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Coprime factors reduction methods for linear parameter varying and uncertain systems ☆

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Abstract

We present a generalization of the coprime factors model reduction method of Meyer and propose a balanced truncation reduction algorithm for a class of systems containing linear parameter varying and uncertain system models. A complete derivation of coprime factorizations for this class of systems is also given. The reduction method proposed is thus applicable to linear parameter varying and uncertain system realizations that do not satisfy the structured ℓ_2-induced stability constraint required in the standard nonfactored case. Reduction error bounds in the ℓ_2-induced norm of the factorized mapping are given.

Keywords: Coprime factors; Balanced truncation; Linear parameter-varying systems; Uncertain systems

1. Introduction

Model reduction methods with guaranteed error bounds have previously been established for linear fractional and uncertain systems [2,14,37,17,9,22,13,47]. A simplistic view of most of these existing reduction results is as generalizations of the balanced truncation and singular perturbation approximation techniques developed for standard state-space models [23,1,16,20,34,30]. As such, an appropriate generalization of state-space type realizations is typically used to describe the system. In order to apply the reduction methods, the generalized state-space system models are then required to satisfy a *robust* or a *structured* stability condition.

In this paper, we propose a method for the reduction of a class of generalized state-space systems containing linear parameter varying (LPV) and uncertain systems that do not satisfy the structured stability constraints required by the existing methods. In particular, we consider an extension of the coprime factors approach proposed by Meyer for standard state-space systems [29]; a complete derivation of coprime factorizations for this class of systems thus is presented as well. The systems we consider are therefore only required to be stabilizable and detectable in the sense defined by Lu et al. [26]. Error bounds are given in the ℓ_2-induced norm of the factorized linear fractional mapping, where this norm is computed over a unity norm-bounded set. These error bounds are thus useful for stability robustness analysis when interpreted in the robustness framework for coprime factors, or in a gap-metric framework [27,46,19,45].

We begin this paper with a brief overview of the linear fractional framework now commonly used to represent uncertain systems and linear parameter varying systems, and more recently, linear time-varying systems and spatial array systems; for simplicity we will collectively refer to the systems we consider as LFT systems. This overview is followed by an outline of stability conditions for these systems, and a review of existing model reduction results; this material is found in Section 2.

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The proposed factorized reduction method and the development of coprime factorizations for LFT systems is then presented in Section 3, and a basic computational reduction approach is given. Note that in the derivation of our results, we focus on dynamical systems evolving in discrete-time. The results presented herein are based on a preliminary reduction algorithm given in [7].

2. Preliminaries

Matrices in the real and complex numbers will be written $\mathbf{R}^{n\times m}$ and $\mathbf{C}^{n\times m}$, respectively; the $n \times n$ identity matrix is denoted by I_n. For a matrix $A \in \mathbf{C}^{n\times m}$, A^* denotes the complex conjugate transpose, and more generally for operators the adjoint. The dimensions of a matrix A are denoted $\dim(A)$. When a matrix A has only real eigenvalues we will use $\lambda_{\min}(A)$ to indicate the smallest of these. For notational convenience, dimensions will not be given unless pertinent to the discussion.

In this paper $\ell(\mathscr{X})$ denotes the linear space of sequences indexed by $\{0, 1, 2, \ldots\}$ taking values in the Euclidean space $\mathscr{X}$. The subspace $\ell_2(\mathscr{X})$ contains the sequences that are square summable; it has the usual norm

$$\|x\|_2 := (|x(0)|^2 + |x(1)|^2 + |x(2)|^2 + \cdots)^{1/2},$$

where $|\cdot|$ denotes the Euclidean norm on $\mathscr{X}$. We will often abbreviate these denotations by ℓ and ℓ_2 when the base space $\mathscr{X}$ is clear from the context or not relevant to the discussion.

The vector space of linear mappings on ℓ will be denoted by $\mathscr{L}(\ell)$. Note that $\mathscr{L}(\ell)$ includes maps between spaces with different base spaces, but this is not explicitly represented in our notation. It will be useful in the sequel to refer to the infinite block-matrix associated with a mapping $\mathscr{G} \in \mathscr{L}(\ell)$: we will use the notation $[\mathscr{G}]_{ij}$ to refer to the matrix entries of this representation with respect to the standard basis for ℓ.

Throughout the paper λ will denote the standard shift or delay mapping on ℓ. The *causal* subset $\mathscr{L}_{\mathrm{c}}(\ell)$ of the linear mappings consists of the operators in $\mathscr{L}(\ell)$ which commute with λ; namely, this set consists of mappings which have lower block-triangular infinite-matrix representations with respect to the standard basis for ℓ. Similarly, we define $\mathscr{L}_{\mathrm{c}}(\ell_2)$ to be the *bounded* linear mappings on ℓ_2 that are causal. The induced norm of an operator $\mathscr{G} \in \mathscr{L}_{\mathrm{c}}(\ell_2)$ is given by

$$\|\mathscr{G}\|_{\ell_2\to\ell_2} := \sup_{x\in\ell_2, x\neq 0} \frac{\|\mathscr{G}x\|_2}{\|x\|_2}.$$

Given any element $\mathscr{G}$ in $\mathscr{L}_{\mathrm{c}}(\ell_2)$ we can extend its domain to all of ℓ using its infinite matrix representation. Thus we can properly regard $\mathscr{L}_{\mathrm{c}}(\ell_2)$ as a subspace of the vector space $\mathscr{L}_{\mathrm{c}}(\ell)$. An important property of the subspace $\mathscr{L}_{\mathrm{c}}(\ell) \subset \mathscr{L}(\ell)$ is that the inverse of any element, if it exists, will also be in $\mathscr{L}_{\mathrm{c}}(\ell)$; that is, if $\mathscr{G} \in \mathscr{L}_{\mathrm{c}}(\ell)$ has an inverse in $\mathscr{L}(\ell)$ it must be causal.

Given a matrix A in $\mathbf{R}^{n\times m}$ it clearly defines a memoryless mapping in $\mathscr{L}_{\mathrm{c}}(\ell_2)$ by pointwise multiplication; in the paper we will not distinguish between this mapping and the matrix A, and will just refer to this memoryless mapping as a "matrix".

2.1. Linear fractional transformations

The LFT paradigm, described below and pictured in Fig. 1, traditionally has allowed for a mathematical representation of uncertainty in system models.

In the systems we consider, we assume G is a matrix and Δ could represent any of the following: repeated scalar uncertainty structures, exogenous time-varying parameters in linear parameter varying systems, temporal and spatial transform variables in spatial array systems. For specific examples of physical systems leading to these types of models see [5,44,10,11,3,35].

The mapping Δ will be parametrized in a special way in terms of an operator p-tuple denoted by $\overline{\delta} = (\delta_1, \delta_2, \ldots, \delta_p)$, where each δ_i is in $\mathscr{L}_{\mathrm{c}}(\ell_2(\mathbf{R}))$. Given the p-tuple of dimensions $\overline{m} = (m_1, \ldots, m_p)$ we associate with $\overline{\delta}$ the operator

$$\Delta(\overline{\delta}) = \mathrm{diag}[\delta_1 I_{m_1}, \ldots, \delta_p I_{m_p}]. \tag{1}$$

As in Fig. 1 we will often suppress the explicit dependence on $\overline{\delta}$ in our notation. Here the notation $\delta_i I_{m_i}$ is used to signify the operator in $\mathscr{L}_{\mathrm{c}}(\ell_2(\mathbf{R}^{m_i}))$ whose action on any element $x \in \ell_2(\mathbf{R}^{m_i})$ is defined by

$$(\delta_i I_{m_i})x := (\delta_i x_1, \delta_i x_2, \ldots, \delta_i x_{m_i}),$$

where the scalar sequences x_j are the channels of x; more precisely, $x =: (x_1, x_2, \ldots, x_{m_i})$. Thus Δ is a member of $\mathscr{L}_{\mathrm{c}}(\ell_2(\mathbf{R}^m))$, where $m := m_1 + \cdots + m_p$.

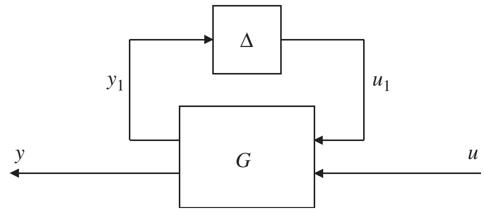

Fig. 1. Multi-dimensional/uncertain system.

