Antennas and Wave Propagation

Lecture 3:
Basic Electromagnetic Analysis

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Previous Lecture

1. Radiation patterns
2. Radiation power density
3. Radiation intensity
4. Beam efficiency
5. Beam width
6. Antenna efficiency
7. Directivity and gain
8. Polarization
9. Bandwidth
10. Input impedance
11. Radiation efficiency

Outline

Maxwell's equations

Power density and Poynting vector

Vector Potentials

Wave Equations

Radiation Boundary Condition (RBC)

Ideal (Hertzian) Dipole

Antenna Theory Problems

Analysis Problem

Given an antenna structure or source current distribution, how does the antenna radiate?

Focus of this course (more mature and developed)

Synthesis problem

Given the desired operational characteristics (like the radiation pattern), find the antenna structure or source current that will generate this.

Challenging! Much less developed.

Topic for research.

General Antenna Analysis

Problem Statement

Given:

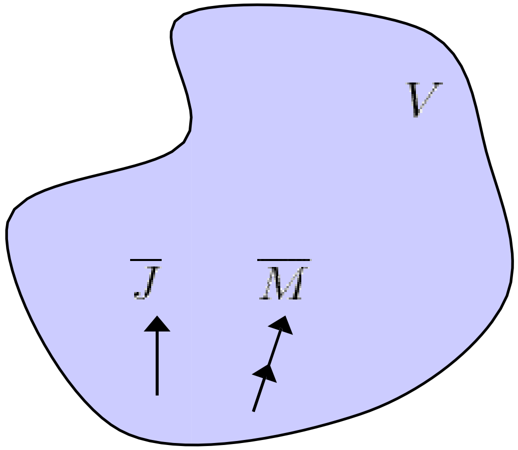
Arbitrary volume *V*

Filled with sources

$\overline{J}$ = electric currents (A/m²)

$\overline{M}$ = magnetic currents (V/m²)

(a current composed of fictitious moving magnetic monopoles.)



Fields:
$\overline{E}$ $\overline{H}$

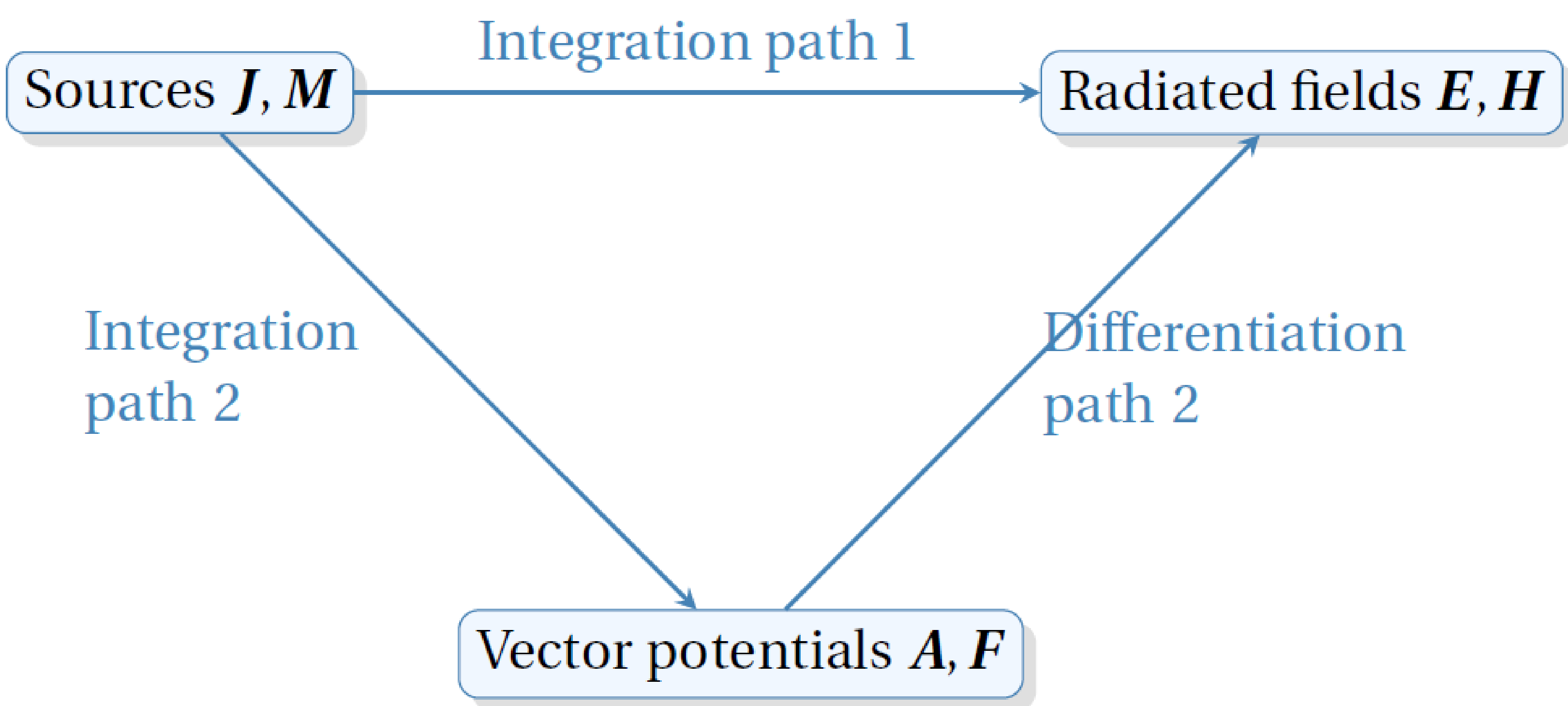
Problem:

Compute fields E and H generated by currents

Solution:

Maxwell's equations gives exact solution

General Antenna Analysis



General Antenna Analysis

The Divergence

The basic definition of divergence is

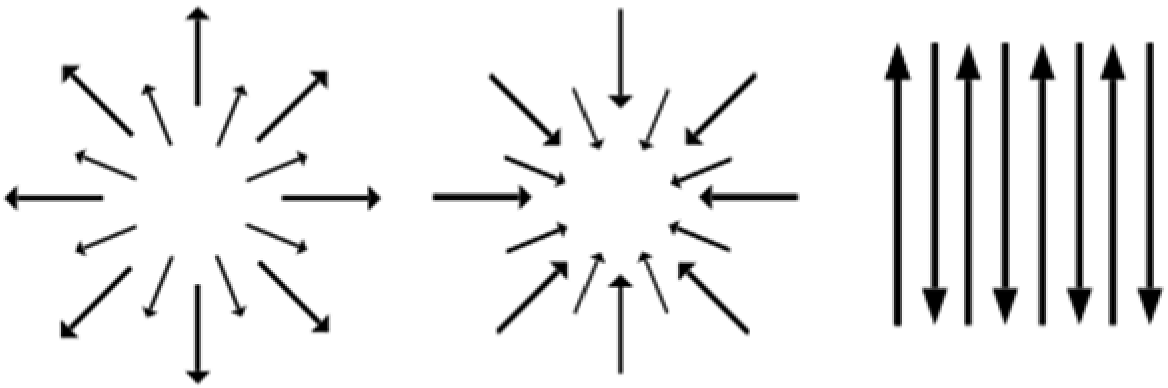
$$\nabla \cdot \boldsymbol{A} = \lim_{\Delta v \to 0} \left[\frac{\oint_S \boldsymbol{A} \cdot d\boldsymbol{S}}{\Delta v} \right].$$

The expansion of divergence in Cartesian coordinates is

$$\nabla \cdot \boldsymbol{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}.$$

General Antenna Analysis

Physical Meaning of Divergence



$\partial/\partial x(\mathbf{A}_x) > 0$	$\partial/\partial x(\mathbf{A}_x) < 0$	$\partial/\partial x(\mathbf{A}_x) = 0$
$\partial/\partial y(\mathbf{A}_y) > 0$	$\partial/\partial y(\mathbf{A}_y) < 0$	$\partial/\partial y(\mathbf{A}_y) = 0$
$\nabla \cdot (\mathbf{A}) > 0$	$\nabla \cdot (\mathbf{A}) < 0$	$\nabla \cdot (\mathbf{A}) = 0$

In **physical** terms, the **divergence** of a vector field is the extent to which the vector field flux behaves like a source at a given point. It is a local measure of its "outgoingness" – the extent to which there is more of the field vectors exiting an infinitesimal region of space than entering it.

General Antenna Analysis

The Curl

The basic definition of curl is

$$\nabla \times \boldsymbol{A} = \lim_{\Delta S \to 0} \left[\frac{\oint_C \boldsymbol{A} \cdot d\boldsymbol{l}}{\Delta S} \right]_{\max} \boldsymbol{i}_n .$$

The expansion of curl in Cartesian coordinates is

$$\nabla \times \boldsymbol{A} = \begin{vmatrix} \boldsymbol{i}_x & \boldsymbol{i}_y & \boldsymbol{i}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} .$$

General Antenna Analysis

The Curl

The basic definition of curl is

$$\nabla \times \boldsymbol{A} = \lim_{\Delta S \to 0} \left[\frac{\oint_C \boldsymbol{A} \cdot d\boldsymbol{l}}{\Delta S} \right]_{\max} \boldsymbol{i}_n.$$

The expansion of curl in Cartesian coordinates is

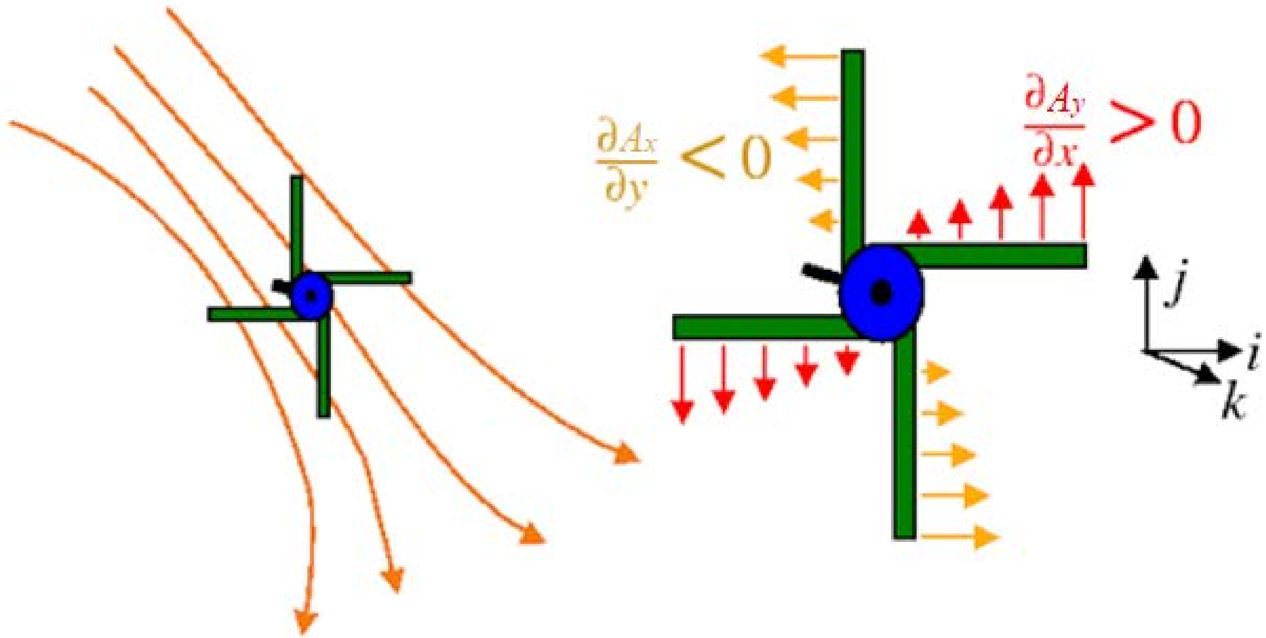
$$\nabla \times \mathbf{A} = \hat{\mathbf{a}}_x \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) + \hat{\mathbf{a}}_y \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) + \hat{\mathbf{a}}_z \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right)$$

$$|A_x \quad A_y \quad A_z|$$

General Antenna Analysis

Video 10’27”

Physical Meaning of Curl



a turning **paddle wheel** indicates that the field is "uneven" and not symmetric

Curl is a measure of how much a vector field circulates or rotates about a given point. when the flow is counter-clockwise, **curl** is considered to be positive and when it is clock-wise, **curl** is negative.

