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Reduced linear fractional representation of nonlinear systems for stability analysis ⋆

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Abstract: Based on symbolic and numeric manipulations, a model simplification technique is proposed in this paper for the linear fractional representation (LFR) and for the differential algebraic representation introduced by Trofino and Dezuo (2013). This representation is needed for computational Lyapunov stability analysis of uncertain rational nonlinear systems. The structure of the parameterized rational Lyapunov function is generated from the linear fractional representation (LFR) of the system model. The developed method is briefly compared to the n-D order reduction technique known from the literature. The proposed model transformations does not affect the structure of Lyapunov function candidate, preserves the well-posedness of the LFR and guarantees that the resulting uncertainty block is at most the same dimensional as the initial one. The applicability of the proposed method is illustrated on two examples.

Keywords: Linear fractional representation, computational methods, stability analysis, Lyapunov functions, model simplification

1. INTRODUCTION

Finding or at least approximating the domain of attraction (DOA) of nonlinear dynamical systems is an important task in model analysis and controller design/evaluation, and numerous works have been devoted to this issue, see, e.g. Vannelli and Vidyasagar (1985); Rozgonyi et al. (2010); Ohta et al. (1993); Giesl and Hafstein (2012). In the last two decades, the theory of linear matrix inequalities (LMI) and convex optimization alongside with nonlinear system modeling have made a considerable progress, which provide powerful and efficient tools to model and solve robust control, stability and filtering problems through LMI techniques. Ghaoui and Scorletti (1996) used quadratic Lyapunov functions (LF) and linear fractional transformation (LFT) to represent a rational nonlinear system and defined convex conditions for stability analysis and state feedback design. Trofino et al. (2013) considered uncertain rational LFs, moreover, affine annihilators and Finsler's lemma was used to formulate parameter dependent LMI conditions for the stability of uncertain rational nonlinear systems. The obtained LMIs were solved within a bounded polytopic subset of the state space.

Although the above mentioned optimization based techniques are advantageous for DOA computation, they are less attractive from a computational point of view. In general, they result in a high-dimensional optimization model, which is difficult to solve. In order to make these procedures numerically tractable, several dimension reduction techniques have been developed. Trofino et al. (2013) showed that omitting certain irrelevant nonlinear terms from the structure of the LF may result in a good estimate of the DOA with less computational effort. Polcz et al. (2017) used LFT and further automatic algebraic simplification steps to generate a reduced set of nonlinear terms considered in the LF. Both techniques operate on the differential-algebraic system representation needed for DOA computation and introduced by Trofino et al. (2013). At the same time, LFT provides several dimension reduction possibilities, e.g. Marcos et al. (2005); Hecker and Varga (2005, 2006); Hecker (2008) proposed symbolic preprocessing and matrix conversion techniques for low-order LFT modeling, such as Horner factorization, continued-fraction form, enhanced tree decomposition, Morton's method, enhanced variable splitting factorization, etc. These new symbolic manipulation techniques are implemented in the `sym2lfr` function of Enhanced LFR-toolbox for Matlab (Hecker et al. (2004)), henceforth is referred to as the *LFR-toolbox*.

After the LFT, the obtained linear fraction representation (LFR) can be considered as a generalized state-space model, on which further numerical order reduction techniques can be applied. Lambrechts et al. (1993) proposed a standard 1-D order reduction (1-DOR) for the LFR, which removes the unobservable/uncontrollable eigenvalues from each subsystem of the LFR corresponding to each

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uncertainty block. Based on a Kalman like decomposition, D'Andrea and Khatri (1997) proposed the n-D order reduction (n-DOR), which is proved to be less conservative than 1-DOR, since it considers all uncertainty blocks at the same time. The n-DOR is implemented in the LFR-toolbox function `minlfr`. According to Magni (2006), numerical order reduction techniques of the LFR-toolbox consider strong input/output equivalence of the LFR models. Based on this equivalence relation of LFRs introduced by Doyle et al. (1996), Magni (2006) defined the notion of *minimality* and *relative-minimality* of an LFR model. An LFR is said to be *minimal* if there is no equivalent representation with a smaller uncertainty block (Δ). Furthermore, an LFR is said to be *relative-minimal* if there is no similarity transformation, such that some states can be eliminated without modifying the input/output characteristics of the LFR. It is shown by D'Andrea and Khatri (1997) that the n-DOR leads to a *relative-minimal* representation.

In this paper, we consider uncertain rational nonlinear systems in the lower LFR (Lambrechts et al. (1993)), that is generated using the systematic symbolic manipulation techniques contained in the function `sym2lfr` of the LFR-toolbox. In general, the obtained model is reducible, therefore, we have the possibility to apply the available order reduction techniques. The structure of the LF candidate is given by a set of rational uncertain terms, which are generated from the LFR by using symbolic computations. Nevertheless, it is stated by Varga and Looye (1999), that *"the interlaced symbolic and numeric manipulations are hazardous since they involve many tolerance dependent rank decisions"*. Additionally, due to the "fraction of integers" symbolic representation of floating point numbers, numerical reduction may lead to very complex symbolic expressions in the LF. On the other hand, after a numerical reduction, the obtained LFR may generate a *reduced* set of rational functions, hence eliminating certain (maybe important) rational terms from the structure of the LF. As a possible resolution, we propose a symbolic model simplification method, which, compared to n-DOR, generates an LFR containing a possibly higher dimensional uncertainty block, but having advantageous properties for DOA estimate computation. And, importantly, it preserves the structure of the LF candidate.

