Intelligent Systems Programming

Lecture 4: Propositional Logic

The Importance of Logic

- Knowledge Representations
  - Sentences in propositional logic
  - Binary decision diagrams
  - Constraint propagation rules
  - Linear inequalities
- Reasoning and Optimization Methods
  - Heuristic search in implicit search spaces
  - Game playing
  - SAT-solving
  - Binary decision diagram construction
  - Configuration
  - Constraint programming
  - Local search
  - Linear programming

Today's Program

- **Propositional Logic [10:00-10:50]**
  - Fundamental Concepts in Logic
  - Syntax and Semantics
  - Inference
    - Entailment
    - Logical equivalence
    - Inference rules
    - Formal proofs
- **Efficient Inference Algorithms [11:00-11:50]**
  - Resolution
  - Inference with Horn clauses
  - Efficient SAT checking
    - DPLL
    - WalkSat
  - Phase transition
- **Exercises [12:00-14:00]**

Fundamental Concepts of Logic

The Purpose of Logics

- Logics are formal languages for representing information such that conclusions can be drawn
- Natural language is too ambiguous

  "John saw the diamond through the window and stole it"

  Reading 1: John stole the diamond

  Reading 2: John stole the window

- Sentences in logics are assertions about a world that are either true or false

Logic: Syntax and Semantics

- Syntax defines the written form of legal sentences in the language
- Semantics define the truth-value of sentences in a world
- World is the setting or environment in which you derive the truth of sentences

- E.g., the language of inequalities
  – $x+2 \geq y$ is a sentence; $x2+y > \{\}$ is not a sentence
  – $x+2 \geq y$ is true in a world where the number $x+2$ is no less than the number y
  – $x+2 \geq y$ is true in a world where $x = 7$, $y = 1$
  – $x+2 \geq y$ is false in a world where $x = 0$, $y = 6$

Entailment and Inference

- Entailment means that one thing follows from another:

$$KB \models \alpha$$

$\models$ is a meta symbol, not a part of the syntax!

- **Knowledge base *KB* entails sentence α if and only if α is true in all worlds where *KB* is true**
  - E.g., *KB* containing *the-apple-is-red* and *the-apple-is-sweet* entails *the-apple-is-sweet*
  - E.g., *KB* containing "$y \geq 4$", "$y \leq 4$" entails $y = 4$
- Inference is to **decide** whether $KB \models \alpha$

Models

- A model is a formal description of a possible world used to decide truth-value of sentences

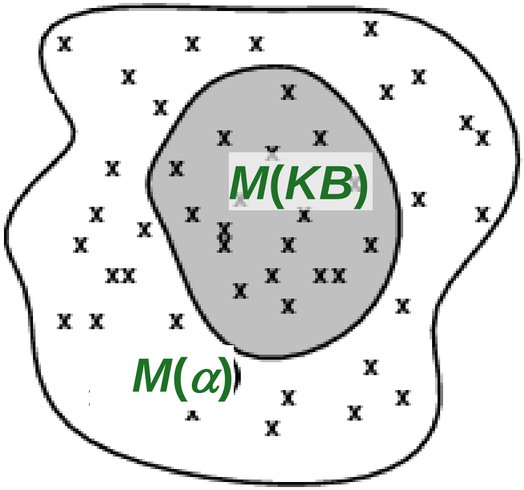
  Example:
  Possible world = state of nuclear power plant
  Model= state {*broken*, *hot*, *cold*} of pipe A, B, and C

- We say m is a model of a sentence α if α is true in m

- $M(s)$ is the set of all models of s

Models

- ***KB*** $\models$ ***α*** **if and only if** ***M(KB)*** $\subseteq$ ***M(α)***



Model Checking is to explicitly check this relationship

Inference Algorithms

- ***KB*** $\vdash_i$ ***α*** : sentence *α* can be derived from *KB* by procedure *i*

