1$ and $C_2$ give the same value. Note that the definition of $\mathbf{F} \cdot d\mathbf{r}$ shows that changing the direction along a curve reverses the sign of the line integral.

# Proof of Theorem 3




**Proof that Part 2 ⇒ Part 1** We want to show that the integral of $\mathbf{F} \cdot d\mathbf{r}$ is zero over any closed loop $C$. We pick two points $A$ and $B$ on $C$ and use them to break $C$ into two pieces: $C_1$ from $A$ to $B$ followed by $C_2$ from $B$ back to $A$ (Figure 16.28). Then

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_A^B \mathbf{F} \cdot d\mathbf{r} - \int_A^B \mathbf{F} \cdot d\mathbf{r} = 0. \qquad \blacksquare$$

**EXAMPLE 2** Find the work done by the conservative field

$$\mathbf{F} = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k} = \nabla f, \quad \text{where} \quad f(x, y, z) = xyz,$$

in moving an object along any smooth curve $C$ joining the point $A(-1, 3, 9)$ to $B(1, 6, -4)$.

**Solution** With $f(x, y, z) = xyz$, we have

$$\begin{aligned}
\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_A^B \nabla f \cdot d\mathbf{r} && \mathbf{F} = \nabla f \text{ and path independence} \\
&= f(B) - f(A) && \text{Theorem 1} \\
&= xyz\big|_{(1,6,-4)} - xyz\big|_{(-1,3,9)} \\
&= (1)(6)(-4) - (-1)(3)(9) \\
&= -24 + 27 = 3.
\end{aligned}$$

# Conservative Fields

**Two questions arise**:

1. **How do we know** whether a given vector field **F** is **conservative**?

2. If F is in fact conservative, **how do we find** a **potential function** $f$ (so that $\mathbf{F} = \nabla f$)?

# Finding Potentials for Conservative Fields

**Component Test for Conservative Fields**

Let $\mathbf{F} = M(x, y, z)\mathbf{i} + N(x, y, z)\mathbf{j} + P(x, y, z)\mathbf{k}$ be a field on an open simply connected domain whose component functions have continuous first partial derivatives. Then, $\mathbf{F}$ is conservative if and only if

$$\frac{\partial P}{\partial y} = \frac{\partial N}{\partial z}, \qquad \frac{\partial M}{\partial z} = \frac{\partial P}{\partial x}, \qquad \text{and} \qquad \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}. \tag{2}$$

- We can view the component test as saying that on a simply connected region, the **vector**

$$\left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)\mathbf{i} + \left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\right)\mathbf{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mathbf{k}$$

- **is zero if and only if F is conservative**.
- This interesting vector is called the **curl of F**

# Finding Potentials for Conservative Fields

- Once we know that **F** is conservative, we often want to **find a potential function** for **F**.

- This requires **solving the equation** $\nabla f = F$ or

$$\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$$

- for $f$.
- We accomplish this by **integrating the three equations**

$$\frac{\partial f}{\partial x} = M, \qquad \frac{\partial f}{\partial y} = N, \qquad \frac{\partial f}{\partial z} = P,$$

# Finding Potentials for Conservative Fields

**EXAMPLE 3** Show that $\mathbf{F} = (e^x \cos y + yz)\mathbf{i} + (xz - e^x \sin y)\mathbf{j} + (xy + z)\mathbf{k}$ is conservative over its natural domain and find a potential function for it.

**Solution** The natural domain of $\mathbf{F}$ is all of space, which is open and simply connected. We apply the test in Equations (2) to

$$M = e^x \cos y + yz, \qquad N = xz - e^x \sin y, \qquad P = xy + z$$

and calculate

$$\frac{\partial P}{\partial y} = x = \frac{\partial N}{\partial z}, \qquad \frac{\partial M}{\partial z} = y = \frac{\partial P}{\partial x}, \qquad \frac{\partial N}{\partial x} = -e^x \sin y + z = \frac{\partial M}{\partial y}.$$

The partial derivatives are continuous, so these equalities tell us that $\mathbf{F}$ is conservative, so there is a function $f$ with $\nabla f = \mathbf{F}$ (Theorem 2).

We find $f$ by integrating the equations

$$\frac{\partial f}{\partial x} = e^x \cos y + yz, \qquad \frac{\partial f}{\partial y} = xz - e^x \sin y, \qquad \frac{\partial f}{\partial z} = xy + z. \tag{3}$$

We integrate the first equation with respect to $x$, holding $y$ and $z$ fixed, to get

$$f(x, y, z) = e^x \cos y + xyz + g(y, z).$$

# Finding Potentials for Conservative Fields

We write the constant of integration as a function of $y$ and $z$ because its value may depend on $y$ and $z$, though not on $x$. We then calculate $\partial f/\partial y$ from this equation and match it with the expression for $\partial f/\partial y$ in Equations (3). This gives

$$-e^x \sin y + xz + \frac{\partial g}{\partial y} = xz - e^x \sin y,$$

so $\partial g/\partial y = 0$. Therefore, $g$ is a function of $z$ alone, and

$$f(x, y, z) = e^x \cos y + xyz + h(z).$$

We now calculate $\partial f/\partial z$ from this equation and match it to the formula for $\partial f/\partial z$ in Equations (3). This gives

$$xy + \frac{dh}{dz} = xy + z, \qquad \text{or} \qquad \frac{dh}{dz} = z,$$

so

$$h(z) = \frac{z^2}{2} + C.$$

Hence,

$$f(x, y, z) = e^x \cos y + xyz + \frac{z^2}{2} + C.$$

We found infinitely many potential functions of $\mathbf{F}$, one for each value of $C$. ■

# Lecture 29: Integrals and Vector Fields.

MA2032 Vector Calculus

Lecturer: Larissa Serdukova

School of Computing and Mathematical Science
University of Leicester

November 25, 2022

# Green's Theorem in the Plane

• If $F$ is a **conservative field**, then we know $F = \nabla f$ for a differentiable function $f$, and we can calculate the **line integral** of $F$ over any path $C$ joining point $A$ to point $B$ as $\int_C \mathbf{F} \cdot d\mathbf{r} = f(B) - f(A)$.

• Today we derive a **method for computing a work or flux integral** over a closed curve C in the plane when the **field F is not conservative**.

• This method comes from **Green's Theorem**, which allows us to **convert the line integral into a double integral** over the region enclosed by C.

• The discussion is given in terms of **velocity fields of fluid flows** (a fluid is a liquid or a gas) because they are easy to visualize.

• However, Green's Theorem **applies to any vector field**, provided the assumptions of the theorem are satisfied.

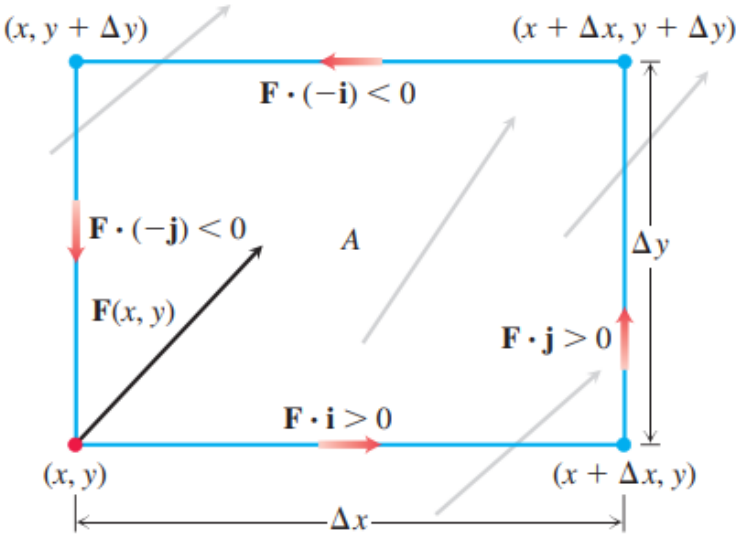
• We introduce two new ideas for Green's Theorem: **circulation density** around an axis perpendicular to the plane and **divergence** (or flux density).

# Spin Around an Axis: The k-Component of Curl

• Suppose that $\mathbf{F}(x, y) = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$ is the **velocity field** of a fluid flowing in the plane.

• The first partial derivatives of M and N are **continuous** at each point of a region R

• Let A be a **small rectangle** lies entirely in R.




• The first idea we use to convey **Green's Theorem** quantifies **the rate at which** a floating paddle wheel, with axis perpendicular to the plane, **spins at a point** in a fluid flowing in a plane region.

• Physicists sometimes refer to this as the **circulation density of a vector field F** at a point.

# Green's Theorem in the Plane

• To approximate the **rate of circulation** at the point $(x, y)$, we calculate the (approximate) **flow rates along each edge** in the directions of the red arrows, sum these rates, and then divide the sum by the area of $A$.

$$\begin{aligned}
&\text{Top:} && \mathbf{F}(x, y + \Delta y) \cdot (-\mathbf{i})\Delta x = -M(x, y + \Delta y)\,\Delta x \\
&\text{Bottom:} && \mathbf{F}(x, y) \cdot \mathbf{i}\,\Delta x = M(x, y)\Delta x \\
&\text{Right:} && \mathbf{F}(x + \Delta x, y) \cdot \mathbf{j}\,\Delta Y = N(x + \Delta x, y)\Delta y \\
&\text{Left:} && \mathbf{F}(x, y) \cdot (-\mathbf{j})\Delta y = -N(x, y)\Delta y
\end{aligned}$$

$$\text{Top and bottom:} \qquad -\big(M(x, y + \Delta y) - M(x, y)\big)\,\Delta x \approx -\left(\frac{\partial M}{\partial y}\Delta y\right)\Delta x$$

$$\text{Right and left:} \qquad \big(N(x + \Delta x, y) - N(x, y)\big)\,\Delta y \approx \left(\frac{\partial N}{\partial x}\Delta x\right)\Delta y.$$

# Green's Theorem in the Plane

• **Taking the limit** as $\Delta x \to 0$ and $\Delta y \to 0$ gives the **rate of the circulation per unit area**.

$$\text{Circulation rate around rectangle} \approx \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\Delta x\,\Delta y.$$

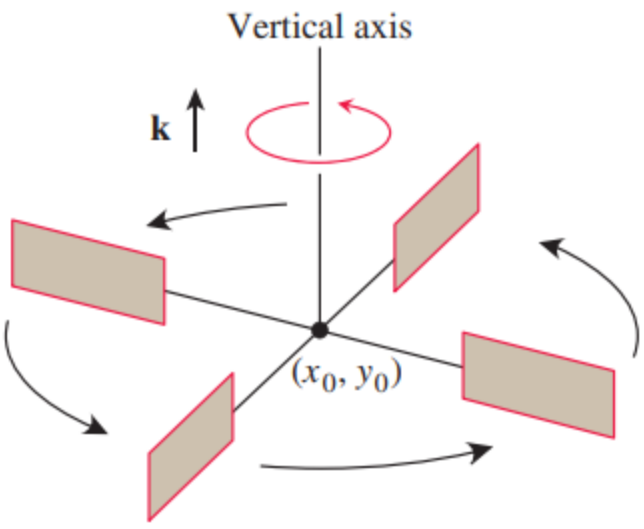
$$\frac{\text{Circulation around rectangle}}{\text{rectangle area}} \approx \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}.$$

# Green's Theorem in the Plane

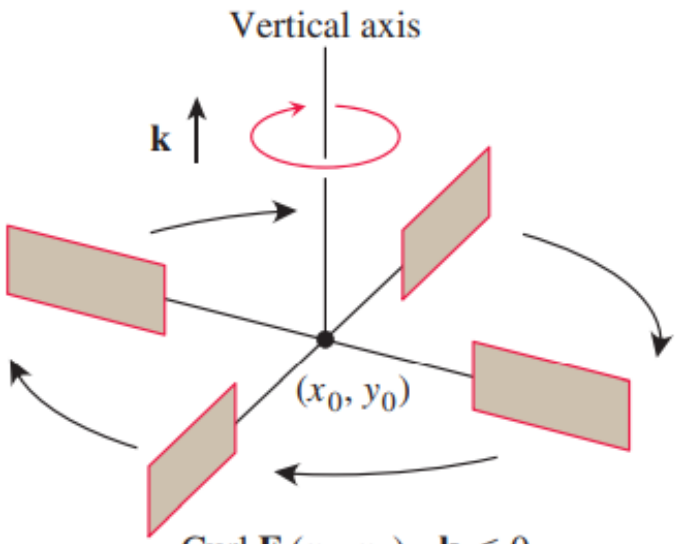
**DEFINITION** The **circulation density** of a vector field $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$ at the point $(x, y)$ is the scalar expression

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}. \tag{1}$$

This expression is also called **the k-component of the curl**, denoted by $(\text{curl } \mathbf{F}) \cdot \mathbf{k}$.




Curl $\mathbf{F}$ $(x_0, y_0) \cdot \mathbf{k} > 0$
Counterclockwise circulation




Curl $\mathbf{F}$ $(x_0, y_0) \cdot \mathbf{k} < 0$
Clockwise circulation

# Green's Theorem in the Plane

**EXAMPLE 1** The following vector fields represent the velocity of a gas flowing in the $xy$-plane. Find the circulation density of each vector field and interpret its physical meaning. Figure 16.31 displays the vector fields.

**(a)** *Uniform expansion or compression:* $\mathbf{F}(x, y) = cx\mathbf{i} + cy\mathbf{j}$ $c$ a constant

**(b)** *Uniform rotation:* $\mathbf{F}(x, y) = -cy\mathbf{i} + cx\mathbf{j}$

**(c)** *Shearing flow:* $\mathbf{F}(x, y) = y\mathbf{i}$

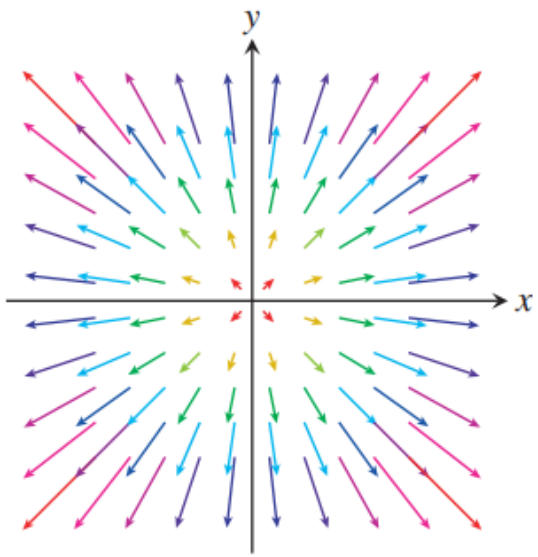
**(d)** *Whirlpool effect:* $\mathbf{F}(x, y) = \dfrac{-y}{x^2 + y^2}\mathbf{i} + \dfrac{x}{x^2 + y^2}\mathbf{j}$

# Solution Example 1

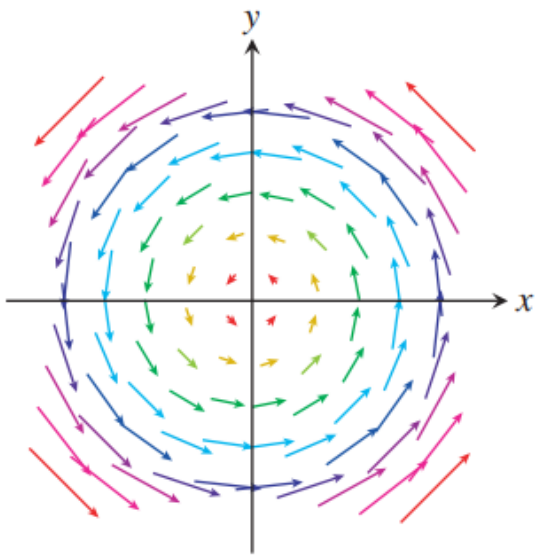
## Solution

**(a)** *Uniform expansion:* $(\text{curl } \mathbf{F}) \cdot \mathbf{k} = \frac{\partial}{\partial x}(cy) - \frac{\partial}{\partial y}(cx) = 0$. The gas is not circulating at very small scales.

**(b)** *Rotation:* $(\text{curl } \mathbf{F}) \cdot \mathbf{k} = \frac{\partial}{\partial x}(cx) - \frac{\partial}{\partial y}(-cy) = 2c$. The constant circulation density indicates rotation around every point. If $c > 0$, the rotation is counterclockwise; if $c < 0$, the rotation is clockwise.




(a)




(b)

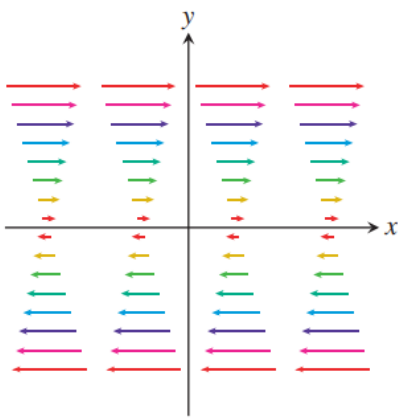
# Solution Example 1

**(c)** *Shear:* $(\text{curl } \mathbf{F}) \cdot \mathbf{k} = -\frac{\partial}{\partial y}(y) = -1$. The circulation density is constant and negative, so a paddle wheel floating in water undergoing such a shearing flow spins clockwise. The rate of rotation is the same at each point. The average rotational effect of the fluid flow is to push fluid clockwise around each of the small circles shown in Figure 16.32.

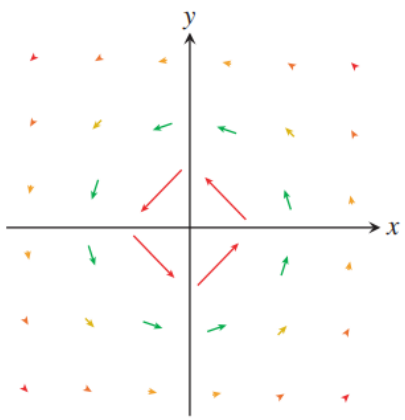
**(d)** *Whirlpool:*

$$(\text{curl } \mathbf{F}) \cdot \mathbf{k} = \frac{\partial}{\partial x}\left(\frac{x}{x^2 + y^2}\right) - \frac{\partial}{\partial y}\left(\frac{-y}{x^2 + y^2}\right) = \frac{y^2 - x^2}{(x^2 + y^2)^2} - \frac{y^2 - x^2}{(x^2 + y^2)^2} = 0.$$

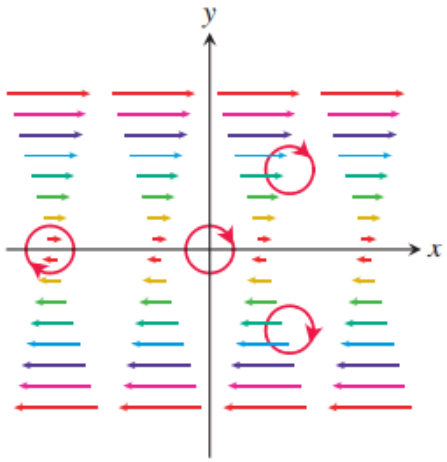
The circulation density is 0 at every point away from the origin (where the vector field is undefined and the whirlpool effect is taking place), and the gas is not circulating at any point for which the vector field is defined. ■




(c)



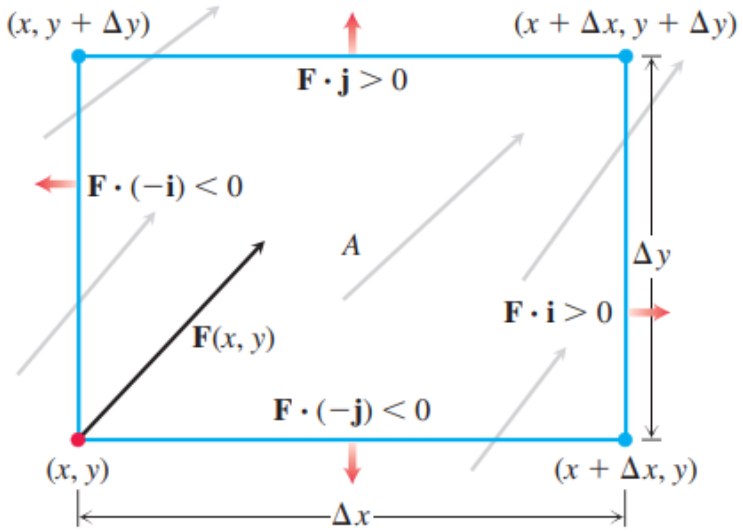

(d)



# Divergence

• Consider again the **velocity field** $\mathbf{F}(x, y) = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$ in a domain containing the **rectangle A**, as shown in Figure.

• As before, we assume the field components **do not change sign** throughout a small region containing the rectangle A.




• Our interest now is to determine **the rate at which the fluid leaves A** by flowing across its boundary.

• To approximate the flow rate at the point $(x, y)$, we calculate the (approximate) **flow rates across each edge** in the directions of the red arrows, **sum these rates**, and then **divide the sum by the area of A**.

# Divergence

| | | |
|---|---|---|
| **Fluid Flow Rates:** | Top: | $\mathbf{F}(x, y + \Delta y) \cdot \mathbf{j}\, \Delta x = N(x, y + \Delta y)\, \Delta x$ |
| | Bottom: | $\mathbf{F}(x, y) \cdot (-\mathbf{j}) \Delta x = -N(x, y)\, \Delta x$ |
| | Right: | $\mathbf{F}(x + \Delta x, y) \cdot \mathbf{i}\, \Delta y = M(x + \Delta x, y)\, \Delta y$ |
| | Left: | $\mathbf{F}(x, y) \cdot (-\mathbf{i}) \Delta y = -M(x, y)\, \Delta y$ |

Summing opposite pairs gives

$$\text{Top and bottom:} \qquad (N(x, y + \Delta y) - N(x, y))\, \Delta x \approx \left(\frac{\partial N}{\partial y} \Delta y\right) \Delta x$$

$$\text{Right and left:} \qquad (M(x + \Delta x, y) - M(x, y))\, \Delta y \approx \left(\frac{\partial M}{\partial x} \Delta x\right) \Delta y.$$

$$\text{Flux across rectangle boundary} \approx \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) \Delta x\, \Delta y.$$

$$\frac{\text{Flux across rectangle boundary}}{\text{rectangle area}} \approx \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right).$$

# Divergence

• **Taking the limit** as $\Delta x \to 0$ and $\Delta y \to 0$ gives the **flow rate per unit area**.

**DEFINITION** The **divergence (flux density)** of a vector field $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$ at the point $(x, y)$ is

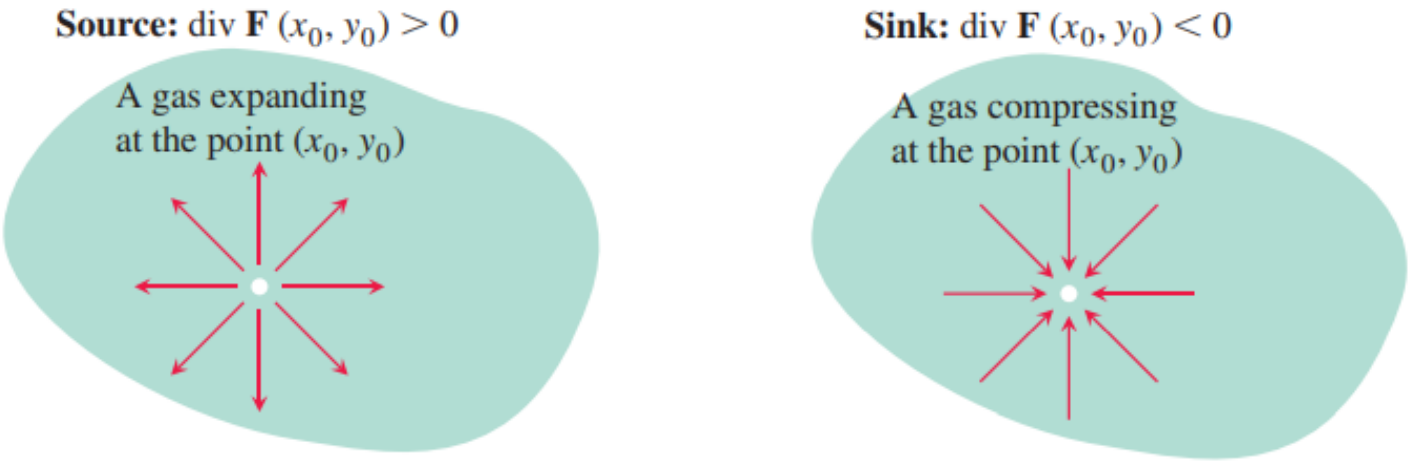
$$\operatorname{div} \mathbf{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}. \tag{2}$$

# Divergence

• A gas is **compressible**, unlike a liquid, and the **divergence of its velocity field** measures to what **extent it is expanding or compressing** at each point.