1.1 The studied uncertain system class

We consider nonlinear systems in the quasi linear parameter varying (quasi-LPV) form

$$\dot{x}(t) = \mathcal{A}(x(t),\delta)x(t), \quad x(t)\in\mathbb{R}^n,\ x_0\in\mathcal{X},\ \delta\in\mathcal{D} \quad (1)$$

where $x(t)\in\mathbb{R}^n$ is the state vector, $x(0)=x_0$ is the initial condition, and $\delta\in\mathbb{R}^d$ is a vector of *constant* uncertain parameters. $\mathcal{X}\subset\mathbb{R}^n$ and $\mathcal{D}\subset\mathbb{R}^d$ are bounded polytopes given a priori. Polytope $\mathcal{X}$ is the set of initial states considered in the stability analysis. We assume that $\mathcal{A}(x,\delta)$ is a matrix of well-defined scalar uncertain *rational* functions $a_{ij}(x,\delta)$, $i,j=\overline{1,n}$. Secondly, we assume that the origin $x^*=0_n\in\mathcal{X}$ is an asymptotically stable equilibrium point of (1) for all $\delta\in\mathcal{D}$. From now on, the time argument of x will be suppressed. We denote the bounded s-dimensional polytope $\mathcal{X}\times\mathcal{D}\subset\mathbb{R}^s$ by Ω, where $s:=n+d$. We use the vector valued variable $\omega=\left(\begin{smallmatrix}x\\ \delta\end{smallmatrix}\right)\in\Omega$ as a shorthand for the arguments of an arbitrary function f depending both on x and δ, namely $f(\omega):=f(x,\delta)$. Finally, let $0_n\in\mathbb{R}^n$, and $I_n\in\mathbb{R}^{n\times n}$ denote the zero vector and the identity matrix, respectively.

1.2 Model representation and Lyapunov functions

Similarly to Trofino et al. (2013), we propose to start from the following differential algebraic representation of (1) needed for stability analysis:

$$\dot{x} = \mathcal{A}(\omega)x = Ax + B\pi(\omega), \qquad \pi(\omega)\in\mathbb{R}^p, \qquad (2a)$$

where $A\in\mathbb{R}^{n\times n}$, $B\in\mathbb{R}^{n\times p}$ are constant matrices, and $\pi(\omega)\in\mathbb{R}^p$ is a vector of well-defined uncertain rational nonlinear functions of ω. Furthermore, we consider the algebraic constraint:

$$0_p = G(\omega)x + F(\omega)\pi(\omega), \qquad \forall\omega\in\Omega, \qquad (2b)$$

which represents the algebraic coupling between the state variables and the nonlinear uncertain terms of the system equation $\mathcal{A}(\omega)x$ collected in vector $\pi(\omega)$. $G(\omega)\in\mathbb{R}^{p\times n}$ and $F(\omega)\in\mathbb{R}^{p\times p}$ are matrices of *affine* functions of ω. Matrix $F(\omega)$ is assumed to be *non-singular* for all $\omega\in\Omega$.

With reference to Trofino et al. (2013), a suitable LF is searched in the form

$$V(\omega) = \pi_b^T(\omega)P\pi_b(\omega),\ \pi_b(\omega) = \left(\begin{smallmatrix}x\\ \pi(\omega)\end{smallmatrix}\right)\in\mathbb{R}^{m:=n+p} \quad (3)$$

where $P\in\mathbb{R}^{m\times m}$ is a (not necessarily positive definite) symmetric matrix. The combined vector $\pi_b(x,\delta)$ contains the rational terms to be considered in the LF. The necessary Lyapunov conditions for local stability, namely

$$v_l(\|x\|) \le V(\omega) \le v_u(\|x\|) \qquad \forall\omega\in\Omega, \quad (4a)$$

$$\dot{V}(\omega) := \partial V(\omega)/\partial x\ \mathcal{A}(\omega)x \le -v_d(\|x\|) \quad \forall\omega\in\Omega, \quad (4b)$$

are ensured using sufficient LMI conditions, where $v_l(\cdot)$, $v_u(\cdot)$ and $v_d(\cdot)$ are continuous strictly increasing functions, being zero in $x=0$. Trofino et al. (2013) introduced further LMI conditions to ensure that the unitary level set of the LF is entirely inside of $\mathcal{X}$ for all $\delta\in\mathcal{D}$. Additionally, a linear objective function is proposed and meant to be minimized in order to enlarge the unitary level set as much as possible. After the LF computation, the maximal stability domain inside $\mathcal{X}$ can be characterized by the following two regions

$$\begin{aligned}\mathfrak{I} &= \{x\in\mathcal{X} \mid \forall\delta\in\mathcal{D}: V(x,\delta)\le 1\},\\ \mathfrak{U} &= \{x\in\mathcal{X} \mid \exists\delta\in\mathcal{D}: V(x,\delta)\le 1\},\ \mathfrak{I}\subseteq\mathfrak{U}.\end{aligned} \quad (5)$$

Due to the Lyapunov conditions it is ensured that any trajectory with an initial condition from $\mathfrak{I}$ will not leave $\mathfrak{U}$.

The proposed LMI optimization problem of Trofino et al. (2013) in equation (91) can be efficiently solved by the available numerical tools, although, the sizes of the obtained LMIs explode combinatorially as the number of coordinates of $\pi(\omega)\in\mathbb{R}^p$ increase. Therefore, the dimension reduction of the optimization model needs to be addressed in order that the method be adoptable on complex and/or higher dimensional systems. At the same time, there is a trade-off between the model's dimension and the conservatism of the obtained estimate, since any new term in $\pi(\omega)$ may result in a better LF with a larger stability region estimate $\mathfrak{I}$ (see e.g. Section 5.2).

