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Frequency Domain Model Reduction Method for Parameter-Dependent Systems

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Abstract— **In this paper a modification of a recently proposed model simplification method for linear time invariant parameterized models is presented. It is able to obtain models with explicit parameter dependence. The method is based on convex optimization with semidefinite constraints. Computation of the reduced model requires only the frequency samples of the full one, hence it is possible to apply the methods to large-scale models. Numerical examples showing the drawbacks and the advantages of the method are also presented.**

I. Introduction

Parameterized model order reduction is a well-known problem and reduced order models are required in many applications. For example, the design and simulation of radio frequency (RF) inductors, resistor inductance capacitor (RLC) lines, or micro-electromechanical resonators is expensive due to high-order models, obtained by discretization of integral-differential equations. The low-order models would significantly reduce the simulation time. There are many approaches in the literature to obtain low-order approximations. However, most of them fail to guarantee one or more of the desired properties: stability, passivity, rational fit, and so forth.

One of the most common approaches to parameterized reduction is moment matching which is based on Krylov subspace projection techniques. The major bottleneck is preserving stability of the reduced models. Preserving passivity, however, can be accomplished efficiently using the PRIMA algorithm ([1]) and its extension to parameterized reduction ([2]). Also only the moments of models are matched, therefore the overall ($\mathcal{H}_\infty$) performance can be far from the original model. Rational fit algorithms are also presented in the literature ([3], [4], [5]). However, there are problems with enforcing extra constraints without extra *a priori* information.

A recently proposed optimization-based algorithm ([6]) addresses these problems. It is based on matching of frequency samples, and thus provides a low order model close to the original one in $\mathcal{H}_\infty$ norm. With a convenient choice of decision variables, stability and passivity become convex constraints. However, the output of the method is a lookup table of transfer functions for every instance of parameters. An explicit parameterized model is hardly obtainable since multivariate spectral factorization of positive trigonometric polynomials generally does not have a solution in finite degree polynomials. For simplicity the quasi-convex optimization approach from [6] will be denoted as a QCO method.

In this paper a modification of [6] is proposed, that bypasses the spectral factorization problem. In order to do so, the search of reduced models is restricted to those with positive real denominators which also guarantees stability of the obtained model. This restriction may result in higher order approximations than ones obtained by the QCO method, but the obtained models will be simplified with respect to parameter dependence, as well. These issues are illustrated by applications, such as the reduction of a highly-resonant deformable mirror model and a construction of an RF inductor parameterized model. The method presented in this paper will be referred to as the Positive Real Denominator (PRD) method.

A number of applications of the approach can be specified. A straightforward application is reduction of linear multivariate (or multidimensional) models, i.e. instead of having one complex domain variable z, models have a vector of variables $z_1, \dots, z_n$. Here, only the bounded input bounded output stability can be guaranteed. The method is also applicable to identification of linear parameter varying (LPV) systems. The proposed method can be characterized as a local approach to LPV identification. More details may be found in [7].

The paper is organized as follows. In Sec. II the approach is presented. A discussion on the limitations and advantages is made in Sec. III. Sec. IV is devoted to implementation issues as obtaining finite-dimensional problem, computational complexity and stability constraint. The numerical examples comparing the QCO and the PRD methods are provided in Sec. V. We do not, however, compare the method to other techniques, since the objectives are generally different and a comparison of the QCO method to other approaches was performed in [6]. Finally, the extension to multi-input-multi-output models is performed in Appendix.

Notation

Notation $\mathcal{H}_\infty$ will be used to denote the space of discrete-time stable transfer functions. Operation $^\sim$ denotes an adjoint in $\mathcal{H}_\infty$ space: $G^\sim(z) = G^T(1/z)$. $G(\omega)$ stands for the frequency response $G(e^{j\omega})$ to $\omega \in [0, \pi]$. The infinity norm is computed as $\|G\|_\infty = \sup_w \overline{\sigma}(G(\omega))$, where $\overline{\sigma}(G(\omega))$

denotes the maximal singular value of $G(\omega)$. The unit circle in the complex plane is denoted as $\mathbb{D} = \{z \big| |z| = 1\}$.