General Antenna Analysis

Vector Identities

div curl = 0 $\quad$ $\nabla \cdot \nabla \times \boldsymbol{A} = 0$

curl grad = 0 $\quad$ $\nabla \times \nabla \phi = 0$

Electromagnetic Quantities

E electric field (V/m)

H magnetic field (A/m)

B magnetic flux density (tesla, T or N/m/A)

D electric displacement, also known as electric flux density, (C/m^2)

J electric current density (A/m^2)

M magnetic current density (V/m^2)

ρ electric charge density (C/m^3)

ε permittivity (F/m)

μ permeability (H/m)

η Characteristic impedance

Electromagnetic Quantities

E electric field (V/m)

An **electric field** surrounds an electric charge, and exerts force on other charges in the field, attracting or repelling them.

H magnetic field (A/m)

B magnetic flux density (tesla, T or N/m/A)

D electric displacement, also known as electric flux density, (C/m^2)

J electric current density (A/m^2)

M magnetic current density (V/m^2)

ρ electric charge density (C/m^3)

ε permittivity (F/m)

μ permeability (H/m)

η Characteristic impedance

Electromagnetic Quantities

E electric field (V/m)

H magnetic field (A/m)

A vector field in the neighborhood of a magnet, electric current, or changing electric field, in which magnetic forces are observable.

B magnetic flux density (tesla, T or N/m/A)

D electric displacement, also known as electric flux density, (C/m^2)

J electric current density (A/m^2)

M magnetic current density (V/m^2)

ρ electric charge density (C/m^3)

ε permittivity (F/m)

μ permeability (H/m)

η Characteristic impedance

Electromagnetic Quantities

E electric field (V/m)

H magnetic field (A/m)

B magnetic flux density (tesla, T or N/m/A)

A measurement of the total **magnetic** field which passes through a given area. It is a useful tool for helping describe the effects of the **magnetic** force on something occupying a given area. The measurement of **magnetic flux** is tied to the particular area chosen.

D electric displacement, also known as electric flux density, (C/m^2)

J electric current density (A/m^2)

M magnetic current density (V/m^2)

ρ electric charge density (C/m^3)

ε permittivity (F/m)

μ permeability (H/m)

η Characteristic impedance

Electromagnetic Quantities

E electric field (V/m)

H magnetic field (A/m)

B magnetic flux density (tesla, T or N/m/A)

D electric displacement, also known as electric flux density, (C/m^2)

The charge per unit area that would be **displaced** across a layer of conductor placed across an **electric** field. This describes also the charge density on an extended surface that could be causing the field.

J electric current density (A/m^2)

M magnetic current density (V/m^2)

ρ electric charge density (C/m^3)

ε permittivity (F/m)

μ permeability (H/m)

η Characteristic impedance

Electromagnetic Quantities

E electric field (V/m)

H magnetic field (A/m)

B magnetic flux density (tesla, T or N/m/A)

D electric displacement, also known as electric flux density, (C/m^2)

J electric current density (A/m^2)

A vector whose magnitude is the **electric current** per cross-sectional area at a given point in space, its direction being that of the motion of the charges at this point.

M magnetic current density (V/m^2)

ρ electric charge density (C/m^3)

ε permittivity (F/m)

μ permeability (H/m)

η Characteristic impedance

Electromagnetic Quantities

E electric field (V/m)

H magnetic field (A/m)

B magnetic flux density (tesla, T or N/m/A)

D electric displacement, also known as electric flux density, (C/m^2)

J electric current density (A/m^2)

M magnetic current density (V/m^2)

A current composed of **fictitious** moving magnetic monopoles through an area. It is used for simplification of certain electromagnetic field problems.

ρ electric charge density (C/m^3)

ε permittivity (F/m)

μ permeability (H/m)

η Characteristic impedance

Electromagnetic Quantities

E electric field (V/m)

H magnetic field (A/m)

B magnetic flux density (tesla, T or N/m/A)

D electric displacement, also known as electric flux density, (C/m^2)

J electric current density (A/m^2)

M magnetic current density (V/m^2)

ρ electric charge density (C/m^3)

charge density is a measure of **electric charge** per unit volume of space, in one, two or three dimensions.

ε permittivity (F/m)

μ permeability (H/m)

η Characteristic impedance

Electromagnetic Quantities

E electric field (V/m)

H magnetic field (A/m)

B magnetic flux density (tesla, T or N/m/A)

D electric displacement, also known as electric flux density, (C/m^2)

J electric current density (A/m^2)

M magnetic current density (V/m^2)

ρ electric charge density (C/m^3)

ε permittivity (F/m)

a measure of the electric polarizability of a dielectric. A material with high permittivity polarizes more in response to an applied electric field than a material with low permittivity, thereby storing more energy in the electric field.

μ permeability (H/m)

η Characteristic impedance

Electromagnetic Quantities

- E electric field (V/m)
- H magnetic field (A/m)
- B magnetic flux density (tesla, T or N/m/A)
- D electric displacement, also known as electric flux density, (C/m^2)
- J electric current density (A/m^2)
- M magnetic current density (V/m^2)
- ρ electric charge density (C/m^3)
- ε permittivity (F/m)
- μ permeability (H/m)

the measure of the resistance of a material against the formation of a magnetic field. Hence, it is the degree of magnetization that a material obtains in response to an applied magnetic field.

- η Characteristic impedance

Electromagnetic Quantities

E electric field (V/m)

H magnetic field (A/m)

B magnetic flux density (tesla, T or N/m/A)

D electric displacement, also known as electric flux density, (C/m^2)

J electric current density (A/m^2)

M magnetic current density (V/m^2)

ρ electric charge density (C/m^3)

ε permittivity (F/m)

μ permeability (H/m)

η Characteristic impedance

In transmission line theory, characteristic impedance of a uniform transmission line is the ratio of the amplitudes of voltage and current of a single wave propagating along the line; that is, a wave travelling in one direction in the absence of reflections in the other direction.

Electromagnetic Quantities

E electric field (V/m)

H magnetic field (A/m)

B magnetic flux density (tesla, T or N/m/A)

D electric displacement, also known as electric flux density, (C/m^2)

J electric current density (A/m^2)

M magnetic current density (V/m^2)

ρ electric charge density (C/m^3)

ε permittivity (F/m)

μ permeability (H/m)

η Characteristic impedance

The characteristic impedance of free space is equal to the square root of the ratio of permeability of free space (in henrys per meter) to the permeability of free space (in farads per meter). It works out to about 377 Ω, and that is the characteristic impedance of the universe.

$$\eta = \sqrt{\frac{\mu}{\epsilon}} \text{ (Ohm)}$$

Maxwell's Equations

Equations (Differential Form)

$$\nabla \cdot \overline{D} = \rho_v$$ Gauss' Law (electric field)

$$\nabla \cdot \overline{B} = 0$$ Gauss' Law (magnetic field)

$$\nabla \times \overline{E} = -\frac{\partial \overline{B}}{\partial t} - \overline{M}$$ Faraday's Law

$$\nabla \times \overline{H} = \frac{\partial \overline{D}}{\partial t} + \overline{J}$$ Ampere's Law

Constitutive Relationships

$$\overline{D} = \overline{\overline{\epsilon}}\, \overline{E}$$

$$\overline{B} = \overline{\overline{\mu}}\, \overline{H}$$

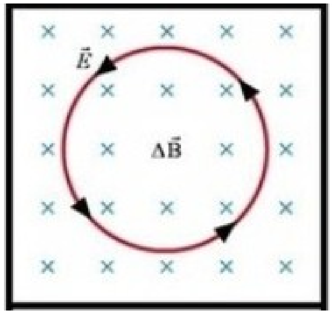
Maxwell's Equations

(a) the law of induction (Faraday's law):

$$\boxed{-\nabla \times \mathbf{E} = \frac{\partial \mathbf{B}}{\partial t} + \mathbf{M}}^{\clubsuit} \qquad (2.1)$$

$$\oint_c \mathbf{E} \cdot d\mathbf{c} = -\frac{\partial}{\partial t} \iint_{S_{[c]}} \mathbf{B} \cdot d\mathbf{s} \quad \Leftrightarrow \quad e = -\frac{\partial \Psi}{\partial t} \qquad (2.1\text{-i})$$

$\mathbf{E}$ (V/m)	electric field (electric field intensity)
$\mathbf{B}$ (T=Wb/m^2)	magnetic flux density
$\mathbf{M}$ (V/m^2)	magnetic current density$^{\clubsuit}$
Ψ (Wb=V·s)	magnetic flux
e (V)	electromotive force



- *Faraday – Lenz Law:*
  A time varying B field induces a rotation in E field

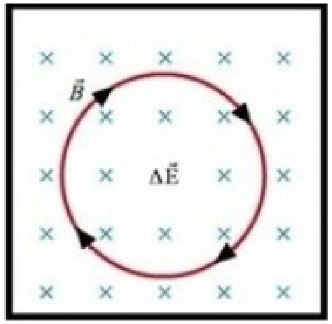
Maxwell's Equations

(b) Ampere's law, generalized by Maxwell to include the displacement current $\partial \mathbf{D} / \partial t$:

$$\boxed{\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{J}} \qquad (2.2)$$

$$\oint_c \mathbf{H} \cdot d\mathbf{c} = \iint_{S_{[c]}} \left(\frac{\partial \mathbf{D}}{\partial t} + \mathbf{J} \right) \cdot d\mathbf{s} \quad \Leftrightarrow \quad I = \oint_c \mathbf{H} \cdot d\mathbf{c} \qquad (2.2\text{-i})$$

$\mathbf{H}$ (A/m) magnetic field (magnetic field intensity)
$\mathbf{D}$ (C/m^2) electric flux density (electric displacement)
$\mathbf{J}$ (A/m^2) electric current density
I (A) electric current



- *Ampère – Maxwell Law:* A time varying E field induces rotation in B field

Maxwell's Equations

(c) Gauss' electric law:

$$\nabla \cdot \mathbf{D} = \rho \tag{2.3}$$

$$\oiint_S \mathbf{D} \cdot d\mathbf{s} = \iiint_{V[s]} \rho dv = Q \tag{2.3-i}$$

ρ (C/m^3)	electric charge density
Q (C)	electric charge

Equation (2.3) follows from equation (2.2) and the continuity relation:

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t}. \tag{2.4}$$

Hint: Take the divergence of both sides of (2.2).

Maxwell's Equations

(d) Gauss' magnetic law:

$$\nabla \cdot \mathbf{B} = \rho_m \text{ ♣♣} \tag{2.5}$$

The equation $\nabla \cdot \mathbf{B} = 0$ follows from equation (2.1), provided that $\mathbf{M} = 0$.

Maxwell's equations alone are insufficient to solve for the four vector quantities: **E**, **D**, **H**, and **B** (twelve scalar quantities). Two additional vector equations are needed.