The matrix G is viewed as the system realization matrix, which we partition as

$$G=\begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

with A an $m \times m$ matrix; the number of system inputs and outputs determine the remaining dimensions of B, C and D. We will be interested in the system defined by the equations

$$\begin{bmatrix} y_1 \\ y \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} u_1 \\ u \end{bmatrix}, \quad u_1 = \Delta y_1, \tag{2}$$

where the variables u, u_1, y_1 and y are in ℓ. That is, we regard the system in Fig. 1 as the representation of a mapping $u \mapsto y$ on ℓ when well-defined. We will restrict the operators δ_i to reside in the prespecified sets $\boldsymbol{\delta}_i \subset \mathscr{L}_c(\ell_2(\mathbf{R}^{m_i}))$. Thus

$$\overline{\delta} \in \boldsymbol{\delta}_1 \times \boldsymbol{\delta}_2 \times \cdots \times \boldsymbol{\delta}_p =: \overline{\boldsymbol{\delta}}.$$

Throughout the paper we make the standing assumption the $\boldsymbol{\delta}_i$ are subsets of the *closed unit ball* in $\mathscr{L}_c(\ell_2)$.

We will use $\mathscr{G} = (\overline{m}, \overline{\boldsymbol{\delta}}, G)$ to denote an *LFT system*; namely, all the maps given by

$$\mathscr{G}(\overline{\delta}) := D + C\Delta(I - A\Delta)^{-1}B \tag{3}$$

when $\overline{\delta} \in \overline{\boldsymbol{\delta}}$. So that we can directly refer to the operators Δ associated with an LFT system $\mathscr{G}$, without explicitly mentioning $\overline{\delta}$, it will be convenient to define the set

$$\boldsymbol{\Delta}_{\mathscr{G}} = \{\Delta \in \mathscr{L}_c(\ell_2(\mathbf{R}^m)) : \Delta = \mathrm{diag}[\delta_1 I_{m_1}, \ldots, \delta_p I_{m_p}],\ \delta_i \in \boldsymbol{\delta}_i\}$$

which contains exactly the operators swept out by the parametrization of (1).

The input/output (I/O) mapping from u to y for the system denoted by Fig. 1 and (2) is given by the LFT mapping $\mathscr{G}(\overline{\delta})$, or equivalently

$$u \mapsto y = D + C\Delta(I - A\Delta)^{-1}B =: \mathscr{G}(\Delta) \tag{4}$$

for $\Delta \in \boldsymbol{\Delta}_{\mathscr{G}}$, whenever the algebraic inverse is well-defined as a mapping in $\mathscr{L}(\ell)$. Note that all such maps are members of $\mathscr{L}_c(\ell)$.

Example 1. Standard one-dimensional (1D) systems may be represented in this framework by setting $\overline{\boldsymbol{\delta}} = \{\lambda\}$, thus, giving $\boldsymbol{\Delta}_{\mathscr{G}} = \{\lambda I\}$ and so $\mathscr{G}(\Delta) = D + C\lambda(I - A\lambda)^{-1}B$. An interesting variant when considering model reduction that preserves topological structure arises when each set $\boldsymbol{\delta}_i = \{\lambda\}$; see the recent paper [25].

Example 2. An important more general class is linear parameter varying (LPV) systems, that is systems of the form

$$x(t+1) = \tilde{A}(\theta(t))x(t) + \tilde{B}(\theta(t))u(t), \quad y(t) = \tilde{C}(\theta(t))x(t) + \tilde{D}(\theta(t))u(t), \tag{5}$$

where $\tilde{A}(\cdot), \tilde{B}(\cdot), \tilde{C}(\cdot), \tilde{D}(\cdot)$ are matrix-valued functions, dependent on the vector-valued parameter function $\theta(t) = (\theta_1(t), \ldots, \theta_k(t))$. In the standard setup of [32] we have that the parameter functions are known only to satisfy $-1 \leqslant \theta_i(t) \leqslant 1$, and that functions $\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}$ are LFT functions of the matrix $\Theta(t) = \mathrm{diag}[\theta_1(t)I_{m_1}, \ldots, \theta_{p-1}(t)I_{m_{p-1}}]$, where the dimensions m_i are appropriately defined. Thus we can write (5) in the form of (2) by setting $\boldsymbol{\delta}_i$ equal to the sets of memoryless multiplication operators generated by the respective functions $\theta_i(t)$, for $1 \leqslant i \leqslant p-1$, and $\boldsymbol{\delta}_p = \{\lambda\}$.

In fact, a major motivation for the current work is LPV systems as in (5). Currently, there is no predictable technique available for model reducing these systems if the time-varying matrix $\tilde{A}(\theta(t))$ is not guaranteed to generate an exponentially stable state transition matrix. In plain words, unstable LPV systems cannot be systematically model reduced by current methods. In this paper we will provide a systematic method. Note that although model reduction methods do not exist, results in [6] provide an important state-space *minimality* theory for such systems, and can be used to find minimal representations for unstable LPV and more generally LFT models.

The results presented in this paper are also applicable to uncertain systems; a specific physically motivated uncertain system provided the original motivation for the preliminary algorithm given in [7].

Having precisely defined the class of systems we consider and provided some motivation for our work, we move on to properties of LFT systems.

2.2. Stability and reduced order LFT systems

We begin by defining the basic notions of well-posedness and stability for the LFT systems we have now introduced.

Definition 1. Given an LFT system $\mathscr{G} = (\overline{m}, \overline{\boldsymbol{\delta}}, G)$ we say that

(i) the system is *well-posed* if $I - A\varDelta$ is invertible in $\mathscr{L}(\ell)$ for all $\varDelta$ in $\boldsymbol{\Delta}_{\mathscr{G}}$
(ii) $\mathscr{G}$ is *stable* if $I - A\varDelta$ is invertible in $\mathscr{L}_{\mathrm{c}}(\ell_2)$ for all $\varDelta$ in $\boldsymbol{\Delta}_{\mathscr{G}}$.

When $\mathscr{G}$ is well-posed but not stable we say it is unstable.

Thus when the system is well-posed the input/output map $\mathscr{G}(\varDelta)$ is a well-defined mapping on ℓ for every $\varDelta \in \boldsymbol{\Delta}_{\mathscr{G}}$. Similarly when the system is stable the map is bounded on ℓ_2.

Determining well-posedness of a system in the context of this paper amounts to checking a matrix singularity condition associated with the infinite matrix representation.

Theorem 2. *Suppose $\mathscr{G} = (\overline{m}, \overline{\boldsymbol{\delta}}, G)$ is an LFT system. Then $\mathscr{G}$ is well-posed if and only if for each $\varDelta \in \boldsymbol{\Delta}_{\mathscr{G}}$ the matrix*

$$I - A[\varDelta]_{ii} \text{ is nonsingular for all } i \geqslant 0,$$

where $[\varDelta]_{ii}$ are the matrices defined from the infinite block-matrix representation of $\varDelta$.

Proof. Given any $\varDelta \in \boldsymbol{\Delta}_{\mathscr{G}}$ we need to show that $I - A\varDelta$ has an inverse in $\mathscr{L}(\ell)$: observe that this operator is in $\mathscr{L}_{\mathrm{c}}(\ell)$ since both A and $\varDelta$ are. Thus, it is invertible exactly when the block-diagonal entries $[I - A\varDelta]_{ii}$ of its infinite matrix representation are all nonsingular; these entries are equal to $I - A[\varDelta]_{ii}$. □

Throughout the sequel, unless otherwise stated, we make the *standing assumption* that any systems referred to are well-posed. Note that the LPV systems of Example 2 are always well-posed provided that the state-space matrix functions are well-defined in terms of the parameter θ.

For characterization of stability we associate with $\mathscr{G}$ the set $\mathscr{T}_{\mathscr{G}}$ of *allowable transformations* defined by

$$\mathscr{T}_{\mathscr{G}} = \{T \in \mathbf{C}^{m\times m} : T = \mathrm{diag}(T_1, \ldots, T_p) \text{ where } T_i \in \mathbf{C}^{m_i\times m_i} \text{ and } \det(T_i) \neq 0\}.$$

It is not hard to show that $\mathscr{T}_{\mathscr{G}}$ consists of nonsingular maps in $\mathscr{L}(\ell_2)$ which commute with all members of $\boldsymbol{\Delta}_{\mathscr{G}}$; that is, given $T \in \mathscr{T}_{\mathscr{G}}$ in $\mathscr{L}(\ell_2)$ the equality $\Delta T = T\varDelta$ holds for all $\varDelta \in \boldsymbol{\Delta}_{\mathscr{G}}$.

Proposition 3. *Suppose $\mathscr{G} = (\overline{m}, \overline{\boldsymbol{\delta}}, G)$ is a given LFT system. If there exists a symmetric positive definite $Y \in \mathscr{T}_{\mathscr{G}}$ such that*

$$AYA^* - Y < 0, \tag{6}$$

then $\mathscr{G}$ is stable.