II. A Positive Real Denominator Approach

A. Non-Parameterized Setup.

The non-parameterized model order reduction problem can be formulated as a minimization problem in $\mathcal{H}_\infty$ norm:

$$\min_{p,q} \left\| G - \frac{p}{q} \right\|_\infty \quad \text{with} \quad p = \sum_{i=0}^{k} p_i z^{-i}, q = \sum_{i=0}^{k} q_i z^{-i}$$

and q is a minimum phase transfer function.

where $q_i, p_i \in \mathbb{R}$ are the decision variables. Minimum phase condition means that all the poles and zeros of q lie inside the unit circle $\mathbb{D}$. Hence it is a stability constraint for p/q. However, this formulation is known to be not convex in decision variables (due to stability constraint in the first place) and there exists no polynomial time method to obtain an optimal solution. While one may consider a relaxation, as in [6], to obtain an approximation, we will go another way: make a restriction on q, that allows a convex problem formulation.

First reformulate the original problem, denoting as $G(\omega)$ the behavior of G on the unit circle $z = e^{\jmath\omega}$ with $\omega \in [0, \pi]$. A similar notation is made for p and q,

$$\begin{aligned} \min_{p,q,\gamma} \quad & \gamma \\ \text{subject to} \quad & |G(\omega)q(\omega) - p(\omega)| < \gamma |q(\omega)| \quad \forall \omega \in [0, \pi] \\ & q \text{ is minimum phase} \end{aligned}$$

Now replace the minimum phase condition with the strict positive realness. This restriction guarantees the asymptotic stability and, more importantly, it is convex in the decision variables. As the reader may notice the right hand side now is the only reason why the problem is not convex. If $q(\omega) > 0 \ \forall \omega$, then a second order cone program for every value of γ can be derived. Since the denominator q is already restricted to positive real functions, it seems reasonable to substitute $|q(\omega)|$ with $\mathrm{Re}(q(\omega))$, i.e. solve instead the following problem:

$$\min_{\gamma>0,p,q} \gamma \quad \text{s.t} \quad |G(\omega)q(\omega) - p(\omega)| < \gamma \mathrm{Re}(q(\omega)) \quad \forall \omega \quad (1)$$

This substitution is valid since the minimal γ will be an upper bound on the optimal solution of the original problem. This is true since $|\mathrm{Re}(q(\omega))| \leq |q(\omega)|$, for all q and ω.

A semidefinite formulation is also available, using the Schur complement:

$$\begin{aligned} &\min_{\gamma>0,p,q} \gamma \quad \text{subject to} \\ &\begin{pmatrix} \gamma \mathrm{Re}(q(\omega)) & G(\omega)q(\omega) - p(\omega) \\ * & \gamma \mathrm{Re}(q(\omega)) \end{pmatrix} > 0 \quad \forall \omega \in [0, \pi], \end{aligned}$$

where the asterisk stands for the Hermitian transpose of upper right corner. In realization the described constraints will be imposed only on a finite frequency grid. The most computationally heavy part will be calculating the frequency samples of the full model, which is relatively cheap for many systems (see, for details Sec. IV).

B. Parameterized Model Simplification.

In this setup simplification is performed in two directions: the order of the system is reduced, and the parameter dependence is simplified. Assume the original model G depends on a parameter $\theta = [\theta_1, \ldots, \theta_n]$ where $\underline{\theta}_i \leq \theta_i \leq \overline{\theta}_i$, $\forall i = 1, \ldots, n$. Now formulate the reduction problem as:

$$\min_{p,q} \max_{\theta} \left\| G(z_0, \theta) - \frac{p(z_0, \theta)}{q(z_0, \theta)} \right\|_\infty,$$

where $p(z_0, \theta) = \sum_{i=0}^{k_0} p_i(\theta) z_0^{-i}$, $q(z_0, \theta) = \sum_{i=0}^{k_0} q_i(\theta) z_0^{-i}$ and q is a minimum phase transfer function in z_0 variable (all the poles and zeros are inside the unit circle $\mathbb{D}$) for all values of θ.

