Notes for the Control IV Class
(Robot Control)

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Disclaimer. Although the authors have made every effort to ensure that the information in this work is correct, they do not accept any responsibility of any error as it is not deliberate. Nevertheless, we will appreciate provision of accurate information to improve our work.

This work is intended to be support material for the *Control IV (Robot Control)* class that is taught at the University of Guadalajara, Mexico, for Robot Engineering students.
The course syllabus is deeply influenced by the collaboration that E. Nuño has had with R. Ortega since 2007 and this is why most of the course exploits the *Energy Shaping plus Damping Injection* control technique. The interested reader is invited to read *the red bible* [11] for an exhaustive and reliable treatment of this control technique. In fact, the results reported here, using this methodology, can entirely be obtained from [11].
This recollection of previous works follows a similar structure as the fundamental book in robot control of Kelly et al. [6].
The readers are encouraged to read the books of Spong et al. [17] and Siciliano et al. [14] to complement their control of robots knowledge.

Contents

Part I

Introduction and Background

Chapter 1

Algebra and Functional Analysis

Most of these definitions have been borrowed from [1] and [6]

1.1 Numbers

Definition 1 (Real Numbers) *Rational and irrational numbers together form the set of real numbers, which is denoted by* $\mathbb{R}$*, i.e.,* $\mathbb{R} := [-\infty, \infty]$.

Definition 2 (Complex Numbers) *The algebraic form of a complex number is*

$$x = a + ib,$$

where $a, b \in \mathbb{R}$*. Number* a *is the real part and number* b *is the imaginary part. The set of all possible complex numbers is denoted as* $\mathbb{C}$.

Definition 3 (Absolute Value) *The absolute value of a real number* x*, denoted* $|x|$*, is the non-negative value of* x *without regard to its sing*

1.2 Vectors

Definition 4 (Column Vector) *Matrices of size* $(n, 1)$ *are column vectors of dimension* n*.*

$$\mathbf{x} := \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

Definition 5 (Row Vector) *Matrices of size* $(1, n)$ *are row vectors of dimension* n*.*

$$\mathbf{x} := [x_1, \ldots, x_n]$$

Definition 6 (Dot Product) *The scalar product or dot product of two vectors* $\mathbf{x}$ *and* $\mathbf{y}$ *is defined by the equation.*

$$\mathbf{x} \cdot \mathbf{y} = \mathbf{x}^\top \mathbf{y} = [x_1, \ldots, x_n] \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \sum_{i=1}^{n} x_i y_i$$

Definition 7 (Vector Norm) *The norm of a real valued vector* $\mathbf{x} \in \mathbb{R}^n$ *can be considered as a generalization of the absolute value (magnitude) of numbers and it is defined as*

$$|\mathbf{x}|_p := \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p},$$

for any integer $p \in [1, \infty)$*.*

Definition 8 (Vector 1-Norm)

$$|\mathbf{x}|_1 := \sum_{i=1}^{n} |x_i|$$

Definition 9 (Vector 2-Norm (Euclidean Norm))

$$|\mathbf{x}|_2 := \sqrt{\sum_{i=1}^{n} x_i^2}$$

Definition 10 (Vector ∞-Norm)

$$|\mathbf{x}|_\infty := \max_{1 \le i \le n} |x_i|$$

Lemma 1 (Monotonicity (Decreasing) of Norms)

$$|\mathbf{x}|_1 \ge |\mathbf{x}|_2 \ge \cdots \ge |\mathbf{x}|_\infty$$

$\diamond$

Definition 11 (Vector Valued Function Norm) *When the real valued vector* $\mathbf{x}(t) \in \mathbb{R}^n$ *is a function of time, then its norm is defined as*

$$\begin{aligned}\|\mathbf{x}(t)\|_p &:= \left(\int_0^\infty |\mathbf{x}(\sigma)|_p^p d\sigma\right)^{1/p} \\ &= \left(\int_0^\infty \sum_{i=1}^n |x_i(\sigma)|^p d\sigma\right)^{1/p}\end{aligned}$$

Definition 12 (Vector Valued Function 1-Norm)

$$\|\mathbf{x}(t)\|_1 = \int_0^\infty |\mathbf{x}(\sigma)|_1 d\sigma$$

Definition 13 (Vector Valued Function 2-Norm)

$$\begin{aligned}\|\mathbf{x}(t)\|_2 &= \sqrt{\int_0^\infty |\mathbf{x}(\sigma)|_2^2 d\sigma} \\ &= \sqrt{\int_0^\infty \mathbf{x}^\top(\sigma)\mathbf{x}(\sigma) d\sigma}\end{aligned}$$

Definition 14 (Vector Valued Function ∞-Norm)

$$\|\mathbf{x}(t)\|_\infty := \sup_{t\geq 0}\{|x_i(t)|\}$$

Definition 15 ($\mathcal{L}_p$ Vector Spaces) *The set* $\mathcal{L}_p$*, for any integer* $p \in [1, \infty]$*, contains all the functions with finite p-Norm, i.e.,*

$$\mathcal{L}_p := \{\mathbf{x}(t) \in \mathbb{R}^n : \|\mathbf{x}(t)\|_p < \infty\}$$

Definition 16 ($\mathcal{L}_2$ Vector Space)

$$\mathcal{L}_2 := \{\mathbf{x}(t) \in \mathbb{R}^n : \|\mathbf{x}(t)\|_2 < \infty\}$$

Definition 17 ($\mathcal{L}_\infty$ Vector Space)

$$\mathcal{L}_\infty := \{\mathbf{x}(t) \in \mathbb{R}^n : \|\mathbf{x}(t)\|_\infty < \infty\}$$

In this work, for any vector $\mathbf{x} \in \mathbb{R}^n$, the Euclidian norm (Vector 2-Norm) is written as $\|\mathbf{x}\|$ and the Vector Valued p-norm as $\|\mathbf{x}(t)\|_p$ (using the explicit subindex p). For a scalar $x \in \mathbb{R}$, its absolute value is written as $|x|$. In some signals, We also omit their explicit dependence of time when clear from the context.

1.3 Matrices

Definition 18 (Real Matrix) *A matrix is an array of m rows times n columns with real elements.*

Definition 19 (Symmetric Matrix) *A square matrix $\mathbf{A}$ is symmetric if*

$$\mathbf{A} = \mathbf{A}^\top$$

Definition 20 (Antisymmetric Matrix) *A square matrix $\mathbf{A}$ is antisymmetric if*

$$\mathbf{A} = -\mathbf{A}^\top$$

Theorem 1 (Matrix Decomposition) *Every square matrix $\mathbf{A}$ can be decomposed into the sum of a symmetric matrix $\mathbf{A}_s$ and an antisymmetric matrix $\mathbf{A}_{as}$ such that*

$$\mathbf{A} = \mathbf{A}_s + \mathbf{A}_{as}$$

where $\mathbf{A}_s = \frac{1}{2}(\mathbf{A} + \mathbf{A}^\top)$ and $\mathbf{A}_{as} = \frac{1}{2}(\mathbf{A} - \mathbf{A}^\top)$. $\diamond$

Definition 21 (Matrix Square Form) *The (scalar) square form of matrix $\mathbf{A}$ is*

$$s(\mathbf{A}) := \mathbf{x}^\top \mathbf{A}\mathbf{x}.$$

Definition 22 (Positive Definite Matrix) *A square, not necessarily symmetric, matrix $\mathbf{A}$ is positive definite if*

$$\mathbf{x}^\top \mathbf{A}\mathbf{x} = 0 \Leftrightarrow \mathbf{x} = \mathbf{0}$$

and

$$\mathbf{x}^\top \mathbf{A}\mathbf{x} > 0 \Leftrightarrow \mathbf{x} \neq \mathbf{0}.$$

Definition 23 (Positive Semi-Definite Matrix) *A square, not necessarily symmetric, matrix $\mathbf{A}$ is positive definite if*

$$\mathbf{x}^\top \mathbf{A}\mathbf{x} \geq 0, \forall \mathbf{x} \in \mathbb{R}^n$$

Definition 24 (Negative Definite Matrix) *Matrix $\mathbf{A}$ is negative definite if matrix $-\mathbf{A}$ is positive definite.*

Definition 25 (Negative Semi-Definite Matrix) *Matrix $\mathbf{A}$ is negative semi-definite if matrix $-\mathbf{A}$ is positive semi-definite.*

Theorem 2 (Antisymmetric Matrix Square Form) *The square form of an antisymmetric matrix* $\mathbf{A}$ *is always zero, i.e., if* $\mathbf{A} = -\mathbf{A}^\top$ *then*

$$\mathbf{x}^\top \mathbf{A}\mathbf{x} = 0, \forall \mathbf{x} \in \mathbb{R}^n.$$

$\diamond$

Fact 1 *From Theorem 3 it holds that the square form of* **any** *matrix* $\mathbf{A} \in \mathbb{R}^{n\times n}$ *is equal to the square form of its symmetric part, i.e.,*

$$\mathbf{x}^\top \mathbf{A}\mathbf{x} = \frac{1}{2}\mathbf{x}^\top(\mathbf{A} + \mathbf{A}^\top)\mathbf{x} = \mathbf{x}^\top \mathbf{A}_s \mathbf{x}.$$

Definition 26 (Matrix Spectrum (Eigenvalues)) *Matrix* $\mathbf{A} \in \mathbb{R}^{n\times n}$ *has* n *eigenvalues that correspond to the solutions of*

$$\det(\lambda \mathbf{I} - \mathbf{A}) = 0,$$

where λ *is called the spectrum of* $\mathbf{A}$.

Definition 27 (Matrix Eigenvectors) *vector* $\mathbf{x}_i \neq \mathbf{0}$ *is called and eigenvector of matrix* $\mathbf{A}$ *associated to the eigenvalue* λ_i *if it holds that*

$$\mathbf{A}\mathbf{x}_i = \lambda_i \mathbf{x}_i,$$

or, what is the same

$$(\mathbf{A} - \lambda_i \mathbf{I})\mathbf{x}_i = \mathbf{0}.$$

Theorem 3 (Positive Definite Matrix) *Matrix* $\mathbf{A} \in \mathbb{R}^{n\times n}$ *is positive definite if and only if: 1) it is symmetric; and 2) all its eigenvalues are strictly positive.* $\diamond$

Definition 28 (Induced 2-Norm Matrix) *The induced 2-norm of matrix* $\mathbf{A}$ *is*

$$\|\mathbf{A}\|_2 := \sqrt{\lambda_M(\mathbf{A}^\top \mathbf{A})}$$

where λ_M *is the maximum eigenvalue of matrix* $\mathbf{A}^\top \mathbf{A}$.

Definition 29 (Diagonalizable Matrix) *Matrix* $\mathbf{A} \in \mathbb{R}^{n\times n}$ *is called diagonalizable if there exist square matrices* $\mathbf{T}$ *and* $\mathbf{D}$ *such that*

$$\mathbf{A} = \mathbf{T}\mathbf{D}\mathbf{T}^{-1},$$

where $\mathbf{D}$ *is a diagonal matrix.*

Theorem 4 (Diagonalizable Matrix Condition) *A matrix* $\mathbf{A}$ *is diagonalizable if its spectrum is composed of different eigenvalues.* $\diamond$

Definition 30 (Jordan Block and Jordan Matrix) *Let* $\lambda \in \mathbb{C}$. *A Jordan block* $\mathbf{J}_k(\lambda)$ *is a* $k \times k$ *upper triangular matrix of the form*

$$\mathbf{J}_k(\lambda) = \begin{bmatrix} \lambda & 1 & 0 & \dots & 0 \\ 0 & \lambda & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda & 1 \end{bmatrix}.$$

A Jordan matrix is any, square $n \times n$, *matrix of the form*

$$\mathbf{J} = \begin{bmatrix} \mathbf{J}_{n_1}(\lambda_1) & & 0 \\ & \ddots & \\ 0 & & \mathbf{J}_{n_k}(\lambda_k) \end{bmatrix},$$

where $\mathbf{J}_{n_i}$ *are Jordan blocks and* $n = n_1 + \cdots + n_k$.

Definition 31 (Jordan Canonical Form) *Any matrix* $\mathbf{A} \in \mathbb{R}^{n\times n}$ *can always be brought to a form*

$$\mathbf{A} = \mathbf{SJS}^{-1},$$

where $\mathbf{S}$ *is a nonsingular matrix and* $\mathbf{J}$ *is a Jordan matrix that is unique (up to permutations of the Jordan blocks).*

Definition 32 (Matrix Exponential) *Consider* $\mathbf{A} \in \mathbb{R}^{n\times n}$ *then* $e^{\mathbf{A}}$ *is defined as*

$$e^{\mathbf{A}} := \sum_{k=0}^{\infty} \frac{1}{k!}\mathbf{A}^k = \mathbf{I} + \mathbf{A} + \frac{1}{2!}\mathbf{A}^2 + \frac{1}{3!}\mathbf{A}^3 + ...$$

Theorem 5 (Matrix Exponential) *Consider any matrix* $\mathbf{A} \in \mathbb{R}^{n\times n}$. *If* $\mathbf{A}$ *is diagonalizable, then* $e^{\mathbf{A}}$ *is given by*

$$e^{\mathbf{A}} = \mathbf{T}e^{\mathbf{D}}\mathbf{T}^{-1}.$$

If $\mathbf{A}$ *is not diagonalizable, then there always exists a nonsingular matrix* $\mathbf{S}$, *such that* $e^{\mathbf{A}}$ *is given by*

$$e^{\mathbf{A}} = \mathbf{S}e^{\mathbf{J}}\mathbf{S}^{-1},$$

where $\mathbf{J}$ *is a Jordan matrix.*

Theorem 6 (Matrix Exponential (Permutation)) *Consider any two matrices* $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{n\times n}$. *If* $\mathbf{AB} = \mathbf{BA}$, *then*

$$e^{\mathbf{A}+\mathbf{B}} = e^{\mathbf{A}}e^{\mathbf{B}}.$$

1.4 Calculus

Definition 33 (Partial Derivative of a Function) *Given an scalar function $f(\mathbf{x},\mathbf{y},t) \in \mathbb{R}$, where $\mathbf{x} \in \mathbb{R}^n$; $\mathbf{y} \in \mathbb{R}^m$ and $t \geq 0$, then*

$$\frac{\partial}{\partial \mathbf{x}} f(\mathbf{x},\mathbf{y},t) = \left[\frac{\partial}{\partial x_1} f, \ldots, \frac{\partial}{\partial x_n} f\right]$$

and

$$\frac{\partial}{\partial \mathbf{y}} f(\mathbf{x},\mathbf{y},t) = \left[\frac{\partial}{\partial y_1} f, \ldots, \frac{\partial}{\partial y_m} f\right].$$

Definition 34 (Total Derivative of a Function) *For a scalar function $f(\mathbf{x},\mathbf{y},t) \in \mathbb{R}$, where $\mathbf{x} \in \mathbb{R}^n$; $\mathbf{y} \in \mathbb{R}^m$ and $t \geq 0$, then*

$$\frac{d}{dt} f(\mathbf{x},\mathbf{y},t) = \dot{f}(\mathbf{x},\mathbf{y},t) = \left[\frac{\partial}{\partial \mathbf{x}} f(\mathbf{x},\mathbf{y},t)\right]^\top \frac{\partial \mathbf{x}}{\partial t} + \left[\frac{\partial}{\partial \mathbf{y}} f(\mathbf{x},\mathbf{y},t)\right]^\top \frac{\partial \mathbf{y}}{\partial t} + \frac{\partial f}{\partial t}.$$

Therefore

$$\frac{d}{dt} f(\mathbf{x},\mathbf{y},t) = \dot{f}(\mathbf{x},\mathbf{y},t) = \left[\frac{\partial}{\partial \mathbf{x}} f(\mathbf{x},\mathbf{y},t)\right]^\top \dot{\mathbf{x}} + \left[\frac{\partial}{\partial \mathbf{y}} f(\mathbf{x},\mathbf{y},t)\right]^\top \dot{\mathbf{y}} + \frac{\partial f}{\partial t}.$$

Definition 35 (Gradient) *The gradient of a scalar function $f(\mathbf{x}) \in \mathbb{R}$, with $\mathbf{x} \in \mathbb{R}^n$, is*

$$\nabla_{\mathbf{x}} f(\mathbf{x}) := \left[\frac{\partial}{\partial \mathbf{x}} f(\mathbf{x})\right]^\top = \left[\frac{\partial}{\partial x_1} f, \ldots, \frac{\partial}{\partial x_n} f\right]^\top$$

In this work we use $\frac{\partial}{\partial \mathbf{x}} f(\mathbf{x})$ as a *column vector* and thus $\frac{\partial}{\partial \mathbf{x}} f(\mathbf{x})$ and $\nabla_{\mathbf{x}} f(\mathbf{x})$ are used *indistinctly*.

Definition 36 (Hessian) *The Hessian of a scalar function $f(\mathbf{x}) \in \mathbb{R}$, with $\mathbf{x} \in \mathbb{R}^n$, is*

$$\nabla^2_{\mathbf{x}} f(\mathbf{x}) := \frac{\partial^2}{\partial \mathbf{x}^2} f(\mathbf{x}) = \begin{bmatrix} \frac{\partial^2}{\partial x_1^2} f & \frac{\partial^2}{\partial x_1 \partial x_2} f & \cdots & \frac{\partial^2}{\partial x_1 \partial x_n} f \\ \frac{\partial^2}{\partial x_2 \partial x_1} f & \frac{\partial^2}{\partial x_2^2} f & \cdots & \frac{\partial^2}{\partial x_2 \partial x_n} f \\ \vdots & \vdots & \ddots & \\ \frac{\partial^2}{\partial x_n \partial x_1} f & \frac{\partial^2}{\partial x_n \partial x_2} f & \cdots & \frac{\partial^2}{\partial x_n^2} f \end{bmatrix}$$

If $f(\mathbf{x})$ is continuous, then $\nabla^2_{\mathbf{x}} f(\mathbf{x}) = \left[\nabla^2_{\mathbf{x}} f(\mathbf{x})\right]^\top$.

Definition 37 (Solution of a LTI Differential Equation) *Consider the following differential equation*

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x},$$

where $\mathbf{A} \in \mathbb{R}^{n \times n}$. *Then*

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0)$$

is its only possible solution.

Definition 38 (Matrix Exponential (cont.)) *If* $\mathbf{A}$ *is diagonalizable then*

$$e^{\mathbf{A}t} = \mathbf{T}e^{\mathbf{D}t}\mathbf{T}^{-1}.$$

Definition 39 (Matrix Exponential (cont.)) *If* $\mathbf{A}$ *is written in Jordan Canonical Form, such that* $\mathbf{A} = \mathbf{S}\mathbf{J}\mathbf{S}^{-1}$, *then*

$$e^{\mathbf{A}t} = \mathbf{S}e^{\mathbf{J}t}\mathbf{S}^{-1}.$$

Lemma 2 (Positive Definite Function Test via the Hessian) *Let* $f(\mathbf{x}) \in \mathbb{R}$ *and* $\mathbf{x} \in \mathbb{R}^n$. *Assume that* $\nabla f(\mathbf{x}) \in \mathbb{R}^n$ *and* $\nabla^2 f(\mathbf{x}) \in \mathbb{R}^{n \times n}$ *exist and they are continuous. If*

1. $f(\mathbf{0}) = 0$;
2. $\nabla f(\mathbf{0}) = \mathbf{0}$;
3. $\nabla^2 f(\mathbf{x}) > 0$, *for all* $\mathbf{x} \in \mathbb{R}^n$,

then $f(\mathbf{x})$ *is positive definite and radially unbounded with a unique global minimum at* $\mathbf{x} = \mathbf{0}$.

Lemma 3 (Positive Definite Function Test) *Let* $f(\mathbf{x}) \in \mathbb{R}$ *and* $\mathbf{x} \in \mathbb{R}^n$. *Assume that* $\nabla f(\mathbf{x}) \in \mathbb{R}^n$ *exists and that it is continuous. If*

1. $f(\mathbf{x}) > 0$, *for all* $\mathbf{x} \neq \mathbf{0}$ *and* $f(\mathbf{0}) = 0$;
2. $\mathbf{x}^\top \nabla f(\mathbf{x}) > 0$, *for all* $\mathbf{x} \neq \mathbf{0}$;
3. $f(\mathbf{x}) \to \infty$ *if* $\|\mathbf{x}\| \to \infty$,

then $f(\mathbf{x})$ *is positive definite and radially unbounded with a unique global minimum at* $\mathbf{x} = \mathbf{0}$ *[9].*

Theorem 7 (Mean Value Theorem) *Consider the bounded function $y = f(x) \in \mathbb{R}$, with $x \in \mathbb{R}$. Suppose that it is continuous in the domain $x \in [a, b]$ and differentiable in the domain $x \in (a, b)$. Then, there exists at least one point $c \in (a, b)$ such that*

$$\frac{f(b) - f(a)}{b - a} = \frac{\partial}{\partial x} f(c).$$

Hence

$$f(b) - f(a) = \frac{\partial}{\partial x} f(c)(b - a).$$

Theorem 8 (Leibniz Theorem) *Suppose that $f(x) \in \mathbb{R}$, with $x \in \mathbb{R}$, is continuous with continuous derivative. Consider $a(x) \in \mathbb{R}$ and $b(x) \in \mathbb{R}$ to be continuous with continuous derivatives. Then*

$$\frac{d}{dx}\left(\int_{a(x)}^{b(x)} f(t)dt\right) = \left(\frac{d}{dx} b(x)\right) f(b(x)) - \left(\frac{d}{dx} a(x)\right) f(a(x)).$$

1.5 Inequalities

Inequality 1 (Triangle's Inequality) *For any $a, b \in \mathbb{R}$,*

$$|a| - |b| \leq |a + b| \leq |a| + |b|.$$

Inequality 2 (Cauchy-Schwarz' Inequality) *For any $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$,*

$$|a_1 b_1 + \cdots + a_n b_n| = |\mathbf{a}^\top \mathbf{b}| \leq \|\mathbf{a}\| \|\mathbf{b}\| = \sqrt{a_1^2 + \cdots + a_n^2} \sqrt{b_1^2 + \cdots + b_n^2}.$$

Inequality 3 (Young's Inequality) *For any $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$ and any $c > 0$ it holds that*

$$\mathbf{a}^\top \mathbf{b} \leq \frac{c}{2} \|\mathbf{a}\|^2 + \frac{1}{2c} \|\mathbf{b}\|^2$$

Chapter 2

Fundamentals of Stability

2.1 Function Properties

This chapter presents the mathematical notions used throughout this manual. It has been gathered from several sources, among them it is worth to mention [6] and [19].

Definition 40 (Continuity) *A function* $f : [0, \infty) \to \mathbb{R}$ *is continuous on* $[0, \infty)$ *if for any given* $\epsilon_0 > 0$ *there exists a* $\delta(\epsilon_0, t_0)$ *such that* $\forall t_0, t \in [0, \infty)$ *for which* $|t - t_0| < \delta(\epsilon_0, t_0)$ *we have* $|f(t) - f(t_0)| < \epsilon_0$.

Definition 41 (Lipschitz) *A function* $f : [a, b] \to \mathbb{R}$ *is Lipschitz on* $[a, b]$ *if*

$$|f(x_1) - f(x_2)| \leq k|x_1 - x_2| \quad \forall x_1, x_2 \in [a, b]$$

where $k \geq 0$ *is a constant referred to as the Lipschitz constant.*

Definition 42 (Locally and Globally Positive Definiteness) *Function* V *is locally positive definite in* $B \subset \mathbb{R}^n$ *if* $V(\mathbf{0}) = 0$ *and* $V(\mathbf{x}) > 0$ $\forall \mathbf{x} \in B, \mathbf{x} \neq \mathbf{0}$. *If in addition* $V(\mathbf{x}) > 0$, $\forall \mathbf{x} \in \mathbb{R}^n$, *then* V *is said to be globally positive definite. For time dependent functions like* $W(\mathbf{x}, t)$, *then* W *is locally or globally positive definite if* $W(\mathbf{0}, t) = 0$ *and* $W(\mathbf{x}, t) \geq V(\mathbf{x})$, $\forall t \geq 0$, *where* $V(\mathbf{x})$ *is a locally or globally positive definite function.*

Definition 43 (Negative Definiteness) *The function* $V(\mathbf{x})$ *is negative definite if* $-V$ *is positive definite.*

Definition 44 (Positive and Negative Semi-Definiteness) *The function* $V(\mathbf{x})$ *is positive semi-definite if* $V(\mathbf{0}) = 0$ *and* $V(\mathbf{x}) \geq 0$ *for* $\mathbf{x} \neq 0$. $V(\mathbf{x})$ *is negative semi-definite if* $-V(\mathbf{x})$ *is positive semi-definite.*

Definition 45 (Radially Unboundedness) *A continuous function $V(\mathbf{x})$ is radially unbounded if $V(\mathbf{x}) \to \infty$ when $|\mathbf{x}| \to \infty$. $W(\mathbf{x}, t)$ is radially unbounded if $W(\mathbf{x}, t) \geq V(\mathbf{x})$, $\forall t \geq 0$, where $V(\mathbf{x})$ is radially unbounded.*

Definition 46 (Global and Isolated Minimum) *A continuous function $V(\mathbf{x})$ has a global and isolated minimum at $\mathbf{x} = \mathbf{0}$ if*

1. $\nabla_{\mathbf{x}} V(\mathbf{x}) = \mathbf{0} \Leftrightarrow \mathbf{x} = \mathbf{0}$.
2. $\nabla^2_{\mathbf{x}} V(\mathbf{x}) > 0$, $\forall \mathbf{x} \in \mathbb{R}^n$.

2.2 Lyapunov Theory

These are the basic concepts of stability theory for autonomous systems and non-autonomous systems. It is intended to be a quick search reference complement for the stability notions used in this document. All Theorems and Lemmas are stated without proofs. This section has been extracted from Ch. 3 and 4 of [7], Ch. 3 and 4 of [16] and Ch. 1 and 2 of [23].

Definition 47 (Equilibrium Point) *Consider the non-autonomous system.*

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t) \tag{2.1}$$

where $\mathbf{f} : \mathbb{R}^n \times \mathbb{R}^+_0 \to \mathbb{R}^n$. If $\mathbf{f}$ does not depend explicitly on time, i.e.,

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \tag{2.2}$$

the system (2.2) is called autonomous. The equilibrium points $\mathbf{x}^*$, for a non-autonomous and for an autonomous system, are defined by $\mathbf{f}(\mathbf{x}^*, t) \equiv \mathbf{0}\ \forall t \geq t_0$ and $\mathbf{f}(\mathbf{x}^*) \equiv \mathbf{0}\ \forall t \geq 0$, respectively.

Definition 48 (Stability) *Let the origin $\mathbf{x} = \mathbf{0}$ be an equilibrium point for (2.1). This equilibrium point is stable if $\forall \epsilon > 0,\ t_0 \geq 0\ \exists\ \delta = \delta(\epsilon, t_0) > 0$ s.t.*

$$\|\mathbf{x}(t_0)\| < \delta \Rightarrow \|\mathbf{x}(t)\| < \epsilon \quad \forall t \geq t_0 \geq 0.$$

Correspondingly, $\mathbf{x} = \mathbf{0}$ is a stable equilibrium point for (2.2) if $\forall \epsilon > 0,\ \exists\ \delta = \delta(\epsilon) > 0$ s.t.

$$\|\mathbf{x}(0)\| < \delta \Rightarrow \|\mathbf{x}(t)\| < \epsilon \quad \forall t \geq 0.$$

Definition 49 (Instability) *Let the origin be an equilibrium point of (2.1) or (2.2). This equilibrium point is unstable if it is not stable.*

Definition 50 (Asymptotic Stability) *An equilibrium point* $\mathbf{x} = \mathbf{0}$ *of (2.1) is Asymptotically Stable (AS) if it is stable, and if in addition* $\forall t_0 \geq 0,\ \exists \delta' = \delta'(t_0) > 0$ *s.t.*

$$\|\mathbf{x}(t_0)\| < \delta' \Rightarrow \|\mathbf{x}(t)\| \to 0\ t \to \infty.$$

The same equilibrium point for (2.2) is AS if it is stable and if $\exists\ \delta'$ *s.t.* $|\mathbf{x}(0)| < \delta' \Rightarrow |\mathbf{x}(t)| \to 0$ *as* $t \to \infty$.

Definition 51 (Global Asymptotic Stability) *The origin is a Globally Asymptotically Stable (GAS) equilibrium of (2.1) or of (2.2) if the origin is stable and if*

$$\|\mathbf{x}(t)\| \to 0 \quad t \to \infty, \ \ \forall \mathbf{x}(t_0) \in \mathbb{R}^n,\ t_0 \geq 0.$$

Definition 52 (Exponential Stability) *The origin is an Exponentially Stable (ES) equilibrium of (2.2) if* $\exists\ \alpha, \beta > 0$, *s.t.*

$$\|\mathbf{x}(t)\| \leq \beta e^{-\alpha t} \|\mathbf{x}(0)\|$$

Definition 53 (Global Exponential Stability) *The origin is a GES equilibrium of (2.2) if* $\exists\ \alpha, \beta > 0$, *s.t.*

$$\|\mathbf{x}(t)\| \leq \beta e^{-\alpha t} \|\mathbf{x}(0)\|,\ \forall \mathbf{x}(0) \in \mathbb{R}^n,\ t \geq 0.$$

Definition 54 (Lyapunov Function Candidate) *Consider (2.2), then* $V \in \mathbb{R}$ *is a Lyapunov function candidate if* $V(\mathbf{x})$ *is positive definite and* $\frac{\partial V(\mathbf{x})}{\partial \mathbf{x}}$ *is continuous.*

Definition 55 (Lyapunov Function) *The Lyapunov function candidate* $V(\mathbf{x})$ *is a Lyapunov function for (2.2) if*

$$\dot{V}(\mathbf{x}) = \left[\frac{\partial V(\mathbf{x})}{\partial \mathbf{x}}\right]^\top \dot{\mathbf{x}} = \left[\frac{\partial V(\mathbf{x})}{\partial \mathbf{x}}\right]^\top \mathbf{f}(\mathbf{x}) < 0 \quad \forall \mathbf{x} \neq \mathbf{0}.$$

Theorem 9 (Lyapunov Local Stability) *The origin is a stable equilibrium of (2.2) if there exists a Lyapunov function candidate* $V(\mathbf{x})$ *s.t.* $\dot{V}(\mathbf{x})$ *is locally negative semi-definite, and is locally AS if* $\dot{V}(\mathbf{x})$ *is locally negative definite.* $\diamond$

Theorem 10 (Lyapunov Global Asymptotic Stability) *The origin of (2.2) is GAS if there exists a radially unbounded Lyapunov function candidate* $V(\mathbf{x})$ *s.t.* $\dot{V}(\mathbf{x})$ *is negative definite.* $\diamond$

Theorem 11 (Lyapunov Exponential Stability) *The origin of (2.2) is ES if* $\exists\ \alpha_1, \alpha_2, \alpha_3 > 0$ *s.t.* $0 < \alpha_1\|\mathbf{x}\|^2 \leq V(\mathbf{x}) \leq \alpha_2\|\mathbf{x}\|^2$ *and* $\dot{V}(\mathbf{x}) \leq -\alpha_3\|\mathbf{x}\|^2$ ◇

Theorem 12 (Lyapunov Equation) *Consider the linear time-invariant system*

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x},$$

if there exists matrices $\mathbf{P}$ *and* $\mathbf{Q}$ *such that*

$$\mathbf{P}\mathbf{A} + \mathbf{A}^\top\mathbf{P} = -\mathbf{Q},$$

for $\mathbf{P} = \mathbf{P}^\top > 0$ *and* $\mathbf{Q} = \mathbf{Q}^\top > 0$ *then* $\mathbf{x} = \mathbf{0}$ *is GES.* ◇

Theorem 13 ((Barbashin-Krasovskii) LaSalle Invariance Theorem) *Consider the autonomous system (2.2) and suppose that* $\mathbf{x} = \mathbf{0}$ *is an equilibrium point. Assume that there exists a radially unbounded Lyapunov function candidate* $V(\mathbf{x})$, *s.t.* $\dot{V}(\mathbf{x}) \leq 0,\ \forall \mathbf{x} \in \mathbb{R}^n$. *Define the set*

$$\Omega = \{\mathbf{x} \in \mathbb{R}^n : V(\mathbf{x}) = 0\}.$$

If $\mathbf{x}(0) = \mathbf{0}$ *is the only initial state in* Ω *whose corresponding solition* $\mathbf{x}(t)$ *remains forever in* Ω, *then* $\mathbf{x}(0) = \mathbf{0}$ *is GAS.* ◇

Lemma 4 (Barbălat's Lemma) *Consider function* $f(t) \in \mathbb{R}$. *If* $\dot{f} \in \mathcal{L}_\infty$ *and*

$$\lim_{t\to\infty} \int_0^t f(\sigma)d\sigma < \infty$$

exists then $f(t) \to 0$ *as* $t \to \infty$. ◇

Lemma 5 (Barbălat's Lemma) *Consider function* $f(t) \in \mathbb{R}$. *If* $f, \dot{f} \in \mathcal{L}_\infty$ *and* $f \in \mathcal{L}_p$ *for some* $p \in [1, \infty)$, *then* $f(t) \to \mathbf{0}$ *as* $t \to \infty$. ◇

Lemma 6 (Barbălat's Lemma) *Consider function* $f(t) \in \mathbb{R}$. *If* $f \in \mathcal{L}_2$ *and* $\dot{f} \in \mathcal{L}_\infty$, *then* $f(t) \to \mathbf{0}$ *as* $t \to \infty$. ◇

Chapter 3

Robot Kinematics

A robot manipulator can be represented, from a mechanical viewpoint, as a kinematic chain of *links* connected by *joints*. The first link in the manipulator is attached to the base, while the last link is know as the *end-effector*. The full structure motion is obtained by composition of each link with respect to the previous one. This is know as the *direct kinematics* [14].

