$\boldsymbol{Chapter\ 2}$. *The heat equation*

Fourier's law of the heat conduction: it says that the amount of heat passing through the piece during a short period dt, which is denoted by dQ , is formulated by

$$dQ = -k\ \frac{\partial u}{\partial n} dS\ dt,$$

Where $k(x, y, z)$ *is a coefficient measuring the thermal conductivity of the medium.*

For a given infinitesimally small area dS, the rate in the amount of heat passing through it should be calculated by $\frac{dQ}{dt} = -k\frac{\partial u}{\partial n} dS$.

The law of heat capacity: it says that the heat units needed for altering the temperature of an objective are proportional to its mass, and the temperature difference, i.e.

$$dQ = c\rho dxdydzdu,$$

Where the proportionality c is the heat capacity, and ρ *is the material mass density.*

To derive a PDE for u, we consider an arbitrary domain Ω *during a time interval* (t_1, t_2)*. As from the law of heat capacity, the heat units required to bring about a temperature change from* $u|_{t=t_1}$ *to* $u|_{t=t_2}$ *is formulated by*

$$\Delta Q = \iiint_\Omega \ c\rho u|_{t_1}^{t_2} dxdydz = \int_{t_1}^{t_2} \iiint_\Omega \ c\rho \frac{\partial u}{\partial t} dxdydz \ . \quad (1)$$

According to Fourier's law of heat transfer, the net heat units flowing out of the system boundary $\partial\Omega$ *(along the outer normal direction* $\boldsymbol{n}$*) is calculated by*

$$\Delta Q = \int_{t_1}^{t_2} \iint_{\partial\Omega} -k \frac{\partial u}{\partial n} dS\, dt = -\int_{t_1}^{t_2} \iiint_{\Omega} k \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) dxdydz\, dt\,. \quad (2)$$

As the total thermal energy is conserved, we have the heat equation as follows

$$\frac{\partial u}{\partial t} = \mathrm{a}^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) + \boxed{f(x, y, z, t)},\ where\ a^2 = \frac{k}{\rho c}.$$

The one dimensional heat equation is

$$u_t = a^2 u_{xx} + f(x, t).$$

A revisit to the method of separation of variable.

Here we are focused on solving an initial boundary value problem for the heat equation given by

$$\begin{cases} u_t - a^2 u_{xx} = 0, (x,t) \in (0,l) \times R^+ \\ u|_{t=0} = \phi(x) \\ u|_{x=0} = 0, \quad \left(\frac{\partial u}{\partial x} + hu\right)|_{x=l} = 0. \end{cases} \qquad (1)$$

Solution . We look for solutions taking the form of $u(x,t) = X(x)T(t)$.

Inserting it into the heat equation, we have $\frac{T'}{a^2 T} = \frac{X''}{X} = -\lambda,$

where λ *is a constant to be determined. Thus, a pair of ODEs are established to be*

$T' + a^2 \lambda T = 0 \;\; and \;\; X'' + \lambda X = 0.$

Combining with the boundary conditions gives a eigenvalue problem of

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In order to get a non-trivial solution, we establish an equation for the eigenvalue λ*:*

$$\tan\gamma = \alpha\gamma, \gamma = \beta l, \alpha = -\frac{1}{hl}.$$

For $\gamma > 0$*, there are infinitely many intersecting points between the two sets of curves. We use* $0 < \gamma_1 < \gamma_2 \dots \gamma_n \dots$ *to denote the intersecting points.*

Then, the eigenvalues of can be calculated by $\lambda_n = \beta_n^2 = \frac{\gamma_n^2}{l^2}, n = 1,2, \dots$. *And the corresponding eigenfunctions are also derived to be* $X_n(x) = B_n \sin\beta_n x\,, n = 1,2, \dots$.

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$$\phi(x) = u(x,0) = \sum_{n=1}^{\infty} C_n \sin \beta_n x \,.$$

Here we need to employ the orthogonality of $\{\sin \beta_n x\}_{n=1}^{\infty}$*, and have*

$$C_n = \frac{\int_0^l \phi(x) \sin \beta_n x \, dx}{\int_0^l \sin^2 \beta_n x \, dx}, n = 1,2, \dots .$$

Exercise.

Use the method of separation of variable to solve the following initial boundary problems.

1. $$\begin{cases} u_t = a^2 u_{xx}, & 0 < x < l, t > 0 \\ u(0,t) = 0, u(l,t) = 0, & t > 0 \\ u(x,0) = \varphi(x), & 0 \le x \le l \end{cases}$$

2. $$\begin{cases} u_t = a^2 u_{xx}, & 0 < x < l, t > 0 \\ u_x(0,t) = 0, u_x(l,t) = 0, & t > 0 \\ u(x,0) = \varphi(x), & 0 \le x \le l \end{cases}$$

Fourier transform

For a given function $y = f(x)$ *that is integrable over the whole space of* R*, its Fourier transform, denoted by*

$$\boldsymbol{\mathcal{F}[f] = \int_{-\infty}^{\infty} f(x) e^{-ix\xi} dx = \tilde{f}(\xi).}$$

It can also be read from the prerequisite subsection that Fourier transform can be reversed by

$$\boldsymbol{\mathcal{F}^{-1}[\tilde{f}] = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{f}(\xi) e^{ix\xi} d\xi = f(x).}$$

The properties of Fourier transform.

1. Linearity

$\forall\ \alpha, \beta \in C, f(x), g(x)$ *are two integrable functions, then* $\boldsymbol{\mathcal{F}[\alpha f + \beta g] = \alpha \mathcal{F}[f] + \beta \mathcal{F}[g]}$***.***

2. Differentiation $\boldsymbol{(|x| \to +\infty, f(x) \to 0)}$

$$\boldsymbol{\mathcal{F}[f'] = i\xi \mathcal{F}[f], \qquad \mathcal{F}\left[f^{(n)}\right] = (i\xi)^n \mathcal{F}[f].}$$

Proof. $\boldsymbol{\mathcal{F}[f'] = f(x) e^{-ix\xi}|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(x) de^{-ix\xi} = i\xi \int_{-\infty}^{\infty} f(x) e^{-ix\xi} dx = i\xi \mathcal{F}[f]}$.

3. $\mathcal{F}[f(x-c)] = \int_{-\infty}^{\infty} f(x-c)e^{-ix\xi}dx = \int_{-\infty}^{\infty} f(t)e^{-i(t+c)\xi}dt = e^{-ic\xi}\mathcal{F}[f(x)]$.

Fourier transform can be defined for high-dimensional cases. We define

$$\mathcal{F}[f] = \int_{R^n} f(r)e^{-ir\cdot\xi}dr = \tilde{f}(\xi).$$

Where $r \cdot \xi$ *denotes the inner product of two n-dimensional vectors. The corresponding inverse Fourier transform is given by*

$$\mathcal{F}^{-1}[\tilde{f}] = \frac{1}{(2\pi)^n}\int_{R^n} \tilde{f}(\xi)e^{ir\cdot\xi}d\xi = f(r).$$

Example. $\mathcal{F}\left[e^{-\alpha|x|}\right] = \int_{-\infty}^{\infty} e^{-\alpha|x|}e^{-ix\xi}dx = \int_{-\infty}^{0} e^{x(\alpha-i\xi)}dx + \int_{0}^{\infty} e^{x(-\alpha-i\xi)}dx = \frac{1}{\alpha-i\xi} + \frac{1}{\alpha+i\xi}$.

Convolution and Fourier transform

Given two unary functions both of which are defined in the whole space of R, their convolution, denoted by $f * g(x)$ *is defined to be*

$$\boldsymbol{f * g(x) = \int_{-\infty}^{\infty} f(x-t)g(t)dt}\,.$$

Upon Fourier transform, the convolution of two functions equals the product of the Fourier transforms of the two individual functions, i.e.

The properties of convolution.

1. *Commutative law.* $\boldsymbol{f} * \boldsymbol{g}(\boldsymbol{x}) = \boldsymbol{g} * \boldsymbol{f}(\boldsymbol{x})$

2. *Associative law.* $(\boldsymbol{f} * \boldsymbol{g}) * \boldsymbol{h}(\boldsymbol{x}) = \boldsymbol{g} * (\boldsymbol{f} * \boldsymbol{h})(\boldsymbol{x})$

3. *Distributive law.* $\boldsymbol{f} * (\boldsymbol{g} + \boldsymbol{h})(\boldsymbol{x}) = \boldsymbol{f} * \boldsymbol{g}(\boldsymbol{x}) + \boldsymbol{f} * \boldsymbol{h}(\boldsymbol{x})$

Convolution theorem. $\mathcal{F}[\boldsymbol{f} * \boldsymbol{g}(\boldsymbol{x})] = \mathcal{F}[\boldsymbol{f}(\boldsymbol{x})]\mathcal{F}[\boldsymbol{g}(\boldsymbol{x})]$,

$$\mathcal{F}^{-1}(\mathcal{F}[\boldsymbol{f}(\boldsymbol{x})]\mathcal{F}[\boldsymbol{g}(\boldsymbol{x})]) = \mathcal{F}^{-1}[\mathcal{F}[\boldsymbol{f}(\boldsymbol{x})] * \mathcal{F}^{-1}[\mathcal{F}[\boldsymbol{g}(\boldsymbol{x})] = (\boldsymbol{f} * \boldsymbol{g})(\boldsymbol{x}).$$

Proof. $\mathcal{F}[f * g(x)] = \int_{-\infty}^{\infty} e^{-i\alpha x}[\int_{-\infty}^{\infty} f(x-\xi)g(\xi)d\xi]dx = \int_{-\infty}^{\infty} g(\xi)e^{-i\alpha\xi}\left[\int_{-\infty}^{\infty} f(x-\xi)e^{-i\alpha(x-\xi)}dx\right]d\xi = \int_{-\infty}^{\infty} g(\xi)e^{-i\alpha\xi}d\xi\left[\int_{-\infty}^{\infty} f(\eta)e^{-i\alpha\eta}d\eta\right] = \mathcal{F}[f(x)]\mathcal{F}[g(x)]$.

Convolution theorem. $\mathcal{F}[\boldsymbol{f}(\boldsymbol{x})\boldsymbol{g}(\boldsymbol{x})] = \frac{\mathbf{1}}{\mathbf{2}\boldsymbol{\pi}}\mathcal{F}[\boldsymbol{f}(\boldsymbol{x})] * \mathcal{F}[\boldsymbol{g}(\boldsymbol{x})]$,

$$\mathcal{F}^{-\mathbf{1}}(\mathcal{F}[\boldsymbol{f}(\boldsymbol{x})] * \mathcal{F}[\boldsymbol{g}(\boldsymbol{x})]) = \mathbf{2}\boldsymbol{\pi}\boldsymbol{f}(\boldsymbol{x})\boldsymbol{g}(\boldsymbol{x}).$$

Proof. $\mathcal{F}[f(x)g(x)] = \int_{-\infty}^{\infty} f(x)g(x)e^{-i\xi x}dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\int_{-\infty}^{\infty} G(\lambda)e^{i\lambda x}e^{-i\xi x}d\lambda\, dx =$

$\frac{1}{2\pi}\int_{-\infty}^{\infty} G(\lambda)\int_{-\infty}^{\infty} f(x)e^{-ix(\xi-\lambda)}dx\, d\lambda = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\lambda)F(\xi-\lambda)d\lambda = \frac{1}{2\pi}G * F = \frac{1}{2\pi}\mathcal{F}[f(x)] * \mathcal{F}[g(x)]$.

Parseval equality. ***If*** $\boldsymbol{F}(\boldsymbol{\xi}) = \boldsymbol{\mathcal{F}}[\boldsymbol{f}(\boldsymbol{x})]$, ***then*** $\int_{-\infty}^{\infty} |\boldsymbol{f}(\boldsymbol{x})|^{\mathbf{2}}\boldsymbol{dx} = \frac{\mathbf{1}}{\mathbf{2}\boldsymbol{\pi}}\int_{-\infty}^{\infty} |\boldsymbol{F}(\boldsymbol{\xi})|^{\mathbf{2}}\boldsymbol{d\xi}$.

Exercise .

1. *Prove* $\mathcal{F}[f(x-c)] = e^{-i\xi c}\mathcal{F}[f(x)], where\ c \in R.$

2. *Set* $F(\xi) = \mathcal{F}[f(x)], prove\ \mathcal{F}[xf(x)] = i\frac{d}{d\xi}F(\xi).$

3. *Set* $F(\alpha) = \mathcal{F}[f(x)], prove\ \mathcal{F}[f(x)cos\ \omega x] = \frac{1}{2}[F(\alpha-\omega) + F(\alpha+\omega)]$

$and\ \mathcal{F}[f(x)sin\ \omega x] = \dfrac{1}{2i}[F(\alpha-\omega) - F(\alpha+\omega)].$

Solution for Cauchy problems.

Now we consider the Cauchy problem as follows

$$\begin{cases} \boldsymbol{u_t - a^2 u_{xx} = 0, (x,t) \in R \times R^+} \\ \boldsymbol{u\Big|_{t=0} = \phi(x).} \end{cases}$$

For its solution, the method of integral transform can be implemented as follows.

First, we apply Fourier transform with respect to x *on the governing equation of the problem,*

leading to $\mathcal{F}[u_t] - a^2\mathcal{F}[u_{xx}] = 0$, *with the differentiation rule for Fourier transform, we have*

$$\frac{d\mathcal{F}[u]}{dt} + a^2\xi^2\mathcal{F}[u] = 0.$$

Fourier transform is also applied to the initial condition, giving rise to

$$\mathcal{F}[u]|_{t=0} = \mathcal{F}[\phi](\xi).$$

Solving the ODE problem gives

$$\mathcal{F}[u] = \mathcal{F}[\phi](\xi)e^{-a^2\xi^2 t}.$$

Suppose u can be expressed in a convolution form, say, $u = \phi * h$. *If one manages to find the inverse*

Fourier transform of $e^{-a^2\xi^2 t}$, $h(x,t) = \mathcal{F}^{-1}\left[e^{-a^2\xi^2 t}\right] = \frac{1}{2a\sqrt{\pi t}}e^{-\frac{x^2}{4a^2 t}}$.

Then , the solution $u(x,t)$ *is calculated by*

$$u(x,t) = \phi * h = \int_{-\infty}^{\infty}\phi(s)h(x-s,t)ds = \frac{1}{2a\sqrt{\pi t}}\int_{-\infty}^{\infty}\phi(s)e^{-\frac{(x-s)^2}{4a^2 t}}ds.$$

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Now we consider the Cauchy problem as follows

$$\begin{cases} \boldsymbol{u_{tt} - a^2 u_{xx} = 0, (x,t) \in R \times R^+} \\ \boldsymbol{u(x,0) = \phi(x), u_t(x,0) = \psi(x).} \end{cases}$$

First, we apply Fourier transform with respect to x *on the governing equation of the problem, and*

we set $U(\alpha,t) = \mathcal{F}[u](\alpha,t), \Phi(\alpha) = \mathcal{F}[\phi](\alpha), \Psi(\alpha) = \mathcal{F}[\psi](\alpha).$

With the differentiation rule for Fourier transform, we have

$$U_{tt} + a^2\alpha^2 U = 0, -\infty < \alpha < +\infty, t > 0$$

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$$U(\alpha,0) = \Phi(\alpha), U_t(\alpha,0) = \Psi(\alpha).$$

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$$U(\alpha, t) = \Phi(\alpha)\cos a\alpha t + \frac{1}{a\alpha}\Psi(\alpha)\sin a\alpha t.$$

Then , the solution $u(x,t)$ *is calculated by*

$$u(x,t) = \mathcal{F}^{-1}[U] = \mathcal{F}^{-1}[\Phi(\alpha)\cos a\alpha t] + \mathcal{F}^{-1}[\frac{1}{a\alpha}\Psi(\alpha)\sin a\alpha t].$$

Using the convolution theorem, we have

$$\mathcal{F}^{-1}[\Phi(\alpha)\cos a\alpha t] = \frac{1}{2\pi}\int_{-\infty}^{\infty}\Phi(\alpha)\cos a\alpha t\, e^{i\alpha x}d\alpha = \frac{1}{4\pi}\int_{-\infty}^{\infty}\Phi(\alpha)(e^{-i\alpha a t} + e^{i\alpha a t})e^{i\alpha x}d\alpha =$$

$$\frac{1}{4\pi}\int_{-\infty}^{\infty}\Phi(\alpha)(e^{i\alpha(x-at)} + e^{i\alpha(x+at)})d\alpha = \frac{1}{2}\big(\phi(x+at) + \phi(x-at)\big).$$

$$\mathcal{F}^{-1}\left[\frac{1}{a\alpha}\Psi(\alpha)\sin a\alpha t\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty}\Psi(\alpha)\frac{\sin\alpha a t}{\alpha a}e^{i\alpha x}d\alpha$$

$$= \frac{1}{4\pi}\int_{-\infty}^{\infty}\Psi(\alpha)\frac{-i}{\alpha a}(e^{i\alpha(x+at)} - e^{i\alpha(x-at)})d\alpha = \frac{1}{4a\pi}\int_{-\infty}^{\infty}\Psi(\alpha)\int_{x-at}^{x+at}e^{ia\xi}d\xi\, d\alpha$$

$$= \frac{1}{4a\pi}\int_{x-at}^{x+at}\int_{-\infty}^{\infty}\Psi(\alpha)e^{ia\xi}d\alpha\, d\xi = \frac{1}{2a}\int_{x-at}^{x+at}\psi(\xi)d\xi\,.$$

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$$u(x,t) = \frac{1}{2}\big(\phi(x+at) + \phi(x-at)\big) + \frac{1}{2a}\int_{x-at}^{x+at}\psi(\xi)d\xi\,.$$

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Solving the ODE problem gives

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$$\mathcal{F}^{-1}\left[\Phi(\alpha)e^{-a^2\alpha^2 t}\right] = \mathcal{F}^{-1}[\Phi(\alpha)] * \mathcal{F}^{-1}[e^{-a^2\alpha^2 t}] = \frac{1}{2a\sqrt{\pi t}} \int_{-\infty}^{\infty} \phi(\xi) \exp\left(-\frac{(x-\xi)^2}{4a^2 t}\right) d\,\xi.$$

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Exercise .

1. *Apply the method of Fourier transform to solve the Cauchy problem given by*

$$\begin{cases} u_t + au_x = f(x,t), -\infty < x < +\infty, t > 0, \\ \qquad u(x,0) = \phi(x), -\infty < x < +\infty. \end{cases}$$

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2. Differentiation $\boldsymbol{(|x| \to +\infty, f(x) \to 0)}$

$$\boldsymbol{\mathcal{F}[f'] = i\xi \mathcal{F}[f], \qquad \mathcal{F}\left[f^{(n)}\right] = (i\xi)^n \mathcal{F}[f].}$$

Proof. $\boldsymbol{\mathcal{F}[f'] = f(x) e^{-ix\xi}|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(x) de^{-ix\xi} = i\xi \int_{-\infty}^{\infty} f(x) e^{-ix\xi} dx = i\xi \mathcal{F}[f]}$.

3. $\mathcal{F}[f(x-c)] = \int_{-\infty}^{\infty} f(x-c)e^{-ix\xi}dx = \int_{-\infty}^{\infty} f(t)e^{-i(t+c)\xi}dt = e^{-ic\xi}\mathcal{F}[f(x)]$.

Fourier transform can be defined for high-dimensional cases. We define

$$\mathcal{F}[f] = \int_{R^n} f(r)e^{-ir\cdot\xi}dr = \tilde{f}(\xi).$$

Where $r \cdot \xi$ *denotes the inner product of two n-dimensional vectors. The corresponding inverse Fourier transform is given by*

$$\mathcal{F}^{-1}[\tilde{f}] = \frac{1}{(2\pi)^n}\int_{R^n} \tilde{f}(\xi)e^{ir\cdot\xi}d\xi = f(r).$$

Example. $\mathcal{F}\left[e^{-\alpha|x|}\right] = \int_{-\infty}^{\infty} e^{-\alpha|x|}e^{-ix\xi}dx = \int_{-\infty}^{0} e^{x(\alpha-i\xi)}dx + \int_{0}^{\infty} e^{x(-\alpha-i\xi)}dx = \frac{1}{\alpha-i\xi} + \frac{1}{\alpha+i\xi}$.

Convolution and Fourier transform

Given two unary functions both of which are defined in the whole space of R, their convolution, denoted by $f * g(x)$ *is defined to be*

$$\boldsymbol{f * g(x) = \int_{-\infty}^{\infty} f(x-t)g(t)dt}\,.$$

Upon Fourier transform, the convolution of two functions equals the product of the Fourier transforms of the two individual functions, i.e.

The properties of convolution.

1. *Commutative law.* $\boldsymbol{f} * \boldsymbol{g}(\boldsymbol{x}) = \boldsymbol{g} * \boldsymbol{f}(\boldsymbol{x})$

2. *Associative law.* $(\boldsymbol{f} * \boldsymbol{g}) * \boldsymbol{h}(\boldsymbol{x}) = \boldsymbol{g} * (\boldsymbol{f} * \boldsymbol{h})(\boldsymbol{x})$

3. *Distributive law.* $\boldsymbol{f} * (\boldsymbol{g} + \boldsymbol{h})(\boldsymbol{x}) = \boldsymbol{f} * \boldsymbol{g}(\boldsymbol{x}) + \boldsymbol{f} * \boldsymbol{h}(\boldsymbol{x})$

Convolution theorem. $\mathcal{F}[\boldsymbol{f} * \boldsymbol{g}(\boldsymbol{x})] = \mathcal{F}[\boldsymbol{f}(\boldsymbol{x})]\mathcal{F}[\boldsymbol{g}(\boldsymbol{x})]$,

$$\mathcal{F}^{-1}(\mathcal{F}[\boldsymbol{f}(\boldsymbol{x})]\mathcal{F}[\boldsymbol{g}(\boldsymbol{x})]) = \mathcal{F}^{-1}[\mathcal{F}[\boldsymbol{f}(\boldsymbol{x})] * \mathcal{F}^{-1}[\mathcal{F}[\boldsymbol{g}(\boldsymbol{x})] = (\boldsymbol{f} * \boldsymbol{g})(\boldsymbol{x}).$$

Proof. $\mathcal{F}[f * g(x)] = \int_{-\infty}^{\infty} e^{-i\alpha x}[\int_{-\infty}^{\infty} f(x-\xi)g(\xi)d\xi]dx = \int_{-\infty}^{\infty} g(\xi)e^{-i\alpha\xi}\left[\int_{-\infty}^{\infty} f(x-\xi)e^{-i\alpha(x-\xi)}dx\right]d\xi = \int_{-\infty}^{\infty} g(\xi)e^{-i\alpha\xi}d\xi\left[\int_{-\infty}^{\infty} f(\eta)e^{-i\alpha\eta}d\eta\right] = \mathcal{F}[f(x)]\mathcal{F}[g(x)]$.

Convolution theorem. $\mathcal{F}[\boldsymbol{f}(\boldsymbol{x})\boldsymbol{g}(\boldsymbol{x})] = \frac{\mathbf{1}}{\mathbf{2}\boldsymbol{\pi}}\mathcal{F}[\boldsymbol{f}(\boldsymbol{x})] * \mathcal{F}[\boldsymbol{g}(\boldsymbol{x})]$,

$$\mathcal{F}^{-\mathbf{1}}(\mathcal{F}[\boldsymbol{f}(\boldsymbol{x})] * \mathcal{F}[\boldsymbol{g}(\boldsymbol{x})]) = \mathbf{2}\boldsymbol{\pi}\boldsymbol{f}(\boldsymbol{x})\boldsymbol{g}(\boldsymbol{x}).$$

Proof. $\mathcal{F}[f(x)g(x)] = \int_{-\infty}^{\infty} f(x)g(x)e^{-i\xi x}dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\int_{-\infty}^{\infty} G(\lambda)e^{i\lambda x}e^{-i\xi x}d\lambda\, dx =$

$\frac{1}{2\pi}\int_{-\infty}^{\infty} G(\lambda)\int_{-\infty}^{\infty} f(x)e^{-ix(\xi-\lambda)}dx\, d\lambda = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\lambda)F(\xi-\lambda)d\lambda = \frac{1}{2\pi}G * F = \frac{1}{2\pi}\mathcal{F}[f(x)] * \mathcal{F}[g(x)]$.

Parseval equality. ***If*** $\boldsymbol{F}(\boldsymbol{\xi}) = \boldsymbol{\mathcal{F}}[\boldsymbol{f}(\boldsymbol{x})]$, ***then*** $\int_{-\infty}^{\infty} |\boldsymbol{f}(\boldsymbol{x})|^{\mathbf{2}}\boldsymbol{dx} = \frac{\mathbf{1}}{\mathbf{2}\boldsymbol{\pi}}\int_{-\infty}^{\infty} |\boldsymbol{F}(\boldsymbol{\xi})|^{\mathbf{2}}\boldsymbol{d\xi}$.

Exercise .

