MA3071 Financial Mathematics - DLI, Year 2023-2024

Exercises for Feedback session 6

Exercise 1 *Calculate* $\mathbb{E}\left[\left(\int_0^t 5u + B_u^2\,dB_u\right)\left(\int_0^t u^2 + B_u\,dB_u\right)\right]$.

Exercise 2 *Let $\{B_t,\, t \geqslant 0\}$ be a standard Brownian motion. Consider a continuous time market with one non-risky bond with a fixed interest rate of $\rho = 0.2$ and one risky asset. The asset price is modelled by a stochastic process $\{S_t,\, t \geqslant 0\}$, which is defined as a solution to the following stochastic differential equation*

$$dS_t = \mu S_t dt + 0.5 S_t dB_t$$

And the current asset price is $S_0 = 2$.

i) *Show that the solution of the SDE is a GBM.*

ii) *Find the value of μ such that $e^{-\rho t}S_t$ is a martingale.*

iii) *Derive the distribution of S_t.*

iv) *Find the expectation and variance of S_t.*

Exercise 3 *Find $\mathbb{E}[S_T^k|\mathcal{F}_t]$, where $S_t = S_0 e^{at+\sigma B_t}$ and k is constant.*

Solution 1 Using the General Ito Isometry,

$$\begin{aligned}
\mathbb{E}\left[\left(\int_0^t 5u + B_u^2\,dB_u\right)\left(\int_0^t u^2 + B_u\,dB_u\right)\right] &= \mathbb{E}\left[\int_0^t (5u + B_u^2)(u^2 + B_u)du\right] \\
&= \int_0^t \mathbb{E}\left[5u^3 + 5uB_u + u^2B_u^2 + B_u^3\right] du \\
&= \int_0^t 5u^3 + 0 + u^3 + 0\,du \\
&= \int_0^t 6u^3 du \\
&= \frac{6}{4}u^4\Big|_0^t \\
&= \frac{3}{2}t^4
\end{aligned}$$

Solution 2 i) Method 1: Assume $S_t = S_0 e^{xt+yB_t}$ and let $f(t, B_t) = S_0 e^{xt+yB_t}$,

$$\begin{aligned} f'_t &= S_0 e^{xt+yB_t} \cdot x = xS_t \\ f'_{B_t} &= S_0 e^{xt+yB_t} \cdot y = yS_t \\ f''_{B_tB_t} &= S_0 e^{xt+yB_t} \cdot y \cdot y = y^2 S_t \end{aligned}$$

Using the classical Ito's lemma,

$$\begin{aligned} df(t, B_t) &= f'_{B_t} dB_t + f'_t dt + \frac{1}{2} f''_{B_tB_t} dt \\ &= yS_t dB_t + xS_t dt + \frac{1}{2} y^2 S_t dt \\ &= (x + \frac{1}{2} y^2) S_t dt + yS_t dB_t \end{aligned}$$

We know $dS_t = \mu S_t dt + 0.5 S_t dB_t$, so we have

$$\begin{cases} x + \frac{1}{2}y^2 = \mu \\ y = 0.5 \end{cases} \quad \Rightarrow \quad \begin{cases} x = \mu - \frac{1}{8} \\ y = 0.5 \end{cases}$$

Thus,

$$S_t = S_0 e^{(\mu - \frac{1}{8})t + 0.5B_t}$$

Method 2: Let $f(t, S_t) = \log(S_t)$.

$$d[\log(S_t)] = df(t, S_t) = f'_t dt + f'_{S_t} dS_t + \frac{1}{2} f''_{S_tS_t} (dS_t)^2$$

Applying

$$\begin{aligned} f'_{S_t} &= \frac{1}{S_t} \\ f''_{S_tS_t} &= -\frac{1}{S_t^2} \\ f'_t &= 0 \\ dS_t &= \mu S_t dt + \sigma S_t dB_t \\ (dS_t)^2 &= \sigma^2 S_t^2 dt \end{aligned}$$