3.1 Pose

A rigid body can be fully described in the space by its (*pose*) position and orientation with respect to a frame of reference. If a minimal representation is employed, then the pose is given by six values, three to describe its position (p_x, p_y, p_z) and three for its orientation (α, β, γ). However, in general, the orientation is represented by a rotation matrix.

3.2 Rotation Matrix

A rotation matrix $\mathbf{R} \in \mathbb{R}^{3\times 3}$ represents the rotation of a body with respect to a reference frame and it is given as

$$\mathbf{R} := \begin{bmatrix} \mathbf{r}_x & \mathbf{r}_y & \mathbf{r}_z, \end{bmatrix}$$

where $\mathbf{r}_i \in \mathbb{R}^3$ is the unit vector of the body reference frame, along the $i-$axis, with respect to the reference frame and $i \in \{x, y, z\}$. Clearly, since $\mathbf{r}_i$ are unit vectors and they are orthogonal one from another, then $\mathbf{R}\mathbf{R}^\top = \mathbf{R}^\top\mathbf{R} = \mathbf{I}$.

When the rotation happens on a single axis the resulting (basic) rotation matrix $\mathbf{R}$ is given as

$$\mathbf{R_x}(\alpha) := \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\alpha) & -\sin(\alpha) \\ 0 & \sin(\alpha) & \cos(\alpha) \end{bmatrix}$$

$$\mathbf{R_y}(\beta) := \begin{bmatrix} \cos(\beta) & 0 & -\sin(\beta) \\ 0 & 1 & 0 \\ \sin(\beta) & 0 & \cos(\beta) \end{bmatrix}$$

$$\mathbf{R_z}(\gamma) := \begin{bmatrix} \cos(\gamma) & -\sin(\gamma) & 0 \\ \sin(\gamma) & \cos(\gamma) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

3.3 Homogeneous Transformations

The position with respect to a frame of reference is represented by vector $\mathbf{p} \in \mathbb{R}^3$, while the rotation matrix $\mathbf{R}$ describes the *orientation*. Then, the *pose* can be represented as a *Homogeneous Transformation* matrix $\mathbf{T} \in \mathbb{R}^{4\times4}$ given by

$$\mathbf{T} := \begin{bmatrix} \mathbf{R} & \mathbf{p} \\ \mathbf{0} & 1 \end{bmatrix}$$

3.4 Direct Kinematics

The *direct kinematics* objective is to compute the *end-effector* pose as a function of the n joint position variables $\mathbf{q} \in \mathbb{R}^n$. This can be done following the *Denavit-Hartenberg Convention* and calculating the product of every $\mathbf{T}$ matrix in ascending order:

$$\mathbf{T}_0^n(\mathbf{q}) = \mathbf{T}_0^1(\mathbf{q})\mathbf{T}_1^2(\mathbf{q})\mathbf{T}_2^3(\mathbf{q}) \dots \mathbf{T}_{n-1}^n(\mathbf{q})$$

Refer to Section 2.8.2 of [14] for the Denavit-Hartenberg Convention.

When a minimal representation $\boldsymbol{\phi} \in \mathbb{R}^3$ of the orientation of the end-effector is employed. Then, its pose is given by $\mathbf{x} = [\mathbf{p}^\top, \boldsymbol{\phi}^\top]^\top \in \mathbb{R}^6$. In such a case, the direct kinematics equation is given by

$$\mathbf{x} = \mathbf{k}(\mathbf{q}),$$

where $\mathbf{k}(\mathbf{q})$ represents the pose as a function of the joint position, i.e., $\mathbf{k}(\mathbf{q}) := [\mathbf{p}^\top(\mathbf{q}), \boldsymbol{\phi}^\top(\mathbf{q})]^\top$.

3.5 Differential Kinematics

The *differential kinematics* gives the relationship between joint velocities and the corresponding end-effector linear and angular velocity. This relationship is described by the *geometric Jacobian* matrix, which depends of the manipulator configuration, or the *analytical Jacobian* obtained differentiating the direct kinematics function.

3.5.1 Geometric Jacobian

The *end-effector* linear velocity $\mathbf{v} := \dot{\mathbf{p}} \in \mathbb{R}^3$ and angular velocity $\boldsymbol{\omega} \in \mathbb{R}^3$ can be obtained, respectively, as a function of the joint velocities $\dot{\mathbf{q}} = \frac{d}{dt}\mathbf{q} \in \mathbb{R}^n$ from

$$\mathbf{v} := \mathbf{J}_v(\mathbf{q})\dot{\mathbf{q}}$$

and

$$\boldsymbol{\omega} := \mathbf{J}_\omega(\mathbf{q})\dot{\mathbf{q}}.$$

Matrices $\mathbf{J}_v(\mathbf{q}) \in \mathbb{R}^{3\times n}$ and $\mathbf{J}_\omega(\mathbf{q}) \in \mathbb{R}^{3\times n}$ relate the contribution of $\dot{\mathbf{q}}$ to the linear and angular velocities, respectively.

The *geometric Jacobian* $\mathbf{J}_G \in \mathbb{R}^{6\times n}$ is then composed as

$$\mathbf{J}_G(\mathbf{q}) := \begin{bmatrix} \mathbf{J}_v(\mathbf{q}) \\ \mathbf{J}_\omega(\mathbf{q}) \end{bmatrix}$$

Refer to Section 3.1.3 of [14] for the explicit computation of $\mathbf{J}_G$.

3.5.2 Analytical Jacobian

When the pose of the end-effector is represented with a minimal number of parameters, then the linear velocity satisfies

$$\mathbf{v} = \frac{\partial \mathbf{p}(\mathbf{q})}{\partial \mathbf{q}}\dot{\mathbf{q}} = \mathbf{J}_v(\mathbf{q})\dot{\mathbf{q}}$$

While the rotational velocity comes from the time derivate of the minimal orientation representation $\boldsymbol{\phi} \in \mathbb{R}^3$. Hence

$$\dot{\boldsymbol{\phi}} = \frac{\partial \boldsymbol{\phi}(\mathbf{q})}{\partial \mathbf{q}}\dot{\mathbf{q}} = \mathbf{J}_\phi(\mathbf{q})\dot{\mathbf{q}}.$$

In this case, the differential kinematics equation is

$$\dot{\mathbf{x}} := \begin{bmatrix} \mathbf{J}_v(\mathbf{q}) \\ \mathbf{J}_\phi(\mathbf{q}) \end{bmatrix} \dot{\mathbf{q}} = \mathbf{J}_A(\mathbf{q})\dot{\mathbf{q}},$$

where $\mathbf{J}_A(\mathbf{q})$ is called the analytical Jacobian.

Chapter 4

Robot Dynamics

4.1 The Euler-Lagrange Equations of Motion

The *Lagrangian* $L(\mathbf{q}, \dot{\mathbf{q}}) \in \mathbb{R}$ of a mechanical system, with $n-$DoF, is defined as

$$L(\mathbf{q}, \dot{\mathbf{q}}) := \mathcal{K}(\mathbf{q}, \dot{\mathbf{q}}) - \mathcal{U}(\mathbf{q}),$$

where $\mathcal{K}(\mathbf{q}, \dot{\mathbf{q}}) \in \mathbb{R}$ is the kinetic energy and $\mathcal{U}(\mathbf{q}) \in \mathbb{R}$ is the potential energy. The *generalized* position and velocity are given by $\mathbf{q} \in \mathbb{R}^n$ and $\dot{\mathbf{q}} \in \mathbb{R}^n$, respectively.

The *kinetic energy* is the energy associated to the motion of the system and it satisfies

$$\mathcal{K}(\mathbf{q}, \dot{\mathbf{q}}) = \sum_{i=1}^{n} \mathcal{K}_i(\mathbf{q}, \dot{\mathbf{q}}),$$

where $\mathcal{K}_i(\mathbf{q}, \dot{\mathbf{q}})$ is the kinetic energy of each DoF, which is given by

$$\mathcal{K}_i(\mathbf{q}, \dot{\mathbf{q}}) := \frac{1}{2} m_i \mathbf{v}_i^\top \mathbf{v}_i + \frac{1}{2} \boldsymbol{\omega}_i^\top \mathbf{R}_i \mathbf{I}_i \mathbf{R}_i^\top \boldsymbol{\omega}_i,$$

where $m_i > 0$ is the mass; $\mathbf{I}_i$ is the moment of inertia matrix; $\mathbf{R}_i$ is the rotation matrix from the base of the link to its center of mass; $\mathbf{v}_i$ is the linear velocity; and $\boldsymbol{\omega}_i$ is the angular velocity of the ith-link. Using the geometric Jacobian definition we have that

$$\mathcal{K}_i(\mathbf{q}, \dot{\mathbf{q}}) := \frac{1}{2} m_i \dot{\mathbf{q}}^\top \mathbf{J}_{vi}(\mathbf{q})^\top \mathbf{J}_{vi}(\mathbf{q}) \dot{\mathbf{q}} + \frac{1}{2} \dot{\mathbf{q}}^\top \mathbf{J}_{\omega i}(\mathbf{q})^\top \mathbf{R}_i \mathbf{I}_i \mathbf{R}_i^\top \mathbf{J}_{\omega i}(\mathbf{q}) \dot{\mathbf{q}}.$$

Hence $\mathcal{K}(\mathbf{q}, \dot{\mathbf{q}})$ satisfies

$$\mathcal{K}(\mathbf{q}, \dot{\mathbf{q}}) = \frac{1}{2} \dot{\mathbf{q}}^\top \mathbf{M}(\mathbf{q}) \dot{\mathbf{q}}, \tag{4.1}$$

where the *inertia matrix* $\mathbf{M}(\mathbf{q})$ is

$$\mathbf{M}(\mathbf{q}) := \sum_{i=1}^{n} \left[m_i \mathbf{J}_{vi}(\mathbf{q})^\top \mathbf{J}_{vi}(\mathbf{q}) + \mathbf{J}_{\omega i}(\mathbf{q})^\top \mathbf{R}_i \mathbf{I}_i \mathbf{R}_i^\top \mathbf{J}_{\omega i}(\mathbf{q}) \right].$$

If the potential energy is only due to gravity, then

$$\mathcal{U}(\mathbf{q}) := \sum_{i=1}^{n} \mathcal{U}_i(\mathbf{q}) = g \sum_{i=1}^{n} m_i h_i(\mathbf{q}),$$

where $h_i(\mathbf{q})$ is the *height* of the ith-mass and g is the acceleration of gravity constant.

Remark 1 *There are other sources of potential energy, such as* linear *springs. The energy stored in a spring, due to Hooke's Law, is*

$$\mathcal{U}_R(\mathbf{q}) := \frac{1}{2} k \|\mathbf{q} - \mathbf{q}_*\|^2,$$

where $k > 0$ is the (constant) string stiffness*, $\mathbf{q}_* \in \mathbb{R}^n$ is the string* equilibrium *position.* ◇

If the robot is *fully-actuated* and it does not contain any *energy dissipation* element, then its dynamical behavior can be described by the Euler-Lagrange equations of motion for *conservative* systems. These are given by

$$\frac{d}{dt}\left(\frac{\partial}{\partial \dot{\mathbf{q}}} L(\mathbf{q}, \dot{\mathbf{q}})\right) - \frac{\partial}{\partial \mathbf{q}} L(\mathbf{q}, \dot{\mathbf{q}}) = \boldsymbol{\tau}, \tag{4.2}$$

where $\boldsymbol{\tau} \in \mathbb{R}^n$ represents the external forces.

Since $\frac{\partial}{\partial \dot{\mathbf{q}}} L(\mathbf{q}, \dot{\mathbf{q}}) = \mathbf{M}(\mathbf{q})\dot{\mathbf{q}}$, then (4.2) can be compactly written as

$$\mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} + \dot{\mathbf{M}}(\mathbf{q})\dot{\mathbf{q}} - \frac{1}{2}\frac{\partial}{\partial \mathbf{q}} \dot{\mathbf{q}}^\top \mathbf{M}(\mathbf{q})\dot{\mathbf{q}} + \frac{\partial}{\partial \mathbf{q}} \mathcal{U}(\mathbf{q}) = \boldsymbol{\tau}.$$

Defining

$$\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} := \dot{\mathbf{M}}(\mathbf{q})\dot{\mathbf{q}} - \frac{1}{2}\frac{\partial}{\partial \mathbf{q}} \dot{\mathbf{q}}^\top \mathbf{M}(\mathbf{q})\dot{\mathbf{q}}$$

and

$$\mathbf{g}(\mathbf{q}) := \frac{\partial}{\partial \mathbf{q}} \mathcal{U}(\mathbf{q}),$$

we get

$$\mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} + \mathbf{g}(\mathbf{q}) = \boldsymbol{\tau}, \tag{4.3}$$

where vector $\mathbf{C}(\mathbf{q},\dot{\mathbf{q}})\dot{\mathbf{q}} \in \mathbb{R}^n$ represents the Coriolis and centrifugal forces and $\mathbf{g}(\mathbf{q}) \in \mathbb{R}^n$ stands for the gravity effects.

Model (4.3) has some important *properties* that are listed below [6].

P1. The inertia matrix is symmetric, i.e., $\mathbf{M}(\mathbf{q}) = \mathbf{M}(\mathbf{q})^\top$.

P2. The inertia matrix is positive definite, i.e., $\exists\, m_1 > 0$ such that $\mathbf{M}(\mathbf{q}) \geq m_1\mathbf{I} > 0$.

P3. (*Passivity Property*) The Coriolis and inertia matrices are related, for all $\mathbf{q}$ and all $\dot{\mathbf{q}}$, by

$$\dot{\mathbf{q}}^\top \left[\dot{\mathbf{M}}(\mathbf{q}) - 2\mathbf{C}(\mathbf{q},\dot{\mathbf{q}})\right]\dot{\mathbf{q}} = 0. \tag{4.4}$$

P4. (*Skewsymmetric Property*) If the Coriolis matrix $\mathbf{C}(\mathbf{q},\dot{\mathbf{q}}) = \left[C_{kj}(\mathbf{q},\dot{\mathbf{q}})\right]$ is obtained using the Christoffel symbols of the first-kind $C_{kj}(\mathbf{q},\dot{\mathbf{q}}) = \sum_{i=1}^n c_{ijk}(\mathbf{q})\dot{q}_i$, where

$$c_{ijk}(\mathbf{q}) = \frac{1}{2}\left[\frac{\partial M_{kj}(\mathbf{q})}{\partial q_i} + \frac{\partial M_{ki}(\mathbf{q})}{\partial q_j} - \frac{\partial M_{ij}(\mathbf{q})}{\partial q_k}\right],$$

then, matrix $\left[\dot{\mathbf{M}}(\mathbf{q}) - 2\mathbf{C}(\mathbf{q},\dot{\mathbf{q}})\right]$ is skewsymmetric. Thus, for all $\mathbf{y} \in \mathbb{R}^n$,

$$\mathbf{y}^\top \left[\dot{\mathbf{M}}(\mathbf{q}) - 2\mathbf{C}(\mathbf{q},\dot{\mathbf{q}})\right]\mathbf{y} = 0.$$

Moreover

$$\dot{\mathbf{M}}(\mathbf{q}) = \mathbf{C}(\mathbf{q},\dot{\mathbf{q}}) + \mathbf{C}(\mathbf{q},\dot{\mathbf{q}})^\top.$$

For robots having *only* revolute joints, the following holds [6]:

P5. There exist constants $k_g, L_g > 0$, such that, for all $\mathbf{q} \in \mathbb{R}^n$,

$$\|\mathbf{g}(\mathbf{q})\| \leq k_g,$$

and

$$\left\| \frac{\partial}{\partial \mathbf{q}} \mathbf{g}(\mathbf{q}) \right\| \leq L_g.$$

P6. There exists $k_{gi} > 0$, such that $|g_i(\mathbf{q})| \leq k_{gi}$, for all $\mathbf{q} \in \mathbb{R}^n$ and all $i \in \{1, \ldots, n\}$.

P7. There exists $m_2 > 0$ such that $\mathbf{M}(\mathbf{q}) \leq m_2 \mathbf{I}$, for all $\mathbf{q} \in \mathbb{R}^n$. In fact,

$$m_2 = n \left(\max_{i,j,\mathbf{q}} \left\{ |M_{ij}(\mathbf{q})| \right\} \right).$$

P8. There exists $k_c > 0$ such that $\|\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}}\| \leq k_c \|\dot{\mathbf{q}}\|^2$, for all $\mathbf{q} \in \mathbb{R}^n$. In fact,

$$k_c = n^2 \left(\max_{i,j,k,\mathbf{q}} \left\{ |c_{ijk}(\mathbf{q})| \right\} \right),$$

and the elements $c_{ijk}(\mathbf{q})$ are the Christofel symbols defined in Property P4.

Remark 2 *Although vector* $\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}}$ *is unique, matrix* $\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})$ *is not. However, all matrices* $\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})$ *satisfy Property P3.* ◇

Remark 3 *It is customary, under a proper definition of the origin of coordinates, that the minimum heigh is equal to zero. Therefore, in general and for all* $\mathbf{q} \in \mathbb{R}^n$, *it holds that* $\mathcal{U}(\mathbf{q}) \geq 0$. ◇

Remark 4 *Constant* $k_g > 0$ *can be found as*

$$k_g = \| \max_{\forall \mathbf{q} \in \mathbb{R}^n} \mathbf{g}(\mathbf{q}) \|.$$

Constants $k_{gi} > 0$ *are given by*

$$k_{gi} = \max_{\forall \mathbf{q} \in \mathbb{R}^n} \{|g_i(\mathbf{q})|\}.$$

Finally, $L_g > 0$ *can be calculated as*

$$L_g = \sqrt{\lambda_M \{\mathbf{G}^\top \mathbf{G}\}},$$

where

$$\mathbf{G} := \max_{\forall \mathbf{q} \in \mathbb{R}^n} \frac{\partial}{\partial \mathbf{q}} \mathbf{g}(\mathbf{q}).$$

For example, suppose that

$$\mathbf{g}(\mathbf{q}) = \begin{bmatrix} p_1 \sin(q_1) + p_2 \sin(q_1 + q_2) \\ p_2 \sin(q_1 + q_2) \end{bmatrix},$$

where $p_1 > 0$ and $p_2 > 0$ are some physical parameters.

Then $\max_{\forall \mathbf{q} \in \mathbb{R}^n} \mathbf{g}(\mathbf{q}) = [p_1 + p_2,\, p_2]^\top$, thus

$$k_g = \sqrt{p_1^2 + 2p_1p_2 + 2p_2^2},$$

and $k_{g1} = p_1 + p_2$, $k_{g2} = p_2$.

Since

$$\frac{\partial}{\partial \mathbf{q}} \mathbf{g}(\mathbf{q}) = \begin{bmatrix} p_1 \cos(q_1) + p_2 \cos(q_1 + q_2) & p_2 \cos(q_1 + q_2) \\ p_2 \cos(q_1 + q_2) & p_2 \cos(q_1 + q_2) \end{bmatrix},$$

then

$$\max_{\forall \mathbf{q} \in \mathbb{R}^n} \frac{\partial}{\partial \mathbf{q}} \mathbf{g}(\mathbf{q}) = \begin{bmatrix} p_1 + p_2 & p_2 \\ p_2 & p_2 \end{bmatrix},$$

thus

$$L_g = \sqrt{\lambda_M \left\{ \begin{bmatrix} (p_1 + p_2)^2 + p_2^2 & p_1p_2 + 2p_2^2 \\ p_1p_2 + 2p_2^2 & 2p_2^2 \end{bmatrix} \right\}}$$

$\diamond$

The following fact is well-known in the literature of robot control and it is the cornerstone of the *passivity based control* of mechanical systems [12, 11].

Fact 2 *The robot system (4.3) is* passive *from (input) force to (output) velocity. More precisely, it holds that*

$$\dot{\mathcal{H}}(\mathbf{q}, \dot{\mathbf{q}}) = \dot{\mathbf{q}}^\top \boldsymbol{\tau},$$

where $\mathcal{H}(\mathbf{q}, \dot{\mathbf{q}})$ is the (total energy) Hamiltonian function

$$\mathcal{H}(\mathbf{q}, \dot{\mathbf{q}}) = \mathcal{K}(\mathbf{q}, \dot{\mathbf{q}}) + \mathcal{U}(\mathbf{q}). \tag{4.5}$$

$\diamond$

Proof. Using the value of the kinetic energy (4.1), the Hamiltonian can be written as

$$\mathcal{H}(\mathbf{q},\dot{\mathbf{q}}) = \frac{1}{2}\dot{\mathbf{q}}^\top \mathbf{M}(\mathbf{q})\dot{\mathbf{q}} + \mathcal{U}(\mathbf{q}).$$

From Properties P1 and P2 and Remark 3 it holds that $\mathcal{H}(\mathbf{q},\dot{\mathbf{q}}) \geq 0$.

The time derivative of $\mathcal{H}(\mathbf{q},\dot{\mathbf{q}})$, using P1, is

$$\dot{\mathcal{H}}(\mathbf{q},\dot{\mathbf{q}}) = \dot{\mathbf{q}}^\top \mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} + \frac{1}{2}\dot{\mathbf{q}}^\top \dot{\mathbf{M}}(\mathbf{q})\dot{\mathbf{q}} + \left[\frac{\partial}{\partial \mathbf{q}}\mathcal{U}(\mathbf{q})\right]^\top \dot{\mathbf{q}}.$$

Using (4.3) in the previous equation yields

$$\dot{\mathcal{H}}(\mathbf{q},\dot{\mathbf{q}}) = -\dot{\mathbf{q}}^\top \left[\mathbf{C}(\mathbf{q},\dot{\mathbf{q}})\dot{\mathbf{q}} + \mathbf{g}(\mathbf{q}) - \boldsymbol{\tau}\right] + \frac{1}{2}\dot{\mathbf{q}}^\top \dot{\mathbf{M}}(\mathbf{q})\dot{\mathbf{q}} + \left[\frac{\partial}{\partial \mathbf{q}}\mathcal{U}(\mathbf{q})\right]^\top \dot{\mathbf{q}}.$$

From P3 and the definition of the gravity vector we finally get

$$\dot{\mathcal{H}}(\mathbf{q},\dot{\mathbf{q}}) = \dot{\mathbf{q}}^\top \boldsymbol{\tau}.$$

◁

Remark 5 *Fact 2 implies that the robot cannot generate energy by itself. This can be easily inferred from the fact that*

$$\mathcal{H}(t) = \mathcal{H}(0) + \int_0^t \dot{\mathbf{q}}(\sigma)^\top \boldsymbol{\tau}(\sigma)d\sigma \tag{4.6}$$

where $\mathcal{H}(t) := \mathcal{H}(\mathbf{q}(t),\dot{\mathbf{q}}(t))$ *and* $\mathcal{H}(0) := \mathcal{H}(\mathbf{q}(0),\dot{\mathbf{q}}(0))$.

Equation (4.6) *means that* the energy at any time instant t is equal to the initial energy plus the energy provided by the external input.

Therefore, if $\boldsymbol{\tau} = \mathbf{0}$ *then* $\mathcal{H}(t) = \mathcal{H}(0)$ *and hence we have established the* energy conservation *law for the robot model* (4.3). ◇

When the robot exhibits *friction*, then (4.2) does not reflect such behavior. In this case one requires to employ the *dissipative* version of (4.2). For, let us define a *Rayleigh dissipation function* $\mathcal{R} \in \mathbb{R}$ as a function of the robot's velocity such that

$$\frac{\partial}{\partial \dot{\mathbf{q}}}\mathcal{R}(\mathbf{0}) = \mathbf{0}$$

and

$$\dot{\mathbf{q}}^\top \frac{\partial}{\partial \dot{\mathbf{q}}}\mathcal{R}(\dot{\mathbf{q}}) \geq 0.$$

Remark 6 *An example of a Rayleigh dissipation function is*

$$\mathcal{R}(\dot{\mathbf{q}}) = \frac{1}{2}\|\dot{\mathbf{q}}\|^2,$$

which satisfies $\frac{\partial}{\partial \dot{\mathbf{q}}}\mathcal{R}(\dot{\mathbf{q}}) = \dot{\mathbf{q}}$. *Therefore it holds that* $\frac{\partial}{\partial \dot{\mathbf{q}}}\mathcal{R}(\mathbf{0}) = \mathbf{0}$ *and* $\dot{\mathbf{q}}^\top \frac{\partial}{\partial \dot{\mathbf{q}}}\mathcal{R}(\dot{\mathbf{q}}) > 0$. $\diamond$

Modeling the friction effects via $\mathcal{R}(\dot{\mathbf{q}})$ yields the *dissipative* Euler-Lagrange equations of motion

$$\frac{d}{dt}\left(\frac{\partial}{\partial \dot{\mathbf{q}}}L(\mathbf{q},\dot{\mathbf{q}})\right) - \frac{\partial}{\partial \mathbf{q}}L(\mathbf{q},\dot{\mathbf{q}}) + \frac{\partial}{\partial \dot{\mathbf{q}}}\mathcal{R}(\dot{\mathbf{q}}) = \boldsymbol{\tau}$$

and thus, the robot dynamic behavior is given by

$$\mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{C}(\mathbf{q},\dot{\mathbf{q}})\dot{\mathbf{q}} + \mathbf{g}(\mathbf{q}) + \frac{\partial}{\partial \dot{\mathbf{q}}}\mathcal{R}(\dot{\mathbf{q}}) = \boldsymbol{\tau}. \tag{4.7}$$

Fact 3 *The robot system (4.7) is* output-strictly passive *from (input) force to (output) velocity. More precisely, it holds that*

$$\dot{\mathcal{H}}(\mathbf{q},\dot{\mathbf{q}}) = -\dot{\mathbf{q}}^\top \frac{\partial}{\partial \dot{\mathbf{q}}}\mathcal{R}(\dot{\mathbf{q}}) + \dot{\mathbf{q}}^\top \boldsymbol{\tau},$$

where $\mathcal{H}(\mathbf{q},\dot{\mathbf{q}})$ *is defined in* (4.5). $\diamond$

Proof. The proof follows as the proof of Fact 2 with the additional fact that $\dot{\mathbf{q}}^\top \frac{\partial}{\partial \dot{\mathbf{q}}}\mathcal{R}(\dot{\mathbf{q}})$ is not negative. $\triangleleft$

Remark 7 *Fact 3 ensures that energy is* dissipated *through friction. In this case, after integration of* $\dot{\mathcal{H}}(\mathbf{q},\dot{\mathbf{q}})$, *we get*

$$\mathcal{H}(t) = \mathcal{H}(0) - \int_0^t \dot{\mathbf{q}}(\sigma)^\top \frac{\partial}{\partial \dot{\mathbf{q}}}\mathcal{R}(\dot{\mathbf{q}}(\sigma))d\sigma + \int_0^t \dot{\mathbf{q}}(\sigma)^\top \boldsymbol{\tau}(\sigma)d\sigma, \tag{4.8}$$

which reads the energy at any time instant t is equal to the initial energy minus the energy dissipated through friction plus the energy provided by the external input. $\diamond$

Since friction injects *damping* that decreases the energy of the robot, then it usually helps in the set-point stabilization setting. Therefore, in the rest of this manuscript we consider that friction is negligible and hence we work only with the conservative system (4.3).

Now, defining the *state* of the robot as $\mathbf{x} = [\mathbf{q}^\top, \dot{\mathbf{q}}^\top]^\top \in \mathbb{R}^{2n}$, we have the following *state-space* representation of (4.3)

$$\begin{aligned}\dot{\mathbf{x}}_1 &= \mathbf{x}_2,\\ \dot{\mathbf{x}}_2 &= -\,\mathbf{M}(\mathbf{x}_1)^{-1}\left[\mathbf{C}(\mathbf{x}_1,\mathbf{x}_2)\mathbf{x}_2 + \mathbf{g}(\mathbf{x}_1) - \boldsymbol{\tau}\right].\end{aligned} \tag{4.9}$$

Clearly, the equilibrium points $\bar{\mathbf{x}}$ of (4.9) must satisfy

$$\begin{aligned}\bar{\mathbf{x}}_2 &= \mathbf{0},\\ \mathbf{g}(\bar{\mathbf{x}}_1) &= \boldsymbol{\tau},\end{aligned}$$

and in *open-loop*, i.e., when $\boldsymbol{\tau} = \mathbf{0}$, then

$$\begin{aligned}\bar{\mathbf{x}}_2 &= \mathbf{0},\\ \mathbf{g}(\bar{\mathbf{x}}_1) &= \mathbf{0}.\end{aligned}$$

This last equation means that all the possible position equilibria are only due to the gravity term.