Let p, q be polynomials both in θ and z_0. q has to be a (trigonometric) polynomial in θ_j (ω_j) in order to enforce the positivity constraints. One can assume, however, more general structures on p. Since every θ_j is bounded, re-parameterize it as $\theta_j = \frac{\overline{\theta}_j + \underline{\theta}_j}{2} + \frac{\overline{\theta}_j - \underline{\theta}_j}{2} \cos(\omega_j)$, where $\omega_j \in [0, \pi], \forall j = 1, \ldots, n$. Thus obtain multivariate trigonometric (pseudo) polynomials:

$$\begin{aligned} q(\boldsymbol{\omega}) &= \sum_{i_0=0}^{k} q_{i_0}(\omega_1, \ldots, \omega_n) e^{-\mathrm{j}\omega_0 i_0}, \\ p(\boldsymbol{\omega}) &= \sum_{i_0=0}^{k} p_{i_0}(\omega_1, \ldots, \omega_n) e^{-\mathrm{j}\omega_0 i_0}, \quad \text{where} \end{aligned}$$

$q_{i_0} = \sum_{\mathbf{i}=\mathbf{0}}^{\mathbf{k}} q_{i_0,\mathbf{i}} \prod_{j=1}^{n} \cos(i_j \omega_j)$, $p_{i_0} = \sum_{\mathbf{i}=\mathbf{0}}^{\mathbf{k}} p_{i_0,\mathbf{i}} \prod_{j=1}^{n} \cos(i_j \omega_j)$, and multi-indexes $\mathbf{i} = \{i_1, \ldots, i_n\}$, $\mathbf{k} = \{k_1, \ldots, k_n\}$.

Again slightly abuse the notation by denoting $p(\boldsymbol{\omega})$ the frequency response of p to $\boldsymbol{\omega} = \{\omega_0, \ldots, \omega_n\}$. Following the same pattern as in the non-parameterized case, a semidefinite program is obtained:

$$\begin{aligned} &\min_{\gamma>0,p,q} \gamma \quad \text{subject to} \\ &\begin{pmatrix} \gamma \mathrm{Re}(q(\boldsymbol{\omega})) & G(\boldsymbol{\omega})q(\boldsymbol{\omega}) - p(\boldsymbol{\omega}) \\ * & \gamma \mathrm{Re}(q(\boldsymbol{\omega})) \end{pmatrix} > 0 \quad \forall \boldsymbol{\omega} \in [0, \pi]^{n+1} \end{aligned} \quad (2)$$

III. Theoretical Limitations and Advantages

The main limitation when applying this method is a quite restrictive property $\mathrm{Re}(q(\omega)) > 0$. The property implies that the phase of the denominator is restricted to the interval $[-\pi/2, \pi/2]$. It is possible to construct numerical examples, where the proposed method fails to fit the original model, without any reduction, i.e. the order of approximation is the same as the one of the original model. The main attribute of such models is a fast change in phase of G. However, one can always approximate a rational function with an FIR filter of sufficiently big order. And an FIR filter, in its turn, can always be approximated by the PRD method (simply by letting $q \equiv 1$). Basically by raising the order of approximation one can always achieve the desired quality. For the simplification of large scale models it means that the reduced model will be of a higher degree, than one provided by QCO, for example. This issue will be illustrated by examples.

There is another approach to address this restriction. It is clear that the poles and zeros near the unit circle have a great impact on frequency response in the form of peaks and troughs in magnitude. Surely in the approximation these poles and zeros should be preserved. Hence it seems to be natural to keep them out of the reduction procedure and reduce only the rest of the model. Thus first identify the dominant poles and zeros of the system G (for example, by method from [9]). Denote $G_d(z,\theta)$ the transfer function with all the dominant poles and zeros. Finally, apply the method to,

$$\min_{\hat{G}} \|G - G_d\hat{G}\|_\infty$$

While using this approach for parameterized reduction the algorithm becomes cheaper. Identification of these poles and zeros reduces the dimensions of the positivity constraint of $\mathrm{Re}(q)$ (this constraint is used to guarantee stability of the reduced model, for details see Sec. IV). It is also easier to impose the stability constraint on G_d, since each pole is considered individually. Surely the overall accuracy may be an issue, thus there is a need in an accurate procedure to identify the dominant part of the model. Also it should be noted, that it is not always possible to identify those or they do not exist. Basically, the decision of such a identification is left to user and made by inspection of the frequency response.