We derive

$$\begin{aligned} d[\log(S_t)] &= 0 + \frac{1}{S_t}(\mu S_t dt + \sigma S_t dB_t) - \frac{1}{2S_t^2} \sigma^2 S_t^2 dt \\ &= \left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma dB_t \end{aligned}$$

Then, we have

$$\begin{aligned} \log(S_t) &= \log(S_0) + \int_0^t \left(\mu - \frac{1}{2}\sigma^2\right) du + \int_0^t \sigma dB_u \\ &= \log(S_0) + \left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma B_t \end{aligned}$$

Hence, we obtain the geometric Brownian motion,

$$S_t = S_0 e^{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma B_t}$$

Since $dS_t = \mu S_t dt + 0.5 S_t dB_t$ and $\sigma = 0.5$, it can be written as

$$S_t = S_0 e^{(\mu - \frac{1}{8})t + 0.5 B_t}$$

ii) Let $f(t, S_t) = e^{-\rho t} S_t$

$$\begin{aligned}
&f'_t = e^{-\rho t} S_t(-\rho) = -\rho e^{-\rho t} S_t \\
&f'_{S_t} = e^{-\rho t} \\
&f''_{S_t S_t} = 0
\end{aligned}$$

$$\begin{aligned}
df(t, S_t) &= f'_t dt + f'_{S_t} dS_t + \frac{1}{2} f''_{S_t S_t} (dS_t)^2 \\
&= -\rho e^{-\rho t} S_t dt + e^{-\rho t} dS_t + 0 \\
&= -\rho e^{-\rho t} S_t dt + e^{-\rho t}(\mu S_t dt + 0.5 S_t dB_t) \\
&= e^{-\rho t}(\mu - \rho) S_t dt + e^{-\rho t} 0.5 S_t dB_t \\
&= 0 + e^{-\rho t} 0.5 S_t dB_t
\end{aligned}$$

for martingale, there should be no dt term, so iff $\mu - \rho = 0$, which implies $\mu = \rho = 0.2$.

iii) As B_t has a normal distribution, then S_t follows a log-normal distribution,

$$\log\left(\frac{S_t}{S_0}\right) = \log(S_t) - \log(S_0) \sim N\left[\left(\mu - \frac{\sigma^2}{2}\right)t, \sigma^2 t\right]$$

$$\log(S_t) \sim N\left[\left(\mu - \frac{\sigma^2}{2}\right)t + \log(S_0), \sigma^2 t\right]$$

We know $\sigma = 0.5$, $\mu = 0.2$, $S_0 = 2$,

$$\left(\mu - \frac{\sigma^2}{2}\right)t = \left(0.2 - \frac{0.5^2}{2}\right)t = 0.075t$$

$$\sigma^2 t = 0.5^2 t = 0.25t$$

$$\log(S_0) = \log(2)$$

Thus,

$$\log(S_t) \sim N(0.075t + \log(2), 0.25t)$$

iv)

$$\mathbb{E}\left[S_t\right] = S_0 e^{\mu t} = 2e^{0.2t}$$

$$Var\left[S_t\right] = S_0^2 e^{2\mu t}\left(e^{\sigma^2 t} - 1\right) = 4e^{0.4t}\left(e^{0.25t} - 1\right)$$

Solution 3

$$\begin{aligned}
\mathbb{E}\left[S_T^k \middle| \mathcal{F}_t\right] &= \mathbb{E}\left[\left(S_t e^{a(T-t)+\sigma(B_T-B_t)}\right)^k \middle| \mathcal{F}_t\right] \\
&= \mathbb{E}\left[S_t^k \left(e^{a(T-t)+\sigma(B_T-B_t)}\right)^k \middle| \mathcal{F}_t\right] \\
&= S_t^k \mathbb{E}\left[\left(e^{a(T-t)+\sigma(B_T-B_t)}\right)^k \middle| \mathcal{F}_t\right] \\
&= S_t^k e^{\left(ka+\frac{k^2\sigma^2}{2}\right)(T-t)}
\end{aligned}$$