1. *Prove* $\mathcal{F}[f(x-c)] = e^{-i\xi c}\mathcal{F}[f(x)], where\ c \in R.$

2. *Set* $F(\xi) = \mathcal{F}[f(x)], prove\ \mathcal{F}[xf(x)] = i\frac{d}{d\xi}F(\xi).$

3. *Set* $F(\alpha) = \mathcal{F}[f(x)], prove\ \mathcal{F}[f(x)cos\ \omega x] = \frac{1}{2}[F(\alpha-\omega) + F(\alpha+\omega)]$

$and\ \mathcal{F}[f(x)sin\ \omega x] = \dfrac{1}{2i}[F(\alpha-\omega) - F(\alpha+\omega)].$

Solution for Cauchy problems.

Now we consider the Cauchy problem as follows

$$\begin{cases} \boldsymbol{u_t - a^2 u_{xx} = 0, (x,t) \in R \times R^+} \\ \boldsymbol{u\Big|_{t=0} = \phi(x).} \end{cases}$$

For its solution, the method of integral transform can be implemented as follows.

First, we apply Fourier transform with respect to x *on the governing equation of the problem,*

leading to $\mathcal{F}[u_t] - a^2\mathcal{F}[u_{xx}] = 0$, *with the differentiation rule for Fourier transform, we have*

$$\frac{d\mathcal{F}[u]}{dt} + a^2\xi^2\mathcal{F}[u] = 0.$$

Fourier transform is also applied to the initial condition, giving rise to

$$\mathcal{F}[u]|_{t=0} = \mathcal{F}[\phi](\xi).$$

Solving the ODE problem gives

$$\mathcal{F}[u] = \mathcal{F}[\phi](\xi)e^{-a^2\xi^2 t}.$$

Suppose u can be expressed in a convolution form, say, $u = \phi * h$. *If one manages to find the inverse*

Fourier transform of $e^{-a^2\xi^2 t}$, $h(x,t) = \mathcal{F}^{-1}\left[e^{-a^2\xi^2 t}\right] = \frac{1}{2a\sqrt{\pi t}}e^{-\frac{x^2}{4a^2 t}}$.

Then , the solution $u(x,t)$ *is calculated by*

$$u(x,t) = \phi * h = \int_{-\infty}^{\infty}\phi(s)h(x-s,t)ds = \frac{1}{2a\sqrt{\pi t}}\int_{-\infty}^{\infty}\phi(s)e^{-\frac{(x-s)^2}{4a^2 t}}ds.$$

Solution for Cauchy problems.

Now we consider the Cauchy problem as follows

$$\begin{cases} \boldsymbol{u_{tt} - a^2 u_{xx} = 0, (x,t) \in R \times R^+} \\ \boldsymbol{u(x,0) = \phi(x), u_t(x,0) = \psi(x).} \end{cases}$$

First, we apply Fourier transform with respect to x *on the governing equation of the problem, and*

we set $U(\alpha,t) = \mathcal{F}[u](\alpha,t), \Phi(\alpha) = \mathcal{F}[\phi](\alpha), \Psi(\alpha) = \mathcal{F}[\psi](\alpha).$

With the differentiation rule for Fourier transform, we have

$$U_{tt} + a^2\alpha^2 U = 0, -\infty < \alpha < +\infty, t > 0$$

Fourier transform is also applied to the initial condition, giving rise to

$$U(\alpha,0) = \Phi(\alpha), U_t(\alpha,0) = \Psi(\alpha).$$

.

Solving the ODE problem gives

$$U(\alpha, t) = \Phi(\alpha)\cos a\alpha t + \frac{1}{a\alpha}\Psi(\alpha)\sin a\alpha t.$$

Then , the solution $u(x,t)$ *is calculated by*

$$u(x,t) = \mathcal{F}^{-1}[U] = \mathcal{F}^{-1}[\Phi(\alpha)\cos a\alpha t] + \mathcal{F}^{-1}[\frac{1}{a\alpha}\Psi(\alpha)\sin a\alpha t].$$

Using the convolution theorem, we have

$$\mathcal{F}^{-1}[\Phi(\alpha)\cos a\alpha t] = \frac{1}{2\pi}\int_{-\infty}^{\infty}\Phi(\alpha)\cos a\alpha t\, e^{i\alpha x}d\alpha = \frac{1}{4\pi}\int_{-\infty}^{\infty}\Phi(\alpha)(e^{-i\alpha a t} + e^{i\alpha a t})e^{i\alpha x}d\alpha =$$

$$\frac{1}{4\pi}\int_{-\infty}^{\infty}\Phi(\alpha)(e^{i\alpha(x-at)} + e^{i\alpha(x+at)})d\alpha = \frac{1}{2}\big(\phi(x+at) + \phi(x-at)\big).$$

$$\mathcal{F}^{-1}\left[\frac{1}{a\alpha}\Psi(\alpha)\sin a\alpha t\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty}\Psi(\alpha)\frac{\sin\alpha a t}{\alpha a}e^{i\alpha x}d\alpha$$

$$= \frac{1}{4\pi}\int_{-\infty}^{\infty}\Psi(\alpha)\frac{-i}{\alpha a}(e^{i\alpha(x+at)} - e^{i\alpha(x-at)})d\alpha = \frac{1}{4a\pi}\int_{-\infty}^{\infty}\Psi(\alpha)\int_{x-at}^{x+at}e^{ia\xi}d\xi\, d\alpha$$

$$= \frac{1}{4a\pi}\int_{x-at}^{x+at}\int_{-\infty}^{\infty}\Psi(\alpha)e^{ia\xi}d\alpha\, d\xi = \frac{1}{2a}\int_{x-at}^{x+at}\psi(\xi)d\xi\,.$$

Then, the solution $u(x,t)$ *is*

$$u(x,t) = \frac{1}{2}\big(\phi(x+at) + \phi(x-at)\big) + \frac{1}{2a}\int_{x-at}^{x+at}\psi(\xi)d\xi\,.$$

Solution for Cauchy problems.

Now we consider the Cauchy problem as follows

$$\begin{cases} \boldsymbol{u_t - a^2 u_{xx} = f(x,t), (x,t) \in R \times R^+} \\ \boldsymbol{u\Big|_{t=0} = \phi(x).} \end{cases}$$

First, we apply Fourier transform with respect to x *on the governing equation of the problem, and*

we set $F(\alpha,t) = \mathcal{F}[f](\alpha,t), U(\alpha,t) = \mathcal{F}[u](\alpha,t), \Phi(\alpha) = \mathcal{F}[\phi](\alpha)$.

leading to $\mathcal{F}[u_t] - a^2\mathcal{F}[u_{xx}] = 0$, *with the differentiation rule for Fourier transform, we have*

$$U_t + a^2\alpha^2 U = F(\alpha,t).$$

Fourier transform is also applied to the initial condition, giving rise to

$$U(\alpha,0) = \Phi(\alpha).$$

Convolution theorem. $\mathcal{F}[\boldsymbol{f} * \boldsymbol{g}(\boldsymbol{x})] = \mathcal{F}[\boldsymbol{f}(\boldsymbol{x})]\mathcal{F}[\boldsymbol{g}(\boldsymbol{x})]$,

$\mathcal{F}^{-1}(\mathcal{F}[\boldsymbol{f}(\boldsymbol{x})]\mathcal{F}[\boldsymbol{g}(\boldsymbol{x})]) = \mathcal{F}^{-1}[\mathcal{F}[\boldsymbol{f}(\boldsymbol{x})] * \mathcal{F}^{-1}[\mathcal{F}[\boldsymbol{g}(\boldsymbol{x})] = (\boldsymbol{f} * \boldsymbol{g})(\boldsymbol{x})$.

Solving the ODE problem gives

$$U(\alpha, t) = \Phi(\alpha)e^{-\alpha^2 a^2 t} + \int_0^t F(\alpha, \tau)e^{-\alpha^2 a^2 (t-\tau)} d\tau.$$

Then, the solution $u(x, t)$ *is calculated by*

$$u(x, t) = \mathcal{F}^{-1}[U] = \mathcal{F}^{-1}[\Phi(\alpha)e^{-a^2\alpha^2 t}] + \mathcal{F}^{-1}[\int_0^t F(\alpha, \tau)e^{-\alpha^2 a^2 (t-\tau)} d\tau].$$

Using the convolution theorem, we have

$$\mathcal{F}^{-1}\left[\Phi(\alpha)e^{-a^2\alpha^2 t}\right] = \mathcal{F}^{-1}[\Phi(\alpha)] * \mathcal{F}^{-1}[e^{-a^2\alpha^2 t}] = \frac{1}{2a\sqrt{\pi t}} \int_{-\infty}^{\infty} \phi(\xi) \exp\left(-\frac{(x-\xi)^2}{4a^2 t}\right) d\,\xi.$$

$$\mathcal{F}^{-1}\left[\int_0^t F(\alpha, \tau)e^{-\alpha^2 a^2 (t-\tau)} d\tau\right] = \int_0^t \mathcal{F}^{-1}[F(\alpha, \tau)e^{-\alpha^2 a^2 (t-\tau)}] d\tau = \int_0^t \int_{-\infty}^{\infty} \frac{f(\xi, \tau)}{2a\sqrt{\pi(t-\tau)}} exp\left(-\frac{(x-\xi)^2}{4a^2(t-\tau)}\right) d\xi\, d\tau\,.$$

Exercise .

1. *Apply the method of Fourier transform to solve the Cauchy problem given by*

$$\begin{cases} u_t + au_x = f(x,t), -\infty < x < +\infty, t > 0, \\ \qquad u(x,0) = \phi(x), -\infty < x < +\infty. \end{cases}$$

$\boldsymbol{Chapter}$ 3. *The Poisson's equation*

In this chapter, we examine PDEs taking the following form:

Two-dimensional case: $\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y);$

Three-dimensional case: $\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = f(x, y, z).$

As a special case when $f = 0$*, the resulting homogeneous Poisson's equation is called Laplace's equation .*

A revisit to the method of separation of variable.

Here we are focused on solving an initial boundary value problem for the heat equation given by

$$\begin{cases} u_{yy} + u_{xx} = 0, 0 < x < a, 0 < y < \mathrm{b}, \\ u(x,0) = f(x), u(x,b) = g(x), 0 \le x \le a \ (1) \\ u_x(0,y) = u_x(a,y) = 0.0 \le y \le b \end{cases}$$

Solution . We look for solutions taking the form of $u(x,t) = X(x)Y(y)$.

Inserting it into the heat equation, we have $-\frac{Y''}{Y} = \frac{X''}{X} = -\lambda,$

where λ *is a constant to be determined. Thus, a pair of ODEs are established to be*

$Y'' - \lambda Y = 0 \ \ and \ \ X'' + \lambda X = 0.$

Combining with the boundary conditions gives a eigenvalue problem of

$$\begin{cases} X'' + \lambda X = 0, 0 < x < a; \\ X'(0) = X'(a) = 0. \end{cases}$$

Then, the eigenvalues of can be calculated by $\lambda_n = \beta_n^2 = \left(\frac{n\pi}{a}\right)^2, n = 0,1,2, \dots.$

And the corresponding eigenfunctions are also derived to be $X_n(x) = A_n \cos\frac{n\pi x}{a}, n = 0,1,2, \dots.$

On using $\lambda = \lambda_n$, *the ODE for temporal function Y(y) becomes* $Y_n'' - \left(\frac{n\pi}{a}\right)^2 Y_n = 0, 0 < y < b.$

Its general solution is given by $Y_0 = C_0 + D_0,\ Y_n(y) = C_n e^{\frac{n\pi y}{a}} + D_n e^{-\frac{n\pi y}{a}}, n = 1,2, \dots.$

The solution of problem (1) is thus expressed by means of a series given by

$$u(x,t) = \sum_{n=0}^{\infty} X_n(x) Y_n(y) = a_0 + b_0 y + \sum_{n=1}^{\infty} \left(a_n e^{\frac{n\pi y}{a}} + b_n e^{-\frac{n\pi y}{a}}\right) \cos\frac{n\pi x}{a}.$$

To determine the coefficients, we insert into the initial conditions $u(x,0) = f(x), u(x,b) = g(x)$, *and obtain*

$$f(x) = u(x,0) = a_0 + \sum_{n=1}^{\infty} (a_n + b_n) \cos \frac{n\pi x}{a},$$

$$g(x) = u(x,b) = a_0 + bb_0 + \sum_{n=1}^{\infty} \left(a_n e^{\frac{n\pi b}{a}} + b_n e^{-\frac{n\pi b}{a}} \right) \cos \frac{n\pi x}{a}$$

Then , we have $a_0 = \frac{1}{a}\int_0^a f(x)dx$, $a_0 + bb_0 = \frac{1}{a}\int_0^a g(x)dx$, $a_n + b_n = \frac{2}{a}\int_0^a f(x)cos\frac{n\pi x}{a}dx$ *and*

$$a_n e^{\frac{n\pi b}{a}} + b_n e^{-\frac{n\pi b}{a}} = \frac{2}{a}\int_0^a g(x)cos\frac{n\pi x}{a}dx.$$

Laplace's equation in polar system.

Here we consider rewriting Laplace's equation $\Delta u = 0$ *with polar coordinates. And the key is to transform Laplacian operator from Cartesian to polar system.*

In two dimensional case, we introduce $\begin{cases} r = \sqrt{x^2 + y^2}; \\ \theta = \arctan\frac{y}{x}. \end{cases}$

The Laplacian operator in two dimensional polar system as

$$\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}.$$

This situation is now generalized to express Laplace's equation with spherical coordinated, which are related to the Cartesian coordinates by

$$\begin{cases} x = r\sin\theta\cos\phi\,; \\ y = r\sin\theta\sin\phi; \\ z = r\cos\theta\,, \end{cases}$$

Where the radius $r > 0$, *the latitude angle* $\theta \in [0, \pi)$, *the longitude angle* $\phi \in [0, 2\pi)$.

Using the multivariable chain rule, we can obtain

$$\Delta u = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2}.$$

Use the method of separation of variables to solve the following Laplace's equation

$$\begin{cases} u_{yy} + u_{xx} = 0, x^2 + y^2 < a^2 \\ u(x,y) = f(x,y), x^2 + y^2 = a^2. \end{cases} \quad (2)$$

Solution . In two dimensional polar system, the Laplace's equation can be written as

$$\begin{cases} \dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2} = 0, r < a, 0 \le \theta \le 2\pi, \\ u(a,\theta) = f(\theta), \qquad 0 \le \theta \le 2\pi, \end{cases}$$

where $u(r,\theta) = u(r\cos\theta, r\sin\theta), f(\theta) = f(a\cos\theta, a\sin\theta)$, *and* $|u(0,\theta)| < +\infty, u(r,\theta) = u(r,\theta + 2\pi)$.

We look for solutions taking the form of $u(r,\theta) = R(r)\Phi(\theta)$.

Inserting it into the heat equation, we have $\frac{r^2R''+rR'}{R} = -\frac{\Phi''}{\Phi} = \lambda$.

Thus, a pair of ODEs are established to be

$$r^2R'' + rR' - \lambda R = 0 \ \ and \ \ \Phi'' + \lambda\Phi = 0.$$

Combining with the boundary conditions gives a eigenvalue problem of

$$\begin{cases} \Phi'' + \lambda\Phi = 0, 0 \le \theta \le 2\pi, \\ \Phi(\theta) = \Phi(\theta + 2\pi), 0 \le \theta \le 2\pi, \end{cases} and \begin{cases} r^2R'' + rR' - \lambda R = 0, 0 \le r \le a, \\ |R(0)| < +\infty. \end{cases}$$

Then, the eigenvalues of can be calculated by $\lambda_n = \beta_n^2 = n^2, n = 0,1,2, \dots$.

And the corresponding eigenfunctions are also derived to be

$$\Phi_n(\theta) = A_n \cos n\theta + B_n \sin n\theta \, , R_n(r) = C_n r^n \, , n = 0,1,2, \dots.$$

The solution of problem (2) is thus expressed by means of a series given by

$$u(r,\theta) = \sum_{n=0}^{\infty} R_n(r)\Phi_n(\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^n(a_n \cos n\theta + b_n \sin n\theta)\,.$$

To determine the coefficients, we insert into the initial conditions $u(x,0) = f(x), u(x,b) = g(x)$, *and obtain*

$$f(\theta) = u(a,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a^n(a_n \cos n\theta + b_n \sin n\theta)$$

$$\begin{cases} a_n = \dfrac{1}{a^n \pi}\displaystyle\int_0^{2\pi} f(t)\cos nt\, dt\,, n = 0,1,2,\dots, \\ b_n = \dfrac{1}{a^n \pi}\displaystyle\int_0^{2\pi} f(t)\sin nt\, dt\,, n = 1,2,\dots. \end{cases}$$

Thus, $u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} f(t)\,(1 + 2\sum_{n=1}^{\infty}\frac{r^n}{a^n}\cos(\theta - t))\, dt.$

$$\begin{cases} \dfrac{1}{\rho}\dfrac{\partial}{\partial\rho}\left(\rho\dfrac{\partial u}{\partial\rho}\right)+\dfrac{1}{\rho^2}\dfrac{\partial^2 u}{\partial\theta^2}=0, & a<\rho<b, 0\le\theta\le 2\pi \\ u(a,\theta)=0, u(b,\theta)=1, & 0\le\theta\le 2\pi \end{cases}$$

Solution. $u(\rho,\theta)=u(\rho,\theta+2\pi)$

$$u(\rho,\theta)=\mathrm{P}(\rho)\Theta(\theta)$$

$$\mathrm{P}''\Theta+\frac{1}{\rho}\mathrm{P}'\Theta+\frac{1}{\rho^2}\mathrm{P}\Theta''=0$$

$$\frac{\rho^2\mathrm{P}''+\rho\mathrm{P}'}{\mathrm{P}}=-\frac{\Theta''}{\Theta}=\lambda$$

$$\rho^2\mathrm{P}''+\rho\mathrm{P}'-\lambda\mathrm{P}=0$$

$$\Theta''+\lambda\Theta=0$$

$$\Theta(\theta)\mathrm{P}(\rho)=\Theta(\theta+2\pi)\mathrm{P}(\rho)$$

$$\Theta(\theta)=\Theta(\theta+2\pi)$$

$$\begin{cases} \Theta''+\lambda\Theta=0, \\ \Theta(\theta)=\Theta(\theta+2\pi), \end{cases}$$

$$\lambda_n=n^2, n=0,1,2,3,\cdots$$

$$\Theta_n=A_n\cos n\theta+B_n\sin n\theta$$

$$\rho^2 \mathrm{P}'' + \rho \mathrm{P}' - \lambda \mathrm{P} = 0$$

$$\lambda_n = n^2, n = 0,1,2,3,\cdots$$

$$\Theta_n = A_n \cos n\theta + B_n \sin n\theta$$

$$\lambda = 0 \qquad \rho^2 \mathrm{P}_0'' + \rho \mathrm{P}_0' = 0 \qquad \mathrm{P}_0 = C_0 + D_0 \ln \rho$$

$$u_0 = \Theta_0 \mathrm{P}_0 = A_0 \left(C_0 + D_0 \ln \rho \right) = E_0 + F_0 \ln \rho$$

$$\lambda = n^2 > 0 \qquad \rho^2 \mathrm{P}_n'' + \rho \mathrm{P}_n' - n^2 \mathrm{P} = 0 \qquad \mathrm{P}_n = C_n \rho^n + D_n \rho^{-n}$$

$$u_n = \Theta_n \mathrm{P}_n = \left(A_n \cos n\theta + B_n \sin n\theta \right) \cdot \left(C_n \rho^n + D_n \rho^{-n} \right)$$

$$= \left(E_n \rho^n + F_n \rho^{-n} \right) \cos n\theta + \left(G_n \rho^n + H_n \rho^{-n} \right) \sin n\theta$$

$$u = \sum_{n=0}^{\infty} u_n = E_0 + F_0 \ln \rho + \sum_{n=1}^{\infty} \left[\left(E_n \rho^n + F_n \rho^{-n} \right) \cos n\theta + \left(G_n \rho^n + H_n \rho^{-n} \right) \sin n\theta \right]$$

$$\begin{cases} \dfrac{1}{\rho}\dfrac{\partial}{\partial\rho}\left(\rho\dfrac{\partial u}{\partial\rho}\right)+\dfrac{1}{\rho^2}\dfrac{\partial^2 u}{\partial\theta^2}=0, & a<\rho<b, 0\le\theta\le 2\pi \\ u(a,\theta)=0, u(b,\theta)=1, & 0\le\theta\le 2\pi \end{cases}$$

$$u=\sum_{n=0}^{\infty}u_n=E_0+F_0\ln\rho+\sum_{n=1}^{\infty}\left[\left(E_n\rho^n+F_n\rho^{-n}\right)\cos n\theta+\left(G_n\rho^n+H_n\rho^{-n}\right)\sin n\theta\right]$$

$$u(a,\theta)=E_0+F_0\ln a+\sum_{n=1}^{\infty}\left[\left(E_n a^n+F_n a^{-n}\right)\cos n\theta+\left(G_n a^n+H_n a^{-n}\right)\sin n\theta\right]=0$$

$$u(b,\theta)=E_0+F_0\ln b+\sum_{n=1}^{\infty}\left[\left(E_n b^n+F_n b^{-n}\right)\cos n\theta+\left(G_n b^n+H_n b^{-n}\right)\sin n\theta\right]=1$$

$$E_0+F_0\ln a=0 \qquad G_n a^n+H_n a^{-n}=0 \qquad E_n a^n+F_n a^{-n}=0$$

$$E_0+F_0\ln b=1 \qquad G_n b^n+H_n b^{-n}=0 \qquad E_n b^n+F_n b^{-n}=0$$

$$E_0=-\ln a\Big/\ln\frac{b}{a} \qquad F_0=1\Big/\ln\frac{b}{a}$$

$$u=-\ln a\Big/\ln\frac{b}{a}+\ln\rho\Big/\ln\frac{b}{a}=\ln\frac{\rho}{a}\Big/\ln\frac{b}{a}$$

$$\begin{cases} \dfrac{1}{\rho}\dfrac{\partial}{\partial\rho}\left(\rho\dfrac{\partial u}{\partial\rho}\right)+\dfrac{1}{\rho^2}\dfrac{\partial^2 u}{\partial\theta^2}=0, & \rho<1, \qquad 0\le\theta\le\pi/3 \\ u(1,\theta)=\sin 6\theta, & 0\le\theta\le\pi/3 \\ u(\rho,0)=u(\rho,\pi/3)=0, & \rho<1 \end{cases}$$

Solution. $|u(0,\theta)|<+\infty$

$$u(\rho,\theta)=\mathrm{P}(\rho)\Theta(\theta)$$

$$\mathrm{P}''\Theta+\frac{1}{\rho}\mathrm{P}'\Theta+\frac{1}{\rho^2}\mathrm{P}\Theta''=0$$

$$\frac{\rho^2\mathrm{P}''+\rho\mathrm{P}'}{\mathrm{P}}=-\frac{\Theta''}{\Theta}=\lambda$$

$$\rho^2\mathrm{P}''+\rho\mathrm{P}'-\lambda\mathrm{P}=0$$

$$\Theta''+\lambda\Theta=0$$

$$\Theta(0)\mathrm{P}(\rho)=\Theta(\pi/3)\mathrm{P}(\rho)=0$$

$$\Theta(0)=\Theta(\pi/3)=0$$

$$\begin{cases} \Theta''+\lambda\Theta=0, & 0\le\theta\le\pi/3 \\ \Theta(0)=\Theta(\pi/3)=0 \end{cases}$$

$$\lambda_n=9n^2, n=1,2,3,\cdots$$

$$\Theta_n=B_n\sin 3n\theta$$

$$\begin{cases} \dfrac{1}{\rho}\dfrac{\partial}{\partial\rho}\left(\rho\dfrac{\partial u}{\partial\rho}\right)+\dfrac{1}{\rho^2}\dfrac{\partial^2 u}{\partial\theta^2}=0, & \rho<1, \qquad 0\le\theta\le\pi/3 \\ u(1,\theta)=\sin 6\theta, & 0\le\theta\le\pi/3 \\ u(\rho,0)=u(\rho,\pi/3)=0, & \rho<1 \end{cases}$$

$$\lambda_n = 9n^2, n=1,2,3,\cdots$$

$$\Theta_n = B_n \sin 3n\theta$$

$$\rho^2 \mathrm{P}'' + \rho \mathrm{P}' - \lambda \mathrm{P} = 0 \qquad \rho^2 \mathrm{P}_n'' + \rho \mathrm{P}_n' - 9n^2 \mathrm{P} = 0$$

$$\mathrm{P}_n = C_n \rho^{3n} + D_n \rho^{-3n} = C_n \rho^{3n}$$

$$u_n = \Theta_n \mathrm{P}_n = B_n \sin 3n\theta \cdot C_n \rho^{3n} = E_n \sin 3n\theta \rho^{3n}$$

$$u = \sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} E_n \sin 3n\theta \rho^{3n}$$

$$u(1,\theta) = \sin 6\theta = \sum_{n=1}^{\infty} E_n \sin 3n\theta$$

$$E_2 = 1, E_n = 0, n \ne 2 \qquad u = \sin 6\theta \cdot \rho^6$$

$$\begin{cases} X''(x)+\lambda X(x)=0 \\ X(0)=0, X(l)=0 \end{cases}$$

$$\lambda_n=(\frac{n\pi}{l})^2, n=1,2,\cdots,$$

$$X_n(x)=\sin\frac{n\pi x}{l}, n=1,2,\cdots.$$

$$\begin{cases} X''(x)+\lambda X(x)=0 \\ X(0)=0, X'(l)=0 \end{cases}$$

$$\lambda_n=(\frac{(2n+1)\pi}{2l})^2, n=0,1,2,\cdots,$$

$$X_n(x)=\sin\frac{(2n+1)\pi x}{2l}, n=0,1,2,\cdots.$$

$$\begin{cases} X''(x)+\lambda X(x)=0 \\ X'(0)=0, X'(l)=0 \end{cases}$$

$$\lambda_n=(\frac{n\pi}{l})^2, n=0,1,2,\cdots,$$

$$X_n(x)=\cos\frac{n\pi x}{l}, n=0,1,2,\cdots.$$

$$\begin{cases} X''(x)+\lambda X(x)=0 \\ X'(0)=0, X(l)=0 \end{cases}$$

$$\lambda_n=(\frac{(2n+1)\pi}{2l})^2, n=0,1,2,\cdots,$$

$$X_n(x)=\cos\frac{(2n+1)\pi x}{2l}, n=0,1,2,\cdots.$$

$$\begin{cases} \Theta''+\lambda\Theta=0, & \lambda_n=n^2, n=0,1,2,3,\cdots \\ \Theta(\theta)=\Theta(\theta+2\pi), & \Theta_n=A_n\cos n\theta+B_n\sin n\theta \end{cases}$$

Sturm-Liouville system.