• Intuitively, if a gas is **expanding** at the point $(x_0, y_0)$, the lines of flow would diverge there (hence the name) and, since the gas would be flowing out of a small rectangle about $(x_0, y_0)$, the divergence of F at $(x_0, y_0)$ would be **positive**.

• If the gas were **compressing** instead of expanding, the divergence would be **negative**.



# Divergence

**EXAMPLE 2** Find the divergence, and interpret what it means, for each vector field in Example 1 representing the velocity of a gas flowing in the $xy$-plane.

**Solution**

**(a)** $\operatorname{div} \mathbf{F} = \frac{\partial}{\partial x}(cx) + \frac{\partial}{\partial y}(cy) = 2c$: If $c > 0$, the gas is undergoing uniform expansion; if $c < 0$, it is undergoing uniform compression.

**(b)** $\operatorname{div} \mathbf{F} = \frac{\partial}{\partial x}(-cy) + \frac{\partial}{\partial y}(cx) = 0$: The gas is neither expanding nor compressing.

**(c)** $\operatorname{div} \mathbf{F} = \frac{\partial}{\partial x}(y) = 0$: The gas is neither expanding nor compressing.

**(d)** $\operatorname{div} \mathbf{F} = \frac{\partial}{\partial x}\left(\frac{-y}{x^2 + y^2}\right) + \frac{\partial}{\partial y}\left(\frac{x}{x^2 + y^2}\right) = \frac{2xy}{(x^2 + y^2)^2} - \frac{2xy}{(x^2 + y^2)^2} = 0$: Again, the divergence is zero at all points in the domain of the velocity field. ■

# Lecture 30: Integrals and Vector Fields.

MA2032 Vector Calculus

Lecturer: Larissa Serdukova

School of Computing and Mathematical Science
University of Leicester

November 28, 2022

# Two Forms for Green's Theorem

• A **simple closed curve C** can be traversed in **two possible directions**. (Recall that a curve is simple if it does not cross itself.)
• The curve is **traversed counterclockwise**, and said to be T**positively oriented**, if the region it encloses is always to the **left** when moving along the curve.
• If the curve is **traversed clockwise** then the enclosed region is on the **right** when moving along the curve and the curve is said to be **negatively oriented**.
• The line integral of a vector field F along C **reverses sign if we change the orientation**.
• We use the notation

$$\oint_C \mathbf{F}(x, y) \cdot d\mathbf{r}$$

• for the line integral when the simple closed curve C is traversed **counterclockwise**, with its **positive orientation**.

# Two Forms for Green's Theorem

- In **one form, Green's Theorem** says that the **counterclockwise circulation** of a vector field around a simple closed curve is the double integral of the **k-component of the curl** of the field over the region enclosed by the curve.

**Circulation and Curl**

$$\text{Circulation around } C = \oint_C \mathbf{F} \cdot \mathbf{T}\, ds$$

$$(\text{curl } \mathbf{F}) \cdot \mathbf{k} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$$

**THEOREM 4—Green's Theorem (Circulation-Curl or Tangential Form)**
Let $C$ be a piecewise smooth, simple closed curve enclosing a region $R$ in the plane. Let $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$ be a vector field with $M$ and $N$ having continuous first partial derivatives in an open region containing $R$. Then the counterclockwise circulation of $\mathbf{F}$ around C equals the double integral of $(\text{curl } \mathbf{F}) \cdot \mathbf{k}$ over $R$.

$$\underbrace{\oint_C \mathbf{F} \cdot \mathbf{T}\, ds}_{\text{Counterclockwise circulation}} = \oint_C M\, dx + N\, dy = \underbrace{\iint_R \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\, dy}_{\text{Curl integral}} \qquad (3)$$

# Two Forms for Green's Theorem

• A **second form of Green's Theorem** says that the **outward flux** of a vector field across a simple closed curve in the plane equals the **double integral of the divergence** of the field over the region enclosed by the curve.

**Flux and Divergence**

$$\text{Flux of } \mathbf{F} \text{ across } C = \oint_C \mathbf{F} \cdot n \, ds$$

$$\text{div } \mathbf{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}$$

**THEOREM 5—Green's Theorem (Flux-Divergence or Normal Form)**

Let $C$ be a piecewise smooth, simple closed curve enclosing a region $R$ in the plane. Let $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$ be a vector field with $M$ and $N$ having continuous first partial derivatives in an open region containing $R$. Then the outward flux of $\mathbf{F}$ across $C$ equals the double integral of div $\mathbf{F}$ over the region $R$ enclosed by $C$.

$$\underbrace{\oint_C \mathbf{F} \cdot \mathbf{n} \, ds}_{\text{Outward flux}} = \oint_C M \, dy - N \, dx = \underbrace{\iint_R \left( \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dx \, dy}_{\text{Divergence integral}} \qquad (4)$$

# Two Forms for Green's Theorem

• The two forms of Green's Theorem are **equivalent**.

• Applying Equation (3) to the field $G_1 = -N_i + M_j$ gives Equation (4), and applying Equation (4) to $G_2 = N_i - M_j$ gives Equation (3).

• Both forms of Green's Theorem can be viewed as **two-dimensional generalizations** of the Fundamental Theorem of Calculus.

• The **counterclockwise circulation of F around C**, defined by the **line integral** on the left-hand side of Equation (3), is the integral of its **rate of change (circulation density)** over the region R enclosed by C, which is the double integral on the right-hand side of Equation (3).

• Likewise, the **outward flux of F across C**, defined by the **line integral** on the left-hand side of Equation (4), is the integral of its **rate of change (flux density)** over the region R enclosed by C, which is the double integral on the right-hand side of Equation (4).

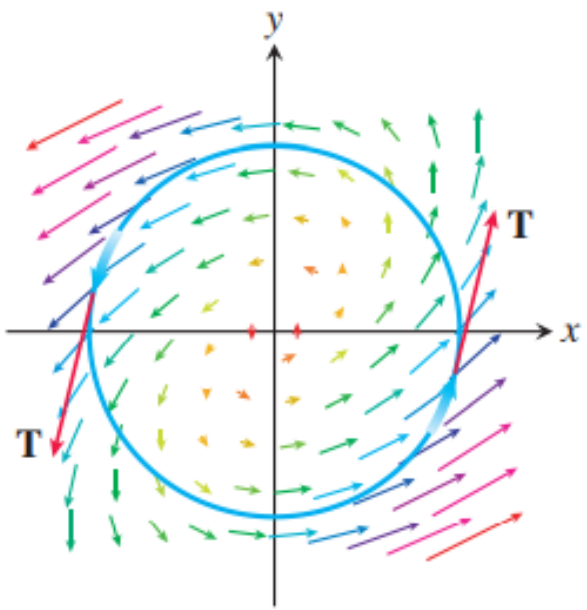
# Two Forms for Green's Theorem

**EXAMPLE 3** Verify both forms of Green's Theorem for the vector field

$$\mathbf{F}(x, y) = (x - y)\mathbf{i} + x\mathbf{j}$$

and the region $R$ bounded by the unit circle

$$C: \quad \mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j}, \quad 0 \le t \le 2\pi.$$



# Two Forms for Green's Theorem

**Solution** First we evaluate the counterclockwise circulation of $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$ around $C$. On the curve $C$ we have $x = \cos t$ and $y = \sin t$. Evaluating $\mathbf{F}(\mathbf{r}(t))$ and computing the partial derivatives of the components of $\mathbf{F}$, we have

$$M = x - y = \cos t - \sin t, \qquad dx = d(\cos t) = -\sin t\, dt,$$

$$N = x = \cos t, \qquad dy = d(\sin t) = \cos t\, dt.$$

Therefore,

$$\oint_C \mathbf{F}\cdot\mathbf{T}\, ds = \oint_C M\, dx + N\, dy$$

$$= \int_{t=0}^{t=2\pi} (\cos t - \sin t)(-\sin t)\, dt + (\cos t)(\cos t)\, dt$$

$$= \int_0^{2\pi} (-\sin t \cos t + 1)\, dt = 2\pi.$$

This gives the left side of Equation (3). Next we find the curl integral, the right side of Equation (3). Since $M = x - y$ and $N = x$, we have

$$\frac{\partial M}{\partial x} = 1, \qquad \frac{\partial M}{\partial y} = -1, \qquad \frac{\partial N}{\partial x} = 1, \qquad \frac{\partial N}{\partial y} = 0.$$

# Two Forms for Green's Theorem

Therefore,

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{dy}\right) dx\, dy = \iint_R (1 - (-1))\, dx\, dy$$

$$= 2 \iint_R dx\, dy = 2(\text{area inside the unit circle}) = 2\pi.$$

Thus, the right and left sides of Equation (3) both equal $2\pi$, as asserted by the circulation-flux version of Green's Theorem.

Figure 16.35 displays the vector field and circulation around $C$.

Now we compute the two sides of Equation (4) in the flux-divergence form of Green's Theorem, starting with the outward flux:

$$\oint_C M\, dy - N\, dx = \int_{t=0}^{t=2\pi} (\cos t - \sin t)(\cos t\, dt) - (\cos t)(-\sin t\, dt)$$

$$= \int_0^{2\pi} \cos^2 t\, dt = \pi.$$

Next we compute the divergence integral:

$$\iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx\, dy = \iint_R (1 + 0)\, dx\, dy = \iint_R dx\, dy = \pi.$$

Hence the right and left sides of Equation (4) both equal $\pi$, as asserted by the flux-divergence version of Green's Theorem. ■

# Using Green's Theorem to Evaluate Line Integrals

**EXAMPLE 4** Evaluate the line integral

$$\oint_C xy\,dy - y^2\,dx,$$

where $C$ is the square cut from the first quadrant by the lines $x = 1$ and $y = 1$.

**Solution** We can use either form of Green's Theorem to change the line integral into a double integral over the square, where $C$ is the square's boundary and $R$ is its interior.

**1.** *With the Tangential Form* Equation (3): Taking $M = -y^2$ and $N = xy$ gives the result:

$$\oint_C -y^2\,dx + xy\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \iint_R (y - (-2y))\,dx\,dy$$

$$= \int_0^1 \int_0^1 3y\,dx\,dy = \int_0^1 \Big[3xy\Big]_{x=0}^{x=1} dy = \int_0^1 3y\,dy = \frac{3}{2}y^2\Big]_0^1 = \frac{3}{2}.$$

**2.** *With the Normal Form* Equation (4): Taking $M = xy$, $N = y^2$, gives the same result:

$$\oint_C xy\,dy - y^2\,dx = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx\,dy = \iint_R (y + 2y)\,dx\,dy = \frac{3}{2}.$$

# Using Green's Theorem to Evaluate Line Integrals

**EXAMPLE 5** Calculate the outward flux of the vector field $\mathbf{F}(x, y) = 2e^{xy}\mathbf{i} + y^3\mathbf{j}$ across the square bounded by the lines $x = \pm 1$ and $y = \pm 1$.

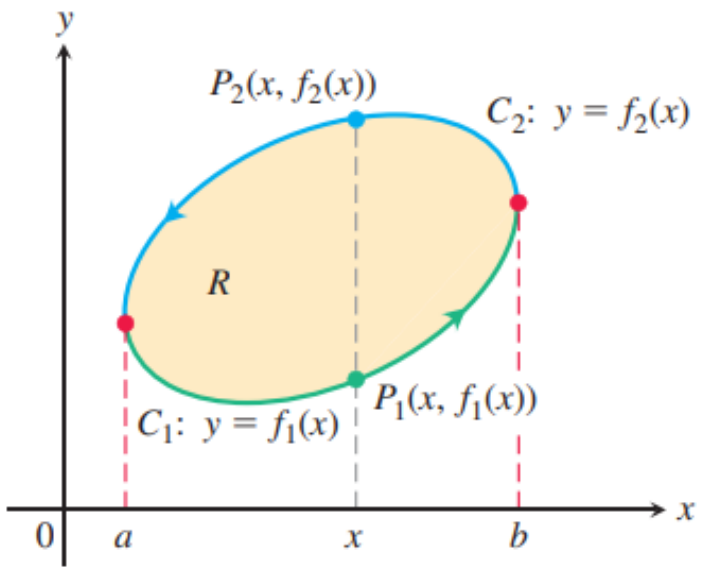
**Solution** Calculating the flux with a line integral would take four integrations, one for each side of the square. With Green's Theorem, we can change the line integral to one double integral. With $M = 2e^{xy}$, $N = y^3$, $C$ the square, and $R$ the square's interior, we have

$$\begin{aligned}
\text{Flux} &= \oint_C \mathbf{F}\cdot\mathbf{n}\,ds = \oint_C M\,dy - N\,dx \\
&= \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx\,dy \qquad \text{Green's Theorem, Eq. (4)} \\
&= \int_{-1}^{1}\int_{-1}^{1} (2ye^{xy} + 3y^2)\,dx\,dy = \int_{-1}^{1} \Big[2e^{xy} + 3xy^2\Big]_{x=-1}^{x=1} dy \\
&= \int_{-1}^{1} (2e^{y} + 6y^2 - 2e^{-y})\,dy = \Big[2e^{y} + 2y^3 + 2e^{-y}\Big]_{-1}^{1} = 4.
\end{aligned}$$

# Proof of Green's Theorem for Special Regions

• Let $C$ be a **smooth simple closed curve** in the $xy$-plane with the **property** that lines parallel to the axes cut it at no more than two points.

• Let $R$ be the region enclosed by $C$ and suppose that $M$, $N$, and their **first partial derivatives** are **continuous** at every point of some open region containing $C$ and $R$.




• We want to prove the **circulation-curl form** of Green's Theorem,

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy.$$

• Figure shows C made up of **two directed parts**

$$C_1: \quad y = f_1(x), \quad a \le x \le b, \qquad C_2: \quad y = f_2(x), \quad b \ge x \ge a.$$

# Proof of Green's Theorem for Special Regions

- For any $x$ between $a$ and $b$, we can **integrate** $\partial M/\partial y$ **with respect to** $y$ from $y = f_1(x)$ to $y = f_2(x)$ and obtain

$$\int_{f_1(x)}^{f_2(x)} \frac{\partial M}{\partial y} dy = M(x, y)\Big]_{y=f_1(x)}^{y=f_2(x)} = M(x, f_2(x)) - M(x, f_1(x)).$$

- We can then **integrate** this **with respect to** $x$ from $a$ to $b$:

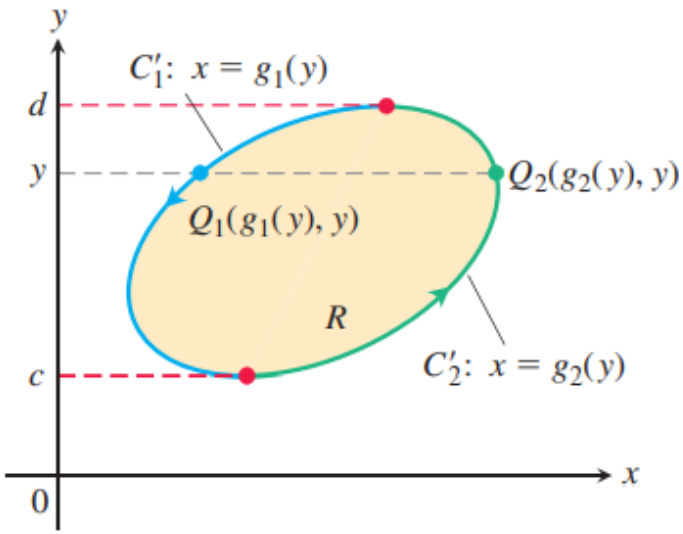
$$\begin{aligned}
\int_a^b \int_{f_1(x)}^{f_2(x)} \frac{\partial M}{\partial y} dy\, dx &= \int_a^b \left[ M(x, f_2(x)) - M(x, f_1(x)) \right] dx \\
&= -\int_b^a M(x, f_2(x))\, dx - \int_a^b M(x, f_1(x))\, dx \\
&= -\int_{C_2} M\, dx - \int_{C_1} M\, dx \\
&= -\oint_C M\, dx.
\end{aligned}$$

# Proof of Green's Theorem for Special Regions

- Therefore, **reversing the order of the equations**, we have

$$\oint_C M\,dx = \iint_R \left(-\frac{\partial M}{\partial y}\right) dx\,dy.$$

- These Equation is **half the result** we need. We derive the other half by **integrating** $\partial N/\partial x$ **first with respect to** $x$ and then **with respect to** $y$, as suggested by Figure.
- The curve $C$ of Figure decomposed into the **two directed parts**




$$C_1'\colon x = g_1(y), d \geq y \geq c \text{ and } C_2'\colon x = g_2(y), c \leq y \leq d.$$

- The result of this double integration is

$$\oint_C N\,dy = \iint_R \frac{\partial N}{\partial x} dx\,dy.$$

# Lecture 31: Integrals and Vector Fields.

MA2032 Vector Calculus

Lecturer: Larissa Serdukova

School of Computing and Mathematical Science
University of Leicester

November 30, 2022

# Surfaces and Area

We have **defined curves** in the plane (space) in **three different ways**:

1. **Explicit** form: $y = f(x)$ $(z = f(x, y))$.

2. **Implicit** form: $F(x, y) = 0$ $(F(x, y, z) = 0)$.

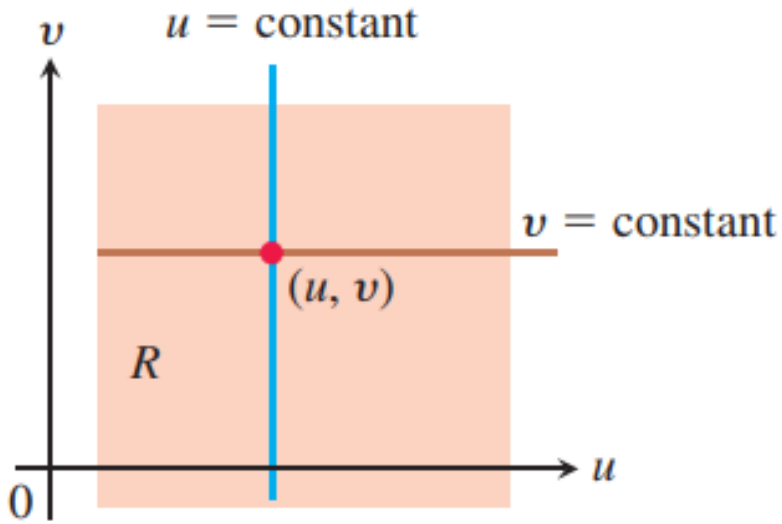
3. **Parametric vector** form: $\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j}$
$(\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k})$, $a \leq t \leq b$.

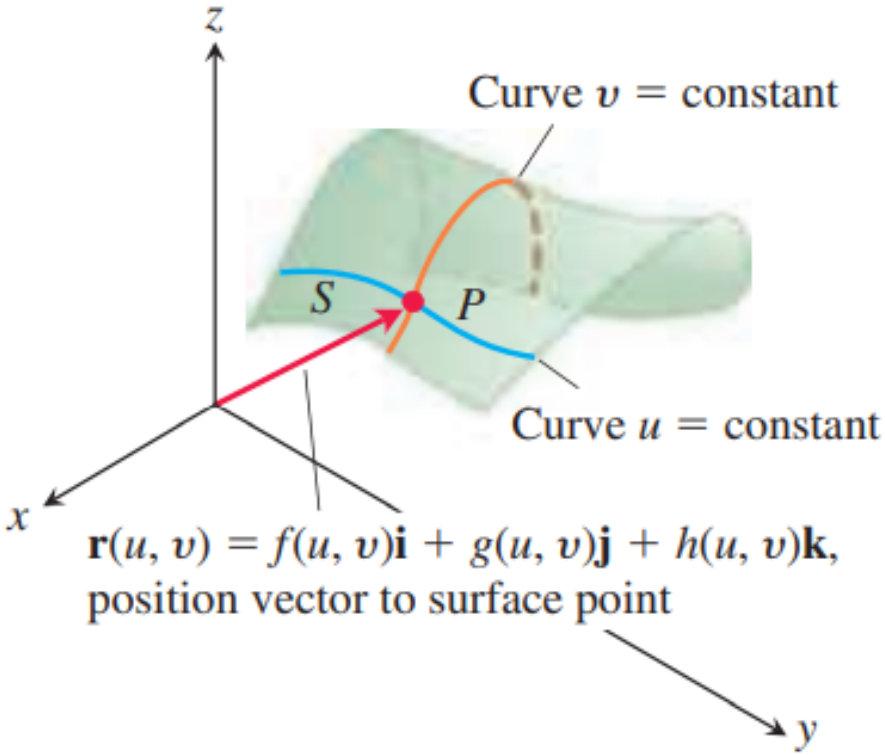
• There is also a **parametric form for surfaces** that gives the position of a point on the surface as a **vector function of two variables**.

• We apply the form to obtain the **area of a surface** as a double integral.

# Parametrizations of Surfaces

• Suppose $\mathbf{r}(u, y) = f(u, v)\,\mathbf{i} + g(u, v)\,\mathbf{j} + h(u, v)\,\mathbf{k}$ (Eq. 1) is a **continuous vector function** that is defined on a region $R$ in the $uv$-plane and **one-to-one** on the interior of $R$.




• **The range** of **r** is the **surface** $S$ defined or traced by **r**.

# Parametrizations of Surfaces

- (Eq. 1) together with the domain $R$ constitutes a **parametrization of the surface**.

- The variables $u$ and $v$ are the **parameters**, and $R$ is the **parameter domain**.

- $R$ is rectangle defined by $a \leq u \leq b$, $c \leq y \leq d$.

- The requirement that $\mathbf{r}$ be **one-to-one** on the interior of $R$ ensures that $S$ **does not cross itself**.