Proof. The result follows from the fact that the matrix inequality implies $\|Y^{1/2}AY^{-1/2}\|_{\ell_2\to\ell_2} < 1$. Thus, using the fact that $Y^{1/2}$ commutes with every $\varDelta$ in $\boldsymbol{\Delta}_{\mathscr{G}}$, and the submultiplicative inequality, we have $\|Y^{1/2}\Delta AY^{-1/2}\|_{\ell_2\to\ell_2} < 1$. This means the spectral radius of ΔA is less than one and so the power series $I + (\Delta A) + (\Delta A)^2 + (\Delta A)^3 + \cdots$ converges in $\mathscr{L}(\ell_2)$, and therefore must be $(I - \Delta A)^{-1}$. This inverse is in $\mathscr{L}_{\mathrm{c}}(\ell_2)$ since each term in the series is causal. □

The methods developed herein focus on exploiting the LMI condition of (6). For convenience of exposition, we will refer to systems that satisfy the LMI condition expressed by (6) as being *strongly stable*.

Remark 4. It is possible to show that if each set $\boldsymbol{\delta}_i$ is the closed unit ball in $\mathscr{L}_c(\ell_2(\mathbf{R}))$ then stability is *equivalent* to strong stability. That is the LMI condition in Proposition 3 is both necessary and sufficient for stability. This result does not seem to be directly available in the literature, but can be proved by a careful and lengthy application of published results on so-called full-block uncertainty structures [28,38,33].

In the sequel our focus will be model reduction of LFT systems. Given two well-posed LFT systems

$$\mathscr{G} = \left(\overline{m}, \overline{\boldsymbol{\delta}}, \begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) \quad \text{and} \quad \mathscr{G}_r = \left(\overline{m}_r, \overline{\boldsymbol{\delta}}, \begin{bmatrix} A_r & B_r \\ C_r & D_r \end{bmatrix}\right),$$

with $\dim(D) = \dim(D_r)$, we say that $\mathscr{G}_r$ is a *model reduction* of $\mathscr{G}$ if

$$m_i \geqslant m_{ri} \quad \text{for each} \quad 1 \leqslant i \leqslant p.$$

That is to say, a system is a reduction of another if each of its dimensions is no larger than that of the other system, and it maps between the same input/output spaces. In this paper we are interested in finding such systems so that $\mathscr{G}_r$ is in some way an approximation to $\mathscr{G}$.

In the next section we will review model reduction of strongly stable LFT systems, before moving on to model reduction of systems that are *not* strongly stable.

2.3. Model reduction for strongly stable LFT systems

The purpose of this section is to quickly review results from [9] on model reduction of strongly stable LFT systems. For standard state-space systems the role of controllability/observability Gramians and Lyapunov equations in balanced model reduction, and the related error bound computations, are well documented; see for instance [23] for a treatment of standard discrete time models. In [9] it is shown that a more general version of these concepts hold for LFT models; an application of these methods to a pressurized water reactor is discussed in [11].

For a stable system $\mathscr{G}$ and a stable approximation $\mathscr{G}_r$ we define the mismatch or error between them by the function

$$\text{error } (\mathscr{G}, \mathscr{G}_r) := \sup_{\overline{\delta} \in \overline{\boldsymbol{\delta}}} \|\mathscr{G}(\overline{\delta}) - \mathscr{G}_r(\overline{\delta})\|_{\ell_2 \to \ell_2}.$$

Namely, the error is given by the least upper bound for the induced norm of $\mathscr{G}(\overline{\delta}) - \mathscr{G}_r(\overline{\delta})$ when $\overline{\delta}$ ranges over $\overline{\boldsymbol{\delta}}$. Using this measure of merit we can state the following result from [9].

Theorem 5. *Suppose $\mathscr{G} = (\overline{m}, \overline{\boldsymbol{\delta}}, G)$ is a strongly stable LFT system, X and Y are positive definite matrices in $\mathscr{T}_{\mathscr{G}}$ satisfying*

(i) $AXA^* - X + BB^* < 0$,
(ii) $A^*YA - Y + C^*C < 0$,

and that $\lambda_{\min}$ is the minimum eigenvalue of XY with $\mu_{\min}$ its multiplicity. Then there exists a model reduction $\mathscr{G}_r = (\overline{m}_r, \overline{\boldsymbol{\delta}}, G_r)$ of $\mathscr{G}$ such that

(a) $\text{error}(\mathscr{G}, \mathscr{G}_r) \leqslant 2\sqrt{\lambda_{\min}}$; *and*
(b) *the degree of $\mathscr{G}_r$ satisfies $m_r \leqslant m - \mu_{\min}$.*

The theorem states that $\mathscr{G}$ can be reduced in dimension equal to the multiplicity of the smallest eigenvalue with an error of $2\sqrt{\lambda_{\min}(XY)}$. The structured matrices Y and X satisfying (i) and (ii) are referred to as *structured Gramians*, and by homogeneity and scaling, one can easily deduce that such LMI solutions always exist when $\mathscr{G}$ is strongly stable. These Lyapunov inequalities may have many solutions, and they can be found using semidefinite programming methods (for example see [48,40,41,18]).

An LFT system is said to be *balanced* when structured Gramians exist such that $Y = X = \Sigma$, with $\Sigma > 0$ and diagonal. Given a strongly stable LFT system and specific solutions X and Y, a balancing transformation can always be found by computing an allowable transformation T_{bal} satisfying $T_{\text{bal}} Y T_{\text{bal}}^* = (T_{\text{bal}}^*)^{-1} X (T_{\text{bal}})^{-1} = \Sigma$; T_{bal} can be found using established simultaneous diagonalization techniques and related procedures [43,36,24].

To derive reduction error bounds we can apply Theorem 5 recursively, resulting in an additive error bound. In particular, if we consider a balanced realization, that is, with $X = Y = \Sigma > 0$ and diagonal, then the reduction error bound is determined by summing twice the distinct elements of Σ corresponding to the submatrices of A, B, and C that are eliminated in the reduction process, giving the familiar "two-times-the-sum-of-the-tail" error bounds. Note that although we assume that the generalized Gramians X and Y are strictly positive definite, i.e., that the systems we consider are essentially "minimal", the methods discussed herein can be applied to nonminimal systems after first eliminating any nonminimal parts via the application of either [6,4] or [39]. A simple numerical example applying Theorem 5 is now given for illustrative purposes.

2.3.1. Multi-dimensional balanced truncation example

Consider the system realization $\mathcal{G} = ((3, 2), \overline{\boldsymbol{\delta}}, G)$ given by

$$A = \left[\begin{array}{ccc|cc} 0.5034 & 0.1768 & -0.2340 & -0.1406 & 0.5814 \\ 0.0096 & 0.5498 & -0.0362 & -0.6744 & 2.2496 \\ 0.0337 & 0.2546 & 0.0984 & -0.4051 & 1.3599 \\ \hline -0.2709 & 0.1470 & 0.3249 & 0.0484 & 0.6356 \\ -0.0909 & 0.0491 & 0.1075 & -0.1019 & 0.5681 \end{array}\right], \quad B = \left[\begin{array}{cc} 0.3306 & 0.1700 \\ 0.8951 & 0.3442 \\ 0.5487 & 0.2143 \\ \hline 0.8748 & 0.8821 \\ 0.5217 & 0.4479 \end{array}\right],$$

$$C = \left[\begin{array}{ccc|cc} 3.0622 & -0.9986 & -0.7126 & 6.4339 & -10.4291 \\ 3.0396 & -0.9913 & -0.7073 & 5.2369 & -8.4887 \end{array}\right], \tag{7}$$

and

$$\Delta_{\mathcal{G}} = \left[\begin{array}{c|c} \delta_1 I_3 & 0 \\ \hline 0 & \delta_2 I_2 \end{array}\right].$$

Employing a simple trace minimization algorithm that utilizes the LMI Toolbox `mincx` command, solutions to the Lyapunov inequalities given in Theorem 5 have been found, i.e., X and Y both in $\mathcal{T}_{\mathcal{G}}$, with the following numerical properties:

$$\mathrm{eig}(X) = \{1.3896, 4.4123, 117.3544, 3.6771, \; 434.0172\},$$

$$\mathrm{eig}(Y) = \{0.0000, 0.0425, 10.3292, 0.0187, 6.8346\}.$$

Note that the block structure of $\Delta_{\mathcal{G}}$ has been maintained in these listings, with the first 3 eigenvalues corresponding to the δ_1 block and the last 2 eigenvalues corresponding to the δ_2 block. Based on these values, it appears that from 1 to 3 dimensions may be reduced with a relatively small amount of error. The true reducibility of this system is determined upon examining $\sqrt{\mathrm{eig}(XY)}$, that is the generalized singular values:

$$\sqrt{\mathrm{eig}(XY)} = \{\sigma_1, \ldots, \sigma_5\} = \{0.00000061, 0.02750005, 2.65387576, 0.00656662, 1.36124876\}.$$

Based on these values, we can balance and truncate the system to dimensions of 1 in each δ_i variable, that is,

$$\Delta_{\mathcal{G}_r} = \begin{bmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{bmatrix}$$

with error less than or equal to $2 \times (\sigma_1 + \sigma_2 + \sigma_4) = 0.06813457$; this represents a relative error of *approximately* 0.84%.