(e) Constitutive relationships

The constitutive relationships describe the properties of matter with respect to electric and magnetic forces.

$$\mathbf{D} = \vec{\varepsilon} \cdot \mathbf{E} \tag{2.6}$$

$$\mathbf{B} = \vec{\mu} \cdot \mathbf{H}. \tag{2.7}$$

(In vacuum, which is isotropic, the permittivity and the permeability are constants $\varepsilon = \varepsilon^0 = 8.8542 \text{ x } 10^{-12}$ F/m; $\mu^0 = 4\pi \text{ x } 10^{-7}$ H/m)

(f) Time-harmonic field analysis

In harmonic analysis of EM fields, the field phasors are introduced:

$$\boldsymbol{e}(x, y, z, t) = \operatorname{Re}\left\{\boldsymbol{E}(x, y, z)e^{j\omega t}\right\}$$
$$\boldsymbol{h}(x, y, z, t) = \operatorname{Re}\left\{\boldsymbol{H}(x, y, z)e^{j\omega t}\right\}. \qquad (2.8)$$

For example, the phasor of

$$e(x, y, z, t) = E_m(x, y, z)\cos(\omega t + \varphi_E)$$

is

$$E(x, y, z) = E_m e^{j\varphi_E}.$$

For clarity, from this point on, we will denote **time-dependent field quantities** with **lower-case letters (bold for vectors)**, while **their phasors** will be denoted with **upper-case letters**. **Complex-conjugate phasors** will be denoted with an **asterisk ***.

Time Harmonic Fields

Time Harmonics and Phasor Notation

Using Euler's identity

$$e^{j(wt+\varphi)} = \cos(wt+\varphi) + j\sin(wt+\varphi)$$

The time harmonic fields can be written as

$$\vec{\mathcal{E}}(x,y,z;t) = \mathrm{Re}\left[\hat{e}\,\mathcal{E}_o(x,y,z)e^{j(wt+\phi_o)}\right]$$

$$= \mathrm{Re}\left[\underbrace{\hat{e}\,\mathcal{E}_o(x,y,z)e^{j\phi_o}}_{\text{Phasor notation}}\, e^{jwt}\right]$$

Time Harmonic Fields

The frequency-domain Maxwell equations are obtained from the time-dependent equations using the following correspondences:

$$f(x,y,z,t) \doteq F(x,y,z)$$

$$\frac{\partial f_{(x,y,z,t)}}{\partial t} \doteq j\omega F(x,y,z)$$

$$\frac{\partial f}{\partial \xi} \doteq \frac{\partial F}{\partial \xi} \quad , \quad \xi = x,y,z \ .$$

Time Harmonic Fields

Assuming

- Time-harmonic fields, or exp($j\omega t$) variation
- Linear, isotropic media

Maxwell's equations become

$$\nabla \cdot \overline{D} = \rho_v \qquad \overline{D} = \epsilon \overline{E}$$

$$\nabla \cdot \overline{B} = 0 \qquad \overline{B} = \mu \overline{H}$$

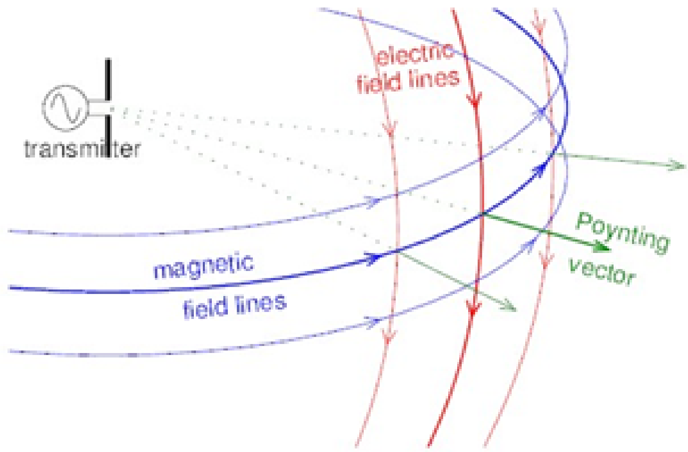
$$\nabla \times \overline{E} = -j\omega \overline{B} - \overline{M}$$

$$\nabla \times \overline{H} = j\omega \overline{D} + \overline{J}$$

where $\omega = 2\pi f$ is the circular frequency (rad/s).

Power Density, Poynting Vector, Radiated Power

Poynting vector (time-domain analysis)



$$\mathbf{p}(t) = \mathbf{e}(t) \times \mathbf{h}(t), \text{ W/m}^2.$$

As follows from Poynting's theorem, **p** is a vector representing

- the power density
- the direction of the EM power flow

the total power leaving certain volume V is obtained as

$$\Pi(t) = \oiint_{S_{[V]}} \mathbf{p}(t) \cdot d\mathbf{s}, \text{ W}.$$

Vector Potential

Basic Antenna Analysis Problem

Given currents $\bar{J}$ and $\bar{M}$, compute fields $\bar{E}$ and $\bar{H}$

Option 1: Direct

Option 2: Use potential

Static Problems

Nice to use electric *scalar* potential (V =voltage) instead of $\bar{E}$

Why? Can solve *scalar* equations instead of *vector* equations

Dynamic Problems

Use the *vector* potential $\bar{A}$ instead of $\bar{B}$

Can solve *vector* equations
instead of complicated *dyadic* (matrix) equations

Definition of $\bar{A}$ chosen to make analysis as simple as possible

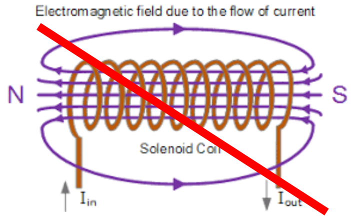
Exploit physical and vectorial properties

Vector Potential for Electric Current Source *J*

- The vector potential $\boldsymbol{A}$ is useful in solving for the EM field generated by a given harmonic electric current $\boldsymbol{J}$.
- The magnetic flux B is always solenoidal; that is, $\nabla \cdot \boldsymbol{B} = 0$.

In vector calculus a **solenoidal** vector field (also known as an incompressible vector field, a divergence-free vector field, or a transverse vector field) is a vector field v **with divergence zero at all points in the field**: A common way of expressing this property is to say that the field has no sources or sinks.

Not to confused with solenoid



Vector Potential for Electric Current Source *J*

- The vector potential $\boldsymbol{A}$ is useful in solving for the EM field generated by a given harmonic electric current $\boldsymbol{J}$.
- The magnetic flux B is always solenoidal; that is, $\nabla \cdot \boldsymbol{B} = 0$.

Therefore, it can be represented as the curl of another vector because it obeys the vector identity

$$\nabla \cdot \nabla \times \boldsymbol{A} = 0.$$

where $\boldsymbol{A}$ is an arbitrary vector. Thus we define

$$\boldsymbol{B}_A = \mu \boldsymbol{H}_A = \nabla \times \boldsymbol{A}. \tag{1}$$

Or

$$\boxed{\mathbf{H}_A = \frac{1}{\mu} \boldsymbol{\nabla} \times \mathbf{A}}$$

Vector Potential for Electric Current Source *J*

Let

$$\nabla \cdot \boldsymbol{A} = -j\omega\epsilon\mu\phi_e,$$

which is known as the Lorenz (gauge) condition.

The scalar function ϕ_e represents an arbitrary electric scalar potential which is a function of position.

In electromagnetism, the Lorenz gauge condition or Lorenz gauge (sometimes mistakenly called the Lorentz gauge) is a partial gauge fixing (denotes a mathematical procedure for coping with redundant degrees of freedom in field variables) of the electromagnetic vector potential. We are free to choose any statement about ∇·A without changing the physics.

Vector Potential for Electric Current Source *J*

Let

$$\nabla \cdot \boldsymbol{A} = -j\omega\epsilon\mu\phi_e,$$

which is known as the Lorenz (gauge) condition.

The scalar function ϕ_e represents an arbitrary electric scalar potential which is a function of position.

Lorenz condition and Maxwell equations result in inhomogeneous Helmholtz equation

$$\boxed{\nabla^2 \boldsymbol{A} + k^2 \boldsymbol{A} = -\mu \boldsymbol{J}}.$$

where $k^2 = \omega^2 \mu\varepsilon$, $\nabla^2 f = \nabla \cdot \nabla f$ is Laplace operator (a second-order differential operator, the divergence of the gradient.

Vector Potential for Electric Current Source J

Once $\mathbf{A}$ is known, $\mathbf{H}_A$ and $\mathbf{E}_A$ can be found from

$$\boxed{\boldsymbol{E}_A = -\nabla\phi_e - j\omega\boldsymbol{A} = -j\omega\boldsymbol{A} - j\frac{1}{\omega\mu\epsilon}\nabla(\nabla\cdot\boldsymbol{A}).}$$

$$\boxed{\mathbf{H}_A = \frac{1}{\mu}\nabla\times\mathbf{A}}$$

Summarizing

We have chosen definition of $\bar{A}$ so vector properties ensure certain physical conditions are automatically satisfied

Simplifies analysis!

Vector Potential for Magnetic Current Source *M*

- Although magnetic currents appear to be physically unrealizable, equivalent magnetic currents arise when we use the volume or the surface equivalence theorems.
- The fields generated by a harmonic magnetic current in a homogeneous region, with $\boldsymbol{J} = 0$ but $\boldsymbol{M} \neq 0$, must satisfy $\nabla \cdot \boldsymbol{D} = 0$.

Vector Potential for Magnetic Current Source *M*

- Therefore, $\boldsymbol{E}_F$ can be expressed as the curl of the vector potential $\boldsymbol{F}$ by

$$\boldsymbol{E}_F = -\frac{1}{\epsilon}\nabla \times \boldsymbol{F}. \tag{3}$$

- By letting

$$\nabla \cdot \boldsymbol{F} = -j\omega\mu\epsilon\phi_m \tag{4}$$

we can obtain

$$\nabla^2 \boldsymbol{F} + k^2 \boldsymbol{F} = -\epsilon \boldsymbol{M}.$$

Once $\boldsymbol{F}$ is known, $\boldsymbol{E}_F$ can be found from Eq. 3 and $\boldsymbol{H}_F$ with $\boldsymbol{M} = 0$. It will be shown later how to find $\boldsymbol{F}$ once $\boldsymbol{M}$ is known. It will be a solution to the inhomogeneous *Helmholtz eauation* of 4.

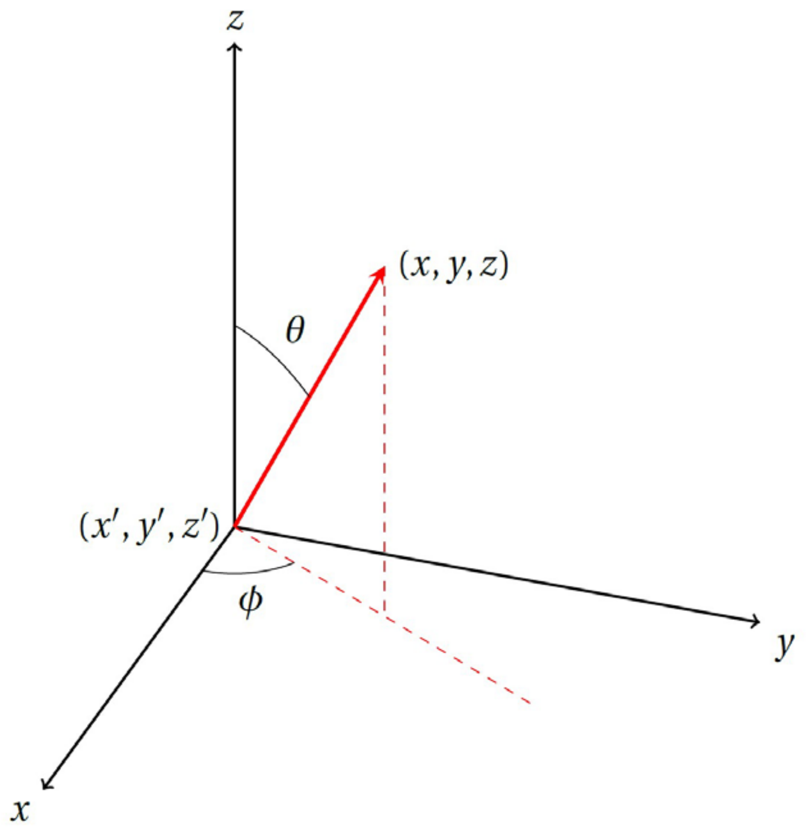
Vector Potential for Total Field

Now, we have developed equations that can be used to find the electric and magnetic fields generated by an electric current source $\boldsymbol{J}$ and a magnetic current source $\boldsymbol{M}$. The procedure requires that the auxiliary potential functions $\boldsymbol{A}$ and $\boldsymbol{F}$ generated, respectively, by $\boldsymbol{J}$ and $\boldsymbol{M}$ are found first. In turn, the corresponding electric and magnetic fields are then determined ($\boldsymbol{E}_A, \boldsymbol{H}_A$ due to $\boldsymbol{A}$ and $\boldsymbol{E}_F, \boldsymbol{H}_F$ due to $\boldsymbol{F}$). The total fields are then obtained by the superposition of the individual fields due to $\boldsymbol{A}$ and $\boldsymbol{F}$ ($\boldsymbol{J}$ and $\boldsymbol{M}$).