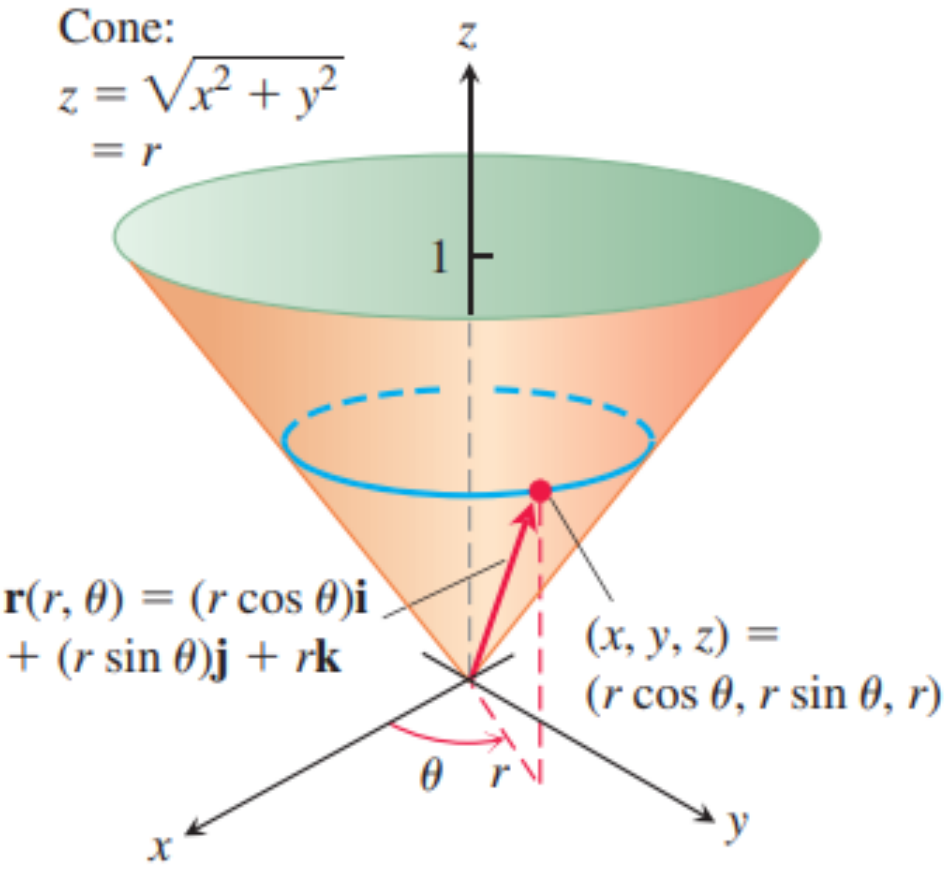
- (Eq. 1) is the vector equivalent of **three parametric equations**:

$$x = f(u, v), \quad y = g(u, v), \quad z = h(u, v).$$

# Parametrizations of Surfaces

**EXAMPLE 1** Find a parametrization of the cone

$$z = \sqrt{x^2 + y^2}, \qquad 0 \le z \le 1.$$



# Solution Example 1

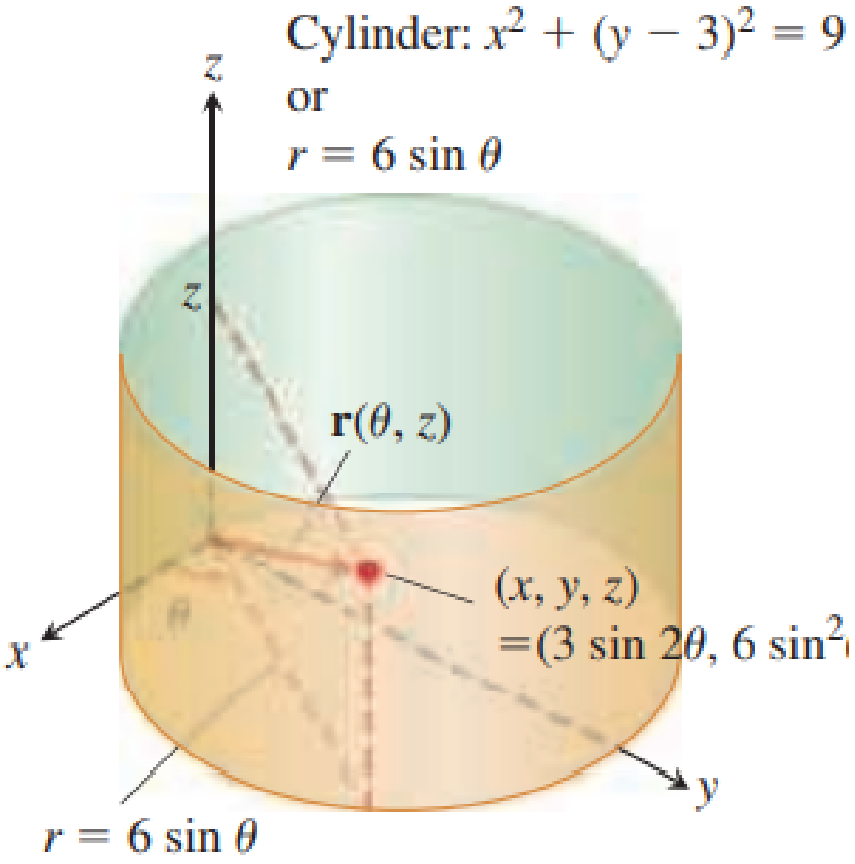
**Solution** Here, cylindrical coordinates provide a parametrization. A typical point $(x, y, z)$ on the cone (Figure 16.40) has $x = r\cos\theta$, $y = r\sin\theta$, and $z = \sqrt{x^2 + y^2} = r$, with $0 \le r \le 1$ and $0 \le \theta \le 2\pi$. Taking $u = r$ and $v = \theta$ in Equation (1) gives the parametrization

$$\mathbf{r}(r, \theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + r\mathbf{k}, \qquad 0 \le r \le 1, \quad 0 \le \theta \le 2\pi.$$

The parametrization is one-to-one on the interior of the domain $R$, though not on the boundary tip of its cone where $r = 0$. ■

**EXAMPLE 3** Find a parametrization of the cylinder

$$x^2 + (y - 3)^2 = 9, \qquad 0 \le z \le 5.$$



# Solution Example 3

**Solution** In cylindrical coordinates, a point $(x, y, z)$ has $x = r\cos\theta$, $y = r\sin\theta$, and $z = z$. For points on the cylinder $x^2 + (y-3)^2 = 9$ (Figure 16.42), the equation is the same as the polar equation for the cylinder's base in the $xy$-plane:

$$x^2 + (y^2 - 6y + 9) = 9$$

$$r^2 - 6r\sin\theta = 0 \qquad x^2 + y^2 = r^2,\ y = r\sin\theta$$

or

$$r = 6\sin\theta, \qquad 0 \le \theta \le \pi.$$

A typical point on the cylinder therefore has

$$\begin{aligned} x &= r\cos\theta = 6\sin\theta\cos\theta = 3\sin 2\theta \\ y &= r\sin\theta = 6\sin^2\theta \\ z &= z. \end{aligned}$$

Taking $u = \theta$ and $v = z$ in Equation (1) gives the one-to-one parametrization

$$\mathbf{r}(\theta, z) = (3\sin 2\theta)\mathbf{i} + (6\sin^2\theta)\mathbf{j} + z\mathbf{k}, \quad 0 \le \theta \le \pi, \quad 0 \le z \le 5.$$

# Surface Area

• Our goal is to find a double integral for calculating the **area of a curved surface** $S$ based on the parametrization

$$\mathbf{r}(u,v) = f(u,v)\,\mathbf{i} + g(u,v)\,\mathbf{j} + h(u,v)\,\mathbf{k}, \quad a \le u \le b,\ c \le v \le d$$

• We need $S$ **to be smooth** for the construction we are about to carry out.

• The definition of smoothness involves the **partial derivatives** of $\mathbf{r}$ with respect to $u$ and $v$:

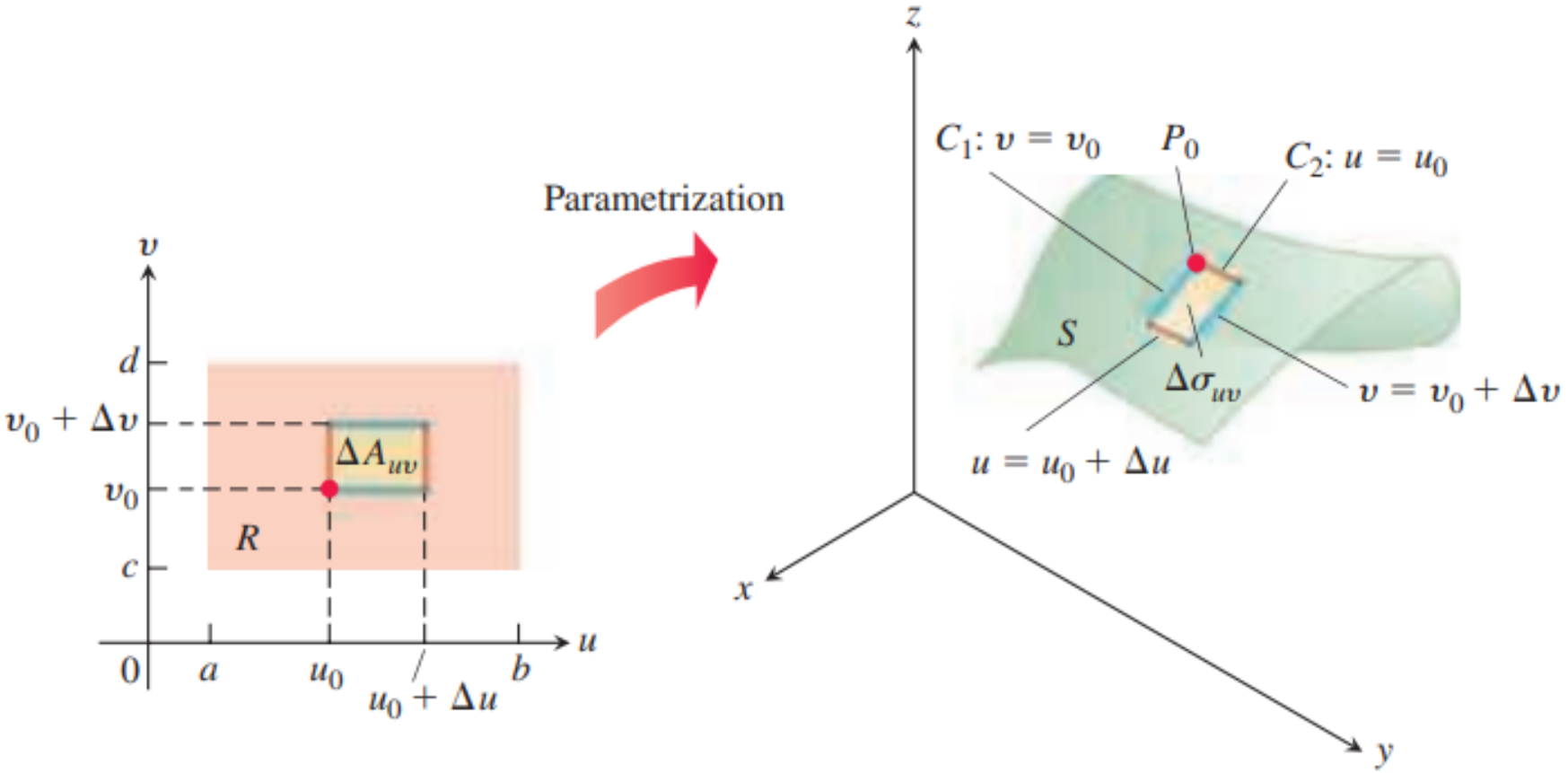
$$\mathbf{r}_u = \frac{\partial \mathbf{r}}{\partial u} = \frac{\partial f}{\partial u}\mathbf{i} + \frac{\partial g}{\partial u}\mathbf{j} + \frac{\partial h}{\partial u}\mathbf{k}$$

$$\mathbf{r}_v = \frac{\partial \mathbf{r}}{\partial v} = \frac{\partial f}{\partial v}\mathbf{i} + \frac{\partial g}{\partial v}\mathbf{j} + \frac{\partial h}{\partial v}\mathbf{k}.$$

**DEFINITION** A parametrized surface $\mathbf{r}(u,v) = f(u,v)\mathbf{i} + g(u,v)\mathbf{j} + h(u,v)\mathbf{k}$ is **smooth** if $\mathbf{r}_u$ and $\mathbf{r}_v$ are continuous and $\mathbf{r}_u \times \mathbf{r}_v$ is never zero on the interior of the parameter domain.

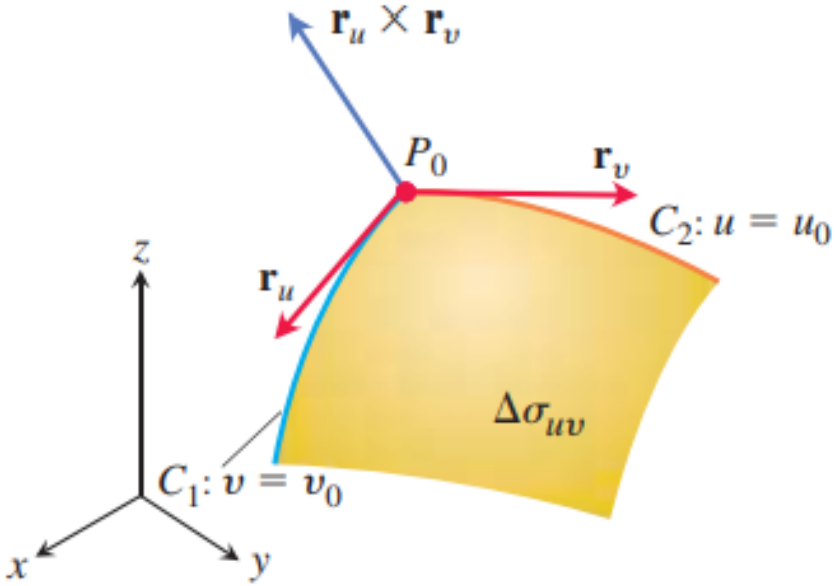
# Surface Area

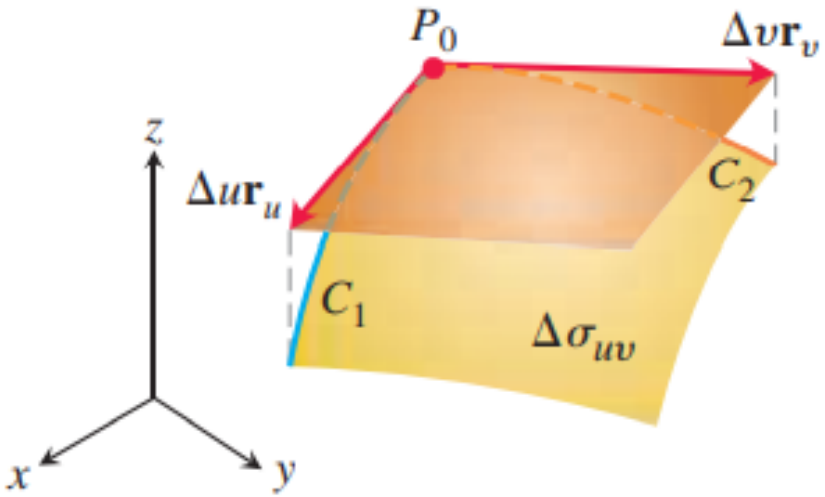
• The **condition** that $\mathbf{r}_u \times \mathbf{r}_v$ is **never the zero vector** in the definition of smoothness means that the two vectors $r_u$ and $r_v$ are nonzero and never lie along the same line, so they **always determine a plane tangent to the surface**.



# Surface Area

- Consider a small **rectangle** $\Delta A_{uv}$ in $R$
- Each side of $\Delta A_{uv}$ maps to a curve on the surface S, and together these four curves bound a **"curved patch element"** $\Delta\sigma_{uv}$.
- The **partial derivative vector** $\mathbf{r}_u(u_0, v_0)$ **is tangent** to $C_1$ at $P_0$. Likewise, $\mathbf{r}_v(u_0, v_0)$ is tangent to $C_2$ at $P_0$ .
- The **cross product** $\mathbf{r}_u \times \mathbf{r}_v$ is **normal** to the surface at $P_0$.



# Surface Area

• We next **approximate** the surface patch element $\Delta\sigma_{uv}$ **by the parallelogram** on the tangent plane whose area is

$$\left|\Delta u\,\mathbf{r}_u \times \Delta v\,\mathbf{r}_v\right| = \left|\mathbf{r}_u \times \mathbf{r}_v\right| \Delta u\,\Delta v.$$

• A **partition** of the surface $S$ into surface patch elements **approximates** the area of $S$ by the sum

$$\sum_n \left|\mathbf{r}_u \times \mathbf{r}_v\right| \Delta u\,\Delta v.$$

# Surface Area

• This sum **in limit** sense ($\Delta u \to 0,\ \Delta v \to 0,\ n \to \infty$) approaches the double integral

**DEFINITION** The **area** of the smooth surface

$$\mathbf{r}(u, v) = f(u, v)\mathbf{i} + g(u, v)\mathbf{j} + h(u, v)\mathbf{k}, \qquad a \le u \le b, \quad c \le v \le d$$

is

$$A = \iint_R |\mathbf{r}_u \times \mathbf{r}_v|\, dA = \int_c^d \int_a^b |\mathbf{r}_u \times \mathbf{r}_v|\, du\, dv. \tag{4}$$

# Surface Area

**EXAMPLE 4** Find the surface area of the cone in Example 1

**Solution** In Example 1, we found the parametrization

$$\mathbf{r}(r, \theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + r\mathbf{k}, \qquad 0 \le r \le 1, \quad 0 \le \theta \le 2\pi.$$

To apply Equation (4), we first find $\mathbf{r}_r \times \mathbf{r}_\theta$:

$$\mathbf{r}_r \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta & \sin\theta & 1 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix}$$

$$= -(r\cos\theta)\mathbf{i} - (r\sin\theta)\mathbf{j} + \underbrace{(r\cos^2\theta + r\sin^2\theta)}_{r}\mathbf{k}.$$

Thus, $|\mathbf{r}_r \times \mathbf{r}_\theta| = \sqrt{r^2\cos^2\theta + r^2\sin^2\theta + r^2} = \sqrt{2r^2} = \sqrt{2}r$. The area of the cone is

$$A = \int_0^{2\pi}\int_0^1 |\mathbf{r}_r \times \mathbf{r}_\theta|\, dr\, d\theta \qquad \text{Eq. (4) with } u = r,\ v = \theta$$

$$= \int_0^{2\pi}\int_0^1 \sqrt{2}r\, dr\, d\theta = \int_0^{2\pi} \frac{\sqrt{2}}{2} d\theta = \frac{\sqrt{2}}{2}(2\pi) = \pi\sqrt{2} \text{ square units.}$$

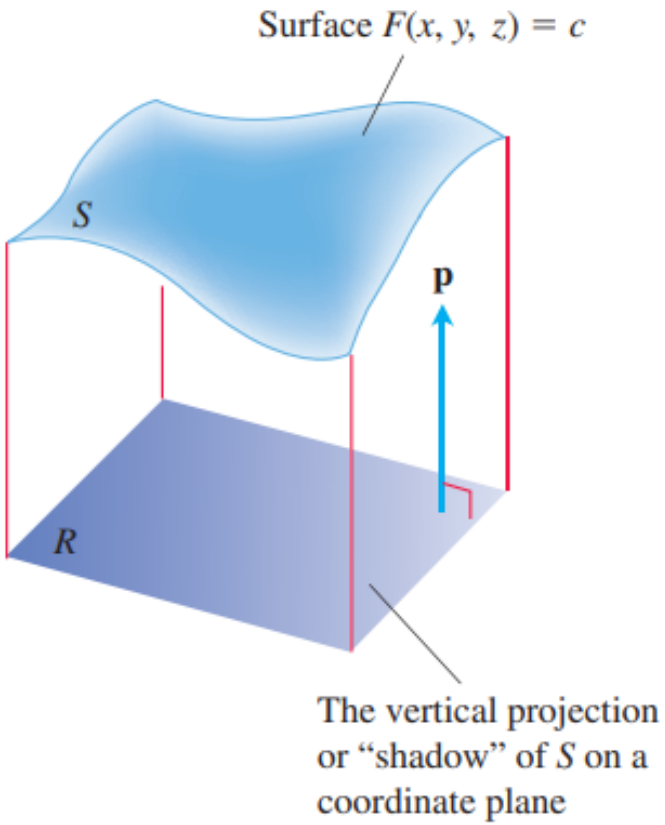
# Implicit Surfaces

• Surfaces are often presented as **level sets of a function**, described by an equation such as $F(x, y, z) = c$, for some constant $c$.
• It may be **difficult to find explicit formulas** for the functions $f$, $g$, and $h$ that describe the surface in the form
$\mathbf{r}(u, v) = f(u, v)\,\mathbf{i} + g(u, v)\,\mathbf{j} + h(u, v)\,\mathbf{k}$.

• We choose $\mathbf{p}$ to be a **unit vector** (it can be $\mathbf{k}$) **normal** to the plane region $R$ ($\in$ $xy$-plane).
• By assumption, we then have

$$\nabla F \cdot \mathbf{p} = \nabla F \cdot \mathbf{k} = F_z \neq 0 \text{ on } S.$$

• The Implicit Function Theorem implies that $S$ is then the **graph of a differentiable function** $z = h(x, y)$.
• Defining $u = x$ and $v = y$, then $z = h(u, v)$ and $\mathbf{r}(u, v) = u\,\mathbf{i} + v\,\mathbf{j} + h(u, v)\,\mathbf{k}$ gives a **parametrization of the surface** $S$.



# Implicit Surfaces

- We use previous results to **find the area** of $S$:

Calculating the partial derivatives of $\mathbf{r}$, we find

$$\mathbf{r}_u = \mathbf{i} + \frac{\partial h}{\partial u}\mathbf{k} \qquad \text{and} \qquad \mathbf{r}_v = \mathbf{j} + \frac{\partial h}{\partial v}\mathbf{k}.$$

Applying the Chain Rule for implicit differentiation (see Equation (2) in Section 14.4) to $F(x, y, z) = c$, where $x = u$, $y = v$, and $z = h(u, v)$, we obtain the partial derivatives

$$\frac{\partial h}{\partial u} = -\frac{F_x}{F_z} \qquad \text{and} \qquad \frac{\partial h}{\partial v} = -\frac{F_y}{F_z}. \qquad F_z \neq 0$$

Substitution of these derivatives into the derivatives of $\mathbf{r}$ gives

$$\mathbf{r}_u = \mathbf{i} - \frac{F_x}{F_z}\mathbf{k} \qquad \text{and} \qquad \mathbf{r}_v = \mathbf{j} - \frac{F_y}{F_z}\mathbf{k}.$$

From a routine calculation of the cross product we find

# Implicit Surfaces

$$\mathbf{r}_u \times \mathbf{r}_v = \frac{F_x}{F_z}\mathbf{i} + \frac{F_y}{F_z}\mathbf{j} + \mathbf{k}$$

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -F_x/F_z \\ 0 & 1 & -F_y/F_z \end{vmatrix}$$

cross product of $\mathbf{r}_u$ $\mathbf{r}_v$

$$= \frac{1}{F_z}(F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k})$$

$$= \frac{\nabla F}{F_z} = \frac{\nabla F}{\nabla F \cdot \mathbf{k}}$$

$$= \frac{\nabla F}{\nabla F \cdot \mathbf{p}}.$$

$\mathbf{p} = \mathbf{k}$

Therefore, the surface area differential is given by

$$d\sigma = |\mathbf{r}_u \times \mathbf{r}_v|\, du\, dv = \frac{|\nabla F|}{|\nabla F \cdot \mathbf{p}|} dx\, dy.$$

$u = x$ and $v = y$

We obtain similar calculations if instead the vector $\mathbf{p} = \mathbf{j}$ is normal to the $xz$-plane when $F_y \neq 0$ on $S$, or if $\mathbf{p} = \mathbf{i}$ is normal to the $yz$-plane when $F_x \neq 0$ on $S$. Combining these results with Equation (4) then gives the following general formula.