Remark 6. The error bounds given in Theorem 5 apply directly to balanced truncation or balanced singular perturbation methods; in the former case it should also be noted that $D_r = D$. These bounds can be improved slightly, by a factor of 2, by utilizing LMI synthesis algorithms and machinery instead; see [2,9] for details.

3. Model reduction for unstable LFT systems

In the preceding section we reviewed results for the reduction of strongly stable LFT systems. If instead we have a model that is not strongly stable, namely, the LMI condition in Eq. (6) cannot be satisfied, then we would like to devise alternatives to enable us to systematically reduce the model and to maintain some means of evaluating the error resulting from the reduction process. This section is devoted to the development of such a method. Specifically, the use of coprime factors methods will be pursued, generalizing those proposed in [29] to LFT systems. In this case, the LFT system under consideration is both stabilizable and detectable, in a sense to be defined in the following.

3.1. Right coprime factors representations

We will now show that a notion of a right-coprime factorization (RCF) for a LFT system, $\mathscr{G} = (\overline{m}, \overline{\boldsymbol{\delta}}, G)$, exists if this LFT system is *strongly stabilizable* and *strongly detectable*, as defined below. The resulting *factors* in such a factorization are then strongly stable, thus it will be possible to reduce these with guaranteed error bounds using the reduction results cited in Section 2.3. A reduced LFT model, $\mathscr{G}_r = (\overline{m}_r, \overline{\boldsymbol{\delta}}, G_r)$, may then be obtained from the reduced coprime factors realization. To begin we now proceed to adapt the stabilizability and detectability results from [26] to our current setting; the derivation and construction of RCFs for LFT systems is then presented.

Definition 7. A LFT system $\mathscr{G} = (\overline{m}, \overline{\boldsymbol{\delta}}, G)$ is said to be strongly *stabilizable* if (a) it is well-posed; and (b) there exist a matrix F and a symmetric positive definite matrix $P \in \mathscr{T}_{\mathscr{G}}$ such that

$$(A + BF)P(A + BF)^* - P < 0. \tag{8}$$

Similarly, we say that $\mathscr{G}$ is strongly *detectable* if $(\overline{m}, \overline{\boldsymbol{\delta}}, G^*)$ is strongly stabilizable.

A condition for determining whether the above stabilizability LMI can be solved is given in the following theorem [26]. Note that it is assumed the matrix B is full rank.

Theorem 8. *Given a positive definite matrix $P \in \mathscr{T}_{\mathscr{G}}$ the following are equivalent*:

(i) *There exists a matrix F so that* (8) *is satisfied*;
(ii) *The following matrix inequality holds*

$$APA^* - P - BB^* < 0. \tag{9}$$

Furthermore, *if $F = -(B^*P^{-1}B)^{-1}B^*P^{-1}A$ is well-defined then $A + BF$ is a strongly stable matrix*.

We would like to use this result to define factorizations for well-posed LFT systems. To start, given two operators R and S in $\mathscr{L}_c(\ell_2)$ we say they are *right-coprime* in $\mathscr{L}_c(\ell_2)$ if

$$\text{there exists } Z \in \mathscr{L}_c(\ell_2) \text{ such that } Z\begin{bmatrix} R \\ S \end{bmatrix} = I.$$

We use this to define the following notion of factorization for mappings in $\mathscr{L}_c(\ell)$. Given a mapping G in $\mathscr{L}_c(\ell)$ we say that the operator pair $(N, M) \in \mathscr{L}_c(\ell_2) \times \mathscr{L}_c(\ell_2)$ is a *right-coprime factorization* (RCF) in $\mathscr{L}_c(\ell_2)$ of G if

(a) M is invertible in $\mathscr{L}_c(\ell)$;
(b) M and N are right-coprime in $\mathscr{L}_c(\ell_2)$;
(c) $G = NM^{-1}$.

We now extend this notion to LFT systems: two stable LFT systems $\mathscr{N}$ and $\mathscr{M}$ are said to be *right-coprime* if for each $\overline{\delta} \in \overline{\boldsymbol{\delta}}$ the operators $\mathscr{N}(\overline{\delta})$ and $\mathscr{M}(\overline{\delta})$ are right-coprime in $\mathscr{L}_c(\ell_2)$. Note that this is equivalent to the existence of a function $\mathscr{Z} : \overline{\boldsymbol{\delta}} \to \mathscr{L}_c(\ell_2)$ satisfying

$$\mathscr{Z}(\overline{\delta})\begin{bmatrix} \mathscr{N}(\overline{\delta}) \\ \mathscr{M}(\overline{\delta}) \end{bmatrix} = I \quad \text{for each } \overline{\delta} \in \overline{\boldsymbol{\delta}}.$$

If $\mathscr{G}$ is a well-posed LFT system we say that the pair $(\mathscr{N}, \mathscr{M})$ of *stable* LFT systems is a right-coprime factorization of $\mathscr{G}$ if for each fixed $\overline{\delta} \in \overline{\boldsymbol{\delta}}$

$$\text{the operator pair } (\mathscr{N}(\overline{\delta}), \mathscr{M}(\overline{\delta})) \text{ is a RCF in } \mathscr{L}_c(\ell_2) \text{ of the mapping } \mathscr{G}(\overline{\delta}).$$

If (9) is satisfied by some $P > 0$, $P \in \mathscr{T}_{\mathscr{G}}$, then a RCF representation for a LFT system can be constructed in a manner similar to that for standard state-space realizations. We have the following result which states that a RCF exists for any LFT system which is both strongly stabilizable and detectable. The proof is motivated by the standard case given in [31].

Theorem 9. *Suppose*

$$\mathscr{G} = \left(\overline{m}, \overline{\boldsymbol{\delta}}, \begin{bmatrix} A & B \\ C & D \end{bmatrix}\right)$$

is a strongly stabilizable and detectable LFT system, and that the matrix F is a corresponding stabilizing feedback. Then the LFT system pair

$$\mathscr{N}=\left(\overline{m},\overline{\boldsymbol{\delta}},\begin{bmatrix}A_F & B\\ C_F & D\end{bmatrix}\right),\quad \mathscr{M}=\left(\overline{m},\overline{\boldsymbol{\delta}},\begin{bmatrix}A_F & B\\ F & I\end{bmatrix}\right),$$

where $A_F = A + BF$ and $C_F = C + DF$, is a right-coprime factorization for $\mathscr{G}$.

Proof. Fix $\overline{\delta}\in\overline{\boldsymbol{\delta}}$, and let Δ be the corresponding element in $\boldsymbol{\Delta}_{\mathscr{G}}$. Our goal is to show that $\mathscr{G}(\overline{\delta})=\mathscr{N}(\overline{\delta})(\mathscr{M}(\overline{\delta}))^{-1}$ holds. First, we must establish that these objects are well-defined; clearly by the well-posedness assumption $\mathscr{G}(\overline{\delta})$, $\mathscr{N}(\overline{\delta})$, and $\mathscr{M}(\overline{\delta})$ are all members of $\mathscr{L}_{\mathrm{c}}(\ell)$.