We consider a second-order linear ODE for the unknow $y = y(x)$*, given by*

$$\frac{d}{dx}\left(k(x)\frac{dy}{dx}\right) - q(x)y(x) + \lambda\rho(x)y(x) = 0, x \in (a, b), \qquad (3)$$

where the given functions satisfy $k(x) > 0, q(x) \geq 0, \rho(x) > 0$ *for all* $x \in (a, b)$; λ *is a parameter actually representing the eigenvalue of the resulting ODE problem. Here the equation (3) is a Strum-Liouville equation. A Sturm-Liouville equation along with boundary conditions defines a Strum-Liouville system. For a Sturm-Liouville equation , it is naturally associated with an inner product define by* $< f, g \geq \int_a^b \rho(x)f(x)g(x)\,dx$.

Sturm-Liouville equation.

1. $y'' + \lambda y = 0,\ 0 < x < L, k(x) = 1, q(x) = 0, \rho(x) = 1. a = 0, b = L.$

2. $\frac{d}{dx}\left(x\frac{dy}{dx}\right) - \frac{n^2}{x}y(x) + \lambda x y(x) = 0, 0 < x < b, k(x) = x, q(x) = \frac{n^2}{x}, \rho(x) = x,$

$x^2y'' + xy' + (\lambda x^2 - n^2)y = 0, 0 < x < b$ *is nth-order Bessel equation.*

3. $(1 - x^2)y'' - 2xy' + \lambda y = 0, 0 < x < 1\ , k(x) = 1 - x^2, q(x) = 0, \rho(x) = 1,$ *is Legendre equation.*

Example. *Show* $y'' + xy' + \lambda y = 0, x \in (-1,1)$ *is a Strum-Liouville equation. Write down the inner product induced by it.*

solution. We multiply the equation by a function $\mu(x)$*, and obtain*

$$\mu(x)y'' + x\mu(x)y' + \lambda\mu(x)y = 0, x \in (-1,1).$$

If we require $\frac{d\mu}{dx} = x\mu(x), \mu(x) = e^{\frac{x^2}{2}}$ *being one of its solutions.*

Then the equation can be expressed in the form of Strum-Liouville equation

$$\frac{d}{dx}\left(e^{\frac{x^2}{2}}\frac{dy}{dx}\right) + \lambda e^{\frac{x^2}{2}}y(x) = 0, x \in (-1,1),$$

And the inner product induced by $< f, g > = \int_{-1}^{1} e^{\frac{x^2}{2}} f(x)g(x)dx$.

Excersise. *Show* $xy'' + y' + (\lambda x - \frac{1}{x})y = 0, x \in (0,1)$ *is a Strum-Liouville equation. Write down the inner product induced by it.*

Conclusion from a Strum-Liouville system.

1. Existence of non-negative eigenvalues. For a Strum-Liouville system, there exist infinitely many non-negative eigenvalues, say, $0 \leq \lambda_1 < \lambda_2 < \cdots$.

For each λ_k*, the Sturm-Liouville system yields a non-trivial solution, denoted by* $y = y_k(x)$*, which is the eigenfunction corresponding to the eigen value* λ_k*.*

2. Orthogonality . When $k \neq m$*, the corresponding eigenfunctions* y_k *and* y_m *are orthogonal about the inner product defined as* $\int_a^b \rho(x) y_k(x) y_m(x) dx = 0$.

3. Completeness . For any continuous function $f(x)$ with $x \in (a, b)$, which satisfied the corresponding boundary conditions, it can always be expressed as a linear combination of the eigenfunctions ,i.e.

$$f(x) = \sum_{k=1}^{\infty} A_k y_k(x),$$

Where the coefficient A_k is given by $A_k = \frac{\int_a^b \rho(x) f(x) y_k(x) dx}{\int_a^b \rho(x) (y_k(x))^2 dx}$.

In summary, a Strum-Liouville system should result in a set of non-positive eigenvalues, and further their corresponding eigenfunctions. These eigenfunctions naturally form a complete set of orthogonal functions.

Laplace transform

For a given function $y = f(t), t \in [0, +\infty)$*, its Laplace transform, denoted by*

$$\mathcal{L}[\boldsymbol{f}(\boldsymbol{t})] = \int_{\mathbf{0}}^{+\infty} \boldsymbol{f}(\boldsymbol{t})\boldsymbol{e}^{-\boldsymbol{st}}\boldsymbol{dt} = \tilde{\boldsymbol{f}}(\boldsymbol{s}),$$

Where $s = \beta + i\lambda,\ Re(s) = \beta, Im(s) = \lambda$*. The Laplace transform can be reversed by*

$$\mathcal{L}^{-\mathbf{1}}\left[\tilde{\boldsymbol{f}}(\boldsymbol{s})\right] = \frac{\mathbf{1}}{\mathbf{2}\boldsymbol{\pi i}} \int_{\boldsymbol{\beta}-\boldsymbol{i}\infty}^{\boldsymbol{\beta}+\boldsymbol{i}\infty} \tilde{\boldsymbol{f}}(\boldsymbol{s})\boldsymbol{e}^{\boldsymbol{st}}\boldsymbol{ds} = \boldsymbol{f}(\boldsymbol{t}).$$

The properties of Fourier transform.

1. Linearity

$\forall\ c_1, c_2 \in C, f(x), g(x)$ *are two integrable functions, then* $\mathcal{L}[\boldsymbol{c_1 f} + \boldsymbol{c_2 g}] = \boldsymbol{c_1}\mathcal{L}[\boldsymbol{f}] + \boldsymbol{c_2}\mathcal{L}[\boldsymbol{g}]$*.*

2. Differentiation

$$\mathcal{L}[\boldsymbol{f}'] = \boldsymbol{s}\mathcal{L}[\boldsymbol{f}] - \boldsymbol{f}(\mathbf{0}), \qquad \mathcal{L}\left[\boldsymbol{f}^{(\boldsymbol{n})}\right] = \boldsymbol{s}^{\boldsymbol{n}}\mathcal{L}[\boldsymbol{f}] - \boldsymbol{s}^{\boldsymbol{n}-\mathbf{1}}\boldsymbol{f}(\mathbf{0}) - \boldsymbol{s}^{\boldsymbol{n}-\mathbf{2}}\boldsymbol{f}'(\mathbf{0}) - \cdots - \boldsymbol{s}\boldsymbol{f}^{(\boldsymbol{n}-\mathbf{2})}(\mathbf{0}) - \boldsymbol{f}^{(\boldsymbol{n}-\mathbf{1})}(\mathbf{0}).$$

3. $\boldsymbol{\mathcal{L}}[\boldsymbol{e}^{\boldsymbol{at}}\boldsymbol{f}(\boldsymbol{t})] = \tilde{\boldsymbol{f}}(\boldsymbol{s} - \boldsymbol{a}),\ \ \boldsymbol{\mathcal{L}}^{-\mathbf{1}}\left[\tilde{\boldsymbol{f}}(\boldsymbol{s} - \boldsymbol{a})\right] = \boldsymbol{e}^{\boldsymbol{at}}\boldsymbol{f}(\boldsymbol{t})$*.*

Convolution and Laplace transform

Given two unary functions both of which are defined in $[0, +\infty)$*, their convolution, denoted by* $f * g(t)$ *is defined to be*

$$\boldsymbol{f * g(t) = \int_0^t f(t-\tau)g(\tau)d\tau}\,.$$

The properties of convolution.

Convolution theorem. $\mathcal{L}[\boldsymbol{f * g(t)}] = \mathcal{L}[\boldsymbol{f(t)}]\mathcal{L}[\boldsymbol{g(t)}]$

$\mathcal{L}^{-1}(\mathcal{L}[\boldsymbol{f(t)}]\mathcal{L}[\boldsymbol{g(t)}]) = (\boldsymbol{f * g})(\boldsymbol{t})$.

1. ***Commutative law.*** $\boldsymbol{f * g(t) = g * f(t)}$
2. ***Associative law.*** $\boldsymbol{(f * g) * h(t) = g * (f * h)(t)}$
3. ***Distributive law.*** $\boldsymbol{f * (g + h)(t) = f * g(t) + f * h(t)}$

$$\begin{cases} u_t = a^2 u_{xx} + u, & 0 < x < l, t > 0 \\ u(0,t) = u(l,t) = 0, & t > 0 \\ u(x,0) = \varphi(x), & 0 \le x \le l \end{cases}$$

Method 1. $u = e^t v$

$$e^t v_t + e^t v = e^t a^2 v_{xx} + e^t v$$

$$u(0,t) = e^t v(0,t) = 0$$
$$u(l,t) = e^t v(l,t) = 0$$
$$u(x,0) = v(x,0)$$

$$\begin{cases} v_t = a^2 v_{xx} \\ v(0,t) = 0 \\ v(l,t) = 0 \\ v(x,0) = \varphi(x) \end{cases}$$

Method2. $u(x,t) = X(x)T(t)$

$$XT' = a^2 X''T + XT$$

$$\frac{T'}{a^2 T} - \frac{1}{a^2} = \frac{X''}{X} = -\lambda$$

$$X'' + \lambda X = 0 \qquad T'' + \left(a^2\lambda - 1\right)T = 0$$

$$u(0,t) = X(0)T(t) = 0$$
$$u(l,t) = X(l)T(t) = 0$$
$$X(0) = 0, X(l) = 0$$

$$\begin{cases} X'' + \lambda X = 0, & 0 < x < l \\ X(0) = 0, X(l) = 0 \end{cases}$$

$$\lambda_n = \left(\frac{n\pi}{l}\right)^2, n = 1,2,3,\cdots$$

$$X_n = B_n \sin\frac{n\pi}{l}x$$

$$\begin{cases} u_t = a^2 u_{xx} + u, & 0 < x < l, t > 0 \\ u(0,t) = u(l,t) = 0, & t > 0 \\ u(x,0) = \varphi(x), & 0 \le x \le l \end{cases}$$

$$\lambda_n = \left(\frac{n\pi}{l}\right)^2, n = 1,2,3,\cdots$$

$$X_n = B_n \sin\frac{n\pi}{l}x$$

$$T'' + \left(a^2\lambda - 1\right)T = 0 \qquad T_n' + \left(\frac{a^2n^2\pi^2}{l^2} - 1\right)T_n = 0 \qquad T_n = A_n e^{\left(1-\frac{a^2n^2\pi^2}{l^2}\right)t}$$

$$u_n = X_n T_n = A_n e^{\left(1-\frac{a^2n^2\pi^2}{l^2}\right)t} \cdot B_n \sin\frac{n\pi}{l}x = C_n e^{\left(1-\frac{a^2n^2\pi^2}{l^2}\right)t} \sin\frac{n\pi}{l}x$$

$$u = \sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} C_n e^{\left(1-\frac{a^2n^2\pi^2}{l^2}\right)t} \sin\frac{n\pi}{l}x$$

$$u(x,0) = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi}{l}x = \varphi(x) \qquad C_n = \frac{2}{l}\int_0^l \varphi(x)\sin\frac{n\pi}{l}x\,\mathrm{d}x$$

Example. *Use the method of separation of variables to solve the following PDE problem:*

$$\begin{cases} u_{tt} = a^2 u_{xx} + \boxed{f(x,t),} & 0 < x < l, t > 0, \\ u(0,t) = u(l,t) = 0, & t \geq 0, \\ u(x,0) = \varphi(x), u_t(x,0) = \psi(x), & 0 \leq x \leq l. \end{cases}$$

Solution . It can be readily verified that it solution u is equal to the superposition of two functions:

$u = v + w$, *where* v *and* w *satisfy*

$$\begin{cases} \dfrac{\partial^2 W}{\partial t^2} = a^2 \dfrac{\partial^2 W}{\partial x^2} \\ W(0,t) = W(l,t) = 0, \\ W(x,0) = \varphi(x), \dfrac{\partial W(x,0)}{\partial t} = \psi(x) \end{cases} \qquad \begin{cases} \dfrac{\partial^2 V}{\partial t^2} = a^2 \dfrac{\partial^2 V}{\partial x^2} + f(x,t), & 0 < x < l, t > 0, \\ V(0,t) = V(l,t) = 0, & t > 0, \\ V(x,0) = \dfrac{\partial V(x,0)}{\partial t} = 0, & 0 \leq x \leq l, \end{cases}$$

respectively. Upon such decomposition, we can use the method of separation variables to solve the first problem.

So now we pay attention to the second nonhomogeneous problem.

$$\begin{cases} X''(x)+\lambda X(x)=0 \\ X(0)=0, X(l)=0 \end{cases}$$

$$\lambda_n=(\frac{n\pi}{l})^2, n=1,2,\cdots,$$

$$X_n(x)=\sin\frac{n\pi x}{l}, n=1,2,\cdots.$$

$$\begin{cases} X''(x)+\lambda X(x)=0 \\ X(0)=0, X'(l)=0 \end{cases}$$

$$\lambda_n=(\frac{(2n+1)\pi}{2l})^2, n=0,1,2,\cdots,$$

$$X_n(x)=\sin\frac{(2n+1)\pi x}{2l}, n=0,1,2,\cdots.$$

$$\begin{cases} X''(x)+\lambda X(x)=0 \\ X'(0)=0, X'(l)=0 \end{cases}$$

$$\lambda_n=(\frac{n\pi}{l})^2, n=0,1,2,\cdots,$$

$$X_n(x)=\cos\frac{n\pi x}{l}, n=0,1,2,\cdots.$$

$$\begin{cases} X''(x)+\lambda X(x)=0 \\ X'(0)=0, X(l)=0 \end{cases}$$

$$\lambda_n=(\frac{(2n+1)\pi}{2l})^2, n=0,1,2,\cdots,$$

$$X_n(x)=\cos\frac{(2n+1)\pi x}{2l}, n=0,1,2,\cdots.$$

$$\begin{cases} \Theta''+\lambda\Theta=0, & \lambda_n=n^2, n=0,1,2,3,\cdots \\ \Theta(\theta)=\Theta(\theta+2\pi), & \Theta_n=A_n\cos n\theta+B_n\sin n\theta \end{cases}$$

$$\begin{cases} V_{tt} = a^2 V_{xx} + f(x,t), & 0 < x < l, t > 0, \\ V(0,t) = V(l,t) = 0, & t \geq 0, \\ V(x,0) = 0, V_t(x,0) = 0), & 0 \leq x \leq l. \end{cases}$$

Set $V = \sum_{n=1}^{\infty} v_n(t) \sin \frac{n\pi}{l} x$ *, we obtain*

$$\sum_{n=1}^{\infty} v_n''(t) \sin \frac{n\pi}{l} x = \sum_{n=1}^{\infty} [-(\frac{an\pi}{l})^2 v_n(t) \sin \frac{n\pi}{l} x] + f(x,t) \longrightarrow \sum_{n=1}^{\infty} [v_n''(t) + (\frac{an\pi}{l})^2 v_n(t)] \sin \frac{n\pi}{l} x = f(x,t)$$

$$f(x,t) = \sum_{n=1}^{\infty} f_n(t) \sin \frac{n\pi}{l} x \longrightarrow f_n(t) = \frac{2}{l} \int_0^l f(x,t) \sin \frac{n\pi}{l} x \mathrm{d}x$$

$$\sum_{n=1}^{\infty} [v_n''(t) + (\frac{an\pi}{l})^2 v_n(t) - f_n(t)] \sin \frac{n\pi}{l} x = 0$$

$$\downarrow$$

$$v_n''(t) + (\frac{an\pi}{l})^2 v_n(t) - f_n(t) = 0$$

$$V=\sum_{n=1}^{\infty} v_n(t)\sin\frac{n\pi}{l}x \qquad v_n''(t)+\left(\frac{an\pi}{l}\right)^2 v_n(t)-f_n(t)=0$$

$$V(x,0)=\sum_{n=1}^{\infty} v_n(0)\sin\frac{n\pi}{l}x=0 \qquad v_n(0)=0 \qquad v_n'(0)=0$$

$$\frac{\partial V(x,0)}{\partial t}=\sum_{n=1}^{\infty} v_n'(0)\sin\frac{n\pi}{l}x=0$$

$$v_n(t)\leftrightarrow V_n(p)$$

$$f_n(t)\leftrightarrow F_n(p)$$

$$v_n''(t)\leftrightarrow p^2V_n(p)-pv_n(0)-v_n'(0)=p^2V_n(p) \qquad \frac{k}{p^2+k^2}\leftrightarrow \sin kt$$

$$p^2V_n(p)+(an\pi/l)^2V_n(p)-F_n(p)=0$$

$$V_n(p)=\frac{1}{p^2+(an\pi/l)^2}\cdot F_n(p)$$

$$v_n(t)=\frac{l}{n\pi a}\sin\frac{n\pi a}{l}t * f_n(t)=\frac{l}{n\pi a}\int_0^t f_n(\tau)\sin\frac{n\pi a}{l}(t-\tau)\mathrm{d}\tau$$

Example. *Use the method of separation of variables to solve the following PDE problem:*

$$\begin{cases} u_t = a^2 u_{xx} + \sin \omega t, \\ u_x(0,t) = u_x(l,t) = 0, \\ u(x,0) = 0, 0 \le x \le l \end{cases}$$

Solution . Consider the related homogeneous problem : $\begin{cases} u_t = a^2 u_{xx}, 0 < x < l, t > 0 \\ u_x(0,t) = u_x(l,t) = 0, t \ge 0 \\ u(x,0) = 0, 0 \le x \le l \end{cases}$

We obtain the eigenvalues and eigenfunctions are $\lambda_n = (n\pi / l)^2, n = 0,1,2,\cdots \quad X_n = B_n \cos \frac{n\pi}{l} x$

Set $\quad u = \sum_{n=0}^{\infty} v_n(t) \cos \frac{n\pi}{l} x$

$$u=\sum_{n=0}^{\infty} v_n(t)\cos\frac{n\pi}{l}x \longrightarrow \sum_{n=0}^{\infty}\left[v_n'(t)+a^2\frac{n^2\pi^2}{l^2}v_n(t)\right]\cos\frac{n\pi}{l}x=\sin\omega t$$

$$u(x,0)=\sum_{n=0}^{\infty} v_n(0)\cos\frac{n\pi}{l}x=0 \longrightarrow v_n(0)=0$$

$n=0$ $\quad v_0'(t)=\sin\omega t \qquad v_0(0)=0$

$$v_0(t)=-\frac{1}{\omega}\cos\omega t+C \qquad v_0(t)=\frac{1}{\omega}(1-\cos\omega t)$$

$n\neq 0$ $\quad v_n'(t)+a^2\frac{n^2\pi^2}{l^2}v_n(t)=0 \qquad v_n(0)=0$

$$v_n(t)=Ce^{-a^2\frac{n^2\pi^2}{l^2}t} \qquad v_n(t)=0$$

$$u=\frac{1}{\omega}(1-\cos\omega t)\cos\frac{n\pi}{l}$$

Example. *Use the method of separation of variables to solve the following PDE problem:*

$$\begin{cases} u_{tt} = a^2 u_{xx} + \sin\frac{2\pi}{l}x \sin\frac{2a\pi}{l}t & 0 < x < l, t > 0 \\ u(0,t) = u(l,t) = 0, & t > 0 \\ u(x,0) = 0, u_t(x,0) = 0, & 0 \le x \le l \end{cases}$$

Solution . Consider the related homogeneous problem : $\begin{cases} u_{tt} = a^2 u_{xx}, 0 < x < l, t > 0 \\ u(0,t) = u(l,t) = 0, t \ge 0 \\ u(x,0) = 0, u_t(x,0) = 0, 0 \le x \le l \end{cases}$

We obtain the eigenvalues and eigenfunctions are $\lambda_n = (n\pi / l)^2, n = 1,2,\cdots \quad X_n(x) = B_n \sin\frac{n\pi}{l}x$

Set $\quad u = \sum_{n=1}^{\infty} v_n(t) \sin\frac{n\pi}{l}x$

$$\sum_{n=1}^{\infty}[v_n''(t)+a^2\frac{n^2\pi^2}{l^2}v_n(t)]\sin\frac{n\pi}{l}x \quad =\sin\frac{2\pi}{l}x\sin\frac{2a\pi}{l}t$$

$$u(x,0)=\sum_{n=1}^{\infty}v_n(0)\sin\frac{n\pi}{l}x=0 \qquad \frac{\partial u(x,0)}{\partial t}=\sum_{n=1}^{\infty}v_n'(0)\sin\frac{n\pi}{l}x=0$$

$$\sum_{n=1}^{\infty}[v_n''(t)+a^2\frac{n^2\pi^2}{l^2}v_n(t)]\sin\frac{n\pi}{l}x$$
$$=\sin\frac{2\pi}{l}x\sin\frac{2a\pi}{l}t$$

$$u(x,0)=\sum_{n=1}^{\infty}v_n(0)\sin\frac{n\pi}{l}x=0$$

$$\frac{\partial u(x,0)}{\partial t}=\sum_{n=1}^{\infty}v_n'(0)\sin\frac{n\pi}{l}x=0$$

$n\neq 2$

$$v_n''(t)+a^2\frac{n^2\pi^2}{l^2}v_n(t)=0$$

$$v_n(0)=0 \qquad v_n'(0)=0$$

$$v_n(t)=A\cos a\frac{n\pi}{l}t+B\sin a\frac{n\pi}{l}t$$

$$v_n(t)=0$$

$n=2$

$$v_2''(t)+a^2\frac{4\pi^2}{l^2}v_2(t)=\sin\frac{2a\pi}{l}t$$

$$v_2(0)=0 \qquad v_2'(0)=0$$

$$v_2''(t) + a^2 \frac{4\pi^2}{l^2} v_2(t) = \sin \frac{2a\pi}{l} t \quad v_2(0) = 0 \quad v_2'(0) = 0$$

$$v_2''(t) \leftrightarrow p^2 V_2(p) - p v_2(0) - v_2'(0) \quad = p^2 V_2(p)$$

$$V_2(p) = \frac{2a\pi / l}{p^2 + (2a\pi / l)^2} \cdot \frac{1}{p^2 + (2a\pi / l)^2}$$

$$\frac{k}{p^2 + k^2} \leftrightarrow \sin kt$$

$$v_2(t) = \frac{l}{2a\pi} \sin \frac{2a\pi}{l} t * \sin \frac{2a\pi}{l} t \quad = \frac{l}{2a\pi} \int_0^t \sin \frac{2a\pi}{l} \tau \sin \frac{2a\pi}{l} (t - \tau) \mathrm{d}\tau$$

$$= -\frac{l}{4a\pi} \int_0^t \left[\cos \frac{2a\pi}{l} t - \cos \frac{2a\pi}{l} (t - 2\tau) \right] \mathrm{d}\tau$$

$$u = \sum_{n=1}^{\infty} v_n(t) \sin \frac{n\pi}{l} x$$

$$= \frac{l}{4a\pi} \left(\frac{l}{2a\pi} \sin \frac{2a\pi}{l} t - t \cos \frac{2a\pi}{l} t \right)$$

$$v_n(t) = 0 \qquad n \neq 2$$

$$u = \frac{l}{4a\pi} \left(\frac{l}{2a\pi} \sin \frac{2a\pi}{l} t - t \cos \frac{2a\pi}{l} t \right) \sin \frac{2\pi}{l} x$$

Example . Now we consider the initial boundary problem as follows

$$\begin{cases} \boldsymbol{u_{tt} = u_{xx}, 0 < x < 1, t > 0,} \\ \boldsymbol{u(0,t) = u(1,t) = 0, t \geq 0,} \\ \boldsymbol{u(x,0) = sin\,\pi x, \qquad u_t(x,0) = -sin\,\pi x\,, 0 \leq x \leq 1.} \end{cases}$$

For its solution, the method of integral transform can be implemented as follows.