$$\boldsymbol{E} = \boldsymbol{E}_A + \boldsymbol{E}_F \qquad = -j\omega \boldsymbol{A} - j\frac{1}{\omega\mu\epsilon}\nabla(\nabla\cdot\boldsymbol{A}) - \frac{1}{\epsilon}\nabla\times\boldsymbol{F} = \frac{1}{j\omega\epsilon}\nabla\times\boldsymbol{H}_A - \frac{1}{\epsilon}\nabla\times\boldsymbol{F}.$$

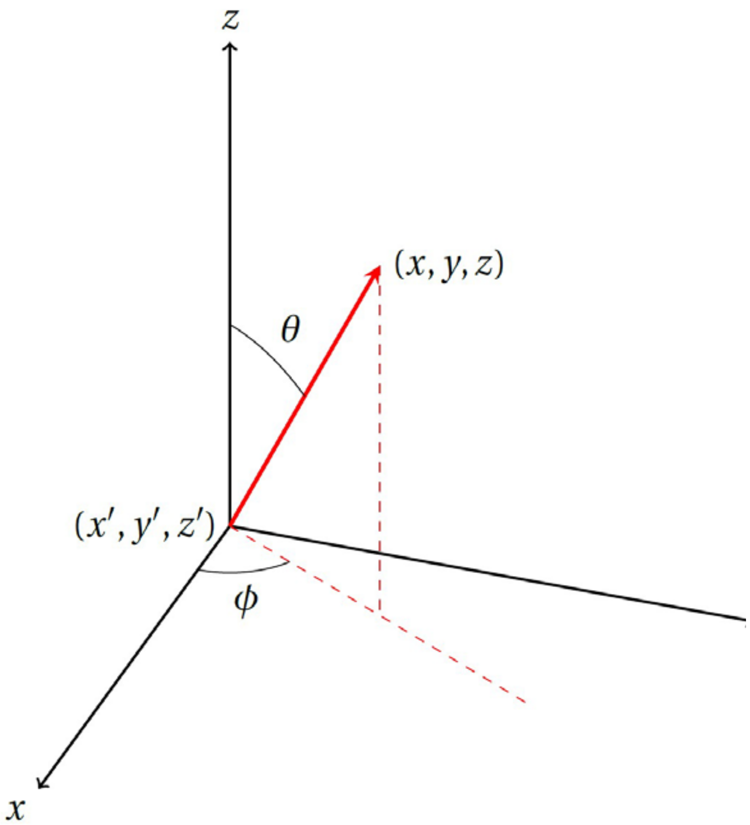
$$\boldsymbol{H} = \boldsymbol{H}_A + \boldsymbol{H}_F \qquad = \frac{1}{\mu}\nabla\times\boldsymbol{A} - j\omega\boldsymbol{F} - j\frac{1}{\omega\mu\epsilon}\nabla(\nabla\cdot\boldsymbol{F}) = \frac{1}{\mu}\nabla\times\boldsymbol{A} - \frac{1}{j\omega\mu}\nabla\times\boldsymbol{E}_F.$$

Solution of the Inhomogeneous Vector Potential Equation



To derive it, let us assume that a source with current density J_z, which in the limit is an infinitesimal source, is placed at the origin of a x, y, z coordinate system.

Solution of the Inhomogeneous Vector Potential Equation



Since the current density is directed along the z-axis (J_z), only an A_z component will exist.

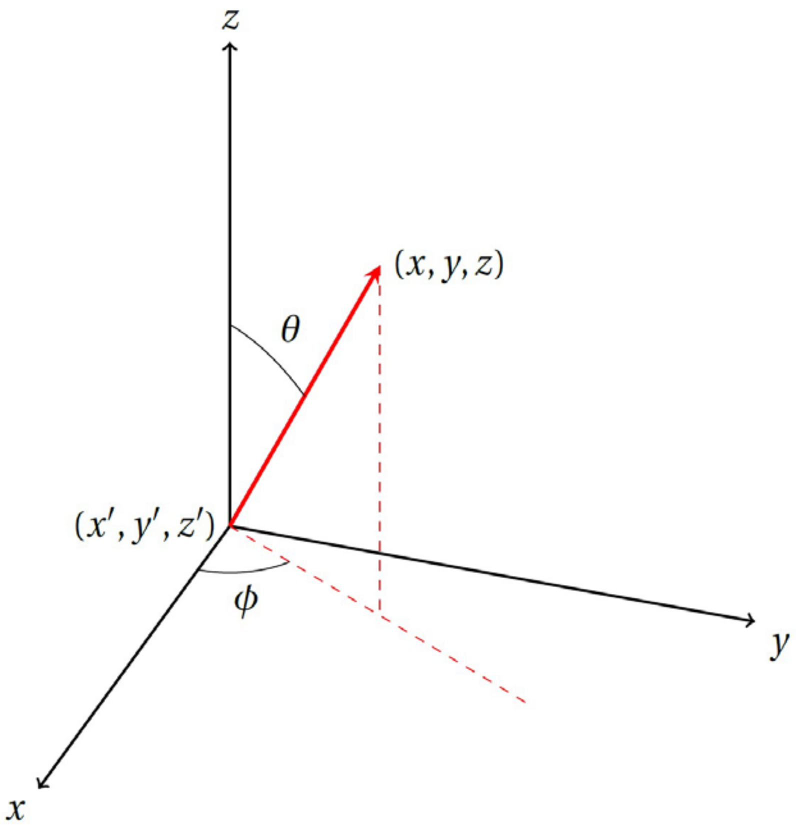
Thus we can write $\nabla^2 \boldsymbol{A} + k^2 \boldsymbol{A} = -\mu \boldsymbol{J}$ as

$$\nabla^2 A_z + k^2 A_z = -\mu J_z. \qquad (5)$$

At points outside of the infinitesimal source,

$$\nabla^2 A_z + k^2 A_z = 0.$$

Solution of the Inhomogeneous Vector Potential Equation



This represents the static solution.

$$A_z = \frac{\mu}{4\pi} \iiint_V \frac{J_z}{r} dv'.$$

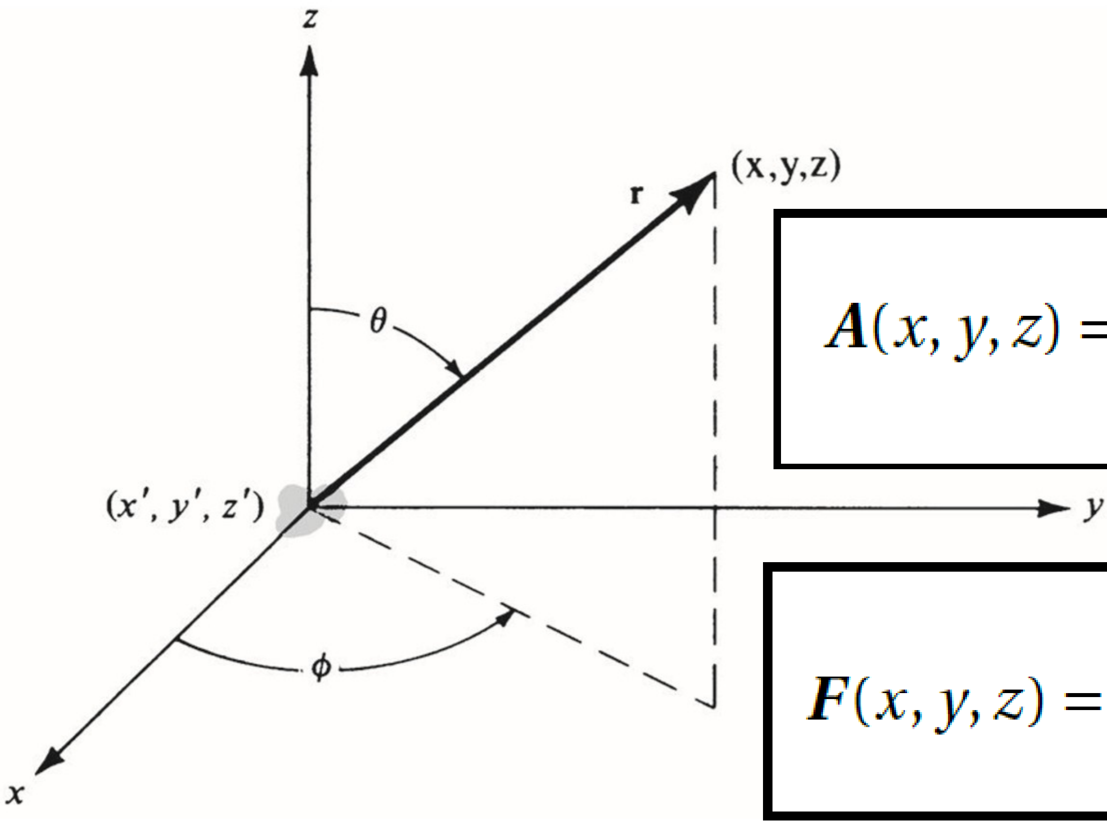
The time-varying solution is

$$A_z = \frac{\mu}{4\pi} \iiint_V J_z \frac{e^{-jkr}}{r} dv'.$$

$k^2 = \omega^2 \mu \varepsilon$ and R is the distance from any point in the source to the observation point.