# Implicit Surfaces

**Formula for the Surface Area of an Implicit Surface**

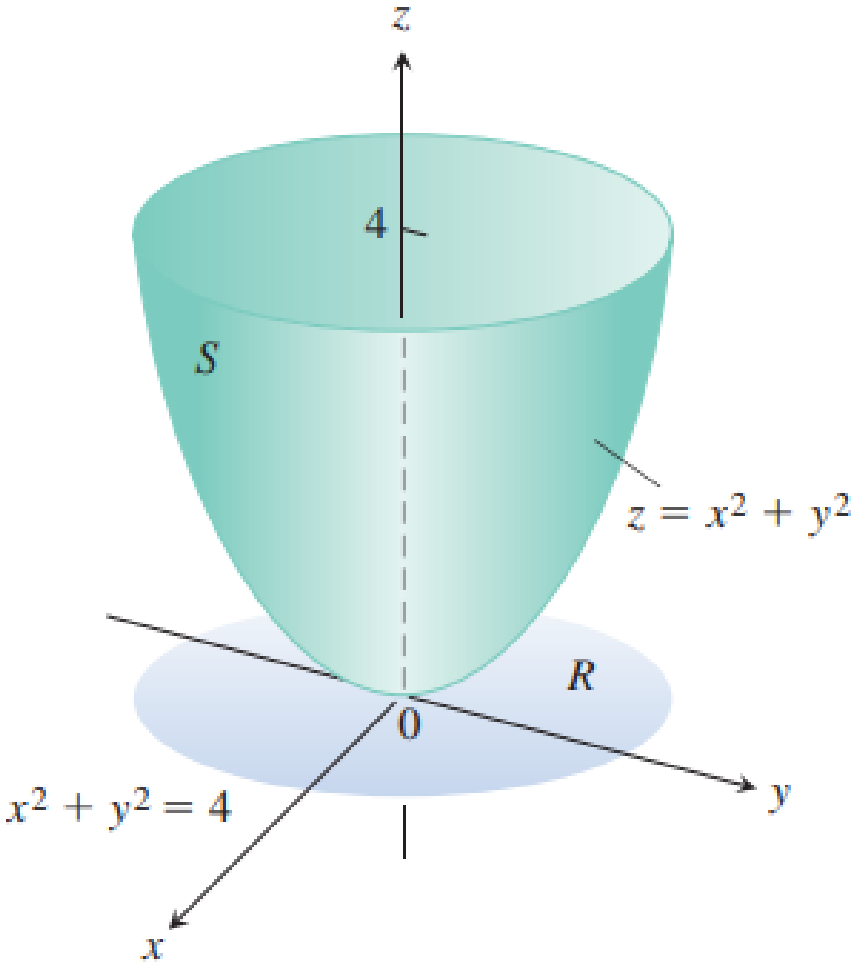
The area of the surface $F(x, y, z) = c$ over a closed and bounded plane region $R$ is

$$\text{Surface area} = \iint_R \frac{|\nabla F|}{|\nabla F \cdot \mathbf{p}|}\, dA, \tag{7}$$

where $\mathbf{p} = \mathbf{i}, \mathbf{j}$, or $\mathbf{k}$ is normal to $R$ and $\nabla F \cdot \mathbf{p} \neq 0$.

# Implicit Surfaces

**EXAMPLE 7** Find the area of the surface cut from the bottom of the paraboloid $x^2 + y^2 - z = 0$ by the plane $z = 4$.



# Solution Example 7

**Solution** We sketch the surface $S$ and the region $R$ below it in the $xy$-plane (Figure 16.48). The surface $S$ is part of the level surface $F(x, y, z) = x^2 + y^2 - z = 0$, and $R$ is the disk $x^2 + y^2 \leq 4$ in the $xy$-plane. To get a unit vector normal to the plane of $R$, we can take $\mathbf{p} = \mathbf{k}$.

At any point $(x, y, z)$ on the surface, we have

$$\begin{aligned} F(x, y, z) &= x^2 + y^2 - z \\ \nabla F &= 2x\mathbf{i} + 2y\mathbf{j} - \mathbf{k} \\ |\nabla F| &= \sqrt{(2x)^2 + (2y)^2 + (-1)^2} \\ &= \sqrt{4x^2 + 4y^2 + 1} \\ |\nabla F \cdot \mathbf{p}| &= |\nabla F \cdot \mathbf{k}| = |-1| = 1. \end{aligned}$$

In the region $R$, $dA = dx\,dy$. Therefore,

# Solution Example 7

$$\text{Surface area} = \iint_R \frac{|\nabla F|}{|\nabla F \cdot \mathbf{p}|} dA \qquad \text{Eq. (7)}$$

$$= \iint_{x^2+y^2 \le 4} \sqrt{4x^2 + 4y^2 + 1}\, dx\, dy$$

$$= \int_0^{2\pi} \int_0^2 \sqrt{4r^2 + 1}\, r\, dr\, d\theta \qquad \text{Polar coordinates}$$

$$= \int_0^{2\pi} \left[ \frac{1}{12}(4r^2 + 1)^{3/2} \right]_0^2 d\theta$$

$$= \int_0^{2\pi} \frac{1}{12}(17^{3/2} - 1)\, d\theta = \frac{\pi}{6}\left(17\sqrt{17} - 1\right).$$

# Lecture 32: Integrals and Vector Fields.

MA2032 Vector Calculus

Lecturer: Larissa Serdukova

School of Computing and Mathematical Science
University of Leicester

December 2, 2022

# Surface Integrals

• **Surface integral** is the two-dimensional **extension** of the line integral concept used to integrate over a one-dimensional curve.

• Like line integrals, surface integrals arise in **two forms**: The first occurs when we **integrate a scalar function** over a surface.

• The second form involves **surface integrals of vector fields**.

• Suppose that the function $G(x, y, z)$ gives the **mass density** (mass per unit area) at each point on a surface $S$.

• Then we can calculate the **total mass** of $S$ as an integral in the following way.

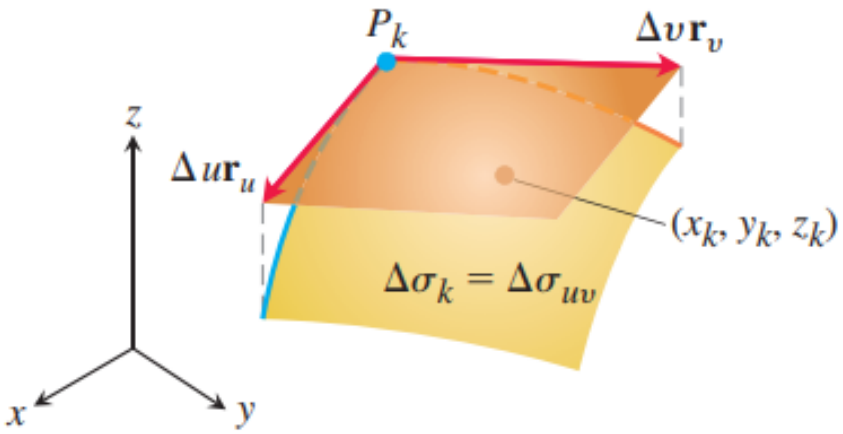
• Assume, that the **surface $S$ is defined parametrically** on a region $R$ in the $uv$-plane.

$$\mathbf{r}(u, v) = f(u, v)\mathbf{i} + g(u, v)\mathbf{j} + h(u, v)\mathbf{k}, \qquad (u, v) \in R.$$

# Surface Integrals

• The surface $S$ is divided into $n$ **curved surface elements** of area

$$\Delta\sigma_{uv} \approx |\mathbf{r}_u \times \mathbf{r}_v|\, du\, dv.$$




• To form a **Riemann sum** over $S$, we choose a point $(x_k, y_k, z_k)$ in the $k$-th patch, multiply the value of the function $G$ at that point by the area $\Delta\sigma_k$, and add together the products:

$$\sum_{k=1}^{n} G(x_k, y_k, z_k)\, \Delta\sigma_k.$$

• Then we take the limit as the number $n$ of surface patches increases, their areas shrink to zero.

# Surface Integrals

• **This limit**, whenever it exists, **defines the surface integral** of $G$ over the surface $S$ as

$$\iint_S G(x, y, z)\, d\sigma = \lim_{n\to\infty} \sum_{k=1}^{n} G(x_k, y_k, z_k)\, \Delta\sigma_k.$$

## Formulas for a Surface Integral of a Scalar Function

**1.** For a smooth surface $S$ defined **parametrically** as $\mathbf{r}(u, v) = f(u, v)\mathbf{i} + g(u, v)\mathbf{j} + h(u, v)\mathbf{k}$, $(u, v) \in R$, and a continuous function $G(x, y, z)$ defined on $S$, the surface integral of $G$ over $S$ is given by the double integral over $R$,

$$\iint_S G(x, y, z)\, d\sigma = \iint_R G(f(u, v), g(u, v), h(u, v))\, |\mathbf{r}_u \times \mathbf{r}_v|\, du\, dv. \qquad (2)$$

## Formulas for a Surface Integral of a Scalar Function

**2.** For a surface $S$ given **implicitly** by $F(x, y, z) = c$, where $F$ is a continuously differentiable function, with $S$ lying above its closed and bounded shadow region $R$ in the coordinate plane beneath it, the surface integral of the continuous function $G$ over $S$ is given by the double integral over $R$,

$$\iint_S G(x, y, z)\, d\sigma = \iint_R G(x, y, z) \frac{|\nabla F|}{|\nabla F \cdot \mathbf{p}|}\, dA, \tag{3}$$

**3.** For a surface $S$ given **explicitly** as the graph of $z = f(x, y)$, where $f$ is a continuously differentiable function over a region $R$ in the $xy$-plane, the surface integral of the continuous function $G$ over $S$ is given by the double integral over $R$,

$$\iint_S G(x, y, z)\, d\sigma = \iint_R G(x, y, f(x, y)) \sqrt{{f_x}^2 + {f_y}^2 + 1}\, dx\, dy. \tag{4}$$

**EXAMPLE 1** Integrate $G(x, y, z) = x^2$ over the cone $z = \sqrt{x^2 + y^2}$, $0 \le z \le 1$.

**Solution** Using Equation (2) and the calculations from Example 4 in Section 16.5, we have $|\mathbf{r}_r \times \mathbf{r}_\theta| = \sqrt{2}r$ and

$$\iint_S x^2\, d\sigma = \int_0^{2\pi}\int_0^1 \left(r^2\cos^2\theta\right)\left(\sqrt{2}r\right) dr\, d\theta \qquad x = r\cos\theta$$

$$= \sqrt{2}\int_0^{2\pi}\int_0^1 r^3\cos^2\theta\, dr\, d\theta$$

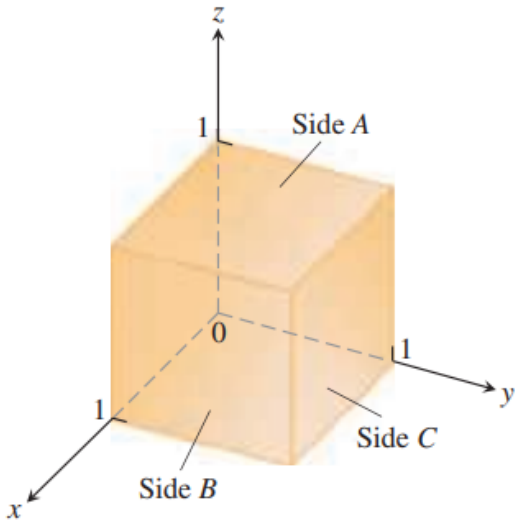
$$= \frac{\sqrt{2}}{4}\int_0^{2\pi}\cos^2\theta\, d\theta = \frac{\sqrt{2}}{4}\left[\frac{\theta}{2} + \frac{1}{4}\sin 2\theta\right]_0^{2\pi} = \frac{\pi\sqrt{2}}{4}.$$

# Surface Integrals

• When $S$ is **partitioned** by smooth curves into a finite number of smooth patches with **non-overlapping interiors** (i.e., if $S$ is piecewise smooth), then the integral over $S$ is the sum of the integrals over the patches:

$$\iint_S G\,d\sigma = \iint_{S_1} G\,d\sigma + \iint_{S_2} G\,d\sigma + \cdots + \iint_{S_n} G\,d\sigma.$$

**EXAMPLE 2** Integrate $G(x, y, z) = xyz$ over the surface of the cube cut from the first octant by the planes $x = 1$, $y = 1$, and $z = 1$



# Surface Integrals

**Solution** We integrate $xyz$ over each of the six sides and add the results. Since $xyz = 0$ on the sides that lie in the coordinate planes, the integral over the surface of the cube reduces to

$$\iint_{\substack{\text{Cube}\\ \text{surface}}} xyz\, d\sigma = \iint_{\text{Side } A} xyz\, d\sigma + \iint_{\text{Side } B} xyz\, d\sigma + \iint_{\text{Side } C} xyz\, d\sigma.$$

Side $A$ is the surface $f(x, y, z) = z = 1$ over the square region $R_{xy}$: $0 \le x \le 1$, $0 \le y \le 1$, in the $xy$-plane. For this surface and region,

$$\mathbf{p} = \mathbf{k}, \qquad \nabla f = \mathbf{k}, \qquad |\nabla f| = 1, \qquad |\nabla f \cdot \mathbf{p}| = |\mathbf{k} \cdot \mathbf{k}| = 1$$

$$d\sigma = \frac{|\nabla f|}{|\nabla f \cdot \mathbf{p}|} dA = \frac{1}{1} dx\, dy = dx\, dy \qquad \text{Eq. (3)}$$

$$xyz = xy(1) = xy$$

and

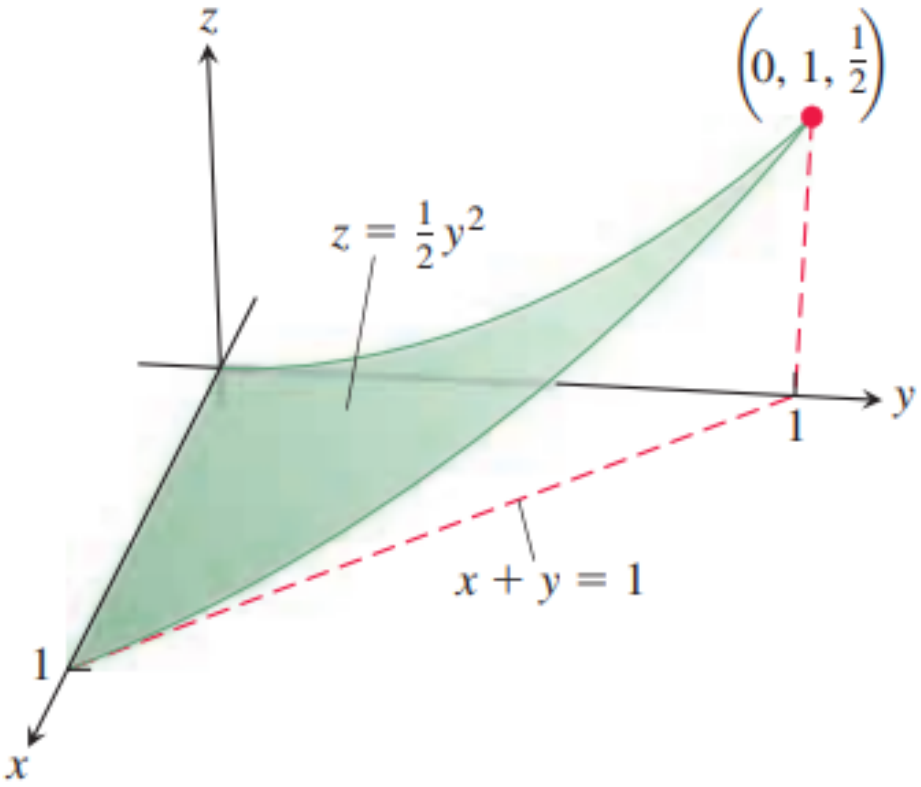
$$\iint_{\text{Side } A} xyz\, d\sigma = \iint_{R_{xy}} xy\, dx\, dy = \int_0^1 \int_0^1 xy\, dx\, dy = \int_0^1 \frac{y}{2} dy = \frac{1}{4}.$$

Symmetry tells us that the integrals of $xyz$ over sides $B$ and $C$ are also $1/4$. Hence,

$$\iint_{\substack{\text{Cube}\\ \text{surface}}} xyz\, d\sigma = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{3}{4}.$$

# Surface Integrals

**EXAMPLE 4** Evaluate $\iint_S \sqrt{x(1+2z)}\, d\sigma$ on the portion of the cylinder $z = y^2/2$ over the triangular region $R$: $x \geq 0$, $y \geq 0$, $x + y \leq 1$ in the $xy$-plane



# Solution Example 4

**Solution** The function $G$ on the surface $S$ is given by

$$G(x, y, z) = \sqrt{x(1 + 2z)} = \sqrt{x}\sqrt{1 + y^2}.$$

With $z = f(x, y) = y^2/2$, we use Equation (4) to evaluate the surface integral:

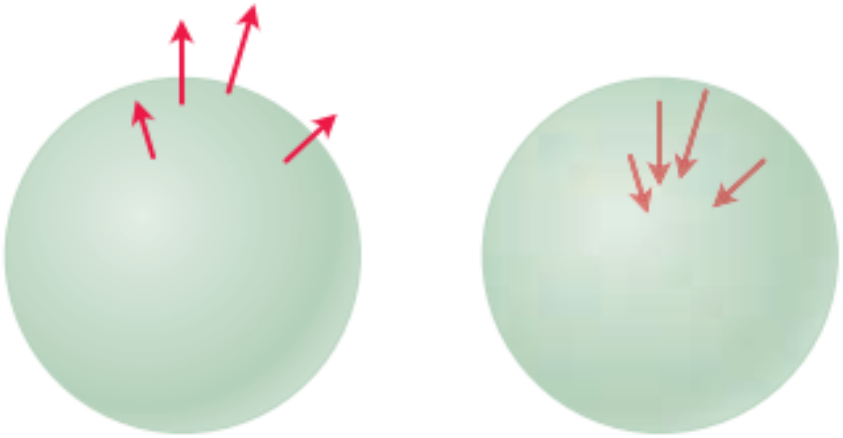
$$d\sigma = \sqrt{f_x{}^2 + f_y{}^2 + 1}\, dx\, dy = \sqrt{0 + y^2 + 1}\, dx\, dy$$

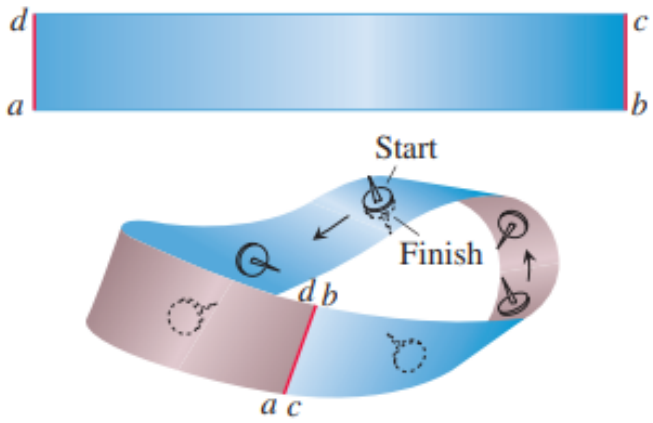
and

$$\begin{aligned}
\iint_S G(x, y, z)\, d\sigma &= \iint_R \left(\sqrt{x}\sqrt{1 + y^2}\right)\sqrt{1 + y^2}\, dx\, dy \\
&= \int_0^1 \int_0^{1-x} \sqrt{x}(1 + y^2)\, dy\, dx \\
&= \int_0^1 \sqrt{x}\left[(1 - x) + \frac{1}{3}(1 - x)^3\right] dx && \text{Integrate and evaluate.} \\
&= \int_0^1 \left(\frac{4}{3}x^{1/2} - 2x^{3/2} + x^{5/2} - \frac{1}{3}x^{7/2}\right) dx && \text{Routine algebra} \\
&= \left[\frac{8}{9}x^{3/2} - \frac{4}{5}x^{5/2} + \frac{2}{7}x^{7/2} - \frac{2}{27}x^{9/2}\right]_0^1 \\
&= \frac{8}{9} - \frac{4}{5} + \frac{2}{7} - \frac{2}{27} = \frac{284}{945} \approx 0.30.
\end{aligned}$$

# Orientation of a Surface

- To specify an **orientation on a surface** in space $S$, we specify a **normal vector** at each point on the surface.
- A parametrization of a surface $\mathbf{r}(u, v)$ gives a **vector $\mathbf{r}_u \times \mathbf{r}_v$ that is normal to the surface**, and so gives an orientation wherever the parametrization applies.
- A **second choice of orientation** is found by taking $-(\mathbf{r}_u \times \mathbf{r}_v)$, giving a vector that points to the **opposite side of the surface** at each point.
- In essence, an orientation is a way of consistently choosing **one of the two sides of a surface**.
- **Not all surfaces have orientations**.





# Surface Integrals of Vector Fields

- We defined the **line integral** of a vector field along a path $C$ as $\int_C \mathbf{F} \cdot \mathbf{T} ds$, where $\mathbf{T}$ is the unit **tangent vector** to the path pointing in the **forward oriented direction**.

- We have a **similar definition for surface integrals**.

**DEFINITION** Let $\mathbf{F}$ be a vector field in three-dimensional space with continuous components defined over a smooth surface $S$ having a chosen field of normal unit vectors $\mathbf{n}$ orienting $S$. Then the **surface integral of $\mathbf{F}$ over $S$** is

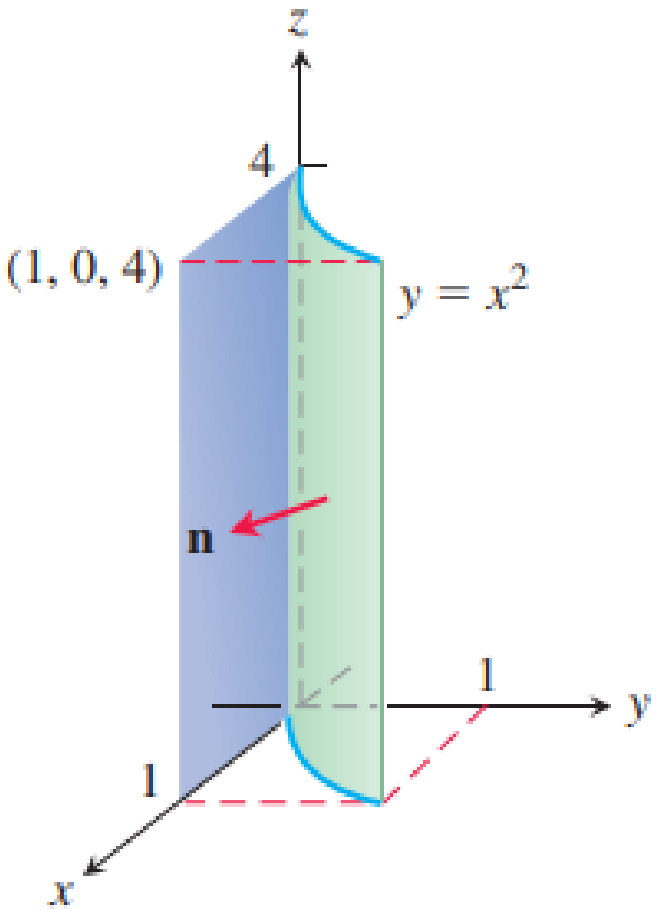
$$\iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma. \tag{5}$$

This integral is also called the **flux** of the vector field $\mathbf{F}$ across $S$.

- If $\mathbf{F}$ is the velocity field of a three-dimensional fluid flow, then the flux of $\mathbf{F}$ across $S$ is the **net rate at which fluid is crossing $S$ per unit time** in the chosen positive direction $\mathbf{n}$ defined by the orientation of $S$.