First, we apply Laplace transform with respect to t *on the governing equation of the problem, leading to*

$$U_{xx} = s^2 U + (1 - s)\sin \pi x,$$

the general solution of the ODE is

$$U(x,s) = A^{sx} + Be^{-sx} + \frac{(s-1)\sin \pi x}{\pi^2 + s^2}$$

Laplace transform is also applied to the boundary conditions, giving rise to

$$U(0,s) = U(1,s) = 0,$$

thus $A = B = 0, \quad U(x,s) = \frac{(s-1)\sin \pi x}{\pi^2+s^2}$.

Then, the solution $u(x,t) = \mathcal{L}^{-1}\left[\frac{(s-1)\sin \pi x}{\pi^2+s^2}\right] = \sin \pi x\,(\cos \pi t - \pi^{-1} \sin \pi t)$.

Example .Now we consider the problem as follows

$$\begin{cases} \boldsymbol{u_t = u_{xx}, 0 < x < L, t > 0,} \\ \boldsymbol{u(x,0) = u_0, 0 \le x \le L,} \\ \boldsymbol{u_x(0,t) = 0, u(L,t) = -u_1, t \ge 0.} \end{cases}$$

For its solution, the method of integral transform can be implemented as follows.

First, we apply Laplace transform with respect to t *on the governing equation of the problem, leading to*

$$U_{xx} = sU - u_0,$$

the general solution of the ODE is

$$U(x,s) = Acosh(\sqrt{s}x) + Bsinh(\sqrt{s}x) + \frac{u_0}{s}$$

Laplace transform is also applied to the boundary conditions, giving rise to

$$U_x(0,s) = 0, U(L,s) = u_1/s$$

thus $U(x,s) = \frac{(u_1 - u_0)\cosh(\sqrt{s}x)}{s\cosh(\sqrt{s}L)} + \frac{u_0}{s}$.

Then, the solution $u(x,t) = u_0 + (u_1 - u_0)\mathcal{L}^{-1}\left[\frac{\cosh(\sqrt{s}x)}{s\cosh(\sqrt{s}L)}\right]$.

Exercise. Apply the method of integral transform to solve the following problems:

1. $$\begin{cases} u_t + au_x = f(x,t), x \in R, t > 0, \\ u(x,0) = \phi(x), x \in R. \end{cases}$$

2. $$\begin{cases} u_t = a^2 u_{xx} - tu, x \in R, t > 0, \\ u(x,0) = f(x), x \in R. \end{cases}$$

3. $$\begin{cases} u_x + \frac{1}{x} u_t = t, x > 0, t > 0, \\ u(x,0) = x, x \geq 0, \\ u(0,t) = 0, t \geq 0. \end{cases}$$

4. $$\begin{cases} u_{xy} = 1, x > 0, y > 0, \\ u(0,y) = y, y \geq 0, \\ u(x,0) = 0, x \geq 0. \end{cases}$$

Example. *Use the method of separation of variables to solve the following PDE problem:*

$$\begin{cases} u_{tt} = a^2 u_{xx} + f(x,t), & 0 < x < l, t > 0 \\ u(0,t) = u_1(t), u(l,t) = u_2(t), & t > 0 \\ u(x,0) = \varphi(x), u_t(x,0) = \psi(x), & 0 \le x \le l \end{cases}$$

Solution. Set $u(x,t) = V(x,t) + W(x,t)$, *where* $W(0,t) = u_1(t) \quad W(l,t) = u_2(t)$

$$\begin{cases} v_{tt} = a^2 v_{xx} + f(x,t) + a^2 w_{xx} - w_{tt}, \\ v(0,t) = 0, v(l,t) = 0, \\ v(x,0) = \varphi(x) - w(x,0), u_t(x,0) = \psi(x) - w_t(x,0) \end{cases}$$

$$W(x,t) = \frac{u_2(t) - u_1(t)}{l} x + u_1(t)$$

$$\begin{cases} v_{tt} = a^2 v_{xx} + f(x,t) - [u_2''(t) - u_1''(t)]x / l - u_1''(t), \\ v(0,t) = 0, v(l,t) = 0, \\ v(x,0) = \varphi(x) - [u_2'(0) - u_1(0)]x / l - u_1(0), \\ u_t(x,0) = \psi(x) - [u_2'(0) - u_1'(0)]x / l - u_1'(0) \end{cases}$$

Example.

$$\begin{cases} u_{tt} = a^2 u_{xx} + p, & 0 < x < l, t > 0 \\ u(0,t) = 0, u(l,t) = q, & t > 0 \\ u(x,0) = \dfrac{q}{l} x, u_t(x,0) = 0, & 0 \le x \le l \end{cases}$$

Method 1 . $u(x,t) = V(x,t) + W(x,t)$ $\qquad W(x,t) = \dfrac{q}{l} x$

$$\begin{cases} V_{tt} = a^2 V_{xx} + p, & 0 < x < l, t > 0 \\ V(0,t) = 0, V(l,t) = 0, & t > 0 \\ V(x,0) = 0, V_t(x,0) = 0, & 0 \le x \le l \end{cases}$$

$$\begin{cases} u_{tt} = a^2 u_{xx} + p, & 0 < x < l, t > 0 \\ u(0,t) = 0, u(l,t) = q, & t > 0 \\ u(x,0) = \dfrac{q}{l} x, u_t(x,0) = 0, & 0 \le x \le l \end{cases}$$

Method 2.

$$u(x,t) = V(x,t) + W(x)$$

$$\frac{\partial^2 V}{\partial t^2} = a^2 \frac{\partial^2 V}{\partial x^2} + p + a^2 W''$$

$$p + a^2 W'' = 0$$

$$W(0) = 0 \qquad W(l) = q$$

$$W = -\frac{p}{2a^2} x^2 + Ax + B$$

$$= -\frac{p}{2a^2} x^2 + \left(\frac{q}{l} + \frac{pl}{2a^2} \right) x$$

$$\begin{cases} \dfrac{\partial^2 V}{\partial t^2} = a^2 \dfrac{\partial^2 V}{\partial x^2}, & 0 < x < l, t > 0 \\ V(0,t) = 0, V(l,t) = 0, & t \ge 0 \\ V(x,0) = \dfrac{px^2}{2a^2} - \dfrac{plx}{2a^2}, & 0 \le x \le l \\ \dfrac{\partial V(x,0)}{\partial t} = 0, & 0 \le x \le l \end{cases}$$

$$\begin{cases} u_t = a^2(u_{xx} + u_{yy}), & 0 < x < p, 0 < y < q, t > 0 \\ u(0,y,t) = u(p,y,t) = 0, & 0 \le y \le q, t > 0 \\ u(x,0,t) = u(x,q,t) = 0, & 0 \le x \le p, t > 0 \\ u(x,y,0) = \varphi(x,y), & 0 < x < p, 0 < y < q \end{cases}$$

Solution.

$$u(x,y,t) = X(x)Y(y)T(t)$$

$$XYT' = a^2\left(X''YT + XY''T\right)$$

$$\frac{X''}{X} + \frac{Y''}{Y} - \frac{1}{a^2}\cdot\frac{T'}{T} = 0$$

$$\frac{X''}{X} = -\mu \qquad \frac{Y''}{Y} = -\nu$$

$$\frac{1}{a^2}\cdot\frac{T'}{T} = -(\mu+\nu)$$

$$X'' + \mu X = 0 \quad Y'' + \nu Y = 0$$

$$T' + (\mu+\nu)a^2 T = 0$$

$$X'' + \mu X = 0 \qquad X(0) = X(p) = 0$$

$$Y'' + \nu Y = 0 \qquad Y(0) = Y(q) = 0$$

$$\mu_n = \frac{n^2\pi^2}{p^2}, n = 1,2,3,\cdots \quad X_n = B_n \sin\frac{n\pi}{p}x$$

$$\nu_m = \frac{m^2\pi^2}{q^2}, m = 1,2,3,\cdots \quad Y_m = C_m \sin\frac{m\pi}{q}y$$

$$T' + (\mu + \nu)a^2 T = 0 \qquad T'_{mn} + (\frac{n^2\pi^2}{p^2} + \frac{m^2\pi^2}{q^2})a^2 T_{mn} = 0 \qquad T_{mn} = D_{mn} e^{-(\frac{n^2\pi^2}{p^2} + \frac{m^2\pi^2}{q^2})a^2 t}$$

$$u_{mn} = X_n Y_m T_{mn} = B_n \sin\frac{n\pi}{p}x \cdot C_m \sin\frac{m\pi}{q}y \cdot D_{mn} e^{-(\frac{n^2\pi^2}{p^2} + \frac{m^2\pi^2}{q^2})a^2 t}$$

$$= E_{mn} \sin\frac{n\pi}{p}x \sin\frac{m\pi}{q}y e^{-(\frac{n^2\pi^2}{p^2} + \frac{m^2\pi^2}{q^2})a^2 t}$$

$$u = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} u_{mn} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} E_{mn} \sin\frac{n\pi}{p}x \sin\frac{m\pi}{q}y e^{-(\frac{n^2\pi^2}{p^2} + \frac{m^2\pi^2}{q^2})a^2 t}$$

$$u(x, y, 0) = \varphi(x, y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} E_{mn} \sin\frac{n\pi}{p}x \sin\frac{m\pi}{q}y$$

$$E_{mn} = \frac{4}{pq}\int_0^q \int_0^p \varphi(x, y) \sin\frac{n\pi}{p}x \sin\frac{m\pi}{q}y \mathrm{d}x\mathrm{d}y$$

$$\begin{cases} u_{xx}+u_{yy}=12(x^2-y^2), & a<\sqrt{x^2+y^2}<b \\ u\big|_{\sqrt{x^2+y^2}=a}=u\big|_{\sqrt{x^2+y^2}=b}=0 \end{cases}$$

Solution.

$$\begin{cases} \rho^{-1}(\rho u_\rho)_\rho+\rho^{-2}u_{\theta\theta}=12\rho^2\cos 2\theta, \\ u(a,\theta)=u(b,\theta)=0, \\ \Theta(\theta)=\Theta(\theta+2\pi), \end{cases}$$

$$\lambda_n=n^2, n=0,1,2,3,\cdots$$

$$\Theta_n=A_n\cos n\theta+B_n\sin n\theta$$

$$u=\sum_{n=0}^{\infty}\left[A_n(\rho)\cos n\theta+B_n(\rho)\sin n\theta\right]$$

$$u(\rho,\theta)=\mathrm{P}(\rho)\Theta(\theta)$$

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\mathrm{P}'\Theta\right)+\frac{1}{\rho^2}\mathrm{P}\Theta''=0$$

$$\mathrm{P}''\Theta+\frac{1}{\rho}\mathrm{P}'\Theta+\frac{1}{\rho^2}\mathrm{P}\Theta''=0$$

$$\frac{\rho^2\mathrm{P}''+\rho\mathrm{P}'}{\mathrm{P}}=-\frac{\Theta''}{\Theta}=\lambda$$

$$\rho^2\mathrm{P}''+\rho\mathrm{P}'-\lambda\mathrm{P}=0$$

$$\Theta''+\lambda\Theta=0$$

$$\begin{cases} \Theta''+\lambda\Theta=0, \\ \Theta(\theta)=\Theta(\theta+2\pi), \end{cases}$$

$$\sum_{n=0}^{\infty}\left[A_n''\cos n\theta+B_n''\sin n\theta+\frac{1}{\rho}A_n'\cos n\theta+\frac{1}{\rho}B_n'\sin n\theta-\frac{n^2}{\rho^2}A_n\cos n\theta-\frac{n^2}{\rho^2}B_n\sin n\theta\right]$$

$$=\sum_{n=0}^{\infty}\left[\left(A_n''+\frac{1}{\rho}A_n'-\frac{n^2}{\rho^2}A_n\right)\cos n\theta+\left(B_n''+\frac{1}{\rho}B_n'-\frac{n^2}{\rho^2}B_n\right)\sin n\theta\right]=12\rho^2\cos 2\theta$$

$$A_2''+\frac{1}{\rho}A_2'-\frac{4}{\rho^2}A_2=12\rho^2 \qquad B_n''+\frac{1}{\rho}B_n'-\frac{n^2}{\rho^2}B_n=0$$

$$A_n''+\frac{1}{\rho}A_n'-\frac{n^2}{\rho^2}A_n=0 \quad n\neq 2 \qquad A_n=0 \quad n\neq 2 \qquad B_n=0$$

$$u(a,\theta)=\sum_{n=0}^{\infty}\left[A_n(a)\cos n\theta+B_n(a)\sin n\theta\right]=0 \qquad A_n(a)=B_n(a)=0$$

$$u(b,\theta)=\sum_{n=0}^{\infty}\left[A_n(b)\cos n\theta+B_n(b)\sin n\theta\right]=0 \qquad A_n(b)=B_n(b)=0$$

$$A_2=C_2\rho^2+D_2\rho^{-2}+\rho^4 \quad C_2=-\frac{a^6-b^6}{a^4-b^4} \quad D_2=-\frac{a^4b^4}{a^2+b^2}$$

$$A_2(\rho)=-\frac{a^6-b^6}{a^4-b^4}\rho^2-\frac{a^4b^4}{a^2+b^2}\rho^{-2}+\rho^4$$

$$u=\left(-\frac{a^6-b^6}{a^4-b^4}\rho^2-\frac{a^4b^4}{a^2+b^2}\rho^{-2}+\rho^4\right)\cos 2\theta$$

$$\begin{cases} \dfrac{\partial u}{\partial t} = a^2 \dfrac{\partial^2 u}{\partial x^2} - b^2 u, & 0 < x < l, t > 0 \\ u(0,t) = 0, u(l,t) = u_1, & t > 0 \\ u(x,0) = \dfrac{u_1}{l^2} x^2, & 0 \le x \le l \end{cases}$$

Solution. $u(x,t) = V(x,t) + W(x)$ $\quad \dfrac{\partial V}{\partial t} = a^2 \dfrac{\partial^2 V}{\partial x^2} - b^2 V + a^2 W'' - b^2 W$

$$a^2 W'' - b^2 W = 0 \quad W(0) = 0 \quad W(l) = u_1$$

$$W(x) = Ae^{\frac{b}{a}x} + Be^{-\frac{b}{a}x} = \frac{u_1}{2\sinh(bl/a)}(e^{\frac{b}{a}x} - e^{-\frac{b}{a}x}) = \frac{u_1 \sinh(bx/a)}{\sinh(bl/a)}$$

$$\begin{cases} \dfrac{\partial V}{\partial t} = a^2 \dfrac{\partial^2 V}{\partial x^2} - b^2 V, & 0 < x < l, t > 0 \\ V(0,t) = 0, V(l,t) = 0, & t \ge 0 \\ V(x,0) = \dfrac{u_1 x^2}{l^2} - \dfrac{u_1 \sinh(bx/a)}{\sinh(bl/a)}, & 0 \le x \le l \end{cases}$$

$$\begin{cases} u_t = a^2 u_{xx} + f(x), & 0 < x < l, t > 0 \\ u(0,t) = 0, u(l,t) = 0, & t > 0 \\ u(x,0) = 0, & 0 \le x \le l \end{cases}$$

Solution. $u(x,t) = V(x,t) + W(x)$

$$f(x) + a^2 W'' = 0$$
$$W(0) = 0 \quad W(l) = 0$$

$$V_t = a^2 V_{xx} + f(x) + a^2 W''$$

$$W = -\frac{1}{a^2}\int_0^x \int_0^\nu f(\mu)\mathrm{d}\mu\mathrm{d}\nu + Ax + B$$

$$W(0) = B = 0$$

$$W(l) = -\frac{1}{a^2}\int_0^l \int_0^\nu f(\mu)\mathrm{d}\mu\mathrm{d}\nu + Al = 0$$

$$A = \frac{1}{a^2 l}\int_0^l \int_0^\nu f(\mu)\mathrm{d}\mu\mathrm{d}\nu$$

$$W = -\frac{1}{a^2}\int_0^x \int_0^\nu f(\mu)\mathrm{d}\mu\mathrm{d}\nu + \frac{x}{a^2 l}\int_0^l \int_0^\nu f(\mu)\mathrm{d}\mu\mathrm{d}\nu$$

$$\begin{cases} V_{tt} = a^2 V_{xx}, \\ V(0,t) = 0, V(l,t) = 0, \\ V(x,0) = \dfrac{1}{a^2}\displaystyle\int_0^x \int_0^\nu f(\mu)\mathrm{d}\mu\mathrm{d}\nu - \dfrac{x}{a^2 l}\int_0^l \int_0^\nu f(\mu)\mathrm{d}\mu\mathrm{d}\nu \end{cases}$$

$$\begin{cases} u_{xx}+u_{yy}=-xy, & x^2+y^2<1 \\ u\big|_{x^2+y^2=1}=0 \end{cases}$$

Solution.

$$\begin{cases} \dfrac{1}{\rho}\dfrac{\partial}{\partial\rho}\left(\rho\dfrac{\partial u}{\partial\rho}\right)+\dfrac{1}{\rho^2}\dfrac{\partial^2 u}{\partial\theta^2}=-\dfrac{1}{2}\rho^2\sin 2\theta, & \rho<1, 0\le\theta\le 2\pi \\ u(1,\theta)=0, |u(0,\theta)|<+\infty, & 0\le\theta\le 2\pi \\ \Theta(\theta)=\Theta(\theta+2\pi), & \rho\le 1 \end{cases}$$

$$u(\rho,\theta)=\mathrm{P}(\rho)\Theta(\theta)$$

$$\mathrm{P}''\Theta+\frac{1}{\rho}\mathrm{P}'\Theta+\frac{1}{\rho^2}\mathrm{P}\Theta''=0$$

$$\frac{\rho^2\mathrm{P}''+\rho\mathrm{P}'}{\mathrm{P}}=-\frac{\Theta''}{\Theta}=\lambda$$

$$\rho^2\mathrm{P}''+\rho\mathrm{P}'-\lambda\mathrm{P}=0 \qquad \Theta''+\lambda\Theta=0$$

$$\begin{cases} \Theta''+\lambda\Theta=0, \\ \Theta(\theta)=\Theta(\theta+2\pi), \end{cases}$$

$$\lambda_n=n^2, n=0,1,2,3,\cdots$$

$$\Theta_n=A_n\cos n\theta+B_n\sin n\theta$$

$$\frac{\partial^2 u}{\partial \rho^2}+\frac{1}{\rho}\frac{\partial u}{\partial \rho}+\frac{1}{\rho^2}\frac{\partial^2 u}{\partial \theta^2}=-\frac{1}{2}\rho^2\sin 2\theta \qquad u=\sum_{n=0}^{\infty}\left[A_n(\rho)\cos n\theta+B_n(\rho)\sin n\theta\right]$$

$$\sum_{n=0}^{\infty}[A_n''\cos n\theta+B_n''\sin n\theta+\frac{1}{\rho}A_n'\cos n\theta+\frac{1}{\rho}B_n'\sin n\theta-\frac{n^2}{\rho^2}A_n\cos n\theta-\frac{n^2}{\rho^2}B_n\sin n\theta]$$

$$=\sum_{n=0}^{\infty}[(A_n''+\frac{1}{\rho}A_n'-\frac{n^2}{\rho^2}A_n)\cos n\theta+(B_n''+\frac{1}{\rho}B'-\frac{n^2}{\rho^2}B_n)\sin n\theta]=-\frac{1}{2}\rho^2\sin 2\theta$$

$$A_n''+\frac{1}{\rho}A_n'-\frac{n^2}{\rho^2}A_n=0$$

$$A_n(1)=0 \mid A_n(0)|<+\infty$$

$$B_n''+\frac{1}{\rho}B_n'-\frac{n^2}{\rho^2}B_n=0 \quad n\neq 2$$

$$B_n(1)=0 \ |B_n(0)|<+\infty$$

$$B_2''+\frac{1}{\rho}B_2'-\frac{4}{\rho^2}B_2=-\frac{1}{2}\rho^2$$

$$B_2(1)=0 \ |B_2(0)|<+\infty$$

$$A_n=0 \qquad B_n=0 \quad n\neq 2$$

$$u(1,\theta)=\sum_{n=0}^{\infty}\left[A_n(1)\cos n\theta+B_n(1)\sin n\theta\right]=0$$

$$\boxed{B_2=C_2\rho^2+D_2\rho^{-2}-\frac{1}{24}\rho^4=C_2\rho^2-\frac{1}{24}\rho^4=\frac{1}{24}\rho^2-\frac{1}{24}\rho^4}$$

$$u=\frac{1}{24}\rho^2(1-\rho^2)\sin 2\theta=\frac{1}{12}(1-x^2-y^2)xy$$

Exercise. Apply the method of integral transform to solve the following problems:

1. $$\begin{cases} u_t + au_x = f(x,t), x \in R, t > 0, \\ u(x,0) = \phi(x), x \in R. \end{cases}$$

2. $$\begin{cases} u_t = a^2 u_{xx} - tu, x \in R, t > 0, \\ u(x,0) = f(x), x \in R. \end{cases}$$

3. $$\begin{cases} u_x + \frac{1}{x} u_t = t, x > 0, t > 0, \\ u(x,0) = x, x \geq 0, \\ u(0,t) = 0, t \geq 0. \end{cases}$$

4. $$\begin{cases} u_{xy} = 1, x > 0, y > 0, \\ u(0,y) = y, y \geq 0, \\ u(x,0) = 0, x \geq 0. \end{cases}$$

$$\begin{cases} u_t = a^2 u_{xx} + f(x,t), & -\infty < x < +\infty, t > 0 \\ u(x,0) = \varphi(x), & -\infty < x < +\infty \end{cases}$$

Solution. Apply Fourier transform with respect to x

$$\begin{cases} \dfrac{\mathrm{d}U(\omega,t)}{\mathrm{d}t} = -a^2\omega^2 U(\omega,t) + F(\omega,t), & t > 0 \\ U(\omega,0) = \Phi(\omega), \end{cases}$$

Apply Laplace transform with respect to x

$$L[e^{at}] = \frac{1}{p-a}$$

$$L[e^{-a^2\omega^2 t}] = \frac{1}{p+a^2\omega^2}$$

$$pU(\omega,p) - \Phi(\omega) = -a^2\omega^2 U(\omega,p) + F(\omega,p)$$

$$U(\omega,p) = \frac{\Phi(\omega)}{p+a^2\omega^2} + F(\omega,p)\frac{1}{p+a^2\omega^2}$$

$$U(\omega,t) = \Phi(\omega)L^{-1}[\frac{1}{p+a^2\omega^2}] + L^{-1}[F(\omega,p)\frac{1}{p+a^2\omega^2}]$$

$$U(\omega,t) = \Phi(\omega)e^{-a^2\omega^2 t} + F(\omega,t) * e^{-a^2\omega^2 t}$$

$$= \Phi(\omega)e^{-a^2\omega^2 t} + \int_0^t F(\omega,\tau)e^{-a^2\omega^2(t-\tau)}d\tau$$

$$\begin{cases} u_t = a^2 u_{xx} + f(x,t), & -\infty < x < +\infty, t > 0 \\ u(x,0) = \varphi(x), & -\infty < x < +\infty \end{cases}$$

$$U(\omega,t) = \Phi(\omega)e^{-a^2\omega^2 t} + \int_0^t F(\omega,\tau)e^{-a^2\omega^2(t-\tau)}d\tau$$

$$U(\omega,t) = F[\varphi(x)]F[\frac{e^{-\frac{x^2}{4a^2t}}}{2a\sqrt{\pi t}}] + \int_0^t F[f(x,\tau)]F[\frac{e^{-\frac{x^2}{4a^2(t-\tau)}}}{2a\sqrt{\pi(t-\tau)}}]\mathrm{d}\tau$$

$$= F[\varphi(x) * \frac{1}{2a\sqrt{\pi t}}e^{-\frac{x^2}{4a^2t}}] + \int_0^t F[f(x,\tau) * \frac{1}{2a\sqrt{\pi(t-\tau)}}e^{-\frac{x^2}{4a^2(t-\tau)}}]\mathrm{d}\tau$$

$$u(x,t) = \varphi(x) * \frac{1}{2a\sqrt{\pi t}}e^{-\frac{x^2}{4a^2t}} + \int_0^t f(x,\tau) * \frac{1}{2a\sqrt{\pi(t-\tau)}}e^{-\frac{x^2}{4a^2(t-\tau)}}\mathrm{d}\tau$$

$$= \frac{1}{2a\sqrt{\pi t}}\int_{-\infty}^{+\infty}\varphi(\xi)e^{-\frac{(x-\xi)^2}{4a^2t}}d\xi + \int_0^t\int_{-\infty}^{+\infty}\frac{f(\xi,\tau)}{2a\sqrt{\pi(t-\tau)}}e^{-\frac{(x-\xi)^2}{4a^2(t-\tau)}}\mathrm{d}\xi\mathrm{d}\tau$$

$$u_{xy}(x,y)=x^2y, x>1, y>0$$
$$u(x,0)=x^2, u(1,y)=\cos y$$

Method 1:

$$\frac{\partial u}{\partial x}=\frac{1}{2}x^2y^2+g(x)$$
$$u=\frac{1}{6}x^3y^2+f_1(x)+f_2(y)$$
$$u(x,0)=f_1(x)+f_2(0)=x^2$$
$$f_1(x)=x^2-f_2(0)$$
$$u(1,y)=\frac{1}{6}y^2+f_1(1)+f_2(y)=\cos y$$
$$f_2(y)=\cos y-\frac{1}{6}y^2-f_1(1)$$
$$u(1,0)=f_1(1)+f_2(0)=1$$
$$u=\frac{1}{6}x^3y^2+x^2+\cos y-\frac{1}{6}y^2-1$$

Method 2: Use Laplace transform to y

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[pU(x,p)-x^2\right]=\frac{x^2}{p^2}$$
$$\frac{\mathrm{d}}{\mathrm{d}x}U(x,p)=\frac{x^2}{p^3}+\frac{2x}{p}$$

$$\boxed{L[\cos t]=\frac{p}{p^2+1}}$$

$$U(x,p)=\frac{x^3}{3p^3}+\frac{x^2}{p}+C$$
$$U(1,p)=\frac{p}{p^2+1}\qquad C=\frac{p}{p^2+1}-\frac{1}{3p^3}-\frac{1}{p}$$
$$U(x,p)=\frac{x^3}{3p^3}+\frac{x^2}{p}+\frac{p}{p^2+1}-\frac{1}{3p^3}-\frac{1}{p}$$
$$u(x,y)=\frac{1}{6}x^3y^2+x^2+\cos y-\frac{1}{6}y^2-1$$

$$\boxed{L[t^2]=\frac{2}{p^3}}\qquad\boxed{L[1]=\frac{1}{p}}$$

$$\begin{cases} u_{xx} + u_{yy} = 0, -\infty < x < +\infty, y > 0, \\ u(x,0) = \varphi(x), -\infty < x < +\infty, \\ \lim\limits_{x^2+y^2\to\infty} u(x,y) = 0. \end{cases}$$

Use Fourier transform to x

$$\begin{cases} \dfrac{d^2}{dy^2} U(\omega, y) - \omega^2 U(\omega, y) = 0, \\ U(\omega,0) = \Phi(\omega), \\ \lim\limits_{y\to+\infty} U(\omega, y) = 0. \end{cases}$$

$$U(\omega, y) = \Phi(\omega) e^{-|\omega| y}.$$

$$F^{-1}[e^{-|\omega| y}] = \frac{y}{\pi(x^2 + y^2)}$$

$$\begin{aligned} u(x,y) &= F^{-1}[U(\omega, y)] = F^{-1}[\Phi(\omega) e^{-|\omega| y}] \\ &= \varphi(x) * \frac{y}{\pi(x^2+y^2)} = \frac{y}{\pi} \int_{-\infty}^{+\infty} \frac{\varphi(\xi) d\xi}{(x-\xi)^2 + y^2}. \end{aligned}$$

The method of Green's function

Ω *is a bounded domain in* R^3 *with smooth boundary.*

Green's formula: $\iiint_\Omega \left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}\right)dxdydz = \iint_{\partial\Omega}[P\cos(n,x)+Q\cos(n,y)+R\cos(n,z)]dS$.