Part II

Position Control

Chapter 5

Fundamentals of Energy Shaping

The control objective behind *position control* is to ensure that the robot is regulated at a given desired position. That is, given a constant desired position $\mathbf{q}_d \in \mathbb{R}^n$, design the controller $\boldsymbol{\tau}$ such that $(\dot{\mathbf{q}}, \tilde{\mathbf{q}}) = (\mathbf{0}, \mathbf{0})$ is GAS, i.e.,

$$\lim_{t\to\infty} \dot{\mathbf{q}}(t) = \mathbf{0}, \qquad \lim_{t\to\infty} \tilde{\mathbf{q}}(t) = \mathbf{0},$$

for all initial conditions $\dot{\mathbf{q}}(0), \tilde{\mathbf{q}}(0) \in \mathbb{R}^n$, where $\tilde{\mathbf{q}} := \mathbf{q} - \mathbf{q}_d$.

In order to achieve such objective first note that, from the previous chapter, the robot dynamics exhibit two important facts: 1) a Rayleigh dissipation function injects damping in the robot dynamics and therefore the energy is dissipated; and 2) the *open-loop* equilibria are only due to the gradient of the potential energy.

These observations are the standing point of the *Energy Shaping plus Damping Injection* (ES+DI) control methodology that is employed in this part of the manual. The main idea behind energy shaping is to **design the controller as a dynamical system with its own kinetic and potential energy such that the sum of both, the robot and the controller, energies has a *global and unique* minimum at the desired point $(\dot{\mathbf{q}}, \tilde{\mathbf{q}}) = (\mathbf{0}, \mathbf{0})$. Then, damping is added to achieve (global) asymptotic stability and, finally, the robot-controller interconnection is given by the gradient of the controller potential energy [11].** As it will become later, this is achieved by using Euler-Lagrange controllers [10].

A Historical Remark. The roots of ES+DI can be traced back to the early work of Lagrange in 1788, more than a century later, Lyapunov published its famous stability theory in 1892 (in Russian) and the LaSalle Invariance Theorem dates back to 1961. However, the first stabilization result for (nonlinear) mechanical systems using ES+DI is due to Takegaki and Arimoto in 1981 [20]. The core behind ES+DI is passivity and it is until 1989 that the term passivity based control is introduced by Ortega and Spong [12] (see also [11]). This work builds upon these inspiring results.

Consider $\boldsymbol{\theta}, \dot{\boldsymbol{\theta}} \in \mathbb{R}^n$ to be the *position* and the *velocity* of the controller. Let $\mathcal{K}_c(\dot{\boldsymbol{\theta}})$ and $\mathcal{U}_c(\mathbf{q}, \boldsymbol{\theta}, \mathbf{q}_d)$ be defined as the controller kinetic and potential energy, respectively; and let $\mathcal{R}_c(\dot{\boldsymbol{\theta}})$ be a controller dissipation function. Then, the (dissipative) EL-equations of motion of the controller are

$$\frac{d}{dt}\left(\frac{\partial}{\partial \dot{\boldsymbol{\theta}}} L_c(\mathbf{q}, \boldsymbol{\theta}, \mathbf{q}_d, \dot{\boldsymbol{\theta}})\right) - \frac{\partial}{\partial \boldsymbol{\theta}} L_c(\mathbf{q}, \boldsymbol{\theta}, \mathbf{q}_d, \dot{\boldsymbol{\theta}}) + \frac{\partial}{\partial \dot{\boldsymbol{\theta}}} \mathcal{R}_c(\dot{\boldsymbol{\theta}}) = \mathbf{0},$$

where $L_c(\mathbf{q}, \boldsymbol{\theta}, \mathbf{q}_d, \dot{\boldsymbol{\theta}})$ is the controller Lagrangian that is given by

$$L_c(\mathbf{q}, \boldsymbol{\theta}, \mathbf{q}_d, \dot{\boldsymbol{\theta}}) := \mathcal{K}_c(\dot{\boldsymbol{\theta}}) - \mathcal{U}_c(\mathbf{q}, \boldsymbol{\theta}, \mathbf{q}_d).$$

Remark 8 *Since the controller is a* virtual *dynamical system, then $\mathcal{K}_c(\dot{\boldsymbol{\theta}})$, $\mathcal{U}_c(\mathbf{q}, \boldsymbol{\theta}, \mathbf{q}_d)$ and $\mathcal{R}_c(\dot{\boldsymbol{\theta}})$ are free to be designed as desired.* ◇

Let us start with the controller design by setting

$$\mathcal{K}_c(\dot{\boldsymbol{\theta}}) := \frac{1}{2} m_c \|\dot{\boldsymbol{\theta}}\|^2,$$

where $m_c \geq 0$ is the *virtual* constant mass of the controller. Note that m_c is a *design* parameter that can be zero.

Then, the compact controller dynamics are

$$m_c \ddot{\boldsymbol{\theta}} + \frac{\partial}{\partial \boldsymbol{\theta}} \mathcal{U}_c(\mathbf{q}, \boldsymbol{\theta}, \mathbf{q}_d) + \frac{\partial}{\partial \dot{\boldsymbol{\theta}}} \mathcal{R}_c(\dot{\boldsymbol{\theta}}) = \mathbf{0}. \tag{5.1}$$

Fact 4 *The dynamic controller (5.1) is* strictly passive *from (input) $\dot{\mathbf{q}}$ to (output) $\frac{\partial}{\partial \mathbf{q}} \mathcal{U}_c(\mathbf{q}, \boldsymbol{\theta}, \mathbf{q}_d)$. More precisely, it holds that*

$$\dot{\mathcal{H}}_c(\mathbf{q}, \boldsymbol{\theta}, \mathbf{q}_d, \dot{\boldsymbol{\theta}}) = -\dot{\boldsymbol{\theta}}^\top \frac{\partial}{\partial \dot{\boldsymbol{\theta}}} \mathcal{R}_c(\dot{\boldsymbol{\theta}}) + \dot{\mathbf{q}}^\top \frac{\partial}{\partial \mathbf{q}} \mathcal{U}_c(\mathbf{q}, \boldsymbol{\theta}, \mathbf{q}_d),$$

where $\mathcal{H}_c(\mathbf{q}, \boldsymbol{\theta}, \mathbf{q}_d, \dot{\boldsymbol{\theta}})$ is the (total energy) Hamiltonian of the controller. ◇

Proof. Consider the controller Hamiltonian

$$\begin{aligned}\mathcal{H}_c(\mathbf{q},\boldsymbol{\theta},\mathbf{q}_d,\dot{\boldsymbol{\theta}}) =& \mathcal{K}_c(\dot{\boldsymbol{\theta}}) + \mathcal{U}_c(\mathbf{q},\boldsymbol{\theta},\mathbf{q}_d)\\ =& \frac{1}{2}m_c|\dot{\boldsymbol{\theta}}|^2 + \mathcal{U}_c(\mathbf{q},\boldsymbol{\theta},\mathbf{q}_d).\end{aligned}$$

Then $\dot{\mathcal{H}}_c(\mathbf{q},\boldsymbol{\theta},\mathbf{q}_d,\dot{\boldsymbol{\theta}})$ yields

$$\dot{\mathcal{H}}_c(\mathbf{q},\boldsymbol{\theta},\mathbf{q}_d,\dot{\boldsymbol{\theta}}) = m_c\dot{\boldsymbol{\theta}}^\top\ddot{\boldsymbol{\theta}} + \dot{\mathbf{q}}^\top\frac{\partial}{\partial\mathbf{q}}\mathcal{U}_c(\mathbf{q},\boldsymbol{\theta},\mathbf{q}_d) + \dot{\boldsymbol{\theta}}^\top\frac{\partial}{\partial\boldsymbol{\theta}}\mathcal{U}_c(\mathbf{q},\boldsymbol{\theta},\mathbf{q}_d) + \dot{\mathbf{q}}_d^\top\frac{\partial}{\partial\mathbf{q}_d}\mathcal{U}_c(\mathbf{q},\boldsymbol{\theta},\mathbf{q}_d).$$

Since $\dot{\mathbf{q}}_d = \mathbf{0}$ then

$$\dot{\mathcal{H}}_c(\mathbf{q},\boldsymbol{\theta},\mathbf{q}_d,\dot{\boldsymbol{\theta}}) = m_c\dot{\boldsymbol{\theta}}^\top\ddot{\boldsymbol{\theta}} + \dot{\mathbf{q}}^\top\frac{\partial}{\partial\mathbf{q}}\mathcal{U}_c(\mathbf{q},\boldsymbol{\theta},\mathbf{q}_d) + \dot{\boldsymbol{\theta}}^\top\frac{\partial}{\partial\boldsymbol{\theta}}\mathcal{U}_c(\mathbf{q},\boldsymbol{\theta},\mathbf{q}_d).$$

Using (5.1) we have that

$$\dot{\mathcal{H}}_c(\mathbf{q},\boldsymbol{\theta},\mathbf{q}_d,\dot{\boldsymbol{\theta}}) = -\dot{\boldsymbol{\theta}}^\top\frac{\partial}{\partial\dot{\boldsymbol{\theta}}}\mathcal{R}_c(\dot{\boldsymbol{\theta}}) + \dot{\mathbf{q}}^\top\frac{\partial}{\partial\mathbf{q}}\mathcal{U}_c(\mathbf{q},\boldsymbol{\theta},\mathbf{q}_d).$$

This completes the proof. $\triangleleft$

Let us now define the *desired energy* $\mathcal{H}_d$ of the *closed-loop* system (4.3) and (5.1) as the sum of the energy of the robot plus the energy of the controller. Hence

$$\mathcal{H}_d(\mathbf{q},\boldsymbol{\theta},\mathbf{q}_d,\dot{\mathbf{q}},\dot{\boldsymbol{\theta}}) := \mathcal{H}(\mathbf{q},\dot{\mathbf{q}}) + \mathcal{H}_c(\mathbf{q},\boldsymbol{\theta},\mathbf{q}_d,\dot{\boldsymbol{\theta}}). \tag{5.2}$$

Desired Energy = Robot Energy + Controller Energy

Using Fact 2 and Fact 4 we have that

$$\dot{\mathcal{H}}_d(\mathbf{q},\boldsymbol{\theta},\mathbf{q}_d,\dot{\mathbf{q}},\dot{\boldsymbol{\theta}}) = \dot{\mathbf{q}}^\top\boldsymbol{\tau} - \dot{\boldsymbol{\theta}}^\top\frac{\partial}{\partial\dot{\boldsymbol{\theta}}}\mathcal{R}_c(\dot{\boldsymbol{\theta}}) + \dot{\mathbf{q}}^\top\frac{\partial}{\partial\mathbf{q}}\mathcal{U}_c(\mathbf{q},\boldsymbol{\theta},\mathbf{q}_d).$$

The ES+DI controller is then designed as

$$\boldsymbol{\tau} = -\frac{\partial}{\partial\mathbf{q}}\mathcal{U}_c(\mathbf{q},\boldsymbol{\theta},\mathbf{q}_d) - \frac{\partial}{\partial\dot{\mathbf{q}}}\mathcal{R}(\dot{\mathbf{q}}), \tag{5.3}$$

where $\mathcal{R}(\dot{\mathbf{q}}) \in \mathbb{R}$ is a dissipation function.

Therefore, using (5.3), we have that $\dot{\mathcal{H}}_d(\mathbf{q}, \boldsymbol{\theta}, \mathbf{q}_d, \dot{\mathbf{q}}, \dot{\boldsymbol{\theta}})$ becomes

$$\dot{\mathcal{H}}_d(\mathbf{q}, \boldsymbol{\theta}, \mathbf{q}_d, \dot{\mathbf{q}}, \dot{\boldsymbol{\theta}}) = -\dot{\mathbf{q}}^\top \frac{\partial}{\partial \dot{\mathbf{q}}} \mathcal{R}(\dot{\mathbf{q}}) - \dot{\boldsymbol{\theta}}^\top \frac{\partial}{\partial \dot{\boldsymbol{\theta}}} \mathcal{R}_c(\dot{\boldsymbol{\theta}}).$$

The following result is a restatement of Proposition 3.1 and Proposition 3.6 in Ortega et al. [11].

Theorem 14 *Consider a robot manipulator* (4.3) *controlled by* (5.3). *Suppose that:*

1) *(Energy Shaping) The desired potential energy* $\mathcal{U}_d(\mathbf{q}, \boldsymbol{\theta}, \mathbf{q}_d) := \mathcal{U}(\mathbf{q}) + \mathcal{U}_c(\mathbf{q}, \boldsymbol{\theta}, \mathbf{q}_d)$ *is positive definite and radially unbounded with regards to* $\tilde{\mathbf{q}}$ *and has a unique and global minimum at* $\tilde{\mathbf{q}} = \mathbf{0}$*; and*

2) *(Damping Injection) The following implication holds*

$$\dot{\mathcal{H}}_d(\mathbf{q}, \boldsymbol{\theta}, \mathbf{q}_d, \dot{\mathbf{q}}, \dot{\boldsymbol{\theta}}) = 0 \quad \Leftrightarrow \quad \dot{\mathbf{q}} = \mathbf{0}.$$

Then, $(\dot{\mathbf{q}}, \tilde{\mathbf{q}}) = (\mathbf{0}, \mathbf{0})$ *is GAS.*

Proof. The resulting closed-loop system is

$$\begin{aligned} \ddot{\mathbf{q}} &= -\mathbf{M}(\mathbf{q})^{-1} \left[\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} + \frac{\partial}{\partial \mathbf{q}} \mathcal{U}_d(\mathbf{q}, \boldsymbol{\theta}, \mathbf{q}_d) + \frac{\partial}{\partial \dot{\mathbf{q}}} \mathcal{R}(\dot{\mathbf{q}}) \right] \\ m_c \ddot{\boldsymbol{\theta}} &= -\frac{\partial}{\partial \boldsymbol{\theta}} \mathcal{U}_c(\mathbf{q}, \boldsymbol{\theta}, \mathbf{q}_d) - \frac{\partial}{\partial \dot{\boldsymbol{\theta}}} \mathcal{R}_c(\dot{\boldsymbol{\theta}}) \end{aligned} \tag{5.4}$$

The desired energy (5.2) can be written as

$$\mathcal{H}_d(\mathbf{q}, \boldsymbol{\theta}, \mathbf{q}_d, \dot{\mathbf{q}}, \dot{\boldsymbol{\theta}}) = \frac{1}{2} \dot{\mathbf{q}}^\top \mathbf{M}(\mathbf{q}) \dot{\mathbf{q}} + \frac{1}{2} m_c \|\dot{\boldsymbol{\theta}}\|^2 + \mathcal{U}_d(\mathbf{q}, \boldsymbol{\theta}, \mathbf{q}_d),$$

and, clearly, it is positive definite and radially unbounded with regards to $\dot{\mathbf{q}}$ and to $\tilde{\mathbf{q}}$. Moreover, we have that

$$\dot{\mathcal{H}}_d(\mathbf{q}, \boldsymbol{\theta}, \mathbf{q}_d, \dot{\mathbf{q}}, \dot{\boldsymbol{\theta}}) = -\dot{\mathbf{q}}^\top \frac{\partial}{\partial \dot{\mathbf{q}}} \mathcal{R}(\dot{\mathbf{q}}) - \dot{\boldsymbol{\theta}}^\top \frac{\partial}{\partial \dot{\boldsymbol{\theta}}} \mathcal{R}_c(\dot{\boldsymbol{\theta}}).$$

Since $\mathcal{H}_d(\mathbf{q}, \boldsymbol{\theta}, \mathbf{q}_d, \dot{\mathbf{q}}, \dot{\boldsymbol{\theta}}) > 0$ and $\dot{\mathcal{H}}_d(\mathbf{q}, \boldsymbol{\theta}, \mathbf{q}_d, \dot{\mathbf{q}}, \dot{\boldsymbol{\theta}}) \leq 0$, then all trajectories of (5.4) are stable.

Now $\dot{\mathcal{H}}_d(\mathbf{q}, \boldsymbol{\theta}, \mathbf{q}_d, \dot{\mathbf{q}}, \dot{\boldsymbol{\theta}}) = 0 \Leftrightarrow \dot{\mathbf{q}} = \mathbf{0}$ ensures that the trajectories that live in the set $\{\dot{\mathcal{H}}_d = 0\}$ are $\dot{\mathbf{q}} = \mathbf{0}$ and $\ddot{\mathbf{q}} = \mathbf{0}$. Therefore, from (5.4), we have that the solutions to

$$\frac{\partial}{\partial \mathbf{q}} \mathcal{U}_d(\mathbf{q}, \boldsymbol{\theta}, \mathbf{q}_d) = \mathbf{0}$$

also live in $\{\dot{\mathcal{H}}_d = 0\}$. Finally, $\mathcal{U}_d(\mathbf{q}, \boldsymbol{\theta}, \mathbf{q}_d)$ has a unique, global minimum at $\tilde{\mathbf{q}} = \mathbf{0}$, hence $\tilde{\mathbf{q}} = \mathbf{0}$ is the only possible solution. The LaSalle Invariance Theorem finishes the proof. $\triangleleft$

In what follows we will design $m_c, \mathcal{U}_c$, $\mathcal{R}_c$ and $\mathcal{R}$ such that the conditions on Theorem 14 hold.

The controllers that we design take into account four different scenarios: 1) the computation burden of the controller; 2) availability of the robot velocity measurements; 3) saturation of the actuators; and 4) uncertainty in the robot physical parameters.

Until today there does not exist a controller that can deal with the four scenarios above !!!

Chapter 6

Energy Shaping for Ideal Actuators

In this chapter we design different controllers that *do not* consider that the actuators have a physical limit (saturation). Therefore, in what follows we suppose that the actuators are ideal.

6.1 PD with gravity cancellation

Suppose that the robot has velocity sensors and that it has a fast and reliable computation interface.

The controller design relies in the following hints:

The gravity vector fixes the position equilibrium points when the robot is in open-loop.

The Rayleigh dissipation function

$$\mathcal{R}(\dot{\mathbf{q}}) = \frac{1}{2} d \|\dot{\mathbf{q}}\|^2,$$

where $d > 0$ is the damping gain, has a linear gradient, which is

$$\frac{\partial}{\partial \dot{\mathbf{q}}} \mathcal{R}(\dot{\mathbf{q}}) = d\dot{\mathbf{q}}.$$

By the Hooke's Law, the energy stored in a linear spring, with stiffness $p > 0$ and position displacement given by $\tilde{\mathbf{q}}$ is

$$\mathcal{U}_s(\mathbf{q}) = \frac{1}{2}p\|\tilde{\mathbf{q}}\|^2 = \frac{1}{2}p\|\mathbf{q} - \mathbf{q}_d\|^2,$$

clearly $\dot{\mathcal{U}}_s(\mathbf{q}) = \mathbf{f}_s^\top \dot{\mathbf{q}}$, where $\mathbf{f}_s := p\tilde{\mathbf{q}}$ is the spring force. Thus

$$\mathcal{U}_s(t) - \mathcal{U}_s(0) = \int_0^t \mathbf{f}_s(\sigma)^\top \dot{\mathbf{q}}(\sigma) d\sigma,$$

more precisely

$$-\int_0^t \mathbf{f}_s(\sigma)^\top \dot{\mathbf{q}}(\sigma) d\sigma \leq \mathcal{U}_s(0),$$

where to obtain this inequality we have employed the fact that $\mathcal{U}_s(t) \geq 0$ for all $t \geq 0$. Therefore the linear spring is passive from force to velocity and, more importantly, $\mathbf{f}_s = \mathbf{0} \Leftrightarrow \tilde{\mathbf{q}} = \mathbf{0}$.

Therefore, we design the potential energy of the controller to cancel out the gravity effects and to add a *virtual* spring between the position of the robot and the desired position. Hence, we set[1]

$$\mathcal{U}_c(\mathbf{q}, \boldsymbol{\theta}, \mathbf{q}_d) = \frac{1}{2}p\|\tilde{\mathbf{q}}\|^2 - \mathcal{U}(\mathbf{q}).$$

This choice yields

$$\begin{aligned}\mathcal{U}_d(\mathbf{q}, \mathbf{q}_d) =& \mathcal{U}(\mathbf{q}) + \mathcal{U}_c(\mathbf{q}, \mathbf{q}_d)\\ =& \frac{1}{2}p\|\tilde{\mathbf{q}}\|^2.\end{aligned}$$

Since $\frac{\partial}{\partial \boldsymbol{\theta}}\mathcal{U}_c(\mathbf{q}, \mathbf{q}_d) = \mathbf{0}$, then (5.1) becomes

$$m_c\ddot{\boldsymbol{\theta}} + \frac{\partial}{\partial \dot{\boldsymbol{\theta}}}\mathcal{R}_c(\dot{\boldsymbol{\theta}}) = \mathbf{0},$$

and we conclude that the dynamic behavior of the controller does not play any role in this case, because such dynamics evolve independently of the robot. Hence, we set $m_c = 0$ and $\mathcal{R}_c(\dot{\boldsymbol{\theta}}) = 0$.

[1] With some abuse of notation and since, in this case, $\mathcal{U}_c(\mathbf{q}, \boldsymbol{\theta}, \mathbf{q}_d)$ does not depend on $\boldsymbol{\theta}$, we will omit this dependence where it is not required.

Finally, the ES+DI controller (5.3) becomes

$$\begin{aligned}\boldsymbol{\tau} =& -\frac{\partial}{\partial \mathbf{q}}\mathcal{U}_c(\mathbf{q},\mathbf{q}_d) - \frac{\partial}{\partial \dot{\mathbf{q}}}\mathcal{R}(\dot{\mathbf{q}}) \\ =& \frac{\partial}{\partial \mathbf{q}}\mathcal{U}(\mathbf{q}) - p\tilde{\mathbf{q}} - d\dot{\mathbf{q}} \\ =& \mathbf{g}(\mathbf{q}) - p\tilde{\mathbf{q}} - d\dot{\mathbf{q}}.\end{aligned} \tag{6.1}$$

The closed-loop is

$$\ddot{\mathbf{q}} = -\mathbf{M}(\mathbf{q})^{-1}\big[\mathbf{C}(\mathbf{q},\dot{\mathbf{q}})\dot{\mathbf{q}} + p\tilde{\mathbf{q}} + d\dot{\mathbf{q}}\big].$$

Defining $\mathbf{x}_1 := \tilde{\mathbf{q}}$ and $\mathbf{x}_2 := \dot{\mathbf{q}}$ then we get

$$\begin{aligned}\dot{\mathbf{x}}_1 =& \mathbf{x}_2, \\ \dot{\mathbf{x}}_1 =& -\mathbf{M}(\mathbf{x}_1 + \mathbf{q}_d)^{-1}\big[\mathbf{C}(\mathbf{x}_1 + \mathbf{q}_d, \mathbf{x}_2)\mathbf{x}_2 + p\mathbf{x}_1 + d\mathbf{x}_2\big],\end{aligned} \tag{6.2}$$

Note that (6.2) is clearly an autonomous system, because it does not depend explicitly on time. The (only) equilibrium point is

$$\bar{\mathbf{x}}_1 = \mathbf{0}, \qquad \bar{\mathbf{x}}_2 = \mathbf{0}.$$

The desired energy (5.2) can now be written as

$$\mathcal{H}_d(\mathbf{x}_1 + \mathbf{q}_d, \mathbf{q}_d, \mathbf{x}_2) = \frac{1}{2}\mathbf{x}_2^\top \mathbf{M}(\mathbf{x}_1 + \mathbf{q}_d)\mathbf{x}_2 + \frac{1}{2}p\|\mathbf{x}_1\|^2,$$

and, clearly, it is positive definite and radially unbounded. Moreover, we have that

$$\dot{\mathcal{H}}_d(\mathbf{x}_1 + \mathbf{q}_d, \mathbf{q}_d, \mathbf{x}_2) = -d\|\mathbf{x}_2\|^2$$

Since $\dot{\mathcal{H}}_d(\mathbf{x}_1 + \mathbf{q}_d, \mathbf{q}_d, \mathbf{x}_2)$ is not strictly negative then the Lyapunov theorem cannot be invoked to prove asymptotical stability of $\mathbf{x} = \mathbf{0}$, instead we invoke LaSalle Invariance Theorem.

For, let us find out which trajectories live in the set $\Omega := \{\mathbf{x} \in \mathbb{R}^{2n} : \dot{\mathcal{H}}_d = 0\}$. Clearly, $\mathbf{x}_2 = \mathbf{0} \in \Omega$, obviously, $\mathbf{x}_2 \neq \mathbf{0} \notin \Omega$. Therefore also $\dot{\mathbf{x}}_2 = \mathbf{0} \in \Omega$ because if $\dot{\mathbf{x}}_2 \neq \mathbf{0}$ and $\mathbf{x}_2 \in \Omega$ then $\mathbf{x}_2$ leaves the set Ω. These implications also ensure, from the closed-loop (6.2), that $\mathbf{x}_1 = \mathbf{0}$ is the only trajectory in Ω. Thus, since $\mathcal{H}_d$ is radially unbounded, $(\dot{\mathbf{q}}, \tilde{\mathbf{q}}) = (\mathbf{0}, \mathbf{0})$ is GAS, as required.

Remark 9 *This controller requires the computation of the gravity vector at all time instants and this is the most time-consuming element of the controller. Hence, when a robot is not equipped with a fast and reliable computer then we should opt to* compensate *the gravity effects instead of cancelling them.* ◇

Controller (6.1) has been originally reported by Takegaki and Arimoto in 1981 [20].

6.2 PD with gravity compensation

We continue with the assumption that velocities are available. However, in this case we compensate for the gravity effects to reduce the computation burden.

Our starting point is the previous PD controller from which we borrow the spring-damper structure and compensate for the gravity forces *only* at the desired equilibrium. Hence, we set

$$\mathcal{R}(\dot{\mathbf{q}}) = \frac{1}{2} d \|\dot{\mathbf{q}}\|^2,$$

$m_c = 0$ and $\mathcal{R}(\dot{\boldsymbol{\theta}}) = 0$.

If we set the controller potential energy as

$$\mathcal{U}_c(\mathbf{q}, \mathbf{q}_d) = \frac{1}{2} p \|\tilde{\mathbf{q}}\|^2 - \mathbf{q}^\top \mathbf{g}(\mathbf{q}_d) + \mathcal{U}_c^*,$$

where $\mathcal{U}_c^* \in \mathbb{R}$ is a constant, then

$$\begin{aligned} \boldsymbol{\tau} =& - \frac{\partial}{\partial \mathbf{q}} \mathcal{U}_c(\mathbf{q}, \mathbf{q}_d) - \frac{\partial}{\partial \dot{\mathbf{q}}} \mathcal{R}(\dot{\mathbf{q}}) \\ =& \mathbf{g}(\mathbf{q}_d) - p \tilde{\mathbf{q}} - d \dot{\mathbf{q}}. \end{aligned} \tag{6.3}$$

This controller compensates the effects of gravity when the system is at the desired position.

Constant $\mathcal{U}_c^*$ is added to ensure that the desired potential energy $\mathcal{U}_d(\mathbf{q}, \mathbf{q}_d)$ is zero when $\mathbf{q} = \mathbf{q}_d$. Hence,

$$\begin{aligned} \mathcal{U}_d(\mathbf{q}_d, \mathbf{q}_d) =& \mathcal{U}(\mathbf{q}_d) + \mathcal{U}_c(\mathbf{q}_d, \mathbf{q}_d) \\ =& \mathcal{U}(\mathbf{q}_d) - \mathbf{q}_d^\top \mathbf{g}(\mathbf{q}_d) + \mathcal{U}_c^*, \end{aligned}$$

setting $\mathcal{U}_c^* := \mathbf{q}_d^\top \mathbf{g}(\mathbf{q}_d) - \mathcal{U}(\mathbf{q}_d)$ implies that $\mathcal{U}_d(\mathbf{q}_d, \mathbf{q}_d) = 0$, as required.

Finally, $\mathcal{U}_d(\mathbf{q}, \mathbf{q}_d)$ is given by

$$\mathcal{U}_d(\mathbf{q}, \mathbf{q}_d) = \mathcal{U}(\mathbf{q}) - \mathcal{U}(\mathbf{q}_d) - \tilde{\mathbf{q}}^\top \mathbf{g}(\mathbf{q}_d) + \frac{1}{2} p \|\tilde{\mathbf{q}}\|^2.$$

Now, the closed-loop dynamics is

$$\ddot{\mathbf{q}} = -\mathbf{M}(\mathbf{q})^{-1} \left[\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}}) \dot{\mathbf{q}} + \mathbf{g}(\mathbf{q}) - \mathbf{g}(\mathbf{q}_d) + p \tilde{\mathbf{q}} + d \dot{\mathbf{q}} \right],$$

or, using the state variables $\mathbf{x}_1 = \tilde{\mathbf{q}}$ and $\mathbf{x}_2 = \dot{\mathbf{q}}$,

$$\begin{aligned}\dot{\mathbf{x}}_1 =& \mathbf{x}_2 \\ \dot{\mathbf{x}}_2 =& - \mathbf{M}(\mathbf{x}_1 + \mathbf{q}_d)^{-1}\big[\mathbf{C}(\mathbf{x}_1 + \mathbf{q}_d, \mathbf{x}_2)\mathbf{x}_2 + \mathbf{g}(\mathbf{x}_1 + \mathbf{q}_d) - \mathbf{g}(\mathbf{q}_d) + p\mathbf{x}_1 + d\mathbf{x}_2\big].\end{aligned} \tag{6.4}$$

The closed-loop (6.4) is autonomous and its equilibrium points satisfy

$$\mathbf{x}_1 = \frac{1}{p}\big[\mathbf{g}(\mathbf{q}_d) - \mathbf{g}(\mathbf{x}_1 + \mathbf{q}_d)\big], \qquad \mathbf{x}_2 = \mathbf{0}.$$

Obviously, $\mathbf{x}_1 = \mathbf{0}$ is one solution to these equations. However, they might have *several* solutions for $\mathbf{x}_1 \neq \mathbf{0}$.

Remark 10 *In order to illustrate the effect, on the position equilibria, of the gravity compensation, consider a single DOF robot with potential function* $\mathcal{U}(q) = mgl(1 + sin(q))$*, for which* $q = 0$ *lies along the horizontal line,* $m, g, l > 0$ *are the mass, the acceleration of gravity constant and the length of the link, respectively. Then, the position equilibria must satisfy*

$$x_1 = \frac{mgl}{p}\big[\cos(q_d) - \cos(x_1 + q_d)\big].$$

Without loss of generality, suppose that $\frac{mgl}{p} = 2$ *and that* $q_d = 0$*. This yields* $x_1 = 2(1 - \cos(x_1))$*, which has* three different solutions*, namely* $x_1 = 0$*,* $x_2 = 1.11$ *and* $x_1 = 3.698$*.* ◇

In fact, the position equilibria, satisfies

$$\frac{\partial}{\partial \mathbf{x}_1}\mathcal{U}_d(\mathbf{x}_1 + \mathbf{q}_d, \mathbf{q}_d) = \mathbf{g}(\mathbf{x}_1 + \mathbf{q}_d) - \mathbf{g}(\mathbf{q}_d) + p\mathbf{x}_1 = \mathbf{0}.$$

Hence, if we ensure that $\mathbf{x}_1 = \mathbf{0}$ is the *only* global minimum of $\mathcal{U}_d(\mathbf{x}_1+\mathbf{q}_d, \mathbf{q}_d)$, then $\mathbf{x}_1 = \mathbf{0}$ will be the only solution of the equilibria equation.