- Soundness: *i* is sound if *KB* $\vdash_i \alpha$ implies *KB* $\models \alpha$
  - Any sentence derived by *i* from *KB* is truth preserving
- Completeness: *i* is complete if *KB* $\models \alpha$ implies *KB* $\vdash_i \alpha$
  - All the sentences entailed by *KB* can be derived by procedure *i*
  - That is, the procedure will answer any question whose answer follows from what is known by the *KB*

Propositional Logic

Syntax

- Atomic sentences
  - Proposition symbols P, Q,… are sentences
  - The two constants *True* and *False* are sentences
- Complex sentences
  - If S_1 and S_2 are sentences then so are (in order of precedence)

$\neg S_1$	*negation*	$\neg$ *not*	$\neg Q$, Q	***literals***
$(S_1 \wedge S_2)$	*conjunction*	$\wedge$ *and*	S_1, S_2	*conjuncts*
$(S_1 \vee S_2)$	*disjunction*	$\vee$ *or*	S_1, S_2	*disjuncts*
$(S_1 \Rightarrow S_2)$	*implication*	$\Rightarrow$ *implies*	S_1	***premise***
			S_2	***conclusion***
$(S_1 \Leftrightarrow S_2)$	*biimplication*	$\Leftrightarrow$ *if-and-only-if*		

Semantics

- **Each model *m* assigns truth value *true* (1) or *false* (0) to each proposition symbol**

  E.g.

	P	Q	R
	false	*true*	*false*

- Rules for evaluating truth with respect to a model *m*:

True	is *true*				
False	is *false*				
$\neg S$	is *true*	iff	S is *false*		
$S_1 \wedge S_2$	is *true*	iff	S_1 is *true*	and	S_2 is *true*
$S_1 \vee S_2$	is *true*	iff	S_1 is *true*	or	S_2 is *true*
$S_1 \Rightarrow S_2$	is *true*	iff	S_1 is *false*	or	S_1, S_2 are *true*
$S_1 \Leftrightarrow S_2$	is *true*	iff	$S_1 \Rightarrow S_2$ is *true*	and	$S_2 \Rightarrow S_1$ is *true*

- Simple recursive process evaluates an arbitrary sentence, e.g.,

$$\neg P \Rightarrow (Q \wedge R) \quad = \quad \neg\, \mathit{false} \Rightarrow (\mathit{true} \wedge \mathit{false}) \quad = \quad \mathit{true} \Rightarrow \mathit{false} \quad = \quad \mathit{false}$$

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Validity

A sentence is valid if it is true in all models

e.g., *True*, $A \vee \neg A$, $A \Rightarrow A$, ...

Validity is connected to entailment via the

Deduction Theorem:

$KB \models \alpha$ if and only if ($KB \Rightarrow \alpha$) is valid

Satisfiability

A sentence is satisfiable if it is true in some model

e.g., $A \lor B$

A sentence is unsatisfiable if it is true in no models

e.g., $A \land \neg A$

Satisfiability is connected to entailment via the following:

$KB \models \alpha$ if and only if $(KB \land \neg \alpha)$ is unsatisfiable

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Inference by Enumeration (Model Checking)

- Inference: decide whether $KB \models \alpha$

```
function TT-ENTAILS?(KB, α) returns true or false
   inputs: KB, the knowledge base, a sentence in propositional logic
           α, the query, a sentence in propositional logic

   symbols ← a list of the proposition symbols in KB and α
   return TT-CHECK-ALL(KB, α, symbols, { })
```

```
function TT-CHECK-ALL(KB, α, symbols, model) returns true or false
   if EMPTY?(symbols) then
      if PL-TRUE?(KB, model) then return PL-TRUE?(α, model)
      else return true //  when KB is false, always return true
   else do
      P ← FIRST(symbols)
      rest ← REST(symbols)
      return (TT-CHECK-ALL(KB, α, rest, model ∪ {P = true})
              and
              TT-CHECK-ALL(KB, α, rest, model ∪ {P = false }))
```

Properties of TT-ENTAILS

- DFS enumeration and check of all models
- Sound? yes checks if α is true when *KB* is true
- Complete? yes checks all models
- For *n* symbols
  - time complexity is $O(2^n)$
  - space complexity is $O(n)$