When the PRD method was applied to parameter-dependent models a major advantage was noticed. The parameterization of the PRD method is more efficient than the one of QCO. Recall, that the latter method solves the problem:

$$\min_{a>0,b} \left\| G - \frac{b}{a} \right\|_\infty \text{ where } \quad a = qq^\sim \quad \frac{b}{a} = \frac{p}{q} + \frac{r}{q^\sim}$$

where p and q are polynomials or order k, r is a polynomial of order $k-1$ and the reduced model is found as p/q. Thus if p and q are parameterized as polynomials of degree $\mathbf{k} = \{k_0,\dots,k_n\}$, then a, b should have to be at least of order $2\mathbf{k} = \{2k_0,\dots,2k_n\}$ to describe the same the set of model. The PRD method requires the polynomials of order $\mathbf{k} = \{k_0,\dots,k_n\}$. The number of decision variables related to the positivity constraint of a and $\mathrm{Re}(q)$ is the same and only the parameterized part of polynomials is affected.

IV. Implementation

The main idea of implementation is to impose the constraints only in a finite number of points, since calculating the frequency samples and imposing the same number of LMI constraints is significantly cheaper than imposing positivity constraint for all frequencies and parameters,

$$\begin{aligned} &\min_{p,q,\gamma_{\mathbf{N}}} \gamma_{\mathbf{N}} \quad \text{subject to} \quad \mathrm{Re}(q(\boldsymbol{\omega})) > 0 \quad \forall \boldsymbol{\omega} \\ &\begin{pmatrix} \gamma_{\mathbf{N}}\mathrm{Re}(q(\boldsymbol{\omega_i})) & G(\boldsymbol{\omega_i})q(\boldsymbol{\omega_i}) - p(\boldsymbol{\omega_i}) \\ * & \gamma_{\mathbf{N}}\mathrm{Re}(q(\boldsymbol{\omega_i})) \end{pmatrix} > 0 \\ &\forall \{\boldsymbol{\omega_i}\}_{\mathbf{i}=0}^{\mathbf{N}} \in [0,\pi]^{n+1}, \end{aligned} \tag{3}$$

where $\mathbf{N} = \{N_0,\dots,N_n\}$. The frequency grid is chosen such that $\{\boldsymbol{\omega_i}\}_{\mathbf{i}=0}^{\mathbf{N}}$ are the first $|\mathbf{N}| = \sum_{i=0}^{n} N_i$ elements of the dense sequence $\{\boldsymbol{\omega_i}\}$ in $[0,\pi]^{n+1}$ (for example, some rational numbers). When $|\mathbf{N}| \to \infty$, (by $|\mathbf{N}| \to \infty$ we mean that $\forall i \quad N_i \to \infty$) given the smoothness of all the functions in (3), it is fairly obvious that $\lim_{|\mathbf{N}|\to\infty} \gamma_{\mathbf{N}} = \gamma$, where γ is the solution to (2).

A. Stability Constraint

While the frequency axis griding is sufficient for the frequency sample matching in (3), it is not sufficient for the condition $\mathrm{Re}(q(\boldsymbol{\omega})) > 0$ in order guarantee stability of the reduced model. The most efficient way to impose the latter constraint for multivariate polynomials is to replace it with a sum of squares constraint. As a justification the following theorem is used:

Theorem 4.1 ([10]): Let $a(z_0,\dots,z_n)$ be a multivariate Laurent polynomial which is strictly positive on the multivariate torus $|z_0| = \cdots = |z_n| = 1$, where $n \geq 1$ is an integer. Then $a(z_0,\dots,z_n)$ can be expressed as a sum of square of magnitudes of polynomials in $z_0,\dots,z_n$

In our case the polynomial $\mathrm{Re}(q)$ can be interpreted as a Laurent polynomial a which satisfies the conditions of the theorem. For more details about the problem, see [11]. Therein sufficient positivity condition of multivariate polynomials is also described. These conditions are imposed on the coefficients of the polynomial. The resulting constraints are semidefinite in nature with $\Pi_{i=0}^{n}(k_i+1)^2$ variables, where k_i is the order of the polynomial corresponding to the z_i variable. A slightly computationally cheaper version can be found in [12, Chapter 3]. The number of variables is halved and the constraints are also semidefinite.