# Surface Integrals of Vector Fields

**EXAMPLE 5** Find the flux of $\mathbf{F} = yz\mathbf{i} + x\mathbf{j} - z^2\mathbf{k}$ through the parabolic cylinder $y = x^2$, $0 \le x \le 1$, $0 \le z \le 4$, in the direction $\mathbf{n}$ indicated in



# Solution. Example 5

**Solution** On the surface we have $x = x$, $y = x^2$, and $z = z$, so we automatically have the parametrization $\mathbf{r}(x, z) = x\mathbf{i} + x^2\mathbf{j} + z\mathbf{k}$, $0 \le x \le 1$, $0 \le z \le 4$. The cross product of tangent vectors is

$$\mathbf{r}_x \times \mathbf{r}_z = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2x & 0 \\ 0 & 0 & 1 \end{vmatrix} = 2x\mathbf{i} - \mathbf{j}.$$

The unit normal vectors pointing outward from the surface as indicated in Figure 16.54 are

$$\mathbf{n} = \frac{\mathbf{r}_x \times \mathbf{r}_z}{|\mathbf{r}_x \times \mathbf{r}_z|} = \frac{2x\mathbf{i} - \mathbf{j}}{\sqrt{4x^2 + 1}}.$$

On the surface, $y = x^2$, so the vector field there is

$$\mathbf{F} = yz\mathbf{i} + x\mathbf{j} - z^2\mathbf{k} = x^2z\mathbf{i} + x\mathbf{j} - z^2\mathbf{k}.$$

Thus,

$$\mathbf{F} \cdot \mathbf{n} = \frac{1}{\sqrt{4x^2 + 1}}((x^2z)(2x) + (x)(-1) + (-z^2)(0)) = \frac{2x^3z - x}{\sqrt{4x^2 + 1}}.$$

# Solution. Example 5

The flux of $\mathbf{F}$ outward through the surface is

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma = \int_0^4\int_0^1 \frac{2x^3z - x}{\sqrt{4x^2+1}}\left|\mathbf{r}_x \times \mathbf{r}_z\right| dx\,dz \qquad d\sigma = \left|\mathbf{r}_x \times \mathbf{r}_z\right| dx\,dz$$

$$= \int_0^4\int_0^1 \frac{2x^3z - x}{\sqrt{4x^2+1}}\sqrt{4x^2+1}\,dx\,dz$$

$$= \int_0^4\int_0^1 (2x^3z - x)\,dx\,dz = \int_0^4 \left[\frac{1}{2}x^4z - \frac{1}{2}x^2\right]_{x=0}^{x=1} dz$$

$$= \int_0^4 \frac{1}{2}(z-1)\,dz = \frac{1}{4}(z-1)^2\Big]_0^4$$

$$= \frac{1}{4}(9) - \frac{1}{4}(1) = 2.$$

# Lecture 33: Infinite Sequences and Series.

MA2032 Vector Calculus

Lecturer: Larissa Serdukova

School of Computing and Mathematical Science
University of Leicester

December 5, 2022

# Sequences. Representing Sequences

• **Sequences are fundamental** to the study of **infinite series** and to many aspects of mathematics.

• A sequence is a **list of numbers**

$$a_1, a_2, a_3, \ldots, a_n, \ldots$$

• **in a given order**. **Order is important!**

• Each of $a_1, a_2, a_3$ and so on **represents a number**. These are the **terms of the sequence**.

• The integer $n$ is called the **index** of $a_n$, and **indicates where $a_n$ occurs in the list**.

• An **infinite sequence of numbers is a function of index** $n$ whose domain is the set of positive integers.

# Sequences. Representing Sequences

- **For example**, the function associated with the sequence

$$2, 4, 6, 8, 10, 12, \ldots, 2n, \ldots$$

- sends 1 to $a_1 = 2$, 2 to $a_2 = 4$, and so on.

- The **general behavior** of this sequence is described by the **formula** $a_n = 2n$.

- Sequences can be **described by writing rules** that specify their terms, such as $a_n = \sqrt{n}$ or **by listing terms**

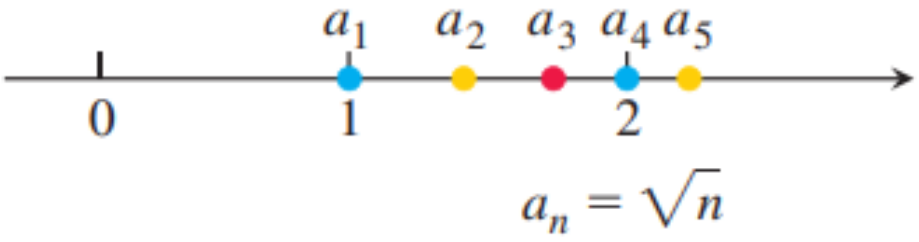
$$\{a_n\} = \{\sqrt{1}, \sqrt{2}, \sqrt{3}, \ldots, \sqrt{n}, \ldots\}$$

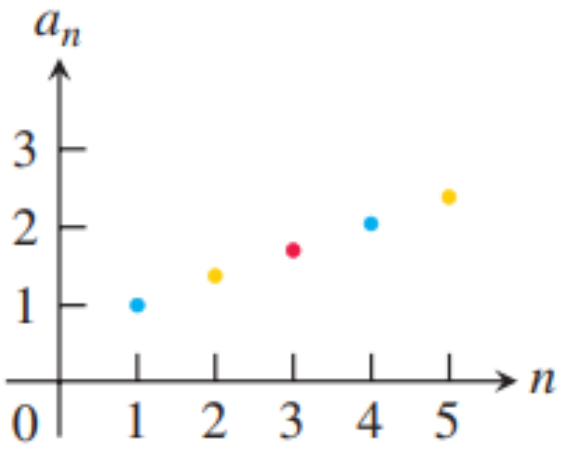
- We also sometimes write a sequence using its rule, as with

$$\{a_n\} = \{\sqrt{n}\}_{n=1}^{\infty}$$

# Sequences. Representing Sequences

- Figure shows two ways to **represent sequences graphically**.



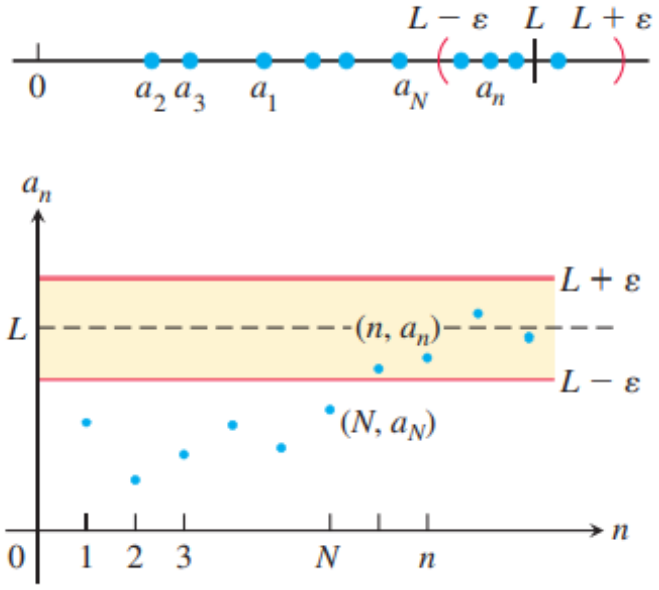
# Convergence and Divergence

**DEFINITIONS** The sequence $\{a_n\}$ **converges** to the number $L$ if for every positive number $\varepsilon$ there corresponds an integer $N$ such that

$$|a_n - L| < \varepsilon \qquad \text{whenever} \qquad n > N.$$

If no such number $L$ exists, we say that $\{a_n\}$ **diverges**.

If $\{a_n\}$ converges to $L$, we write $\lim_{n\to\infty} a_n = L$, or simply $a_n \to L$, and call $L$ the **limit** of the sequence (Figure 10.2).



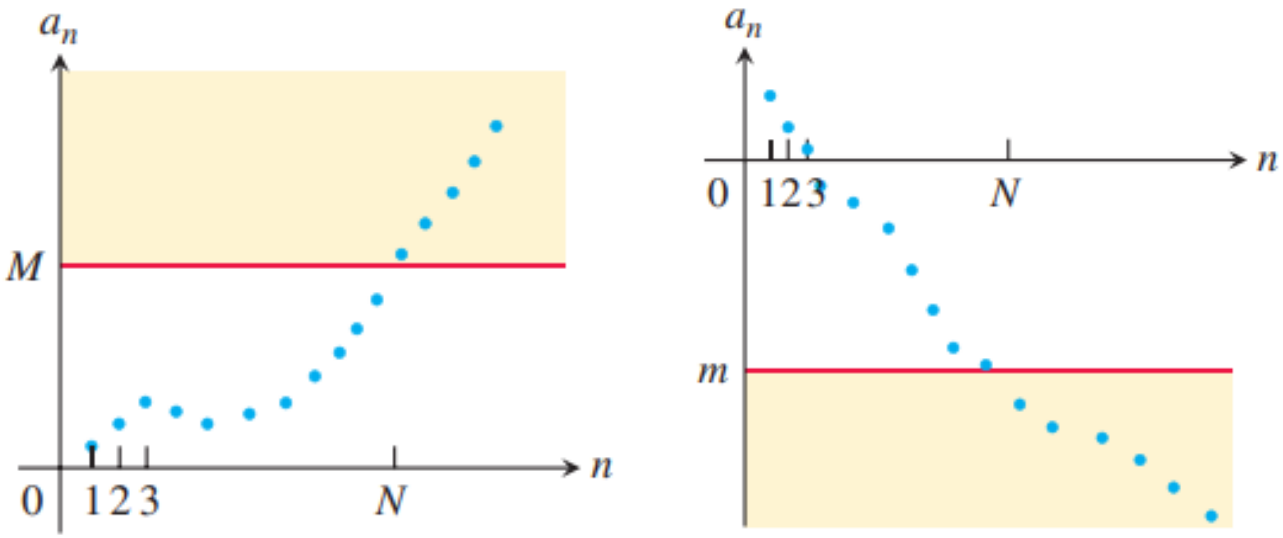
# Convergence and Divergence

**DEFINITION** The sequence $\{a_n\}$ **diverges to infinity** if for every number $M$ there is an integer $N$ such that for all $n$ larger than $N$, $a_n > M$. If this condition holds we write

$$\lim_{n\to\infty} a_n = \infty \qquad \text{or} \qquad a_n \to \infty.$$

Similarly, if for every number $m$ there is an integer $N$ such that for all $n > N$ we have $a_n < m$, then we say $\{a_n\}$ **diverges to negative infinity** and write

$$\lim_{n\to\infty} a_n = -\infty \qquad \text{or} \qquad a_n \to -\infty.$$




(a) (b)

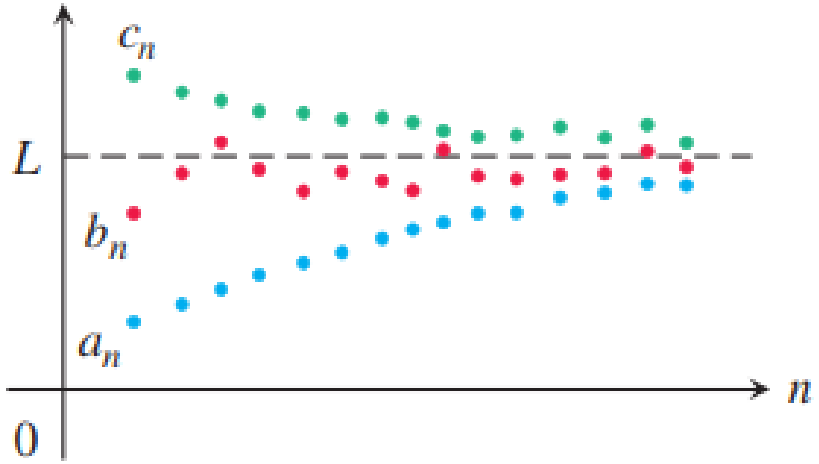
# Calculating Limits of Sequences

**THEOREM 1** Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers, and let $A$ and $B$ be real numbers. The following rules hold if $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$.

1. *Sum Rule:* $\lim_{n\to\infty}(a_n + b_n) = A + B$
2. *Difference Rule:* $\lim_{n\to\infty}(a_n - b_n) = A - B$
3. *Constant Multiple Rule:* $\lim_{n\to\infty}(k \cdot b_n) = k \cdot B$ (any number $k$)
4. *Product Rule:* $\lim_{n\to\infty}(a_n \cdot b_n) = A \cdot B$
5. *Quotient Rule:* $\lim_{n\to\infty}\dfrac{a_n}{b_n} = \dfrac{A}{B}$ if $B \neq 0$

# Calculating Limits of Sequences

**THEOREM 2—The Sandwich Theorem for Sequences**
Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences of real numbers. If $a_n \leq b_n \leq c_n$ holds for all $n$ beyond some index $N$, and if $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$ also.



# Calculating Limits of Sequences

**EXAMPLE** Since $1/n \to 0$, we know that

**(a)** $\dfrac{\cos n}{n} \to 0$ because $-\dfrac{1}{n} \le \dfrac{\cos n}{n} \le \dfrac{1}{n}$;

**(b)** $\dfrac{1}{2^n} \to 0$ because $0 \le \dfrac{1}{2^n} \le \dfrac{1}{n}$;

**(c)** $(-1)^n \dfrac{1}{n} \to 0$ because $-\dfrac{1}{n} \le (-1)^n \dfrac{1}{n} \le \dfrac{1}{n}$.

**(d)** If $|a_n| \to 0$, then $a_n \to 0$ because $-|a_n| \le a_n \le |a_n|$.

# Using L'Hôpital's Rule

**THEOREM 3—The Continuous Function Theorem for Sequences**
Let $\{a_n\}$ be a sequence of real numbers. If $a_n \to L$ and if $f$ is a function that is continuous at $L$ and defined at all $a_n$, then $f(a_n) \to f(L)$.

**THEOREM 4** Suppose that $f(x)$ is a function defined for all $x \geq n_0$ and that $\{a_n\}$ is a sequence of real numbers such that $a_n = f(n)$ for $n \geq n_0$. Then

$$\lim_{n\to\infty} a_n = L \qquad \text{whenever} \qquad \lim_{x\to\infty} f(x) = L.$$

# Using L'Hôpital's Rule

**EXAMPLE** Does the sequence whose $n$th term is

$$a_n = \left(\frac{n+1}{n-1}\right)^n$$

converge? If so, find $\lim_{n\to\infty} a_n$.

**Solution** The limit leads to the indeterminate form $1^\infty$. We can apply l'Hôpital's Rule if we first change the form to $\infty \cdot 0$ by taking the natural logarithm of $a_n$:

$$\ln a_n = \ln\left(\frac{n+1}{n-1}\right)^n = n \ln\left(\frac{n+1}{n-1}\right).$$

Then,

# Solution. Example

$$\lim_{n\to\infty} \ln a_n = \lim_{n\to\infty} n \ln\left(\frac{n+1}{n-1}\right) \qquad \infty \cdot 0 \text{ form}$$

$$= \lim_{n\to\infty} \frac{\ln\left(\frac{n+1}{n-1}\right)}{1/n} \qquad \frac{0}{0} \text{ form}$$

$$= \lim_{n\to\infty} \frac{-2/(n^2-1)}{-1/n^2} \qquad \text{L'Hôpital's Rule: differentiate numerator and denominator.}$$

$$= \lim_{n\to\infty} \frac{2n^2}{n^2-1} = 2. \qquad \text{Simplify and evaluate.}$$

Since $\ln a_n \to 2$ and $f(x) = e^x$ is continuous, Theorem 3 tells us that

$$a_n = e^{\ln a_n} \to e^2.$$

The sequence $\{a_n\}$ converges to $e^2$.

# Commonly Occurring Limits

**THEOREM 5** The following six sequences converge to the limits listed below:

**1.** $\lim_{n\to\infty} \frac{\ln n}{n} = 0$

**2.** $\lim_{n\to\infty} \sqrt[n]{n} = 1$

**3.** $\lim_{n\to\infty} x^{1/n} = 1 \qquad (x > 0)$

**4.** $\lim_{n\to\infty} x^n = 0 \qquad (|x| < 1)$

**5.** $\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x \qquad (\text{any } x)$

**6.** $\lim_{n\to\infty} \frac{x^n}{n!} = 0 \qquad (\text{any } x)$

In Formulas (3) through (6), $x$ remains fixed as $n \to \infty$.

# Infinite Series

**DEFINITIONS** Given a sequence of numbers $\{a_n\}$, an expression of the form

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

is an **infinite series**. The number $a_n$ is the ***n*th term** of the series. The sequence $\{s_n\}$ defined by

$$\begin{aligned} s_1 &= a_1 \\ s_2 &= a_1 + a_2 \\ &\vdots \\ s_n &= a_1 + a_2 + \cdots + a_n = \sum_{k=1}^{n} a_k \\ &\vdots \end{aligned}$$

is the **sequence of partial sums** of the series, the number $s_n$ being the ***n*th partial sum**. If the sequence of partial sums converges to a limit $L$, we say that the series **converges** and that its **sum** is $L$. In this case, we also write

$$a_1 + a_2 + \cdots + a_n + \cdots = \sum_{n=1}^{\infty} a_n = L.$$

If the sequence of partial sums of the series does not converge, we say that the series **diverges**.

# Geometric Series

**Geometric series** are series of the form

$$a + ar + ar^2 + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1}$$

in which $a$ and $r$ are fixed real numbers and $a \neq 0$. The series can also be written as $\sum_{n=0}^{\infty} ar^n$. The **ratio** $r$ can be positive, as in

$$1 + \frac{1}{2} + \frac{1}{4} + \cdots + \left(\frac{1}{2}\right)^{n-1} + \cdots, \qquad r = 1/2, a = 1$$

or negative, as in

$$1 - \frac{1}{3} + \frac{1}{9} - \cdots + \left(-\frac{1}{3}\right)^{n-1} + \cdots. \qquad r = -1/3, a = 1$$

# Geometric Series

If $r = 1$, the $n$th partial sum of the geometric series is

$$s_n = a + a(1) + a(1)^2 + \cdots + a(1)^{n-1} = na,$$

and the series diverges because $\lim_{n\to\infty} s_n = \pm\infty$, depending on the sign of $a$. If $r = -1$, the series diverges because the $n$th partial sums alternate between $a$ and 0 and never approach a single limit. If $|r| \neq 1$, we can determine the convergence or divergence of the series in the following way:

$$\begin{aligned}
s_n &= a + ar + ar^2 + \cdots + ar^{n-1} && \text{Write the } n\text{th partial sum.}\\
rs_n &= ar + ar^2 + \cdots + ar^{n-1} + ar^n && \text{Multiply } s_n \text{ by } r.\\
s_n - rs_n &= a - ar^n && \text{Subtract } rs_n \text{ from } s_n\text{. Most of the terms on the right cancel.}\\
s_n(1 - r) &= a(1 - r^n) && \text{Factor.}\\
s_n &= \frac{a(1 - r^n)}{1 - r}, \qquad (r \neq 1). && \text{We can solve for } s_n \text{ if } r \neq 1.
\end{aligned}$$

If $|r| < 1$, then $r^n \to 0$ as $n \to \infty$ (as in Section 10.1), so $s_n \to a/(1 - r)$ in this case. On the other hand, if $|r| > 1$, then $|r^n| \to \infty$ and the series diverges.

# Geometric Series

If $|r| < 1$, the geometric series $a + ar + ar^2 + \cdots + ar^{n-1} + \cdots$ converges to $a/(1 - r)$:

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1 - r}, \qquad |r| < 1.$$

If $|r| \geq 1$, the series diverges.

# Geometric Series

**EXAMPLE** The geometric series with $a = 1/9$ and $r = 1/3$ is

$$\frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \cdots = \sum_{n=1}^{\infty} \frac{1}{9}\left(\frac{1}{3}\right)^{n-1} = \frac{1/9}{1 - (1/3)} = \frac{1}{6}.$$

**EXAMPLE** The series

$$\sum_{n=0}^{\infty} \frac{(-1)^n 5}{4^n} = 5 - \frac{5}{4} + \frac{5}{16} - \frac{5}{64} + \cdots$$

is a geometric series with $a = 5$ and $r = -1/4$. It converges to

$$\frac{a}{1 - r} = \frac{5}{1 + (1/4)} = 4.$$

# The nth-Term Test for a Divergent Series

- One reason that a series may fail to converge is that its terms don't become small.

We now show that $\lim_{n\to\infty} a_n$ must equal zero if the series $\sum_{n=1}^{\infty} a_n$ converges. To see why, let $S$ represent the series' sum and $s_n = a_1 + a_2 + \cdots + a_n$ the $n$th partial sum. When $n$ is large, both $s_n$ and $s_{n-1}$ are close to $S$, so their difference, $a_n$, is close to zero. More formally,

$$a_n = s_n - s_{n-1} \quad \to \quad S - S = 0.$$

Difference Rule for sequences

This establishes the following theorem.

**THEOREM 7** If $\sum_{n=1}^{\infty} a_n$ converges, then $a_n \to 0$.

Theorem 7 leads to a test for detecting the kind of divergence

**The $n$th-Term Test for Divergence**

$\sum_{n=1}^{\infty} a_n$ diverges if $\lim_{n\to\infty} a_n$ fails to exist or is different from zero.

# Combining Series

**THEOREM 8** If $\sum a_n = A$ and $\sum b_n = B$ are convergent series, then

1. *Sum Rule:* $\sum(a_n + b_n) = \sum a_n + \sum b_n = A + B$
2. *Difference Rule:* $\sum(a_n - b_n) = \sum a_n - \sum b_n = A - B$
3. *Constant Multiple Rule:* $\sum ka_n = k\sum a_n = kA$ (any number $k$).

1. Every nonzero constant multiple of a divergent series diverges.
2. If $\sum a_n$ converges and $\sum b_n$ diverges, then $\sum(a_n + b_n)$ and $\sum(a_n - b_n)$ both diverge.

# Combining Series

**EXAMPLE** Find the sums of the following series.

**(a)**
$$\sum_{n=1}^{\infty} \frac{3^{n-1}-1}{6^{n-1}} = \sum_{n=1}^{\infty}\left(\frac{1}{2^{n-1}} - \frac{1}{6^{n-1}}\right)$$
$$= \sum_{n=1}^{\infty}\frac{1}{2^{n-1}} - \sum_{n=1}^{\infty}\frac{1}{6^{n-1}} \qquad \text{Difference Rule}$$
$$= \frac{1}{1-(1/2)} - \frac{1}{1-(1/6)} \qquad \text{Geometric series with } a = 1 \text{ and } r = 1/2, 1/6$$
$$= 2 - \frac{6}{5} = \frac{4}{5}$$

**(b)**
$$\sum_{n=0}^{\infty}\frac{4}{2^n} = 4\sum_{n=0}^{\infty}\frac{1}{2^n} \qquad \text{Constant Multiple Rule}$$
$$= 4\left(\frac{1}{1-(1/2)}\right) \qquad \text{Geometric series with } a = 1, r = 1/2$$
$$= 8$$

# Lecture 34: Infinite Sequences and Series.

MA2032 Vector Calculus

Lecturer: Larissa Serdukova

School of Computing and Mathematical Science
University of Leicester

December 7, 2022

# The Integral Test

• The most basic **question** we can ask about a series **is whether it converges**.

• We begin to study this question, starting with **series that have nonnegative terms**.

• **Such a series converges if its sequence of partial sums is bounded.**

• If we establish that a given series does converge, **we generally do not have a formula available for its sum**.

• So to get an **estimate for the sum** of a convergent series, we investigate the **error** involved when using a **partial sum to approximate the total sum**.

# The Integral Test

**Corollary of Theorem 6**
A series $\sum_{n=1}^{\infty} a_n$ of nonnegative terms converges if and only if its partial sums are bounded from above.

**DEFINITION** A sequence $\{a_n\}$ is **bounded from above** if there exists a number $M$ such that $a_n \leq M$ for all $n$. The number $M$ is an **upper bound** for $\{a_n\}$. If $M$ is an upper bound for $\{a_n\}$ but no number less than $M$ is an upper bound for $\{a_n\}$, then $M$ is the **least upper bound** for $\{a_n\}$.

A sequence $\{a_n\}$ is **bounded from below** if there exists a number $m$ such that $a_n \geq m$ for all $n$. The number $m$ is a **lower bound** for $\{a_n\}$. If $m$ is a lower bound for $\{a_n\}$ but no number greater than $m$ is a lower bound for $\{a_n\}$, then $m$ is the **greatest lower bound** for $\{a_n\}$.