Since $\mathbf{x}_1 = \mathbf{0}$ implies that $\mathcal{U}_d(\mathbf{x}_1 + \mathbf{q}_d, \mathbf{q}_d) = 0$ and that $\frac{\partial}{\partial \mathbf{x}_1}\mathcal{U}_d(\mathbf{x}_1 + \mathbf{q}_d, \mathbf{q}_d) = \mathbf{0}$, it only rests to prove that $\frac{\partial^2}{\partial \mathbf{x}_1^2}\mathcal{U}_d(\mathbf{x}_1 + \mathbf{q}_d, \mathbf{q}_d) > 0$ everywhere, refer to Lemma 2.

Note that

$$\frac{\partial^2}{\partial \mathbf{x}_1^2}\mathcal{U}_d(\mathbf{x}_1 + \mathbf{q}_d, \mathbf{q}_d) = \frac{\partial}{\partial \mathbf{x}_1}\mathbf{g}(\mathbf{x}_1 + \mathbf{q}_d) + p\mathbf{I}.$$

Therefore, for robots with only revolute joints, it holds that

$$\begin{aligned}\left\|\frac{\partial^2}{\partial \mathbf{x}_1^2}\mathcal{U}_d(\mathbf{x}_1 + \mathbf{q}_d, \mathbf{q}_d)\right\| \geq& \, p - \left\|\frac{\partial}{\partial \mathbf{x}_1}\mathbf{g}(\mathbf{x}_1 + \mathbf{q}_d)\right\| \\ \geq& \, p - L_g.\end{aligned}$$

Setting $p > L_g$ implies that $\left\| \frac{\partial^2}{\partial \mathbf{x}_1^2} \mathcal{U}_d(\mathbf{x}_1 + \mathbf{q}_d, \mathbf{q}_d) \right\| > 0$, for any $\mathbf{x}_1 \in \mathbb{R}^n$.

The desired energy (5.2) transforms to

$$\mathcal{H}_d(\mathbf{x}_1+\mathbf{q}_d, \mathbf{q}_d, \mathbf{x}_2) = \frac{1}{2}\mathbf{x}_2^\top \mathbf{M}(\mathbf{x}_1+\mathbf{q}_d)\mathbf{x}_2 + \mathcal{U}(\mathbf{x}_1+\mathbf{q}_d) - \mathcal{U}(\mathbf{q}_d) - \mathbf{x}_1^\top \mathbf{g}(\mathbf{q}_d) + \frac{1}{2}p\|\mathbf{x}_1\|^2,$$

and, for robots with only revolute joints, is positive definite and radially unbounded if $p > L_g$. In this case

$$\dot{\mathcal{H}}_d(\mathbf{x}_1 + \mathbf{q}_d, \mathbf{q}_d, \mathbf{x}_2) = -d\|\mathbf{x}_2\|^2.$$

Proceeding as with the previous controller, LaSalle Invariance Theorem guarantees that $(\mathbf{x}_1, \mathbf{x}_2) = (\mathbf{0}, \mathbf{0})$ is GAS.

Remark 11 *In this controller, the proportional gain has to dominate the gradient of the gravity vector to ensure that the position error equal to zero is the only possible equilibrium solution. For, the regulation problem is solved only for robots with revolute joints.* $\diamond$

Remark 12 *As the previous PD controller, this gravity compensation scheme also requires the joint velocities to be measurable.* $\diamond$

Controller (6.3) has been originally reported by Takegaki and Arimoto in 1981 [20]. See also [4] for a stability proof using a *strict* Lyapunov function.

6.3 P'D' control with gravity cancellation

When velocity measurements are not available, it is not possible to dissipate the robot energy as with the previous *static* PD controllers. The solution is then to design a *dynamic* controller where the energy dissipation is done in the controller dynamics.

Since velocities cannot be measured, the dissipation function of the robot is equal to zero, i.e., $\mathcal{R}(\dot{\mathbf{q}}) = 0$. As for the dissipation function in the controller, we set $\mathcal{R}_c(\dot{\boldsymbol{\theta}}) = \frac{1}{2}d\|\dot{\boldsymbol{\theta}}\|^2$, with $d > 0$.

Now, the main idea in the design of this controller is to employ a *passive* interconnection between the plant and the controller. For, we employ a virtual (linear) spring. Hence, the potential energy of the controller is designed such that it cancels out the gravity effects and it adds two springs, one for the position error term and the other for the robot-controller interconnection. Hence

$$\mathcal{U}_c(\mathbf{q}, \boldsymbol{\theta}, \mathbf{q}_d) = -\mathcal{U}(\mathbf{q}) + \frac{1}{2}p\|\tilde{\mathbf{q}}\|^2 + \frac{1}{2}k\|\mathbf{q} - \boldsymbol{\theta}\|^2,$$

where $p > 0$ is the proportional to the error gain and $k > 0$ is the robot-controller interconnection gain.

These choices yield

$$\begin{aligned}\boldsymbol{\tau} =& -\frac{\partial}{\partial \mathbf{q}}\mathcal{U}_c(\mathbf{q}, \boldsymbol{\theta}, \mathbf{q}_d) - \frac{\partial}{\partial \dot{\mathbf{q}}}\mathcal{R}(\dot{\mathbf{q}}) \\ =& \, \mathbf{g}(\mathbf{q}) - p\tilde{\mathbf{q}} - k(\mathbf{q} - \boldsymbol{\theta}).\end{aligned} \tag{6.5}$$

Now, as for the controller kinetic energy we have two options, wether $m_c = 0$ or $m_c > 0$. For simplicity, first we set $m_c = 0$.

Therefore, the controller dynamics becomes

$$\dot{\boldsymbol{\theta}} = \frac{k}{d}(\mathbf{q} - \boldsymbol{\theta}). \tag{6.6}$$

The desired potential energy is

$$\mathcal{U}_d(\mathbf{q}, \boldsymbol{\theta}, \mathbf{q}_d) = \frac{1}{2}p\|\tilde{\mathbf{q}}\|^2 + \frac{1}{2}k\|\mathbf{q} - \boldsymbol{\theta}\|^2,$$

which has a single global minimum at $(\mathbf{q}, \boldsymbol{\theta}) = (\mathbf{q}_d, \mathbf{q}_d)$.

Setting $\mathbf{x}_1 = \tilde{\mathbf{q}}$, $\mathbf{x}_2 = \dot{\mathbf{q}}$ and $\mathbf{x}_3 = \mathbf{q} - \boldsymbol{\theta}$, the closed-loop is

$$\begin{aligned}\dot{\mathbf{x}}_1 =& \, \mathbf{x}_2 \\ \dot{\mathbf{x}}_2 =& -\mathbf{M}(\mathbf{x}_1 + \mathbf{q}_d)^{-1}\left[\mathbf{C}(\mathbf{x}_1 + \mathbf{q}_d, \mathbf{x}_2)\mathbf{x}_2 + p\mathbf{x}_1 + k\mathbf{x}_3\right] \\ \dot{\mathbf{x}}_3 =& \, \mathbf{x}_2 - \frac{k}{d}\mathbf{x}_3.\end{aligned} \tag{6.7}$$

The desired energy (5.2) transforms to

$$\mathcal{H}_d(\mathbf{x}_1+\mathbf{q}_d, \mathbf{x}_1-\mathbf{x}_3+\mathbf{q}_d, \mathbf{q}_d, \mathbf{x}_2) = \frac{1}{2}\mathbf{x}_2^\top \mathbf{M}(\mathbf{x}_1+\mathbf{q}_d)\mathbf{x}_2 + \frac{1}{2}p\|\mathbf{x}_1\|^2 + \frac{1}{2}k\|\mathbf{x}_3\|^2.$$

Hence, $\mathcal{H}_d$ is positive definite and radially unbounded.

The time-derivative of $\mathcal{H}_d$ is

$$\dot{\mathcal{H}}_d(\mathbf{x}_1+\mathbf{q}_d, \mathbf{q}_d, \mathbf{x}_2) = \mathbf{x}_2^\top \mathbf{M}(\mathbf{x}_1+\mathbf{q}_d)\dot{\mathbf{x}}_2 + \frac{1}{2}\mathbf{x}_2^\top \dot{\mathbf{M}}(\mathbf{x}_1+\mathbf{q}_d)\mathbf{x}_2 + p\mathbf{x}_1^\top \dot{\mathbf{x}}_1 + k\mathbf{x}_3^\top \dot{\mathbf{x}}_3,$$

and, evaluating this function along (6.7) yields

$$\dot{\mathcal{H}}_d(\mathbf{x}_1+\mathbf{q}_d, \mathbf{q}_d, \mathbf{x}_2) = -\frac{k^2}{d}\|\mathbf{x}_3\|^2.$$

Clearly, $\dot{\mathcal{H}}_d$ is negative semi-definite.

Now, let us construct the set $\Omega := \{\mathbf{x} \in \mathbb{R}^{3n} : \dot{\mathcal{H}}_d = 0\}$. Clearly, $\mathbf{x}_3 = \mathbf{0} \in \Omega$. Therefore also $\dot{\mathbf{x}}_3 = \mathbf{0} \in \Omega$. These implications also ensure, from the closed-loop (6.7), that $\mathbf{x}_2 = \mathbf{0}$ and that $\mathbf{x}_1 = \mathbf{0}$ are the only trajectories in Ω. Thus, since $\mathcal{H}_d$ is radially unbounded, $(\dot{\mathbf{q}}, \tilde{\mathbf{q}}) = (\mathbf{0}, \mathbf{0})$ is GAS, as required.

Remark 13 *Setting the* virtual *mass of the controller to be positive, instead of zero, yields* $\mathcal{K}_c(\dot{\boldsymbol{\theta}}) = \frac{1}{2}m_c|\dot{\boldsymbol{\theta}}|^2$ *and thus the controller dynamics* (6.6) *changes to*

$$\ddot{\boldsymbol{\theta}} = -\frac{1}{m_c}\left[d\dot{\boldsymbol{\theta}} + k(\boldsymbol{\theta} - \mathbf{q})\right].$$

In this case, the proof that the point $(\dot{\mathbf{q}}, \tilde{\mathbf{q}}) = (\mathbf{0}, \mathbf{0})$ *is GAS, follows using the positive definite desired energy function*

$$\mathcal{H}_d(\mathbf{q}, \boldsymbol{\theta}, \mathbf{q}_d, \dot{\mathbf{q}}, \dot{\boldsymbol{\theta}}) = \frac{1}{2}\left[\dot{\mathbf{q}}^\top \mathbf{M}(\mathbf{q})\dot{\mathbf{q}} + m_c\|\dot{\boldsymbol{\theta}}\|^2 + p\|\tilde{\mathbf{q}}\|^2 + k\|\mathbf{q} - \theta\|^2\right].$$

◇

Remark 14 *The P'D' controller analyzed in this section, does not require the velocities to be measurable. However gravity needs to be cancelled out. The next controller compensates for the gravity effects.* ◇

Controller (6.5), using the ES+DI methodology, has been reported by Ortega et al. in 1995 [10]. However, its structure is reported by Kelly in 1993 in [3].

6.4 P'D' with gravity compensation

This scheme borrows the structure of the previous P'D' controller and the gravity compensation idea of the controller in Section 6.2. Hence, we set $\mathcal{R}(\dot{\mathbf{q}}) = 0$, $\mathcal{R}_c(\dot{\boldsymbol{\theta}}) = \frac{1}{2}d\|\dot{\boldsymbol{\theta}}\|^2$ and $m_c = 0$. The potential energy of the controller is then designed as

$$\mathcal{U}_c(\mathbf{q}, \boldsymbol{\theta}, \mathbf{q}_d) = \frac{1}{2}p\|\tilde{\mathbf{q}}\|^2 - \tilde{\mathbf{q}}^\top \mathbf{g}(\mathbf{q}_d) - \mathcal{U}(\mathbf{q}_d) + \frac{1}{2}k\|\mathbf{q} - \boldsymbol{\theta}\|^2.$$

These choices yield

$$\begin{aligned} \boldsymbol{\tau} &= -\frac{\partial}{\partial \mathbf{q}}\mathcal{U}_c(\mathbf{q}, \mathbf{q}_d) - \frac{\partial}{\partial \dot{\mathbf{q}}}\mathcal{R}(\dot{\mathbf{q}}) \\ &= \mathbf{g}(\mathbf{q}_d) - p\tilde{\mathbf{q}} - k(\mathbf{q} - \boldsymbol{\theta}), \end{aligned} \tag{6.8}$$

and the corresponding controller dynamics remains the same as (6.6).

Now, the desired potential energy is

$$\mathcal{U}_d(\mathbf{q}, \boldsymbol{\theta}, \mathbf{q}_d) = \mathcal{U}(\mathbf{q}) - \mathcal{U}(\mathbf{q}_d) + \frac{1}{2}p\|\tilde{\mathbf{q}}\|^2 - \tilde{\mathbf{q}}^\top \mathbf{g}(\mathbf{q}_d) + \frac{1}{2}k\|\mathbf{q} - \boldsymbol{\theta}\|^2.$$

Note that, in this case, the gradient of $\mathcal{U}_d$ is

$$\nabla \mathcal{U}_d(\mathbf{q}, \boldsymbol{\theta}, \mathbf{q}_d) = \begin{bmatrix} \frac{\partial}{\partial \mathbf{q}}\mathcal{U}_d \\ \frac{\partial}{\partial \boldsymbol{\theta}}\mathcal{U}_d \end{bmatrix} = \begin{bmatrix} \mathbf{g}(\mathbf{q}) - \mathbf{g}(\mathbf{q}_d) + p\tilde{\mathbf{q}} + k(\mathbf{q} - \boldsymbol{\theta}) \\ -k(\mathbf{q} - \boldsymbol{\theta}) \end{bmatrix}.$$

The solutions to $\nabla \mathcal{U}_d(\mathbf{q}, \boldsymbol{\theta}, \mathbf{q}_d) = \mathbf{0}$ always have to satisfy $\boldsymbol{\theta} = \mathbf{q}$ and $\mathbf{g}(\mathbf{q}) - \mathbf{g}(\mathbf{q}_d) + p\tilde{\mathbf{q}} = \mathbf{0}$. This last equation, as explained before, might have several solutions different than $\tilde{\mathbf{q}} = \mathbf{0}$. In order to ensure that this is not the case, we have to enforce that $\nabla^2 \mathcal{U}_d(\mathbf{q}, \boldsymbol{\theta}, \mathbf{q}_d)$ is a positive definite matrix for all possible $\mathbf{q} \in \mathbb{R}^n$. This Hessian matrix is

$$\nabla^2 \mathcal{U}_d(\mathbf{q}, \boldsymbol{\theta}, \mathbf{q}_d) = \begin{bmatrix} \frac{\partial}{\partial \mathbf{q}}\mathbf{g}(\mathbf{q}) + p\mathbf{I} + k\mathbf{I} & -k\mathbf{I} \\ -k\mathbf{I} & k\mathbf{I} \end{bmatrix}.$$

A sufficient condition for this matrix to be positive definite is to set p such that it dominates the gradient of the gravity force. Therefore, for robots having only revolute joints, we set $p > L_g$ and hence $\tilde{\mathbf{q}} = \mathbf{0}$ is the only solution to this problem.

Setting $\mathbf{x}_1 = \tilde{\mathbf{q}}$, $\mathbf{x}_2 = \dot{\mathbf{q}}$ and $\mathbf{x}_3 = \mathbf{q} - \boldsymbol{\theta}$, the closed-loop is

$$\begin{aligned} \dot{\mathbf{x}}_1 &= \mathbf{x}_2 \\ \dot{\mathbf{x}}_2 &= -\mathbf{M}(\mathbf{x}_1 + \mathbf{q}_d)^{-1}\left[\mathbf{C}(\mathbf{x}_1 + \mathbf{q}_d, \mathbf{x}_2)\mathbf{x}_2 + p\mathbf{x}_1 + k\mathbf{x}_3 + \mathbf{g}(\mathbf{x}_1 + \mathbf{q}_d) - \mathbf{g}(\mathbf{q}_d)\right] \\ \dot{\mathbf{x}}_3 &= \mathbf{x}_2 - \frac{k}{d}\mathbf{x}_3. \end{aligned} \tag{6.9}$$

The desired energy (5.2) transforms to

$$\mathcal{H}_d = \frac{1}{2}\mathbf{x}_2^\top \mathbf{M}(\mathbf{x}_1+\mathbf{q}_d)\mathbf{x}_2+\mathcal{U}(\mathbf{x}_1+\mathbf{q}_d)-\mathcal{U}(\mathbf{q}_d)-\mathbf{x}_1^\top \mathbf{g}(\mathbf{q}_d)+\frac{1}{2}p\|\mathbf{x}_1\|^2+\frac{1}{2}k\|\mathbf{x}_3\|^2,$$

and evaluating $\dot{\mathcal{H}}_d$ along (6.9) yields

$$\dot{\mathcal{H}}_d(\mathbf{x}_1+\mathbf{q}_d, \mathbf{q}_d, \mathbf{x}_2) = -\frac{k^2}{d}\|\mathbf{x}_3\|^2.$$

Since $\mathcal{H}_d$is positive definite and radially unbounded, provided that $p > L_g$, and $\dot{\mathcal{H}}_d \leq 0$ then, by LaSalle's Invariance Theorem, we can show that $(\dot{\mathbf{q}}, \tilde{\mathbf{q}}) = (\mathbf{0}, \mathbf{0})$ is GAS.

Remark 15 *Adding a* virtual *mass to the controller dynamics, i.e.,* $m_c > 0$ *and* $\mathcal{K}_c(\dot{\boldsymbol{\theta}}) = \frac{1}{2}m_c\|\dot{\boldsymbol{\theta}}\|^2$*, does not change the stability result in this section, provided that* $p > L_g$*.* ◇

Remark 16 *The P'D' controller analyzed in this section, does not require the velocities to be measurable and compensates for the gravity forces only at the desired position. However, large torques might be required and thus the actuators might be saturated.* ◇

Controller (6.8) has been reported by Ortega et al. in 1995 [10].

Chapter 7

Energy Shaping for Actuators with Saturation

The controllers in the previous chapter assume that the actuators are ideal and therefore that they cannot be saturated. However, in reality, the permissible torques are bounded and such bound is defined by the robot's electromechanical design.

This chapter presents the counterparts of the controllers or the previous chapter for the case when the actuators of the robot are not ideal and can be saturated.

In all cases, we assume that the torque that each of the actuators can provide, satisfies $|\tau_i| \leq \bar{\tau}_i$, where $\bar{\tau}_i > 0$ is the *known* bound of the ith-actuator. Moreover, we also assume that the robot is composed of only revolute joints and thus Property P6 holds.

In order to ensure that the equilibrium $(\dot{\mathbf{q}}, \tilde{\mathbf{q}}) = (\mathbf{0}, \mathbf{0})$ is GAS, we require the robot actuators to be able to provide *more* torque than the one required to cancel out the gravity effects. Thus, it holds that $\bar{\tau}_i > k_{gi}$, where k_{gi} is defined in Property P6.

The controllers designed in this section make use of the following *linear saturation* scalar function (see Figure 7.1)

$$\mathrm{sat}_\delta(x) := \begin{cases} x & |x| < \delta \\ \delta \,\mathrm{sign}(x) & |x| \geq \delta \end{cases}, \qquad \forall x \in \mathbb{R}.$$

where

$$\mathrm{sign}(x) := \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases}, \qquad \forall x \in \mathbb{R}.$$

When $\mathbf{x} \in \mathbb{R}^n$, then $\mathrm{sat}_\delta(\mathbf{x}) := [\mathrm{sat}_{\delta_1}(x_1), \ldots, \mathrm{sat}_{\delta_i}(x_n)]^\top$.

Further, the integral of the saturation function is positive definite and radially unbounded. This is concluded from the fact that

$$\int_0^x \mathrm{sign}(y)dy = |x|,$$

and that

$$\int_0^x \mathrm{sat}_\delta(y)dy = \begin{cases} \frac{1}{2}x^2 & |x| < \delta \\ \delta|x| - \frac{1}{2}\delta^2 & |x| \geq \delta. \end{cases}$$

Noting that, for all $|x| \geq \delta$, $\delta|x| - \frac{1}{2}\delta^2 \geq \frac{1}{2}\delta|x|$ (with equality only at $|x| = \delta$), then

$$\int_0^x \mathrm{sat}_\delta(y)dy > 0.$$

Clearly,

$$\frac{\partial}{\partial x}\int_0^x \mathrm{sat}_\delta(y)dy = \mathrm{sat}_\delta(x)$$

From the fact that vector norms are monotonically decreasing (see Lemma 1), for any vector $\mathbf{x} \in \mathbb{R}^n$,

$$|\mathbf{x}|_1 = \sum_{i=1}^n |x_i| \geq \left(\sum_{i=1}^n |x_i|^2\right)^{\frac{1}{2}} = \|\mathbf{x}\|.$$

Therefore

$$\sum_{i=1}^n \int_0^x \mathrm{sat}_{\delta_i}(y)dy \geq \begin{cases} \frac{1}{2}\|\mathbf{x}\|^2 & \|\mathbf{x}\| < \underline{\delta} \\ \underline{\delta}\|\mathbf{x}\| - \frac{1}{2}\underline{\delta}^2 & \|\mathbf{x}\| \geq \underline{\delta}, \end{cases} \tag{7.1}$$

where $\underline{\delta} := \min_{1\leq i\leq n}\{\delta_i\}$.

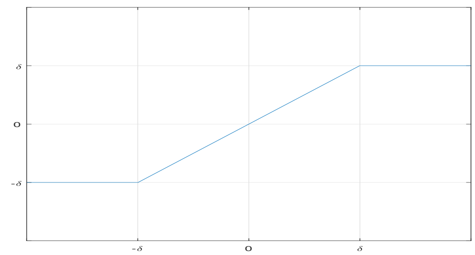

Figure 7.1: Saturation Function

7.1 sPsD with gravity cancellation

This controller of the Saturated-Proportional Saturated-Derivative (sPsD) type. Assuming that velocities are measurable and that gravity can be cancelled out at each sampling time, then the sPsD with gravity cancellation controller is obtained from setting

$$\mathcal{R}(\dot{\mathbf{q}}) = d \sum_{i=1}^{n} \int_0^{\dot{q}_i} \mathrm{sat}_{\delta_{di}}(y) dy,$$

and

$$\mathcal{U}_c(\mathbf{q}, \mathbf{q}_d) = -\mathcal{U}(\mathbf{q}) + p \sum_{i=1}^{n} \int_0^{\tilde{q}_i} \mathrm{sat}_{\delta_{pi}}(y) dy.$$

Since with these definitions $\frac{\partial}{\partial \boldsymbol{\theta}} \mathcal{U}_c(\mathbf{q}, \mathbf{q}_d) = \mathbf{0}$, the dynamics of the controller does not play any role in the stabilization mechanism, thus we set $m_c = 0$ and $\mathcal{R}_c(\dot{\boldsymbol{\theta}}) = 0$.

The controller is then given by

$$\begin{aligned} \boldsymbol{\tau} =& -\frac{\partial}{\partial \mathbf{q}} \mathcal{U}_c(\mathbf{q}, \mathbf{q}_d) - \frac{\partial}{\partial \dot{\mathbf{q}}} \mathcal{R}(\dot{\mathbf{q}}) \\ =& \mathbf{g}(\mathbf{q}) - p \,\mathrm{sat}_{\delta_p}(\tilde{\mathbf{q}}) - d \,\mathrm{sat}_{\delta_d}(\dot{\mathbf{q}}). \end{aligned} \tag{7.2}$$

Each ith-element of the proposed controller is thus given by

$$\tau_i = g_i(\mathbf{q}) - p\,\mathrm{sat}_{\delta_{pi}}(\tilde{q}_i) - d\,\mathrm{sat}_{\delta_{di}}(\dot{q}_i).$$

Therefore

$$\begin{aligned}|\tau_i| =&|g_i(\mathbf{q}) - p\,\mathrm{sat}_{\delta_{pi}}(\tilde{q}_i) - d\,\mathrm{sat}_{\delta_{di}}(\dot{q}_i)|,\\ \leq&|g_i(\mathbf{q})| + |p\,\mathrm{sat}_{\delta_{pi}}(\tilde{q}_i)| + |d\,\mathrm{sat}_{\delta_{di}}(\dot{q}_i)|,\\ \leq&k_{gi} + p\delta_{pi} + d\delta_{di},\end{aligned}$$

where k_{gi} is the bound of each of the gravity vector elements that is given in Property P6; δ_{pi} and δ_{di} are the bounds of the error and velocity saturation functions, respectively; and p and d are the proportional and derivative gains, respectively.
This bound is independent of the values of the position error and of the velocity.

Since we need to satisfy $|\tau_i| \leq \bar{\tau}_i$, then

$$|\tau_i| \leq k_{gi} + p\delta_{pi} + d\delta_{di} < \bar{\tau}_i.$$

Therefore, setting

$$p\delta_{pi} + d\delta_{di} < \bar{\tau}_i - k_{gi} \tag{7.3}$$

ensures that the actuators are *never* saturated.

The closed-loop system is

$$\ddot{\mathbf{q}} = -\mathbf{M}(\mathbf{q})^{-1}\big[\mathbf{C}(\mathbf{q},\dot{\mathbf{q}})\dot{\mathbf{q}} + p\,\mathrm{sat}_{\delta_p}(\tilde{\mathbf{q}}) + d\,\mathrm{sat}_{\delta_d}(\dot{\mathbf{q}})\big].$$

Defining $\mathbf{x}_1 := \tilde{\mathbf{q}}$ and $\mathbf{x}_2 := \dot{\mathbf{q}}$, we obtain

$$\begin{aligned}\dot{\mathbf{x}}_1 =&\mathbf{x}_2,\\ \dot{\mathbf{x}}_1 =&-\mathbf{M}(\mathbf{x}_1 + \mathbf{q}_d)^{-1}\big[\mathbf{C}(\mathbf{x}_1 + \mathbf{q}_d, \mathbf{x}_2)\mathbf{x}_2 + p\,\mathrm{sat}_{\delta_p}(\mathbf{x}_1) + d\,\mathrm{sat}_{\delta_d}(\mathbf{x}_2)\big].\end{aligned} \tag{7.4}$$

Since $\mathrm{sat}_{\delta_p}(\mathbf{x}_1) = \mathbf{0} \Leftrightarrow \mathbf{x}_1 = \mathbf{0}$ and $\mathrm{sat}_{\delta_d}(\mathbf{x}_2) = \mathbf{0} \Leftrightarrow \mathbf{x}_2 = \mathbf{0}$ then the (only) equilibrium point is

$$(\bar{\mathbf{x}}_1, \bar{\mathbf{x}}_2) = (\mathbf{0}, \mathbf{0}).$$

The desired energy (5.2) is

$$
\begin{aligned}
\mathcal{H}_d(\mathbf{x}_1+\mathbf{q}_d,\mathbf{q}_d,\mathbf{x}_2) =& \frac{1}{2}\mathbf{x}_2^\top \mathbf{M}(\mathbf{x}_1+\mathbf{q}_d)\mathbf{x}_2 + p\sum_{i=1}^n \int_0^{x_{1i}} \mathrm{sat}_{\delta_{pi}}(y)dy \\
=& \frac{1}{2}\mathbf{x}_2^\top \mathbf{M}(\mathbf{x}_1+\mathbf{q}_d)\mathbf{x}_2 + p\sum_{i=1}^n \begin{cases} \frac{1}{2}|x_{1i}|^2 & |x_{1i}|<\delta_{pi} \\ \delta_{pi}|x_{1i}| - \frac{1}{2}\delta_{pi}^2 & |x_{1i}| \geq \delta_{pi}, \end{cases}
\end{aligned} \tag{7.5}
$$

Using (7.1), it holds that

$$
\sum_{i=1}^n \int_0^{x_{1i}} \mathrm{sat}_{\delta_{pi}}(y)dy \geq \begin{cases} \frac{1}{2}\|\mathbf{x}_1\|^2 & \|\mathbf{x}_1\| < \underline{\delta}_p \\ \underline{\delta}_p\|\mathbf{x}_1\| - \frac{1}{2}\underline{\delta}_p^2 & \|\mathbf{x}_1\| \geq \underline{\delta}_p, \end{cases} \geq \frac{1}{2}\begin{cases} \|\mathbf{x}_1\|^2 & \|\mathbf{x}_1\| < \underline{\delta}_p \\ \underline{\delta}_p\|\mathbf{x}_1\| & \|\mathbf{x}_1\| \geq \underline{\delta}_p, \end{cases}
$$

where $\underline{\delta}_p := \min\limits_{1\leq i\leq n}\{\delta_{pi}\}$. Hence $\mathcal{H}_d(\mathbf{x}_1+\mathbf{q}_d,\mathbf{q}_d,\mathbf{x}_2)$ is positive definite and radially unbounded. Moreover, we have that

$$
\begin{aligned}
\dot{\mathcal{H}}_d(\mathbf{x}_1+\mathbf{q}_d,\mathbf{q}_d,\mathbf{x}_2) =& -d\mathbf{x}_2^\top \mathrm{sat}_{\delta_d}(\mathbf{x}_2) \\
=& -d\sum_{i=1}^n x_{2i}\mathrm{sat}_{\delta_{di}}(x_{2i}) \\
=& -d\sum_{i=1}^n \begin{cases} x_{2i}^2 & |x_{2i}|<\delta_{di} \\ \delta_{di}x_{2i}\mathrm{sign}(x_{2i}) & |x_{2i}|\geq\delta_{di} \end{cases} \\
=& -d\sum_{i=1}^n \begin{cases} x_{2i}^2 & |x_{2i}|<\delta_{di} \\ \delta_{di}|x_{2i}| & |x_{2i}|\geq\delta_{di}. \end{cases}
\end{aligned}
$$

Therefore, $\mathcal{H}_d(\mathbf{x}_1+\mathbf{q}_d,\mathbf{q}_d,\mathbf{x}_2)$ is positive definite and radially unbounded and $\dot{\mathcal{H}}_d(\mathbf{x}_1+\mathbf{q}_d,\mathbf{q}_d,\mathbf{x}_2)=0$ only when $\mathbf{x}_2=\mathbf{0}$ and $\dot{\mathcal{H}}_d(\mathbf{x}_1+\mathbf{q}_d,\mathbf{q}_d,\mathbf{x}_2)\leq 0$, otherwise. The trajectories that live in the set $\Omega := \{\mathbf{x}\in\mathbb{R}^{2n} : \dot{\mathcal{H}}_d = 0\}$ are $\mathbf{x}_2=\mathbf{0}$, $\dot{\mathbf{x}}_2=\mathbf{0}$ and, from (7.4), $\mathbf{x}_1=\mathbf{0}$. Thus, using LaSalle Invariance Theorem, $(\dot{\mathbf{q}},\tilde{\mathbf{q}})=(\mathbf{0},\mathbf{0})$ is GAS, as required.