B. Numerical Complexity

There are two main contributors to the complexity of the PRD method. The first one is computing the frequency samples which is $O(m^3)$ for each frequency point, with m being the order of the original model. In many cases it can be lowered to $O(m\log(m))$ even for the dense models (for example, [13]). The second one is the cost of the optimization algorithm. It does not exceed $O(N_1^2N_2^2)$ with N_1 being the number of decision variables and N_2 the number of constraints, when the method is implemented with SeDuMi. In non-parameterized reduction $N_1 = O(k^2)$ (if the stability constraints are imposed with KYP lemma) and $N_2 = 2N$, where $k << m$ is the order of the reduced model and N is the number of points in the grid.

In parameterized reduction the positive real constraint adds many more decision variables. Indeed the number of variables N_1 will be $O((k_0+1)\cdots(k_n+1))^2$. So the computational complexity rises significantly with the order of the reduced system, thus being the heaviest part. The number of constraints is the same. Then the total cost will be $O(m\log(m)N) + O(((k_0+1)\cdots(k_n+1))^4N^2)$.

The conventional PRIMA method ([1]) has computational complexity $O(m^2)$, since the most computationally heavy part, an orthogonal basis of the Krylov subspace, requires $O(m^2)$ floating point operations for calculating every column of the basis. In the parameterized case, the parameters have

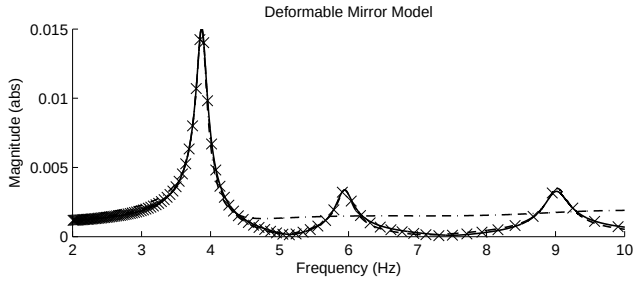

Fig. 1. Non-parameterized reduction. Frequency responses of models. Solid - reduced by PRD to order 15, dotted - reduced by QCO to order 10 (coincides with the solid line), dash-dotted - reduced by PRD to order 10, crosses - original frequency data.

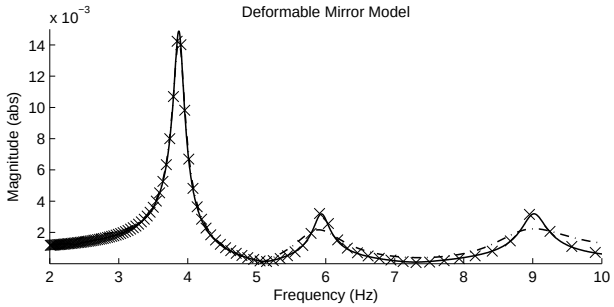

Fig. 2. Parameterized Reduction. Frequency responses of models. Approximations orders $k = [15, 1]$, Dash-dotted - reduced by QCO, solid - reduced by PRD, crosses - original frequency data. The friction coefficient $\alpha = 0.02$

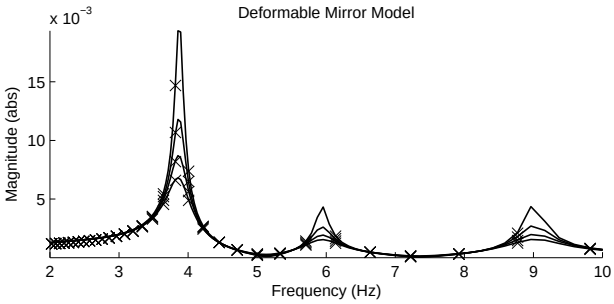

Fig. 3. Parameterized Reduction by PRD using dominant pole algorithm. Frequency responses of models. Approximations orders $k = [10, 2]$, The friction coefficient $\alpha = 0.015,\ 0.025,\ 0.035,\ 0.045$.

to be taken into account. Then the total cost is equal to $O(m^2(k_0+1)\cdots(k_n+1))$. Note, however, that the passivity will not be enforced for all instances of parameters.