Solution of the Inhomogeneous Vector Potential Equation

If the source placed at a position represented by (x', y', z'),



$$\boxed{\boldsymbol{A}(x,y,z) = \frac{\mu}{4\pi}\iiint_V \boldsymbol{J}(x',y',z')\frac{e^{-jkR}}{R}dv'.}\ *$$

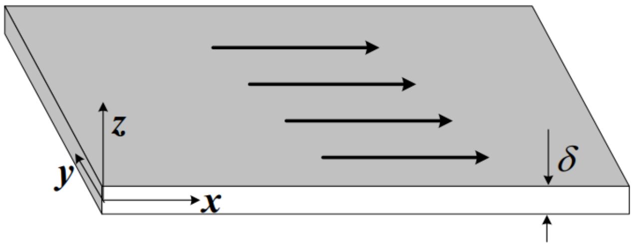
$$\boxed{\boldsymbol{F}(x,y,z) = \frac{\epsilon}{4\pi}\iiint_V \boldsymbol{M}(x',y',z')\frac{e^{-jkR}}{R}dv'.}$$

Solution of the Inhomogeneous Vector Potential Equation

If $\boldsymbol{J}_s$ and $\boldsymbol{M}_s$ represent surface current densities

$$\mathbf{A}(x,y,z) = \frac{\mu}{4\pi}\iint_S \boldsymbol{J}_s(x',y',z')\frac{e^{-jkR}}{R}ds'.$$

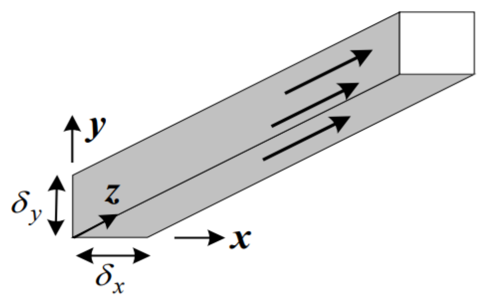
$$\boldsymbol{F}(x,y,z) = \frac{\epsilon}{4\pi}\iint_S \boldsymbol{M}_s(x',y',z')\frac{e^{-jkR}}{R}ds'.$$



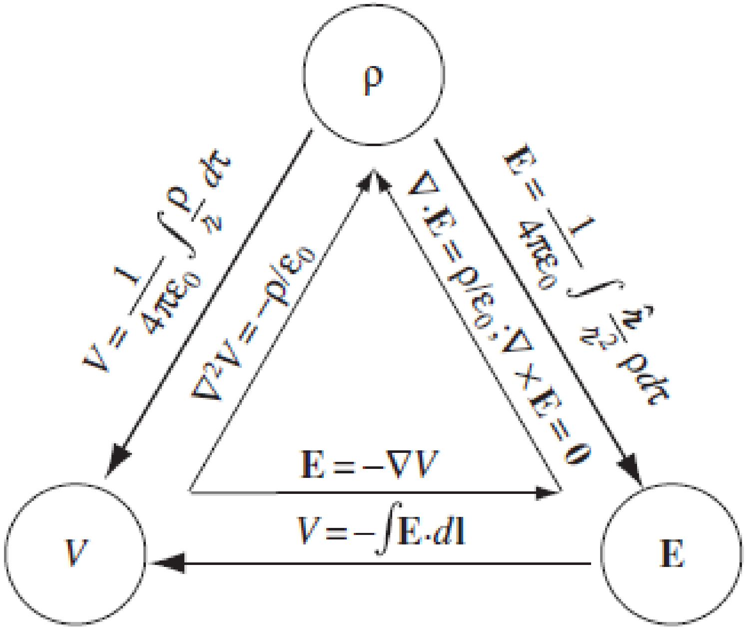
If $\boldsymbol{I}_e$ and $\boldsymbol{I}_m$ represent linear currents on a thin wire

$$\mathbf{A}(x,y,z) = \frac{\mu}{4\pi}\int_C \boldsymbol{I}_e(x',y',z')\frac{e^{-jkR}}{R}dl'.$$

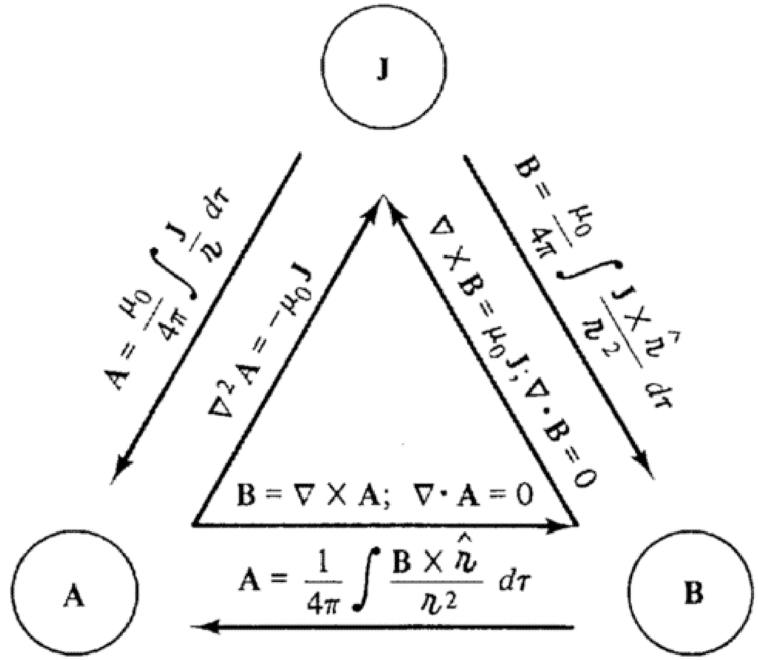
$$\boldsymbol{F}(x,y,z) = \frac{\epsilon}{4\pi}\int_C \boldsymbol{I}_m(x',y',z')\frac{e^{-jkR}}{R}dl'.$$



Vector Potential Summary



electrostatics



electrodynamics

The laws of electrostatics in a “triangle diagram” relating the source (ρ), the field (E), and the potential (V). Construct the analogous diagram for electrodynamics, with sources ρ and J (constrained by the continuity equation), fields E and B, and potentials V and A (constrained by the Lorenz gauge condition)

Radiation Boundary Condition

EM sources (currents and charges on the antenna) are more or less accurately known.

radiate in unbounded space and the resulting EM field is determined as integrals over the currents on the antenna

Such problems, where the field sources are known and the resulting field is to be determined are called analysis (forward, direct) problems

Radiation Boundary Condition

To ensure the uniqueness of the solution in an unbounded analysis problem, we have to impose the radiation boundary condition (RBC) on the EM field vectors, i.e., for distances far away from the source (r → ∞)

$$r(\mathbf{E} - \eta \mathbf{H} \times \hat{\mathbf{r}}) \to 0,$$

$$r(\mathbf{H} - \frac{1}{\eta} \hat{\mathbf{r}} \times \mathbf{E}) \to 0$$

The above RBC is known as the Sommerfeld vector RBC or the Silver-Müller RBC. η is the intrinsic impedance of the medium, in vacuum

$$\eta = \sqrt{\mu_0 / \varepsilon_0} \approx 377\ \Omega$$

Far-Field Radiation Vector Potential $r \geq 2\frac{D^2}{\lambda}$

Spherical wave general solution for far field
(far away, regardless the shape of antenna)

$$\mathbf{A} \approx \underbrace{\left[\hat{\mathbf{r}}A_r(\theta,\varphi) + \hat{\boldsymbol{\theta}}A_\theta(\theta,\varphi) + \hat{\boldsymbol{\varphi}}A_\varphi(\theta,\varphi)\right]}\underbrace{\frac{e^{-jkr}}{r}}, \quad r \to \infty.$$

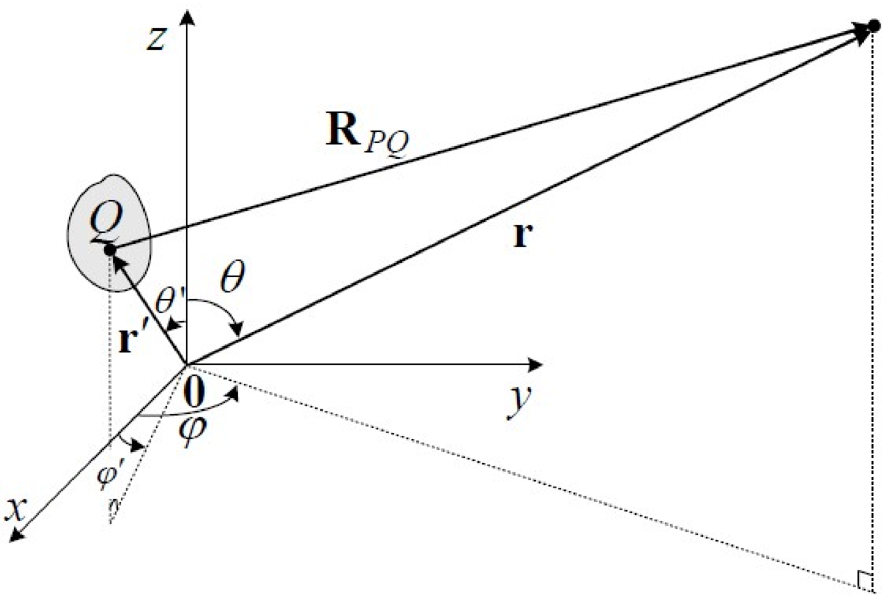
$(\hat{\mathbf{r}},\hat{\boldsymbol{\theta}},\hat{\boldsymbol{\varphi}})$ are the unit vectors of the spherical coordinate system (SCS)

$k = \omega\sqrt{\mu\varepsilon}$ is the wave number (or the phase constant)

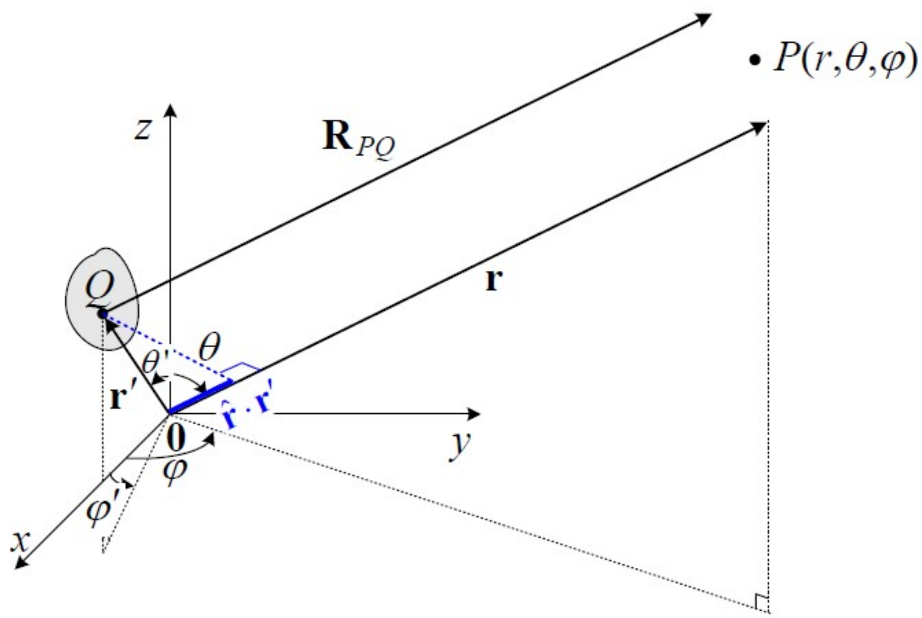
The term e^{-jkr} shows propagation along $\hat{\mathbf{r}}$ away from the antenna at the speed of light. The term $1/r$ shows the spherical spread of the potential in space, which results in a decrease of its magnitude with distance.

Notice an important feature of the far-field potential: the dependence on the distance r is separable from the dependence on the observation angle (θ,φ), and it is the same for any antenna: e^{-jkr}/r.

Far-Field Radiation Vector Potential



(a) original problem



(b) far-field approximation of the original problem

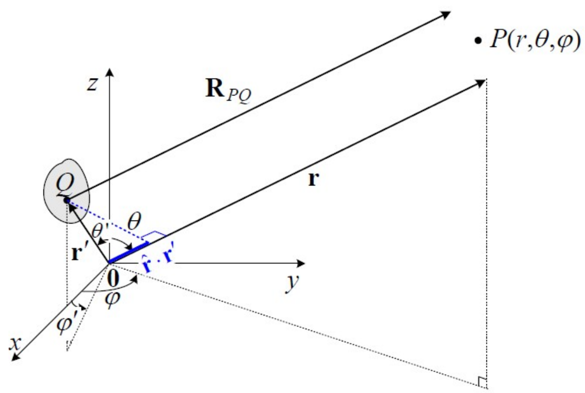
$$\frac{e^{-jkR_{PQ}}}{R_{PQ}} \approx \frac{e^{-jk(r-\hat{\mathbf{r}}\cdot\mathbf{r}')}}{r}$$

Far-Field Radiation Vector Potential

We now apply the far-field approximation to the vector potential in (*):

$$\mathbf{A}(P) = \frac{e^{-jkr}}{4\pi r} \cdot \iiint_{v_Q} \mu \mathbf{J}(Q) e^{jk\hat{\mathbf{r}}\cdot\mathbf{r}'} dv_Q .$$

The integrand no longer depends on the distance between the source and the observation point. It depends only on the current distribution of the source and the angle between the position vector of the integration point $\mathbf{r}'$ and the unit position vector of the observation point $\hat{\mathbf{r}}$.