In order to reduce the dimension of $\pi(\omega)$ appearing in (2), Polcz et al. (2017) proposed systematic algebraic simplification steps, which resulted in a significant dimension reduction of the optimization problem. However, the proposed algebraic manipulations do not preserve the non-singularity of matrix $F(x)$, and it is generally not assured that $F(x)$ remains a square matrix. Nevertheless, Theorem 4.1 of Trofino et al. (2013), related to local stability, requires the non-singularity of the *square* matrix $F(\omega)$. Furthermore, in certain cases, the proposed simplification steps may result in an even higher dimensional model as the initial one.

2. LINEAR FRACTIONAL TRANSFORMATION

LFT plays an important role in modeling uncertain rational systems, and it is often used in literature, as presented by Ghaoui and Scorletti (1996). Using the LFT any quasi-LPV system of the form (1) can be represented as a linear time invariant (LTI) system:

$$\dot{x} = Ax + B\pi, \qquad A \in \mathbb{R}^{n\times n}, \qquad B \in \mathbb{R}^{n\times p}, \tag{6a}$$

$$y = Cx + D\pi, \qquad C \in \mathbb{R}^{p\times n}, \qquad D \in \mathbb{R}^{p\times p}, \tag{6b}$$

with a nonlinear input characterized by the operator $\Delta(\omega)$:

$$\pi = \pi(\omega) = \Delta(\omega)\, y, \quad \Delta : \mathbb{R}^{n+d} \to \mathbb{R}^{p\times p}. \tag{6c}$$

In representation (6), A, B, C, D are constant matrices, $x \in \mathbb{R}^n$, π, $y \in \mathbb{R}^p$ are the state, input, and output vectors, respectively. Henceforth, (6) will be shortly denoted by $\mathcal{F}_p(A, B, C, D; \Delta)$, which is a lower LFR of matrix $\mathcal{A}(\omega)$. Subscript p emphasizes the dimension of the uncertainty block $\Delta(\omega)$ of the LFR.

Multiplying (6b) by $\Delta(\omega)$ from the left, and using (6c), we obtain an algebraic relation (2b) between x and $\pi(\omega)$,

$$\text{where } G(\omega) := -\Delta(\omega)C \in \mathbb{R}^{p\times n}, \tag{7a}$$

$$F(\omega) := I_p - \Delta(\omega)D \in \mathbb{R}^{p\times p}. \tag{7b}$$

This LFR is well-posed (i.e. well-defined) since, by assumption, $F(\omega)$ is non-singular for all $\omega \in \Omega$. Consequently, for each well-posed LFR, an explicit formula can be given for the set of rational functions considered in the LF (3):

$$\pi(\omega) = -F(\omega)^{-1}G(\omega)x \in \mathbb{R}^p. \tag{7c}$$

Although, the numerical n-DOR technique preserves the value of $\mathcal{A}(\omega) = A - BF(\omega)^{-1}G(\omega)$, it may also remove certain rational terms from the generated vector $\pi(\omega)$ of the initial LFR. On the other hand, the symbolic calculation of $F(\omega)^{-1}$ for the numerically reduced LFR may be difficult.

In Section 3, we introduce a symbolic preprocessing step, a decomposition of vector $\pi_b(\omega)$, then, in Section 4, we propose a model simplification method for representation (2), which decreases the dimension of $\pi(\omega)$ (if possible), such that the structure of the initial LF (3) remains the same.

3. CANONICAL DECOMPOSITION OF A VECTOR OF RATIONAL FUNCTIONS

In this section, the vector valued variables $\boldsymbol{z}(\omega)$, $\check{\boldsymbol{z}}(\omega), \in \mathbb{R}^m$ and $\boldsymbol{z}_0(\omega) \in \mathbb{R}^K$ will be denoted as bold symbols, in order to emphasize the distinction between them and their (scalar valued) coordinate values $z_i(\omega)$, $\check{z}_i(\omega)$, and $z_{0,j}(\omega)$, where $i = \overline{1, m}$ and $j = \overline{1, K}$.

Let $\boldsymbol{z}(\omega) \in \mathbb{R}^m$, be a vector of *rational* functions, namely

$$\boldsymbol{z}(\omega) = \begin{pmatrix} z_1(\omega) \\ \dots \\ z_m(\omega) \end{pmatrix}, \quad z_i(\omega) = \frac{u_i(\omega)}{v_i(\omega)}, \quad i = \overline{1, m}, \tag{8}$$

Procedure 1. Decomposition of $z(\omega)$.

1: **procedure** DECOMPOSE($z(\omega) \in \mathbb{R}^m$)
2: $\quad v(\omega;\kappa) \leftarrow \langle \kappa, \boldsymbol{z}(\omega)\rangle$, where $\kappa \in \mathbb{R}^{1\times m}$, $\omega \in \mathbb{R}^s$
3: $\quad \check{v}(\omega;\kappa)$, $q(\omega) \leftarrow$ NUMDEN $(v(\omega;\kappa))$
4: $\quad c_j(\kappa)$, $z_{0,j}(\omega) \leftarrow$ COEFFS($\check{v}(\omega;\kappa), \omega$), where $j = \overline{1, K}$
5: $\quad \boldsymbol{z}_0(\omega) \leftarrow (z_{0,1}(\omega)\ \dots\ z_{0,K}(\omega))^T$
6: $\quad \Theta \leftarrow$ EQUATIONSTOMATRIX($c_1(\kappa), ..., c_2(\kappa), \kappa$)
7: $\quad$ **return** Θ, $\boldsymbol{z}_0(\omega)$, $q(\omega)$
8: **end procedure**

where $u_i(\omega)$ and $v_i(\omega)$ are multivariate polynomials of $\omega_1, ..., \omega_s \in \mathbb{R}$ (i.e. in short $\omega \in \mathbb{R}^s$) and $v_i(\omega) \neq 0$ for all $\omega \in \Omega$. We aim to find the following decomposition

$$\boldsymbol{z}(\omega) = \Theta\, \boldsymbol{z}_0(\omega)\, q(\omega)^{-1}, \tag{9}$$

where $\Theta \in \mathbb{R}^{m\times K}$ is a constant *coefficient matrix* of $\boldsymbol{z}(\omega)$, $q(\omega)$ is a *monic* polynomial of ω (with leading coefficient 1), and $\boldsymbol{z}_0(\omega) \in \mathbb{R}^K$ is a vector, in which the coordinates $z_{0,j}(\omega)$ are distinct monic monomials, $j = \overline{1, K}$.