Entailment by Theorem Proving

Logical Equivalence

- Two sentences are logically equivalent iff true in same models: $\alpha \equiv \beta$ iff $\alpha \models \beta$ and $\beta \models \alpha$
- Standard Equivalences:
  - $\alpha \wedge \neg\alpha \equiv False$
  - $\alpha \vee \neg\alpha \equiv True$
  - $\alpha \wedge True \equiv \alpha$
  - $\alpha \vee False \equiv \alpha$
  - $\alpha \wedge False \equiv False$
  - $\alpha \vee True \equiv True$
  - $\alpha \wedge \alpha \equiv \alpha$
  - $\alpha \vee \alpha \equiv \alpha$

How would you prove these equivalences?

More Standard Equivalences

$$
\begin{aligned}
(\alpha \land \beta) &\equiv (\beta \land \alpha) \quad \text{commutativity of } \land \\
(\alpha \lor \beta) &\equiv (\beta \lor \alpha) \quad \text{commutativity of } \lor \\
((\alpha \land \beta) \land \gamma) &\equiv (\alpha \land (\beta \land \gamma)) \quad \text{associativity of } \land \\
((\alpha \lor \beta) \lor \gamma) &\equiv (\alpha \lor (\beta \lor \gamma)) \quad \text{associativity of } \lor \\
\neg(\neg\alpha) &\equiv \alpha \quad \text{double-negation elimination} \\
(\alpha \Rightarrow \beta) &\equiv (\neg\beta \Rightarrow \neg\alpha) \quad \text{contraposition} \\
(\alpha \Rightarrow \beta) &\equiv (\neg\alpha \lor \beta) \quad \text{implication elimination} \\
(\alpha \Leftrightarrow \beta) &\equiv ((\alpha \Rightarrow \beta) \land (\beta \Rightarrow \alpha)) \quad \text{biconditional elimination} \\
\neg(\alpha \land \beta) &\equiv (\neg\alpha \lor \neg\beta) \quad \text{de Morgan} \\
\neg(\alpha \lor \beta) &\equiv (\neg\alpha \land \neg\beta) \quad \text{de Morgan} \\
(\alpha \land (\beta \lor \gamma)) &\equiv ((\alpha \land \beta) \lor (\alpha \land \gamma)) \quad \text{distributivity of } \land \text{ over } \lor \\
(\alpha \lor (\beta \land \gamma)) &\equiv ((\alpha \lor \beta) \land (\alpha \lor \gamma)) \quad \text{distributivity of } \lor \text{ over } \land
\end{aligned}
$$

Formal Proof

- **Determine logical equivalence between sentences using standard rules**

Example

$$
\begin{array}{lll}
 & (a \vee (b \Rightarrow a)) & \\
\equiv & (a \vee (\neg b \vee a)) & \text{impl. elim.} \\
\equiv & \ldots & \\
\equiv & b \Rightarrow a &
\end{array}
$$