We now demonstrate that the inverse of $\mathscr{M}(\overline{\delta})$ exists, observing that since $\mathscr{M}(\overline{\delta})$ is in $\mathscr{L}_{\mathrm{c}}(\ell)$ it is sufficient to show that it has a right inverse. To prove this we first note that $(I-\Delta A)^{-1}$ is well-defined since $\mathscr{G}$ is well-posed, and therefore $Q := I - F(I-\Delta A)^{-1}\Delta B$ is an element in $\mathscr{L}_{\mathrm{c}}(\ell)$. We will now show by direct calculation that Q is the right inverse of $\mathscr{M}(\overline{\delta})$: we have

$$\begin{aligned}\mathscr{M}(\overline{\delta})Q &= \{F(I-\Delta A_F)^{-1}\Delta B + I\}\{I - F(I-\Delta A)^{-1}\Delta B\}\\ &= I - F(I-\Delta A)^{-1}\Delta B + F(I-\Delta A_F)^{-1}(I-\Delta A-\Delta BF)(I-\Delta A)^{-1}\Delta B = I.\end{aligned}$$

Having established that $\mathscr{N}(\overline{\delta})(\mathscr{M}(\overline{\delta}))^{-1}$ is a well-defined element in $\mathscr{L}_{\mathrm{c}}(\ell)$ we now verify that it is equal to $\mathscr{G}(\overline{\delta})$. To start we have that

$$\begin{aligned}\mathscr{N}(\overline{\delta})Q &= \{C_F(I-\Delta A_F)^{-1}\Delta B + D\}\{I - F(I-\Delta A)^{-1}\Delta B\}\\ &= C_F(I-\Delta A_F)^{-1}\{\Delta B - \Delta BF(I-\Delta A)^{-1}\Delta B\} - DF(I-\Delta A)^{-1}\Delta B + D,\end{aligned}$$

which can be checked by direct multiplication, noting that the existence in $\mathscr{L}_{\mathrm{c}}(\ell)$ of all the above objects has already been established. In the first term we now factor out $(I-\Delta A)^{-1}\Delta B$ on the right yielding

$$C_F(I-\Delta A_F)^{-1}\{I-\Delta A-\Delta BF\}(I-\Delta A)^{-1}\Delta B - DF(I-\Delta A)^{-1}\Delta B + D$$

which using the definitions of A_F and C_F reduces directly to $C(I-\Delta A)^{-1}\Delta B + D$ as required.

Finally, to complete the proof we demonstrate that $\mathscr{N}(\overline{\delta})$ and $\mathscr{M}(\overline{\delta})$ are right coprime. Let

$$\mathscr{Z}(\overline{\delta}) := [\mathscr{U}(\overline{\delta})\ \ \mathscr{V}(\overline{\delta})]$$

with

$$\mathscr{U}=\left(\overline{m},\overline{\boldsymbol{\delta}},\begin{bmatrix}A_L & L\\ F & 0\end{bmatrix}\right),\quad \mathscr{V}=\left(\overline{m},\overline{\boldsymbol{\delta}},\begin{bmatrix}A_L & B_L\\ -F & I\end{bmatrix}\right),$$

where $A_L = A + LC$, $B_L = B + LD$, and L is a matrix chosen according to the dual result of Theorem 8, by the assumption of strong detectability, such that $A+LC$ is a strongly stable matrix. Then by extending the usual arguments, as was done previously in establishing $\mathscr{G}(\overline{\delta})=\mathscr{N}(\overline{\delta})(\mathscr{M}(\overline{\delta}))^{-1}$, we can readily see that $\mathscr{U}(\overline{\delta})$ and $\mathscr{V}(\overline{\delta})$ are members of $\mathscr{L}_{\mathrm{c}}(\ell_2)$, and that

$$\mathscr{U}(\overline{\delta})\mathscr{N}(\overline{\delta}) + \mathscr{V}(\overline{\delta})\mathscr{M}(\overline{\delta}) = I \quad \text{for each } \overline{\delta}\in\overline{\boldsymbol{\delta}}. \qquad \square$$

It is straightforward to verify that the RCF given in the theorem satisfies

$$\begin{bmatrix}\mathscr{N}(\overline{\delta})\\ \mathscr{M}(\overline{\delta})\end{bmatrix} = \mathscr{F}(\overline{\delta}) \quad \text{where } \mathscr{F}=\left(\overline{m},\overline{\boldsymbol{\delta}},\begin{bmatrix}A+BF & B\\ F & I\\ C+DF & D\end{bmatrix}\right). \tag{10}$$

The system $\mathscr{F}$ is strongly stable, and therefore can be model reduced via application of the results given in Section 2.3; also the dimension of the A-matrix for $\mathscr{F}$ is the same as that of $\mathscr{G}$. We now show that a reduction of $\mathscr{F}$ yields a coprime factorization of a reduced order system.

3.2. Model reduction

In this section we will develop results for model reduction of unstable LFT systems. Unlike in the stable case we will not be able to use a norm directly to capture the mismatch between the nominal system and the lower order approximation. The measure we use will be norm-based, and related to the closed-loop stability of the two systems. Here we will concentrate on the reduction procedure.

To begin we have a proposition which establishes a situation under which inversion in $\mathscr{L}_{\mathrm{c}}(\ell)$ is guaranteed.

Proposition 10. *If the operator E is in $\mathscr{L}_{\mathrm{c}}(\ell_2)$, then $I - \lambda E$ has an inverse in $\mathscr{L}_{\mathrm{c}}(\ell)$.*

Proof. It can be readily verified that the formal series

$$I + \lambda E + (\lambda E)^2 + (\lambda E)^3 + \cdots$$

uniquely defines an element of $\mathscr{L}_{\mathrm{c}}(\ell)$, and that this is indeed the inverse of $I - \lambda E$. $\square$

This fact motivates us to define the notion of strict causality: given an element $S \in \mathscr{L}_{\mathrm{c}}(\ell_2)$ we say it is *strictly causal* if it can be written as

$$S = \lambda E \quad \text{for some } E \text{ in } \mathscr{L}_{\mathrm{c}}(\ell_2).$$

So the preceding proposition states that if S is strictly causal then $I - S$ is invertible in $\mathscr{L}_{\mathrm{c}}(\ell)$. We will say a set $\mathscr{S} \subset \mathscr{L}_{\mathrm{c}}(\ell_2)$ is strictly causal if each of its elements is. We can now state the first result of this section.

Theorem 11. *Suppose $\mathscr{G} = (\overline{m}, \overline{\boldsymbol{\delta}}, G)$ is a strongly stabilizable and detectable LFT system, and that the set $\overline{\boldsymbol{\delta}}$ is strictly causal. Also, let $X, Y > 0$ and F be matrices satisfying*

$$(A + BF)X(A + BF)^* - X + BB^* < 0, \quad (A + BF)^*Y(A + BF) - Y + C^*C < 0. \tag{11}$$

If $\lambda_{\min}$ is the minimum eigenvalue of $X^{1/2}Y^{1/2}$, with its multiplicity denoted $\mu_{\min}$, then there exists a well-posed LFT system $\mathscr{G}_r = (\overline{m}_r, \overline{\boldsymbol{\delta}}, G_r)$ such that

(i) *the dimensional inequality $m_r \leqslant m - \mu_{\min}$ is satisfied; and*
(ii) *there exist RCFs $(\mathscr{N}, \mathscr{M})$ and $(\mathscr{N}_r, \mathscr{M}_r)$ of $\mathscr{G}$ and $\mathscr{G}_r$, respectively, so that*

$$\operatorname{error}\left(\begin{bmatrix} \mathscr{N} \\ \mathscr{M} \end{bmatrix} - \begin{bmatrix} \mathscr{N}_r \\ \mathscr{M}_r \end{bmatrix}\right) \leqslant 2\lambda_{\min}.$$

Proof. Recalling that X and Y are necessarily of the form $X = \operatorname{diag}(X_1, \ldots, X_p)$ and $Y = \operatorname{diag}(Y_1, \ldots, Y_p)$, define the nonsingular matrices $T_1, \ldots, T_p$ such that

$$T_i X_i T_i^* = (T_i^{-1})^* Y_i (T_i^{-1}) = \Sigma_i \text{ and is diagonal for each } i \in \{1, \ldots, p\}.$$

Without loss of generality we assume that the T_i matrices have been chosen so that each Σ_i has the form

$$\Sigma_i = \begin{bmatrix} \Gamma_i & 0 \\ 0 & \lambda_{\min} I_{v_i} \end{bmatrix},$$

where all the eigenvalues of Γ_i are greater than $\lambda_{\min}$, and the dimension $v_i \geqslant 0$. Note that $v_1 + v_2 + \cdots + v_p = \mu_{\min}$, and we can define the p-tuple of dimensions $\overline{v} = (v_1, v_2, \ldots, v_p)$. Now define the matrices

$$\mathscr{P}_i = [I_{(k_i - v_i)} \;\; 0]$$

such that $\mathscr{P}_i \Sigma_i \mathscr{P}_i^* = \Gamma_i$.