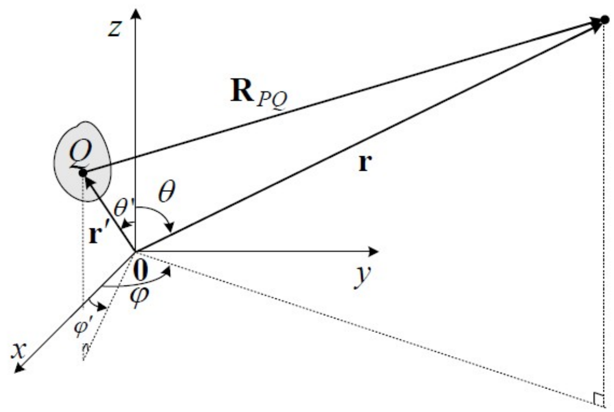


(b) far-field approximation of the original problem

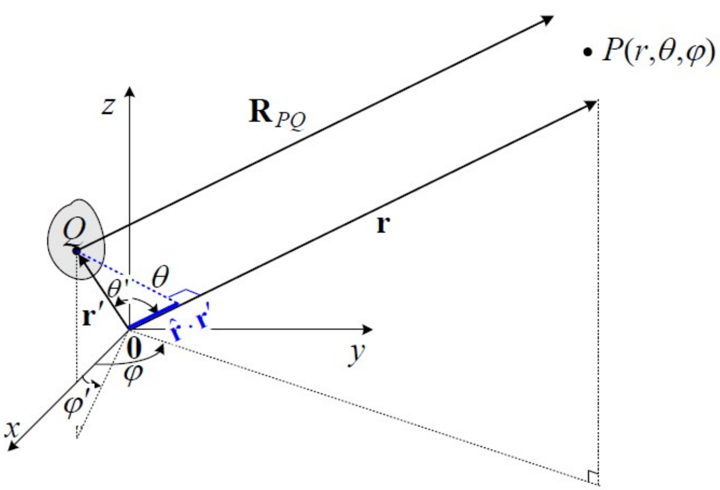
$$\frac{e^{-jkR_{PQ}}}{R_{PQ}} \approx \frac{e^{-jk(r-\hat{\mathbf{r}}\cdot\mathbf{r}')}}{r}$$

Far-Field Radiation Vector Potential

Once we obtain the vector potential at *P*, we can find the fields at *P*



(a) original problem

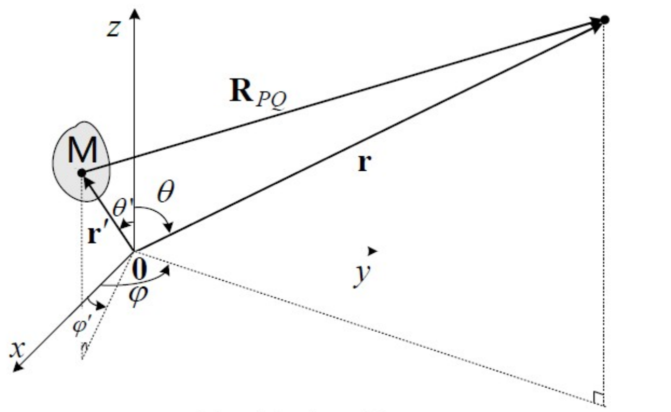


(b) far-field approximation of the original problem

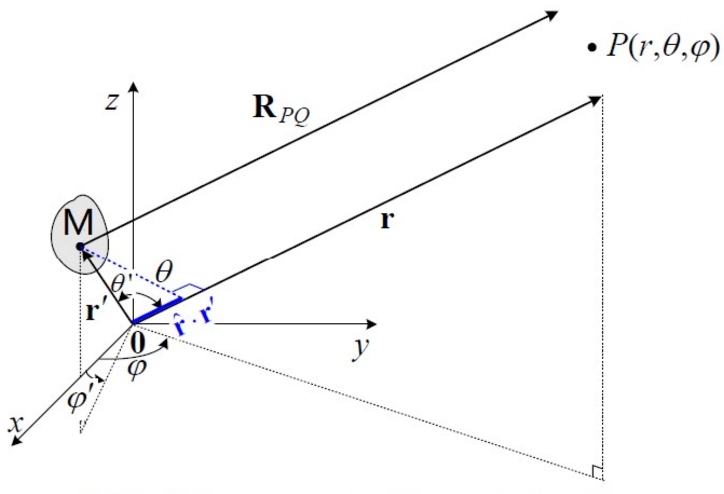
$$\left.\begin{aligned} E_r &\approx 0 \\ E_\theta &\approx -j\omega A_\theta \\ E_\varphi &\approx -j\omega A_\varphi \end{aligned}\right\} \Rightarrow \mathbf{E}^A \approx -j\omega\mathbf{A}, \text{ where } E_r^A \approx 0$$

$$\left.\begin{aligned} H_r &\approx 0 \\ H_\theta &\approx +j\frac{\omega}{\eta}A_\varphi = -\frac{E_\varphi}{\eta} \\ H_\varphi &\approx -j\frac{\omega}{\eta}A_\theta = +\frac{E_\theta}{\eta} \end{aligned}\right\} \Rightarrow \mathbf{H}^A \approx -j\frac{\omega}{\eta}\hat{\mathbf{r}}\times\mathbf{A} = \frac{1}{\eta}\hat{\mathbf{r}}\times\mathbf{E}^A$$

Far-Field Radiation Vector Potential



(a) original problem



(b) far-field approximation of the original problem

far-zone fields due to a magnetic source M (F is corresponding vector potential)

$$\left.\begin{aligned} H_r &\approx 0 \\ H_\theta &\approx -j\omega F_\theta \\ H_\varphi &\approx -j\omega F_\varphi \end{aligned}\right\} \Rightarrow \mathbf{H}^F \approx -j\omega\mathbf{F}, \quad H_r^F \approx 0,$$

$$\left.\begin{aligned} E_r &\approx 0 \\ E_\theta &\approx -j\omega\eta F_\varphi = \eta H_\varphi \\ E_\varphi &\approx +j\omega\eta F_\theta = -\eta H_\theta \end{aligned}\right\} \Rightarrow \mathbf{E}^F \approx j\omega\eta\hat{\mathbf{r}}\times\mathbf{F} = -\eta\hat{\mathbf{r}}\times\mathbf{H}^F$$

Vector Potential Summary

Summarizing...

Transforms complicated wave equation involving $\bar{E}$ and $\bar{H}$

Simpler scalar wave equations for components of $\bar{A}$ and $\bar{F}$

Approach

Given $\bar{J}$ and $\bar{M}$ solve for $\bar{A}$ and $\bar{F}$

When all finished, *then* find $\bar{E}$ and $\bar{H}$

Helps us manage the complexity of EM antenna problems

Vector Potential review video 10’31’’

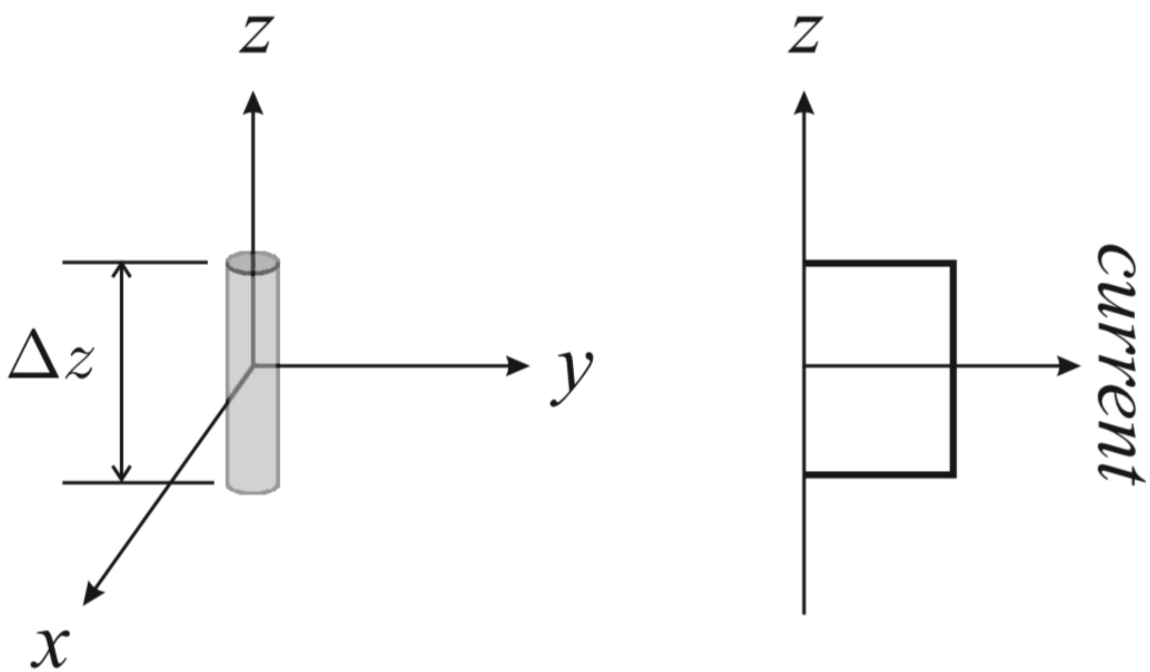
Dual Equations for Electric (J) and Magnetic (M) Current Sources

Electric Sources ($\mathbf{J} \neq 0, \mathbf{M} = 0$)	Magnetic Sources ($\mathbf{J} = 0, \mathbf{M} \neq 0$)
$\nabla \times \mathbf{E}_A = -j\omega\mu\mathbf{H}_A$	$\nabla \times \mathbf{H}_F = j\omega\varepsilon\mathbf{E}_F$
$\nabla \times \mathbf{H}_A = \mathbf{J} + j\omega\varepsilon E_A$	$-\nabla \times \mathbf{E}_F = \mathbf{M} + j\omega\mu\mathbf{H}_F$
$\nabla^2\mathbf{A} + k^2\mathbf{A} = -\mu\mathbf{J}$	$\nabla^2\mathbf{F} + k^2\mathbf{F} = -\varepsilon\mathbf{M}$
$\mathbf{A} = \frac{\mu}{4\pi}\iiint_V \mathbf{J}\frac{e^{-jkR}}{R}dv'$	$\mathbf{F} = \frac{\varepsilon}{4\pi}\iiint_V \mathbf{M}\frac{e^{-jkR}}{R}dv'$
$\mathbf{H}_A = \frac{1}{\mu}\nabla \times \mathbf{A}$	$\mathbf{E}_F = -\frac{1}{\varepsilon}\nabla \times \mathbf{F}$
$\mathbf{E}_A = -j\omega\mathbf{A} - j\frac{1}{\omega\mu\varepsilon}\nabla(\nabla \cdot \mathbf{A})$	$\mathbf{H}_F = -j\omega\mathbf{F} - j\frac{1}{\omega\mu\varepsilon}\nabla(\nabla \cdot \mathbf{F})$