Inference Rules

numerator (premises) $\models$ ***denominator (conclusion)***

$$\frac{\alpha \Rightarrow \beta, \quad \alpha}{\beta}$$

Modus Ponens

$$\frac{\alpha \wedge \beta}{\alpha}$$

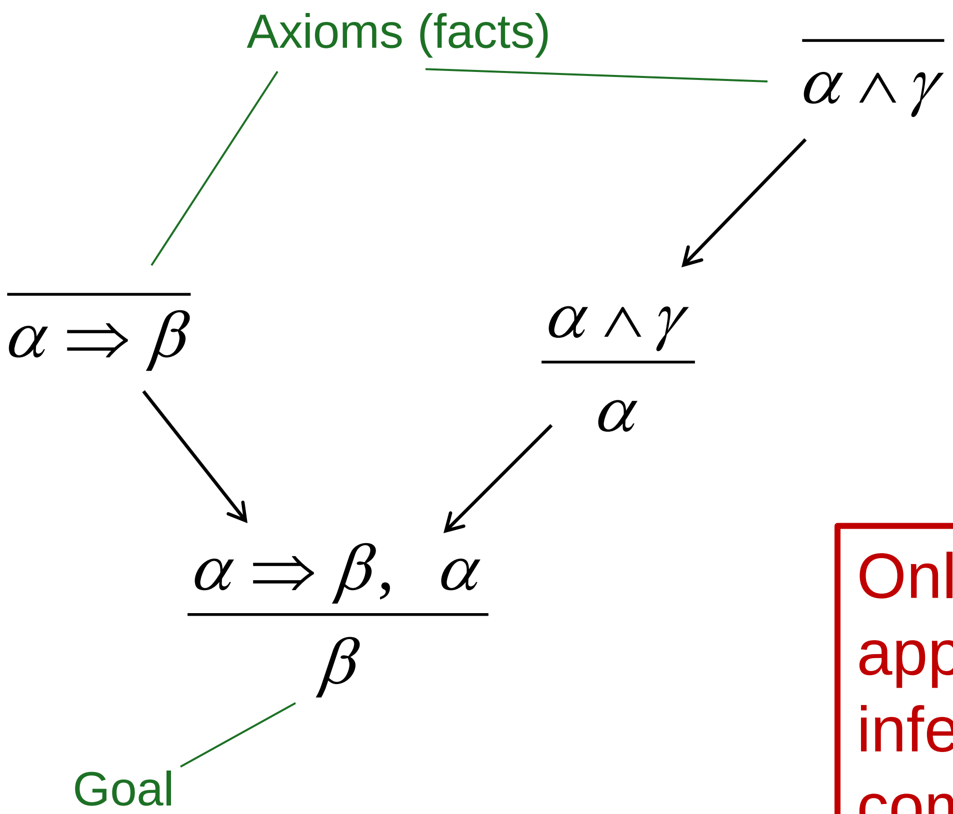
And-Elimination

$$\frac{\alpha \Rightarrow \beta}{\neg\alpha \vee \beta} \qquad \frac{\neg\alpha \vee \beta}{\alpha \Rightarrow \beta}$$

All Equivalence rules

Inference Proof

- Search for inference rules to chain "goal" with axioms



Only complete approach if set of inference rules is complete!

Efficient Inference Algorithms

Conjunctive Normal Form (CNF)

Definition

- A literal is a symbol or a negated symbol
- A clause is a disjunction of literals
- CNF is a conjunction of clauses

E.g., $(A \lor \neg B) \land (B \lor \neg C \lor \neg D)$

Translate Arbitrary Sentence to CNF

1. Eliminate $\Leftrightarrow$, replacing $\alpha \Leftrightarrow \beta$ with $(\alpha \Rightarrow \beta) \land (\beta \Rightarrow \alpha)$
2. Eliminate $\Rightarrow$, replacing $\alpha \Rightarrow \beta$ with $\neg\alpha \lor \beta$
3. Move $\neg$ inwards using de Morgan's rules and double-negation
4. Apply distribution law ($\land$ over $\lor$) and flatten
5. Eliminate symbol duplicates in clauses (factoring)

Resolution: A Complete Inference Rule

$$\frac{l_1 \vee \ldots \vee l_k, \qquad m_1 \vee \ldots \vee m_n}{l_1 \vee \ldots \vee l_{i-1} \vee l_{i+1} \vee \ldots \vee l_k \vee m_1 \vee \ldots \vee m_{j-1} \vee m_{j+1} \vee \ldots \vee m_n}$$

where l_i and m_j are complementary literals ($l_i \equiv \neg m_j$).