Next define the matrices

$$T_{\mathrm{bal}} = \begin{bmatrix} T_1 & & 0 \\ & \ddots & \\ 0 & & T_p \end{bmatrix} \quad \text{and} \quad \mathscr{P} = \begin{bmatrix} \mathscr{P}_1 & & 0 \\ & \ddots & \\ 0 & & \mathscr{P}_p \end{bmatrix}$$

in terms of the objects defined so far, and note that $\mathscr{P}^* \Sigma \mathscr{P} = \operatorname{diag}(\Gamma_1, \ldots, \Gamma_p)$. Clearly the LFT system

$$\mathscr{G}_{\mathrm{bal}} = \left(\overline{m}, \overline{\boldsymbol{\delta}}, \begin{bmatrix} T_{\mathrm{bal}} A T_{\mathrm{bal}}^{-1} & T_{\mathrm{bal}} B \\ C T_{\mathrm{bal}}^{-1} & D \end{bmatrix}\right) \text{ is equivalent to } \mathscr{G}.$$

We now define the reduced order system

$$\mathscr{G}_r = \left(\overline{m} - \overline{v}, \overline{\boldsymbol{\delta}}, \begin{bmatrix} \mathscr{P} T_{\mathrm{bal}} A T_{\mathrm{bal}}^{-1} \mathscr{P}^* & \mathscr{P} T_{\mathrm{bal}} B \\ C T_{\mathrm{bal}}^{-1} \mathscr{P}^* & D \end{bmatrix} \right).$$

Observe that by Proposition 10 this system is guaranteed to be well-posed since the set $\overline{\boldsymbol{\delta}}$ is strictly causal. Also the dimension of $\mathscr{G}_r$ is $m - \mu_{\min}$ as required in (i). It remains to show (ii).

Define the strongly stable LFT system

$$\mathscr{F}_{\mathrm{bal}} = \left(\overline{m}, \overline{\boldsymbol{\delta}}, \begin{bmatrix} T_{\mathrm{bal}}(A + BF) T_{\mathrm{bal}}^{-1} & T_{\mathrm{bal}} B \\ F T_{\mathrm{bal}}^{-1} & I \\ (C + DF) T_{\mathrm{bal}}^{-1} & D \end{bmatrix} \right).$$

This system, by Theorem 9 and the partitioning of (10), provides an RCF of $\mathscr{G}_{\mathrm{bal}}$ and therefore $\mathscr{G}$. From Eqs. (11) and the definition of T_{bal} we see that $\mathscr{G}_{\mathrm{bal}}$ is balanced. Therefore, the balanced truncation procedure related to Theorem 5 directly applies and we get that

$$\text{error } (\mathscr{F}_{\mathrm{bal}} - \mathscr{F}_r) \leqslant 2\lambda_{\min},$$

where

$$\mathscr{F}_r = \left(\overline{m}_r, \overline{\boldsymbol{\delta}}, \begin{bmatrix} \mathscr{P} T_{\mathrm{bal}}(A + BF) T_{\mathrm{bal}}^{-1} \mathscr{P}^* & \mathscr{P} T_{\mathrm{bal}} B \\ F T_{\mathrm{bal}}^{-1} \mathscr{P}^* & I \\ (C + DF) T_{\mathrm{bal}}^{-1} \mathscr{P}^* & D \end{bmatrix} \right)$$

and is strongly stable.

It only remains to show that $\mathscr{F}_r$ provides an RCF for the system $\mathscr{G}_r$. This follows since the A-matrix of the reduced factored system $\mathscr{F}_r$ is $(\mathscr{P} T_{\mathrm{bal}} A T_{\mathrm{bal}}^{-1} \mathscr{P}^*) + (\mathscr{P} T_{\mathrm{bal}} B F T_{\mathrm{bal}}^{-1} \mathscr{P}^*)$ and is strongly stable, and therefore we see that $F T_{\mathrm{bal}}^{-1} \mathscr{P}^*$ is a stabilizing feedback for $\mathscr{G}_r$. We now invoke Theorem 9. □

We now have the following extension of this result.

Corollary 12. *Assume the supposition in Theorem* 11 *is satisfied, and that* $\lambda_1 > \lambda_2 > \cdots > \lambda_q$ *are the distinct ordered eigenvalues of* $X^{1/2} Y^{1/2}$ *with respective multiplicities* $\mu_1, \mu_2, \ldots, \mu_q$.

If j *is an element of* $\{1, 2, \ldots, q\}$, *then there exists a well-posed LFT system* $\mathscr{G}_r = (\overline{m}_r, \overline{\boldsymbol{\delta}}, G_r)$ *such that*

(i) *the dimensional inequality* $m_r \leqslant m - (\mu_j + \cdots \mu_q)$ *is satisfied; and*
(ii) *there exist RCFs* $(\mathscr{N}, \mathscr{M})$ *and* $(\mathscr{N}_r, \mathscr{M}_r)$ *of* $\mathscr{G}$ *and* $\mathscr{G}_r$, *respectively, so that*

$$\text{error}\left(\begin{bmatrix} \mathscr{N} \\ \mathscr{M} \end{bmatrix} - \begin{bmatrix} \mathscr{N}_r \\ \mathscr{M}_r \end{bmatrix} \right) \leqslant 2(\lambda_j + \lambda_{j+1} + \cdots \lambda_q).$$

Proof. This result follows by repeated application of the procedure given in the proof for Theorem 11 noting that balancing only needs to be done once, and that the mapping error$(\cdot)$, although not a norm or metric, satisfies the triangle inequality. □

Both Theorem 11 and Corollary 12 have the requirement that the set $\overline{\boldsymbol{\delta}}$ be strictly causal. This requirement may be too restrictive for some applications, most notably in LPV systems where the elements in $\overline{\boldsymbol{\delta}}$ are of the form $(\delta_1, \ldots, \delta_{m-1}, \lambda)$, with the δ_i being *memoryless* operators corresponding to parameters.

Remark 13. The model reduction procedure described in the proofs of Theorem 11 and Corollary 12 can be applied to an LFT system $\mathscr{G}$ irrespective of whether $\overline{\boldsymbol{\delta}}$ is strictly causal or not. However, the procedure may yield a reduced order system $\mathscr{G}_r$ that is not well-posed. If $\mathscr{G}_r$ is well-posed then the error bounds provided by these results remain valid.

Thus the procedure so far can be used provided that the reduced system $\mathscr{G}_r$ is well-posed. We would now like to develop a related procedure that explicitly guarantees a well-posed system approximation when $\overline{\boldsymbol{\delta}}$ contains elements that are not strictly causal. We begin with the following observation.

Lemma 14. *Suppose that S_1, S_2, S_3 and S_4 are elements of $\mathscr{L}_{\mathrm{c}}(\ell_2)$. If S_3 and S_4 are both strictly causal, and $I - S_1$ is invertible in $\mathscr{L}_{\mathrm{c}}(\ell)$, then*

$$\left\{\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} - \begin{bmatrix} S_1 & S_2 \\ S_3 & S_4 \end{bmatrix}\right\} \text{ is invertible in } \mathscr{L}_{\mathrm{c}}(\ell).$$

Proof. By assumption $(I - S_1)^{-1} \in \mathscr{L}_{\mathrm{c}}(\ell)$ and so we can write the factorization

$$\begin{bmatrix} I - S_1 & -S_2 \\ -S_3 & I - S_4 \end{bmatrix} = \begin{bmatrix} I & 0 \\ -S_3(I - S_1)^{-1} & I \end{bmatrix} \begin{bmatrix} I - S_1 & -S_2 \\ 0 & I - \{S_4 + S_3(I - S_1)^{-1} S_2\} \end{bmatrix}.$$

From this we see that if $I - \{S_4 + S_3(I - S_1)^{-1} S_2\}$ is invertible, then so too is the left-hand side above. The latter object is indeed invertible by Proposition 10, noting that $S_4 + S_3(I - S_1)^{-1} S_2$ is strictly causal since both S_4 and S_3 are. □

We now have the following specialized definition for our purposes.

Definition 15. Given a subset $\{i_0, \ldots, i_s\}$ of the indices $\{1, \ldots, p\}$, we say that the set $\overline{\boldsymbol{\delta}}$ has strict $(\boldsymbol{\delta}_{i_0}, \ldots, \boldsymbol{\delta}_{i_s})$-causality if each set $\boldsymbol{\delta}_{i_k}$ is strictly causal.