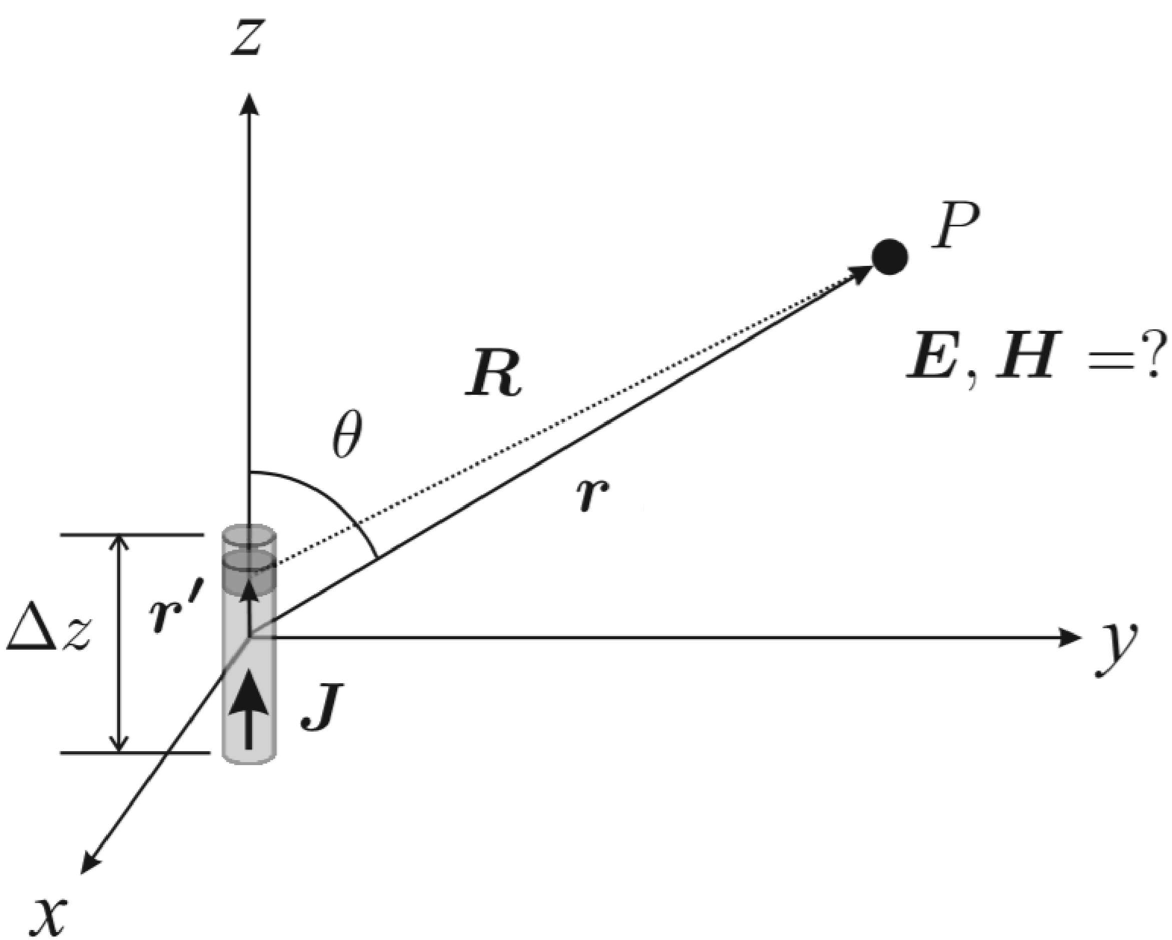
Duality only serves as a guide to form mathematical solutions. It can be used in an abstract manner to explain the motion of magnetic charges giving rise to magnetic currents, when compared to their dual quantities of moving electric charges creating electric currents. It must, however, be emphasized that this is purely mathematical in nature since it is known, as of today, that there are no magnetic charges or currents in nature.

Ideal (Hertzian) Dipole Antenna

- very simple radiating element
- very short (length $<<\lambda$)
- current uniformly distributed along its length

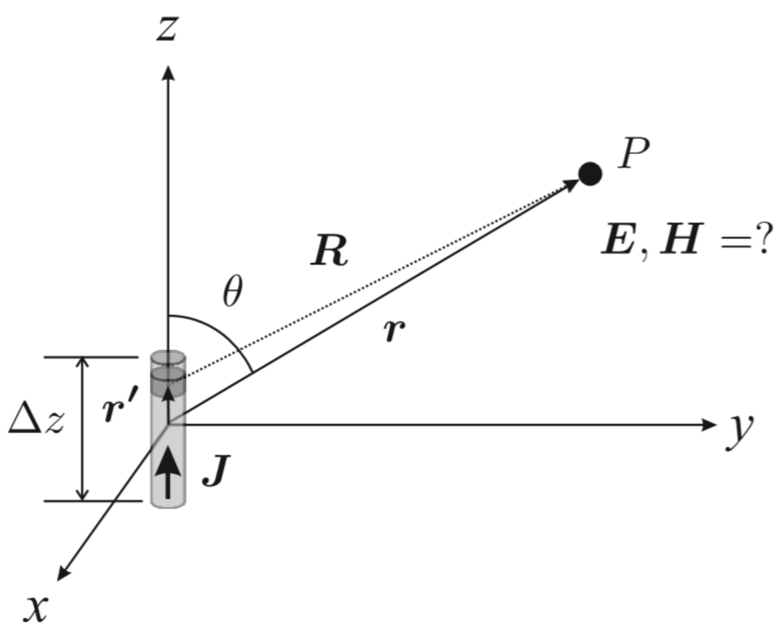


Ideal (Hertzian) Dipole Antenna



- orient the ideal dipole along the z-axis
- the current flowing through the dipole as $\boldsymbol{I}$
- the associated surface current density $\boldsymbol{J}$
- R is the distance from the current element to the field point P
- r is the distance from the origin to P

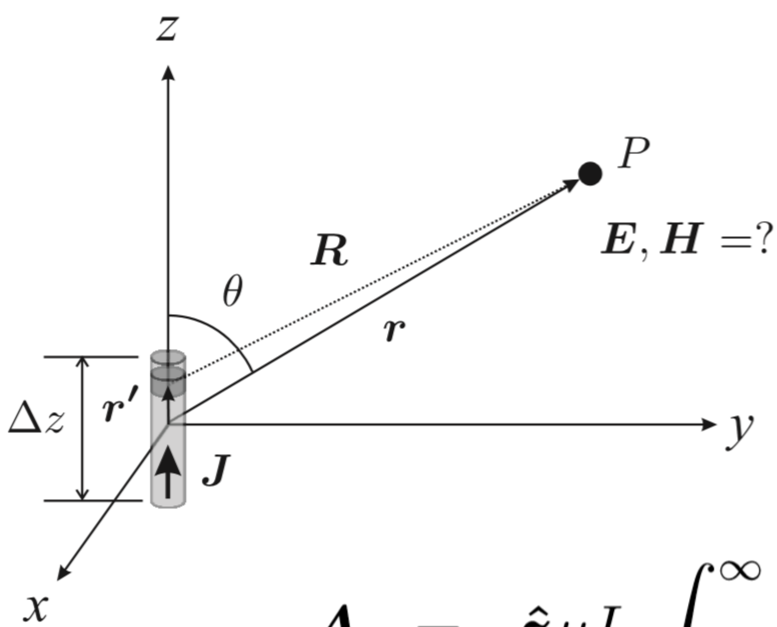
Ideal (Hertzian) Dipole Antenna



- current only has *z*-component, **vector potential *A*** will only have a *z*-component.

$$\boldsymbol{A} = \int_V \mu \boldsymbol{J} \frac{e^{-jkR}}{4\pi R} dv' = \iiint \mu \boldsymbol{J} \frac{e^{-jkR}}{4\pi R} dx' dy' dz'$$

Ideal (Hertzian) Dipole Antenna

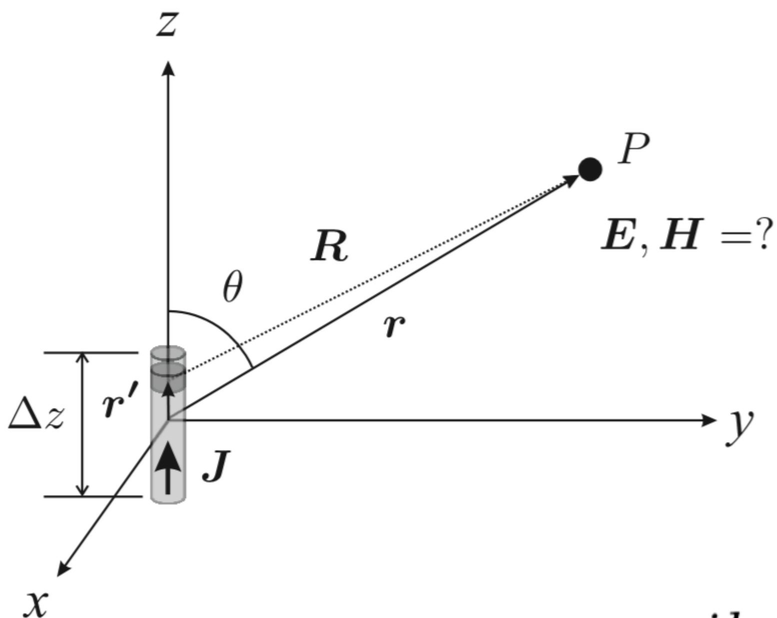


- infinitely thin dipole

$$\boldsymbol{J}(\boldsymbol{r}') = \begin{cases} I_0\delta(x')\delta(y')\hat{\boldsymbol{z}} & \Delta z/2 < z' < \Delta z/2 \\ 0 & \text{elsewhere} \end{cases}$$

$$\begin{aligned} \boldsymbol{A} &= \hat{\boldsymbol{z}}\mu I_0 \int_{-\infty}^{\infty} \delta(x')dx' \int_{-\infty}^{\infty} \delta(y')dy' \int_{\Delta z/2}^{\Delta z/2} \frac{e^{-jkR}}{4\pi R} dz' \\ &= \hat{\boldsymbol{z}}\mu I_0 \int_{\Delta z/2}^{\Delta z/2} \frac{e^{-jkR}}{4\pi R} dz'. \end{aligned}$$

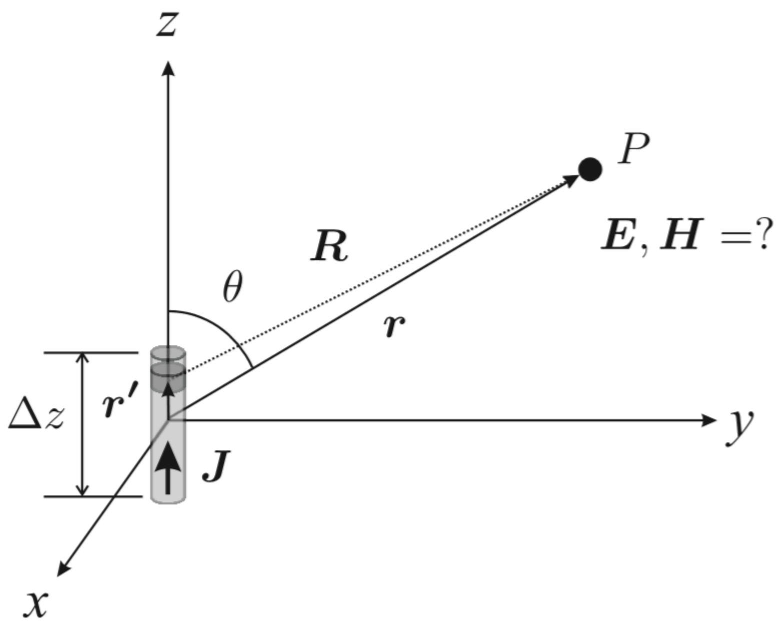
Ideal (Hertzian) Dipole Antenna



- Δz is small
- $r \approx R$.
- R is not a function of z'

$$\boldsymbol{A} = \hat{\boldsymbol{z}}\mu I_0 \frac{e^{-jkr}}{4\pi r}\int_{\Delta z/2}^{\Delta z/2} dz' = \frac{\mu I_0 e^{-jkr}}{4\pi r}\Delta z\,\hat{\boldsymbol{z}}$$