Where n *denotes the direction of along the outer normal direction on the domain boundary* $\partial\Omega$.

The first Green's formula: $\iiint_\Omega (u\Delta v)dxdydz + \iiint_\Omega (\nabla u\cdot\nabla v)dxdydz = \iint_{\partial\Omega} u\frac{\partial v}{\partial n}dS.$

The second Green's formula : $\iiint_\Omega (u\Delta v - v\Delta u)dxdydz = \iint_{\partial\Omega}(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n})dS.$

Theorem 1. *If a function* u *is harmonic in* Ω*, and it is continuous towards the boundary* $\partial\Omega$*, then*

$$\iint_{\partial\Omega}\frac{\partial u}{\partial n}dS = 0$$

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holds.

Exercise. If $u(x, y)$ *is a solution of the following Laplacian equation with Neumann boundary*

$$\textit{conditions} \begin{cases} u_{xx} + u_{yy} = 0, 0 < x < a, 0 < y < b, \\ u_x(0, y) = f_1(y), u_x(a, y) = f_2(y), \quad 0 \le y \le b, \textit{show} \\ u_y(x, 0) = g_1(x), u_y(x, b) = g_2(x), 0 \le x \le a, \end{cases}$$

$$\int_0^a [g_1(x) - g_2(x)]dx + \int_0^b [f_1(y) - f_2(y)]dy = 0.$$

Fundamental solution.

Here the function v *is chosen to be the radial solution to three dimensional Laplace's equation (with respect to the reference point* $M_0 = M_0(x_0, y_0, z_0) \in R^3$*), i.e.*

$$v = -\frac{1}{4\pi\sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}} = -\frac{1}{4\pi r_{MM_0}},$$

where $r_{MM_0} = \sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}$.

Theorem 2. *If* $u = u(x, y, z) \in C^2(\Omega) \cap C^1(\overline{\Omega})$*, then* $\forall M_0 = M_0(x_0, y_0, z_0) \in \Omega$ *we have*

$$u(M_0) = \frac{1}{4\pi} \iint_{\partial\Omega} \left[\frac{1}{r_{MM_0}} \frac{\partial u}{\partial n} - u \frac{\partial}{\partial n}\left(\frac{1}{r_{MM_0}}\right)\right] dS - \frac{1}{4\pi} \iiint_{\Omega} \frac{\Delta u}{r_{MM_0}} dxdydz\,.$$

Proof. *Use the second Green's formula, we have*

$$\iiint_{\Omega_\epsilon}\left[u\Delta\left(\frac{1}{r_{MM_0}}\right)-\frac{1}{r_{MM_0}}\Delta u\right]dxdydz=\iint_{\partial\Omega_\epsilon}\left(u\frac{\partial}{\partial n}\left(\frac{1}{r_{MM_0}}\right)-\frac{1}{r_{MM_0}}\frac{\partial u}{\partial n}\right)dS\,,$$

where $\Omega_\epsilon=\Omega\backslash \mathrm{B}_\epsilon(M_0)$, *and* $\partial\Omega_\epsilon=\partial\Omega\cup\partial B_\epsilon(M_0)$.

$\Delta\left(\frac{1}{r_{MM_0}}\right)=0$ *in* Ω_ϵ, *and* $\frac{\partial}{\partial n}\left(\frac{1}{r_{MM_0}}\right)=\frac{1}{{r_{MM_0}}^2}=\frac{1}{\epsilon^2}$ *holds on* ∂B_ϵ.

According to the mean value theorem of integrals, there exist $M'_\epsilon, M''_\epsilon\in\partial\mathrm{B}_\epsilon(M_0)$ *so that* $M'_\epsilon, M''_\epsilon\to M_0$ *when* $\epsilon\to 0$, *then we get*

$\iint_{\partial\mathrm{B}_\epsilon}u\frac{\partial}{\partial n}(\frac{1}{r_{MM_0}})dS=\frac{1}{\epsilon^2}\iint_{\partial\mathrm{B}_\epsilon}udS=4\pi u(M'_\epsilon)\to 4\pi u(M_0)$;

$\iint_{\partial\mathrm{B}_\epsilon}\frac{1}{r_{MM_0}}\frac{\partial u}{\partial n}dS=\frac{1}{\epsilon}\iint_{\partial\mathrm{B}_\epsilon}\frac{\partial u}{\partial n}dS=4\pi\epsilon\frac{\partial u(M'_\epsilon)}{\partial n}\to 0.\#$

Theorem 2. *If* $u = u(x, y, z) \in C^2(\Omega) \cap C^1(\overline{\Omega})$, *then* $\forall M_0 = M_0(x_0, y_0, z_0) \in \Omega$ *we have*

$$u(M_0) = \frac{1}{4\pi} \iint\limits_{\partial\Omega} \left[\frac{1}{r_{MM_0}} \frac{\partial u}{\partial n} - u \frac{\partial}{\partial n} \left(\frac{1}{r_{MM_0}} \right) \right] dS - \frac{1}{4\pi} \iiint\limits_{\Omega} \frac{\Delta u}{r_{MM_0}} dxdydz\,.$$

Lemma 1. *If* $u = u(x, y, z) \in C^2(\Omega) \cap C^1(\overline{\Omega})$ *and* $\Delta u = 0$, *then* $\forall M_0 = M_0(x_0, y_0, z_0) \in \Omega$ *we have*

$$u(M_0) = \frac{1}{4\pi} \iint\limits_{\partial\Omega} \left[\frac{1}{r_{MM_0}} \frac{\partial u}{\partial n} - u \frac{\partial}{\partial n} \left(\frac{1}{r_{MM_0}} \right) \right] dS\,.$$

Lemma 2. *If* $u = u(x, y, z) \in C^2(\Omega) \cap C^1(\overline{\Omega})$ *and* $\Delta u = f(x, y, z)$, *then* $\forall M_0 = M_0(x_0, y_0, z_0) \in \Omega$ *we have*

$$u(M_0) = \frac{1}{4\pi} \iint\limits_{\partial\Omega} \left[\frac{1}{r_{MM_0}} \frac{\partial u}{\partial n} - u \frac{\partial}{\partial n} \left(\frac{1}{r_{MM_0}} \right) \right] dS - \frac{1}{4\pi} \iiint\limits_{\Omega} \frac{f(x, y, z)}{r_{MM_0}} dxdydz\,.$$

Mean value formula.

Theorem 3. *Suppose that* $u = u(x, y, z) \in C^2(\Omega) \cap C^1(\overline{\Omega})$ *is harmonic. Then for any balls falling inside* $B_a(M_0) \in \Omega$, *the following mean-value formula should hold:*

$$u(M_0) = \frac{1}{4\pi a^2} \iint_{\partial B_a(M_0)} u dS .$$

Proof. On specifying the domain Ω *in lemma 1to be* $B_a(M_0)$, *we have*

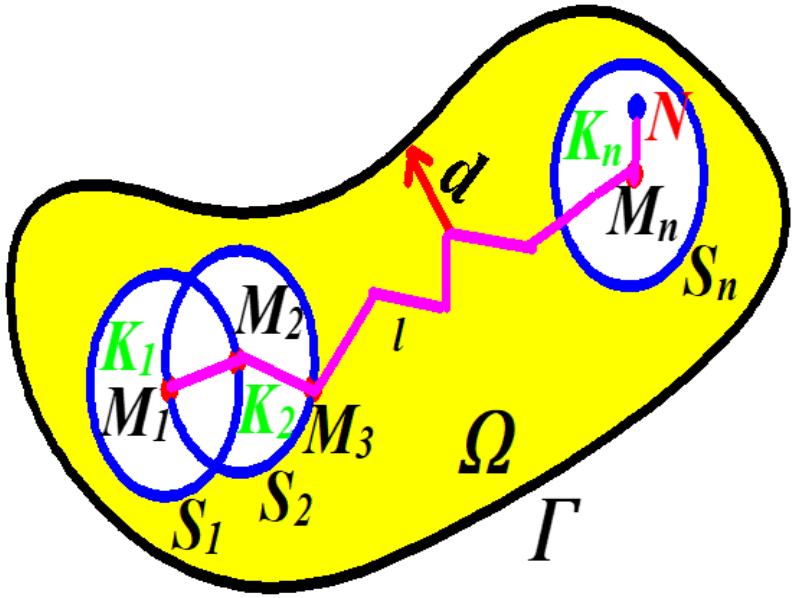
$$u(M_0) = \frac{1}{4\pi} \iint_{\partial\Omega} \left[\frac{1}{r_{MM_0}} \frac{\partial u}{\partial n} - u \frac{\partial}{\partial n} \left(\frac{1}{r_{MM_0}} \right) \right] dS .$$

Note that on $\partial B_a(M_0), r_{MM_0} = a, \frac{\partial}{\partial n}\left(\frac{1}{r_{MM_0}}\right) = -\frac{1}{a^2}$. *Thus, combining the theorem 1, the first term on the right side of the equation above is 0.*

Maximum principle.

The maximum principle for harmonic functions. *If u is non-constant harmonic function in Ω, and u is continuous towards its boundary, then u cannot be maximized in the interior part of Ω.*

Proof. Assume that u can be maximized at an interior point, say $r^* \in \Omega$, and we seek for contradiction. There should exist $a > 0$, such that $B_a(r^*) \subset \Omega$.



Suppose $u(r^*) = M = max\, u$ in Ω. As from the mean-value formula,

$$u(r^*) = \frac{1}{4\pi a^2} \iint_{\partial B_a(r^*)} u(r) dS = M.$$

That implies that the average value of u over $\partial B_a(r^*)$ equals the maximum value M of the whole

domain. Then the only possible is $u|_{\partial B_a(r^*)} = M$. *As the radius* a *varies, one should show that there exists a sphere, say* K_1, *centered at* r^*, *such that the function* u *uniformly equals* M *in* K_1.

Now we show that for any $s \in \Omega, u(s) = M$ *holds. If* $s \in K_1$, *it is already the case. So we are focused on the situation* $s \notin K_1$. *We choose another point, say* r^{**}, *in* K_1. *As* $u(r^{**}) = M$, *repeating the discussion above, we conclude that there exists another sphere centered at* r^{**}, *say* K_2, *such that* u *is uniform equal to* M *in* K_2. K_2 *is now closer to the target point* s *than* K_1. *One may carry on to generate a series of spheres* $K_i, i = 1,2, \dots, n$, *such that* $u|_{K_i} = M$.

This procedure is continued until s *falls inside one ball from the series. Therefore, we have shown that for any* $s \in \Omega, u(s) = M$. *This is contradictory to the fact that* u *is not a constant in* Ω. *Thus, the maximum principle is proved.*

Exercise 1. Prove the uniqueness and stability for the boundary conditions of the solution of problems governed by Poisson's equation with Dirichlet boundary condition in a ***bounded region****.*

Exercise 2. Ω *is a bounded region in* R^3*, set* $\Omega^c = R^3 \backslash \Omega$ *is a complement region, prove the uniqueness and stability for the boundary conditions of the solution of the exterior problems governed by Poisson's equation with Dirichlet boundary condition .*

Exterior boundary value problem

$$\begin{cases} u_t = a^2 u_{xx}, x > 0, t > 0, \\ u(x,0) = 0, x \geq 0, \\ u(0,t) = N, u(+\infty,t) = 0, t > 0 \end{cases}$$

Apply Laplace transform with respect to t

$$\begin{cases} pU(x,p) = a^2 \dfrac{\mathrm{d}^2 U(x,p)}{\mathrm{d}x^2}, \quad x > 0 \\ U(0,p) = \dfrac{N}{p}, U(+\infty,p) = 0 \end{cases}$$

$$U(x,p) = Ae^{\frac{\sqrt{p}}{a}x} + Be^{-\frac{\sqrt{p}}{a}x}$$

$$A + B = \frac{N}{p}, A = 0$$

$$U(x,p) = \frac{N}{p} e^{-\frac{\sqrt{p}}{a}x}$$

$$\mathrm{erfc}\left(\frac{x}{2a\sqrt{t}}\right) \leftrightarrow \frac{1}{p} e^{-\frac{x}{a}\sqrt{p}}$$

$$u(x,t) = N \cdot \mathrm{erfc}\left(\frac{x}{2a\sqrt{t}}\right)$$

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$$

$$\mathrm{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^{+\infty} e^{-t^2} dt$$

$$\begin{cases} u_{xx} + u_{yy} = 0, -\infty < x < +\infty, y > 0, \\ u(x,0) = \varphi(x), -\infty < x < +\infty, \\ \lim\limits_{x^2+y^2\to\infty} u(x,y) = 0. \end{cases}$$

Apply Fourier transform with respect to x

$$\begin{cases} \dfrac{d^2}{dy^2} U(\omega, y) - \omega^2 U(\omega, y) = 0, \\ U(\omega,0) = \Phi(\omega), \\ \lim\limits_{y\to+\infty} U(\omega, y) = 0. \end{cases}$$

$$U(\omega, y) = \Phi(\omega) e^{-|\omega| y}.$$

$$F^{-1}[e^{-|\omega| y}] = \frac{y}{\pi(x^2 + y^2)}$$

$$\begin{aligned} u(x,y) &= F^{-1}[U(\omega, y)] = F^{-1}[\Phi(\omega) e^{-|\omega| y}] \\ &= \varphi(x) * \frac{y}{\pi(x^2+y^2)} = \frac{y}{\pi} \int_{-\infty}^{+\infty} \frac{\varphi(\xi) d\xi}{(x-\xi)^2 + y^2}. \end{aligned}$$

Maximum principle.

Now we consider the heat equation $u_t = c^2 u_{xx}, 0 < x < L, 0 < t \leq T$, *set* $Q = \{(x,t) | \, 0 < x < L, 0 < t \leq T\}$.

The maximum principle for heat equation. *If* $u = u(x,t) \in C^{2,1}(Q) \cap C(\bar{Q})$ *satisfies the heat equation, then* u *cannot be maximized in the interior part of* Q.

Exercise 1 . we consider $\begin{cases} u_t = c^2 u_{xx} + h(x,t), 0 < x < L, t > 0, \\ u(x,0) = f(x), 0 \le x \le L, \\ u(0,t) = p(t), u(L,t) = q(t), t \ge 0. \end{cases}$

Prove the uniqueness and stability of the solution of the initial-boundary problem.

Exercise 2. Consider the following initial-boundary value problem

$$\begin{cases} u_t = c^2 u_{xx} + bu, \qquad 0 < x < L, 0 < t < T, \\ u(x,0) = f(x), \qquad 0 \le x \le L, \\ u(0,t) = p(t), \qquad u(L,t) = q(t), \qquad 0 \le t \le T, \end{cases}$$

where $b \ge 0, |p(t)| \le B, |q(t)| \le B, |f(x)| \le M.$

Show that if $u = u(x,t)$ *is a solution to this problem, then*

$$|u(x,t)| \le \max\{Me^{bt}, Be^{bt}\}, \qquad (x,t) \in [0,L] \times [0,T].$$

Green's function.

Consider the Laplace equation with the Dirichlet boundary condition $\begin{cases} \Delta u = 0, x \in \Omega, \\ u\big|_{\partial\Omega} = f \end{cases}$

If u and v are harmonic functions in Ω*, then we have*

$$u(M_0) = -\frac{1}{4\pi}\iint_{\partial\Omega}[u(\frac{\partial}{\partial n}(\frac{1}{r_{MM_0}}) - \frac{1}{r_{MM_0}}\frac{\partial u}{\partial n}]\, dS \qquad (1)$$

$$0 = \iint_{\partial\Omega}(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n})dS \qquad (2)$$

Eq(2) minus Eq(1), we obtain

$$u(M_0) = -\iint_{\partial\Omega}\left\{u\frac{\partial}{\partial n}(v + \frac{1}{4\pi}\frac{1}{r_{MM_0}}) - (v + \frac{1}{4\pi}\frac{1}{r_{MM_0}})\frac{\partial u}{\partial n}\right\} dS \qquad (3)$$

$$u(M_0) = -\iint_{\partial\Omega} \left\{ u \frac{\partial}{\partial n}\left(v + \frac{1}{4\pi}\frac{1}{r_{MM_0}}\right) - \left(v + \frac{1}{4\pi}\frac{1}{r_{MM_0}}\right)\frac{\partial u}{\partial n} \right\} dS \qquad (3)$$

Set v satisfies $v\big|_{\partial\Omega} = -\frac{1}{4\pi}\frac{1}{r_{MM_0}}\Big|_{\partial\Omega}$ *then*

$$u(M_0) = -\iint_{\partial\Omega} u \frac{\partial}{\partial n}\left(v + \frac{1}{4\pi}\frac{1}{r_{MM_0}}\right) dS. \qquad (4)$$

To this stage, we define

$$G(M, M_0) = v + \frac{1}{4\pi}\frac{1}{r_{MM_0}} \qquad (5)$$

Inserting it into Eq.(4) finally expresses the solution by

$$\boldsymbol{u(M_0) = -\iint_{\partial\Omega} u \frac{\partial G}{\partial n} ds = -\iint_{\partial\Omega} f \frac{\partial G}{\partial n} ds} \qquad (6)$$

Properties of Green's Function

Property 1. *As* $M \to M_0$, G *tends to infinity at a same rate as* $\frac{1}{4\pi \mathrm{r}}$, *i.e.* $G \sim O\left(\frac{1}{4\pi r_{MM_0}}\right)$, *as* $M \to M_0$.

Property 2. $G|_{\partial\Omega} = 0$.

Property 3. *The inputs of* G *are communicative, i.e.* $G(r;s) = G(s;r)$ *holds.*

Property 4. $\iint_{\partial\Omega} \frac{\partial G}{\partial n} dS = -1$

Two dimensional case.

Here the function v *is chosen to be the radial solution to two dimensional Laplace's equation (with respect to the reference point* $M_0 = M_0(x_0, y_0) \in R^2$*), i.e.*

$$v = -\frac{1}{2\pi}\ln\frac{1}{r_{MM_0}},$$

where $r_{MM_0} = \sqrt{(x-x_0)^2 + (y-y_0)^2}$.

The second Green's formula: $\iint_D (u\Delta v - v\Delta u)dxdy = \int_\tau \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right)ds.$

Theorem 4. *If* $u = u(x,y) \in C^2(\mathrm{D}) \cap C^1(\overline{\mathrm{D}})$ *and u is harmonic in* Ω*, then* $\forall M_0 = M_0(x_0, y_0) \in \mathrm{D}$ *we have*

$$u(M_0) = \frac{1}{2\pi}\int_\tau \left(\ln\frac{1}{r_{MM_0}}\frac{\partial u}{\partial n} - u\frac{\partial}{\partial n}(\ln\frac{1}{r_{MM_0}})\right)ds.$$

Mean value formula.

Theorem 5. *Suppose that* $u = u(x, y) \in D$ *is harmonic. Then for any circular falling inside* $C_a(M_0) \subset$ D, *the following mean-value formula should hold:*

$$u(M_0) = \frac{1}{2\pi a} \int_{\tau_a} u(\xi, \eta) ds\,,$$

where $\tau_a = \{(\xi, \eta) | (\xi - x_0)^2 + (\eta - y_0)^2 = a^2\}$.

For two dimensional situation $\begin{cases} \Delta \boldsymbol{u} = \boldsymbol{u_{xx}} + \boldsymbol{u_{yy}} = \boldsymbol{0}, (\boldsymbol{x}, \boldsymbol{y}) \in \boldsymbol{D} \\ \boldsymbol{u}(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y}), (\boldsymbol{x}, \boldsymbol{y}) \in \mathbf{T} = \boldsymbol{\partial D}. \end{cases}$

The Green's function can be constructed by $G(M, M_0) = v + \frac{1}{2\pi} \ln \frac{1}{r_{MM_0}}$, *where* $M_0 = M_0(x_0, y_0), M =$

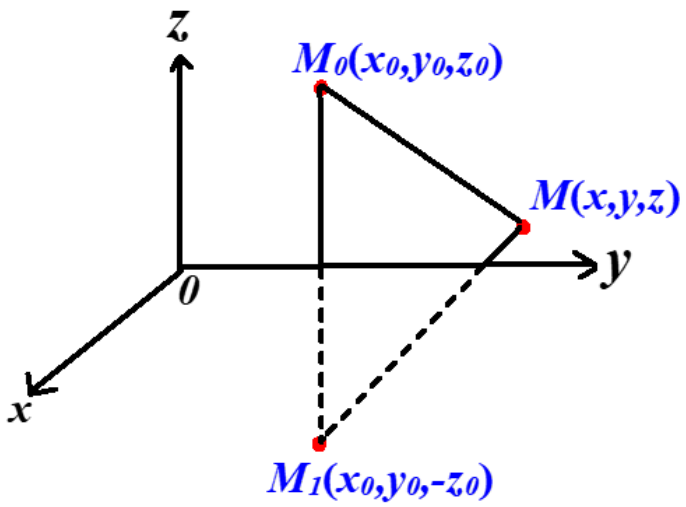
$M(x, y) \in D, r_{MM_0} = \sqrt{(x - x_0)^2 + \left(y_{y0}\right)^2}$, *and* $v = v(x, y)$ *satisfies* $\begin{cases} \Delta v = v_{xx} + v_{yy} = 0, (x, y) \in D \\ v(x, y) = \frac{1}{2\pi} \ln \frac{1}{r_{MM_0}}, (x, y) \in \mathrm{T} = \partial D. \end{cases}$

Then the solution of the original problem can be expressed by

$$u(x, y) = -\int_{\tau} f \frac{\partial G}{\partial n} ds\,.$$

Image method for electric potentials

Let us consider $\begin{cases} u_{xx} + u_{yy} + u_{zz} = 0, z > 0 \\ u(x, y, 0) = f(x, y), -\infty < x, y < +\infty. \end{cases}$ (7)



Then we seek to express the Green's function for the problem.

Our aim is to determine the distribution of the induced charges in the other half plane (z<0), such that the plane of Z=0 is an iso-potential plane.