V. Numerical Examples

In our implementation the optimization problems in (3) were solved using a cutting plane method (see, [14]). Recall that the described method addresses the reduction of discrete-time systems, therefore in order to reduce a continuous time model it is discretized using a bi-linear transformation. After the reduced model is obtained it is mapped back to continuous time.

A. Reduction of a Deformable Mirror Model.

The following model was studied in [15] and obtained by means of a finite element modeling approach that resulted in a system of second-order differential equations:

$$M\ddot{x} + D\dot{x} + Kx = Fu,$$

where matrices M, K, D have high dimensions which results in a state-space model with more than 20000 states. In [15] a heuristic reduction have been performed and the obtained model has 420 sensors and actuators and 2000 states. In the process the system becomes mass-normalized, i.e. $M = I$, and $K = \Lambda^2$, where Λ is a diagonal matrix. Damping is chosen to be a modal one, i.e. $D = \alpha\Lambda$, where $\alpha = 0.02$. Only one entry of the transfer function will be investigated in the frequency range $[0, 60\text{Hz}]$.

Non-parameterized reduction. The QCO and PRD methods are implemented on the frequency grid with 100 samples. In both cases the approximations were obtained in less than 2 minutes (including the calculation of the frequency response). For comparison, Hankel model reduction took around 30 minutes to compute the approximation. It was possible to reduce the model to order 10 on the mentioned frequency interval by means of the QCO method. The PRD method did not provide a good 10-th order approximation on this interval. Supposedly the restriction on the denominator is activated in this example. Nevertheless, by letting the reduction order be 15, the same quality approximation is achieved as in the case of the 10-th order QCO approximation (see frequency responses of the approximations in Fig. 1).

Parameterized reduction. Consider the coefficient α that defines friction. Let α take values $0.01,\ 0.02,\ 0.03,\ 0.04,\ 0.05$. The orders of reduction are set to $k_0 = 15,\ k_1 = 1$. The order for parameter dependence is chosen to be 1, since a slight change in the friction coefficient will not affect the frequency response radically. The results are somewhat surprising. The QCO method could not provide a matching frequency response, even though in the non-parameterized case the order 10 was sufficient to do that. The PRD, however, did better. The overall error for PRD is around 1% and the QCO 5%. Percent are measured with respect to $\|G\|_\infty$ (see frequency responses of the approximations in in Fig. 2).

Identifying the dominant part of the model. After a careful consideration it was decided to identify 6 dominant poles and zeros and remove them from the modeling procedure i.e. those, that correspond to peaks shown in all the figures. The dominant poles are identified by running the non-parameterized QCO algorithm for every parameter in the training grid $\alpha = 0.01,\ 0.02,\ 0.03,\ 0.04,\ 0.05$ and setting the order of approximation to 15. The obtained approximations had errors around $10^{-6}\%$. Also assume, that poles are not intersecting. Here it is a valid assumption, since the parameter dependence is not complicated. However, in general one should keep in mind this problem.

The frequency responses of the resulting approximations with the order of the non-dominant part $k_n = [4, 0]$ (which yields the overall order of approximation $k = [10, 2]$) are

shown in Fig. 3. The overall error is around 1%, so we did not lose anything in accuracy. The parameter dependence is transferred to the dominant poles and zeros. The actual dependence on the parameter is linear thus extra accuracy is not gained by this transfer.

From the above, it is safe to say that the PRD method should be mainly considered in a parameterized framework. In non-parameterized reduction the PRD method has no advantages in comparison with the QCO method.

B. Modeling an RF Inductor

A spiral four-turn RF-Inductor is described by an integral-differential system of equations (can be found in [16]). Two parameters are considered: wire separation D and wire width W. The system is then discretized and the obtained system depends on parameters in a non-linear fashion. The usual way to evaluate the performance of inductors is by using the quality factor and the inductance of inductor:

$$L_{ind} = \frac{\mathrm{Im}(G_{ind})}{\omega}, \quad Q_{ind} = \frac{\mathrm{Im}(G_{ind})}{\mathrm{Re}(G_{ind})},$$

where G_{ind} is the transfer function of the inductor, which has the meaning of impedance of the inductor. One could introduce the corresponding constraints on quality factor and inductance into the program. We, however, will approximate the transfer functions and see how it affects the quality factor and inductance.

