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Homework Sheet 1, available online September 27th, online delivery October 4th before 4:00 pm UK time.

(Vectors and the Geometry of Space)

Problem 1. (3 points) Let $u = \langle 1, 2, 2\rangle$, $v = \langle 1, -1, -1\rangle$, $w = \langle 1, 3, -1\rangle$, and $z = \langle 2, 11, 8\rangle$. Write $z = u_1 + u_2 + u_3$, where u_1 is parallel to u, u_2 is parallel to v, and u_3 is parallel to w. What are u_1, u_2, u_3?

Solution:

let $u_1 = \langle k_1, 2k_1, 2k_1\rangle$ $u_2 = \langle k_2, -k_2, -k_2\rangle$, $u_3 = \langle k_3, 3k_3, -k_3\rangle$

Hence. $z = u_1 + u_2 + u_3 = \langle k_1 + k_2 + k_3, 2k_1 - k_2 + 3k_3, 2k_1 - k_2 - k_3\rangle = \langle 2, 11, 8\rangle$,

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and $k_1 = \frac{10}{3}$, $k_2 = -\frac{25}{12}$, $k_3 = \frac{3}{4}$

We have $u_1 = \langle \frac{10}{3}, \frac{20}{3}, \frac{20}{3}\rangle$, $u_2 = \langle -\frac{25}{12}, \frac{25}{12}, \frac{25}{12}\rangle$, $u_3 = \langle \frac{3}{4}, \frac{9}{4}, -\frac{3}{4}\rangle$

Problem 2. (3 points) Suppose that AB is the diameter of a circle with center O and that C is a point on one of the two arcs joining A and B. Show that $\overrightarrow{CA}$ and $\overrightarrow{CB}$ are orthogonal.

Solution: let O be the original B be the x-axis and orthogonal with $\overrightarrow{AB}$ be the y-axis

the $\overrightarrow{OB} = \langle n, 0\rangle$, $\overrightarrow{OA} = \langle -n, 0\rangle$ $\overrightarrow{OC} = \langle n\cdot\cos\theta, n\cdot\sin\theta\rangle$

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$\overrightarrow{CA} = \overrightarrow{OA} - \overrightarrow{OC} = \langle n(-1-\cos\theta), -n\cdot\sin\theta\rangle$, $\overrightarrow{CB} = \overrightarrow{OB} - \overrightarrow{OC} = \langle n(1-\cos\theta), -n\sin\theta\rangle$

$\langle \overrightarrow{CA}, \overrightarrow{CB}\rangle = n^2(-1-\cos\theta)(1-\cos\theta) + n^2\sin^2\theta = n^2\cos^2\theta + n^2\sin^2\theta - n^2 = n^2 - n^2 = 0$

Hence. $\overrightarrow{CA}$ and $\overrightarrow{CB}$ is orthogonal

Problem 3. **(3 points)** Determine whether the points $A(1,1,1)$, $B(-1,0,4)$, $C(0,2,1)$, and $D(2,-2,3)$ are coplanar.

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Solution: $\vec{AB} = \langle -2, -1, 3 \rangle$ $\cancel{\vec{AC} = \langle -1, 1, 0 \rangle}$. $\vec{AD} = \langle 1, -3, 2 \rangle$

$\vec{n_1} = \vec{AB} \times \vec{AC} = \langle -3, -3, -3 \rangle$

$\vec{n_2} = \vec{AB} \times \vec{AD} = \langle 7, 7, 5 \rangle$

Vol = abs $\begin{vmatrix} -2 & -1 & 3 \\ -1 & 1 & 0 \\ 1 & -3 & 2 \end{vmatrix} = 0 \Rightarrow$ points are coplanar

Since, $\vec{n_1} \parallel \vec{n_2}$ $\vec{AB}$, $\vec{AC}$ and $\vec{AD}$ are coplanar, which means

A, B, C, D are coplanar

Problem 4. **(3 points)** Find a plane through the origin that is perpendicular to the plane $M: \ 2x+3y+z = 12$ in a right angle. How do you know that your plane is perpendicular to M?

Solution: Since plane M_1 through the origin. M_1: $ax+by+z=0$

In addition. M_1 perpendicular to M. which means their norms are also perpendicular that is $\langle (a,b,c), (2,3,1) \rangle = 0$

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let $a=1$, $y=1$, $z=-5$, M_1: $x+y-5z=0$

Any plane $Ax+By+Cz=0$ with $2A+3B+C=0$

Problem 5. **(8=2+1+1+1+1+2 points)** The parallelogram shown here has vertices at $A(2,-1,4)$, $B(1,0,-1)$, $C(1,2,3)$, and D. Find

a) the coordinates of D,

b) the cosine of the interior angle at B,

c) the vector projection of $\overrightarrow{BA}$ onto $\overrightarrow{BC}$,

d) the area of the parallelogram,

e) an equation for the plane of the parallelogram,

f) the areas of the orthogonal projections of the parallelogram on the three coordinate planes.

Solution:

a) $\overrightarrow{BD} = \overrightarrow{BA} + \overrightarrow{BC} = D - B$, Hence, $D = (2, 1, 8)$ 2

b) $\langle \overrightarrow{BA} \cdot \overrightarrow{BC} \rangle = \|\overrightarrow{BA}\| \|\overrightarrow{BC}\| \cdot \cos\theta$, Hence, $\theta = 39.23^\circ$ $\cos\theta = \frac{\sqrt{15}}{5}$ 0

c) The projection $\vec{P} = \frac{\langle BA, BC \rangle}{|BC|^2} \cdot \overrightarrow{BC} = (0, \frac{9}{5}, \frac{18}{5})$ 1

d) $\|\overrightarrow{BA} \times \overrightarrow{BC}\| = 6\sqrt{6}$ 1

e) $\overrightarrow{BA} \times \overrightarrow{BC} = (-14, 4, 2)$, and cross $B(1, 0, -1)$

Hence an equation is $-14(x-1) + 4y - 2(z+1) = 0$

$-7x + 2y - z = -6$ $7x - 2y + z = 8$ 0

f) On x-y plane

$2\|(1, -1, 0) \times (0, 2, 0)\|$

$= 2$

On x-z plane

$\|(1, 0, 5) \times (0, 0, 4)\|$

$= 4$

On y-z plane

$\|(0, -1, 5) \times (0, 2, 4)\|$

$= 14$