If $\{a_n\}$ is bounded from above and below, then $\{a_n\}$ is **bounded**. If $\{a_n\}$ is not bounded, then we say that $\{a_n\}$ is an **unbounded** sequence.

# The Integral Test

**THEOREM 9—The Integral Test**

Let $\{a_n\}$ be a sequence of positive terms. Suppose that $a_n = f(n)$, where $f$ is a continuous, positive, decreasing function of $x$ for all $x \geq N$ ($N$ a positive integer). Then the series $\sum_{n=N}^{\infty} a_n$ and the integral $\int_N^{\infty} f(x)\,dx$ both converge or both diverge.

# The Integral Test

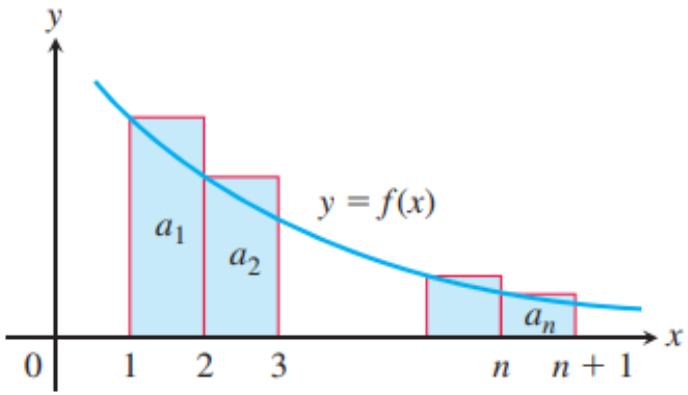
**Proof** We establish the test for the case $N = 1$. The proof for general $N$ is similar.

We start with the assumption that $f$ is a decreasing function with $f(n) = a_n$ for every $n$. This leads us to observe that the rectangles in Figure 10.12a, which have areas $a_1, a_2, \ldots, a_n$, collectively enclose more area than that under the curve $y = f(x)$ from $x = 1$ to $x = n + 1$. That is,

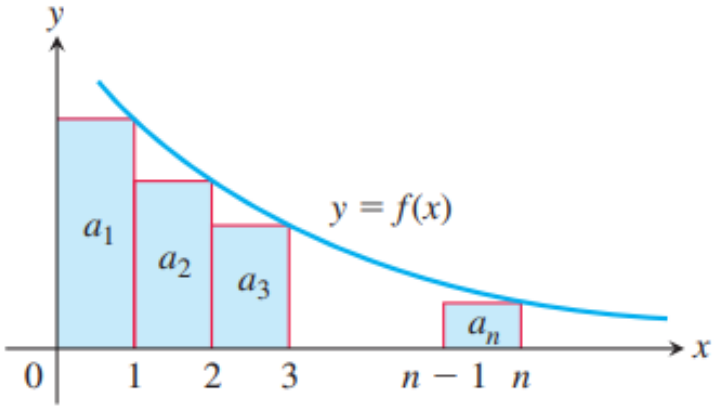
$$\int_1^{n+1} f(x)\,dx \le a_1 + a_2 + \cdots + a_n.$$

In Figure 10.12b the rectangles have been faced to the left instead of to the right. If we momentarily disregard the first rectangle of area $a_1$, we see that

$$a_2 + a_3 + \cdots + a_n \le \int_1^{n} f(x)\,dx.$$




(a)




(b)

# The Integral Test

If we include $a_1$, we have

$$a_1 + a_2 + \cdots + a_n \leq a_1 + \int_1^n f(x)\,dx.$$

Combining these results gives

$$\int_1^{n+1} f(x)\,dx \leq a_1 + a_2 + \cdots + a_n \leq a_1 + \int_1^n f(x)\,dx.$$

These inequalities hold for each $n$, and continue to hold as $n \to \infty$.

If $\int_1^\infty f(x)\,dx$ is finite, the right-hand inequality shows that $\sum a_n$ is finite. If $\int_1^\infty f(x)\,dx$ is infinite, the left-hand inequality shows that $\sum a_n$ is infinite. Hence the series and the integral are either both finite or both infinite. ■

**EXAMPLE 5** Determine the convergence or divergence of the series.

**(a)** $\sum_{n=1}^{\infty} ne^{-n^2}$ **(b)** $\sum_{n=1}^{\infty} \frac{1}{2^{\ln n}}$

**Solutions**

**(a)** We apply the Integral Test and find that

$$\int_1^{\infty} \frac{x}{e^{x^2}}\,dx = \frac{1}{2}\int_1^{\infty} \frac{du}{e^u} \qquad u = x^2,\ du = 2x\,dx$$

$$= \lim_{b\to\infty}\left[-\frac{1}{2}e^{-u}\right]_1^b$$

$$= \lim_{b\to\infty}\left(-\frac{1}{2e^b} + \frac{1}{2e}\right) = \frac{1}{2e}.$$

Since the integral converges, the series also converges.

# Solution. Example

**(b)** Again applying the Integral Test,

$$\int_1^\infty \frac{dx}{2^{\ln x}} = \int_0^\infty \frac{e^u\,du}{2^u} \qquad u = \ln x, x = e^u, dx = e^u\,du$$

$$= \int_0^\infty \left(\frac{e}{2}\right)^u du$$

$$= \lim_{b\to\infty} \frac{1}{\ln\left(\frac{e}{2}\right)}\left(\left(\frac{e}{2}\right)^b - 1\right) = \infty. \qquad (e/2) > 1$$

The improper integral diverges, so the series diverges also.

# Error Estimation

• For some convergent series, such as the geometric series, we can actually find the **total sum of the series**.

• That is, we can find the **limiting value $S$ of the sequence of partial sums**.

• For **most convergent series**, however, **we cannot easily find the total sum**.

• Nevertheless, we can **estimate the sum** by adding the first $n$ terms to get $s_n$, but we need to know **how far off $s_n$ is from the total sum $S$**.

• An approximation to a function or to a number is more useful when it is accompanied by a **bound on the size of the worst possible error** that could occur.

# Error Estimation

**Bounds for the Remainder in the Integral Test**

Suppose $\{a_k\}$ is a sequence of positive terms with $a_k = f(k)$, where $f$ is a continuous positive decreasing function of $x$ for all $x \geq n$, and that $\Sigma a_n$ converges to $S$. Then the remainder $R_n = S - s_n$ satisfies the inequalities

$$\int_{n+1}^{\infty} f(x)\,dx \leq R_n \leq \int_{n}^{\infty} f(x)\,dx. \tag{1}$$

If we add the partial sum $s_n$ to each side of the inequalities in (1), we get

$$s_n + \int_{n+1}^{\infty} f(x)\,dx \leq S \leq s_n + \int_{n}^{\infty} f(x)\,dx \tag{2}$$

since $s_n + R_n = S$. The inequalities in (2) are useful for estimating the error in approximating the sum of a series known to converge by the Integral Test. The error can be no larger than the length of the interval containing $S$, with endpoints given by (2).

# Error Estimation

**EXAMPLE 6** Estimate the sum of the series $\Sigma(1/n^2)$ using the inequalities in (2) and $n = 10$.

**Solution** We have that

$$\int_n^\infty \frac{1}{x^2}dx = \lim_{b\to\infty}\left[-\frac{1}{x}\right]_n^b = \lim_{b\to\infty}\left(-\frac{1}{b}+\frac{1}{n}\right) = \frac{1}{n}.$$

Using this result with the inequalities in (2), we get

$$s_{10} + \frac{1}{11} \le S \le s_{10} + \frac{1}{10}.$$

Taking $s_{10} = 1 + (1/4) + (1/9) + (1/16) + \cdots + (1/100) \approx 1.54977$, these last inequalities give

$$1.64068 \le S \le 1.64977.$$

If we approximate the sum $S$ by the midpoint of this interval, we find that

$$\sum_{n=1}^{\infty}\frac{1}{n^2} \approx 1.6452.$$

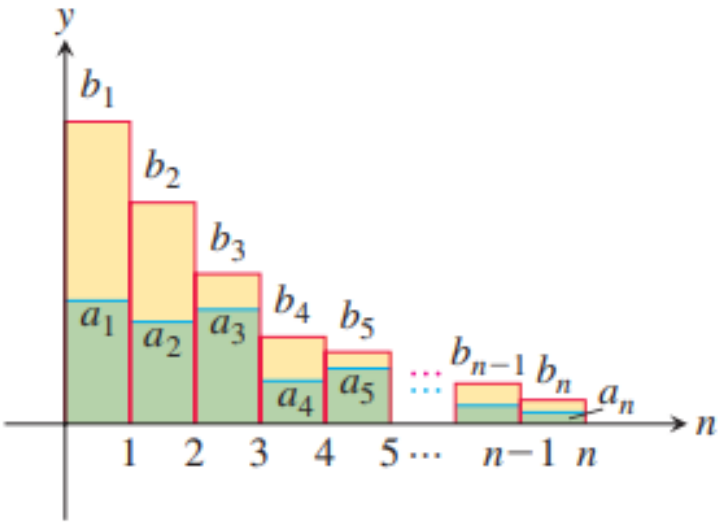
The error in this approximation is then less than half the length of the interval, so the error is less than 0.005. Using a trigonometric *Fourier series* (studied in advanced calculus), it can be shown that $S$ is equal to $\pi^2/6 \approx 1.64493$. ■

# Comparison Tests

• We can **test the convergence** of many more series **by comparing their terms** to those of a series whose **convergence is already known**.

**THEOREM 10—Direct Comparison Test**
Let $\sum a_n$ and $\sum b_n$ be two series with $0 \leq a_n \leq b_n$ for all $n$. Then

1. If $\sum b_n$ converges, then $\sum a_n$ also converges.
2. If $\sum a_n$ diverges, then $\sum b_n$ also diverges.



# Comparison Tests

**Proof** The series $\sum a_n$ and $\sum b_n$ have nonnegative terms. The Corollary of Theorem 6 stated in Section 10.3 tells us that the series $\sum a_n$ and $\sum b_n$ converge if and only if their partial sums are bounded from above.

In Part (1) we assume that $\sum b_n$ converges to some number $M$. The partial sums $\sum_{n=1}^{N} a_n$ are all bounded from above by $M = \sum b_n$, since

$$s_N = a_1 + a_2 + \cdots + a_N \leq b_1 + b_2 + \cdots + b_N \leq \sum_{n=1}^{\infty} b_n = M.$$

Since the partial sums of $\sum a_n$ are bounded from above, the Corollary of Theorem 6 implies that $\sum a_n$ converges. We conclude that when $\sum b_n$ converges, then so does $\sum a_n$. Figure 10.12 illustrates this result, with each term of each series interpreted as the area of a rectangle.

In Part (2), where we assume that $\sum a_n$ diverges, the partial sums of $\sum_{n=1}^{\infty} b_n$ are not bounded from above. If they were, the partial sums for $\sum a_n$ would also be bounded from above, since

$$a_1 + a_2 + \cdots + a_N \leq b_1 + b_2 + \cdots + b_N,$$

and this would mean that $\sum a_n$ converges. We conclude that if $\sum a_n$ diverges, then so does $\sum b_n$. ■

# The Limit Comparison Test

**THEOREM 11—Limit Comparison Test**

Suppose that $a_n > 0$ and $b_n > 0$ for all $n \geq N$ ($N$ an integer).

1. If $\lim_{n\to\infty} \frac{a_n}{b_n} = c$ and $c > 0$, then $\sum a_n$ and $\sum b_n$ both converge or both diverge.
2. If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges.
3. If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.

# The Limit Comparison Test

**Proof** We will prove Part 1. Parts 2 and 3 are left as Exercises 57a and b.

Since $c/2 > 0$, there exists an integer $N$ such that

$$\left|\frac{a_n}{b_n} - c\right| < \frac{c}{2} \qquad \text{whenever} \qquad n > N.$$

Limit definition with $\varepsilon = c/2$, $L = c$, and $a_n$ replaced by $a_n/b_n$

Thus, for $n > N$,

$$-\frac{c}{2} < \frac{a_n}{b_n} - c < \frac{c}{2},$$

$$\frac{c}{2} < \frac{a_n}{b_n} < \frac{3c}{2},$$

$$\left(\frac{c}{2}\right)b_n < a_n < \left(\frac{3c}{2}\right)b_n.$$

If $\sum b_n$ converges, then $\sum(3c/2)b_n$ converges and $\sum a_n$ converges by the Direct Comparison Test. If $\sum b_n$ diverges, then $\sum(c/2)b_n$ diverges and $\sum a_n$ diverges by the Direct Comparison Test. ■

# The Limit Comparison Test

**EXAMPLE 3** Does $\sum_{n=1}^{\infty} \frac{\ln n}{n^{3/2}}$ converge?

**Solution** Because $\ln n$ grows more slowly than $n^c$ for any positive constant $c$ (Section 10.1, Exercise 115), we can compare the series to a convergent $p$-series. To get the $p$-series, we see that

$$\frac{\ln n}{n^{3/2}} < \frac{n^{1/4}}{n^{3/2}} = \frac{1}{n^{5/4}}$$

for $n$ sufficiently large. Then taking $a_n = (\ln n)/n^{3/2}$ and $b_n = 1/n^{5/4}$, we have

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{\ln n}{n^{1/4}} \\
&= \lim_{n\to\infty} \frac{1/n}{(1/4)n^{-3/4}} \qquad \text{l'Hôpital's Rule} \\
&= \lim_{n\to\infty} \frac{4}{n^{1/4}} = 0.
\end{aligned}$$

Since $\sum b_n = \sum(1/n^{5/4})$ is a $p$-series with $p > 1$, it converges. Therefore $\sum a_n$ converges by Part 2 of the Limit Comparison Test. ■

# Lecture 35: Infinite Sequences and Series.

MA2032 Vector Calculus

Lecturer: Larissa Serdukova

School of Computing and Mathematical Science
University of Leicester

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# Absolute Convergence

• For a **general series** with both positive and negative terms, we can apply the **tests for convergence studied before** to the **series of absolute values of its terms**.

• In doing so, we are led naturally to the following concept.

**DEFINITION** A series $\sum a_n$ **converges absolutely** (is **absolutely convergent**) if the corresponding series of absolute values, $\sum |a_n|$, converges.

# Absolute Convergence

**THEOREM 12—The Absolute Convergence Test**

If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

**Proof** For each $n$,

$$-|a_n| \le a_n \le |a_n|, \qquad \text{so} \qquad 0 \le a_n + |a_n| \le 2|a_n|.$$

If $\Sigma_{n=1}^{\infty} |a_n|$ converges, then $\Sigma_{n=1}^{\infty} 2|a_n|$ converges and, by the Direct Comparison Test, the nonnegative series $\Sigma_{n=1}^{\infty} (a_n + |a_n|)$ converges. The equality $a_n = (a_n + |a_n|) - |a_n|$ now lets us express $\Sigma_{n=1}^{\infty} a_n$ as the difference of two convergent series:

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (a_n + |a_n| - |a_n|) = \sum_{n=1}^{\infty} (a_n + |a_n|) - \sum_{n=1}^{\infty} |a_n|.$$

Therefore, $\sum_{n=1}^{\infty} a_n$ converges. ■

# Absolute Convergence

**EXAMPLE 1** This example gives two series that converge absolutely.

**(a)** For $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^2} = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \cdots$, the corresponding series of absolute values is the convergent series

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots.$$

The original series converges because it converges absolutely.

**(b)** For $\sum_{n=1}^{\infty}\frac{\sin n}{n^2} = \frac{\sin 1}{1} + \frac{\sin 2}{4} + \frac{\sin 3}{9} + \cdots$, which contains both positive and negative terms, the corresponding series of absolute values is

$$\sum_{n=1}^{\infty}\left|\frac{\sin n}{n^2}\right| = \frac{|\sin 1|}{1} + \frac{|\sin 2|}{4} + \cdots,$$

which converges by comparison with $\sum_{n=1}^{\infty}(1/n^2)$ because $|\sin n| \leq 1$ for every $n$. The original series converges absolutely; therefore it converges. ■

# The Ratio Test

• The Ratio Test measures **the rate of growth (or decline)** of a series by examining the ratio $a_{n+1}/a_n$.

**THEOREM 13—The Ratio Test**
Let $\sum a_n$ be any series and suppose that

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \rho.$$

Then **(a)** the series *converges absolutely* if $\rho < 1$, **(b)** the series *diverges* if $\rho > 1$ or $\rho$ is infinite, **(c)** the test is *inconclusive* if $\rho = 1$.

# The Ratio Test

**Proof**

**(a)** $\boldsymbol{\rho < 1}$**.** Let $r$ be a number between $\rho$ and 1. Then the number $\varepsilon = r - \rho$ is positive. Since

$$\left|\frac{a_{n+1}}{a_n}\right| \to \rho,$$

$|a_{n+1}/a_n|$ must lie within $\varepsilon$ of $\rho$ when $n$ is large enough, say, for all $n \geq N$. In particular,

$$\left|\frac{a_{n+1}}{a_n}\right| < \rho + \varepsilon = r, \qquad \text{when } n \geq N.$$

Hence

$$\begin{aligned}
|a_{N+1}| &< r|a_N|, \\
|a_{N+2}| &< r|a_{N+1}| < r^2|a_N|, \\
|a_{N+3}| &< r|a_{N+2}| < r^3|a_N|, \\
&\vdots \\
|a_{N+m}| &< r|a_{N+m-1}| < r^m|a_N|.
\end{aligned}$$

# The Ratio Test

Therefore,

$$\sum_{m=N}^{\infty} |a_m| = \sum_{m=0}^{\infty} |a_{N+m}| \leq \sum_{m=0}^{\infty} |a_N| \, r^m = |a_N| \sum_{m=0}^{\infty} r^m.$$

The geometric series on the right-hand side converges because $0 < r < 1$, so the series of absolute values $\sum_{m=N}^{\infty} |a_m|$ converges by the Direct Comparison Test. Because adding or deleting finitely many terms in a series does not affect its convergence or divergence property, the series $\sum_{n=1}^{\infty} |a_n|$ also converges. That is, the series $\sum a_n$ is absolutely convergent.

**(b) $1 < \rho \leq \infty$.** From some index $M$ on,

$$\left| \frac{a_{n+1}}{a_n} \right| > 1 \qquad \text{and} \qquad |a_M| < |a_{M+1}| < |a_{M+2}| < \cdots.$$

The terms of the series do not approach zero as $n$ becomes infinite, and the series diverges by the $n$th-Term Test.

# The Ratio Test

**(c)** $\boldsymbol{\rho = 1}$**.** The two series

$$\sum_{n=1}^{\infty} \frac{1}{n} \qquad \text{and} \qquad \sum_{n=1}^{\infty} \frac{1}{n^2}$$

show that some other test for convergence must be used when $\rho = 1$.

$$\text{For } \sum_{n=1}^{\infty} \frac{1}{n}: \quad \left|\frac{a_{n+1}}{a_n}\right| = \frac{1/(n+1)}{1/n} = \frac{n}{n+1} \to 1.$$

$$\text{For } \sum_{n=1}^{\infty} \frac{1}{n^2}: \quad \left|\frac{a_{n+1}}{a_n}\right| = \frac{1/(n+1)^2}{1/n^2} = \left(\frac{n}{n+1}\right)^2 \to 1^2 = 1.$$

In both cases, $\rho = 1$, yet the first series diverges, whereas the second converges. ■

# The Ratio Test

• The Ratio Test is often **effective** when the terms of a series **contain factorials** of expressions involving $n$ or expressions **raised to a power involving** $n$.

**EXAMPLE 2** Investigate the convergence of the following series.

**(a)** $\displaystyle\sum_{n=0}^{\infty} \frac{2^n + 5}{3^n}$ **(b)** $\displaystyle\sum_{n=1}^{\infty} \frac{(2n)!}{n!n!}$

# The Ratio Test

**Solution** We apply the Ratio Test to each series.

**(a)** For the series $\sum_{n=0}^{\infty}(2^n + 5)/3^n$,

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(2^{n+1} + 5)/3^{n+1}}{(2^n + 5)/3^n} = \frac{1}{3}\cdot\frac{2^{n+1} + 5}{2^n + 5} = \frac{1}{3}\cdot\left(\frac{2 + 5\cdot 2^{-n}}{1 + 5\cdot 2^{-n}}\right) \to \frac{1}{3}\cdot\frac{2}{1} = \frac{2}{3}.$$

The series converges absolutely (and thus converges) because $\rho = 2/3$ is less than 1. This does *not* mean that $2/3$ is the sum of the series. In fact,

$$\sum_{n=0}^{\infty}\frac{2^n + 5}{3^n} = \sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^n + \sum_{n=0}^{\infty}\frac{5}{3^n} = \frac{1}{1 - (2/3)} + \frac{5}{1 - (1/3)} = \frac{21}{2}.$$

**(b)** If $a_n = \dfrac{(2n)!}{n!n!}$, then $a_{n+1} = \dfrac{(2n + 2)!}{(n + 1)!(n + 1)!}$ and

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{n!n!(2n + 2)(2n + 1)(2n)!}{(n + 1)!(n + 1)!(2n)!}$$

$$= \frac{(2n + 2)(2n + 1)}{(n + 1)(n + 1)} = \frac{4n + 2}{n + 1} \to 4.$$

The series diverges because $\rho = 4$ is greater than 1.

# The Root Test

**THEOREM 14—The Root Test**

Let $\sum a_n$ be any series and suppose that

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \rho.$$

Then **(a)** the series *converges absolutely* if $\rho < 1$, **(b)** the series *diverges* if $\rho > 1$ or $\rho$ is infinite, **(c)** the test is *inconclusive* if $\rho = 1$.

# The Root Test

**Proof**

**(a)** $\boldsymbol{\rho < 1}$**.** Choose an $\varepsilon > 0$ so small that $\rho + \varepsilon < 1$. Since $\sqrt[n]{|a_n|} \to \rho$, the terms $\sqrt[n]{|a_n|}$ eventually get to within $\varepsilon$ of $\rho$. So there exists an index $M$ such that

$$\sqrt[n]{|a_n|} < \rho + \varepsilon \qquad \text{when } n \geq M.$$

Then it is also true that

$$|a_n| < (\rho + \varepsilon)^n \qquad \text{for } n \geq M.$$

Now, $\sum_{n=M}^{\infty}(\rho + \varepsilon)^n$ is a geometric series with ratio $(\rho + \varepsilon) < 1$ and therefore converges. By the Direct Comparison Test, $\sum_{n=M}^{\infty} |a_n|$ converges. Adding finitely many terms to a series does not affect its convergence or divergence, so the series

$$\sum_{n=1}^{\infty} |a_n| = |a_1| + \cdots + |a_{M-1}| + \sum_{n=M}^{\infty} |a_n|$$

also converges. Therefore, $\sum a_n$ converges absolutely.

**(b)** $\boldsymbol{1 < \rho \leq \infty}$**.** For all indices beyond some integer $M$, we have $\sqrt[n]{|a_n|} > 1$, so that $|a_n| > 1$ for $n > M$. The terms of the series do not converge to zero. The series diverges by the $n$th-Term Test.

**(c)** $\boldsymbol{\rho = 1}$**.** The series $\sum_{n=1}^{\infty}(1/n)$ and $\sum_{n=1}^{\infty}(1/n^2)$ show that the test is not conclusive when $\rho = 1$. The first series diverges and the second converges, but in both cases $\sqrt[n]{|a_n|} \to 1$. ■

# The Root Test

**EXAMPLE 3** Consider again the series with terms $a_n = \begin{cases} n/2^n, & n \text{ odd} \\ 1/2^n, & n \text{ even}. \end{cases}$

Does $\sum a_n$ converge?