This example was considered in [14], where the QCO method was applied to obtain reduced order models and compare the effects of reduction on the quality factor and inductance. Here we check the performance of the PRD method in a similar manner. The training grid is chosen as $W = [1, 2, 3, 4, 5]\mu m, D = [1, 2, 3, 4, 5]\mu m$ and, for every instance of parameters, 25 frequency samples are calculated. The validation grid is chosen as $W = [1.5, 2.5, 3.5, 4.5]\mu m, D = [1.5, 2.5, 3.5, 4.5]\mu m$ and, for every instance of parameters, 20 frequency samples are computed. The reduced model is calculated using the data on the training grid only. The order of the reduced models is set to $k_0 = 5, \quad k_1 = 4, \quad k_2 = 4$.

QCO approximation procedure took about 7 hours and the lower bound was $2.2652 \cdot 10^{-3}\%$. The PRD method provided the reduced model with the upper bound $2.5905 \cdot 10^{-3}\%$. The optimization took about 3 hours. For the non-parameterized version of QCO the actual error usually is very close to the lower bound, thus performance of the methods on the training grid is expected to be similar.

The approximations on the training grid, i.e. in the points where the constraints were enforced, are not really surprising. Both PRD and QCO approximations give almost an exact fit in quality factor and inductance, which is expected given the good transfer function approximation in the training points.

The most interesting is comparing the data on the validation grid, i.e. in the points where the constraints were not enforced. The quality factor matching was good for both methods and therefore the comparison is omitted. Comparing the inductance for different values of parameters (Fig. 4 and 5) shows a considerable difference between the PRD approximation (solid line) and the QCO (dashed line). We can also notice, that the PRD approximation follows the validation data, which is represented by crosses, better than the QCO one.

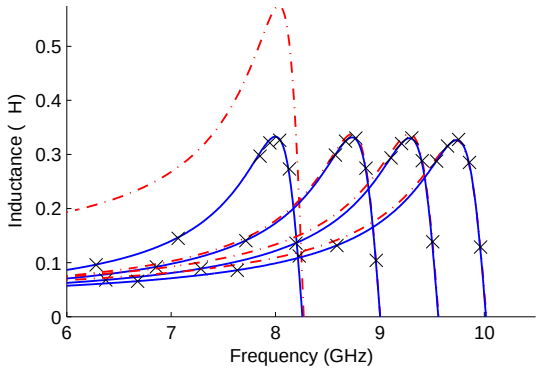

Fig. 4. Inductance on the validation grid for $W = 1.5\mu m$, $D = 1.5$, 2.5, 3.5, $4.5\mu m$. Crosses represent characteristics of the original model, solid lines - the PRD approximations, dash-dotted lines - the QCO ones.

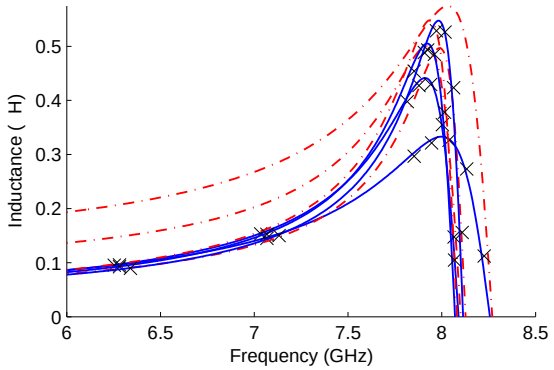

Fig. 5. Inductance on the validation grid for $D = 1.5\mu m$, $W = 1.5$, 2.5, 3.5, $4.5\mu m$. Crosses represent characteristics of the original model, solid lines - the PRD approximations, dash-dotted lines - the QCO ones.

In both figures we can also notice that the QCO method does not provide a better approximation, although there is no restriction of the denominator and there is no explicit transfer function dependence. The reduced model obtained by the PRD method clearly has more advantages in comparison with the one obtained by the QCO method. A further investigation was performed in [17, Chapter 6].

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VII. Conclusion

A new method for model reduction of linear systems was presented. The advantages of the method can be seen only

in the parameterized setup, e.g. we can obtain an explicit parameter-dependent model or simplify the parameter dependence. Also the parameterization is more efficient than in analogous methods, e.g. [14]. The limitations of the introduced algorithm are also addressed, providing a heuristic solution to bypass them. Known advantages and drawbacks of the method were illustrated on industrial models.