In order to perform this decomposition, first we determine the smallest degree common monic denominator $q(\omega)$ of rational functions $z_1(\omega), ..., z_m(\omega)$. We introduce vector $\check{\boldsymbol{z}}(\omega) := \boldsymbol{z}(\omega)\, q(\omega)$, in which the coordinates functions are polynomials, and let $\boldsymbol{z}_0(\omega)$ contain every distinct monomial term, which appears in each coordinate of $\check{\boldsymbol{z}}(\omega)$. Then, for each $\check{z}_i$ there exist real values $\vartheta_{ij} \in \mathbb{R}$ such that

$$\check{z}_i(\omega) = \sum\nolimits_{j=1}^K \vartheta_{ij}\, z_{0,j}(\omega), \quad i = \overline{1, m}. \tag{10}$$

Finally, $\check{\boldsymbol{z}}(\omega)$ can be written as:

$$\check{\boldsymbol{z}}(\omega) = \Theta\, \boldsymbol{z}_0(\omega), \ \Theta := \begin{pmatrix} \vartheta_{11} & \dots & \vartheta_{1K} \\ & \dots & \\ \vartheta_{m1} & \dots & \vartheta_{mK} \end{pmatrix} \in \mathbb{R}^{m\times K}, \tag{11}$$

which gives the decomposition (9) of vector $\boldsymbol{z}(\omega)$.

The proposed decomposition described in Procedure 1 can be efficiently produced using Matlab's Symbolic Math Toolbox (SMT). In order to compute $q(\omega)$, we introduce m number of auxiliary symbolic (indeterminate) scalar variables $\kappa_1, ..., \kappa_m$, collected in a row vector $\kappa = (\kappa_1\ \dots\ \kappa_m) \in \mathbb{R}^{1\times m}$. The operations of Procedure 1 are explained below in the following list, where the numbers correspond to the line numbers of Procedure 1:

2. Evaluate the dot product of $\langle \kappa, \boldsymbol{z}(\omega)\rangle =: v(\omega;\kappa)$. Both κ and ω are vectors of symbolic variables. $\boldsymbol{z}(\omega) \in \mathbb{R}^m$ is a vector of ω-dependent symbolic rational expressions. The resulting scalar valued object $v(\omega, \kappa)$ is a rational expression of indeterminates ω and κ.
3. Reduce $v(\omega;\kappa)$ into an irreducible fractional form, then determine the resulting numerator $\check{v}(\omega;\kappa) := \langle \kappa, \check{\boldsymbol{z}}(\omega)\rangle$ and denominator $q(\omega)$. These operations can be automated using the SMT function `numden`.
4. Collect and extract the multipliers of the common monomial terms $(z_{0,j}(\omega))$ of the numerator $\check{v}(\omega;\kappa)$ with respect to variables $\omega_1, ..., \omega_s$:
$$\check{v}(\omega;\kappa) = \sum\nolimits_{i=1}^m \kappa_i\, \check{z}_i(\omega) = \sum\nolimits_{j=1}^K c_j(\kappa)\, z_{0,j}(\omega), \tag{12}$$
where the multipliers $c_j(\kappa) = \sum_{i=1}^m \kappa_i \vartheta_{ij}$ are linear functions of indeterminates $\kappa_1, ..., \kappa_m$, furthermore, $z_{0,j}(\omega)$ is the corresponding monomial term. These pairs can be extracted using SMT function `coeffs`.
5. Let the coordinates of $\boldsymbol{z}_0(\omega)$ be the resulting monomials $z_{0,j}(\omega)$ in the same order as it was returned by SMT functions `coeffs`.

6. Using SMT function `equationsToMatrix`, determine the coefficients $\vartheta_{1j}, ..., \vartheta_{mj}$ for each multiplier $c_j(\kappa)$ returned by SMT functions `coeffs`, then construct the coefficient matrix $\Theta \in \mathbb{R}^{m\times K}$ as presented in (11).
7. This procedure will return matrix Θ, vector $\boldsymbol{z}_0(\omega)$ with distinct monic monomial coordinates functions and the denominator $q(\omega)$, which together will give the desired decomposition of $\boldsymbol{z}(\omega)$.

Note that for a fixed monomial order, this decomposition is unique.

4. MODEL SIMPLIFICATION

In this section, we propose a linear (similarity) transformation of the LFR, which imply that some variables in vectors π and y and the corresponding equations in (6b) can be eliminated from the LFR without modifying both the value of the uncertain matrix $\mathcal{A}(\omega)$, and the set of uncertain rational terms of the LF candidate.