Corollary 16. *Suppose $\mathscr{G} = (\overline{m}, \overline{\boldsymbol{\delta}}, G)$ is a strongly stabilizable and detectable LFT system, and that*

(i) *the integer h satisfies $0 \leqslant h \leqslant p$;*
(ii) *the set $\overline{\boldsymbol{\delta}}$ has strict $(\boldsymbol{\delta}_{h+1}, \ldots, \boldsymbol{\delta}_p)$-causality;*
(iii) *the matrices $X, Y \in \mathscr{T}_{\mathscr{G}}$ are positive definite, F is a matrix, and they satisfy the inequalities in* (11);
(iv) *the scalars $\lambda_1 > \lambda_2 > \cdots > \lambda_q$ are the distinct ordered eigenvalues of $X^{1/2} Y^{1/2}$ with respective multiplicities $\mu_1, \mu_2, \ldots, \mu_q$;*
(v) *the matrix X further satisfies*

$$\begin{bmatrix} A_{11} & \cdots & A_{1h} \\ \vdots & \ddots & \vdots \\ A_{h1} & \cdots & A_{hh} \end{bmatrix}^* \begin{bmatrix} X_1 & & 0 \\ & \ddots & \\ 0 & & X_h \end{bmatrix} \begin{bmatrix} A_{11} & \cdots & A_{1h} \\ \vdots & \ddots & \vdots \\ A_{h1} & \cdots & A_{hh} \end{bmatrix} - \begin{bmatrix} X_1 & & 0 \\ & \ddots & \\ 0 & & X_h \end{bmatrix} < 0.$$

If j is an element of $\{1, 2, \ldots, q\}$, then there exists a well-posed LFT system $\mathscr{G}_r = (\overline{m}_r, \overline{\boldsymbol{\delta}}, G_r)$ such that

(i) *the dimensional inequality $k_r \leqslant k - (\mu_j + \cdots \mu_q)$ is satisfied; and*
(ii) *there exist RCFs $(\mathscr{N}, \mathscr{M})$ and $(\mathscr{N}_r, \mathscr{M}_r)$ of $\mathscr{G}$ and $\mathscr{G}_r$, respectively, so that*

$$\operatorname{error}\left(\begin{bmatrix} \mathscr{N} \\ \mathscr{M} \end{bmatrix} - \begin{bmatrix} \mathscr{N}_r \\ \mathscr{M}_r \end{bmatrix}\right) \leqslant 2(\lambda_j + \lambda_{j+1} + \cdots \lambda_q).$$

Proof. We will make the simplifying assumption that $j = q$, so that we can follow the proof of Theorem 11. Extension of the proof to the general case stated above follows from repeated application of this simplified case together with some cosmetic modifications.

We will follow the proof of Theorem 11 setting $\lambda_{\min} := \lambda_q$, and then following the same steps define

$$\mathscr{G}_r = \left(\overline{m} - \overline{v}, \overline{\boldsymbol{\delta}}, \begin{bmatrix} \mathscr{P} T_{\mathrm{bal}} A T_{\mathrm{bal}}^{-1} \mathscr{P}^* & \mathscr{P} T_{\mathrm{bal}} B \\ C T_{\mathrm{bal}}^{-1} \mathscr{P}^* & D \end{bmatrix}\right).$$

To complete the proof it is sufficient to prove that $\mathscr{G}_r$ is a well-posed LFT system, since then the remainder of the proof of Theorem 11 can be used. Namely, we need to show that

$$I - \Delta \mathscr{P} T_{\mathrm{bal}} A T_{\mathrm{bal}}^{-1} \mathscr{P}^* \text{ is invertible in } \mathscr{L}_{\mathrm{c}}(\ell),$$

for all Δ in $\boldsymbol{\Delta}_{\mathscr{G}_r}$.

To prove this define $\overline{A} := T_{\mathrm{bal}} A T_{\mathrm{bal}}^{-1}$, which therefore has block entries $\overline{A}_{ij} := (T_{\mathrm{bal}})_i A_{ij} (T_{\mathrm{bal}}^{-1})_j$. Then by the assumption given in (v) we can conclude that

$$\begin{bmatrix} \overline{A}_{11} & \cdots & \overline{A}_{1h} \\ \vdots & \ddots & \vdots \\ \overline{A}_{h1} & \cdots & \overline{A}_{hh} \end{bmatrix}^* \begin{bmatrix} \Sigma_1 & & 0 \\ & \ddots & \\ 0 & & \Sigma_h \end{bmatrix} \underbrace{\begin{bmatrix} \overline{A}_{11} & \cdots & \overline{A}_{1h} \\ \vdots & \ddots & \vdots \\ \overline{A}_{h1} & \cdots & \overline{A}_{hh} \end{bmatrix}}_{=:\overline{A}_{FF}} - \begin{bmatrix} \Sigma_1 & & 0 \\ & \ddots & \\ 0 & & \Sigma_h \end{bmatrix} < 0.$$

Therefore, we have that $(\overline{\mathscr{P}}_F^* \overline{A}_{FF} \overline{\mathscr{P}}_F)^* \overline{\Gamma}_F (\overline{\mathscr{P}}_F^* \overline{A}_{FF} \overline{\mathscr{P}}_F) - \overline{\Gamma}_F < 0$, where $\overline{\mathscr{P}}_F = \mathrm{diag}(\mathscr{P}_1, \ldots, \mathscr{P}_h)$ and $\overline{\Gamma}_F = \mathrm{diag}(\Gamma_1, \ldots, \Gamma_h)$. We conclude that $\overline{\mathscr{P}}_F^* \overline{A}_{FF} \overline{\mathscr{P}}_F$ is a strongly stable matrix with respect to the parameter set $\boldsymbol{\delta}_1 \times \boldsymbol{\delta}_2 \times \cdots \times \boldsymbol{\delta}_h$. Now notice that given a mappingΔ in $\boldsymbol{\Delta}_{\mathscr{G}_r}$

$$\Delta \mathscr{P} T_{\mathrm{bal}} A T_{\mathrm{bal}}^{-1} \mathscr{P}^* = \begin{bmatrix} \Delta_F & 0 \\ 0 & \Delta_S \end{bmatrix} \begin{bmatrix} \overline{\mathscr{P}}_F^* \overline{A}_{FF} \overline{\mathscr{P}}_F & \overline{\mathscr{P}}_F^* \overline{A}_{FS} \overline{\mathscr{P}}_S \\ \overline{\mathscr{P}}_S^* \overline{A}_{SF} \overline{\mathscr{P}}_F & \overline{\mathscr{P}}_S^* \overline{A}_{SS} \overline{\mathscr{P}}_S \end{bmatrix},$$

where $\overline{\mathscr{P}}_S = \mathrm{diag}(\mathscr{P}_{h+1}, \ldots, \mathscr{P}_p)$, and A_{SF}, A_{FS}, and A_{SS} are defined in the obvious way given the definition of A_{FF} above. In the partition of Δ it is straightforward to verify that Δ_S is strictly causal from assumption (ii). Set

$$S := \begin{bmatrix} S_1 & S_2 \\ S_3 & S_4 \end{bmatrix} := \text{RHS above}$$

and thus we have that both S_3 and S_4 are strictly causal operators since Δ_S is. Further, $S_1 = \Delta_F \overline{\mathscr{P}}_F^* \overline{A}_{FF} \overline{\mathscr{P}}_F$ and so from above we know that the inverse of $(I - \Delta_F \overline{\mathscr{P}}_F^* \overline{A}_{FF} \overline{\mathscr{P}}_F)$ exists in $\mathscr{L}_{\mathrm{c}}(\ell_2)$, since $\overline{\mathscr{P}}_F^* \overline{A}_{FF} \overline{\mathscr{P}}_F$ is strongly stable.

We now invoke Lemma 14 to conclude that $I - S$ has an inverse in $\mathscr{L}_{\mathrm{c}}(\ell)$; namely, $(I - \Delta \mathscr{P} T_{\mathrm{bal}} A T_{\mathrm{bal}}^{-1} \mathscr{P}^*)^{-1}$ exists as a mapping in $\mathscr{L}_{\mathrm{c}}(\ell)$. Thus we have that $\mathscr{G}_r$ is well-posed as required. □

Note that the error bounds are always given in terms of the distance between the coprime factors realizations, and thus may be interpreted in the graph or gap metric settings with connotations for control design robustness analysis, or more generally for robustness of interconnections. We now state a simple algorithm for carrying out balanced truncation of coprime factors realizations for multi-dimensional systems.

3.3. A right coprime factors reduction algorithm

A particularly straightforward (but nonoptimal) computational approach summarizing the preceding discussions is given below.