Ideal (Hertzian) Dipole Antenna



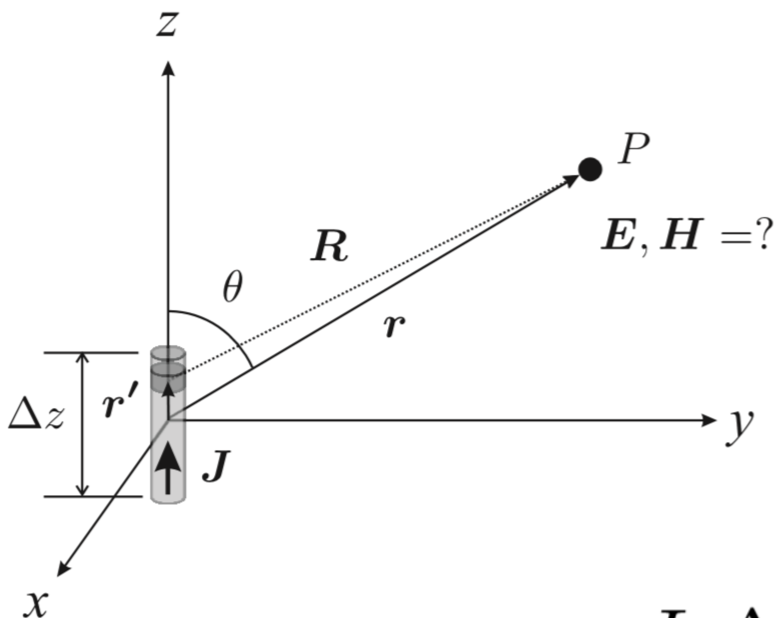
- radiated magnetic field of the dipole

$$\boldsymbol{H} = \frac{1}{\mu}\boldsymbol{\nabla} \times \boldsymbol{A} = \frac{1}{\mu}\boldsymbol{\nabla} \times A_z\,\hat{\boldsymbol{z}}$$

- solution is spherical wave
- best to evaluate this curl in spherical coordinates
- convert $\boldsymbol{A}$ to spherical coordinate

$$\begin{aligned} A_r &= \boldsymbol{A} \cdot \hat{\boldsymbol{r}} = A_z\,\hat{\boldsymbol{z}} \cdot \hat{\boldsymbol{r}} = A_z \cos\theta \\ A_\theta &= \boldsymbol{A} \cdot \hat{\boldsymbol{\theta}} = A_z\,\hat{\boldsymbol{z}} \cdot \hat{\boldsymbol{\theta}} = -A_z \sin\theta \\ A_\phi &= \boldsymbol{A} \cdot \hat{\boldsymbol{\phi}} = A_z\,\hat{\boldsymbol{z}} \cdot \hat{\boldsymbol{\phi}} = 0. \end{aligned}$$

Ideal (Hertzian) Dipole Antenna



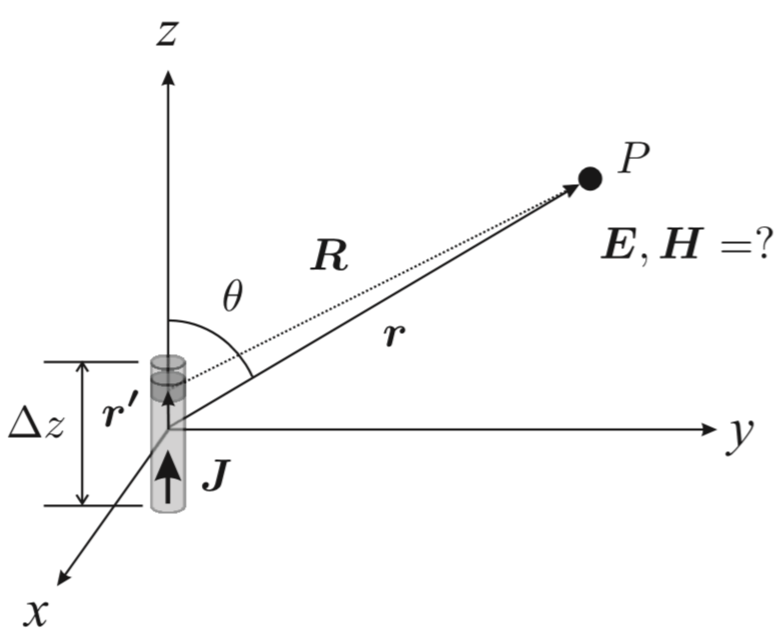
- radiated magnetic field of the dipole

$$\boldsymbol{H} = \frac{1}{\mu}\boldsymbol{\nabla} \times \boldsymbol{A} = \frac{1}{\mu}\boldsymbol{\nabla} \times A_z\,\hat{\boldsymbol{z}}$$

$$\begin{aligned} A_r &= \boldsymbol{A}\cdot\hat{\boldsymbol{r}} = A_z\,\hat{\boldsymbol{z}}\cdot\hat{\boldsymbol{r}} = A_z\cos\theta \\ A_\theta &= \boldsymbol{A}\cdot\hat{\boldsymbol{\theta}} = A_z\,\hat{\boldsymbol{z}}\cdot\hat{\boldsymbol{\theta}} = -A_z\sin\theta \\ A_\phi &= \boldsymbol{A}\cdot\hat{\boldsymbol{\phi}} = A_z\,\hat{\boldsymbol{z}}\cdot\hat{\boldsymbol{\phi}} = 0. \end{aligned}$$

$$\boldsymbol{H} = \frac{I_0\Delta z}{4\pi} jk\left(1 + \frac{1}{jkr}\right)\frac{e^{-jkr}}{r}\sin\theta\,\hat{\boldsymbol{\phi}}$$

Ideal (Hertzian) Dipole Antenna



- radiated magnetic field of the dipole

$$\boldsymbol{H} = \frac{1}{\mu}\boldsymbol{\nabla} \times \boldsymbol{A} = \frac{1}{\mu}\boldsymbol{\nabla} \times A_z\,\hat{\boldsymbol{z}}$$

- electric field from Maxwell's curl equation

$$\boldsymbol{E} = \frac{1}{j\omega\varepsilon}\boldsymbol{\nabla} \times \boldsymbol{H}$$

$$\boldsymbol{E} = \frac{I_0\Delta z}{2\pi}\eta\left(\frac{1}{r} - \frac{j}{kr^2}\right)\frac{e^{-jkr}}{r}\cos\theta\,\hat{\boldsymbol{r}} + \frac{I_0\Delta z j\omega\mu}{4\pi}\left[1 + \frac{1}{jkr} - \frac{1}{(kr)^2}\right]\frac{e^{-jkr}}{r}\sin\theta\,\hat{\boldsymbol{\theta}}.$$

Ideal (Hertzian) Dipole Antenna

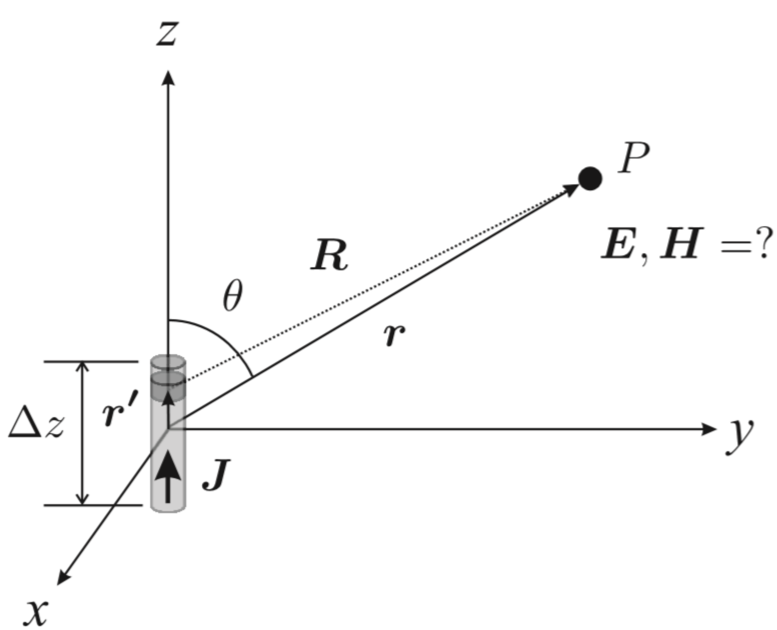
radiated field of the dipole

$$\boldsymbol{H} = \frac{I_0 \Delta z}{4\pi} jk \left(1 + \frac{1}{jkr}\right) \frac{e^{-jkr}}{r} \sin\theta \, \hat{\boldsymbol{\phi}}$$

$$\boldsymbol{E} = \frac{I_0 \Delta z}{2\pi} \eta \left(\frac{1}{r} - \frac{j}{kr^2}\right) \frac{e^{-jkr}}{r} \cos\theta \, \hat{\boldsymbol{r}}$$

$$+ \frac{I_0 \Delta z j\omega\mu}{4\pi} \left[1 + \frac{1}{jkr} - \frac{1}{(kr)^2}\right] \frac{e^{-jkr}}{r} \sin\theta \, \hat{\boldsymbol{\theta}}.$$

Ideal (Hertzian) Dipole Antenna



- far field of the antenna ($r >> \lambda$)
- all the terms with r in the denominator tend to zero
- solutions become

$$\boldsymbol{E}_{\text{ff}} = \frac{I_0 \Delta z j \omega \mu}{4\pi} \frac{e^{-jkr}}{r} \sin\theta \,\hat{\boldsymbol{\theta}}$$

$$\boldsymbol{H}_{\text{ff}} = \frac{I_0 \Delta z}{4\pi} jk \frac{e^{-jkr}}{r} \sin\theta \,\hat{\boldsymbol{\phi}}.$$

Ideal (Hertzian) Dipole Antenna

$$\boxed{\begin{aligned} \boldsymbol{E}_{\text{ff}} &= \frac{I_0 \Delta z j\omega\mu}{4\pi} \frac{e^{-jkr}}{r} \sin\theta\, \hat{\boldsymbol{\theta}} \\ \boldsymbol{H}_{\text{ff}} &= \frac{I_0 \Delta z}{4\pi} jk \frac{e^{-jkr}}{r} \sin\theta\, \hat{\boldsymbol{\phi}}. \end{aligned}}$$

- *E* has no a radial component in the far field, it is totally polarized in the $\hat{\boldsymbol{\theta}}$ direction;
- *E* and *H* are orthogonal to each other and the direction of propagation and hence the resulting wave is TEM (as we expect for a spherical wave);
- The ratio of $E_\theta/H\varphi$ is $\frac{E_\theta}{H_\phi} = \frac{\omega\mu}{k} = \sqrt{\frac{\mu}{\varepsilon}} = \eta$

Ideal (Hertzian) Dipole Antenna

$$\boldsymbol{E}_{\text{ff}} = \underbrace{\frac{I_0 \Delta z}{4\pi} j\omega\mu}_{\text{strength factor}} \cdot \underbrace{\frac{e^{-jkr}}{r}}_{\text{distance factor}} \cdot \underbrace{\sin\theta}_{\text{shape/element factor}} \cdot \hat{\boldsymbol{\theta}}.$$

- Strength factor – determined solely by material parameters, magnitude of excitation current, and dipole length
- Distance factor – purely the amplitude decay and phase shift incurred with distance
- Shape factor – determined the radiation pattern of the antenna, or the part that is a function of θ, φ.

Ideal (Hertzian) Dipole Antenna

- Poynting vector of the far fields components

$$\boldsymbol{P} = \frac{1}{2}\boldsymbol{E} \times \boldsymbol{H}^* = \frac{1}{2}E_\theta H_\phi^* \,\hat{\boldsymbol{r}}$$

- Power density (real part of poynting vector)

$$\boldsymbol{P} \quad = \quad \frac{I_0^2 \Delta z^2 \omega \mu k}{2(4\pi r)^2} \sin^2\theta \,\hat{\boldsymbol{r}}.$$

- Directivity

$$D_{\text{max}} = P_{max}/P_{\text{avg}} = 1.5$$

Homework

- Prove vector field $3yz^2 i + 4xz j - 3xy k$ is a solenoidal field.
- Show directivity of Hertzian dipole is 1.5
- Plot 2D and 3D patterns of Hertzian dipole