**Solution** We apply the Root Test, finding that

$$\sqrt[n]{|a_n|} = \begin{cases} \sqrt[n]{n}/2, & n \text{ odd} \\ 1/2, & n \text{ even}. \end{cases}$$

Therefore,

$$\frac{1}{2} \leq \sqrt[n]{|a_n|} \leq \frac{\sqrt[n]{n}}{2}.$$

Since $\sqrt[n]{n} \to 1$ (Section 10.1, Theorem 5), we have $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1/2$ by the Sandwich Theorem. The limit is less than 1, so the series converges absolutely by the Root Test. ■

# The Root Test

**EXAMPLE 4** Which of the following series converge, and which diverge?

**(a)** $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ **(b)** $\sum_{n=1}^{\infty} \frac{2^n}{n^3}$ **(c)** $\sum_{n=1}^{\infty} \left(\frac{1}{1+n}\right)^n$

**Solution** We apply the Root Test to each series, noting that each series has positive terms.

**(a)** $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ converges because $\sqrt[n]{\frac{n^2}{2^n}} = \frac{\sqrt[n]{n^2}}{\sqrt[n]{2^n}} = \frac{\left(\sqrt[n]{n}\right)^2}{2} \to \frac{1^2}{2} < 1.$

**(b)** $\sum_{n=1}^{\infty} \frac{2^n}{n^3}$ diverges because $\sqrt[n]{\frac{2^n}{n^3}} = \frac{2}{\left(\sqrt[n]{n}\right)^3} \to \frac{2}{1^3} > 1.$

**(c)** $\sum_{n=1}^{\infty} \left(\frac{1}{1+n}\right)^n$ converges because $\sqrt[n]{\left(\frac{1}{1+n}\right)^n} = \frac{1}{1+n} \to 0 < 1.$ ■

# Lecture 36: Infinite Sequences and Series.

MA2032 Vector Calculus

Lecturer: Larissa Serdukova

School of Computing and Mathematical Science
University of Leicester

December 12, 2022

# Alternating Series and Conditional Convergence

• A series in which the terms are alternately positive and negative is an **alternating series**.

• Here are three **examples**:

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots + \frac{(-1)^{n+1}}{n} + \cdots \tag{1}$$

$$-2 + 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots + \frac{(-1)^n 4}{2^n} + \cdots \tag{2}$$

$$1 - 2 + 3 - 4 + 5 - 6 + \cdots + (-1)^{n+1} n + \cdots \tag{3}$$

• We see from these examples that the $n$-**th term** of an alternating series **is of the form**

$$a_n = (-1)^{n+1} u_n \qquad \text{or} \qquad a_n = (-1)^n u_n$$

• where $u_n = |a_n|$ is a positive number.

# Alternating Series and Conditional Convergence

• Series (1), called the **alternating harmonic series**, **converges**, as we will see in a moment.

• Series (2), a **geometric series** with ratio $r = -1/2$, **converges** to $-2/[1 + (1/2)] = -4/3$.

• Series (3) **diverges** because the $n$-th term does not approach zero.

• We prove the convergence of the alternating harmonic series by applying the **Alternating Series Test**.

• This test is for convergence of an alternating series and **cannot be used to conclude** that such a **series diverges**.

# Alternating Series and Conditional Convergence

**THEOREM 15—The Alternating Series Test**

The series

$$\sum_{n=1}^{\infty}(-1)^{n+1}u_n = u_1 - u_2 + u_3 - u_4 + \cdots$$

converges if the following conditions are satisfied:

1. The $u_n$'s are all positive.
2. The $u_n$'s are eventually nonincreasing: $u_n \geq u_{n+1}$ for all $n \geq N$, for some integer $N$.
3. $u_n \to 0$.

# Alternating Series and Conditional Convergence

**Proof** We look at the case where $u_1, u_2, u_3, u_4, \ldots$ is nonincreasing, so that $N = 1$. If $n$ is an even integer, say $n = 2m$, then the sum of the first $n$ terms is

$$\begin{aligned} s_{2m} &= (u_1 - u_2) + (u_3 - u_4) + \cdots + (u_{2m-1} - u_{2m}) \\ &= u_1 - (u_2 - u_3) - (u_4 - u_5) - \cdots - (u_{2m-2} - u_{2m-1}) - u_{2m}. \end{aligned}$$

The first equality shows that $s_{2m}$ is the sum of $m$ nonnegative terms, since each term in parentheses is positive or zero. Hence $s_{2m+2} \geq s_{2m}$, and the sequence $\{s_{2m}\}$ is nondecreasing. The second equality shows that $s_{2m} \leq u_1$. Since $\{s_{2m}\}$ is nondecreasing and bounded from above, it has a limit, say

$$\lim_{m \to \infty} s_{2m} = L. \qquad \text{Theorem 6} \tag{4}$$

If $n$ is an odd integer, say $n = 2m + 1$, then the sum of the first $n$ terms is $s_{2m+1} = s_{2m} + u_{2m+1}$. Since $u_n \to 0$,

$$\lim_{m \to \infty} u_{2m+1} = 0$$

and, as $m \to \infty$,

$$s_{2m+1} = s_{2m} + u_{2m+1} \to L + 0 = L. \tag{5}$$

Combining the results of Equations (4) and (5) gives $\lim_{n \to \infty} s_n = L$ ■

# Alternating Series and Conditional Convergence

• **Rather than directly verifying** the definition $u_n \geq u_{n+1}$, a **second way** to show that the sequence $\{u_n\}$ **is nonincreasing** is to **define a differentiable function** $f(x)$ satisfying $f(n) = u_n$.

• That is, the values of $f$ match the values of the sequence at every positive integer $n$.

• If $f'(x) \leq 0$ for all $x$ greater than or equal to some positive integer $N$, then $f(x)$ **is nonincreasing** for $x \geq N$.

• It follows that $(n) \geq (n+1)$, or $u_n \geq u_{n+1}$, for $n \geq N$.

**EXAMPLE** We show that the sequence $u_n = 10n/(n^2 + 16)$ is eventually nonincreasing. Define $f(x) = 10x/(x^2 + 16)$. Then from the Derivative Quotient Rule,

$$f'(x) = \frac{10(16 - x^2)}{(x^2 + 16)^2} \leq 0 \qquad \text{whenever } x \geq 4.$$

It follows that $u_n \geq u_{n+1}$ for $n \geq 4$. That is, the sequence $\{u_n\}$ is nonincreasing for $n \geq 4$. ■

# Alternating Series and Conditional Convergence

• A graphical interpretation of the partial sums **shows how an alternating series converges** to its limit L when the three conditions of Theorem 15 are satisfied with $N = 1$.

• The **limit $L$ lies between any two successive sums $s_n$ and $s_{n+1}$** and hence differs from $s_n$ by an amount less than $u_{n+1}$.

• Because $|L - s_n| \leq u_{n+1}$ for $n \geq N$, we can make **useful estimates of the sums of convergent alternating series**.

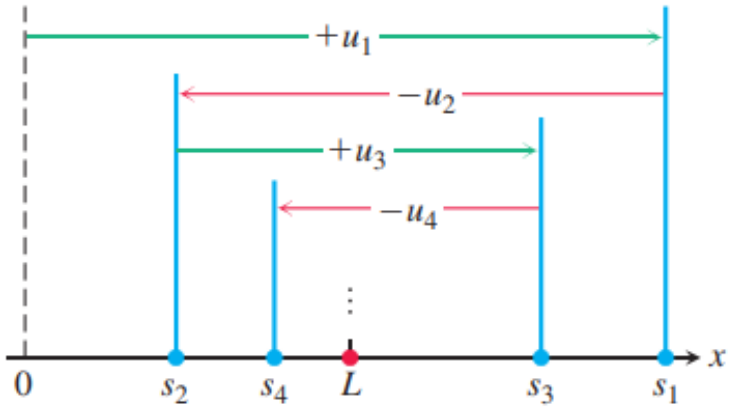



**FIGURE 10.15** The partial sums of an alternating series that satisfies the hypotheses of Theorem 15 for $N = 1$ straddle the limit from the beginning.

# Alternating Series and Conditional Convergence

**THEOREM 16—The Alternating Series Estimation Theorem**
If the alternating series $\sum_{n=1}^{\infty}(-1)^{n+1}u_n$ satisfies the three conditions of Theorem 15, then for $n \geq N$,

$$s_n = u_1 - u_2 + \cdots + (-1)^{n+1}u_n$$

approximates the sum $L$ of the series with an error whose absolute value is less than $u_{n+1}$, the absolute value of the first unused term. Furthermore, the sum $L$ lies between any two successive partial sums $s_n$ and $s_{n+1}$, and the remainder, $L - s_n$, has the same sign as the first unused term.

# Alternating Series and Conditional Convergence

**EXAMPLE** We try Theorem 16 on a series whose sum we know:

$$\sum_{n=0}^{\infty}(-1)^n\frac{1}{2^n} = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \frac{1}{32} + \frac{1}{64} - \frac{1}{128} + \frac{1}{256} - \cdots.$$

The theorem says that if we truncate the series after the eighth term, we throw away a total that is positive and less than $1/256$. The sum of the first eight terms is $s_8 = 0.6640625$ and the sum of the first nine terms is $s_9 = 0.66796875$. The sum of the geometric series is

$$\frac{1}{1-(-1/2)} = \frac{1}{3/2} = \frac{2}{3},$$

and we note that $0.6640625 < (2/3) < 0.66796875$. The difference, $(2/3) - 0.6640625 = 0.0026041666\ldots$, is positive and is less than $(1/256) = 0.00390625$. ■

# Conditional Convergence

• If we **replace all the negative terms** in the alternating series in Example (slide 10), **changing them to positive terms** instead, we obtain the geometric series $\sum 1/2^n$.

• The **original series and the new series of absolute values both converge** (although to different sums).

• For an **absolutely convergent series**, changing infinitely many of the negative terms in the series to positive values **does not change its property of still being a convergent series**.

• **Other convergent series may behave differently**.

# Conditional Convergence

• The **convergent alternating harmonic series** has infinitely many negative terms, but if we change its negative terms to positive values, **the resulting series is the divergent harmonic series**.

• So the presence of **infinitely many negative terms is essential to the convergence** of the alternating harmonic series.

• The following **terminology distinguishes these two types of convergent series**.

DEFINITION A series that is convergent but not absolutely convergent is called **conditionally convergent**.

# Conditional Convergence

**EXAMPLE 4** If $p$ is a positive constant, the sequence $\{1/n^p\}$ is a decreasing sequence with limit zero. Therefore, the alternating $p$-series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^p} = 1 - \frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} + \cdots, \quad p > 0$$

converges.

If $p > 1$, the series converges absolutely as an ordinary $p$-series. If $0 < p \leq 1$, the series converges conditionally by the alternating series test. For instance,

$$\text{Absolute convergence } (p = 3/2): \quad 1 - \frac{1}{2^{3/2}} + \frac{1}{3^{3/2}} - \frac{1}{4^{3/2}} + \cdots$$

$$\text{Conditional convergence } (p = 1/2): \quad 1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \cdots$$

# Rearranging Series

• We can always **rearrange the terms of a finite collection of numbers** without changing their sum.

• The same result **is true for an infinite series that is absolutely convergent**.

**THEOREM 17—The Rearrangement Theorem for Absolutely Convergent Series**
If $\sum_{n=1}^{\infty} a_n$ converges absolutely, and $b_1, b_2, \ldots, b_n, \ldots$ is any arrangement of the sequence $\{a_n\}$, then $\sum b_n$ converges absolutely and

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} a_n.$$

• On the other hand, **if we rearrange the terms of a conditionally convergent series, we can get different results**.

# Conditional Convergence

• Next Example shows that **we cannot rearrange the terms of a conditionally convergent series** and expect the new series to be the same as the original one.

• When we use a **conditionally convergent series**, the **terms must be added together in the order they are given** to obtain a correct result.

• **In contrast, Theorem 17** guarantees that the terms of an absolutely convergent series can be summed in any order without affecting the result.

# Conditional Convergence

**EXAMPLE** We know that the alternating harmonic series $\sum_{n=1}^{\infty}(-1)^{n+1}/n$ converges to some number $L$. Moreover, by Theorem 16, $L$ lies between the successive partial sums $s_2 = 1/2$ and $s_3 = 5/6$, so $L \neq 0$. If we multiply the series by 2 we obtain

$$2L = 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n} = 2\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + \frac{1}{11} - \cdots\right)$$

$$= 2 - 1 + \frac{2}{3} - \frac{1}{2} + \frac{2}{5} - \frac{1}{3} + \frac{2}{7} - \frac{1}{4} + \frac{2}{9} - \frac{1}{5} + \frac{2}{11} - \cdots.$$

Now we change the order of this last sum by grouping each pair of terms with the same odd denominator, but leaving the negative terms with the even denominators as they are

# Conditional Convergence

placed (so the denominators are the positive integers in their natural order). This rearrangement gives

$$
\begin{aligned}
(2-1)-\frac{1}{2}+\left(\frac{2}{3}-\frac{1}{3}\right)-\frac{1}{4}+\left(\frac{2}{5}-\frac{1}{5}\right)-\frac{1}{6}+\left(\frac{2}{7}-\frac{1}{7}\right)-\frac{1}{8}+\cdots \\
=\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}+\frac{1}{9}-\frac{1}{10}+\frac{1}{11}-\cdots\right) \\
=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}=L.
\end{aligned}
$$

So by rearranging the terms of the conditionally convergent series $\sum_{n=1}^{\infty}2(-1)^{n+1}/n$, the series becomes $\sum_{n=1}^{\infty}(-1)^{n+1}/n$, which is the alternating harmonic series itself. If the two series are the same, it would imply that $2L = L$, which is clearly false since $L \neq 0$. ■

# Summary of Tests to Determine Convergence or Divergence

1. **The $n$th-Term Test for Divergence:** Unless $a_n \to 0$, the series diverges.
2. **Geometric series:** $\sum ar^n$ converges if $|r| < 1$; otherwise it diverges.
3. **$p$-series:** $\sum 1/n^p$ converges if $p > 1$; otherwise it diverges.
4. **Series with nonnegative terms:** Try the Integral Test or try comparing to a known series with the Direct Comparison Test or the Limit Comparison Test. Try the Ratio or Root Test.
5. **Series with some negative terms:** Does $\sum |a_n|$ converge by the Ratio or Root Test, or by another of the tests listed above? Remember, absolute convergence implies convergence.
6. **Alternating series:** $\sum a_n$ converges if the series satisfies the conditions of the Alternating Series Test.

# Lecture 37: Infinite Sequences and Series.

MA2032 Vector Calculus

Lecturer: Larissa Serdukova

School of Computing and Mathematical Science
University of Leicester

December 14, 2022

# Power Series and Convergence

• Now that we can test many infinite series of numbers for convergence, we can study **sums that look like "infinite polynomials."**

• We call these sums **power series** because they are defined as infinite series of powers of some variable, in our case $x$.

• Like polynomials, power series **can be added, subtracted, multiplied, differentiated, and integrated** to give new power series.

• With power series we can **extend the methods of calculus** to a vast array of functions, making the techniques of calculus applicable in an even wider setting.

# Power Series and Convergence

• We begin with the **formal definition**, which specifies the **notation and terminology** used for power series.

**DEFINITIONS** **A power series about $x = 0$** is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots. \tag{1}$$

**A power series about $x = a$** is a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n + \cdots \tag{2}$$

in which the **center** $a$ and the **coefficients** $c_0, c_1, c_2, \ldots, c_n, \ldots$ are constants.

• We will see that a **power series defines a function** $f(x)$ on a certain interval where it converges.

• Moreover, this function will be shown **to be continuous and differentiable** over the interior of that interval.

**EXAMPLE 1** Taking all the coefficients to be 1 in Equation (1) gives the geometric power series

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots + x^n + \cdots.$$

This is the geometric series with first term 1 and ratio $x$. It converges to $1/(1-x)$ for $|x| < 1$. We express this fact by writing

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots, \qquad -1 < x < 1. \tag{3}$$

■

• We think of the partial sums of the series on the right as **polynomials** $P_n(x)$ that **approximate the function** on the left.

# Power Series and Convergence

- For **values of** $x$ **near zero**, we need take only a **few terms of the series to get a good approximation**.

- As we **move toward** $x = 1$, or $-1$, we must **take more terms**.

- The function $f(x) = 1/(1-x)$ is **not continuous** on intervals containing $x = 1$, where it has a vertical asymptote.

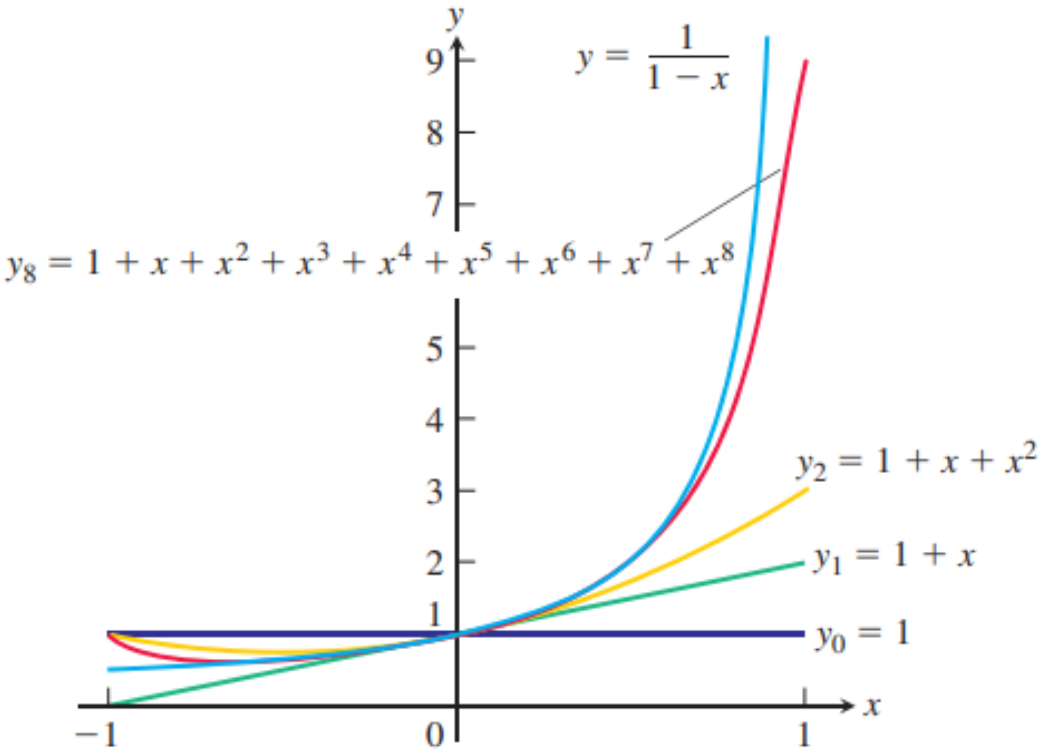



**FIGURE** The graphs of $f(x) = 1/(1-x)$ in Example 1 and four of its polynomial approximations.

- The **approximations do not apply** when $x \geq 1$.

# Power Series and Convergence

• The following example illustrates **how we test a power series for convergence** by using the Ratio Test to see where it converges and diverges.

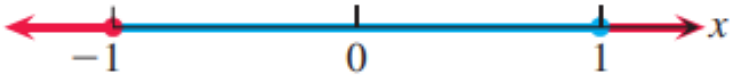
**EXAMPLE 3** For what values of $x$ do the following power series converge?

**(a)** $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$

**Solution** Apply the Ratio Test to the series $\sum |u_n|$, where $u_n$ is the $n$th term of the power series in question.

**(a)** $\displaystyle\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{x^{n+1}}{n+1}\cdot\frac{n}{x}\right| = \frac{n}{n+1}|x| \to |x|.$

By the Ratio Test, the series converges absolutely for $|x| < 1$ and diverges for $|x| > 1$. At $x = 1$, we get the alternating harmonic series $1 - 1/2 + 1/3 - 1/4 + \cdots$, which converges. At $x = -1$, we get $-1 - 1/2 - 1/3 - 1/4 - \cdots$, the negative of the harmonic series, which diverges. Series (a) converges for $-1 < x \le 1$ and diverges elsewhere.



# Power Series and Convergence

• We will see that **this series converges to the function** $\ln(1+x)$ on the interval $(-1, 1]$

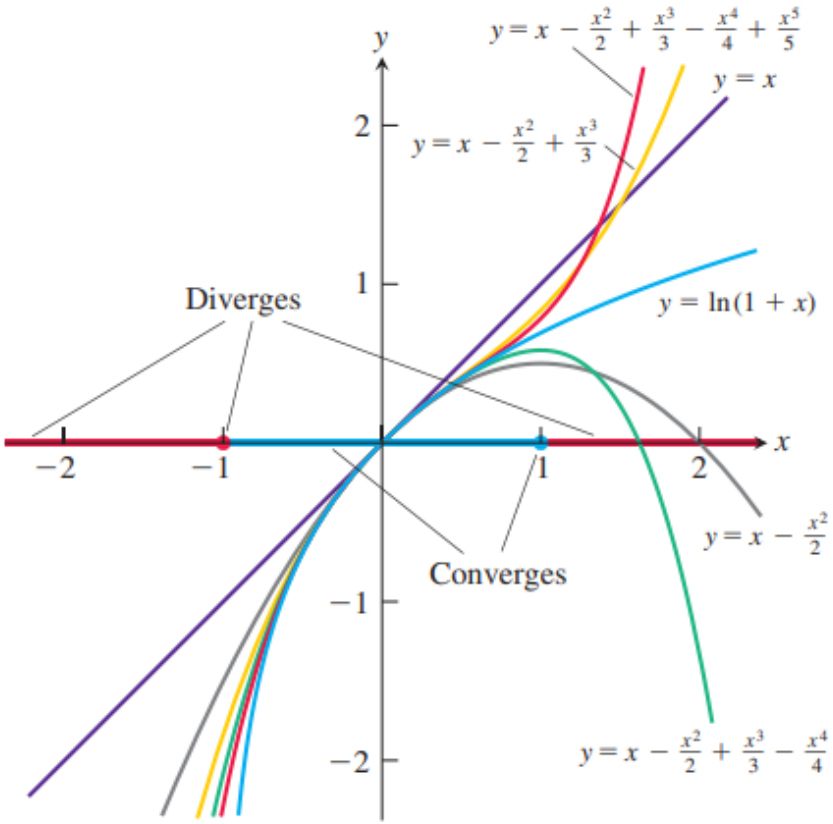



**FIGURE** The power series $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$ converges on the interval $(-1, 1]$.

# Power Series and Convergence

• The next result shows that if a power series converges at more than one value, then it **converges over an entire interval of values**.

• The interval might be **finite or infinite** and contain one, both, or none of its endpoints.

• We will see that **each endpoint** of a finite interval **must be tested independently** for convergence or divergence.

**THEOREM 18—The Convergence Theorem for Power Series**

If the power series

$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$ converges at $x = c \neq 0$, then it converges absolutely for all $x$ with $|x| < |c|$. If the series diverges at $x = d$, then it diverges for all $x$ with $|x| > |d|$.