$\forall\ M_0 = M_0(x_0, y_0, z_0) \in \Omega$, *the image point is given by* $M_1 = M_1(x_0, y_0, z_0)$, *and the induced (negative unit) charge there exerts an electric potential of* $\boldsymbol{v = \dfrac{1}{4\pi\, r_{MM_1}}}$.

Then the Green's function can be given by $G(M, M_0) = \dfrac{1}{4\pi}\left(\dfrac{1}{r_{MM_1}} - \dfrac{1}{r_{MM_0}}\right)$.

On the z=0 plane, $dS = dxdy$, and the outer normal vector lies in the same direction of –z.

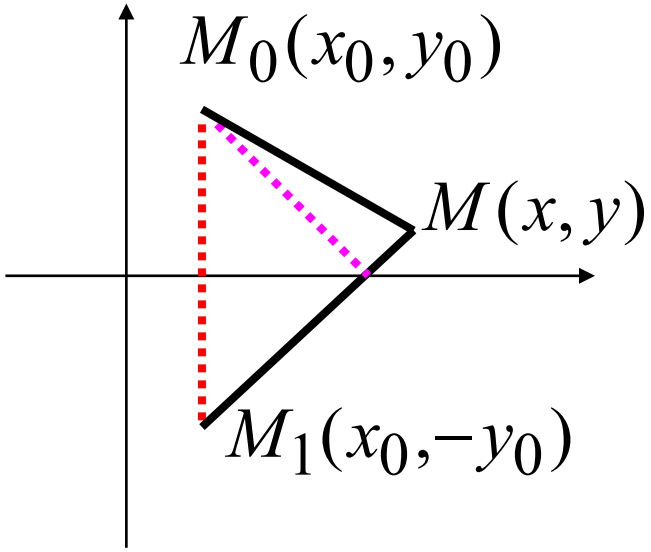
Thus, we have $\frac{\partial G}{\partial n}|_{z=0} = -\frac{z_0}{2\pi}[(x-x_0)^2 + (y-y_0)^2 + z_0^2]^{-\frac{3}{2}}$

Inserting it into Eq.(6) finally expresses the solution for problem (7) by

$$u(x_0, y_0, z_0) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{z_0 f(x,y)}{[(x-x_0)^2 + (y-y_0)^2 + z_0^{\,2}]^{3/2}}dxdy$$

The case in two dimension

Let us consider $\begin{cases} \Delta u(x,y) = u_{xx} + u_{yy} = 0, y > 0, \\ u(x,0) = f(x), -\infty < x < +\infty \end{cases}$



Then the Green's function can be given by

$$G(M, M_0) = \frac{1}{2\pi}\left(\ln\frac{1}{r_{MM_1}} - \ln\frac{1}{r_{MM_0}}\right)$$

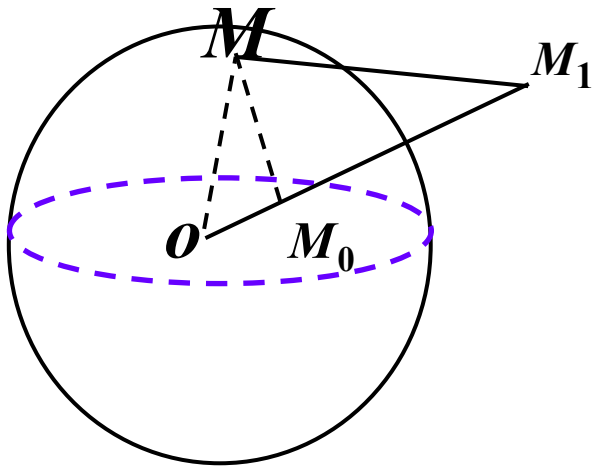
Thus, we have

$$\left.\frac{\partial G}{\partial n}\right|_{y=0} = -\left.\frac{\partial G}{\partial y}\right|_{y=0} = \frac{y_0}{\pi[(x-x_0)^2 + y_0^2]}$$

The solution of the problem is given by $u(x_0, y_0) = \frac{y_0}{\pi}\int_{-\infty}^{+\infty}\frac{f(x)dx}{(x-x_0)^2 + y_0^2}$

The case in a spherical domain

When the domain of definition becomes a spherical region, say, $\Omega = \{r \| r| < R\}$, *the corresponding interior boundary value problem can be formulated by* $\begin{cases} u_{xx} + u_{yy} + u_{zz} = 0, (x, y, z) \in \Omega \\ u|_{\partial\Omega} = f(x, y, z) \end{cases}$



We derive the corresponding Green's formula with reference to an arbitrary reference point inside the sphere, say M_0.

We then extended the vector OM_0 *outside the sphere to point* M_1, *such that* $\rho_0\rho_1 = R^2$, *where* $\rho_0 = |\overrightarrow{OM_0}|, \rho_1 = |\overrightarrow{OM_1}|$. *Here the point* M_1 *is called the* ***inversion point*** *of* M_0 *about the spherical surface.*

Let us suppose that the Green's formula in this scenario has the form of

$$G(M, M_0) = \frac{1}{4\pi}\left(\frac{1}{r_{MM_0}} - \frac{q}{r_{MM_1}}\right)$$

We need to determine q, such that the spherical surface becomes iso-potential. To this end, a point

P on $\partial\Omega$ *is arbitrary chosen. We can conclude that* $\Delta OM_0M \sim \Delta OM_1M$ *,hence* $\frac{r_{M_1M}}{r_{M_0M}} = \frac{R}{\rho_0}$, $\Rightarrow$ $q = \frac{R}{\rho_0}$.

Therefore, the Green's function is expressed by

$$G(M, M_0) = \frac{1}{4\pi}\left(\frac{1}{r_{MM_0}} - \frac{R}{\rho_0}\frac{1}{r_{MM_1}}\right)$$

Then solution of the initial boundary problem $\begin{cases} u_{xx} + u_{yy} + u_{zz} = 0, (x, y, z) \in \Omega \\ u|_{\partial\Omega} = f(x, y, z) \end{cases}$

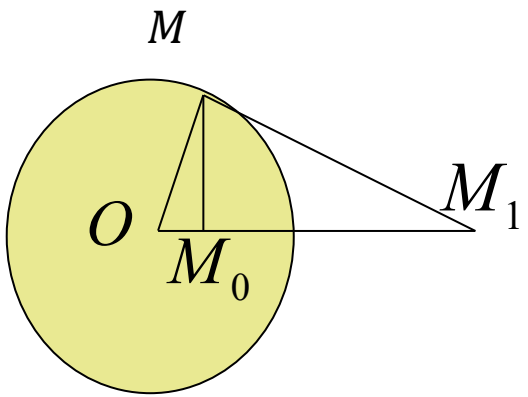
is calculated by $u(x_0, y_0, z_0) = -\iint_{\partial\Omega} f(x, y, z)\frac{\partial G}{\partial n} dS$

The case in a circular domain

When the domain of definition becomes a circular region, say, $\Omega = \{r \,|\, |r| < R\}$, *the corresponding interior boundary value problem can be formulated by*

$$\begin{cases} \Delta u(x,y) = u_{xx} + u_{yy} = 0, (x,y) \in \Omega \\ u|_{\partial\Omega} = f(x,y), (x,y) \in \partial\Omega \end{cases}$$

The Green's function is expressed by $G(M, M_0) = \frac{1}{2\pi}\left(\ln\frac{1}{r_{MM_0}} - \ln\frac{R}{r_{OM_0}}\frac{1}{r_{MM_1}}\right)$



Exercise 1 . Express the Green's function in $\Omega = \{(x, y) | x, y > 0\}$*, and use the Green's function to solve the case for Laplace's equation with Dirichlet condition as follows:*

$$\begin{cases} u_{xx} + u_{yy} = 0, 0 < x, 0 < y, \\ u(0, y) = 0, y \geq 0, \\ u(x, 0) = f(x), x \geq 0. \end{cases}$$

Classification of second-order linear PDEs with two variables.

The general form: $a_{11}u_{xx} + 2a_{12}u_{xy} + a_{22}u_{yy} + a_1u_x + b_1u_y + cu = f$, ***(1)***

Where $a_{11}, a_{12}, a_{22}, a_1, b_1, c, f$ *can all be known functions of* x *and* y.

Our aim is to rewrite the eq(1) in a new coordinate system in a simplified form. We introduce

$$\begin{cases} \xi = \xi(x, y), \\ \eta = \eta(x, y), \end{cases}$$

To make the transformation is nonsingular, the Jacobean determinant should not vanish, i.e.

$$\xi_x\eta_y - \xi_y\eta_x \neq 0.$$

On using the multivariable chain rule, eq(1) can be expressed by

$$A_{11}u_{\xi\xi} + 2A_{12}u_{\xi\eta} + A_{22}u_{\eta\eta} + A_1u_\xi + B_1u_\eta + Cu + F = 0, \textit{(2)}$$

$$\textit{where} \begin{cases} A_{11} = a_{11}\xi_x^2 + 2a_{12}\xi_x\xi_y + a_{22}\xi_y^2, \\ A_{12} = a_{11}\xi_x\eta_x + a_{12}\left(\xi_x\eta_y + \xi_y\eta_x\right) + a_{22}\xi_y\eta_y, \\ A_{22} = a_{11}\eta_x^2 + 2a_{12}\eta_x\eta_y + a_{22}\eta_y^2, \end{cases}$$

Classification of second-order linear PDEs with two variables.

The general form: $a_{11}u_{xx} + 2a_{12}u_{xy} + a_{22}u_{yy} + a_1u_x + b_1u_y + cu = f,$

Where $a_{11}, a_{12}, a_{22}, a_1, b_1, c, f$ *can all be known functions of* x *and* y.

At any point falling inside its domain of definition, say (x_0, y_0), *it can be classified into one of the following three groups.*

Case 1. $\boldsymbol{a_{12}^2 - a_{11}a_{22} > 0}$, *the equation is hyperbolic at* $(\boldsymbol{x_0}, \boldsymbol{y_0})$.*(the wave equation)*

Case 2. $\boldsymbol{a_{12}^2 - a_{11}a_{22} = 0}$, *the equation is parabolic at* $(\boldsymbol{x_0}, \boldsymbol{y_0})$.*(the heat equation)*

Case 3. $\boldsymbol{a_{12}^2 - a_{11}a_{22} < 0}$, *the equation is elliptic at* $(\boldsymbol{x_0}, \boldsymbol{y_0})$.*(the Possion's equation)*

$$\begin{cases} A_{11} = a_{11}\xi_x^2 + 2a_{12}\xi_x\xi_y + a_{22}\xi_y^2, \\ A_{12} = a_{11}\xi_x\eta_x + a_{12}\left(\xi_x\eta_y + \xi_y\eta_x\right) + a_{22}\xi_y\eta_y, \\ A_{22} = a_{11}\eta_x^2 + 2a_{12}\eta_x\eta_y + a_{22}\eta_y^2. \end{cases}$$

We consider another PDE for the binary function $\phi(x, y)$ *governed by*

$$a_{11}\phi_x^2 + 2a_{12}\phi_x\phi_y + a_{22}\phi_y^2 = 0. \ (3)$$

If we can find two mutually independent solutions to eq (3), say, $\phi = \phi_1(x, y), \phi = \phi_2(x, y)$*, then*

$A_{11} = A_{22} = 0$ *would be derived by using the mapping rule by* $\begin{cases} \xi = \phi_1(x, y), \\ \eta = \phi_2(x, y). \end{cases}$

Let $\phi_y \neq 0$*, divide* ϕ_y^2 *on both sides of eq(3) leads to*

$$a_{11}\left(\frac{\phi_x}{\phi_y}\right)^2 + 2a_{12}\frac{\phi_x}{\phi_y} + a_{22} = 0.$$

$$a_{11}\left(\frac{\phi_x}{\phi_y}\right)^2 + 2a_{12}\frac{\phi_x}{\phi_y} + a_{22} = 0. \ (4)$$

Eq (4) can be rewritten as $a_{11}(dy)^2 - 2a_{12}dxdy + a_{22}(dx)^2 = 0. (5)$

Eq (5) is the characteristic equation of eq (1).

The integral curves of eq(4) are characteristic curves of eq(1) which can be obtained from

$$\frac{dy}{dx} = -\frac{\phi_x}{\phi_y} = \frac{a_{12} \pm \sqrt{a_{12}^2 - a_{11}a_{22}}}{a_{11}}. \ (6)$$

Case 1. $\Delta = \boldsymbol{a_{12}^2 - a_{11}a_{22} > 0}$, *the equation is hyperbolic.*

In this case, the eq(5) yields two real families of parametric curves $\phi_1(x,y) = C_1, \phi_2(x,y) = C_2$.

Eq (1) is simplified to be $u_{\xi\eta} = A_2 u_\xi + B_2 u_\eta + C_2 u + F_2$ *(7)*

by using the mapping rule by $\begin{cases} \xi = \phi_1(x,y), \\ \eta = \phi_2(x,y). \end{cases}$

.

$u_{\xi\eta} = A_2 u_\xi + B_2 u_\eta + C_2 u + F_2$. *(7)*

Eq (7``) is called the ***first canonical form of second order linear hyperbolic PDEs****.*

If we further introduce $\begin{cases} \alpha = \xi + \eta \\ \beta = \xi - \eta \end{cases}$ *the eq (7) can be rewritten by*

$u_{\alpha\alpha} - u_{\beta\beta} = A_3 u_\alpha + B_3 u_\beta + C_3 u + F_3$. *(8)*

Eq (8) is called the ***second canonical form of second order linear hyperbolic PDEs****.*

Case 2. $\Delta = \boldsymbol{a_{12}^2 - a_{11}a_{22} = 0}$*, the equation is parabolic.*

In this case, eq(6) becomes $\frac{dy}{dx} = \frac{a_{12}}{a_{11}}$.

Eq (1) is simplified to be $u_{\eta\eta} = A_4 u_\xi + B_4 u_\eta + C_4 u + F_4$ *(9),*

by using the mapping rule by $\xi = \phi_1(x, y)$.

Eq (9) is called the ***canonical form of second order linear parabolic PDEs****.*

Case 3. $\Delta = \boldsymbol{a_{12}^2 - a_{11}a_{22} < 0}$, *the equation is elliptic.*

In this case, eq(6) yields no real integral curves .

Assume that eq (6) can be solved by $\phi(x, y) = \phi_1(x, y) \pm i\phi_2(x, y)$.

Eq (1) is simplified to be $u_{\xi\xi} + u_{\eta\eta} = A_5 u_\xi + B_5 u_\eta + C_5 u + F_5$ *(10),*

by using the mapping rule by $\begin{cases} \xi = \phi_1(x, y), \\ \eta = \phi_2(x, y). \end{cases}$

Eq (10) is called the ***canonical form of second order linear elliptic PDEs****.*

$$\begin{cases} u_{xx} + 2u_{xy} - 3u_{yy} = 0, & y > 0, -\infty < x < +\infty \\ u(x,0) = e^{-x^2}, u_y(x,0) = 0, & -\infty < x < +\infty \end{cases}$$

Characteristic equation $(dy)^2 - 2dxdy - 3(dx)^2 = 0$ $\quad (dy - 3dx)(dy + dx) = 0$

Characteristic lines $y - 3x = C_1, y + x = C_2$

Set $\xi = y - 3x \quad \eta = y + x \longrightarrow \dfrac{\partial^2 u}{\partial \xi \partial \eta} = 0 \quad u = f_1(\xi) + f_2(\eta)$

The general solution is $u(x,y) = f_1(y - 3x) + f_2(y + x)$

$$\begin{cases} u(x,0) = e^{-x^2} = f_1(-3x) + f_2(x) \\ u_y(x,0) = 0 = f_1'(-3x) + f_2'(x) \end{cases} \qquad \begin{cases} f_1(-3x) + f_2(x) = e^{-x^2} \\ -\dfrac{1}{3} f_1(-3x) + f_2(x) = C \end{cases}$$

$$\begin{cases} f_1(-3x) = \dfrac{3}{4} e^{-x^2} - \dfrac{3}{4} C \\ f_2(x) = \dfrac{1}{4} e^{-x^2} + \dfrac{3}{4} C \end{cases} \qquad \begin{cases} f_1(x) = \dfrac{3}{4} e^{-x^2/9} - \dfrac{3}{4} C \\ f_2(x) = \dfrac{3}{4} e^{-x^2} + \dfrac{3}{4} C \end{cases}$$

$$u(x,y) = \frac{3}{4} e^{-(y-3x)^2/9} - \frac{3}{4} C + \frac{3}{4} e^{-(y+x)^2} + \frac{3}{4} C = \frac{3}{4} e^{-(y-3x)^2/9} + \frac{3}{4} e^{-(y+x)^2}$$

Classification of second-order linear PDEs with two variables.

The general form: $a_{11}u_{xx} + 2a_{12}u_{xy} + a_{22}u_{yy} + a_1u_x + b_1u_y + cu = f,$

Where $a_{11}, a_{12}, a_{22}, a_1, b_1, c, f$ *can all be known functions of* x *and* y.

At any point falling inside its domain of definition, say (x_0, y_0), *it can be classified into one of the following three groups.*

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Case 3. $\boldsymbol{a_{12}^2 - a_{11}a_{22} < 0}$, *the equation is elliptic at* $(\boldsymbol{x_0}, \boldsymbol{y_0})$.*(the Possion's equation)*

Exercise 1. Judge the type of the Euler-Tricomi equation: $yu_{xx} + u_{yy} = 0$.

Exercise 2. transform the wave equation $u_{tt} - a^2 u_{xx} = 0$ *into its canonical form.*

Energy integration method.

Theorem 1. If there are smooth solutions for the following problem given by

$$\begin{cases} u_t = c^2\Delta u + h(x,y,z,t), (x,y,z) \in \Omega, 0 < t \le T, \\ u(x,y,z,0) = f(x,y,z), (x,y,z) \in \Omega, \\ u(x,y,z,t) = g(x,y,z,t), (x,y,z) \in \partial\Omega, 0 \le t \le T \end{cases} ,(1)$$

then the solution is unique.

Proof. If u_1, u_2 *are both solutions for problem (1), then we introduce* $u = u_1 - u_2$*, which satisfies*

$$\begin{cases} u_t = c^2\Delta u, (x,y,z) \in \Omega, 0 < t \le T, \\ u(x,y,z,0) = 0, (x,y,z) \in \Omega, \\ u(x,y,z,t) = 0, (x,y,z) \in \partial\Omega, 0 \le t \le T \end{cases} .(2)$$

We multiply both sides of $u_t = c^2\Delta u$ *by* u*, and integrate the equation with respect to spatial variables in* Ω*. As from Green's formula, we have*

$$\frac{1}{2}\frac{d}{dt}\iiint_{\Omega} u^2 dxdydz + c^2 \iiint_{\Omega} |\nabla u|^2 dxdydz = c^2 \iint_{\partial\Omega} u\frac{\partial u}{\partial n} dS = 0,$$

where $|\nabla u|^2 = u_x^2 + u_y^2 + u_z^2$.

$$\frac{1}{2}\frac{d}{dt}\iiint_{\Omega} u^2 dxdydz + c^2 \iiint_{\Omega} |\nabla u|^2 dxdydz = c^2 \iint_{\partial\Omega} u\frac{\partial u}{\partial n} dS = 0,$$

where $|\nabla u|^2 = u_x^2 + u_y^2 + u_z^2$.

We define the energy function as $E(t) = \frac{1}{2}\iiint_{\Omega} u^2(x,y,z,t)dxdydz\,, 0 \le t \le T$.

As from $E'(t) \le 0, 0 \le t \le T$ *and* $E(0) = \frac{1}{2}\iiint_{\Omega} u^2(x,y,z,0)dxdydz = 0$, *we have*

$E(t) = 0, 0 \le t \le T$. *Then* $u(x,y,z,t) = 0$.

Theorem 2. If there are smooth solutions for the following problem given by

$$\begin{cases} u_t = c^2 u_{xx} + h(x,t), 0 < x < L, 0 < t, \\ \quad u(x,0) = f(x), 0 \le x \le L, \\ u(0,t) = p(t), u_x(L,t) = q(t), 0 \le t \end{cases} ,(3)$$

then the solution is unique.

Proof. If u_1, u_2 *are both solutions for problem (3), then we introduce* $u = u_1 - u_2$*, which satisfies*

$$\begin{cases} u_t = c^2 u_{xx}, 0 < x < L, 0 < t, \\ \quad u(x,0) = 0, 0 \le x \le L, \\ u(0,t) = 0, u_x(L,t) = 0, 0 \le t \end{cases} ,(4)$$

We multiply both sides of $u_t = c^2 u_{xx}$ *by* u*, and integrate the equation with respect to spatial variables in* $[0, \mathrm{L}]$*. We have*

$$\frac{1}{2}\frac{d}{dt}\int_0^L u^2 dx = \int_0^L u u_t dx = c^2 \int_0^L u u_{xx} dx = c^2 \int_0^L (u_x u)_x dx - c^2 \int_0^L u_x^2 dx\,.$$

Then we obtain $\frac{1}{2}\frac{d}{dt}\int_0^L u^2 dx = -c^2 \int_0^L u_x^2 dx \le 0$,

That implies $u = 0, 0 \le x \le L, t \ge 0.$

Theorem 3. If there are smooth solutions for the following problem given by

$$\begin{cases} u_{xx} + u_{yy} + u_{zz} = h(x,y,z), (x,y,z) \in \Omega \\ \alpha u + \beta \frac{\partial u}{\partial n} = g(x,y,z), (x,y,z) \in \partial\Omega \end{cases} \quad (5)$$

where $\alpha > 0, \beta \geq 0$ *, then the solution is unique.*

Proof. If u_1, u_2 *are both solutions for problem (5), then we introduce* $u = u_1 - u_2$*, which satisfies*

$$\begin{cases} u_{xx} + u_{yy} + u_{zz} = 0, (x,y,z) \in \Omega \\ \alpha u + \beta \frac{\partial u}{\partial n} = 0, (x,y,z) \in \partial\Omega \end{cases} \quad (6)$$

We multiply both sides of $u_{xx} + u_{yy} + u_{zz} = 0$ *by* u*, and integrate the equation with respect to spatial variables in* Ω*. As from the Green's formula, we have*

$$\iiint_{\Omega} |\nabla u|^2 dxdydz - \iint_{\partial\Omega} u \frac{\partial u}{\partial n} dS = 0.$$

Then we obtain $\iiint_{\Omega} |\nabla u|^2 dxdydz + \iint_{\partial\Omega} \frac{\beta}{\alpha} \left(\frac{\partial u}{\partial n}\right)^2 dS = 0.$

,

That implies $u = 0$ *in* Ω.

Theorem 4. If there are smooth solutions for the following problem given by

$$\begin{cases} u_{tt} = c^2\Delta u + h(x,y,z,t), (x,y,z) \in \Omega, 0 < t, \\ u(x,y,z,0) = f(x,y,z), u_t(x,y,z,0) = g(x,y,z), (x,y,z) \in \Omega, \\ \alpha u + \beta \frac{\partial u}{\partial n} = \phi(x,y,z), (x,y,z) \in \partial\Omega, 0 \le t \end{cases}, \quad (7)$$

where $\alpha > 0, \beta \ge 0$ *, then the solution is unique.*

Proof. If u_1, u_2 *are both solutions for problem (1), then we introduce* $u = u_1 - u_2$*, which satisfies*

$$\begin{cases} u_{tt} = c^2\Delta u, (x,y,z) \in \Omega, 0 < t, \\ u(x,y,z,0) = 0, u_t(x,y,z,0) = 0, (x,y,z) \in \Omega, \\ \alpha u + \beta \frac{\partial u}{\partial n} = 0, (x,y,z) \in \partial\Omega, 0 \le t \end{cases}, \quad (7)$$

We multiply both sides of $u_{tt} = c^2\Delta u$ *by* u_t*, and integrate the equation with respect to spatial variables in* Ω*. As from Green's formula, we have*

$$\frac{1}{2}\frac{d}{dt}\iiint_{\Omega} [u_t^2 + c^2|\nabla u|^2]dxdydz = c^2 \iint_{\partial\Omega} u_t \frac{\partial u}{\partial n} dS,$$

Where $\iint_{\partial\Omega} u_t \frac{\partial u}{\partial n} dS = -\iint_{\partial\Omega} \frac{\alpha}{\beta} u u_t dS = -\frac{1}{2}\frac{d}{dt}\iint_{\partial\Omega} \frac{\alpha}{\beta} u^2 dS.$

$$\frac{1}{2}\frac{d}{dt}\iiint_{\Omega}[u_t^2 + c^2|\nabla u|^2]dxdydz - c^2\iint_{\partial\Omega} u_t\frac{\partial u}{\partial n}dS = 0,$$

Where $\iint_{\partial\Omega} u_t\frac{\partial u}{\partial n}dS = -\iint_{\partial\Omega}\frac{\alpha}{\beta}uu_t dS = -\frac{1}{2}\frac{d}{dt}\iint_{\partial\Omega}\frac{\alpha}{\beta}u^2 dS.$

We define the energy function as $E(t) = \frac{1}{2}\iiint_{\Omega}[u_t^2 + c^2|\nabla u|^2]dxdydz + c^2\frac{1}{2}\iint_{\partial\Omega}\frac{\alpha}{\beta}u^2 dS$.