We consider a system in representation (6) with the vector $\pi_b(\omega) = \left(\begin{smallmatrix} x \\ \pi(\omega)\end{smallmatrix}\right)$. As presented in (9), we produce the decomposition of $\pi_b(\omega)$ as follows:

$$\pi_b(\omega) = \Theta\, \pi_0(\omega)\, q(\omega)^{-1} \in \mathbb{R}^{m=n+p}. \tag{13}$$

If $\Theta \in \mathbb{R}^{m\times K}$ is full row-rank, our method cannot reduce the model's dimension. Let the rank of Θ be $n + k < m$, than Θ can be written in the following block-matrix form:

$$\Theta = \begin{pmatrix} \Theta_x \\ \Theta_{\pi_1} \\ \hline \Theta_{\pi_2} \end{pmatrix} \in \mathbb{R}^{m\times K}, \quad \begin{matrix} \Theta_x \in \mathbb{R}^{n\times K},\ \Theta_{\pi_1} \in \mathbb{R}^{k\times K} \\ \Theta_{\pi_2} \in \mathbb{R}^{(p-k)\times K}. \end{matrix} \tag{14}$$

Proposition 1. Matrix $\Theta_x \in \mathbb{R}^{n\times K}$ is full row-rank.

Proof. Suppose that $\Theta_x = \left(\begin{smallmatrix} \boldsymbol{\vartheta}_1 \\ \vdots \\ \boldsymbol{\vartheta}_n \end{smallmatrix}\right)$ is rank deficient. Then, there exists an index j and real values β_i, such that

$$\boldsymbol{\vartheta}_j = \textstyle\sum_{i=1,i\neq j}^n \beta_i \boldsymbol{\vartheta}_i, \quad 1 \leq j \leq n. \tag{15}$$

Considering the dot product of both sides of (15) with $\pi_0(\omega)q(\omega)^{-1}$ we obtain that $x_j = \sum_{i=1,i\neq j}^n \beta_i x_i$, which is a contradiction, since generally, there is no linear dependence between the state variables. Consequently, Θ_x must be a full row-rank matrix.

After an appropriate permutation of coordinates in vector $\pi(\omega)$ of model (6), we may assume, without the loss of generality, that Θ_{π_1} is full row-rank. Consequently, there exist matrices $\Gamma_1 \in \mathbb{R}^{(p-k)\times n}$ and $\Gamma_2 \in \mathbb{R}^{(p-k)\times k}$ such that

$$\Theta_{\pi_2} = \Gamma_1 \Theta_x + \Gamma_2 \Theta_{\pi_1}. \tag{16}$$

Namely, the rows of Θ_{π_2} can be expressed as the linear combination of the rows in matrices Θ_x and Θ_{π_1}.

Let us introduce the following decompositions of (6):

$$\begin{aligned} \dot{x} &= Ax + B_1\pi_1 + B_2\pi_2, && \text{(17a)} & \pi_1 &= \Delta_1(\omega)y_1, && \text{(17d)} \\ y_1 &= C_1x + D_{11}\pi_1 + D_{12}\pi_2, && \text{(17b)} & \pi_2 &= \Delta_2(\omega)y_2, && \text{(17e)} \\ y_2 &= C_2x + D_{21}\pi_1 + D_{22}\pi_2, && \text{(17c)} \end{aligned}$$

where $y_1 \in \mathbb{R}^k$ and $\pi_1 \in \mathbb{R}^k$, $k < p$ are the output and the nonlinear input, respectively.

Proposition 2. If we introduce the transformed matrices

$$\begin{aligned} \widehat{A} &:= A + B_2\Gamma_1, & \widehat{B} &:= B_1 + B_2\Gamma_2, \\ \widehat{C} &:= C_1 + D_{12}\Gamma_1, & \widehat{D} &:= D_{11} + D_{12}\Gamma_2, \end{aligned} \tag{18}$$

representation $\mathcal{F}_k(\widehat{A}, \widehat{B}, \widehat{C}, \widehat{D}; \Delta_1)$, with π_1 and y_1 is a dimensionally reduced equivalent of (6). Furthermore, $\mathcal{F}_k(\widehat{A}, \widehat{B}, \widehat{C}, \widehat{D}; \Delta_1)$ is well-posed if (6) is well-posed.

Proof. Due to (16), there exists a matrix $S \in \mathbb{R}^{m\times m}$,

$$S = \left(\begin{array}{cc|c} I_n & 0 & 0 \\ 0 & I_k & 0 \\ \hline -\Gamma_1 & -\Gamma_2 & I_{p-k} \end{array}\right), \quad \text{s.t. } S\cdot\Theta = \begin{pmatrix} \Theta_x \\ \Theta_{\pi_1} \\ \hline 0 \end{pmatrix}. \tag{19}$$

Multiplying both sides of the second equation of (19) with $\pi_0(\omega)q(\omega)^{-1}$, we obtain a key identity for π_2

$$S \cdot \begin{pmatrix} x \\ \pi_1 \\ \pi_2 \end{pmatrix} = \begin{pmatrix} x \\ \pi_1 \\ 0 \end{pmatrix} \;\Rightarrow\; \pi_2 = \Gamma_1 x + \Gamma_2 \pi_1. \tag{20}$$

Considering (20), the state and output equations (17a,17b) of representation (17) are rewritten as

$$\dot{x} = (A + B_2\Gamma_1)x + (B_1 + B_2\Gamma_2)\pi_1, \tag{21a}$$
$$y_1 = (C_1 + D_{12}\Gamma_1)x + (D_{11} + D_{12}\Gamma_2)\pi_1, \tag{21b}$$
$$\text{with } \pi_1 = \Delta_1(\omega)y_1. \tag{21c}$$

The output equation (17c) of y_2 and (17e) can be suppressed, since the system dynamics (21a) does not depend on π_2 nor on y_2. Representation (21) describes the same dynamics as the original (decomposed) model (17).