Given the strongly stabilizable and detectable system

$$\mathscr{G} = \left(\overline{m}, \overline{\boldsymbol{\delta}}, \begin{bmatrix} A & B \\ C & D \end{bmatrix} \right):$$

1. Find $P > 0$, $P \in \mathscr{T}_{\mathscr{G}}$ satisfying (9), and set $F = -(B^* P^{-1} B)^{-1} B^* P^{-1} A$.
2. Construct a RCF realization, as defined in (10) for example.
3. Find $Y > 0$ and $X > 0$, both in $\mathscr{T}_{\mathscr{G}}$, satisfying the two LMIs given in Theorem 5 for the RCF realization. Note that X may be further required to satisfy (v) in Corollary 16, in order to guarantee that a well-posed approximation results when $\overline{\boldsymbol{\delta}}$ is not strictly causal.
4. Simultaneously diagonalize $Y > 0$ and $X > 0$ and construct a balancing similarity transformation, T_{bal}, for the RCF realization, and the associated diagonal matrix Σ of generalized singular values. (See [11,43,36,24] and the references therein for details).
5. Based on the relative values of the elements of Σ, determine suitable truncation dimensions v_{ji}, and the error bound as given in Theorem 11 and Corollary 12 (or Corollary 16).
6. Define the truncation matrix $\mathscr{P}_T := \mathrm{diag}[[I_{r_1}\ \mathbf{O}],\ [I_{r_2}\ \mathbf{O}], \ldots, [I_{r_p}\ \mathbf{O}]]$ where $r_i \geqslant m_i - \sum_j v_{ji}$, and $\mathbf{O}$ represents the zero matrix with dimensions $r_i \times m_i - r_i$. Compute the reduced LFT system realization G_r as follows:

$$G_r = \left[\begin{array}{c|c} \mathcal{P}_T T_{bal} A T_{bal}^{-1} \mathcal{P}_T^T & \mathcal{P}_T T_{bal} B \\ \hline C T_{bal}^{-1} \mathcal{P}_T^T & D \end{array} \right]$$

Remark 17. The direct balancing procedure referred to in this algorithm, specifically the process of simultaneous diagonalization, can be numerically ill-conditioned. In such a case, a *balancing-free* approach should be used; see [39,43]. Also, note that LMI/SDP solvers at present are not appropriate for very large scale systems; however, this is still an active research area.

An application of this algorithm to the simple numerical example considered in Section 2.3 is given here strictly for illustrative purposes and so that it can be contrasted with the earlier methods; see [7] for an application to a physically motivated example.

3.3.1. Multi-dimensional RCF balanced truncation example

Consider the system realization given in (7), with block structure $\Delta = \mathrm{diag}[\delta_1 I_3, \ \ \delta_2 I_2]$.

Steps 1, 2: Solving the stabilizability LMI given in (9) for a feasible P matrix and computing the resulting stabilizing gain matrix gives

$$F = \begin{bmatrix} -0.4218 & -0.7233 & 0.3335 & 1.2690 & -3.5277 \\ 0.6882 & 0.6235 & -0.6582 & -1.2906 & 2.8093 \end{bmatrix}$$

we can then form the RCF realization as in (10):

$$\mathcal{F} = \left(\overline{m}, \overline{\boldsymbol{\delta}}, \left[\begin{array}{c|c} A+BF & B \\ \hline F & I \\ C+DF & D \end{array}\right]\right) = \left[\begin{array}{ccccc|cc} 0.4810 & 0.0437 & -0.2356 & 0.0596 & -0.1074 & 0.3306 & 0.1700 \\ -0.1312 & 0.1169 & 0.0358 & 0.0174 & 0.0587 & 0.8951 & 0.3442 \\ -0.0503 & -0.0087 & 0.1404 & 0.0147 & 0.0261 & 0.5487 & 0.2143 \\ -0.0329 & 0.0641 & 0.0361 & 0.0201 & 0.0275 & 0.8748 & 0.8821 \\ -0.0028 & -0.0490 & -0.0133 & -0.0179 & -0.0140 & 0.5217 & 0.4479 \\ \hline -0.4218 & -0.7233 & 0.3335 & 1.2690 & -3.5277 & 1 & 0 \\ 0.6882 & 0.6235 & -0.6582 & -1.2906 & 2.8093 & 0 & 1 \\ 3.0622 & -0.9986 & -0.7126 & 6.4339 & -10.4291 & 0 & 0 \\ 3.040 & -0.9913 & -0.7073 & 5.2369 & -8.4887 & 0 & 0 \end{array}\right].$$

Steps 3, 4: Using the same trace minimization algorithm as in the preceding example, the Lyapunov inequalities for the RCF realization given above are solved and the solutions X and Y are found which have the following numerical properties:

$$\mathrm{eig}(X) = \{1.7299, 2.5990, 8.2492, 114.2547, 374.5947\},$$

$$\mathrm{eig}(Y) = \{0.0000, 0.0212, 0.0305, 2.9871, 3.6051\}.$$

Again, the block structure of Δ has been maintained in these listings: the first 3 eigenvalues correspond to the δ_1 block and the last 2 eigenvalues correspond to the δ_2 block. Based on these values, it appears that from 1 to 3 dimensions may be reduced.

Step 5: As before, the true reducibility of the RCF realization is determined upon simultaneously diagonalizing X and Y and examining the generalized singular values:

$$\sqrt{\mathrm{eig}(XY)} = \{\sigma_1, \ldots, \sigma_5\} = \{3.5894, 1.4760, 0.0000, 3.9328, 2.6304\}.$$

Based on these values, we can balance and truncate the system to dimensions of 2 in each δ_i variable, that is,

$$\Delta_r = \begin{bmatrix} \delta_1 I_2 & 0 \\ 0 & \delta_2 I_2 \end{bmatrix}$$

with approximately no error (i.e., $2 \times \sigma_3 \approx 0$); however, reducing this RCF realization further results in a more substantial possible relative error than in the original example (e.g., $2 \times (\sigma_2 + \sigma_3) = 2.9520$ represents approximately as much as 25% relative error).

Step 6: The resulting truncation matrix $\mathscr{P}_T$ for this RCF realization is:

$$\mathscr{P}_T := \begin{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} & \\ & \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{bmatrix},$$

which should then be applied to the *balanced* RCF realization.

Recall that the error bounds resulting from balancing and truncating a RCF are related to stability robustness in a graph-metric type setting, and cannot necessarily be expected to be the same as the error bounds for the original realization. This example also brings to light the obvious question of whether a better RCF can be found, for example, can we optimally (or suboptimally) solve for P and/or F in Steps 1 and 2 of the procedure to achieve better reduction results? Note that jointly solving for X and F, and Y and F that optimize the *bilinear* matrix inequalities (BMIs) resulting from considering the LMIs of (11) for the RCF realization of (10) is clearly a nonconvex problem. However, heuristic approaches such as those discussed in [42,21] may provide acceptable computational solutions to this class of problems.

4. Robustness implications and conclusions

It has been well-established in the standard 1D case (i.e., $\Delta = \lambda I$) that if the interconnection of two systems, for example a feedback interconnection of some *nominal* system G_0 with coprime factors (N, M) and a fixed system K, is internally stable, and

$$\|M^{-1}(I - KG_0)^{-1}[K \;\; I]\|_\infty \leqslant \frac{1}{\varepsilon},$$

then a stable interconnection results for *any* system G belonging to the set $\tilde{G}$ defined by

$$\tilde{G} := \left\{ (N + E_N)(M + E_M)^{-1} : \left\| \begin{matrix} E_N \\ E_M \end{matrix} \right\|_\infty < \varepsilon \right\}.$$

See [27,45] for details.

These coprime factor robustness results give us explicit information regarding how far we can reduce a coprime factor representation of G_0 and still remain stable under the given interconnection. Namely, the difference between the full and reduced RCF realizations, that is, the reduction error, may be represented by (E_N, E_M) and must be bounded by ε. This type of bound can be extended to the class of systems we have considered herein by applying existing results. Namely, existing multi-dimensional versions of the KYP Lemma and the small gain theorem (see, for example, Chapter 11 of [15] and the references therein) may be used to derive an extension of the above results to an interconnection of LFT systems $\mathscr{G}_0 = (\overline{m}, \overline{\boldsymbol{\delta}}, G_0) = \mathscr{N}(\delta)M(\delta)^{-1}$ and $\mathscr{K} = (\overline{k}, \overline{\boldsymbol{\delta}}, K)$.

Note that in the standard state-space case, if the coprime factors used in the reduction algorithm are *normalized*, then the least conservative robustness conditions result, where the error term is interpreted in a gap metric sense [19,46]. Unfortunately, constructing normalized coprime factors for structured LFT systems is quite difficult due to the spatial structure required of the allowable transformations we use. In this case, *contractive* and *expansive* coprime factors realizations may be considered instead; see [8] for a discussion relating to this topic.

In summary, we have presented a coprime factors approach to the reduction of LFT systems models. The proposed reduction method allows for the reduction of LPV and uncertain systems that are stabilizable and detectable in a specific multi-dimensional sense, but are *not strongly stable*. The focus of our efforts is concentrated most specifically on LPV systems, where some care must be taken if the somewhat restrictive assumption of strict causality, placed on the operators over which the system is parameterized, is to be relaxed. The reduction algorithm and resulting error bounds given herein may also be applied to spatial array systems [5], or to classical multi-dimensional systems [12], or to systems where all operators δ_i represent the same shift operator, but a structured partitioning of the system realization is induced due to topological constraints [25].

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