As from $E'(t) = 0, 0 \leq t$ *and* $E(0) = 0$, *we have* $E(t) = 0, 0 \leq t$. *Then* $u(x, y, z, t) = 0$.

Example. Use the energy integration method to demonstrate the following problem given by

$$\begin{cases} u_{xx} + u_{yy} = 0, \qquad -1 < x < 1, -1 < y < 1, \\ u(-1, y) = u(1, y) = 0, \qquad -1 \le y \le 1, \\ \left(u_x - u_y\right)\Big|_{|y|=1} = 0, \qquad -1 \le x \le 1. \end{cases}$$

simply yields a trivial solution.

Proof. We multiply both sides of $u_{xx} + u_{yy} = 0$ *by* u*, and integrate the equation with respect to spatial variables in* $[-1,1] \times [-1,1]$*. Make the use of the boundary conditions, we have*

$$0 = \int_{-1}^{1}\int_{-1}^{1}(uu_{xx} + uu_{yy})dxdy = \int_{-1}^{1}\left(uu_x|_{-1}^{1} - \int_{-1}^{1} u_x^2\, dx\right)dy + \int_{-1}^{1}\left(uu_y|_{-1}^{1} - \int_{-1}^{1} u_y^2\, dy\right)dx =$$

$$-\int_{-1}^{1}\int_{-1}^{1}\left(u_x^2 + u_y^2\right)dxdy + \int_{-1}^{1}[u(x,1)u_x(x,1) - u(x,-1)u_x(x,-1)]dx = -\int_{-1}^{1}\int_{-1}^{1}\left(u_x^2 + u_y^2\right)dxdy +$$

$$\frac{1}{2}u^2(x,1)|_{-1}^{1} - \frac{1}{2}u^2(x,-1)|_{-1}^{1} = -\int_{-1}^{1}\int_{-1}^{1}\left(u_x^2 + u_y^2\right)dxdy.$$

That implies $u_x = u_y = 0$*, also* $u(-1, y) = u(1, y) = 0$ *holds, thus* $u = 0$.

Exercise 1. Use the energy integration method to demonstrate the solution of the following initial boundary value problem given by

$$\begin{cases} u_{tt} + c^2 u_{xxxx} = f(x,t), \ \ 0 < x < L, t > 0, \\ u(x,0) = \phi(x), u_t(x,0) = \psi(x), \ \ 0 \le x \le L, \\ u(0,t) = u(L,t) = u_{xx}(0,t) = u_{xx}(L,t) = 0, \ \ t \ge 0 \end{cases}$$

is unique.

$$\begin{cases} \dfrac{\partial u}{\partial t} = a^2 \dfrac{\partial^2 u}{\partial x^2}, & 0 < x < l, t > 0 \\ \dfrac{\partial u(0,t)}{\partial x} = 0, \dfrac{\partial u(l,t)}{\partial x} = 0, & t > 0 \\ u(x,0) = x, & 0 \le x \le l \end{cases}$$

$$u(x,t) = X(x)T(t)$$

$$T'X = a^2 TX''$$

$$\frac{T'}{a^2 T} = \frac{X''}{X} = -\lambda$$

$$X'' + \lambda X = 0$$

$$T' + a^2 \lambda T = 0$$

$$\frac{\partial u(0,t)}{\partial x} = X'(0)T(t) = 0$$

$$\frac{\partial u(l,t)}{\partial x} = X'(l)T(t) = 0$$

$$X'(0) = 0, X'(l) = 0$$

$$\begin{cases} X'' + \lambda X = 0 & 0 < x < l \\ X'(0) = 0, & X'(l) = 0 \end{cases}$$

$$\begin{cases} X'' + \lambda X = 0 & 0 < x < l \\ X'(0) = 0, & X'(l) = 0 \end{cases}$$

$\lambda = 0$ $\quad X'' = 0 \quad X = Ax + B \quad X = B_0$

$\lambda = \beta^2 > 0$ $\quad X'' + \beta^2 X = 0 \quad X = A\sin\beta x + B\cos\beta x$

$$X'(0) = A\beta = 0 \qquad X'(l) = -B\beta\sin\beta l = 0$$

$$\beta_n = \frac{n\pi}{l} \quad \lambda_n = \beta_n^2 = \left(\frac{n\pi}{l}\right)^2, n = 1,2,3,\cdots \; X_n = B_n\cos\frac{n\pi}{l}x$$

$$T' + a^2\lambda T = 0$$

$\lambda = 0$ $\quad T_0' = 0 \quad T_0 = B_0 t + A_0 = A_0 \quad u_0 = X_0 T_0 = B_0 A_0 = C_0$

$\lambda > 0$ $\quad T_n' + \dfrac{a^2 n^2 \pi^2}{l^2} T_n = 0 \quad T_n = A_n e^{-\frac{a^2 n^2 \pi^2}{l^2} t}$

$$u_n = X_n T_n = A_n B_n e^{-\frac{a^2 n^2 \pi^2}{l^2} t}\cos\frac{n\pi}{l}x = C_n e^{-\frac{a^2 n^2 \pi^2}{l^2} t}\cos\frac{n\pi}{l}x$$

$$u = \sum_{n=0}^{\infty} u_n = C_0 + \sum_{n=1}^{\infty} C_n e^{-\frac{a^2 n^2 \pi^2}{l^2} t}\cos\frac{n\pi}{l}x$$

$$u = C_0 + \sum_{n=1}^{\infty} C_n e^{-\frac{a^2 n^2 \pi^2}{l^2} t} \cos \frac{n\pi}{l} x$$

$$u(x,0) = x = C_0 + \sum_{n=1}^{\infty} C_n \cos \frac{n\pi}{l} x \qquad C_0 = \frac{1}{l}\int_0^l x\mathrm{d}x = \frac{l}{2}$$

$$n \neq 0$$

$$C_n = \frac{2}{l}\int_0^l x \cos \frac{n\pi}{l} x\mathrm{d}x = \frac{2}{l}\frac{l}{n\pi}\int_0^l x\mathrm{d}\sin \frac{n\pi}{l} x = \frac{2}{n\pi} x \sin \frac{n\pi}{l} x \Big|_0^l - \frac{2}{n\pi}\int_0^l \sin \frac{n\pi}{l} x\mathrm{d}x$$

$$= \frac{2l}{n^2\pi^2} \cos \frac{n\pi}{l} x \Big|_0^l = \frac{2l}{n^2\pi^2}\left[(-1)^n - 1\right] = \begin{cases} 0, & n \text{ is } an\ even \\ -\dfrac{4l}{n^2\pi^2}, & n\ is\ an\ odd \end{cases}$$

$$u = \frac{l}{2} - \sum_{n=1}^{\infty} \frac{4l}{(2n-1)^2\pi^2} e^{-\frac{(2n-1)^2\pi^2 a^2}{l^2} t} \cos \frac{(2n-1)\pi}{l} x$$

$$\begin{cases} \Delta u = 0, & 0 < \theta < \alpha, r < a \\ u(r,0) = u(r,\alpha) = 0, & r \le a \\ u(a,\theta) = f(\theta), & 0 \le \theta \le \alpha \end{cases}$$

$$|u(0,\theta)| < +\infty$$

$$u(r,\theta) = \mathrm{P}(r)\Theta(\theta)$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\mathrm{P}'\Theta\right) + \frac{1}{r^2}\mathrm{P}\Theta'' = 0$$

$$\mathrm{P}''\Theta + \frac{1}{r}\mathrm{P}'\Theta + \frac{1}{r^2}\mathrm{P}\Theta'' = 0$$

$$\frac{r^2\mathrm{P}'' + r\mathrm{P}'}{\mathrm{P}} = -\frac{\Theta''}{\Theta} = \lambda$$

$$r^2\mathrm{P}'' + r\mathrm{P}' - \lambda\mathrm{P} = 0$$

$$\Theta'' + \lambda\Theta = 0$$

$$\Theta(0)\mathrm{P}(r) = \Theta(\alpha)\mathrm{P}(r) = 0$$

$$\Theta(0) = \Theta(\alpha) = 0$$

$$\begin{cases} \Theta'' + \lambda\Theta = 0, & 0 \le \theta \le \alpha \\ \Theta(0) = \Theta(\alpha) = 0 \end{cases}$$

$$\lambda_n = \left(\frac{n\pi}{\alpha}\right)^2, n = 1,2,3,\cdots$$

$$\Theta_n = B_n \sin\frac{n\pi}{\alpha}\theta$$

$$\lambda_n = \left(\frac{n\pi}{\alpha}\right)^2, n = 1,2,3,\cdots \quad \Theta_n = B_n \sin\frac{n\pi}{\alpha}\theta \qquad r^2\mathrm{P}'' + r\mathrm{P}' - \lambda\mathrm{P} = 0$$

$$r^2\mathrm{P}_n'' + r\mathrm{P}_n' - \left(\frac{n\pi}{\alpha}\right)^2\mathrm{P} = 0 \qquad \mathrm{P}_n = C_n r^{\frac{n\pi}{\alpha}} + D_n r^{-\frac{n\pi}{\alpha}} = C_n r^{\frac{n\pi}{\alpha}}$$

$$u_n = \Theta_n \mathrm{P}_n = B_n \sin\frac{n\pi}{\alpha}\theta \cdot C_n r^{\frac{n\pi}{\alpha}} = E_n \sin\frac{n\pi}{\alpha}\theta r^{\frac{n\pi}{\alpha}}$$

$$u = \sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} E_n \sin\frac{n\pi}{\alpha}\theta r^{\frac{n\pi}{\alpha}} \qquad u(a,\theta) = f(\theta) = \sum_{n=1}^{\infty} E_n \sin\frac{n\pi}{\alpha}\theta \cdot a^{\frac{n\pi}{\alpha}}$$

$$E_n = \frac{\int_0^{\alpha} f(\theta)\sin\frac{n\pi}{\alpha}\theta \mathrm{d}\theta}{a^{n\pi/\alpha}\int_0^{\alpha}\sin^2\frac{n\pi}{\alpha}\theta \mathrm{d}\theta} = \frac{\int_0^{\alpha} f(\theta)\sin\frac{n\pi}{\alpha}\theta \mathrm{d}\theta}{a^{n\pi/\alpha}\frac{\alpha}{2}}$$

$$u = \frac{2}{\alpha}\sum_{n=1}^{\infty}\int_0^{\alpha} f(\theta)\sin\frac{n\pi}{\alpha}\theta \mathrm{d}\theta \left(\frac{r}{a}\right)^{n\pi/\alpha} \sin\frac{n\pi}{\alpha}\theta$$

$$\begin{cases} u_{xy} = 1, & x > 0, y > 0 \\ u(0, y) = y + 1, & y \geq 0 \\ u(x, 0) = 1, & x \geq 0 \end{cases}$$

Method 1. $L_y[u(x, y)] = U(x, p)$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[pU(x, p) - 1\right] = \frac{1}{p} \qquad p\frac{\mathrm{d}}{\mathrm{d}x}U(x, p) = \frac{1}{p} \qquad U(x, p) = \frac{x}{p^2} + C$$

$$U(0, p) = L_y[y + 1] = \frac{1}{p^2} + \frac{1}{p} \qquad U(x, p) = \frac{x}{p^2} + \frac{1}{p^2} + \frac{1}{p}$$

$$u(x, y) = xy + y + 1$$

$$\begin{cases} u_{xy} = 1, & x > 0, y > 0 \\ u(0, y) = y + 1, & y \geq 0 \\ u(x, 0) = 1, & x \geq 0 \end{cases}$$

Method 2.

$$\frac{\partial u}{\partial x} = y + f(x) \qquad u = xy + f_1(x) + f_2(y)$$

$$u(0, y) = f_1(0) + f_2(y) = y + 1 \qquad u(x, 0) = f_1(x) + f_2(0) = 1$$

$$f_2(y) = y + 1 - f_1(0) \qquad f_1(x) = 1 - f_2(0)$$

$$u = xy + 1 - f_2(0) + y + 1 - f_1(0) = xy + y + 2 - f_2(0) - f_1(0)$$

$$u(0,0) = 2 - f_2(0) - f_1(0) = 1$$

$$u = xy + y + 1$$

$$\begin{cases} u_{tt} = 4u_{xx}, & -\infty < x < +\infty, t > 0 \\ u\big|_{t=0} = x^2, & u_t\big|_{t=0} = \cos x \end{cases}$$

Method 1. $u(x,t) = \dfrac{1}{2}[\phi(x+at)+\phi(x-at)] + \dfrac{1}{2a}\displaystyle\int_{x-at}^{x+at} \psi(\xi)d\xi$

$$u(x,t) = \frac{1}{2}[(x+2t)^2 + (x-2t)^2] + \frac{1}{4}\int_{x-2t}^{x+2t} \cos\xi d\xi$$

$$= x^2 + 4t^2 - \frac{1}{4}[\sin(x+2t) - \sin(x-2t)]$$

$$= x^2 + 4t^2 + \frac{1}{2}\cos x \sin 2t$$

Method 2.

$$\begin{cases} \dfrac{\mathrm{d}^2U(\omega,t)}{\mathrm{d}t^2} = -4\omega^2 U(\omega,t), & t>0 \\ U(\omega,0) = F(x^2), \dfrac{\mathrm{d}U(\omega,0)}{\mathrm{d}t} = F(\cos x), \end{cases}$$

$$f(x) \leftrightarrow F(\omega)$$
$$f'(x) \leftrightarrow \mathrm{i}\omega F(\omega)$$
$$f''(x) \leftrightarrow -\omega^2 F(\omega)$$
$$f(x-\xi) \leftrightarrow F(\omega)e^{-\mathrm{i}\omega\xi}$$
$$\int_{-\infty}^{x} f(\xi)\mathrm{d}\xi \leftrightarrow \frac{F(\omega)}{\mathrm{i}\omega}$$

$$U(\omega,t) = A\cos 2\omega t + B\sin 2\omega t$$

$$U(\omega,0) = A = F(x^2) \quad B = F(\cos x)/2\omega$$

$$U(\omega,t) = F(x^2)\cos 2\omega t + \frac{F(\cos x)}{2\omega}\sin 2\omega t$$

$$= F(x^2)\frac{e^{\mathrm{i}2\omega t}+e^{-\mathrm{i}2\omega t}}{2} + \frac{F(\cos x)}{2\omega}\frac{e^{\mathrm{i}2\omega t}-e^{-\mathrm{i}2\omega t}}{2\mathrm{i}}$$

$$= \frac{1}{2}\left[F(x^2)e^{\mathrm{i}2\omega t} + F(x^2)e^{-\mathrm{i}2\omega t}\right]$$

$$+\frac{1}{4}\left[\frac{F(\cos x)}{\mathrm{i}\omega}e^{\mathrm{i}2\omega t} - \frac{F(\cos x)}{\mathrm{i}\omega}e^{-\mathrm{i}2\omega t}\right]$$

$$u(x,t) = \frac{1}{2}[(x+2t)^2 + (x-2t)^2]$$

$$+\frac{1}{4}[\int_{-\infty}^{x+2t}\cos\xi\mathrm{d}\xi - \int_{-\infty}^{x-2t}\cos\xi)\mathrm{d}\xi]$$

$$= x^2 + 4t^2 + \frac{1}{2}\cos x\sin 2t$$

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = a^2 \dfrac{\partial^2 u}{\partial x^2} \\ u(0,t) = u(\pi,t) = 0 \\ u(x,0) = 2x(\pi - x), \quad \dfrac{\partial u}{\partial t}\Big|_{t=0} = 3\sin 2x \end{cases}$$

$$u(x,t) = X(x)T(t)$$

$$T''X = a^2 TX''$$

$$\begin{cases} T'' + \lambda a^2 T = 0 \\ X'' + \lambda X = 0 \\ X(0) = X(\pi) = 0 \end{cases}$$

$$\lambda_n = \left(\frac{n\pi}{\pi}\right)^2 = n^2, \ X_n = \sin nx$$

$$u_n = (C_n \cos ant + D_n \sin ant)\sin nx \qquad u(x,t) = \sum_{n=1}^{\infty}(C_n \cos ant + D_n \sin ant)\sin nx$$

$$C_n = \frac{2}{n\pi}\int_0^{\pi} x(\pi - x)\sin nx dx = \frac{8[1-(-1)^n]}{n^3\pi} \quad D_n = \frac{2}{an\pi}\int_0^{\pi} 3\sin 2x \sin nx dx = \begin{cases} 0 & (n \neq 2) \\ \dfrac{3}{a} & (n = 2) \end{cases}$$

$$u(x,t) = \frac{3}{a}\sin 2at \sin 2x + \sum_{n=1}^{\infty}\frac{8[1-(-1)^n]}{n^3\pi}\cos ant \sin nx$$

1

$$\begin{cases} u_{tt} = 9u_{xx}, \quad -\infty < x < +\infty, t > 0 \\ u\big|_{t=0} = x^3 + 2, \quad u_t\big|_{t=0} = \sin x \end{cases}$$

2

$$\begin{cases} u_{xy}(x, y) = xy, x > 0, y > 0 \\ u(x, 0) = x, u(0, y) = \cos y - 1 \end{cases}$$

3

$$\begin{cases} u_{xx} + u_{yy} = 0, \quad y > 0 \\ u\,|_{y=0} = f(x). \end{cases}$$

$$\begin{cases} u_{tt} = 9u_{xx}, \quad -\infty < x < +\infty, t > 0 \\ u\big|_{t=0} = x^3 + 2, \quad u_t\big|_{t=0} = \sin x \end{cases}$$

$$a = 3, \quad \phi(x) = x^3 + 2, \quad \psi(x) = \sin x$$

$$u(x,t) = \frac{1}{2}[\phi(x+at) + \phi(x-at)] + \frac{1}{2a}\int_{x-at}^{x+at} \psi(\xi)d\xi$$

$$u(x,t) = \frac{1}{2}[(x+3t)^3 + 2 + (x-3t)^3 + 2] + \frac{1}{4}\int_{x-3t}^{x+3t} \sin\xi d\xi$$

$$= x^3 + 27xt^2 + 2 - \frac{1}{4}[\cos(x+3t) - \cos(x-3t)]$$

$$= x^2 + 4t^2 + \frac{1}{2}\sin x \sin 3t$$

$$\begin{cases} u_{xy}(x,y) = xy, x > 0, y > 0 \\ u(x,0) = x, u(0,y) = \cos y - 1 \end{cases}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}[pU(x,p) - x] = \frac{x}{p^2} \qquad \frac{\mathrm{d}}{\mathrm{d}x}U(x,p) = \frac{x}{p^3} + \frac{1}{p}$$

$$U(x,p) = \frac{x^2}{2p^3} + \frac{x}{p} + C$$

$$U(0,p) = \frac{p}{p^2+1} - \frac{1}{p}$$

$$U(x,p) = \frac{x^2}{2p^3} + \frac{x}{p} + \frac{p}{p^2+1} - \frac{1}{p}$$

$$u(x,y) = \frac{1}{4}x^2y^2 + x + \cos y - 1$$

$$\begin{cases} u_{xx} + u_{yy} = 0, & y > 0 \\ u\big|_{y=0} = f(x). \end{cases}$$

$$G(M, M_0) = \frac{1}{2\pi}\left(\ln\frac{1}{r_{MM_1}} - \ln\frac{1}{r_{MM_0}}\right)$$

$$= \frac{1}{2\pi}\left\{\left[\ln(x-x_0)^2 + (y-y_0)^2\right] - \ln\left[(x-x_0)^2 + (y+y_0)^2\right]\right\}$$

$$\frac{\partial G}{\partial n}\Big|_{y=0} = -\frac{\partial G}{\partial y}\Big|_{y=0}$$

$$= -\frac{1}{2\pi}\left[\frac{(y-y_0)}{(x-x_0)^2 + (y-y_0)^2} - \frac{(y+y_0)}{(x-x_0)^2 + (y+y_0)^2}\right]\Bigg|_{y=0}$$

$$= \frac{y_0}{\pi} \cdot \frac{1}{(x-x_0)^2 + y_0^2}$$

$$u(x_0, y_0) = \int_\Gamma u \frac{\partial G}{\partial n} dl = \frac{y_0}{\pi}\int_{-\infty}^{+\infty} f(x) \frac{1}{(x-x_0)^2 + y_0^2} dx$$

$$\begin{cases} u_{tt} = a^2 u_{xx} + f(x,t), & 0 < x < l, t > 0 \\ u(0,t) = u_1(t), u(l,t) = u_2(t), & t > 0 \\ u(x,0) = \varphi(x), u_t(x,0) = \psi(x), & 0 \le x \le l \end{cases}$$

$$\begin{cases} u_{tt} = a^2 u_{xx} + f(x,t), & 0 < x < l, t > 0, \\ u(0,t) = u(l,t) = 0, & t \ge 0, \\ u(x,0) = \varphi(x), u_t(x,0) = \psi(x), & 0 \le x \le l. \end{cases}$$

$$\begin{cases} u_t = a^2 (u_{xx} + u_{yy}), & 0 < x < p, 0 < y < q, t > 0 \\ u(0,y,t) = u(p,y,t) = 0, & 0 \le y \le q, t > 0 \\ u(x,0,t) = u(x,q,t) = 0, & 0 \le x \le p, t > 0 \\ u(x,y,0) = \varphi(x,y), & 0 < x < p, 0 < y < q \end{cases}$$

Target problem

- Solve

$$-\frac{d}{dx}\left(c(x)\frac{du(x)}{dx}\right) = f(x), \ a < x < b,$$
$$u(a) = g_a, u(b) = g_b$$

  for $u(x)$.

- Why do we start from this problem?
- An easy look at the basic idea of the finite element method.
- Numerical methods for partial differential equations: finite element method, finite difference method, finite volume method, boundary element method, etc., which use different techniques to discretize partial differential equations.

Weak formulation

- First, multiply a function $v(x)$ on both sides of the original equation,

$$
\begin{aligned}
& -\frac{d}{dx}\left(c(x)\frac{du(x)}{dx}\right) = f(x), \;\; a < x < b \\
\Rightarrow \quad & -\frac{d}{dx}\left(c(x)\frac{du(x)}{dx}\right) v(x) = f(x)v(x), \;\; a < x < b \\
\Rightarrow \quad & -\int_a^b \frac{d}{dx}\left(c(x)\frac{du(x)}{dx}\right) v(x)\, dx = \int_a^b f(x)v(x)\, dx.
\end{aligned}
$$

- $u(x)$ is called a trial function and $v(x)$ is called a test function.

Weak formulation

- Second, using integration by parts, we obtain

$$
\begin{aligned}
& \int_a^b \frac{d}{dx}\left(c(x)\frac{du(x)}{dx}\right) v(x)\ dx \\
= & \int_a^b (cu')'\, v\ dx \\
= & \int_a^b v\ d(cu') \\
= & \ cu'v|_a^b - \int_a^b cu'\ dv \\
= & \ c(b)u'(b)v(b) - c(a)u'(a)v(a) - \int_a^b cu'v'\ dx.
\end{aligned}
$$

Weak formulation

- Then

$$-c(b)u'(b)v(b) + c(a)u'(a)v(a) + \int_a^b cu'v' \, dx = \int_a^b fv \, dx.$$

- Since the solution at $x = a$ and $x = b$ are given by $u(a) = g_a, u(b) = g_b$, then we can choose the test function $v(x)$ such that $v(a) = v(b) = 0$.
- Hence

$$\int_a^b cu'v' \, dx = \int_a^b fv \, dx.$$

- What spaces should u and v belong to? Sobolev spaces!

1D Sobolev spaces

Definition

$C_0^\infty(I)$ is the set of all functions that are infinitely differentiable on I and compactly supported in I.

- Recall integration by parts:

$$\begin{aligned}\int_a^b u'v\ dx &= uv|_a^b - \int_a^b uv'\ dv \\ &= u(b)v(b) - u(a)v(a) - \int_a^b uv'\ dx.\end{aligned}$$

- For $v \in C_0^\infty(I)$, we have $v(a) = v(b) = 0$. Then

$$\int_a^b u'v\ dx = -\int_a^b uv'\ dx.$$

1D Sobolev spaces

Definition (L^2 space)

$$L^2(I) = \{v : I \to \mathbf{R} : \int_a^b v^2 \, dx < \infty\}$$

where $I = (a, b)$.

Definition (H^1 space)

$$H^1(I) = \{v \in L^2(I) : v' \in L^2(I)\}$$

where $I = (a, b)$.

Definition (H_0^1 space)

$$H_0^1(I) = \{v \in H^1(I) : v(a) = v(b) = 0\}$$

where $I = (a, b)$.

Weak formulation

- Weak formulation: find $u \in H^1(I)$ such that

$$\int_a^b cu'v' \, dx = \int_a^b fv \, dx.$$

for any $v \in H_0^1(I)$ where $I = (a, b)$.

- Let $a(u, v) = \int_a^b cu'v' \, dx$ and $(f, v) = \int_a^b fv \, dx$.

- Weak formulation: find $u \in H^1(I)$ such that

$$a(u, v) = (f, v)$$

for any $v \in H_0^1(I)$ where $I = (a, b)$.