# The Radius of Convergence of a Power Series

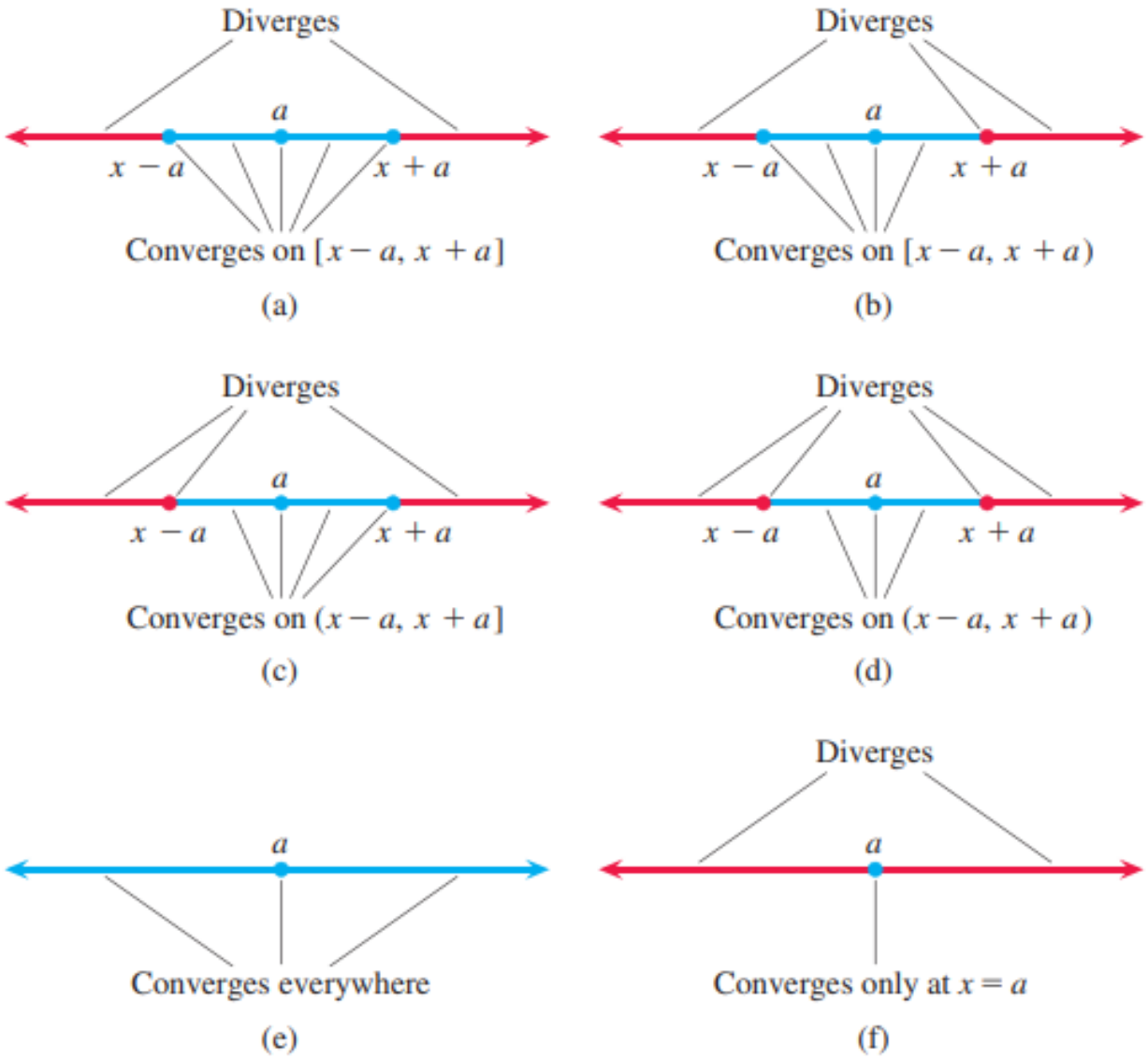



**FIGURE** The six possibilities for an interval of convergence.

# Power Series and Convergence

**Corollary to Theorem 18**

The convergence of the series $\sum c_n(x-a)^n$ is described by one of the following three cases:

1. There is a positive number $R$ such that the series diverges for $x$ with $|x-a| > R$ but converges absolutely for $x$ with $|x-a| < R$. The series may or may not converge at either of the endpoints $x = a - R$ and $x = a + R$.
2. The series converges absolutely for every $x$ ($R = \infty$).
3. The series converges at $x = a$ and diverges elsewhere ($R = 0$).

• R is the **radius of convergence** of the power series, and the interval of radius R centered at $x = a$ is the **interval of convergence**.

• The interval of convergence may be open, closed, or half-open.

• At points $x$ with $|x - a| < R$, the series **converges absolutely**.

• If the series converges for all values of $x$, then $R$ **is infinite**. If it converges only at $x = a$, then $R = 0$.

# Power Series and Convergence

### How to Test a Power Series for Convergence

1. Use the Ratio Test (or Root Test) to find the largest open interval where the series converges absolutely,

$$|x - a| < R \qquad \text{or} \qquad a - R < x < a + R.$$

2. If $R$ is finite, test for convergence or divergence at each endpoint, as in Examples 3a and b. Use a Comparison Test, the Integral Test, or the Alternating Series Test.
3. If $R$ is finite, the series diverges for $|x - a| > R$ (it does not even converge conditionally) because the $n$th term does not approach zero for those values of $x$.

# Operations on Power Series

**THEOREM 19—Series Multiplication for Power Series**
If $A(x) = \sum_{n=0}^{\infty} a_n x^n$ and $B(x) = \sum_{n=0}^{\infty} b_n x^n$ converge absolutely for $|x| < R$, and

$$c_n = a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \cdots + a_{n-1} b_1 + a_n b_0 = \sum_{k=0}^{n} a_k b_{n-k},$$

then $\sum_{n=0}^{\infty} c_n x^n$ converges absolutely to $A(x)B(x)$ for $|x| < R$:

$$\left(\sum_{n=0}^{\infty} a_n x^n\right)\left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty} c_n x^n.$$

**THEOREM 20** If $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely for $|x| < R$ and $f$ is a continuous function, then $\sum_{n=0}^{\infty} a_n (f(x))^n$ converges absolutely on the set of points $x$ where $|f(x)| < R$.

# Operations on Power Series

**THEOREM 21—Term-by-Term Differentiation**

If $\sum c_n(x-a)^n$ has radius of convergence $R > 0$, it defines a function

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n \qquad \text{on the interval} \qquad a - R < x < a + R.$$

This function $f$ has derivatives of all orders inside the interval, and we obtain the derivatives by differentiating the original series term by term:

$$f'(x) = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1},$$

$$f''(x) = \sum_{n=2}^{\infty} n(n-1)c_n(x-a)^{n-2},$$

and so on. Each of these derived series converges at every point of the interval $a - R < x < a + R$.

# Operations on Power Series

**EXAMPLE** Find series for $f'(x)$ and $f''(x)$ if

$$f(x) = \frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots + x^n + \cdots$$

$$= \sum_{n=0}^{\infty} x^n, \qquad -1 < x < 1.$$

**Solution** We differentiate the power series on the right term by term:

$$f'(x) = \frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots + nx^{n-1} + \cdots$$

$$= \sum_{n=1}^{\infty} nx^{n-1}, \qquad -1 < x < 1;$$

$$f''(x) = \frac{2}{(1-x)^3} = 2 + 6x + 12x^2 + \cdots + n(n-1)x^{n-2} + \cdots$$

$$= \sum_{n=2}^{\infty} n(n-1)x^{n-2}, \qquad -1 < x < 1.$$

# Operations on Power Series

**THEOREM 22—Term-by-Term Integration**

Suppose that

$$f(x) = \sum_{n=0}^{\infty} c_n (x - a)^n$$

converges for $a - R < x < a + R (R > 0)$. Then

$$\sum_{n=0}^{\infty} c_n \frac{(x - a)^{n+1}}{n + 1}$$

converges for $a - R < x < a + R$ and

$$\int f(x)\, dx = \sum_{n=0}^{\infty} c_n \frac{(x - a)^{n+1}}{n + 1} + C$$

for $a - R < x < a + R$.

# Operations on Power Series

**EXAMPLE** Identify the function

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots, \qquad -1 \le x \le 1.$$

**Solution** We differentiate the original series term by term and get

$$f'(x) = 1 - x^2 + x^4 - x^6 + \cdots, \qquad -1 < x < 1. \qquad \text{Theorem 21}$$

This is a geometric series with first term 1 and ratio $-x^2$, so

$$f'(x) = \frac{1}{1-(-x^2)} = \frac{1}{1+x^2}.$$

We can now integrate $f'(x) = 1/(1+x^2)$ to get

$$\int f'(x)\,dx = \int \frac{dx}{1+x^2} = \tan^{-1} x + C.$$

The series for $f(x)$ is zero when $x = 0$, so $C = 0$. Hence

$$f(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots = \tan^{-1} x, \qquad -1 < x < 1. \tag{6}$$

It can be shown that the series also converges to $\tan^{-1} x$ at the endpoints $x = \pm 1$, but we omit the proof. ■

# Operations on Power Series

**EXAMPLE** The series

$$\frac{1}{1+t} = 1 - t + t^2 - t^3 + \cdots$$

converges on the open interval $-1 < t < 1$. Therefore,

$$\ln(1+x) = \int_0^x \frac{1}{1+t}dt = t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \cdots \Big]_0^x \qquad \text{Theorem 22}$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

or

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}, \qquad -1 < x < 1.$$

It can also be shown that the series converges at $x = 1$ to the number $\ln 2$, but that was not guaranteed by the theorem. A proof of this is outlined in Exercise . ■

# Lecture 38: Infinite Sequences and Series.

MA2032 Vector Calculus

Lecturer: Larissa Serdukova

School of Computing and Mathematical Science
University of Leicester

December 16, 2022

# Fourier Series

• We can express a given function as an **infinite sum of sines and/or cosines**.

• These trigonometric series are called **Fourier series**;

• they are somewhat analogous to Taylor series in that both types of series provide a means of **expressing quite complicated functions in terms of certain familiar elementary functions**.

# Fourier Series

• We begin with a series of the form Eq. (1)

$$\frac{a_0}{2} + \sum_{m=1}^{\infty} \left( a_m \cos \frac{m\pi x}{L} + b_m \sin \frac{m\pi x}{L} \right).$$

• On the set of points where the **series (1) converges**, it defines a function $f$, whose value at each point is the sum of the series for that value of $x$.

• In this case the series (1) is said to be the **Fourier series for** $f$.

• Our **immediate goals** are to determine **what functions** can be represented by a Fourier series and to **find the coefficients** $a_0$**,** $a_m$ **and** $b_m$.

• Fourier series are useful in **partial differential equations**, in the analysis of **mechanical or electrical systems** acted on by periodic external forces, and etc.

# Periodicity of the Sine and Cosine Functions

- To discuss Fourier series, it is necessary to develop **certain properties of the trigonometric functions** $\sin(m\pi x/L)$ and $\cos(m\pi x/L)$, where $m$ is a positive integer.

- The **first property** is their **periodic character**.

- A function $f$ is said to be **periodic with period** $T > 0$ if the domain of $f$ contains $x + T$ whenever it contains $x$, and if Eq. (2)

$$f(x + T) = f(x)$$

- for every value of $x$.

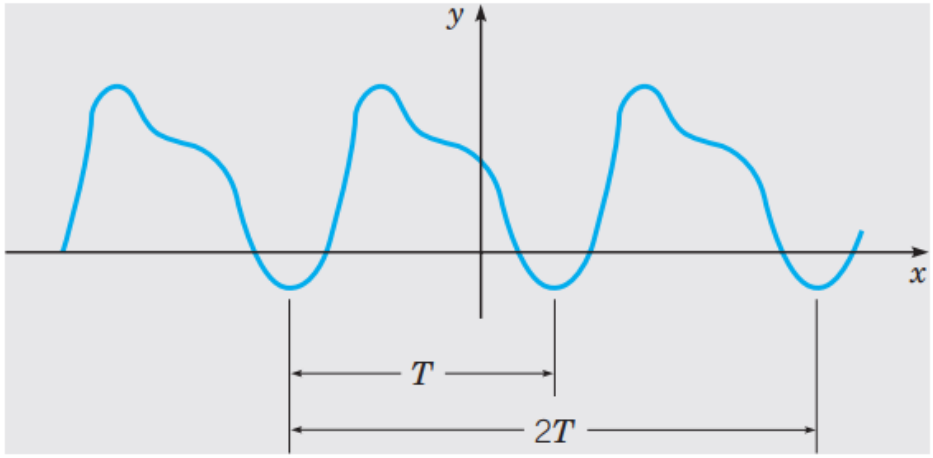



**FIGURE** A periodic function of period $T$.

# Periodicity of the Sine and Cosine Functions

• If $T$ is a period of $f$, then $2T$ is also a period, and so indeed is **any multiple of** $T$.

• The smallest value of $T$ for which Eq. (2) holds is called the **fundamental period of** $f$.

• If $f$ and $g$ are any two periodic functions with common period $T$, then any **linear combination** $c_1 f + c_2 g$ is **also periodic** with period $T$.

• To prove this statement, let $F(x) = c_1 f(x) + c_2 g(x)$; then, for any $x$,

$$F(x+T) = c_1 f(x+T) + c_2 g(x+T) = c_1 f(x) + c_2 g(x) = F(x).$$

• Moreover, it can be shown that the **sum of any finite number**, or even the sum of a convergent infinite series, of functions of period $T$ is also periodic with period $T$.

In a similar way, you can show that the **product** $f\ g$ **is periodic** with period $T$.

# Orthogonality of the Sine and Cosine Functions

• In particular, the functions $\sin(m\pi x/L)$ and $\cos(m\pi x/L)$, $m = 1, 2, 3, \ldots$, are periodic with **fundamental period** $T = 2L/m$ and **common period** $2L$.

• To describe a second essential property of the functions $\sin(m\pi x/L)$ and $\cos(m\pi x/L)$, we **generalize the concept of orthogonality** of vectors.

• The **standard inner product** $(u, v)$ of two real-valued functions $u$ and $v$ on the interval $\alpha \leq x \leq \beta$ is defined by

$$(u, v) = \int_{\alpha}^{\beta} u(x)v(x)\,dx.$$

• The functions $u$ and $v$ are said to be **orthogonal** on $\alpha \leq x \leq \beta$ if their inner **product is zero**, that is, if

$$\int_{\alpha}^{\beta} u(x)v(x)\,dx = 0.$$

# Orthogonality of the Sine and Cosine Functions

• A set of functions is said to be **mutually orthogonal** if each distinct pair of functions in the set is orthogonal.

$$\int_{-L}^{L} \cos\frac{m\pi x}{L}\cos\frac{n\pi x}{L}\,dx = \begin{cases} 0, & m \neq n, \\ L, & m = n; \end{cases}$$

$$\int_{-L}^{L} \cos\frac{m\pi x}{L}\sin\frac{n\pi x}{L}\,dx = 0, \qquad \text{all } m, n;$$

$$\int_{-L}^{L} \sin\frac{m\pi x}{L}\sin\frac{n\pi x}{L}\,dx = \begin{cases} 0, & m \neq n, \\ L, & m = n. \end{cases}$$

# Orthogonality of the Sine and Cosine Functions

• These results can be obtained by **direct integration**. For example, to derive last Equation

$$\int_{-L}^{L} \sin\frac{m\pi x}{L}\sin\frac{n\pi x}{L}\,dx = \frac{1}{2}\int_{-L}^{L}\left[\cos\frac{(m-n)\pi x}{L} - \cos\frac{(m+n)\pi x}{L}\right]dx$$

$$= \frac{1}{2}\frac{L}{\pi}\left\{\frac{\sin[(m-n)\pi x/L]}{m-n} - \frac{\sin[(m+n)\pi x/L]}{m+n}\right\}\Bigg|_{-L}^{L}$$

$$= 0$$

• as long as $m+n$ and $m-n$ are not zero.

• Since $m$ and $n$ are positive, $m+n \neq 0$.

# Orthogonality of the Sine and Cosine Functions

• On the other hand, if $mn = 0$, then $m = n$, and the integral must be **evaluated in different way**.

$$\begin{aligned}
\int_{-L}^{L} \sin\frac{m\pi x}{L}\sin\frac{n\pi x}{L}\,dx &= \int_{-L}^{L}\left(\sin\frac{m\pi x}{L}\right)^2 dx \\
&= \frac{1}{2}\int_{-L}^{L}\left[1-\cos\frac{2m\pi x}{L}\right]dx \\
&= \frac{1}{2}\left\{x-\frac{\sin(2m\pi x/L)}{2m\pi/L}\right\}\Bigg|_{-L}^{L} \\
&= L.
\end{aligned}$$

# The Euler–Fourier Formulas

***The Euler–Fourier Formulas.*** Now let us suppose that a series of the form (1) converges, and let us call its sum $f(x)$:

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} \left( a_m \cos \frac{m\pi x}{L} + b_m \sin \frac{m\pi x}{L} \right). \tag{9}$$

The coefficients $a_m$ and $b_m$ can be related to $f(x)$ as a consequence of the orthogonality conditions (6), (7), and (8). First multiply Eq. (9) by $\cos(n\pi x/L)$, where $n$ is a *fixed* positive integer ($n > 0$), and integrate with respect to $x$ from $-L$ to $L$. Assuming that the integration can be legitimately carried out term by term,[3] we obtain

$$\int_{-L}^{L} f(x) \cos \frac{n\pi x}{L}\, dx = \frac{a_0}{2} \int_{-L}^{L} \cos \frac{n\pi x}{L}\, dx + \sum_{m=1}^{\infty} a_m \int_{-L}^{L} \cos \frac{m\pi x}{L} \cos \frac{n\pi x}{L}\, dx$$

$$+ \sum_{m=1}^{\infty} b_m \int_{-L}^{L} \sin \frac{m\pi x}{L} \cos \frac{n\pi x}{L}\, dx. \tag{10}$$

# The Euler–Fourier Formulas

Keeping in mind that $n$ is fixed whereas $m$ ranges over the positive integers, it follows from the orthogonality relations (6) and (7) that the only nonzero term on the right side of Eq. (10) is the one for which $m = n$ in the first summation. Hence

$$\int_{-L}^{L} f(x) \cos \frac{n\pi x}{L}\, dx = La_n, \qquad n = 1, 2, \ldots. \tag{11}$$

To determine $a_0$, we can integrate Eq. (9) from $-L$ to $L$, obtaining

$$\begin{aligned} \int_{-L}^{L} f(x)\, dx &= \frac{a_0}{2} \int_{-L}^{L} dx + \sum_{m=1}^{\infty} a_m \int_{-L}^{L} \cos \frac{m\pi x}{L}\, dx + \sum_{m=1}^{\infty} b_m \int_{-L}^{L} \sin \frac{m\pi x}{L}\, dx \\ &= La_0, \end{aligned} \tag{12}$$

# The Euler–Fourier Formulas

since each integral involving a trigonometric function is zero. Thus

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx, \qquad n = 0, 1, 2, \ldots. \tag{13}$$

By writing the constant term in Eq. (9) as $a_0/2$, it is possible to compute all the $a_n$ from Eq. (13). Otherwise, a separate formula would have to be used for $a_0$.

A similar expression for $b_n$ may be obtained by multiplying Eq. (9) by $\sin(n\pi x/L)$, integrating termwise from $-L$ to $L$, and using the orthogonality relations (7) and (8); thus

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx, \qquad n = 1, 2, 3, \ldots. \tag{14}$$

Equations (13) and (14) are known as the Euler–Fourier formulas for the coefficients in a Fourier series. Hence, if the series (9) converges to $f(x)$, and if the series can be integrated term by term, then the coefficients *must be given* by Eqs. (13) and (14).

# Fourier series. Example

Assume that there is a Fourier series converging to the function $f$ defined by

$$f(x) = \begin{cases} -x, & -2 \le x < 0, \\ x, & 0 \le x < 2; \end{cases} \qquad f(x+4) = f(x). \tag{15}$$

Determine the coefficients in this Fourier series.

This function represents a triangular wave (see Figure 10.2.2) and is periodic with period 4. Thus in this case $L = 2$, and the Fourier series has the form

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} \left( a_m \cos \frac{m\pi x}{2} + b_m \sin \frac{m\pi x}{2} \right), \tag{16}$$

where the coefficients are computed from Eqs. (13) and (14) with $L = 2$. Substituting for $f(x)$ in Eq. (13) with $m = 0$, we have

$$a_0 = \frac{1}{2} \int_{-2}^{0} (-x)\, dx + \frac{1}{2} \int_{0}^{2} x\, dx = 1 + 1 = 2. \tag{17}$$

# Fourier series. Example

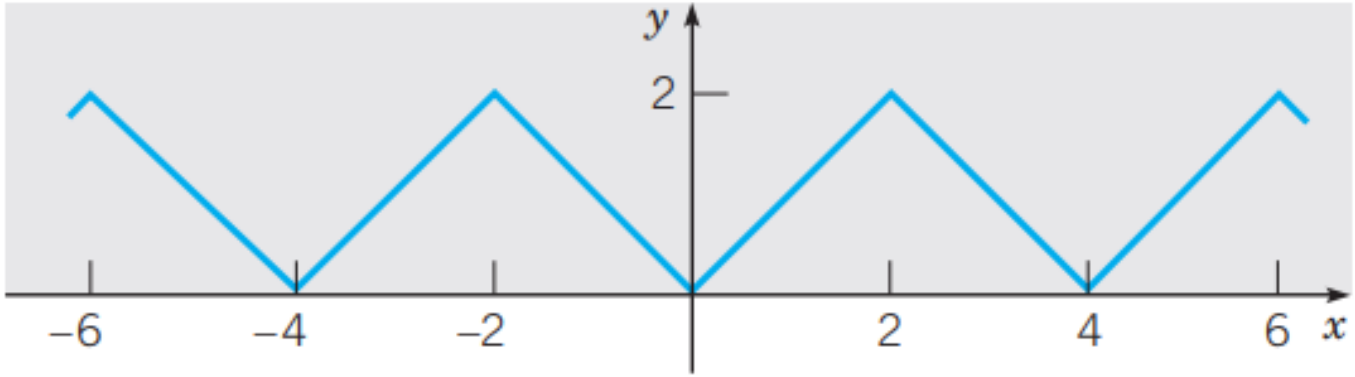



**FIGURE** The triangular wave in Example 1.

For $m > 0$, Eq. (13) yields

$$a_m = \frac{1}{2}\int_{-2}^{0}(-x)\cos\frac{m\pi x}{2}\,dx + \frac{1}{2}\int_{0}^{2} x\cos\frac{m\pi x}{2}\,dx.$$

These integrals can be evaluated through integration by parts, with the result that

# Fourier series. Example

$$a_m = \frac{1}{2}\left[-\frac{2}{m\pi}x\sin\frac{m\pi x}{2} - \left(\frac{2}{m\pi}\right)^2\cos\frac{m\pi x}{2}\right]\Bigg|_{-2}^{0}$$

$$+\frac{1}{2}\left[\frac{2}{m\pi}x\sin\frac{m\pi x}{2} + \left(\frac{2}{m\pi}\right)^2\cos\frac{m\pi x}{2}\right]\Bigg|_{0}^{2}$$

$$= \frac{1}{2}\left[-\left(\frac{2}{m\pi}\right)^2 + \left(\frac{2}{m\pi}\right)^2\cos m\pi + \left(\frac{2}{m\pi}\right)^2\cos m\pi - \left(\frac{2}{m\pi}\right)^2\right]$$

$$= \frac{4}{(m\pi)^2}(\cos m\pi - 1), \qquad m = 1, 2, \ldots$$

$$= \begin{cases} -8/(m\pi)^2, & m \text{ odd}, \\ 0, & m \text{ even}. \end{cases} \tag{18}$$

# Fourier series. Example

Finally, from Eq. (14) it follows in a similar way that

$$b_m = 0, \qquad m = 1, 2, \ldots. \tag{19}$$

By substituting the coefficients from Eqs. (17), (18), and (19) in the series (16), we obtain the Fourier series for $f$:

$$\begin{aligned} f(x) &= 1 - \frac{8}{\pi^2}\left(\cos\frac{\pi x}{2} + \frac{1}{3^2}\cos\frac{3\pi x}{2} + \frac{1}{5^2}\cos\frac{5\pi x}{2} + \cdots\right) \\ &= 1 - \frac{8}{\pi^2}\sum_{m=1,3,5,\ldots}^{\infty}\frac{\cos(m\pi x/2)}{m^2} \\ &= 1 - \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{\cos(2n-1)\pi x/2}{(2n-1)^2}. \end{aligned} \tag{20}$$