Using the inverse of S, vector $\pi_b(\omega)$ can be given as follows:

$$\pi_b(\omega) = S^{-1}\begin{pmatrix} x \\ \pi_1 \\ 0 \end{pmatrix}, \text{ with } S^{-1} = \left(\begin{array}{cc|c} I_n & 0 & 0 \\ 0 & I_k & 0 \\ \hline \Gamma_1 & \Gamma_2 & I_{p-k} \end{array}\right). \tag{22}$$

On the other hand, considering the block-matrix decomposition of matrices $G(\omega)$ and $F(\omega)$ of (7), we have that

$$\begin{array}{r} \\ k \updownarrow \\ p-k \updownarrow \end{array} \begin{pmatrix} \overset{n}{\longleftrightarrow} & \overset{k}{\longleftrightarrow} & \overset{p-k}{\longleftrightarrow} \\ G_1(\omega) & F_{11}(\omega) & F_{12}(\omega) \\ G_2(\omega) & F_{21}(\omega) & F_{22}(\omega) \end{pmatrix} \begin{pmatrix} x \\ \pi_1(\omega) \\ \pi_2(\omega) \end{pmatrix} = 0. \tag{23}$$

Using (22), and considering only the upper part (first k rows) of identity (23), we reach to the following identity:

$$\Big(G_1(\omega) + F_{12}(\omega)\Gamma_1\Big)x + \Big(F_{11}(\omega) + F_{12}(\omega)\Gamma_2\Big)\pi_1(\omega) = 0.$$

Note that $(F_{11}(\omega)\ F_{12}(\omega))$ and $\left(\begin{smallmatrix} I_k \\ \Gamma_2 \end{smallmatrix}\right)$ are full row-, and column-rank matrices, respectively, thus their product $\widehat{F}(\omega) := F_{11}(\omega) + F_{12}(\omega)\Gamma_2$ is non-singular, hence invertible, and $\pi_1(\omega) = -\widehat{F}(\omega)^{-1}\widehat{G}(\omega)x$ with $\widehat{G}(\omega) = G_1(\omega) + F_{12}(\omega)\Gamma_1$, which completes the proof.

Proposition 3. (Structure invariance of the LF). Let

$$\pi_b(\omega) = \begin{pmatrix} x \\ \pi_1(\omega) \\ \pi_2(\omega) \end{pmatrix}, \quad \text{and} \quad \widehat{\pi}_b(\omega) = \begin{pmatrix} x \\ \pi_1(\omega) \end{pmatrix}. \tag{24}$$

Then, for every matrix $P \in \mathbb{R}^{m\times m}$ there exists a matrix $\widehat{P} \in \mathbb{R}^{(n+k)\times(n+k)}$, such that

$$V(\omega) = \pi_b^T(\omega) P \pi_b(\omega) = \widehat{\pi}_b^T(\omega)\widehat{P}\widehat{\pi}_b(\omega), \forall \omega \in \mathbb{R}^{n+d}. \tag{25}$$

Proof. We introduce the block-matrix decomposition of matrix P of the LF $V(\omega)$ in (25):

$$P = \left(\begin{array}{c|c} P_{11} & P_{12} \\ \hline P_{21} & P_{22} \end{array}\right), \text{ where } P_{11} \in \mathbb{R}^{(n+k)\times(n+k)}. \tag{26}$$

Considering (22), the LF (25) can be altered as follows:

$$\begin{aligned} \pi_b^T P \pi_b &= \left(\widehat{\pi}_b^T \,|\, 0\right) S^{-T} \left(\begin{array}{c|c} P_{11} & P_{12} \\ \hline P_{21} & P_{22} \end{array}\right) S^{-1} \begin{pmatrix} \widehat{\pi}_b \\ 0 \end{pmatrix} \qquad (27) \\ &= \widehat{\pi}_b^T \left(P_{11} + \Gamma^T P_{21} + P_{12}\Gamma + \Gamma^T P_{22}\Gamma\right)\widehat{\pi}_b =: \widehat{\pi}_b^T \widehat{P} \widehat{\pi}_b, \end{aligned}$$

where $\Gamma := (\Gamma_1\ \Gamma_2)$. Consequently, we obtained that P with π_b and $\widehat{P}$ with $\widehat{\pi}_b$ satisfies (25) for every $\omega \in \mathbb{R}^s$.

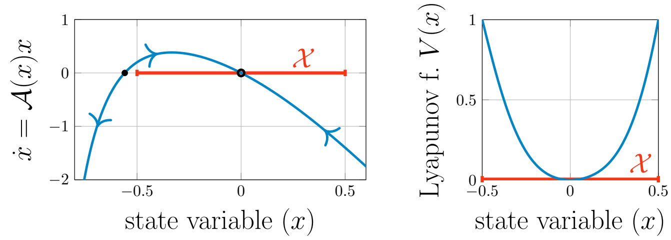

Fig. 1. Plot of $\dot{x}$ versus x for 1D system (28) (left). $x^* = 0$ is a (locally) stable equilibrium point, since the graph crosses axis x with a negative slope. The obtained rational LF for (28) is illustrated in the right figure. The red line segment highlights polytope $\mathcal{X}$.

To conclude, the proposed linear transformation (18) results in a simplified LFR model (i.e. with a smaller uncertainty bock Δ_1). Furthermore, the obtained LFR $\mathcal{F}_k(\widehat{A}, \widehat{B}, \widehat{C}, \widehat{D}; \Delta_1)$ generates the same rational terms to be considered in the LF as the initial LFR model.

5. ILLUSTRATIVE NUMERICAL EXAMPLES

In this section, we illustrate the applicability of the approach presented above through different numerical examples. The results presented in this section were computed in the Matlab environment. For symbolic computations, we applied Matlab's built-in Symbolic Math Toolbox based on Mupad. To model and solve semidefinite optimization (SDP) problems, we used YALMIP, Löfberg (2004), with Mosek solver, MOSEK ApS (2015). We used LFT software tools provided by the LFR-toolbox.