Galerkin formulation

- Assume there is a finite dimensional subspace $U_h \subset H^1[a, b]$. Define U_{h0} to be the space which consists of the functions of U_h with value 0 on the Dirichlet boundary.
- Galerkin formulation: find $u_h \in U_h$ such that

$$a(u_h, v_h) = (f, v_h)$$
$$\Leftrightarrow \quad \int_a^b c u_h' v_h' \, dx = \int_a^b f v_h \, dx$$

  for any $v_h \in U_{h0}$.
- Basic idea of Galerkin formulation: use finite dimensional space to approximate infinite dimensional space.
- Question: How to obtain U_h?

Mesh

- Assume that we have a uniform partition of $[a, b]$ into N elements with mesh size $h = \frac{b-a}{N}$.
- Let $x_i = a + (i-1)h$ $(i = 1, \cdots, N+1)$ denote the mesh nodes.
- Let $E_n = [x_n, x_{n+1}]$ $(n = 1, \cdots, N)$ denote the mesh elements.

1D linear finite element space

- Define 1D linear finite element space:

$$\begin{aligned} U_h &= \{\phi \in C[a,b] : \ \phi(x) \text{ is linear on each } [x_n, x_{n+1}] \\ &\qquad (n = 1, 2, \cdots, N)\}. \end{aligned}$$

- U_h is actually a piecewise linear function space based on the mesh generated in the previous section.

1D linear finite element space

Theorem (I)

U_h is an $(N+1)-$dimensional subspace of $C[a,b]$.

Proof:

- First, it is easy to verify that U_h is a subspace of $C[a,b]$.
- If we can find a continuous piecewise linear basis of $N+1$ functions for U_h, then the proof is completed.
- Consider $\phi_j(x) \in U_h$ such that

$$\phi_j(x_i) = \delta_{ij} = \begin{cases} 0, & \text{if } j \neq i, \\ 1, & \text{if } j = i. \end{cases}$$

for $i, j = 1, \cdots, N+1$.

1D linear finite element space

Continued proof:

- In fact,

$$\phi_1(x) = \begin{cases} \frac{x_2-x}{h}, & \text{if } x_1 \le x \le x_2, \\ 0, & \text{otherwise}, \end{cases}$$

$$\phi_j(x) = \begin{cases} \frac{x-x_{j-1}}{h}, & \text{if } x_{j-1} \le x \le x_j, \\ \frac{x_{j+1}-x}{h}, & \text{if } x_j \le x \le x_{j+1}, \\ 0, & \text{otherwise}, \end{cases}$$

$$(j = 2, \cdots, N)$$

$$\phi_{N+1}(x) = \begin{cases} \frac{x-x_N}{h}, & \text{if } x_N \le x \le x_{N+1}, \\ 0, & \text{otherwise}, \end{cases}$$

- In order to show that $\phi_j(x)$ $(i = 1, \cdots, N+1)$ form a basis of U_h, we need to show the linear independence of $\{\phi_j\}_{j=1}^{N+1}$ and $U_h = span\{\phi_j\}_{j=1}^{N+1}$.

1D linear finite element space

Continued proof:

- Linear independence: consider

$$\sum_{j=1}^{N+1} c_j \phi_j(x) = 0$$

  for any $x \in [a, b]$.

- Let $x = x_i$ $(i = 1, \cdots, N+1)$, then

$$\phi_j(x_i) = \delta_{ij} = \begin{cases} 0, & \text{if } j \neq i, \\ 1, & \text{if } j = i. \end{cases}$$

$$\Rightarrow \quad c_i = 0 \ (i = 1, \cdots, N+1)$$

- So $\phi_j(x)$ $(j = 1, \cdots, N+1)$ are linearly independent.

1D linear finite element space

Continued proof:

- Span: Given any $f \in U_h$, let $c_j = f(x_j)$ and consider

$$g(x) = \sum_{j=1}^{N+1} c_j \phi_j(x).$$

- First, $g(x_i) = c_i = f(x_i)$ $(i = 1, \cdots, N+1)$.
- Second, both $f(x)$ and $g(x)$ are linear in each piece $[x_i, x_{i+1}]$ $(j = 1, \cdots, N)$.
- Hence $f(x) = g(x)$ in each piece $[x_i, x_{i+1}]$ $(i = 1, \cdots, N)$.
- Then $f(x) = g(x) = \sum_{j=1}^{N+1} c_j \phi_j(x)$.
- This implies $U_h = span\{\phi_j\}_{j=1}^{N+1}$.
- Therefore $\phi_j(x)$ $(j = 1, \cdots, N+1)$ form a basis of U_h.

Discretization formulation

- Recall the Galerkin formulation: find $u_h \in U_h$ such that

$$a(u_h, v_h) = (f, v_h)$$
$$\Leftrightarrow \quad \int_a^b c u_h' v_h' \, dx = \int_a^b f v_h \, dx$$

for any $v_h \in U_{h0}$.

- For an easier implementation, we use the following Galerkin formulation (without considering the Dirichlet boundary condition, which will be handled later): find $u_h \in U_h$ such that

$$a(u_h, v_h) = (f, v_h)$$
$$\Leftrightarrow \quad \int_a^b c u_h' v_h' \, dx = \int_a^b f v_h \, dx$$

  for any $v_h \in U_h$.

- Since $u_h \in U_h = span\{\phi_j\}_{j=1}^{N+1}$, then

$$u_h = \sum_{j=1}^{N+1} u_j \phi_j$$

  for some coefficients u_j $(j = 1, \cdots, N+1)$.

Discretization formulation

- If we can set up a linear algebraic system for u_j $(j = 1, \cdots, N+1)$ and solve it, then we can obtain the finite element solution u_h.
- Therefore, we choose the test function $v_h = \phi_i$ $(i = 1, \cdots, N+1)$. Then the finite element formulation gives

$$\int_a^b c \left(\sum_{j=1}^{N+1} u_j \phi_j \right)' \phi_i' \, dx = \int_a^b f \phi_i \, dx, \ i = 1, \cdots, N+1$$

$$\Rightarrow \sum_{j=1}^{N+1} u_j \left[\int_a^b c \phi_j' \phi_i' \, dx \right] = \int_a^b f \phi_i \, dx, \ i = 1, \cdots, N+1.$$

Discretization formulation

- Define the stiffness matrix

$$A = [a_{ij}]_{i,j=1}^{N+1} = \left[\int_a^b c\phi_j'\phi_i' \, dx \right]_{i,j=1}^{N+1}.$$

- Define the load vector

$$\vec{b} = [b_i]_{i=1}^{N+1} = \left[\int_a^b f\phi_i \, dx \right]_{i=1}^{N+1}.$$

- Define the unknown vector

$$\vec{X} = [u_j]_{j=1}^{N+1}.$$

- Then we obtain the linear algebraic system

$$A\vec{X} = \vec{b}.$$

- Here A is symmetric positive-definite if the original elliptic equation is symmetric positive-definite.

Assembly of the stiffness matrix

- In this section we will first introduce the matrix and vector assembly by using a special method. In the later section "FE method", we will discuss a different universal framework.
- From the definition of ϕ_j $(j = 1, \cdots, N+1)$, we can see that ϕ_j are non-zero only on the elements adjacent to the node x_j, but 0 on all the other elements.
- This observation motivates us to think about

$$a_{ij} = \int_a^b c\phi_j'\phi_i' \, dx = \sum_{n=1}^{N} \int_{x_n}^{x_{n+1}} c\phi_j'\phi_i' \, dx, \;\; i, j = 1, \cdots, N+1.$$

- It is easy to see that most of $\int_{x_n}^{x_{n+1}} c\phi_j'\phi_i' \, dx$ will be 0.
- So we only need to use numerical integration to compute those nonzero integrals.

Assembly of the stiffness matrix

- Case 1: when $|i-j|>1$, x_i and x_j are not neighboring mesh nodes.
- Then on any element $[x_n, x_{n+1}]$ $(n=1,\cdots,N)$, at least one of ϕ_j and ϕ_i is 0.
- Hence

$$\int_{x_n}^{x_{n+1}} c\phi_j'\phi_i' \, dx = 0 \ (n=1,\cdots,N)$$

$$\Rightarrow \quad a_{ij} = \sum_{n=1}^{N} \int_{x_n}^{x_{n+1}} c\phi_j'\phi_i' \, dx = 0.$$

Assembly of the stiffness matrix

- Case 2: when $i = j + 1$ ($j = 1, \cdots, N$), the only element, on which both ϕ_j and ϕ_i are not zero, is $[x_j, x_{j+1}]$.
- Hence

$$\int_{x_n}^{x_{n+1}} c\phi_j'\phi_i' \, dx = 0 \ (n = 1, \cdots, j-1, j+1, \cdots, N)$$

$$\Rightarrow \quad a_{ij} = \sum_{n=1}^{N} \int_{x_n}^{x_{n+1}} c\phi_j'\phi_i' \, dx = \int_{x_j}^{x_{j+1}} c\phi_j'\phi_i' \, dx$$

$$\Rightarrow \quad a_{j+1,j} = \int_{x_j}^{x_{j+1}} c(x) \left(\frac{x_{j+1} - x}{h} \right)' \left(\frac{x - x_j}{h} \right)' \, dx$$

$$= -\frac{1}{h^2} \int_{x_j}^{x_{j+1}} c(x) \, dx.$$

Assembly of the stiffness matrix

- Case 3: when $i = j - 1$ ($j = 2, \cdots, N+1$), the only element, on which both ϕ_j and ϕ_i are not zero, is $[x_{j-1}, x_j]$.
- Hence

$$\int_{x_n}^{x_{n+1}} c\phi_j'\phi_i' \, dx = 0 \ (n = 1, \cdots, i-1, i+1, \cdots, N)$$

$$\Rightarrow \quad a_{ij} = \sum_{n=1}^{N} \int_{x_n}^{x_{n+1}} c\phi_j'\phi_i' \, dx = \int_{x_{j-1}}^{x_j} c\phi_j'\phi_i' \, dx$$

$$\Rightarrow \quad a_{j-1,j} = \int_{x_{j-1}}^{x_j} c(x) \left(\frac{x - x_{j-1}}{h}\right)' \left(\frac{x_j - x}{h}\right)' \, dx$$

$$= -\frac{1}{h^2} \int_{x_{j-1}}^{x_j} c(x) \, dx.$$

Assembly of the stiffness matrix

- Case 4: when $i = j$ ($j = 2, \cdots, N$), the only two elements, on which both ϕ_j and ϕ_i are not zero, are $[x_{j-1}, x_j]$ and $[x_j, x_{j+1}]$.
- Hence

$$\int_{x_n}^{x_{n+1}} c\phi_j'\phi_i' \, dx = 0 \ (n = 1, \cdots, j-2, j+1, \cdots, N)$$

$$\Rightarrow \quad a_{ij} = \sum_{n=1}^{N} \int_{x_n}^{x_{n+1}} c\phi_j'\phi_i' \, dx = \int_{x_{j-1}}^{x_j} c\phi_j'\phi_i' \, dx + \int_{x_j}^{x_{j+1}} c\phi_j'\phi_i' \, dx$$

$$\Rightarrow \quad a_{jj} = \int_{x_{j-1}}^{x_j} c(x) \left(\frac{x - x_{j-1}}{h}\right)' \left(\frac{x - x_{j-1}}{h}\right)' \, dx$$

$$+ \int_{x_j}^{x_{j+1}} c(x) \left(\frac{x_{j+1} - x}{h}\right)' \left(\frac{x_{j+1} - x}{h}\right)' \, dx$$

$$= \frac{1}{h^2} \int_{x_{j-1}}^{x_j} c(x) \, dx + \frac{1}{h^2} \int_{x_j}^{x_{j+1}} c(x) \, dx.$$

Assembly of the stiffness matrix

- Case 5: when $i = j = 1$, the only element, on which both ϕ_j and ϕ_i are not zero, is $[x_1, x_2]$.
- Hence

$$\int_{x_n}^{x_{n+1}} c\phi_1'\phi_1' \, dx = 0 \; (n = 2, \cdots, N)$$

$$\Rightarrow \quad a_{11} = \sum_{n=1}^{N} \int_{x_n}^{x_{n+1}} c\phi_1'\phi_1' \, dx = \int_{x_1}^{x_2} c\phi_1'\phi_1' \, dx$$

$$\Rightarrow \quad a_{11} = \int_{x_1}^{x_2} c(x) \left(\frac{x_2 - x}{h}\right)' \left(\frac{x_2 - x}{h}\right)' \, dx$$

$$= \frac{1}{h^2} \int_{x_1}^{x_2} c(x) \, dx.$$

Assembly of the stiffness matrix

- Case 6: when $i = j = N + 1$, the only element, on which both ϕ_j and ϕ_i are not zero, is $[x_N, x_{N+1}]$.
- Hence

$$\int_{x_n}^{x_{n+1}} c\phi'_{N+1}\phi'_{N+1}\, dx = 0 \ (n = 1, \cdots, N-1)$$

$$\Rightarrow \quad a_{N+1,N+1} = \sum_{n=1}^{N} \int_{x_n}^{x_{n+1}} c\phi'_{N+1}\phi'_{N+1}\, dx = \int_{x_N}^{x_{N+1}} c\phi'_{N+1}\phi'_{N+1}\, dx$$

$$\Rightarrow \quad a_{N+1,N+1} = \int_{x_N}^{x_{N+1}} c(x) \left(\frac{x - x_N}{h}\right)' \left(\frac{x - x_N}{h}\right)' dx$$

$$= \frac{1}{h^2} \int_{x_N}^{x_{N+1}} c(x)\, dx.$$

Assembly of the stiffness matrix

- From the above discussion, we can see that most of the elements a_{ij} $(i, j = 1, \cdots, N+1)$ are 0.
- Hence the stiffness matrix A is called a sparse matrix.
- We can also see that we only need to compute the integrals on local elements instead of the whole domain, which later will lead to the "local assembly" idea of finite elements.

Assembly of the stiffness matrix

Algorithm I:

- Initialize the matrix: $A = sparse(N+1, N+1)$;
- Compute the integrals and assemble them into A:

$FOR\ j = 1, \cdots, N+1$:

$\quad IF\ j \leq N,\ THEN$

$\quad\quad$ Compute $A(j+1, j) = -\frac{1}{h^2} \int_{x_j}^{x_{j+1}} c(x)\ dx$;

$\quad END$

$\quad IF\ j \geq 2,\ THEN$

$\quad\quad$ Compute $A(j-1, j) = -\frac{1}{h^2} \int_{x_{j-1}}^{x_j} c(x)\ dx$;

$\quad END$

$\quad IF\ 2 \leq j \leq N,\ THEN$

$\quad\quad$ Compute $A(j, j) = \frac{1}{h^2} \int_{x_{j-1}}^{x_j} c(x)\ dx + \frac{1}{h^2} \int_{x_j}^{x_{j+1}} c(x)\ dx$;

$\quad END$

END

Compute $A(1, 1) = \frac{1}{h^2} \int_{x_1}^{x_2} c(x)\ dx$;

Compute $A(N+1, N+1) = \frac{1}{h^2} \int_{x_N}^{x_{N+1}} c(x)\ dx$;

Assembly of the load vector

- The idea for the assembly of the load vector is similar. We have

$$b_i = \int_a^b f\phi_i \, dx \quad = \quad \sum_{n=1}^{N} \int_{x_n}^{x_{n+1}} f\phi_i \, dx, \;\; i = 1, \cdots, N+1,$$

- Case 1: when $2 \leq i \leq N$, the only two elements, on which ϕ_i is not zero, are $[x_{i-1}, x_i]$ and $[x_i, x_{i+1}]$. Then

$$\int_{x_n}^{x_{n+1}} f\phi_i \, dx = 0 \; (n = 1, \cdots, i-2, i+1, \cdots, N)$$

$$\Rightarrow \quad b_i = \sum_{n=1}^{N} \int_{x_n}^{x_{n+1}} f\phi_i \, dx = \int_{x_{i-1}}^{x_i} f\phi_i \, dx + \int_{x_i}^{x_{i+1}} f\phi_i \, dx$$

$$= \int_{x_{i-1}}^{x_i} f(x) \frac{x - x_{i-1}}{h} \, dx + \int_{x_i}^{x_{i+1}} f(x) \frac{x_{i+1} - x}{h} \, dx.$$

Assembly of the load vector

- Case 2: when $i = 1$, the only element, on which ϕ_1 is not zero, is $[x_1, x_2]$. Then

$$\int_{x_n}^{x_{n+1}} f\phi_1 \, dx = 0 \; (n = 2, \cdots, N)$$

$$\Rightarrow \quad b_1 = \sum_{n=1}^{N} \int_{x_n}^{x_{n+1}} f\phi_1 \, dx$$

$$= \int_{x_1}^{x_2} f\phi_1 \, dx = \int_{x_1}^{x_2} f(x) \frac{x_2 - x}{h} \, dx.$$

Assembly of the load vector

- Case 3: when $i = N + 1$, the only element, on which ϕ_{N+1} is not zero, is $[x_N, x_{N+1}]$. Then

$$\int_{x_n}^{x_{n+1}} f\phi_{N+1}\, dx = 0 \ (n = 1, \cdots, N-1)$$

$$\Rightarrow \quad b_{N+1} = \sum_{n=1}^{N} \int_{x_n}^{x_{n+1}} f\phi_{N+1}\, dx = \int_{x_N}^{x_{N+1}} f\phi_{N+1}\, dx$$

$$= \int_{x_N}^{x_{N+1}} f(x)\frac{x - x_N}{h}\, dx.$$

Assembly of the load vector

Algorithm II:

- Initialize the matrix: $b = zeros(N+1, 1)$;
- Compute the integrals and assemble them into $\vec{b}$:
  $FOR\ i = 2, \cdots, N$:
  Compute
  $b(i) = \int_{x_{i-1}}^{x_i} f(x)\frac{x-x_{i-1}}{h}\ dx + \int_{x_i}^{x_{i+1}} f(x)\frac{x_{i+1}-x}{h}\ dx$;
  END
  Compute $b(1) = \int_{x_1}^{x_2} f(x)\frac{x_2-x}{h}\ dx$;
  Compute $b(N+1) = \int_{x_N}^{x_{N+1}} f(x)\frac{x-x_N}{h}\ dx$;

Dirichlet boundary condition

- Basically, the Dirichlet boundary condition $u(a) = g_a, u(b) = g_b$ give the solutions at $x_1 = a$ and $x_{N+1} = b$.
- Since the coefficient u_j in the finite element solution $u_h = \sum_{j=1}^{N+1} u_j \phi_j$ is actually the numerical solution at the node x_j $(j = 1, \cdots, N+1)$, we actually know that $u_1 = u(a) = g_a$ and $u_{N+1} = u(b) = g_b$.
- Therefore, we don't really need the first and last equations in the linear system since they are set up for u_1 and u_{N+1} by using ϕ_1 and ϕ_{N+1}.

Dirichlet boundary condition

- One way to impose the Dirichlet boundary condition is to replace the first and last equations in the linear system by the following two equations

$$u_1 = g_a \Rightarrow 1 \cdot u_1 + 0 \cdot u_2 + \cdots + 0 \cdot u_{N+1} = g_a,$$
$$u_{N+1} = g_b \Rightarrow 0 \cdot u_1 + \cdots + 0 \cdot u_N + 1 \cdot u_{N+1} = g_b.$$

- That is, the first and last rows of the matrix A should become

$$(1, 0, \cdots, 0)$$

and

$$(0, \cdots, 0, 1)$$

respectively.

- And the first and last elements of the vector $\vec{b}$ should become g_a and g_b respectively.

Dirichlet boundary condition

Algorithm III:

- Deal with the Drichlet boundary conditions:

$$A(1,:) = 0;$$
$$A(1,1) = 1;$$
$$A(N+1,:) = 0;$$
$$A(N+1,N+1) = 1;$$
$$b(1) = g_a;$$
$$b(N+1) = g_b;$$

Basic algorithm

- Input a, b, and N. Compute $h = \frac{b-a}{N}$ and $x_j = a + (j-1)h$ $(j = 1, \cdots, N+1)$.
- Assemble the stiffness matrix A by using Algorithm I.
- Assemble the load vector $\vec{b}$ by using Algorithm II.
- Deal with the Drichlet boundary condition by using Algorithm III.
- Solve $A\vec{X} = \vec{b}$ for $\vec{X}$ by using a direct or iterative method.

Remark

The above algorithm uses the Algorithms I, II, and III, which are designed for some particular cases with a special method. It is not general enough to deal with different types of PDEs. Therefore, we will discuss a more universal framework in the following.

Numerical example

- Example 1: Use the 1D linear finite element method to solve the following equation:

$$
\begin{aligned}
&-\frac{d}{dx}\left(e^{x}\frac{du(x)}{dx}\right)\\
&= -e^{x}[cos(x) - 2sin(x) - x\ cos(x) - x\ sin(x)]\ (0 \le x \le 1),\\
&u(0) = 0, u(1) = cos(1).
\end{aligned}
$$

- The analytic solution of this problem is $u = x\ cos(x)$, which can be used to compute the error of the numerical solution.
- Let's code for the linear finite element method for 1D elliptic equation together!
- Open your Matlab!

Numerical example

h	maximum absolute error at all nodes
$1/4$	2.3340×10^{-3}
$1/8$	5.8317×10^{-4}
$1/16$	1.4645×10^{-4}
$1/32$	3.6675×10^{-5}
$1/64$	9.1700×10^{-6}
$1/128$	2.2929×10^{-6}

Table: The maximum numerical errors at all mesh nodes.

1D quadratic finite element space

- In fact,

$$\phi_j|_{E_n} = \begin{cases} \psi_{n1}, & \text{if } j = 2n-1, \\ \psi_{n2}, & \text{if } j = 2n+1, \\ \psi_{n3}, & \text{if } j = 2n, \\ 0, & \textit{otherwise}, \end{cases}$$

for $j = 1, \cdots, 2N+1$ and $n = 1, \cdots, N$.

General 1D finite element method

- Example 2: Use the 1D quadratic finite element method to solve the following equation:

$$
\begin{aligned}
&-\frac{d}{dx}\left(e^{x}\frac{du(x)}{dx}\right) \\
&= -e^{x}[cos(x) - 2sin(x) - x\ cos(x) - x\ sin(x)]\ (0 \le x \le 1), \\
&u(0) = 0, u(1) = cos(1).
\end{aligned}
$$

- The analytic solution of this problem is $u = x\ cos(x)$, which can be used to compute the error of the numerical solution.
- Let's code for the quadratic finite element method for 1D elliptic equation together!
- Open your Matlab!

General 1D finite element method

h	maximum absolute error at all nodes
$1/4$	4.6597×10^{-5}
$1/8$	2.9918×10^{-6}
$1/16$	1.8901×10^{-7}
$1/32$	1.1869×10^{-8}
$1/64$	7.4356×10^{-10}
$1/128$	4.6623×10^{-11}

Table: The maximum numerical errors at all mesh nodes.

1D cubic finite element space

- We consider the reference cubic basis functions on the reference interval $[\hat{A}_1, \hat{A}_2] = [0, 1]$.
- Define four reference quadratic basis functions

$$
\begin{aligned}
\hat{\psi}_1(\hat{x}) &= a_1\hat{x}^3 + b_1\hat{x}^2 + c_1\hat{x} + d_1, \\
\hat{\psi}_2(\hat{x}) &= a_2\hat{x}^3 + b_2\hat{x}^2 + c_2\hat{x} + d_2, \\
\hat{\psi}_3(\hat{x}) &= a_3\hat{x}^3 + b_3\hat{x}^2 + c_3\hat{x} + d_3, \\
\hat{\psi}_4(\hat{x}) &= a_4\hat{x}^3 + b_4\hat{x}^2 + c_4\hat{x} + d_4,
\end{aligned}
$$

such that

$$
\begin{aligned}
&\hat{\psi}_1(\hat{A}_1) = 1, \hat{\psi}_1(\hat{A}_2) = 0, \hat{\psi}_1'(\hat{A}_1) = 0, \hat{\psi}_1'(\hat{A}_2) = 0; \\
&\hat{\psi}_2(\hat{A}_1) = 0, \hat{\psi}_2(\hat{A}_2) = 1, \hat{\psi}_2'(\hat{A}_1) = 0, \hat{\psi}_2'(\hat{A}_2) = 0; \\
&\hat{\psi}_3(\hat{A}_1) = 0, \hat{\psi}_3(\hat{A}_2) = 0, \hat{\psi}_3'(\hat{A}_1) = 1, \hat{\psi}_3'(\hat{A}_2) = 0; \\
&\hat{\psi}_4(\hat{A}_1) = 0, \hat{\psi}_4(\hat{A}_2) = 0, \hat{\psi}_4'(\hat{A}_1) = 0, \hat{\psi}_4'(\hat{A}_2) = 1.
\end{aligned}
$$

1D cubic finite element space

- This is a Hermite type of finite elements since the definition of the basis functions involves with the derivatives. The linear and quadratic elements discussed before are Lagrange type of finite elements since the definition of the basis functions involves with the nodal values only.