Remark 17 *As the controller in Section 6.1, this controller also requires the computation of the gravity vector at all time instants and it relies in the use of velocity measurements.* ◇

Controller (7.2) has been originally reported by Kelly and Santibañez in 1996 [5].

7.2 sPsD with gravity compensation

The idea here is to extend controller (6.3) to its saturated version, as in (7.2). So, since we assume that robot velocities are measurable, we design

$$\mathcal{R}(\dot{\mathbf{q}}) = d\sum_{i=1}^{n}\int_0^{\dot{q}_i} \mathrm{sat}_{\delta_{di}}(y)dy.$$

Moreover, we want to compensate for the gravity effects, thus we also borrow the structure of $\mathcal{U}_c$ of controller (6.3) and design the controller potential energy as

$$\mathcal{U}_c(\mathbf{q},\mathbf{q}_d) = -\mathcal{U}(\mathbf{q}_d) - \tilde{\mathbf{q}}^\top \mathbf{g}(\mathbf{q}_d) + p\sum_{i=1}^{n}\int_0^{\tilde{q}_i} \mathrm{sat}_{\delta_{pi}}(y)dy.$$

Since with these definitions $\frac{\partial}{\partial \boldsymbol{\theta}}\mathcal{U}_c(\mathbf{q},\mathbf{q}_d) = \mathbf{0}$, the dynamics of the controller does not play any role in the stabilization mechanism, thus we set $m_c = 0$ and $\mathcal{R}_c(\dot{\boldsymbol{\theta}}) = 0$.

The resulting controller is

$$\boldsymbol{\tau} = \mathbf{g}(\mathbf{q}_d) - p\,\mathrm{sat}_{\delta_p}(\tilde{\mathbf{q}}) - d\,\mathrm{sat}_{\delta_d}(\dot{\mathbf{q}}). \tag{7.6}$$

Similar to the previous case, each ith-element of the proposed controller is

$$\tau_i = g_i(\mathbf{q}_d) - p\,\mathrm{sat}_{\delta_{pi}}(\tilde{q}_i) - d\,\mathrm{sat}_{\delta_{di}}(\dot{q}_i).$$

Therefore

$$|\tau_i| \le k_{gi} + p\delta_{pi} + d\delta_{di}.$$

where k_{gi} is the bound of each of the gravity vector elements that is given in Property P6; δ_{pi} and δ_{di} are the bounds of the error and velocity saturation functions, respectively; and p and d are the proportional and derivative gains, respectively.

This bound is independent of the values of the position error and of the velocity.

Since we are required to satisfy $|\tau_i| \le \bar{\tau}_i$, then

$$|\tau_i| \le k_{gi} + p\delta_{pi} + d\delta_{di} < \bar{\tau}_i.$$

Therefore, setting the saturation limits and the controller gains such that (7.3) holds, ensures that the actuators are *never* saturated.

The desired potential energy is then given by

$$\mathcal{U}_d(\mathbf{q},\mathbf{q}_d) = \mathcal{U}(\mathbf{q}) - \mathcal{U}(\mathbf{q}_d) - \tilde{\mathbf{q}}^\top \mathbf{g}(\mathbf{q}_d) + p\sum_{i=1}^{n}\int_0^{\tilde{q}_i} \mathrm{sat}_{\delta_{pi}}(y)dy. \tag{7.7}$$

However such function is not, in general, positive definite (as that in the case of controller (6.3)). Recall that, for controller (6.3), we employed the gradient-Hessian method to find the conditions under which $\mathcal{U}_d$ was positive definite. Unfortunately, due to the saturation, function (7.7) is not twice differentiable and hence its Hessian is not well-defined.

In order to observe that (7.7) is not twice differentiable, note that

$$\nabla\mathcal{U}_d(\mathbf{q},\mathbf{q}_d) = \mathbf{g}(\mathbf{q}) - \mathbf{g}(\mathbf{q}_d) + p\,\mathrm{sat}_{\delta_p}(\tilde{\mathbf{q}}). \tag{7.8}$$

Moreover

$$\nabla^2\mathcal{U}_d(\mathbf{q},\mathbf{q}_d) = \nabla\mathbf{g}(\mathbf{q}) + p\nabla\mathrm{sat}_{\delta_p}(\tilde{\mathbf{q}}),$$

however, $\nabla\mathrm{sat}_{\delta_p}(\tilde{\mathbf{q}})$ involves the derivative of $\mathrm{sign}(\tilde{\mathbf{q}})$, which *does not exist*.

In this case we use Lemma 3 to show that $\mathcal{U}_d(\mathbf{q},\mathbf{q}_d)$ is positive definite, radially unbounded, with a unique global minimum at $\mathbf{q}=\mathbf{q}_d$. **This is the hardest technical problem with this controller** and we solve it in what follows.

First note that, when $\mathbf{q}=\mathbf{q}_d$, the desired potential energy is zero, i.e., $\mathcal{U}_d(\mathbf{q}_d,\mathbf{q}_d)=0$.

Now, let us define

$$\mathcal{P}(\mathbf{q},\mathbf{q}_d) := \mathcal{U}(\mathbf{q}) - \mathcal{U}(\mathbf{q}_d) - \tilde{\mathbf{q}}^\top\mathbf{g}(\mathbf{q}_d). \tag{7.9}$$

Using the Mean Value Theorem (see Theorem 7) there exists $\mathbf{q}'\in\mathbb{R}^n$ such that $\mathbf{q}'\in(\mathbf{q},\mathbf{q}_d)$ and

$$\mathcal{U}(\mathbf{q}) - \mathcal{U}(\mathbf{q}_d) = \mathbf{g}(\mathbf{q}')^\top\tilde{\mathbf{q}},$$

where $\mathbf{g}(\mathbf{q}') = \frac{\partial}{\partial\mathbf{q}}\mathcal{U}(\mathbf{q})\Big|_{\mathbf{q}=\mathbf{q}'}$. Hence

$$|\mathcal{U}(\mathbf{q}) - \mathcal{U}(\mathbf{q}_d)| = |\mathbf{g}(\mathbf{q}')^\top\tilde{\mathbf{q}}| \le \|\mathbf{g}(\mathbf{q}')\|\|\tilde{\mathbf{q}}\|.$$

Using Property P5 we have that

$$|\mathcal{U}(\mathbf{q}) - \mathcal{U}(\mathbf{q}_d)| \le k_g\|\tilde{\mathbf{q}}\|.$$

Moreover

$$|\mathcal{P}(\mathbf{q},\mathbf{q}_d)| = |\mathcal{U}(\mathbf{q}) - \mathcal{U}(\mathbf{q}_d) - \tilde{\mathbf{q}}^\top\mathbf{g}(\mathbf{q}_d)| \le |\mathcal{U}(\mathbf{q}) - \mathcal{U}(\mathbf{q}_d)| + |\tilde{\mathbf{q}}^\top\mathbf{g}(\mathbf{q}_d)| \le 2k_g\|\tilde{\mathbf{q}}\|. \tag{7.10}$$

Now, note that

$$\frac{\partial^2}{\partial \tilde{\mathbf{q}}^2}\mathcal{P}(\mathbf{q},\mathbf{q}_d) = \frac{\partial}{\partial \tilde{\mathbf{q}}}\mathbf{g}(\mathbf{q}).$$

Hence, using again Property P5,

$$\left\|\frac{\partial^2}{\partial \tilde{\mathbf{q}}^2}\mathcal{P}(\mathbf{q},\mathbf{q}_d)\right\| = \left\|\frac{\partial}{\partial \tilde{\mathbf{q}}}\mathbf{g}(\mathbf{q})\right\| \leq L_g.$$

Integrating two times the previous inequality yields

$$\int_{\mathbf{0}}^{\tilde{\mathbf{q}}}\int_{\mathbf{0}}^{\mathbf{x}}\left\|\frac{\partial^2}{\partial \mathbf{y}^2}\mathcal{P}(\mathbf{y}+\mathbf{q}_d,\mathbf{q}_d)\right\| d\mathbf{y}d\mathbf{x} \leq \frac{1}{2}L_g\|\tilde{\mathbf{q}}\|^2 \tag{7.11}$$

Now, using the fact that

$$\left\|\int_{\mathbf{0}}^{\tilde{\mathbf{q}}}\int_{\mathbf{0}}^{\mathbf{x}}\frac{\partial^2}{\partial \mathbf{y}^2}\mathcal{P}(\mathbf{y}+\mathbf{q}_d,\mathbf{q}_d)d\mathbf{y}d\mathbf{x}\right\| = \left\|\int_{\mathbf{0}}^{\tilde{\mathbf{q}}}\frac{\partial}{\partial \mathbf{x}}\mathcal{P}(\mathbf{x}+\mathbf{q}_d,\mathbf{q}_d)d\mathbf{x}\right\| = |\mathcal{P}(\mathbf{q},\mathbf{q}_d)|,$$

plus the property that

$$\left\|\int_{\mathbf{0}}^{\tilde{\mathbf{q}}}\int_{\mathbf{0}}^{\mathbf{x}}\frac{\partial^2}{\partial \mathbf{y}^2}\mathcal{P}(\mathbf{y}+\mathbf{q}_d,\mathbf{q}_d)d\mathbf{y}d\mathbf{x}\right\| \leq \int_{\mathbf{0}}^{\tilde{\mathbf{q}}}\int_{\mathbf{0}}^{\mathbf{x}}\left\|\frac{\partial^2}{\partial \mathbf{y}^2}\mathcal{P}(\mathbf{y}+\mathbf{q}_d,\mathbf{q}_d)\right\| d\mathbf{y}d\mathbf{x}.$$

Thus

$$|\mathcal{P}(\mathbf{q},\mathbf{q}_d)| \leq \int_{\mathbf{0}}^{\tilde{\mathbf{q}}}\int_{\mathbf{0}}^{\mathbf{x}}\left\|\frac{\partial^2}{\partial \mathbf{y}^2}\mathcal{P}(\mathbf{y}+\mathbf{q}_d,\mathbf{q}_d)\right\| d\mathbf{y}d\mathbf{x}.$$

clearly, using (7.11), it holds that

$$|\mathcal{P}(\mathbf{q},\mathbf{q}_d)| \leq \frac{1}{2}L_g\|\tilde{\mathbf{q}}\|^2. \tag{7.12}$$

From (7.1), we also obtain

$$\sum_{i=1}^{n}\int_0^{\tilde{q}_i}\mathrm{sat}_{\delta_{pi}}(y)dy \geq \frac{1}{2}\begin{cases}\|\tilde{\mathbf{q}}\|^2 & \|\tilde{\mathbf{q}}\| < \underline{\delta}_p\\ \underline{\delta}_p\|\tilde{\mathbf{q}}\| & \|\tilde{\mathbf{q}}\| \geq \underline{\delta}_p.\end{cases} \tag{7.13}$$

Finally we have that

$$|\mathcal{U}_d(\mathbf{q},\mathbf{q}_d)| = \left|\mathcal{P}(\mathbf{q},\mathbf{q}_d) + p\sum_{i=1}^{n}\int_0^{\tilde{q}_i}\mathrm{sat}_{\delta_{pi}}(y)dy\right| \geq \left|p\sum_{i=1}^{n}\int_0^{\tilde{q}_i}\mathrm{sat}_{\delta_{pi}}(y)dy\right| - |\mathcal{P}(\mathbf{q},\mathbf{q}_d)|.$$

Hence, using (7.10), (7.12) and (7.13), we get

$$|\mathcal{U}_d(\mathbf{q},\mathbf{q}_d)| \geq \frac{1}{2}\begin{cases}(p-L_g)\|\tilde{\mathbf{q}}\|^2 & \|\tilde{\mathbf{q}}\| < \underline{\delta}_p\\ (p\underline{\delta}_p - 4k_g)\|\tilde{\mathbf{q}}\| & \|\tilde{\mathbf{q}}\| \geq \underline{\delta}_p.\end{cases}.$$

> Setting $p > L_g$ and $p > \frac{4K_g}{\underline{\delta}_p}$ ensures that $\mathcal{U}_d(\mathbf{q},\mathbf{q}_d) > 0$ when $\tilde{\mathbf{q}} \neq \mathbf{0}$.
> Therefore Item 1 of Lemma 3 holds.
> These inequalities are satisfied when
>
> $$p > \max\left\{L_g, \frac{4k_g}{\underline{\delta}_p}\right\}. \tag{7.14}$$

In order to prove Item 2 of Lemma 3, from (7.8), we have that

$$\tilde{\mathbf{q}}^\top \nabla \mathcal{U}_d(\mathbf{q},\mathbf{q}_d) = \tilde{\mathbf{q}}^\top \big(\mathbf{g}(\mathbf{q}) - \mathbf{g}(\mathbf{q}_d)\big) + p\tilde{\mathbf{q}}^\top \mathrm{sat}_{\delta_p}(\tilde{\mathbf{q}}).$$

Clearly,

$$\left|\tilde{\mathbf{q}}^\top \nabla \mathcal{U}_d(\mathbf{q},\mathbf{q}_d)\right| = \left|\tilde{\mathbf{q}}^\top \big(\mathbf{g}(\mathbf{q}) - \mathbf{g}(\mathbf{q}_d)\big) + p\tilde{\mathbf{q}}^\top \mathrm{sat}_{\delta_p}(\tilde{\mathbf{q}})\right| \geq p\left|\tilde{\mathbf{q}}^\top \mathrm{sat}_{\delta_p}(\tilde{\mathbf{q}})\right| - \left|\tilde{\mathbf{q}}^\top \big(\mathbf{g}(\mathbf{q}) - \mathbf{g}(\mathbf{q}_d)\big)\right|.$$

Now, on the one hand, we have that

$$\tilde{\mathbf{q}}^\top \mathrm{sat}_{\delta_p}(\tilde{\mathbf{q}}) = \sum_{i=1}^n \tilde{q}_i \mathrm{sat}_{\delta_{pi}}(\tilde{q}_i) = \sum_{i=1}^n \begin{cases} \tilde{q}_i^2 & |\tilde{q}_i| < \delta_{pi} \\ \delta_{pi}|\tilde{q}_i| & |\tilde{q}_i| \geq \delta_{pi} \end{cases} \geq \begin{cases} \|\tilde{\mathbf{q}}\|^2 & \|\tilde{\mathbf{q}}\| < \underline{\delta}_p \\ \underline{\delta}_p\|\tilde{\mathbf{q}}\| & \|\tilde{\mathbf{q}}_1\| \geq \underline{\delta}_p. \end{cases} \tag{7.15}$$

On the other hand,

$$\left|\tilde{\mathbf{q}}^\top \big(\mathbf{g}(\mathbf{q}) - \mathbf{g}(\mathbf{q}_d)\big)\right| \leq \left|\tilde{\mathbf{q}}^\top \mathbf{g}(\mathbf{q})\right| + \left|\tilde{\mathbf{q}}^\top \mathbf{g}(\mathbf{q}_d)\right| \leq 2k_g\|\tilde{\mathbf{q}}\|, \tag{7.16}$$

and

$$\left|\tilde{\mathbf{q}}^\top \big(\mathbf{g}(\mathbf{q}) - \mathbf{g}(\mathbf{q}_d)\big)\right| \leq \|\tilde{\mathbf{q}}\|\|\mathbf{g}(\mathbf{q}) - \mathbf{g}(\mathbf{q}_d)\| \leq L_g\|\tilde{\mathbf{q}}\|^2, \tag{7.17}$$

where, to obtain this last inequality, we have employed Property P5 plus the fact that, using the Mean Value Theorem (see Theorem 7),

$$\mathbf{g}(\mathbf{q}) - \mathbf{g}(\mathbf{q}_d) = \frac{\partial}{\partial \mathbf{q}}\mathbf{g}(\mathbf{q}')^\top \tilde{\mathbf{q}},$$

where $\frac{\partial}{\partial \mathbf{q}}\mathbf{g}(\mathbf{q}') = \frac{\partial^2}{\partial \mathbf{q}^2}\mathcal{U}(\mathbf{q})\Big|_{\mathbf{q}=\mathbf{q}'}$.

Therefore, using (7.15), (7.16) and (7.17), yields

$$\left|\tilde{\mathbf{q}}^\top \nabla \mathcal{U}_d(\mathbf{q},\mathbf{q}_d)\right| \geq \begin{cases} (p - L_g)\|\tilde{\mathbf{q}}\|^2 & \|\tilde{\mathbf{q}}\| < \underline{\delta}_p \\ (p\underline{\delta}_p - 2k_g)\|\tilde{\mathbf{q}}\| & \|\tilde{\mathbf{q}}\| \geq \underline{\delta}_p. \end{cases}$$

Hence, setting p satisfying (7.14) ensures that $\tilde{\mathbf{q}}^\top \nabla \mathcal{U}_d(\mathbf{q},\mathbf{q}_d) > 0$ for all $\tilde{\mathbf{q}} \neq \mathbf{0}$ as required. Finally, Item 3 of of Lemma 3 follows immediately from the the lower bound of $\mathcal{U}_d(\mathbf{q},\mathbf{q}_d)$. Therefore, $\mathcal{U}_d(\mathbf{q},\mathbf{q}_d)$ is positive definite and radially unbounded with a unique global minimum at $\tilde{\mathbf{q}}=\mathbf{0}$, provided that condition (7.14) holds.

The closed-loop system is thus given by

$$\ddot{\mathbf{q}} = -\mathbf{M}(\mathbf{q})^{-1}\big[\mathbf{C}(\mathbf{q},\dot{\mathbf{q}})\dot{\mathbf{q}} + \mathbf{g}(\mathbf{q}) - \mathbf{g}(\mathbf{q}_d) + p\,\mathrm{sat}_{\delta_p}(\tilde{\mathbf{q}}) + d\,\mathrm{sat}_{\delta_d}(\dot{\mathbf{q}})\big].$$

Defining $\mathbf{x}_1 := \tilde{\mathbf{q}}$ and $\mathbf{x}_2 := \dot{\mathbf{q}}$, we obtain

$$\begin{aligned}\dot{\mathbf{x}}_1 =& \mathbf{x}_2,\\ \dot{\mathbf{x}}_1 =& -\mathbf{M}(\mathbf{x}_1+\mathbf{q}_d)^{-1}\big[\mathbf{C}(\mathbf{x}_1+\mathbf{q}_d,\mathbf{x}_2)\mathbf{x}_2 + \mathbf{g}(\mathbf{x}_1+\mathbf{q}_d) - \mathbf{g}(\mathbf{q}_d)\big]\\ &- \mathbf{M}(\mathbf{x}_1+\mathbf{q}_d)^{-1}\big[p\,\mathrm{sat}_{\delta_p}(\mathbf{x}_1) + d\,\mathrm{sat}_{\delta_d}(\mathbf{x}_2)\big].\end{aligned} \tag{7.18}$$

Setting p satisfying (7.14) ensures that $(\bar{\mathbf{x}}_1,\bar{\mathbf{x}}_2)=(\mathbf{0},\mathbf{0})$ is the only equilibrium point.

Moreover, the desired energy (5.2) yields

$$\mathcal{H}_d = \frac{1}{2}\mathbf{x}_2^\top\mathbf{M}(\mathbf{x}_1+\mathbf{q}_d)\mathbf{x}_2 + \mathcal{U}(\mathbf{x}_1+\mathbf{q}_d) - \mathcal{U}(\mathbf{q}_d) - \mathbf{x}_1^\top\mathbf{g}(\mathbf{q}_d) + p\sum_{i=1}^{n}\int_0^{x_{1i}}\mathrm{sat}_{\delta_{pi}}(y)dy,$$

Moreover, $\dot{\mathcal{H}}_d$ evaluated along (7.18) is

$$\begin{aligned}\dot{\mathcal{H}}_d =& -d\mathbf{x}_2^\top\mathrm{sat}_{\delta_d}(\mathbf{x}_2)\\ =& -d\sum_{i=1}^{n}\begin{cases} x_{2i}^2 & |x_{2i}|<\delta_{di}\\ \delta_{di}|x_{2i}| & |x_{2i}|\geq\delta_{di}.\end{cases}\end{aligned}$$

Therefore, if p satisfies (7.14), $\mathcal{H}_d(\mathbf{x}_1+\mathbf{q}_d,\mathbf{q}_d,\mathbf{x}_2)$ is positive definite and radially unbounded and $\dot{\mathcal{H}}_d(\mathbf{x}_1+\mathbf{q}_d,\mathbf{q}_d,\mathbf{x}_2)=0$ only when $\mathbf{x}_2=\mathbf{0}$ and $\dot{\mathcal{H}}_d(\mathbf{x}_1+\mathbf{q}_d,\mathbf{q}_d,\mathbf{x}_2)\leq 0$, otherwise. The trajectories that live in the set $\Omega := \{\mathbf{x}\in\mathbb{R}^{2n} : \dot{\mathcal{H}}_d = 0\}$ are $\mathbf{x}_2=\mathbf{0}$, $\dot{\mathbf{x}}_2=\mathbf{0}$ and, from (7.18), $\mathbf{x}_1=\mathbf{0}$. Thus, using LaSalle Invariance Theorem, $(\dot{\mathbf{q}},\tilde{\mathbf{q}})=(\mathbf{0},\mathbf{0})$ is GAS, as required.

Remark 18 *Although this controller is bounded and does not require the computation of online gravity cancellation, it does rely on the use of velocity measurements.* ◇

Controller (7.6) is a consequence of the work of Loría et al. in 1997 [9].

7.3 sP‘sD’ with gravity cancellation

This controller does not require velocities to be measurable, as controller (6.5) and relies on the saturation functions that have been employed so far to obtain bounded controllers.

The design starts by setting $\mathcal{R}(\dot{\mathbf{q}}) = 0$, because $\dot{\mathbf{q}}$ is not available. Therefore, damping injection has to be done via the controller and thus we define

$$\mathcal{R}_c(\dot{\boldsymbol{\theta}}) = \frac{1}{2} d \|\dot{\boldsymbol{\theta}}\|^2.$$

As for the controller potential energy, we cancel-out the gravity effects and add two *nonlinear* springs, one for the position error and the other for the robot-controller interconnection. The nonlinear springs have to provide a bounded force. Thus we set

$$\mathcal{U}_c(\mathbf{q}, \boldsymbol{\theta}, \mathbf{q}_d) = -\mathcal{U}(\mathbf{q}) + p \sum_{i=1}^{n} \int_0^{\tilde{q}_i} \mathrm{sat}_{\delta_{pi}}(y) dy + k \sum_{i=1}^{n} \int_0^{q_i - \theta_i} \mathrm{sat}_{\delta_{di}}(y) dy.$$

Using Leibniz Theorem (see Theorem 8), we have that

$$\frac{\partial}{\partial \mathbf{q}} \mathcal{U}_c(\mathbf{q}, \boldsymbol{\theta}, \mathbf{q}_d) = -\mathbf{g}(\mathbf{q}) + p \, \mathrm{sat}_{\delta_p}(\tilde{\mathbf{q}}) + k \, \mathrm{sat}_{\delta_d}(\mathbf{q} - \boldsymbol{\theta}),$$

and, from the fact that the derivative with respect to $\boldsymbol{\theta}$ of the upper limit of the last integral is -1, i.e., $\frac{\partial}{\partial \boldsymbol{\theta}_i}(q_i - \theta_i) = -1$, we get

$$\frac{\partial}{\partial \boldsymbol{\theta}} \mathcal{U}_c(\mathbf{q}, \boldsymbol{\theta}, \mathbf{q}_d) = -k \, \mathrm{sat}_{\delta_d}(\mathbf{q} - \boldsymbol{\theta}).$$

At this point we arrive to the question, do we want a second-order dynamical controller or a first order? In any case, the regulation problem is solved. We go for the first order dynamics and set $m_c = 0$ (the second order case is obtained setting $m_c > 0$).

Using these definitions we obtain the controller

$$\boldsymbol{\tau} = \mathbf{g}(\mathbf{q}) - p \, \mathrm{sat}_{\delta_p}(\tilde{\mathbf{q}}) - k \, \mathrm{sat}_{\delta_d}(\mathbf{q} - \boldsymbol{\theta}), \tag{7.19}$$

with the following controller dynamics

$$\dot{\boldsymbol{\theta}} = \frac{k}{d} \mathrm{sat}_{\delta_d}(\mathbf{q} - \boldsymbol{\theta}).$$

Controller (7.19), obviously, satisfies

$$|\tau_i| \leq k_{gi} + p\delta_{pi} + k\delta_{di}.$$

Furthermore, since we are required to satisfy $|\tau_i| \leq \bar{\tau}_i$, the following inequality must hold

$$p\delta_{pi} + k\delta_{di} < \bar{\tau}_i - k_{gi}.$$

Defining the state variables $\mathbf{x}_1 = \tilde{\mathbf{q}}$, $\mathbf{x}_2 = \dot{\mathbf{q}}$ and $\mathbf{x}_3 = \mathbf{q} - \boldsymbol{\theta}$ we obtain

$$\begin{aligned}
\dot{\mathbf{x}}_1 =& \mathbf{x}_2, \\
\dot{\mathbf{x}}_2 =& -\mathbf{M}^{-1}(\mathbf{x}_1 + \mathbf{q}_d)\big[\mathbf{C}(\mathbf{x}_1 + \mathbf{q}_d, \mathbf{x}_2)\mathbf{x}_2 + p\,\mathrm{sat}_{\delta_p}(\mathbf{x}_1) + k\,\mathrm{sat}_{\delta_d}(\mathbf{x}_3)\big], \\
\dot{\mathbf{x}}_3 =& \mathbf{x}_2 - \frac{k}{d}\,\mathrm{sat}_{\delta_d}(\mathbf{x}_3).
\end{aligned} \tag{7.20}$$

It is not difficult to see that $(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = (\mathbf{0}, \mathbf{0}, \mathbf{0})$ is the only equilibrium point of (7.20).

The desired energy is thus given by

$$\mathcal{H}_d = \frac{1}{2}\mathbf{x}_2^\top \mathbf{M}(\mathbf{x}_1 + \mathbf{q}_d)\mathbf{x}_2 + p\sum_{i=1}^{n}\int_0^{x_{1i}} \mathrm{sat}_{\delta_{pi}}(y)dy + k\sum_{i=1}^{n}\int_0^{x_{3i}} \mathrm{sat}_{\delta_{di}}(y)dy,$$

which is positive definite and radially unbounded. Its time-derivative, evaluated along (7.20) yields

$$\dot{\mathcal{H}}_d = -d\|\mathbf{x}_2 - \dot{\mathbf{x}}_3\|^2 = -\frac{k^2}{d}\|\mathrm{sat}_{\delta_d}(\mathbf{x}_3)\|^2.$$

LaSalle Invariance Theorem can be invoked using the following chain of implications $\dot{\mathcal{H}}_d = 0 \Rightarrow \mathbf{x}_3 = \mathbf{0} \Rightarrow \dot{\mathbf{x}}_3 = \mathbf{0}$; from the last equation of (7.20), $\mathbf{x}_3 = \dot{\mathbf{x}}_3 = \mathbf{0} \Rightarrow \mathbf{x}_2 = \mathbf{0} \Rightarrow \dot{\mathbf{x}}_2 = \mathbf{0}$; and, from the second equation of (7.20), $\mathbf{x}_3 = \mathbf{x}_2 = \dot{\mathbf{x}}_2 = \mathbf{0} \Rightarrow \mathbf{x}_1 = \mathbf{0}$. Thus, $(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = (\mathbf{0}, \mathbf{0}, \mathbf{0})$ is GAS, as required.

Remark 19 *This controller is bounded and does not rely on velocity measurements. However it needs to cancel-out the effects of gravity.* $\diamond$

Controller (7.19) has been originally reported by Santibáñez and Kelly in 1997 [13].

7.4 sP'sD' with gravity compensation

This is the last controller obtained using the ES+DI methodology. It corresponds to the case when velocities are not available, gravity is compensated (thus only calculated once) and with bounded control torques. Hence, this *would be* the controller to implement in most robots.

The design of this controller follows as (6.8) and (7.6), for the bounded case. We start by setting the robot dissipation to zero, i.e., $\mathcal{R}(\dot{\mathbf{q}}) = 0$, and the controller dissipation as a quadratic function, i.e.,

$$\mathcal{R}_c(\dot{\boldsymbol{\theta}}) = \frac{1}{2} d \|\dot{\boldsymbol{\theta}}\|^2.$$

The potential energy is designed to compensate for the gravity effects and to interconnect the robot and the controller via a *virtual* spring, as

$$\mathcal{U}_c = -\mathcal{U}(\mathbf{q}_d) - \tilde{\mathbf{q}}^\top \mathbf{g}(\mathbf{q}_d) + p \sum_{i=1}^{n} \int_0^{\tilde{q}_i} \operatorname{sat}_{\delta_{pi}}(y) dy + k \sum_{i=1}^{n} \int_0^{q_i - \theta_i} \operatorname{sat}_{\delta_{di}}(y) dy. \tag{7.21}$$

For the first time, we will implement a second-order controller and hence we set $m_c > 0$, in particular we set $m_c = 1$.

With all these choices we obtain

$$\boldsymbol{\tau} = \mathbf{g}(\mathbf{q}_d) - p \operatorname{sat}_{\delta_p}(\tilde{\mathbf{q}}) - k \operatorname{sat}_{\delta_d}(\mathbf{q} - \boldsymbol{\theta}), \tag{7.22}$$

with the following controller dynamics

$$\ddot{\boldsymbol{\theta}} = -d\dot{\boldsymbol{\theta}} + k \operatorname{sat}_{\delta_d}(\mathbf{q} - \boldsymbol{\theta}).$$

In this case, Controller (7.22) is bounded, for all $\mathbf{q}_d \in \mathbb{R}^n$, as

$$|\tau_i| \leq k_{gi} + p\delta_{pi} + k\delta_{di}.$$

Furthermore, since we are required to satisfy $|\tau_i| \leq \bar{\tau}_i$, the following inequality must hold

$$p\delta_{pi} + k\delta_{di} < \bar{\tau}_i - k_{gi}.$$

Now, as in the case of Controller (7.6), the desired potential energy

$$\mathcal{U}_d = \mathcal{U}(\mathbf{q}) - \mathcal{U}(\mathbf{q}_d) - \tilde{\mathbf{q}}^\top \mathbf{g}(\mathbf{q}_d) + p \sum_{i=1}^{n} \int_0^{\tilde{q}_i} \operatorname{sat}_{\delta_{pi}}(y) dy + k \sum_{i=1}^{n} \int_0^{q_i - \theta_i} \operatorname{sat}_{\delta_{di}}(y) dy,$$

is not positive definite. However, proceeding as in such case, we can write $\mathcal{U}_d$ as

$$\mathcal{U}_d(\mathbf{q},\boldsymbol{\theta},\mathbf{q}_d) = \mathcal{Q}(\mathbf{q},\mathbf{q}_d) + k\sum_{i=1}^{n}\int_0^{q_i-\theta_i} \mathrm{sat}_{\delta_{di}}(y)dy,$$

where

$$\mathcal{Q}(\mathbf{q},\mathbf{q}_d) := \mathcal{P}(\mathbf{q},\mathbf{q}_d) + p\sum_{i=1}^{n}\int_0^{\tilde{q}_i} \mathrm{sat}_{\delta_{pi}}(y)dy,$$

and $\mathcal{P}(\mathbf{q},\mathbf{q}_d)$ is defined in (7.9).