5.1 *One dimensional benchmark system*

In order to transparently illustrate the difference between the n-DOR, the symbolic manipulations of Polcz et al. (2017), and the proposed model reduction, we consider a very simple artificial one dimensional benchmark system:

$$\dot{x} = \mathcal{A}(x)x, \text{ where } \mathcal{A}(x) = \tfrac{1}{x+1} + \tfrac{1}{x^3+x^2+x+1} - 4. \quad (28)$$

If the domain of operation is constrained to $x(t) \in (-1, \infty)$, this system is well-defined and it has a locally stable equilibrium point at $x^* = 0$, for which the DOA can be determined precisely considering the graph of $\dot{x} = \mathcal{A}(x)x$ versus the state variable x (Fig. 1). The reason we still use this 1D model is to demonstrate the operations of the proposed model reduction method, and to show its possible advantages compared to the two other mentioned techniques know from the literature. The set of possible initial conditions considered in the stability analysis is chosen to be $\mathcal{X} = [-0.5, 0.5]$.

Applying LFR-toolbox function `sym2lfr` to $\mathcal{A}(x)$, an initial 4-dimensional LFR model is obtained:

$$\widetilde{A} = -2, \qquad \widetilde{B} = \begin{pmatrix} -1 & 0 & 0 & -1 \end{pmatrix}, \quad (29a)$$

$$\widetilde{C} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \qquad \widetilde{D} = \begin{pmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \qquad \widetilde{\Delta}(x) = xI_4.$$

The generated set of rational terms to appear in the LF is

$$\widetilde{\pi}(x) = \begin{pmatrix} \frac{x_1^4+x_1^3+x_1^2}{x_1^3+x_1^2+x_1+1} & \frac{x_1^2}{x_1^2+1} & \frac{x_1^2}{x_1^3+x_1^2+x_1+1} & \frac{x_1^2}{x_1+1} \end{pmatrix}^T \quad (29b)$$

The numerical n-DOR technique results in a 3-dimensional LFR (A.1a), for which the obtained rational terms are presented in (A.1b). The symbolic simplification steps for model (29) proposed by Polcz et al. (2017) generates a 5-dimensional model (A.2) with a set of rational terms, in which both the denominators and the monomial numerators are monic. Using the proposed model simplification technique presented in Section 4, we obtain a 3-dimensional LFR (A.3), for which the corresponding rational functions in $\pi(\omega)$ are relatively simple compared to the model (A.1a) generated by the n-DOR.

5.2 *An uncertain mass-action kinetic (MAK) system*

We consider the minimal MAK system presented by Wilhelm (2009) with a modified parameter configuration and with an uncertain reaction rate coefficient δ:

$$\begin{aligned} \dot{\bar{x}}_1 &= v_1 - \delta \bar{x}_1 \bar{x}_2^2, \\ \dot{\bar{x}}_2 &= \delta \bar{x}_1 \bar{x}_2^2 - k_2 \bar{x}_2, \end{aligned} \quad \text{where} \quad \begin{aligned} &\delta \in \mathcal{D} = [0.8, 1.2], \\ &k_2 = 2, \ v_1 = 4. \end{aligned} \quad (30)$$

This system with the given parameter values has a locally asymptotically stable equilibrium point at $\bar{x}^*(\delta) = \begin{pmatrix} \frac{1}{\delta} & 2 \end{pmatrix}^T$ with an unbounded DOA. The task is to generate a model (2) and then give a (bounded) estimate for the DOA by solving the optimization problem given in (91) of Trofino et al. (2013).

Introducing the centered state vector $x := \bar{x} - \bar{x}^*$ and considering the numerical values of coefficients v_1 and k_2, we obtain a system $\dot{x} = \mathcal{A}(x)x$, where:

$$\mathcal{A}(\omega) = \begin{pmatrix} -\delta(x_2+2)^2 & -x_2-4 \\ \delta(x_2+2)^2 & x_2+2 \end{pmatrix}. \quad (31)$$

Applying the LFR-toolbox function `sym2lfr` to $\mathcal{A}(\omega)$, we obtain a 8-dimensional LFR model. The numerical n-DOR results in a 3-dimensional LFR. The corresponding set of rational terms is denoted by $\pi'(\omega)$. The proposed LFR transformation produces a 4-dimensional model with $\pi(\omega)$. The values of both vectors $\pi'(\omega)$ and $\pi(\omega)$ are given in (A.4). For this system, the algebraic simplifications of Polcz et al. (2017) generates a 4-dimensional model.

Consider the following two *rectangular* polytopes

$$\mathcal{X}_1 = [-1.4, 1.4] \times [-0.7, 1.3], \ \mathcal{X}_2 = [-1.9, 1.9] \times [-0.7, 1.5]. \quad (32)$$

We solved the DOA computation problem (91) of Trofino et al. (2013) for both polytopes and both (simplified) models (A.4). For model (A.4a) with polytope $\mathcal{X}_2$, the problem is infeasible. The area of the estimated stability region in the different cases is given in Table 1.

Though the optimization is adopted on the centered system, the obtained LF $V(x, \delta)$ is transformed back into the original coordinate system $\bar{V}(\bar{x}, \delta) = V(\bar{x} - \bar{x}^*(\delta), \delta)$. Then the two regions $\mathfrak{I} \subset \mathfrak{U}$ are computed similarly as presented in (5). The obtained areas are illustrated in Figure 2.