APPENDIX

A. Multivariable Extension

Define the reduced model as PQ^{-1}, where $P = \sum_{i=0}^{k} P_i z^{-i}, Q = \sum_{i=0}^{k} Q_i z^{-i}$, $P_i \in \mathbb{R}^{m_1 \times m_2}$, $Q_i \in \mathbb{R}^{m_2 \times m_2}$ for all $i = 1, \ldots, k$, i.e. the model has m_2 inputs and m_1 outputs, and $Q(z)$ is strictly positive real. Depending on the problem one can also parameterize $\hat{G} = Q^{-1}P$. In this case $Q_i \in \mathbb{R}^{m_1 \times m_1}$.

In the matrix case the real and imaginary parts of Q are uniquely defined as $\mathrm{Re}(Q(\omega)) = (Q(\omega) + Q(\omega)^{\sim})/2$, and $\mathrm{Im}(Q(\omega)) = (Q(\omega) - Q(\omega)^{\sim})/(2j)$, and thus are Hermitian.

A multivariable extension can be performed as in [8], and a semidefinite program can be obtained:

$$\begin{aligned} &\min_{Q,P,f,\gamma} \gamma \quad \text{subject to} \quad f(\omega)I \leq \mathrm{Re}(Q(\omega)) \\ &\begin{pmatrix} \gamma f(\omega)I & G(\omega)Q(\omega) - P(\omega) \\ * & \gamma \mathrm{Re}(Q(\omega)) \end{pmatrix} > 0, \quad \forall \omega \in [0, \pi] \end{aligned} \tag{4}$$

where $f \in \mathcal{L}_\infty[0, \pi]$ is a scalar measurable function and it is obtained from the optimization procedure.

A justification for this extension follows from the next lemma:

Lemma 1.1: Assume G, P, Q are transfer functions, f is a measurable function and γ is a positive scalar. Then the following statement holds:

$$\left.\begin{aligned} &f(\omega)I \leq \mathrm{Re}(Q(\omega)) \\ &\begin{pmatrix} \gamma f(\omega)I & G(\omega)Q(\omega) - P(\omega) \\ * & \gamma \mathrm{Re}(Q(\omega)) \end{pmatrix} > 0 \quad \forall \omega \in [0, \pi] \end{aligned}\right\} \Rightarrow$$
$$\Rightarrow \|G - PQ^{-1}\|_\infty \leq \gamma$$

Proof:

$$\begin{pmatrix} \gamma f I & GQ - P \\ * & \gamma \mathrm{Re}(Q) \end{pmatrix} > 0 \overset{(a)}{\Longrightarrow}$$
$$\gamma^2 f I > (GQ - P)(\mathrm{Re}(Q))^{-1}(GQ - P)^{\sim} \overset{(b)}{\Rightarrow}$$
$$\gamma^2 I > (GQ - P)(\mathrm{Re}(Q)f)^{-1}(GQ - P)^{\sim} \overset{(c)}{\geq}$$
$$\geq (GQ - P)(Q^{\sim}Q)^{-1}(GQ - P)^{\sim} =$$
$$= (G - PQ^{-1})(G - PQ^{-1})^{\sim}$$

The conjecture (a) is valid due to the Schur complement. (b) is trivial, when f is scalar. To prove (c) one has to show that $\mathrm{Re}(Q)f \leq Q^{\sim}Q$. Consider inequality, that is valid for all ω :

$$(Q - fI)^{\sim}(Q - fI) \geq 0$$

Now with simple transformations the result is achieved, since the inequality $\mathrm{Re}(Q) \geq fI$ is one of the conditions of this lemma.

$$Q^{\sim}Q - \underbrace{(fQ + fQ^{\sim})}_{=2f\mathrm{Re}(Q)} + f^2 I \geq 0$$
$$Q^{\sim}Q - f\mathrm{Re}(Q) \geq f(\mathrm{Re}(Q) - fI) \geq 0$$

■

Remark 1.1: The lemma also gives insights into the multivariable extension. The conjecture (b) is true if f is scalar. Thus explaining the introduction of this restriction.

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