Hence, we can prove that

$$|\mathcal{Q}(\mathbf{q},\mathbf{q}_d)| \geq \frac{1}{2}\begin{cases}(p-L_g)\|\tilde{\mathbf{q}}\|^2 & \|\tilde{\mathbf{q}}\| < \underline{\delta}_p\\ (p\underline{\delta}_p - 4k_g)\|\tilde{\mathbf{q}}\| & \|\tilde{\mathbf{q}}\| \geq \underline{\delta}_p\end{cases}$$

and that

$$\left|\tilde{\mathbf{q}}^\top\nabla\mathcal{Q}(\mathbf{q},\mathbf{q}_d)\right| \geq \begin{cases}(p-L_g)\|\tilde{\mathbf{q}}\|^2 & \|\tilde{\mathbf{q}}\| < \underline{\delta}_p\\ (p\underline{\delta}_p - 2k_g)\|\tilde{\mathbf{q}}\| & \|\tilde{\mathbf{q}}\| \geq \underline{\delta}_p.\end{cases}$$

Hence, using Lemma 3 and setting p satisfying (7.14) ensures that $\mathcal{Q}(\mathbf{q},\mathbf{q}_d)$ is positive definite and radially unbounded with regards to $\tilde{\mathbf{q}}$.

The closed-loop system is thus given by

$$\begin{aligned}\ddot{\mathbf{q}} =& -\mathbf{M}(\mathbf{q})^{-1}\big[\mathbf{C}(\mathbf{q},\dot{\mathbf{q}})\dot{\mathbf{q}} + \mathbf{g}(\mathbf{q}) - \mathbf{g}(\mathbf{q}_d) + p\,\mathrm{sat}_{\delta_p}(\tilde{\mathbf{q}}) + k\,\mathrm{sat}_{\delta_d}(\mathbf{q}-\boldsymbol{\theta})\big]\\ \ddot{\boldsymbol{\theta}} =& -d\dot{\boldsymbol{\theta}} + k\,\mathrm{sat}_{\delta_d}(\mathbf{q}-\boldsymbol{\theta}).\end{aligned}$$

Defining the state variables $\mathbf{x}_1 = \tilde{\mathbf{q}}$, $\mathbf{x}_2 = \dot{\mathbf{q}}$, $\mathbf{x}_3 = \mathbf{q}-\boldsymbol{\theta}$ and $\mathbf{x}_4 = \dot{\boldsymbol{\theta}}$, we obtain

$$\begin{aligned}\dot{\mathbf{x}}_1 =& \mathbf{x}_2,\\ \dot{\mathbf{x}}_2 =& -\mathbf{M}^{-1}(\mathbf{x}_1+\mathbf{q}_d)\big[\mathbf{C}(\mathbf{x}_1+\mathbf{q}_d,\mathbf{x}_2)\mathbf{x}_2 + \mathbf{g}(\mathbf{x}_1+\mathbf{q}_d) - \mathbf{g}(\mathbf{q}_d)\big]\\ &-\mathbf{M}^{-1}(\mathbf{x}_1+\mathbf{q}_d)\big[p\,\mathrm{sat}_{\delta_p}(\mathbf{x}_1) + k\,\mathrm{sat}_{\delta_d}(\mathbf{x}_3)\big],\\ \dot{\mathbf{x}}_3 =& \mathbf{x}_2 - \mathbf{x}_4,\\ \dot{\mathbf{x}}_4 =& -d\mathbf{x}_4 + k\,\mathrm{sat}_{\delta_d}(\mathbf{x}_3).\end{aligned} \tag{7.23}$$

If the proportional gain p is set such that (7.14) holds, then

$$(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3,\mathbf{x}_4) = (\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0}) \tag{7.24}$$

is the only equilibrium point of (7.23).

The desired energy is thus given by

$$\mathcal{H}_d = \frac{1}{2}\mathbf{x}_2^\top \mathbf{M}(\mathbf{x}_1 + \mathbf{q}_d)\mathbf{x}_2 + \frac{1}{2}|\mathbf{x}_4|^2 + k\sum_{i=1}^{n}\int_0^{x_{3i}} \mathrm{sat}_{\delta_{di}}(y)dy$$
$$+ \mathcal{U}(\mathbf{x}_1 + \mathbf{q}_d) - \mathcal{U}(\mathbf{q}_d) - \mathbf{x}_1^\top \mathbf{g}(\mathbf{q}_d) + p\sum_{i=1}^{n}\int_0^{x_{1i}} \mathrm{sat}_{\delta_{pi}}(y)dy$$

which, under the assumption that p satisfies (7.14), is positive definite and radially unbounded. Its time-derivative, evaluated along (7.23) yields

$$\dot{\mathcal{H}}_d = -d\|\mathbf{x}_4\|^2$$

Since (7.23) is autonomous and $\dot{\mathcal{H}}_d \leq 0$, then LaSalle Invariance Theorem can be invoked using the following chain of implications $\dot{\mathcal{H}}_d = 0 \Rightarrow \mathbf{x}_4 = \mathbf{0} \Rightarrow \dot{\mathbf{x}}_4 = \mathbf{0}$; from the last equation of (7.23), $\mathbf{x}_3 = \mathbf{0}$ and thus $\dot{\mathbf{x}}_3 = \mathbf{0}$. From the second equation of (7.23), $\dot{\mathbf{x}}_2 = \mathbf{0}$, hence $\dot{\mathbf{x}}_2 = \mathbf{0}$; and, from the second equation of (7.23), $\mathbf{x}_3 = \mathbf{x}_2 = \dot{\mathbf{x}}_2 = \mathbf{0} \Rightarrow \mathbf{x}_1 = \mathbf{0}$. Thus, (7.24) is GAS.

Remark 20 *This controller provides bounded torques, it does not rely on velocity measurements and it compensates for the gravity effects. Hence, this is the most suitable for application purposes of all the ES+DI schemes outlined here.* ◇

Controller (7.22) has been originally reported by Loría et al. in 1997 [9].

Chapter 8

PID (Adaptive) Control

All the previous *regulation* schemes, either with gravity cancellation or with gravity compensation, require the exact knowledge of the physical parameters of the gravity term.

We should underscore that the inertia and Coriolis matrices and the gravity vector, are not only a function of the robot position and/or velocity, these are also a function of the physical parameters. These parameters appear linearly in the robot dynamics and thus, for any differentiable vector $\mathbf{q}_r \in \mathbb{R}^n$, we have that

$$\mathbf{M}(\mathbf{q})\dot{\mathbf{q}}_r = \mathbf{M}(\mathbf{q}, \phi)\dot{\mathbf{q}}_r = \mathbf{Y}_M(\mathbf{q}, \dot{\mathbf{q}}_r)\boldsymbol{\phi},$$

$$\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\mathbf{q}_r = \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}}, \phi)\mathbf{q}_r = \mathbf{Y}_C(\mathbf{q}, \dot{\mathbf{q}}, \mathbf{q}_r)\boldsymbol{\phi},$$

and

$$\mathbf{g}(\mathbf{q}) = \mathbf{g}(\mathbf{q}, \phi) = \mathbf{Y}_g(\mathbf{q})\boldsymbol{\phi},$$

where $\boldsymbol{\phi} \in \mathbb{R}^p$ is a vector containing $p > 0$ physical parameters and matrices $\mathbf{Y}_g(\mathbf{q}), \mathbf{Y}_M(\mathbf{q}, \dot{\mathbf{q}}_r), \mathbf{Y}_C(\mathbf{q}, \dot{\mathbf{q}}, \mathbf{q}_r) \in \mathbb{R}^{n\times p}$ contain known functions of the signals $\mathbf{q}, \dot{\mathbf{q}}, \mathbf{q}_r, \dot{\mathbf{q}}_r$.

In fact, it holds that

$$\mathbf{M}(\mathbf{q}, \phi)\dot{\mathbf{q}}_r + \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}}, \phi)\mathbf{q}_r + \mathbf{g}(\mathbf{q}, \phi) = \mathbf{Y}(\mathbf{q}, \dot{\mathbf{q}}, \mathbf{q}_r, \dot{\mathbf{q}}_r)\boldsymbol{\phi}, \tag{8.1}$$

where $\mathbf{Y}(\mathbf{q}, \dot{\mathbf{q}}, \mathbf{q}_r, \dot{\mathbf{q}}_r) \in \mathbb{R}^{n\times p}$ is the *regressor* matrix of known functions that satisfies

$$\mathbf{Y}(\mathbf{q}, \dot{\mathbf{q}}, \mathbf{q}_r, \dot{\mathbf{q}}_r) = \mathbf{Y}_M(\mathbf{q}, \dot{\mathbf{q}}_r) + \mathbf{Y}_C(\mathbf{q}, \dot{\mathbf{q}}, \mathbf{q}_r) + \mathbf{Y}_g(\mathbf{q}).$$

Remark 21 *Consider the Pelican robot model in Section 5.3.2 of Kelly et al. [6] and any differential vector function* $\mathbf{q}_r \in \mathbb{R}^2$*. Then, the inertia and the Coriolis matrices and the gravity vector, respectively, support the following parameterization*

$$\begin{aligned}\mathbf{M}(\mathbf{q},\boldsymbol{\phi})\dot{\mathbf{q}}_r &= \begin{bmatrix} \phi_1 + 2\phi_2\cos(q_2) & \phi_3 + \phi_2\cos(q_2) \\ \phi_3 + \phi_2\cos(q_2) & \phi_3 \end{bmatrix}\dot{\mathbf{q}}_r \\ &= \underbrace{\begin{bmatrix} \dot{q}_{r1} & \cos(q_2)(2\dot{q}_{r1} + \dot{q}_{r2}) & \dot{q}_{r2} & 0 & 0 \\ 0 & \dot{q}_{r1}\cos(q_2) & \dot{q}_{r1} + \dot{q}_{r2} & 0 & 0 \end{bmatrix}}_{\mathbf{Y}_M(\mathbf{q},\dot{\mathbf{q}}_r)}\boldsymbol{\phi},\end{aligned}$$

$$\begin{aligned}\mathbf{C}(\mathbf{q},\dot{\mathbf{q}},\boldsymbol{\phi})\mathbf{q}_r &= \phi_2\sin(q_2)\begin{bmatrix} -\dot{q}_2 & -(\dot{q}_1 + \dot{q}_2) \\ \dot{q}_1 & 0 \end{bmatrix}\mathbf{q}_r \\ &= \underbrace{\begin{bmatrix} 0 & -\sin(q_2)(\dot{q}_2 q_{r1} + \dot{q}_1 q_{r2} + \dot{q}_2 q_{r2}) & 0 & 0 & 0 \\ 0 & \dot{q}_1 q_{r1}\sin(q_2) & 0 & 0 & 0 \end{bmatrix}}_{\mathbf{Y}_C(\mathbf{q},\dot{\mathbf{q}},\mathbf{q}_r)}\boldsymbol{\phi},\end{aligned}$$

and

$$\begin{aligned}\mathbf{g}(\mathbf{q},\boldsymbol{\phi}) &= \begin{bmatrix} \phi_4 g\sin(q_1) + \phi_5 g\sin(q_1 + q_2) \\ \phi_5 g\sin(q_1 + q_2) \end{bmatrix} \\ &= \underbrace{\begin{bmatrix} 0 & 0 & 0 & g\sin(q_1) & g\sin(q_1 + q_2) \\ 0 & 0 & 0 & 0 & g\sin(q_1 + q_2) \end{bmatrix}}_{\mathbf{Y}_g(\mathbf{q})}\boldsymbol{\phi},\end{aligned}$$

where $\boldsymbol{\phi} := [\phi_1, \ldots, \phi_5]^\top$ *and*

$$\begin{aligned}\phi_1 &:= m_1 l_{c1}^2 + m_2(l_1^2 + l_{c2}^2) + I_1 + I_2, \\ \phi_2 &:= m_2 l_1 l_{c2}, \\ \phi_3 &:= m_2 l_{c2}^2 + I_2, \\ \phi_4 &:= m_1 l_{c1} + m_2 l_1, \\ \phi_5 &:= m_2 l_{c2}.\end{aligned}$$

$\diamond$

Suppose now that we want to solve the regulation problem via the PD with gravity cancellation scheme (6.1). However, we do not exactly know the physical parameters for the gravity cancellation term. Instead we only have an *estimation* of $\boldsymbol{\phi}$ given by $\hat{\boldsymbol{\phi}}$, then controller (6.1) transforms to

$$\boldsymbol{\tau} = -p\tilde{\mathbf{q}} - d\dot{\mathbf{q}} + \hat{\mathbf{g}}(\mathbf{q}),$$

where $\hat{\mathbf{g}}(\mathbf{q}) := \mathbf{g}(\mathbf{q}, \hat{\boldsymbol{\phi}})$. Moreover, the closed-loop becomes

$$\ddot{\mathbf{q}} = -\mathbf{M}(\mathbf{q})^{-1}\big[\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} + \mathbf{g}(\mathbf{q}) - \hat{\mathbf{g}}(\mathbf{q}) + p\tilde{\mathbf{q}} + d\dot{\mathbf{q}}\big],$$

and the equilibrium points are now $\dot{\mathbf{q}} = \mathbf{0}$ and

$$\mathbf{g}(\mathbf{q}) - \hat{\mathbf{g}}(\mathbf{q}) + p\tilde{\mathbf{q}} = \mathbf{0}.$$

Despite the fact that this equilibrium equation resembles the one of gravity compensation, in this case $\mathbf{q} = \mathbf{q}_d$ is no longer an equilibrium because, in general, $\mathbf{g}(\mathbf{q}_d) - \hat{\mathbf{g}}(\mathbf{q}_d) \neq \mathbf{0}$ when $\boldsymbol{\phi} \neq \hat{\boldsymbol{\phi}}$.

Therefore, when *parametric uncertainty* arises, the equilibrium is *shifted* from the desired value. Since the shift from the desired equilibrium position is constant, one might think that the easiest way to deal with such constant error is to simply add a linear integral of the position error, such that the linear PID controller could do the job. This linear PID controller is

$$\begin{aligned}\boldsymbol{\tau} &= -\,p\tilde{\mathbf{q}} - d\dot{\mathbf{q}} - i\boldsymbol{\xi},\\ \dot{\boldsymbol{\xi}} &= \tilde{\mathbf{q}}.\end{aligned}$$

However, as it is well explained in Chapter 9 of Kelly et al., [6] and in Chapter 4.3 of Ortega et al., [11], **the linear PID does not guarantee global convergence of position error and velocity to zero !!!**

Nevertheless, the *parametric uncertainty* case can be dealt with an *adaptive controller*, which can be seen as a *nonlinear PID controller*.

For the regulation problem there are, mainly, two possible schemes: one that relies on the *complete* model and another that only employs the gravity part of the model.

8.1 Complete Model Adaptive Controller

First Some Hints...

In order to put the adaptive controller in perspective, recall first the robot manipulator controlled by the PD with gravity cancellation controller (6.1). The closed-loop, in such case, is

$$\ddot{\mathbf{q}} = -\mathbf{M}(\mathbf{q})^{-1}\big[\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} + p\tilde{\mathbf{q}} + d\dot{\mathbf{q}}\big],$$

the desired (total) energy is

$$\mathcal{H}_d = \frac{1}{2}\dot{\mathbf{q}}^\top \mathbf{M}(\mathbf{q})\dot{\mathbf{q}} + \frac{1}{2}p\|\tilde{\mathbf{q}}\|^2,$$

and

$$\dot{\mathcal{H}}_d = -d\|\dot{\mathbf{q}}\|^2.$$

GAS of the point $(\tilde{\mathbf{q}}, \dot{\mathbf{q}}) = (\mathbf{0}, \mathbf{0})$ is ensured by LaSalle arguments with the fact that $\mathcal{H}_d$ is positive definite and radially unbounded and $\dot{\mathcal{H}}_d$ is negative semi-definite.

We want to mimic this procedure for the design of the adaptive controller because its simplicity. For, suppose that *all* the physical parameters are known.

Now the Case when ALL Parameters are Known...

Consider a *free* differentiable vector $\mathbf{s} \in \mathbb{R}^n$ and suppose that the controller is such that we arrive to a closed-loop of the form

$$\dot{\mathbf{s}} = -\mathbf{M}(\mathbf{q})^{-1}\big[\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\mathbf{s} + p\tilde{\mathbf{q}} + d\dot{\mathbf{q}}\big], \tag{8.2}$$

then the desired energy like function is

$$\mathcal{H}_d = \frac{1}{2}\mathbf{s}^\top \mathbf{M}(\mathbf{q})\mathbf{s} + \frac{1}{2}p\|\tilde{\mathbf{q}}\|^2 \tag{8.3}$$

and

$$\dot{\mathcal{H}}_d = -d\mathbf{s}^\top\big(\dot{\mathbf{q}} + \frac{p}{d}\tilde{\mathbf{q}}\big) + p\tilde{\mathbf{q}}^\top\dot{\mathbf{q}},$$

this last equation suggest the value of $\mathbf{s}$, in fact if we set

$$\mathbf{s} = \dot{\mathbf{q}} + \frac{p}{d}\tilde{\mathbf{q}},$$

then

$$\begin{aligned}
\dot{\mathcal{H}}_d &= -\frac{d}{2}\|\mathbf{s}\|^2 - \frac{d}{2}\|\dot{\mathbf{q}} + \frac{p}{d}\tilde{\mathbf{q}}\|^2 + p\tilde{\mathbf{q}}^\top\dot{\mathbf{q}} \\
&= -\frac{d}{2}\|\mathbf{s}\|^2 - \frac{d}{2}\big(\|\dot{\mathbf{q}}\|^2 + 2\frac{p}{d}\tilde{\mathbf{q}}^\top\dot{\mathbf{q}} + \frac{p^2}{d^2}\|\tilde{\mathbf{q}}\|^2\big) + p\tilde{\mathbf{q}}^\top\dot{\mathbf{q}} \\
&= -\frac{d}{2}\|\mathbf{s}\|^2 - \frac{d}{2}\|\dot{\mathbf{q}}\|^2 - \frac{p^2}{2d}\|\tilde{\mathbf{q}}\|^2
\end{aligned}$$

Hence, we have that $\mathcal{H}_d$ is positive definite and radially unbounded and, more importantly, $\dot{\mathcal{H}}_d$ is *negative definite* !!!

> So far, function (8.3) is the only Strict Lyapunov Function designed in this manual and it has been originally proposed by Spong et al., in 1990 [18].

If we define $\mathbf{x}_1 = \tilde{\mathbf{q}}$ and $\mathbf{x}_2 = \dot{\mathbf{q}}$ then we get

$$
\begin{aligned}
\dot{\mathbf{x}}_1 =& \mathbf{x}_2, \\
\dot{\mathbf{x}}_2 =& -\frac{p}{d}\mathbf{x}_2 - \mathbf{M}(\mathbf{x}_1 + \mathbf{q}_d)^{-1}\Big[\mathbf{C}(\mathbf{x}_1 + \mathbf{q}_d, \mathbf{x}_2)\Big(\mathbf{x}_2 + \frac{p}{d}\mathbf{x}_1\Big) + p\mathbf{x}_1 + d\mathbf{x}_2\Big]
\end{aligned}
$$

where we have used the fact that $\dot{\mathbf{s}} = \ddot{\mathbf{q}} + \frac{p}{d}\dot{\mathbf{q}}$. Moreover, $\mathcal{H}_d$ and $\dot{\mathcal{H}}_d$ can be written as

$$
\mathcal{H}_d = \frac{1}{2}\Big(\mathbf{x}_2 + \frac{p}{d}\mathbf{x}_1\Big)^\top \mathbf{M}(\mathbf{x}_1 + \mathbf{q}_d)\Big(\mathbf{x}_2 + \frac{p}{d}\mathbf{x}_1\Big) + \frac{1}{2}p\|\mathbf{x}_1\|^2,
$$

and

$$
\dot{\mathcal{H}}_d = -\frac{d}{2}\|\mathbf{x}_2 + \frac{p}{d}\mathbf{x}_1\|^2 - \frac{d}{2}\|\mathbf{x}_2\|^2 - \frac{p^2}{2d}\|\mathbf{x}_1\|^2,
$$

respectively. Obviously, $\mathcal{H}_d = 0$ and $\dot{\mathcal{H}}_d = 0$ only when $\mathbf{x}_1 = \mathbf{x}_2 = \mathbf{0}$, otherwise $\mathcal{H}_d > 0$ and $\dot{\mathcal{H}}_d < 0$. Moreover, $\mathcal{H}_d$ is radially unbounded. Thus, Lyapunov Theorem (see Theorem 10), ensures that $(\mathbf{x}_1, \mathbf{x}_2) = (\mathbf{0}, \mathbf{0})$ is GAS.

Solving the question: which is the controller?

The controller has to be designed such that the closed-loop yields (8.2). So, let us rewrite (8.2) as

$$
\begin{aligned}
\ddot{\mathbf{q}} =& -\frac{p}{d}\dot{\mathbf{q}} - \mathbf{M}(\mathbf{q})^{-1}\Big[\mathbf{C}(\mathbf{q},\dot{\mathbf{q}})\Big(\dot{\mathbf{q}} + \frac{p}{d}\tilde{\mathbf{q}}\Big) + p\tilde{\mathbf{q}} + d\dot{\mathbf{q}}\Big] \\
=& -\frac{p}{d}\mathbf{M}(\mathbf{q})^{-1}\mathbf{M}(\mathbf{q})\dot{\mathbf{q}} - \mathbf{M}(\mathbf{q})^{-1}\Big[\mathbf{C}(\mathbf{q},\dot{\mathbf{q}})\dot{\mathbf{q}} + \mathbf{g}(\mathbf{q}) - \boldsymbol{\tau} + \boldsymbol{\tau} + \frac{p}{d}\mathbf{C}(\mathbf{q},\dot{\mathbf{q}})\tilde{\mathbf{q}} + p\tilde{\mathbf{q}} + d\dot{\mathbf{q}} - \mathbf{g}(\mathbf{q})\Big] \\
=& -\mathbf{M}(\mathbf{q})^{-1}\Big[\mathbf{C}(\mathbf{q},\dot{\mathbf{q}})\dot{\mathbf{q}} + \mathbf{g}(\mathbf{q}) - \boldsymbol{\tau}\Big] - \mathbf{M}(\mathbf{q})^{-1}\Big[\boldsymbol{\tau} + \frac{p}{d}\mathbf{M}(\mathbf{q})\dot{\mathbf{q}} + \frac{p}{d}\mathbf{C}(\mathbf{q},\dot{\mathbf{q}})\tilde{\mathbf{q}} + p\tilde{\mathbf{q}} + d\dot{\mathbf{q}} - \mathbf{g}(\mathbf{q})\Big] \\
=& -\mathbf{M}(\mathbf{q})^{-1}\Big[\mathbf{C}(\mathbf{q},\dot{\mathbf{q}})\dot{\mathbf{q}} + \mathbf{g}(\mathbf{q}) - \boldsymbol{\tau}\Big] - \mathbf{M}(\mathbf{q})^{-1}\Big[\boldsymbol{\tau} - \mathbf{M}(\mathbf{q})\dot{\mathbf{q}}_r - \mathbf{C}(\mathbf{q},\dot{\mathbf{q}})\mathbf{q}_r - \mathbf{g}(\mathbf{q}) + d\mathbf{s}\Big],
\end{aligned}
$$

where we have defined $\mathbf{q}_r := -\frac{p}{d}\tilde{\mathbf{q}}$ and hence $\mathbf{s} = \dot{\mathbf{q}} - \mathbf{q}_r$. Therefore, the controller is

$$
\boldsymbol{\tau} = \mathbf{M}(\mathbf{q})\dot{\mathbf{q}}_r + \mathbf{C}(\mathbf{q},\dot{\mathbf{q}})\mathbf{q}_r + \mathbf{g}(\mathbf{q}) - d\mathbf{s}. \tag{8.4}
$$

The Uncertain Parameter Case...

We should note that controller (8.4) requires the knowledge of all the physical parameters and when there is uncertainty then the controller (8.4) becomes

$$
\begin{aligned}
\boldsymbol{\tau} =& \mathbf{M}(\mathbf{q},\hat{\boldsymbol{\phi}})\dot{\mathbf{q}}_r + \mathbf{C}(\mathbf{q},\dot{\mathbf{q}},\hat{\boldsymbol{\phi}})\mathbf{q}_r + \mathbf{g}(\mathbf{q},\hat{\boldsymbol{\phi}}) - d\mathbf{s}, \\
=& \hat{\mathbf{M}}(\mathbf{q})\dot{\mathbf{q}}_r + \hat{\mathbf{C}}(\mathbf{q},\dot{\mathbf{q}})\mathbf{q}_r + \hat{\mathbf{g}}(\mathbf{q}) - d\mathbf{s},
\end{aligned}
$$

or, using (8.1),

$$
\boldsymbol{\tau} = \mathbf{Y}(\mathbf{q},\dot{\mathbf{q}},\mathbf{q}_r,\dot{\mathbf{q}}_r)\hat{\boldsymbol{\phi}} - d\mathbf{s}, \tag{8.5}
$$

in fact, using again (8.1), the controller can be written as

$$\begin{aligned}\boldsymbol{\tau} &= \mathbf{Y}(\mathbf{q},\dot{\mathbf{q}},\mathbf{q}_r,\dot{\mathbf{q}}_r)\hat{\boldsymbol{\phi}} - d\mathbf{s}\\ &= \mathbf{Y}(\mathbf{q},\dot{\mathbf{q}},\mathbf{q}_r,\dot{\mathbf{q}}_r)\hat{\boldsymbol{\phi}} - d\mathbf{s} - \mathbf{Y}(\mathbf{q},\dot{\mathbf{q}},\mathbf{q}_r,\dot{\mathbf{q}}_r)\boldsymbol{\phi} + \mathbf{M}(\mathbf{q})\dot{\mathbf{q}}_r + \mathbf{C}(\mathbf{q},\dot{\mathbf{q}})\mathbf{q}_r + \mathbf{g}(\mathbf{q}),\\ &= \mathbf{Y}(\mathbf{q},\dot{\mathbf{q}},\mathbf{q}_r,\dot{\mathbf{q}}_r)\big(\hat{\boldsymbol{\phi}} - \boldsymbol{\phi}\big) - d\mathbf{s} + \mathbf{M}(\mathbf{q})\dot{\mathbf{q}}_r + \mathbf{C}(\mathbf{q},\dot{\mathbf{q}})\mathbf{q}_r + \mathbf{g}(\mathbf{q}),\\ &= \mathbf{Y}(\mathbf{q},\dot{\mathbf{q}},\mathbf{q}_r,\dot{\mathbf{q}}_r)\tilde{\boldsymbol{\phi}} - d\mathbf{s} + \mathbf{M}(\mathbf{q})\dot{\mathbf{q}}_r + \mathbf{C}(\mathbf{q},\dot{\mathbf{q}})\mathbf{q}_r + \mathbf{g}(\mathbf{q}),\end{aligned}$$

where $\tilde{\boldsymbol{\phi}} := \hat{\boldsymbol{\phi}} - \boldsymbol{\phi}$ is the parameter estimation error.

Therefore, the closed-loop becomes

$$\dot{\mathbf{s}} = -\mathbf{M}(\mathbf{q})^{-1}\big[\mathbf{C}(\mathbf{q},\dot{\mathbf{q}})\mathbf{s} + d\mathbf{s} - \mathbf{Y}(\mathbf{q},\dot{\mathbf{q}},\mathbf{q}_r,\dot{\mathbf{q}}_r)\tilde{\boldsymbol{\phi}}\big], \tag{8.6}$$

and the time derivative of the energy function $\mathcal{H}_d$, given in (8.3), becomes

$$\dot{\mathcal{H}}_d = -\frac{d}{2}\|\mathbf{s}\|^2 - \frac{d}{2}\|\dot{\mathbf{q}}\|^2 - \frac{p^2}{2d}\|\tilde{\mathbf{q}}\|^2 + \mathbf{s}^\top\mathbf{Y}(\mathbf{q},\dot{\mathbf{q}},\mathbf{q}_r,\dot{\mathbf{q}}_r)\tilde{\boldsymbol{\phi}}.$$

Hence $\dot{\mathcal{H}}_d$ is no longer negative definite.

If we now define the *energy-like* storage function for the estimation error

$$\mathcal{U}_{\tilde{\boldsymbol{\phi}}} := \frac{1}{2k}\|\tilde{\boldsymbol{\phi}}\|^2,$$

where $k > 0$ is a constant gain, we have that

$$\dot{\mathcal{U}}_{\tilde{\boldsymbol{\phi}}} = \frac{1}{k}\tilde{\boldsymbol{\phi}}^\top\dot{\hat{\boldsymbol{\phi}}},$$

where we have used the fact that $\dot{\tilde{\boldsymbol{\phi}}} = \dot{\hat{\boldsymbol{\phi}}}$ because $\boldsymbol{\phi}$ is constant.

Now, defining function $\mathcal{E}_d$ as

$$\begin{aligned}\mathcal{E}_d &:= \mathcal{H}_d + \mathcal{U}_{\tilde{\boldsymbol{\phi}}}\\ &= \frac{1}{2}\mathbf{s}^\top\mathbf{M}(\mathbf{q})\mathbf{s} + \frac{1}{2}p\|\tilde{\mathbf{q}}\|^2 + \frac{1}{2k}\|\tilde{\boldsymbol{\phi}}\|^2,\end{aligned}$$

we obtain

$$\begin{aligned}\dot{\mathcal{E}}_d &= \dot{\mathcal{H}}_d + \dot{\mathcal{U}}_{\tilde{\boldsymbol{\phi}}}\\ &= -\frac{d}{2}\|\mathbf{s}\|^2 - \frac{d}{2}\|\dot{\mathbf{q}}\|^2 - \frac{p^2}{2d}\|\tilde{\mathbf{q}}\|^2 + \tilde{\boldsymbol{\phi}}^\top\left(\mathbf{Y}(\mathbf{q},\dot{\mathbf{q}},\mathbf{q}_r,\dot{\mathbf{q}}_r)^\top\mathbf{s} + \frac{1}{k}\dot{\hat{\boldsymbol{\phi}}}\right)\end{aligned}$$

Since the *adaptive law* $\dot{\hat{\boldsymbol{\phi}}}$ is free to choose, setting

$$\dot{\hat{\boldsymbol{\phi}}} = -k\mathbf{Y}(\mathbf{q},\dot{\mathbf{q}},\mathbf{q}_r,\dot{\mathbf{q}}_r)^\top\mathbf{s}, \tag{8.7}$$

ensures that

$$\dot{\mathcal{E}}_d = -\frac{d}{2}\|\mathbf{s}\|^2 - \frac{d}{2}\|\dot{\mathbf{q}}\|^2 - \frac{p^2}{2d}\|\tilde{\mathbf{q}}\|^2.$$

Hence, $\dot{\mathcal{E}}_d$ is now negative semi-definite.