6. CONCLUSIONS

In this paper, we presented a model simplification technique for the LFR based on symbolic and numeric manipulations. Using the obtained LFR with a reduced number of input-output pairs (π_i, y_i), a specific system representation is generated, which is used for the DOA estimate computation for rational uncertain nonlinear systems, as proposed by Trofino et al. (2013). The structure of the Lyapunov function candidate is given by a set of rational uncertain terms, which are generated from the dimensionally reduced LFR. Compared to n-DOR, our proposed technique results

Table 1. Area of the estimated stability region using different polytopes and different sets of rational functions considered in the LF. The two values in the 3rd column is the area of the inner and outer regions, respectively, introduced in (5).

Model reduction technique	$\mathcal{X}.$	Area
(A.4a) using n-DOR	$\mathcal{X}_1$	$1.14, 2.44u^2$
(A.4a) using n-DOR	$\mathcal{X}_2$	infeasible
(A.4b) using our proposed method	$\mathcal{X}_1$	$1.61, 3.32u^2$
(A.4b) using our proposed method	$\mathcal{X}_2$	$1.57, 3.17u^2$

Table 2. Number of free decision variables of the optimization problem are given in the 2nd column for each model. The 3rd column gives the dimensions of the LMIs corresponding to the Lyapunov conditions (4).

Model reduction technique	#var	Size of LMIs
(A.4a) using n-DOR	637	$5 \times 5, 18 \times 18$
(A.4b) using our proposed method	913	$6 \times 6, 22 \times 22$

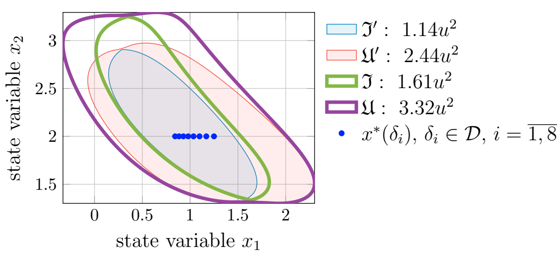

Fig. 2. Computed stability regions for the minimal MAK system using both sets of rational function (A.4).

in a possibly higher dimensional model (i.e. with larger Δ block), but all distinct rational terms of the initial LFR representation are preserved. The illustrative examples show that the method is indeed suitable for DOA estimation.

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Appendix A. NUMERICAL VALUES

Matrices of the LFR and vector $\pi(x)$ after using n-DOR on the initial LFR obtained for the benchmark 1D system:

$$A' = -2, \qquad B' = \begin{pmatrix} 0 & 0 & 1.4142 \end{pmatrix}, \tag{A.1a}$$

$$C' = \begin{pmatrix} -0.8165 \\ -1.1547 \\ -1.4142 \end{pmatrix}, \quad D' = \begin{pmatrix} -0 & -0.3536 & -0.5774 \\ 1.4142 & 0 & -0.8165 \\ -0 & 0.6124 & -1 \end{pmatrix}, \quad \Delta' = xI_3.$$

$$\pi'(x) = \begin{pmatrix} 2.5e31\,x^2(2.0e31\,x - 4.1e31) \\ -1.4e32\,x^2(1.0e31\,x + 1.0e31) \\ -1.4x^2(1.2e63\,x^2 + 6.2e62\,x + 1.2e63) \end{pmatrix} \cdot \frac{1}{q(x)}, \tag{A.1b}$$

$$q(x) = 1.2e63\,x^3 + 1.2e63\,x^2 + 1.2e63\,x + 1.2e63.$$

Simplified modeling as presented in Polcz et al. (2017):

$$\breve{B} = \begin{pmatrix} -1 & 0 & -1 & -1 & -1 \end{pmatrix}, \qquad \breve{F}(x) = \begin{pmatrix} x_1{+}1 & -x_1 & x_1{+}1 & 0 & x_1{+}1 \\ x_1 & 1 & 0 & 0 & x_1 \\ x_1 & 0 & x_1{+}1 & 0 & x_1 \\ 0 & 0 & 0 & x_1{+}1 & 0 \end{pmatrix},$$

$$\breve{\pi}(x) = \begin{pmatrix} \frac{x_1^4}{\breve{q}(x)} & \frac{x_1^2}{x_1^2+1} & \frac{x_1^2}{\breve{q}(x)} & \frac{x_1^2}{x_1+1} & \frac{x_1^3}{\breve{q}(x)} \end{pmatrix}, \ \breve{q}(x) = x_1^3 + x_1^2 + x_1 + 1\,. \tag{A.2}$$

LFR model and vector $\pi(\omega)$ after the proposed model simplification:

$$A = -2,\ B = \begin{pmatrix} -2 & 1 & -1 \end{pmatrix},\ C = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},\ D = \begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}$$

$$\Delta(x) = xI_3,\ \pi(x) = \begin{pmatrix} \frac{x_1^4+x_1^3+x_1^2}{x_1^3+x_1^2+x_1+1} & \frac{x_1^2}{x_1^2+1} & \frac{x_1^2}{x_1^3+x_1^2+x_1+1} \end{pmatrix} \tag{A.3}$$

The set of rational functions considered in the LF for the uncertain MAK system generated by n-DOR (π') and by the proposed model simplification technique (π):

$$\pi'(\omega) = \begin{pmatrix} -d_1 x_1 \sqrt{2} \\ 1122^{-\frac{1}{2}} \cdot (d_1 x_1 x_2 - 8x_2^2 - 8 d_1 x_1 x_2^2) \\ 66^{-\frac{1}{2}} \cdot (2x_2^2 + 8 d_1 x_1 x_2 + 2 d_1 x_1 x_2^2) \end{pmatrix} \tag{A.4a}$$

$$\pi(\omega) = \begin{pmatrix} d_1 x_1 & \frac{d_1 x_1 x_2^2}{4} + \frac{d_1 x_1 x_2}{2} & \frac{d_1 x_1 x_2}{2} & x_2^2 \end{pmatrix}^T \tag{A.4b}$$