UNIVERSITY OF LEICESTER

MA3071
All candidates

Semester 1 Examinations 2024

MA3071 FINANCIAL MATHEMATICS
FORMULA SHEET

(I) Binomial Tree Models

(1) Hedging Portfolio for single period tree:

$$V_0 = \phi S_0 + \psi, \qquad \phi = \frac{C_u - C_d}{S_0(u-d)}, \qquad \psi = \frac{uC_d - dC_u}{(1+\rho)(u-d)}$$

(II) Standard Brownian Motion

(1) $\mathbb{E}[f(t,B_t)|\mathcal{F}_s] = \mathbb{E}[f(t,B_s+(B_t-B_s))|\mathcal{F}_s]$;

(2) $\mathbb{E}[f(s,B_s)|\mathcal{F}_s] = f(s,B_s)$;

(3) $\mathbb{E}[(B_t-B_s)^{2m+1}|\mathcal{F}_s] = 0, \;\; m = 0,1,...;$

(4) $\mathbb{E}[(B_t-B_s)^{2m}|\mathcal{F}_s] = (t-s)^m(2m-1)!!, \;\; m = 0,1,...,;$

(5) If $f(t,B_t) = g(t)h(B_t), \;\; \mathbb{E}[g(t)h(B_t)|\mathcal{F}_s] = g(t)\mathbb{E}[h(B_t)|\mathcal{F}_s]$;

(6) $\mathbb{E}[g(t)|\mathcal{F}_s] = g(t)$ and $Var[g(t)] = 0$;

for all $t > s \geqslant 0$.

(III) Stochastic Integrals

(1) $\int_0^t Y_u dB_u \sim N\left(0, \mathbb{E}\left[\int_0^t Y_u^2 du\right]\right)$;

(2) $Var\left[\int_0^t A_u du\right] = 0$;

(3) If $X_t = X_0 + \int_0^t A_u du + \int_0^t Y_u dB_u$, then $X_t \sim N\left(X_0 + \int_0^t \mathbb{E}[A_u]du, \int_0^t \mathbb{E}\left[Y_u^2\right]du\right)$.

(IV) Ito's Lemma

(1) $df(t,B_t) = f_t' dt + f_{B_t}' dB_t + \frac{1}{2} f_{B_tB_t}'' dt$;

(2) $df(t,X_t) = f_t' dt + f_{X_t}' dX_t + \frac{1}{2} f_{X_tX_t}'' (dX_t)^2$;

(3) If $dX_t = A_t dt + Y_t dB_t$, then $df(t,X_t) = (f_t' + A_t f_{X_t}' + \frac{1}{2} Y_t^2 f_{X_tX_t}'')dt + Y_t f_{X_t}' dB_t$.

(V) Geometric Brownian Motion

(1) $\log\left(\frac{S_t}{S_0}\right) \sim N\left[\left(\mu - \frac{\sigma^2}{2}\right)t, \sigma^2 t\right]$;

(2) $\mathbb{E}[S_t] = S_0 e^{\mu t}$;

(3) $Var[S_t] = S_0^2 e^{2\mu t}\left(e^{\sigma^2 t} - 1\right)$;

(4) $\mathbb{E}[f(S_T)|\mathcal{F}_t] = \mathbb{E}\left[f\left(S_t e^{a(T-t)+\sigma(B_T-B_t)}\right)\middle|\mathcal{F}_t\right]$;

(5) $\mathbb{E}[f(S_t)|\mathcal{F}_t] = f(S_t)$;

(6) $\mathbb{E}[S_T|\mathcal{F}_t] = S_t e^{\mu(T-t)}$;

(7) $\mathbb{E}\left[\left(e^{a(T-t)+\sigma(B_T-B_t)}\right)^k\middle|\mathcal{F}_t\right] = e^{\left(ka+\frac{k^2\sigma^2}{2}\right)(T-t)}$, k is a constant;

(8) $S_t \perp e^{a(T-t)+\sigma(B_T-B_t)}$;

for all $T > t \geqslant 0$.

(VI) Black-Scholes Model

(1) Let $g(t, S_t)$ be the option price at time t and S_t be a GBM, under the no arbitrage condition, the Black-Scholes equation is: $g'_t + \rho S_t g'_{S_t} + \frac{1}{2}\sigma^2 {S_t}^2 g''_{S_t S_t} = \rho g$;

(2) If no dividend payment:

- European call option price at time t: $g(t, S_t) = S_t \Phi(d_1) - K e^{-\rho(T-t)} \Phi(d_2)$;
- European put option price at time t: $g(t, S_t) = K e^{-\rho(T-t)} \Phi(-d_2) - S_t \Phi(-d_1)$.

where $\Phi(x)$ is the cumulative distribution function of a standard normal random variable, and

- $d_1 = \frac{log\left(\frac{S_t}{K}\right) + \left(\rho + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}$,
- $d_2 = d_1 - \sigma\sqrt{T-t}$,

(3) Put-call parity: $P_t + S_t = C_t + K e^{-\rho(T-t)}$;

(4) For European options with no dividend, the exact expressions of the Greeks are

		European Call Option	European Put Option
$\boldsymbol{\Delta}$ (Delta)	$\frac{\partial g}{\partial S_t}$	$\Phi(d_1)$	$-\Phi(-d_1)$
$\boldsymbol{\Gamma}$ (Gamma)	$\frac{\partial^2 g}{\partial S_t^2}$	$\frac{\phi(d_1)}{S_t \sigma \sqrt{T-t}}$	$\frac{\phi(d_1)}{S_t \sigma \sqrt{T-t}}$
$\boldsymbol{\nu}$ (Vega)	$\frac{\partial g}{\partial \sigma}$	$S_t \phi(d_1) \sqrt{T-t}$	$S_t \phi(d_1) \sqrt{T-t}$
$\boldsymbol{\rho}$ (Rho)	$\frac{\partial g}{\partial \rho}$	$K(T-t) e^{-\rho(T-t)} \Phi(d_2)$	$-K(T-t) e^{-\rho(T-t)} \Phi(-d_2)$
$\boldsymbol{\Theta}$ (Theta)	$\frac{\partial g}{\partial t}$	$-\frac{S_t \phi(d_1) \sigma}{2\sqrt{T-t}} - \rho K e^{-\rho(T-t)} \Phi(d_2)$	$-\frac{S_t \phi(d_1) \sigma}{2\sqrt{T-t}} + \rho K e^{-\rho(T-t)} \Phi(-d_2)$

where $\phi(x)$ is the density function of a standard normal distribution such that $\phi(x) = \Phi'(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$.

(VII) Capital Asset Pricing Models

(1) Slope of capital allocation line (CAL): $\frac{\mathbb{E}[R_t] - \rho}{\sigma_t}$;

(2) Equation of capital market line (CML): $\mathbb{E}[R_p] - \rho = \left(\frac{\mathbb{E}[R_M] - \rho}{\sigma_M}\right) \sigma_p$;

(3) Equation relating the return on any individual asset to the return on the market portfolio: $\mathbb{E}[R_i] - \rho = \frac{Cov(R_i, R_M)}{\sigma_M^2} (\mathbb{E}[R_M] - \rho)$.