If we define $\mathbf{x}_1 = \tilde{\mathbf{q}}$, $\mathbf{x}_2 = \dot{\mathbf{q}}$ and $\mathbf{x}_3 = \tilde{\boldsymbol{\phi}}$, we get

$$\begin{aligned}
\dot{\mathbf{x}}_1 =& \mathbf{x}_2,\\
\dot{\mathbf{x}}_2 =& -\frac{p}{d}\mathbf{x}_2 - \mathbf{M}(\mathbf{x}_1+\mathbf{q}_d)^{-1}\Big[\mathbf{C}(\mathbf{x}_1+\mathbf{q}_d,\mathbf{x}_2)\Big(\mathbf{x}_2+\frac{p}{d}\mathbf{x}_1\Big) + p\mathbf{x}_1 + d\mathbf{x}_2\Big]\\
&+ \mathbf{M}(\mathbf{x}_1+\mathbf{q}_d)^{-1}\mathbf{Y}(\mathbf{x}_1+\mathbf{q}_d,\mathbf{x}_2,-\frac{p}{d}\mathbf{x}_1,-\frac{p}{d}\mathbf{x}_2)\mathbf{x}_3,\\
\dot{\mathbf{x}}_3 =& -k\mathbf{Y}(\mathbf{x}_1+\mathbf{q}_d,\mathbf{x}_2,-\frac{p}{d}\mathbf{x}_1,-\frac{p}{d}\mathbf{x}_2)^\top\Big(\mathbf{x}_2+\frac{p}{d}\mathbf{x}_1\Big).
\end{aligned}$$

Clearly this closed-loop is autonomous, because it does not depend explicitly on time. Moreover

$$\mathcal{E}_d = \frac{1}{2}\Big(\mathbf{x}_2+\frac{p}{d}\mathbf{x}_1\Big)^\top\mathbf{M}(\mathbf{x}_1+\mathbf{q}_d)\Big(\mathbf{x}_2+\frac{p}{d}\mathbf{x}_1\Big) + \frac{1}{2}p\|\mathbf{x}_1\|^2 + \frac{1}{2k}\|\mathbf{x}_3\|^2,$$

and

$$\dot{\mathcal{E}}_d = -\frac{d}{2}\|\mathbf{x}_2+\frac{p}{d}\mathbf{x}_1\|^2 - \frac{d}{2}\|\mathbf{x}_2\|^2 - \frac{p^2}{2d}\|\mathbf{x}_1\|^2.$$

Therefore, $\mathcal{E}_d$ is positive definite and radially unbounded and $\dot{\mathcal{E}}_d$ is negative semidefinite. Further $\dot{\mathcal{E}}_d$ only vanishes when $\mathbf{x}_1 = \mathbf{0}$ and $\mathbf{x}_2 = \mathbf{0}$. Thus, by LaSalle's Invariance Theorem, we have that

$$(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3) = \big(\mathbf{0},\mathbf{0},\mathbf{Y}(\mathbf{q}_d,\mathbf{0},\mathbf{0},\mathbf{0})\bar{\mathbf{x}}_3 = \mathbf{0}\big)$$

is GAS. Note that this equilibrium *does not* guarantee that $\mathbf{x}_3 = \mathbf{0}$. Hence **controller** (8.5)**, with** (8.7)**, *does not* ensure exact parameter estimation**.

Wrapping-up, the *dynamic* adaptive controller that solves the regulation problem with parameter uncertainty is given by

$$\begin{aligned}\boldsymbol{\tau} &= \mathbf{Y}(\mathbf{q}, \dot{\mathbf{q}}, \mathbf{q}_r, \dot{\mathbf{q}}_r)\hat{\boldsymbol{\phi}} - d\mathbf{s}, \\ \dot{\hat{\boldsymbol{\phi}}} &= -\, k\mathbf{Y}(\mathbf{q}, \dot{\mathbf{q}}, \mathbf{q}_r, \dot{\mathbf{q}}_r)^\top \mathbf{s},\end{aligned}$$

with

$$\begin{aligned}\mathbf{s} &= \dot{\mathbf{q}} - \mathbf{q}_r, \\ \mathbf{q}_r &= -\, \frac{p}{d}\tilde{\mathbf{q}}, \\ \mathbf{Y}(\mathbf{q}, \dot{\mathbf{q}}, \mathbf{q}_r, \dot{\mathbf{q}}_r) &= \mathbf{Y}_M(\mathbf{q}, \dot{\mathbf{q}}_r) + \mathbf{Y}_C(\mathbf{q}, \dot{\mathbf{q}}, \mathbf{q}_r) + \mathbf{Y}_g(\mathbf{q}),\end{aligned}$$

with any gains $p, d, k > 0$.

Remark 22 *This controller deals with parametric uncertainty. However it requires the knowledge of the complete regressor matrix* $\mathbf{Y}$*.* $\diamond$

Controller (8.5) has been originally reported by Slotine and Li in 1988 [15].

8.2 Adaptive Gravity Compensation Controller

As stated before, a simpler adaptive controller is

$$\boldsymbol{\tau} = -p\tilde{\mathbf{q}} - d\dot{\mathbf{q}} + \hat{\mathbf{g}}(\mathbf{q}),$$

where $\hat{\mathbf{g}}(\mathbf{q}) := \mathbf{g}(\mathbf{q}, \hat{\boldsymbol{\phi}})$. Using the parameterization of the gravity term $\mathbf{g}(\mathbf{q}, \hat{\boldsymbol{\phi}}) = \mathbf{Y}_g(\mathbf{q})\hat{\boldsymbol{\phi}}$ yields

$$\boldsymbol{\tau} = -p\tilde{\mathbf{q}} - d\dot{\mathbf{q}} + \mathbf{Y}_g(\mathbf{q})\hat{\boldsymbol{\phi}}. \tag{8.8}$$

This controller yields the closed-loop

$$\begin{aligned} \ddot{\mathbf{q}} &= -\mathbf{M}(\mathbf{q})^{-1}\big[\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} + p\tilde{\mathbf{q}} + d\dot{\mathbf{q}} - \big(\hat{\mathbf{g}}(\mathbf{q}) - \mathbf{g}(\mathbf{q})\big)\big], \\ &= -\mathbf{M}(\mathbf{q})^{-1}\big[\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} + p\tilde{\mathbf{q}} + d\dot{\mathbf{q}} - \mathbf{Y}_g(\mathbf{q})\tilde{\boldsymbol{\phi}}\big], \end{aligned} \tag{8.9}$$

where $\tilde{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}} - \boldsymbol{\phi}$.

The equilibrium points satisfy $\ddot{\mathbf{q}} = \dot{\mathbf{q}} = \mathbf{0}$ and hence

$$\tilde{\mathbf{q}} = \frac{1}{p}\mathbf{Y}_g(\mathbf{q})\tilde{\boldsymbol{\phi}}.$$

Clearly, this last equation does not ensure that $\tilde{\mathbf{q}} = \mathbf{0}$. Hence, in order to ensure that $\tilde{\mathbf{q}} = \mathbf{0}$ we have to design a Lyapunov function such that its derivative is negative definite with regards to $\tilde{\mathbf{q}}$. This fact can be observed in the following.

First note that, in this case, the energy function

$$\mathcal{H}_d = \frac{1}{2}\dot{\mathbf{q}}^\top\mathbf{M}(\mathbf{q})\dot{\mathbf{q}} + \frac{1}{2}p\|\tilde{\mathbf{q}}\|^2,$$

yields

$$\dot{\mathcal{H}}_d = -d\|\dot{\mathbf{q}}\|^2 + \dot{\mathbf{q}}^\top\mathbf{Y}_g(\mathbf{q})\tilde{\boldsymbol{\phi}}.$$

Similar to the previous adaptive scheme, the *energy-like* storage function for the estimation error

$$\mathcal{U}_{\tilde{\phi}} := \frac{1}{2k}\|\tilde{\boldsymbol{\phi}}\|^2,$$

where $k > 0$ is a constant gain, yields

$$\dot{\mathcal{U}}_{\tilde{\phi}} = \frac{1}{k}\tilde{\boldsymbol{\phi}}^\top\dot{\hat{\boldsymbol{\phi}}},$$

where we have used the fact that $\dot{\tilde{\boldsymbol{\phi}}} = \dot{\hat{\boldsymbol{\phi}}}$ because $\boldsymbol{\phi}$ is constant.

In this case, function $\mathcal{E}_d := \mathcal{H}_d + \mathcal{U}_{\tilde{\phi}}$ is

$$\mathcal{E}_d = \frac{1}{2}\dot{\mathbf{q}}^\top \mathbf{M}(\mathbf{q})\dot{\mathbf{q}} + \frac{1}{2}p\|\tilde{\mathbf{q}}\|^2 + \frac{1}{2k}\|\tilde{\boldsymbol{\phi}}\|^2,$$

and thus

$$\dot{\mathcal{E}}_d = -d\|\dot{\mathbf{q}}\|^2 + \left(\mathbf{Y}_g(\mathbf{q})^\top \dot{\mathbf{q}} + \frac{1}{k}\dot{\hat{\boldsymbol{\phi}}}\right)^\top \tilde{\boldsymbol{\phi}}.$$

Obviously, setting

$$\dot{\hat{\boldsymbol{\phi}}} = -k\mathbf{Y}_g(\mathbf{q})^\top \dot{\mathbf{q}}, \tag{8.10}$$

ensures that

$$\dot{\mathcal{E}}_d = -d\|\dot{\mathbf{q}}\|^2.$$

While the adaptive law (8.10) ensures stability and convergence to zero of velocities, it does not ensure position error convergence, because the equilibrium positions satisfy (as previously explained) $\bar{\mathbf{q}} = \mathbf{q}_d + \frac{1}{p}\mathbf{Y}_g(\bar{\mathbf{q}})\tilde{\boldsymbol{\phi}}$. Therefore (8.10) does not solve the regulation problem.

In order to obtain a negative definite term of the position error in the derivative of the Lyapunov function, let us define the functions

$$\mathbf{h}(\tilde{\mathbf{q}}) = \frac{1}{1+\|\tilde{\mathbf{q}}\|^2}\tilde{\mathbf{q}},$$

and

$$\mathcal{V}(\tilde{\mathbf{q}}, \dot{\mathbf{q}}) := \mathbf{h}(\tilde{\mathbf{q}})^\top \mathbf{M}(\mathbf{q})\dot{\mathbf{q}}.$$

Now, the time derivative of $\mathcal{V}$ yields

$$\dot{\mathcal{V}} = \dot{\mathbf{h}}(\tilde{\mathbf{q}})^\top \mathbf{M}(\mathbf{q})\dot{\mathbf{q}} + \mathbf{h}(\tilde{\mathbf{q}})^\top \dot{\mathbf{M}}(\mathbf{q})\dot{\mathbf{q}} + \mathbf{h}(\tilde{\mathbf{q}})^\top \mathbf{M}(\mathbf{q})\ddot{\mathbf{q}}.$$

Substituting $\ddot{\mathbf{q}}$ in the last term yields

$$\begin{aligned}
\mathbf{h}(\tilde{\mathbf{q}})^\top \mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} &= -\,\mathbf{h}(\tilde{\mathbf{q}})^\top \left[\mathbf{C}(\mathbf{q},\dot{\mathbf{q}})\dot{\mathbf{q}} + p\tilde{\mathbf{q}} + d\dot{\mathbf{q}} - \mathbf{Y}_g(\mathbf{q})\tilde{\boldsymbol{\phi}}\right] \\
&= -\,p\frac{1}{1+\|\tilde{\mathbf{q}}\|^2}\|\tilde{\mathbf{q}}\|^2 - \mathbf{h}(\tilde{\mathbf{q}})^\top \left[\mathbf{C}(\mathbf{q},\dot{\mathbf{q}})\dot{\mathbf{q}} + d\dot{\mathbf{q}} - \mathbf{Y}_g(\mathbf{q})\tilde{\boldsymbol{\phi}}\right].
\end{aligned}$$

Note that the term $-p\frac{1}{1+\|\tilde{\mathbf{q}}\|^2}\|\tilde{\mathbf{q}}\|^2$ is negative definite with regards to $\tilde{\mathbf{q}}$. Therefore, function $\mathcal{V}$ seems a nice candidate to incorporate in the Lyapunov function. Nevertheless function $\mathcal{V}$ is sign indefinite and hence it should be *dominated* by the positive terms.

Consider then the following function $\mathcal{V}_d := \mathcal{H}_d + \mathcal{U}_{\tilde{\phi}} + \gamma\mathcal{V}$. Hence

$$\mathcal{V}_d = \frac{1}{2}\dot{\mathbf{q}}^\top \mathbf{M}(\mathbf{q})\dot{\mathbf{q}} + \frac{1}{2}p\|\tilde{\mathbf{q}}\|^2 + \frac{1}{2k}\|\tilde{\phi}\|^2 + \gamma\mathbf{h}(\tilde{\mathbf{q}})^\top \mathbf{M}(\mathbf{q})\dot{\mathbf{q}}.$$

Now observe that

$$\frac{1}{2}\dot{\mathbf{q}}^\top \mathbf{M}(\mathbf{q})\dot{\mathbf{q}}+\gamma\mathbf{h}(\tilde{\mathbf{q}})^\top \mathbf{M}(\mathbf{q})\dot{\mathbf{q}} = \frac{1}{2}\big(\dot{\mathbf{q}}+\gamma\mathbf{h}(\tilde{\mathbf{q}})\big)^\top \mathbf{M}(\mathbf{q})\big(\dot{\mathbf{q}}+\gamma\mathbf{h}(\tilde{\mathbf{q}})\big) - \frac{1}{2}\gamma^2\mathbf{h}(\tilde{\mathbf{q}})^\top \mathbf{M}(\mathbf{q})\mathbf{h}(\tilde{\mathbf{q}}),$$

where to obtain this equation we added and subtracted the term $\frac{1}{2}\gamma^2\mathbf{h}(\tilde{\mathbf{q}})^\top \mathbf{M}(\mathbf{q})\mathbf{h}(\tilde{\mathbf{q}})$ to complete the perfect square.

Thus, we get

$$\mathcal{V}_d = \frac{1}{2}\big(\dot{\mathbf{q}}+\gamma\mathbf{h}(\tilde{\mathbf{q}})\big)^\top \mathbf{M}(\mathbf{q})\big(\dot{\mathbf{q}}+\gamma\mathbf{h}(\tilde{\mathbf{q}})\big) - \frac{1}{2}\gamma^2\mathbf{h}(\tilde{\mathbf{q}})^\top \mathbf{M}(\mathbf{q})\mathbf{h}(\tilde{\mathbf{q}}) + \frac{1}{2}p\|\tilde{\mathbf{q}}\|^2 + \frac{1}{2k}\|\tilde{\phi}\|^2$$

Now, it holds that

$$\mathbf{h}(\tilde{\mathbf{q}})^\top \mathbf{M}(\mathbf{q})\mathbf{h}(\tilde{\mathbf{q}}) = \frac{1}{\big(1+\|\tilde{\mathbf{q}}\|^2\big)^2}\tilde{\mathbf{q}}^\top \mathbf{M}(\mathbf{q})\tilde{\mathbf{q}} \le m_2\|\tilde{\mathbf{q}}\|^2,$$

where we used Property P7 and the fact that, for all $\tilde{\mathbf{q}} \in \mathbb{R}^n$,

$$\frac{1}{1+\|\tilde{\mathbf{q}}\|^2} \le 1 \quad \Rightarrow \quad \frac{1}{\big(1+\|\tilde{\mathbf{q}}\|^2\big)^2} \le 1.$$

Hence, $\mathcal{V}_d$ satisfies the following bound

$$\mathcal{V}_d \ge \frac{1}{2}\big(\dot{\mathbf{q}}+\gamma\mathbf{h}(\tilde{\mathbf{q}})\big)^\top \mathbf{M}(\mathbf{q})\big(\dot{\mathbf{q}}+\gamma\mathbf{h}(\tilde{\mathbf{q}})\big) + \frac{1}{2}\big(p-\gamma^2 m_2\big)\|\tilde{\mathbf{q}}\|^2 + \frac{1}{2k}\|\tilde{\phi}\|^2.$$

If $p > \gamma^2 m_2$ then $\mathcal{V}_d$ is positive definite and radially unbounded with regards to $\dot{\mathbf{q}}, \tilde{\mathbf{q}}$ and $\tilde{\phi}$. This is satisfied if

$$\gamma < \sqrt{\frac{p}{m_2}}. \tag{8.11}$$

Moreover $\dot{\mathcal{V}}_d$ evaluated along (8.9) yields

$$\begin{aligned}\dot{\mathcal{V}}_d =& -d\|\dot{\mathbf{q}}\|^2 - \gamma p\frac{1}{1+\|\tilde{\mathbf{q}}\|^2}\|\tilde{\mathbf{q}}\|^2 + \Big(\mathbf{Y}_g(\mathbf{q})^\top\dot{\mathbf{q}} + \frac{1}{k}\dot{\tilde{\phi}}\Big)^\top\tilde{\phi} + \gamma\dot{\mathbf{h}}(\tilde{\mathbf{q}})^\top \mathbf{M}(\mathbf{q})\dot{\mathbf{q}} \\ &+ \gamma\mathbf{h}(\tilde{\mathbf{q}})^\top\dot{\mathbf{M}}(\mathbf{q})\dot{\mathbf{q}} - \gamma\mathbf{h}(\tilde{\mathbf{q}})^\top\big[\mathbf{C}(\mathbf{q},\dot{\mathbf{q}})\dot{\mathbf{q}} + d\dot{\mathbf{q}} - \mathbf{Y}_g(\mathbf{q})\tilde{\phi}\big].\end{aligned}$$

Using the fact that $\dot{\mathbf{M}}(\mathbf{q}) = \mathbf{C}(\mathbf{q},\dot{\mathbf{q}}) + \mathbf{C}(\mathbf{q},\dot{\mathbf{q}})^\top$ we get

$$\begin{aligned}\dot{\mathcal{V}}_d =& - d\|\dot{\mathbf{q}}\|^2 - \gamma p\frac{1}{1+\|\tilde{\mathbf{q}}\|^2}\|\tilde{\mathbf{q}}\|^2 + \Big(\mathbf{Y}_g(\mathbf{q})^\top\big(\dot{\mathbf{q}} + \gamma\mathbf{h}(\tilde{\mathbf{q}})\big) + \frac{1}{k}\dot{\hat{\boldsymbol{\phi}}}\Big)^\top\tilde{\boldsymbol{\phi}} \\ &+ \gamma\dot{\mathbf{h}}(\tilde{\mathbf{q}})^\top\mathbf{M}(\mathbf{q})\dot{\mathbf{q}} + \gamma\mathbf{h}(\tilde{\mathbf{q}})^\top\mathbf{C}(\mathbf{q},\dot{\mathbf{q}})^\top\dot{\mathbf{q}} - \gamma d\mathbf{h}(\tilde{\mathbf{q}})^\top\dot{\mathbf{q}}.\end{aligned}$$

Thus, setting the adaptive law $\dot{\hat{\boldsymbol{\phi}}}$ as

$$\dot{\hat{\boldsymbol{\phi}}} = -k\mathbf{Y}_g(\mathbf{q})^\top\big(\dot{\mathbf{q}} + \gamma\mathbf{h}(\tilde{\mathbf{q}})\big), \tag{8.12}$$

returns

$$\dot{\mathcal{V}}_d = -d\|\dot{\mathbf{q}}\|^2 - \gamma p\frac{1}{1+\|\tilde{\mathbf{q}}\|^2}\|\tilde{\mathbf{q}}\|^2 + \gamma\dot{\mathbf{h}}(\tilde{\mathbf{q}})^\top\mathbf{M}(\mathbf{q})\dot{\mathbf{q}} + \gamma\mathbf{h}(\tilde{\mathbf{q}})^\top\mathbf{C}(\mathbf{q},\dot{\mathbf{q}})^\top\dot{\mathbf{q}} - \gamma d\mathbf{h}(\tilde{\mathbf{q}})^\top\dot{\mathbf{q}}.$$

We are now going to dominate the sign indefinite terms in $\dot{\mathcal{V}}_d$. First note that

$$\frac{1}{1+\|\tilde{\mathbf{q}}\|^2} \le 1 \quad \Rightarrow \quad \|\mathbf{h}(\tilde{\mathbf{q}})\| \le \|\tilde{\mathbf{q}}\|,$$

and

$$\|\mathbf{h}(\tilde{\mathbf{q}})\| = \frac{\|\tilde{\mathbf{q}}\|}{1+\|\tilde{\mathbf{q}}\|^2} \le 1.$$

Moreover

$$\dot{\mathbf{h}} = \frac{1-\|\tilde{\mathbf{q}}\|^2}{(1+\|\tilde{\mathbf{q}}\|^2)^2}\dot{\mathbf{q}},$$

thus

$$\|\dot{\mathbf{h}}\| = \left|\frac{1-\|\tilde{\mathbf{q}}\|^2}{(1+\|\tilde{\mathbf{q}}\|^2)^2}\right|\|\dot{\mathbf{q}}\| \le \|\dot{\mathbf{q}}\|.$$

Using these bounds together with Properties P7 and P8, respectively, we get

$$\dot{\mathbf{h}}(\tilde{\mathbf{q}})^\top\mathbf{M}(\mathbf{q})\dot{\mathbf{q}} \le m_2\|\dot{\mathbf{q}}\|^2,$$

and

$$\mathbf{h}(\tilde{\mathbf{q}})^\top\mathbf{C}(\mathbf{q},\dot{\mathbf{q}})^\top\dot{\mathbf{q}} \le k_c\|\dot{\mathbf{q}}\|^2.$$

Using Young's inequality (see Inequality 3), for any $c > 0$, we have that

$$-\mathbf{h}(\tilde{\mathbf{q}})^\top\dot{\mathbf{q}} \le \frac{c}{2}\|\mathbf{h}(\tilde{\mathbf{q}})\|^2 + \frac{1}{2c}\|\dot{\mathbf{q}}\|^2,$$

further

$$\|\mathbf{h}(\tilde{\mathbf{q}})\|^2 = \frac{1}{(1+\|\tilde{\mathbf{q}}\|^2)^2}\|\tilde{\mathbf{q}}\|^2 \le \frac{1}{1+\|\tilde{\mathbf{q}}\|^2}\|\tilde{\mathbf{q}}\|^2,$$

thus

$$-\mathbf{h}(\tilde{\mathbf{q}})^\top \dot{\mathbf{q}} \leq \frac{c}{2}\frac{1}{1+\|\tilde{\mathbf{q}}\|^2}\|\tilde{\mathbf{q}}\|^2 + \frac{1}{2c}\|\dot{\mathbf{q}}\|^2.$$

Finally, we obtain the following bound for $\dot{\mathcal{V}}_d$,

$$\dot{\mathcal{V}}_d \leq -\Big(d - \gamma\big(m_2 + k_c + \frac{d}{2c}\big)\Big)\|\dot{\mathbf{q}}\|^2 - \gamma\Big(p - \frac{dc}{2}\Big)\frac{1}{1+\|\tilde{\mathbf{q}}\|^2}\|\tilde{\mathbf{q}}\|^2.$$

Note that if constant c satisfies

$$c < \frac{2p}{d}, \tag{8.13}$$

and γ is set as

$$\gamma < \frac{d}{m_2 + k_c + \frac{d}{2c}},$$

then $\dot{\mathcal{V}}_d \leq 0$.

In fact, using (8.11), γ must satisfy

$$\gamma < \min\left\{\sqrt{\frac{p}{m_2}},\ \frac{d}{m_2 + k_c + \frac{d}{2c}}\right\}. \tag{8.14}$$

If we define $\mathbf{x}_1 = \tilde{\mathbf{q}}$, $\mathbf{x}_2 = \dot{\mathbf{q}}$ and $\mathbf{x}_3 = \tilde{\boldsymbol{\phi}}$, we get

$$\begin{aligned}
\dot{\mathbf{x}}_1 =& \mathbf{x}_2,\\
\dot{\mathbf{x}}_2 =& -\mathbf{M}(\mathbf{x}_1+\mathbf{q}_d)^{-1}\Big[\mathbf{C}(\mathbf{x}_1+\mathbf{q}_d,\mathbf{x}_2)\mathbf{x}_2 + p\mathbf{x}_1 + d\mathbf{x}_2 - \mathbf{Y}_g(\mathbf{x}_1+\mathbf{q}_d)\mathbf{x}_3\Big]\\
\dot{\mathbf{x}}_3 =& -k\mathbf{Y}_g(\mathbf{x}_1+\mathbf{q}_d)^\top\Big(\mathbf{x}_2 + \gamma\frac{1}{1+\|\mathbf{x}_1\|^2}\mathbf{x}_1\Big).
\end{aligned}$$

Clearly this closed-loop is autonomous, because it does not depend explicitly on time. Moreover

$$\mathcal{V}_d = \frac{1}{2}\mathbf{x}_2^\top\mathbf{M}(\mathbf{x}_1+\mathbf{q}_d)\mathbf{x}_2 + \frac{1}{2}p\|\mathbf{x}_1\|^2 + \frac{1}{2k}\|\mathbf{x}_3\|^2 + \gamma\mathbf{h}(\mathbf{x}_1)^\top\mathbf{M}(\mathbf{x}_1+\mathbf{q}_d)\mathbf{x}_2,$$

and

$$\dot{\mathcal{V}}_d \leq -\Big(d - \gamma\big(m_2 + k_c + \frac{d}{2c}\big)\Big)\|\mathbf{x}_2\|^2 - \gamma\Big(p - \frac{dc}{2}\Big)\frac{1}{1+\|\mathbf{x}_1\|^2}\|\mathbf{x}_1\|^2.$$

Therefore, $\mathcal{V}_d$ is positive definite and radially unbounded and $\dot{\mathcal{V}}_d$ is negative semidefinite provided that (8.13) and (8.14) hold.

Further $\dot{\mathcal{V}}_d$ only vanishes when $\mathbf{x}_1 = \mathbf{0}$ and $\mathbf{x}_2 = \mathbf{0}$. Thus, by LaSalle's Invariance Theorem, we have that

$$(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = \big(\mathbf{0}, \mathbf{0}, \mathbf{Y}_g(\mathbf{q}_d)\bar{\mathbf{x}}_3 = \mathbf{0}\big)$$

is GAS. Note again that this equilibrium *does not* guarantee that $\mathbf{x}_3 = \mathbf{0}$. Hence **controller** (8.8)**, with** (8.12)**, *does not* ensure exact parameter estimation**.

Wrapping-up, the adaptive controller only for the gravity term is given by

$$\begin{aligned}\boldsymbol{\tau} =& -p\tilde{\mathbf{q}} - d\dot{\mathbf{q}} + \mathbf{Y}_g(\mathbf{q})\hat{\boldsymbol{\phi}},\\ \dot{\hat{\boldsymbol{\phi}}} =& -k\mathbf{Y}_g(\mathbf{q})^\top\Big(\dot{\mathbf{q}} + \gamma\frac{1}{1+\|\tilde{\mathbf{q}}\|^2}\tilde{\mathbf{q}}\Big),\end{aligned}$$

with constants c and γ satisfying, for any gains $p, d > 0$ and $k > 0$,

$$\gamma < \min\left\{\sqrt{\frac{p}{m_2}},\ \frac{d}{m_2 + k_c + \frac{d}{2c}}\right\}.$$

and

$$c < \frac{2p}{d}.$$

Remark 23 *This controller deals with parametric uncertainty and only requires to estimate the gravity term.* ◇

- Controller (8.8) has been originally reported by Tomei in 1991 [21].
- The extension to the adaptive gravity compensation case is reported by Kelly in 1993 [2].
- The extension to the case when velocities are not available for measurement and the actuators are considered not ideal (with saturation) is reported by López-Araujo et al. in 2013 [8].

Part III

Position and Velocity Control

Chapter 9

Computed Torque Control

All the previous control schemes are designed to regulate the robot at a desired constant position $\mathbf{q}_d \in \mathbb{R}^n$ and therefore $\dot{\mathbf{q}}_d = \mathbf{0}$.

However several applications require the robot to track a desired trajectory for which $\dot{\mathbf{q}}_d \neq \mathbf{0}$. In such a case the position and velocity control (also known as trajectory tracking control) aims at proving that, for a given desired trajectory $\mathbf{q}_d(t) \in \mathbb{R}^n$ that is *twice continuously differentiable*, i.e., $\mathbf{q}_d(t) \in \mathcal{C}^2$ (this means that $\dot{\mathbf{q}}_d(t), \ddot{\mathbf{q}}_d(t)$ exist and are continuous), the equilibrium $(\dot{\tilde{\mathbf{q}}}, \tilde{\mathbf{q}}) = (\mathbf{0}, \mathbf{0})$ is GAS, i.e.,

$$\lim_{t\to\infty} \dot{\tilde{\mathbf{q}}}(t) = \mathbf{0}, \qquad \lim_{t\to\infty} \tilde{\mathbf{q}}(t) = \mathbf{0},$$

for all initial conditions $\dot{\mathbf{q}}(0), \mathbf{q}(0) \in \mathbb{R}^n$, where $\tilde{\mathbf{q}} := \mathbf{q} - \mathbf{q}_d$ and $\dot{\tilde{\mathbf{q}}} = \dot{\mathbf{q}} - \dot{\mathbf{q}}_d$.

The first scheme that we design for the tracking problem is the Computed Torque Controller. This scheme incorporates a nonlinear PD term plus a term that cancels-out the gravity and the Coriolis effects. Its famous name comes from the following reasoning. Consider the Lyapunov candidate function

$$V(\tilde{\mathbf{q}}, \dot{\tilde{\mathbf{q}}}) = \frac{1}{2}\|\dot{\tilde{\mathbf{q}}}\|^2 + \frac{1}{2}p\|\tilde{\mathbf{q}}\|^2,$$

which is positive definite and radially unbounded. The question is: *which is the controller that ensures that* $\dot{V} = -d\|\dot{\tilde{\mathbf{q}}}\|^2$ *?*

To answer that question, let us first find $\dot{V}$. Clearly

$$\begin{aligned}
\dot{V} &= \dot{\tilde{\mathbf{q}}}^\top \big(\ddot{\tilde{\mathbf{q}}} + p\tilde{\mathbf{q}}\big) \\
&= \dot{\tilde{\mathbf{q}}}^\top \big(\ddot{\mathbf{q}} - \ddot{\mathbf{q}}_d + p\tilde{\mathbf{q}} + d\dot{\tilde{\mathbf{q}}}\big) - d\|\dot{\tilde{\mathbf{q}}}\|^2 \\
&= \dot{\tilde{\mathbf{q}}}^\top \mathbf{M}^{-1}(\mathbf{q})\big(\boldsymbol{\tau} - \mathbf{C}(\mathbf{q},\dot{\mathbf{q}})\dot{\mathbf{q}} - \mathbf{g}(\mathbf{q}) - \mathbf{M}(\mathbf{q})\big(\ddot{\mathbf{q}}_d - p\tilde{\mathbf{q}} + d\dot{\tilde{\mathbf{q}}}\big)\big) - d\|\dot{\tilde{\mathbf{q}}}\|^2.
\end{aligned}$$

Obviously, the controller is

$$\boldsymbol{\tau} = \mathbf{M}(\mathbf{q})\left[\ddot{\mathbf{q}}_d - d\dot{\tilde{\mathbf{q}}} - p\tilde{\mathbf{q}}\right] + \mathbf{C}(\mathbf{q},\dot{\mathbf{q}})\dot{\mathbf{q}} + \mathbf{g}(\mathbf{q}), \tag{9.1}$$

where $p, d > 0$.

Substituting (8.8) in (4.3) yields the closed-loop system

$$\ddot{\tilde{\mathbf{q}}} + d\dot{\tilde{\mathbf{q}}} + p\tilde{\mathbf{q}} = \mathbf{0}.$$

Defining $\mathbf{x}_1 = \tilde{\mathbf{q}}$ and $\mathbf{x}_2 = \dot{\tilde{\mathbf{q}}}$ the closed-loop can be written as the linear-time-invariant system

$$\begin{aligned} \dot{\mathbf{x}}_1 &= \mathbf{x}_2 \\ \dot{\mathbf{x}}_2 &= -\,p\mathbf{x}_1 - d\mathbf{x}_2. \end{aligned} \tag{9.2}$$

Clearly, (9.2) can be expressed as an equation of the form $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$, with

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -p & -d \end{bmatrix} \otimes \mathbf{I},$$

where $\otimes$ is the standard Kronecker product. The only solution to (9.2) is thus

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0).$$

If we compute the matrix exponential we can predict the closed-loop trajectory at any time instant. For, we first obtain the eigenvalues of matrix $\mathbf{A}$. Note that there are two main eigenvalues with multiplicity equal to n. These two main eigenvalues satisfy the fundamental equation

$$\lambda^2 + d\lambda + p = 0,$$

and thus

$$\lambda_1 = -\frac{d}{2} + \frac{1}{2}\sqrt{d^2 - 4p}$$

and

$$\lambda_2 = -\frac{d}{2} - \frac{1}{2}\sqrt{d^2 - 4p}.$$

At this point we note that three type of behaviors can be obtained, depending on the *discriminant*: 1) if $d^2 - 4p = 0$ then the robot is *critically damped*; 2) if $d^2 - 4p < 0$ then the robot is *under damped*; and 3) if $d^2 - 4p > 0$ then the robot is *over damped*.

In the three cases above, the real part of the eigenvalues is always strictly negative for any $p, d > 0$. Therefore, for any $p, d > 0$, **the equilibrium** $(\dot{\tilde{\mathbf{q}}}, \tilde{\mathbf{q}}) = (\mathbf{0}, \mathbf{0})$ **is GES!!!**.

In what follows we design the damping gain to satisfy $d = 2\sqrt{p}$ for the robot to behave as *critically damped.*

Hence, setting $d = 2\sqrt{p}$, yields $\lambda_1 = \lambda_2 = -\sqrt{p}$. Unfortunately, in this case, matrix $\mathbf{A}$ is not diagonalizable but it accepts a Jordan block decomposition of the form

$$\mathbf{A} = \mathbf{TJT}^{-1} \otimes \mathbf{I},$$

where

$$\mathbf{J} = \begin{bmatrix} -\sqrt{p} & 1 \\ 0 & -\sqrt{p} \end{bmatrix}, \qquad \mathbf{T} = \begin{bmatrix} 1 & 0 \\ -\sqrt{p} & 1 \end{bmatrix}, \qquad \mathbf{T}^{-1} = \begin{bmatrix} 1 & 0 \\ \sqrt{p} & 1 \end{bmatrix}.$$

Thus, the matrix exponential satisfies the following

$$e^{\mathbf{A}t} = e^{(\mathbf{TJT}^{-1} \otimes \mathbf{I})t} = e^{\mathbf{TJT}^{-1}t} \otimes \mathbf{I} = \mathbf{T}e^{\mathbf{J}t}\mathbf{T}^{-1} \otimes \mathbf{I}.$$

Now, observe that $\mathbf{J} = \mathbf{B} + \mathbf{C}$, where

$$\mathbf{B} = \begin{bmatrix} -\sqrt{p} & 0 \\ 0 & -\sqrt{p} \end{bmatrix}; \qquad \mathbf{C} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Since $\mathbf{B}$ and $\mathbf{C}$ conmute, i.e., $\mathbf{BC} = \mathbf{CB}$, then

$$e^{\mathbf{J}t} = e^{\mathbf{B}t + \mathbf{C}t} = e^{\mathbf{B}t}e^{\mathbf{C}t} = \begin{bmatrix} e^{-\sqrt{p}t} & 0 \\ 0 & e^{-\sqrt{p}t} \end{bmatrix} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} = e^{-\sqrt{p}t} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}.$$

Finally

$$\begin{aligned} e^{\mathbf{A}t} &= e^{-\sqrt{p}t} \begin{bmatrix} 1 & 0 \\ -\sqrt{p} & 1 \end{bmatrix} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \sqrt{p} & 1 \end{bmatrix} \otimes \mathbf{I} \\ &= e^{-\sqrt{p}t} \begin{bmatrix} 1 + \sqrt{p}t & t \\ -pt & 1 - \sqrt{p}t \end{bmatrix} \otimes \mathbf{I}. \end{aligned}$$

The solution to (9.2) with $d = 2\sqrt{p}$ is

$$\mathbf{x}(t) = e^{-\sqrt{p}t} \left(\begin{bmatrix} 1 + \sqrt{p}t & t \\ -pt & 1 - \sqrt{p}t \end{bmatrix} \otimes \mathbf{I} \right) \mathbf{x}(0),$$

or, what is the same,

$$\tilde{\mathbf{q}}(t) = e^{-\sqrt{p}t} \left((1 + \sqrt{p}t)\tilde{\mathbf{q}}(0) + t\dot{\tilde{\mathbf{q}}}(0) \right),$$

and

$$\dot{\tilde{\mathbf{q}}}(t) = e^{-\sqrt{p}t} \left(-pt\tilde{\mathbf{q}}(0) + (1 - \sqrt{p}t)\dot{\tilde{\mathbf{q}}}(0) \right).$$

Therefore, the robot trajectory is

$$\mathbf{q}(t) = \mathbf{q}_d(t) + e^{-\sqrt{p}t}\left((1+\sqrt{p}t)\tilde{\mathbf{q}}(0) + t\dot{\tilde{\mathbf{q}}}(0)\right), \tag{9.3}$$

and

$$\dot{\mathbf{q}}(t) = \dot{\mathbf{q}}_d(t) + e^{-\sqrt{p}t}\left(-pt\tilde{\mathbf{q}}(0) + (1-\sqrt{p}t)\dot{\tilde{\mathbf{q}}}(0)\right). \tag{9.4}$$

Obviously, as $t \to \infty$, $\tilde{\mathbf{q}}(t) \to \mathbf{0}$ and $\dot{\tilde{\mathbf{q}}}(t) \to \mathbf{0}$ with exponential rate of convergence equal to $\sqrt{p}$.

Remark 24 *This controller ensures that $\mathbf{q}(t) \to \mathbf{q}_d(t)$ and $\dot{\mathbf{q}}(t) \to \dot{\mathbf{q}}_d(t)$ exponentially fast, for any desired trajectory $\mathbf{q}_d(t) \in \mathcal{C}^2$. However, it requires the knowledge of all the robot physical parameters and it does not take advantage of the robot model properties (like the skew-symmetric property).* $\diamond$

Controller (9.1) has been originally reported by Wen and Bayard in 1988 [22].
For a deeper Lyapunov analysis of this scheme the reader is invited to read Chapter 10 of [6].

Chapter 10

PD Control with Compensation

In order to employ the robot dynamical model properties, consider the same controller for the regulation case defined in (8.4), which is

$$\boldsymbol{\tau} = \mathbf{M}(\mathbf{q})\dot{\mathbf{q}}_r + \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\mathbf{q}_r + \mathbf{g}(\mathbf{q}) - d\mathbf{s}.$$

Using the same definition of $\mathbf{s}$, as in the previous case, $\mathbf{s} := \dot{\mathbf{q}} - \mathbf{q}_r$, we have that the closed-loop is

$$\dot{\mathbf{s}} = -\mathbf{M}(\mathbf{q})^{-1}\big[\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\mathbf{s} + d\mathbf{s}\big].$$

In this case, if we define the desired energy like function as

$$\mathcal{H}_d = \frac{1}{2}\mathbf{s}^\top \mathbf{M}(\mathbf{q})\mathbf{s} + \frac{1}{2}p\|\tilde{\mathbf{q}}\|^2,$$

then it holds that

$$\dot{\mathcal{H}}_d = -d\|\mathbf{s}\|^2 + p\tilde{\mathbf{q}}^\top \dot{\tilde{\mathbf{q}}}.$$

Defining

$$\mathbf{q}_r := \dot{\mathbf{q}}_d - \frac{p}{d}\tilde{\mathbf{q}},$$

then

$$\begin{aligned}\mathbf{s} &= \dot{\mathbf{q}} - \mathbf{q}_r \\ &= \dot{\tilde{\mathbf{q}}} + \frac{p}{d}\tilde{\mathbf{q}},\end{aligned}$$

and thus

$$\dot{\mathcal{H}}_d = -\frac{d}{2}\|\mathbf{s}\|^2 - \frac{d}{2}\|\dot{\tilde{\mathbf{q}}} + \frac{p}{d}\tilde{\mathbf{q}}\|^2 + p\tilde{\mathbf{q}}^\top \dot{\tilde{\mathbf{q}}}.$$

From the fact that

$$\|\dot{\tilde{\mathbf{q}}} + \frac{p}{d}\tilde{\mathbf{q}}\|^2 = \|\dot{\tilde{\mathbf{q}}}\|^2 + \frac{2p}{d}\dot{\tilde{\mathbf{q}}}^\top\tilde{\mathbf{q}} + \frac{p^2}{d^2}\|\tilde{\mathbf{q}}\|^2,$$

we obtain

$$\dot{\mathcal{H}}_d = -\frac{d}{2}\|\mathbf{s}\|^2 - \frac{d}{2}\|\dot{\tilde{\mathbf{q}}}\|^2 - \frac{p^2}{2d}\|\tilde{\mathbf{q}}\|^2.$$

Therefore, not surprisingly, we have that $\mathcal{H}_d$ is positive definite and radially unbounded and, more importantly, $\dot{\mathcal{H}}_d$ is *negative definite* !!!

Using the new definition of $\mathbf{q}_r$ we can rewrite the controller as

$$\boldsymbol{\tau} = \mathbf{M}(\mathbf{q})\Big(\ddot{\mathbf{q}}_d - \frac{p}{d}\dot{\tilde{\mathbf{q}}}\Big) + \mathbf{C}(\mathbf{q},\dot{\mathbf{q}})\Big(\dot{\mathbf{q}}_d - \frac{p}{d}\tilde{\mathbf{q}}\Big) + \mathbf{g}(\mathbf{q}) - d\dot{\tilde{\mathbf{q}}} - p\tilde{\mathbf{q}}. \qquad (10.1)$$

The name of the controller comes from its structure, a dynamic compensation term plus a PD scheme.

If we define $\mathbf{x}_1 = \tilde{\mathbf{q}}$ and $\mathbf{x}_2 = \dot{\tilde{\mathbf{q}}}$ then we get

$$\begin{aligned}\dot{\mathbf{x}}_1 =& \mathbf{x}_2,\\ \dot{\mathbf{x}}_2 =& -\frac{p}{d}\mathbf{x}_2 - \mathbf{M}(\mathbf{x}_1 + \mathbf{q}_d(t))^{-1}\Big[\mathbf{C}(\mathbf{x}_1 + \mathbf{q}_d(t), \mathbf{x}_2 + \dot{\mathbf{q}}_d(t))\Big(\mathbf{x}_2 + \frac{p}{d}\mathbf{x}_1\Big) + p\mathbf{x}_1 + d\mathbf{x}_2\Big]\end{aligned}$$

where we have used the fact that $\dot{\mathbf{s}} = \ddot{\tilde{\mathbf{q}}} + \frac{p}{d}\dot{\tilde{\mathbf{q}}} = \dot{\mathbf{x}}_2 + \frac{p}{d}\mathbf{x}_2$. Moreover, $\mathcal{H}_d$ and $\dot{\mathcal{H}}_d$ can be written as

$$\mathcal{H}_d = \frac{1}{2}\Big(\mathbf{x}_2 + \frac{p}{d}\mathbf{x}_1\Big)^\top\mathbf{M}(\mathbf{x}_1 + \mathbf{q}_d)\Big(\mathbf{x}_2 + \frac{p}{d}\mathbf{x}_1\Big) + \frac{1}{2}p\|\mathbf{x}_1\|^2,$$

and

$$\dot{\mathcal{H}}_d = -\frac{d}{2}\|\mathbf{x}_2 + \frac{p}{d}\mathbf{x}_1\|^2 - \frac{d}{2}\|\mathbf{x}_2\|^2 - \frac{p^2}{2d}\|\mathbf{x}_1\|^2,$$

respectively. Obviously, $\mathcal{H}_d = 0$ and $\dot{\mathcal{H}}_d = 0$ only when $\mathbf{x}_1 = \mathbf{x}_2 = \mathbf{0}$, otherwise $\mathcal{H}_d > 0$ and $\dot{\mathcal{H}}_d < 0$. Moreover, $\mathcal{H}_d$ is radially unbounded. Thus, Lyapunov Theorem (see Theorem 10), ensures that $(\mathbf{x}_1, \mathbf{x}_2) = (\mathbf{0}, \mathbf{0})$ is GAS for any $\mathbf{q}_d(t) \in \mathcal{C}^2$.

In fact, for robots having only revolute joints we have that

$$\frac{1}{2}m_1\|\mathbf{x}_2 + \frac{p}{d}\mathbf{x}_1\|^2 + \frac{1}{2}p\|\mathbf{x}_1\|^2 \leq \mathcal{H}_d \leq \frac{1}{2}m_2\|\mathbf{x}_2 + \frac{p}{d}\mathbf{x}_1\|^2 + \frac{1}{2}p\|\mathbf{x}_1\|^2$$

and

$$\dot{\mathcal{H}}_d \leq -\frac{d}{2}\|\mathbf{x}_2 + \frac{p}{d}\mathbf{x}_1\|^2 - \frac{p^2}{2d}\|\mathbf{x}_1\|^2.$$

Defining $\alpha_2 := \frac{1}{2}\max\{m_2, p\}$ and $\alpha_3 := \frac{1}{2}\min\{d, \frac{p^2}{d}\}$, yields

$$\|\mathbf{x}_2 + \frac{p}{d}\mathbf{x}_1\|^2 + \|\mathbf{x}_1\|^2 \geq \frac{1}{\alpha_2}\mathcal{H}_d$$

and

$$\dot{\mathcal{H}}_d \leq -\alpha_3\Big(\|\mathbf{x}_2 + \frac{p}{d}\mathbf{x}_1\|^2 + \|\mathbf{x}_1\|^2\Big).$$

Thus

$$\dot{\mathcal{H}}_d \leq -\frac{\alpha_3}{\alpha_1}\mathcal{H}_d.$$

Therefore, it also holds that

$$\mathcal{H}_d(t) \leq e^{-\frac{\alpha_3}{\alpha_1}t}\mathcal{H}_d(0).$$

Define now $\alpha_1 := \frac{1}{2}\max\{m_1, p\}$, then

$$\mathcal{H}_d \geq \alpha_2\Big(\|\mathbf{x}_2 + \frac{p}{d}\mathbf{x}_1\|^2 + \|\mathbf{x}_1\|^2\Big).$$

Therefore, using $\mathcal{H}_d(t) \leq e^{-\frac{\alpha_3}{\alpha_1}t}\mathcal{H}_d(0)$, it holds that

$$\alpha_1\Big(\|\mathbf{x}_2(t) + \frac{p}{d}\mathbf{x}_1(t)\|^2 + \|\mathbf{x}_1(t)\|^2\Big) \leq e^{-\frac{\alpha_3}{\alpha_1}t}\mathcal{H}_d(0)$$

Finally, we can establish that

$$\|\mathbf{x}_2(t)+\frac{p}{d}\mathbf{x}_1(t)\|^2+\|\mathbf{x}_1(t)\|^2 \leq \frac{1}{2\alpha_2}e^{-\frac{\alpha_3}{\alpha_1}t}\big(m_2\|\mathbf{x}_2(0)+\frac{p}{d}\mathbf{x}_1(0)\|^2+p\|\mathbf{x}_1(0)\|^2\big).$$

Thus, although the closed-loop system is nonlinear, $(\mathbf{x}_1, \mathbf{x}_2) = (\mathbf{0}, \mathbf{0})$ is GES!!!
With exponential rate of convergence, at least, equal to $\frac{\alpha_3}{\alpha_1}$.

Remark 25 *This controller solves the tracking problem by employing the structural properties of the robot dynamics, i.e., the skew-symmetric property of $\dot{\mathbf{M}}(\mathbf{q}) - 2\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})$. However it requires the knowledge of the complete physical parameters of the robot. This last is, of course, difficult to meet in real-life applications.* ◇

Controller (9.1) has been originally reported by Slotine and Li in 1988 [15]. The Lyapunov function $\mathcal{H}_d$ has been reported by Spong et al., in 1990 [18].
A deeper analysis of this controller can be found in Chapter 11 of [6].

Chapter 11

Adaptive Control

The adaptive controller has been previously described in Section 8.1 for the regulation case. The controller is the uncertain version of controller (10.1) and it is given by

$$\boldsymbol{\tau} = \hat{\mathbf{M}}(\mathbf{q})\dot{\mathbf{q}}_r + \hat{\mathbf{C}}(\mathbf{q},\dot{\mathbf{q}})\mathbf{q}_r + \hat{\mathbf{g}}(\mathbf{q}) - d\mathbf{s},$$

where

$$\mathbf{q}_r := \dot{\mathbf{q}}_d - \frac{p}{d}\tilde{\mathbf{q}},$$

and

$$\begin{aligned} \mathbf{s} &= \dot{\mathbf{q}} - \mathbf{q}_r \\ &= \dot{\tilde{\mathbf{q}}} + \frac{p}{d}\tilde{\mathbf{q}}. \end{aligned} \tag{11.1}$$

Hence, we can write

$$\boldsymbol{\tau} = \hat{\mathbf{M}}(\mathbf{q})\Big(\ddot{\mathbf{q}}_d - \frac{p}{d}\dot{\tilde{\mathbf{q}}}\Big) + \hat{\mathbf{C}}(\mathbf{q},\dot{\mathbf{q}})\Big(\dot{\mathbf{q}}_d - \frac{p}{d}\tilde{\mathbf{q}}\Big) + \hat{\mathbf{g}}(\mathbf{q}) - d\mathbf{s}, \tag{11.2}$$

or, using (8.1),

$$\boldsymbol{\tau} = \mathbf{Y}\big(\mathbf{q},\dot{\mathbf{q}},\dot{\mathbf{q}}_d - \frac{p}{d}\tilde{\mathbf{q}},\ddot{\mathbf{q}}_d - \frac{p}{d}\dot{\tilde{\mathbf{q}}}\big)\hat{\boldsymbol{\phi}} - d\mathbf{s}.$$

Defining

$$\tilde{\boldsymbol{\phi}} := \hat{\boldsymbol{\phi}} - \boldsymbol{\phi}$$

as the parameter estimation error, the closed-loop system becomes

$$\dot{\mathbf{s}} = -\mathbf{M}(\mathbf{q})^{-1}\big[\mathbf{C}(\mathbf{q},\dot{\mathbf{q}})\mathbf{s} + d\mathbf{s} - \mathbf{Y}(\mathbf{q},\dot{\mathbf{q}},\mathbf{q}_r,\dot{\mathbf{q}}_r)\tilde{\boldsymbol{\phi}}\big]. \tag{11.3}$$

Using function $\mathcal{E}_d$ as

$$\mathcal{E}_d := \frac{1}{2}\mathbf{s}^\top \mathbf{M}(\mathbf{q})\mathbf{s} + \frac{1}{2}p\|\tilde{\mathbf{q}}\|^2 + \frac{1}{2k}\|\tilde{\boldsymbol{\phi}}\|^2,$$

we obtain

$$\dot{\mathcal{E}}_d = -\frac{d}{2}\|\mathbf{s}\|^2 - \frac{d}{2}\|\dot{\tilde{\mathbf{q}}}\|^2 - \frac{p^2}{2d}\|\tilde{\mathbf{q}}\|^2 + \tilde{\boldsymbol{\phi}}^\top \Big(\mathbf{Y}(\mathbf{q}, \dot{\mathbf{q}}, \mathbf{q}_r, \dot{\mathbf{q}}_r)^\top \mathbf{s} + \frac{1}{k}\dot{\tilde{\boldsymbol{\phi}}}\Big).$$

Setting

$$\dot{\tilde{\boldsymbol{\phi}}} = -k\mathbf{Y}(\mathbf{q}, \dot{\mathbf{q}}, \mathbf{q}_r, \dot{\mathbf{q}}_r)^\top \mathbf{s}, \tag{11.4}$$

ensures that

$$\dot{\mathcal{E}}_d = -\frac{d}{2}\|\mathbf{s}\|^2 - \frac{d}{2}\|\dot{\tilde{\mathbf{q}}}\|^2 - \frac{p^2}{2d}\|\tilde{\mathbf{q}}\|^2. \tag{11.5}$$

Hence, $\dot{\mathcal{E}}_d$ is negative semi-definite.

If we define $\mathbf{x}_1 = \tilde{\mathbf{q}}$, $\mathbf{x}_2 = \dot{\tilde{\mathbf{q}}}$ and $\mathbf{x}_3 = \tilde{\boldsymbol{\phi}}$, we get

$$\begin{aligned}
\dot{\mathbf{x}}_1 =& \, \mathbf{x}_2, \\
\dot{\mathbf{x}}_2 =& -\frac{p}{d}\mathbf{x}_2 - \mathbf{M}(\mathbf{x}_1 + \mathbf{q}_d(t))^{-1}\Big[\mathbf{C}(\mathbf{x}_1 + \mathbf{q}_d(t), \mathbf{x}_2 + \dot{\mathbf{q}}_d(t))\Big(\mathbf{x}_2 + \frac{p}{d}\mathbf{x}_1\Big) + p\mathbf{x}_1 + d\mathbf{x}_2\Big] \\
&+ \mathbf{M}(\mathbf{x}_1 + \mathbf{q}_d(t))^{-1}\mathbf{Y}\big(\mathbf{x}_1 + \mathbf{q}_d(t), \mathbf{x}_2 + \dot{\mathbf{q}}_d(t), \dot{\mathbf{q}}_d(t) - \frac{p}{d}\mathbf{x}_1, \ddot{\mathbf{q}}_d(t) - \frac{p}{d}\mathbf{x}_2\big)\mathbf{x}_3, \\
\dot{\mathbf{x}}_3 =& -k\mathbf{Y}\big(\mathbf{x}_1 + \mathbf{q}_d(t), \mathbf{x}_2 + \dot{\mathbf{q}}_d(t), \dot{\mathbf{q}}_d(t) - \frac{p}{d}\mathbf{x}_1, \ddot{\mathbf{q}}_d(t) - \frac{p}{d}\mathbf{x}_2\big)^\top \Big(\mathbf{x}_2 + \frac{p}{d}\mathbf{x}_1\Big).
\end{aligned}$$

Clearly this closed-loop is **NOT autonomous, because it does depend explicitly on time**. This time-dependence is due to the desired time-varying trajectory.

Hence, LaSalle's Invariance Theorem cannot be employed to show convergence and Lyapunov's Theorem can only ensure stability of the equilibrium equations

$$(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = \big(\mathbf{0}, \mathbf{0}, \mathbf{Y}(\mathbf{q}_d(t), \dot{\mathbf{q}}_d(t), \dot{\mathbf{q}}_d(t), \ddot{\mathbf{q}}_d(t))\bar{\mathbf{x}}_3 = \mathbf{0}\big).$$

The convergence proof will proceed using Barbalat's Lemma

First note that $\dot{\mathcal{E}}_d \leq 0$, thus integrating from 0 to t both sides of this inequality we obtain

$$\mathcal{E}_d(t) \leq \mathcal{E}_d(0).$$

Now, using Property P2 we have that $\mathcal{E}_d(t)$ can be bounded as

$$\frac{1}{2}m_1\|\mathbf{s}(t)\|^2 + \frac{1}{2}p\|\tilde{\mathbf{q}}(t)\|^2 + \frac{1}{2k}\|\tilde{\boldsymbol{\phi}}(t)\|^2 \leq \mathcal{E}_d(t),$$

hence

$$\frac{1}{2}m_1\|\mathbf{s}(t)\|^2 + \frac{1}{2}p\|\tilde{\mathbf{q}}(t)\|^2 + \frac{1}{2k}\|\tilde{\boldsymbol{\phi}}(t)\|^2 \le \mathcal{E}_d(t) \le \mathcal{E}_d(0) < \infty.$$

Thus, for any $t \ge 0$, $\mathbf{s}(t), \tilde{\mathbf{q}}(t)$ and $\tilde{\boldsymbol{\phi}}(t)$ are bounded. Thus, $\mathbf{s}, \tilde{\mathbf{q}}, \tilde{\boldsymbol{\phi}} \in \mathcal{L}_\infty$. Since $\dot{\tilde{\mathbf{q}}} = -\frac{p}{d}\tilde{\mathbf{q}} + \mathbf{s}$, we also have that $\dot{\tilde{\mathbf{q}}} \in \mathcal{L}_\infty$.

Using (11.5) and integrating, from 0 to t, both sides of (11.5) yields

$$\mathcal{E}_d(t) - \mathcal{E}_d(0) = -\frac{d}{2}\int_0^t \|\mathbf{s}(\sigma)\|^2 d\sigma - \frac{d}{2}\int_0^t \|\dot{\tilde{\mathbf{q}}}(\sigma)\|^2 d\sigma - \frac{p^2}{2d}\int_0^t \|\tilde{\mathbf{q}}(t)\|^2 d\sigma.$$

Using the fact that $\mathcal{E}_d(t) \ge 0$ for all $t \ge 0$, we have that

$$\int_0^t \|\mathbf{s}(\sigma)\|^2 d\sigma + \int_0^t \|\dot{\tilde{\mathbf{q}}}(\sigma)\|^2 d\sigma + \frac{p^2}{d^2}\int_0^t \|\tilde{\mathbf{q}}(t)\|^2 d\sigma \le \frac{2}{d}\mathcal{E}_d(0).$$

Since the previous inequality holds for all $t \ge 0$, let us consider the case when $t = \infty$, then

$$\int_0^\infty \|\mathbf{s}(\sigma)\|^2 d\sigma + \int_0^\infty \|\dot{\tilde{\mathbf{q}}}(\sigma)\|^2 d\sigma + \frac{p^2}{d^2}\int_0^\infty \|\tilde{\mathbf{q}}(t)\|^2 d\sigma \le \frac{2}{d}\mathcal{E}_d(0).$$

Clearly $\mathcal{E}_d(0) \le \infty$. Thus $\mathbf{s}, \tilde{\mathbf{q}}, \dot{\tilde{\mathbf{q}}} \in \mathcal{L}_2$.

Invoking Barbalat's Lemma (see Lemma 6) with $\tilde{\mathbf{q}} \in \mathcal{L}_\infty \cap \mathcal{L}_2$ and $\dot{\tilde{\mathbf{q}}} \in \mathcal{L}_\infty$ then

$$\lim_{t\to\infty} \tilde{\mathbf{q}}(t) = \mathbf{0},$$

for any $\mathbf{q}(0), \dot{\mathbf{q}}(0), \hat{\boldsymbol{\phi}}(0)$ and any $\mathbf{q}_d(t) \in \mathcal{C}^2$.

We have also established that with $\dot{\tilde{\mathbf{q}}} \in \mathcal{L}_\infty \cap \mathcal{L}_2$, therefore, in order to prove that $\dot{\tilde{\mathbf{q}}}(t)$ converges to zero, we require to establish that $\ddot{\tilde{\mathbf{q}}} \in \mathcal{L}_\infty$.

Using (11.1) and (11.3), we have that

$$\begin{aligned}\ddot{\tilde{\mathbf{q}}} &= -\frac{p}{d}\dot{\tilde{\mathbf{q}}} + \dot{\mathbf{s}} \\ &= -\frac{p}{d}\dot{\tilde{\mathbf{q}}} - \mathbf{M}(\mathbf{q})^{-1}\big[\mathbf{C}(\mathbf{q},\dot{\mathbf{q}})\mathbf{s} + d\mathbf{s} - \mathbf{Y}(\mathbf{q},\dot{\mathbf{q}},\mathbf{q}_r,\dot{\mathbf{q}}_r)\tilde{\boldsymbol{\phi}}\big].\end{aligned} \tag{11.6}$$

Recall that $\tilde{\mathbf{q}}, \dot{\tilde{\mathbf{q}}}, \mathbf{s}, \tilde{\boldsymbol{\phi}} \in \mathcal{L}_\infty$. Since, from the properties of the Euler-Lagrange model, $\mathbf{M}(\mathbf{q})$, $\mathbf{C}(\mathbf{q},\dot{\mathbf{q}})$ and $\mathbf{Y}(\mathbf{q},\dot{\mathbf{q}},\mathbf{q}_r,\dot{\mathbf{q}}_r)$ are bounded for all $\mathbf{q} \in \mathbb{R}^n$, then the right hand side of (11.6) will be bounded if we prove that $\dot{\mathbf{q}}, \mathbf{q}_r$ and $\dot{\mathbf{q}}_r$ are also bounded.

Since

$$\begin{aligned}\dot{\mathbf{q}} &= \dot{\tilde{\mathbf{q}}} + \dot{\mathbf{q}}_d,\\ \mathbf{q}_r &= \dot{\mathbf{q}}_d - \frac{p}{d}\tilde{\mathbf{q}},\\ \dot{\mathbf{q}}_r &= \ddot{\mathbf{q}}_d - \frac{p}{d}\dot{\tilde{\mathbf{q}}},\end{aligned}$$

these signals are bounded if $\mathbf{q}_d, \dot{\mathbf{q}}_d$ and $\ddot{\mathbf{q}}_d$ are also bounded. Therefore we make the following assumption.

The desired trajectory $\mathbf{q}_d \in \mathcal{C}^2$ has to be designed such that $\mathbf{q}_d, \dot{\mathbf{q}}_d$ and $\ddot{\mathbf{q}}_d$ are bounded.

Thus the assumption that $\mathbf{q}_d, \dot{\mathbf{q}}_d, \ddot{\mathbf{q}}_d \in \mathcal{L}_\infty$ ensures that $\ddot{\tilde{\mathbf{q}}} \in \mathcal{L}_\infty$. Thus, again by Barbalat's Lemma (see Lemma 6),

$$\lim_{t\to\infty} \dot{\tilde{\mathbf{q}}}(t) = \mathbf{0}.$$

Hence, $(\tilde{\mathbf{q}}, \dot{\tilde{\mathbf{q}}}) = (\mathbf{0}, \mathbf{0})$ is GAS.

This controller does not ensure parameter error convergence to zero. It can only be established that the estimation error converges to a *manifold* given by

$$\mathbf{Y}(\mathbf{q}_d(t), \dot{\mathbf{q}}_d(t), \dot{\mathbf{q}}_d(t), \ddot{\mathbf{q}}_d(t))\tilde{\boldsymbol{\phi}} = \mathbf{0}.$$

Controller (11.2), with estimation law (11.4), has been originally reported by Slotine and Li in 1988 [15]. The Lyapunov function $\mathcal{H}_d$ has been reported by Spong et al., in 1990 [18]. A deeper analysis of this controller can be found in Chapter 16 of [6].

Part IV

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