E.g., $$\frac{P \vee Q, \qquad \neg Q \vee R}{P \vee R}$$

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Resolution Algorithm

- Prove $KB \models \alpha$ by contradiction, i.e., show $KB \land \neg\alpha$ is unsatisfiable
- Keep doing resolution on clauses until
  - fixpoint reached with no empty clause ()
    return *false*
  - () (= False) derived
    return *true*

$$\frac{Q, \quad \neg Q}{()}$$

Resolution Algorithm

function PL-RESOLUTION(KB, α) **returns** *true* or *false*
  inputs: KB, the knowledge base, a sentence in propositional logic
    α, the query, a sentence in propositional logic

  $clauses \leftarrow$ the set of clauses in the CNF representation of $KB \wedge \neg\alpha$
  $new \leftarrow \{\ \}$
  loop do
    for each pair of clauses C_i, C_j **in** *clauses* **do**
      $resolvents \leftarrow$ PL-RESOLVE(C_i, C_j)
      if *resolvents* contains the empty clause **then return** *true*
      $new \leftarrow new \cup resolvents$
    if $new \subseteq clauses$ **then return** *false*
    $clauses \leftarrow clauses \cup new$

Properties of the Resolution Algorithm

- **Sound**, yes
- **Complexity**
  - Time, Space: $O(2^n)$, due to blow-up of number of clauses
- **Complete,** yes
  Ground resolution theorem
  if a set of clauses S is unsatisfiable, then the resolution closure of those clauses $RC(S)$ contains the empty clause $() \in RC(S)$

Proof of ground resolution theorem

We must show: **if S is unsatisfiable then () in $RC(S)$**

Prove log. equiv. sentence: **if () not in $RC(S)$ then S satisfiable**

Satisfying assignment to propositions P_1 to P_k

For $i = 1$ to k

- $P_i = F$ if there is a clause $(F \lor \ldots \lor F \lor \neg P_i)$
- $P_i = T$ otherwise

Why is this assignment satisfying?

Horn Form Clauses

- Horn clause: clause with *at most* one positive literal
- Definit clause: *exactly* one positive literal

$P_1 \wedge P_2 \wedge \ldots \wedge P_n \Rightarrow C$

$\Rightarrow F$

head

body

fact

- Modus Ponens (for Definit clause)

$$\frac{\alpha_1, \ldots, \alpha_n, \qquad \alpha_1 \wedge \ldots \wedge \alpha_n \Rightarrow \beta}{\beta}$$

- **Inference possible in linear time!**

Forward chaining inference

KB:

$P \Rightarrow Q$

$L \wedge M \Rightarrow P$

$B \wedge L \Rightarrow M$

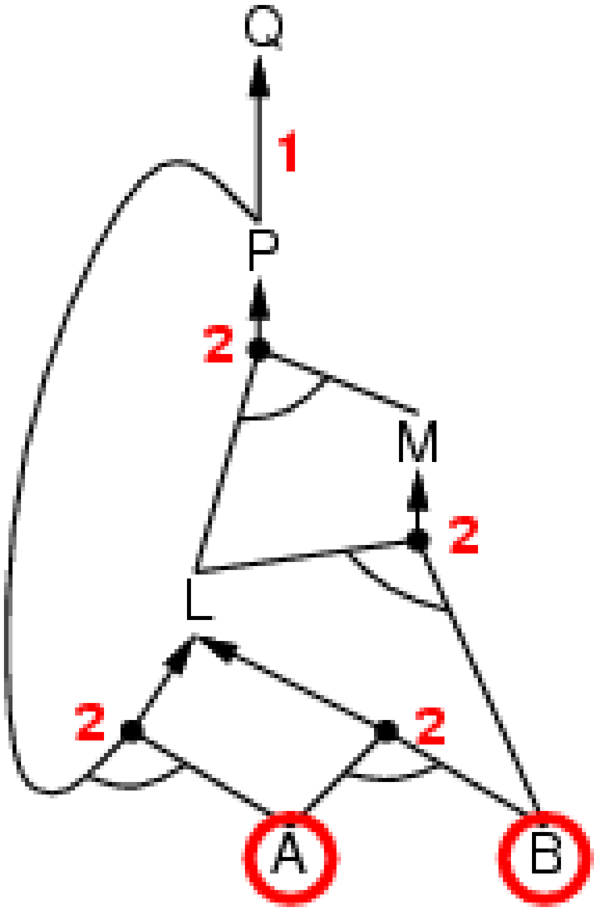
$A \wedge P \Rightarrow L$

$A \wedge B \Rightarrow L$

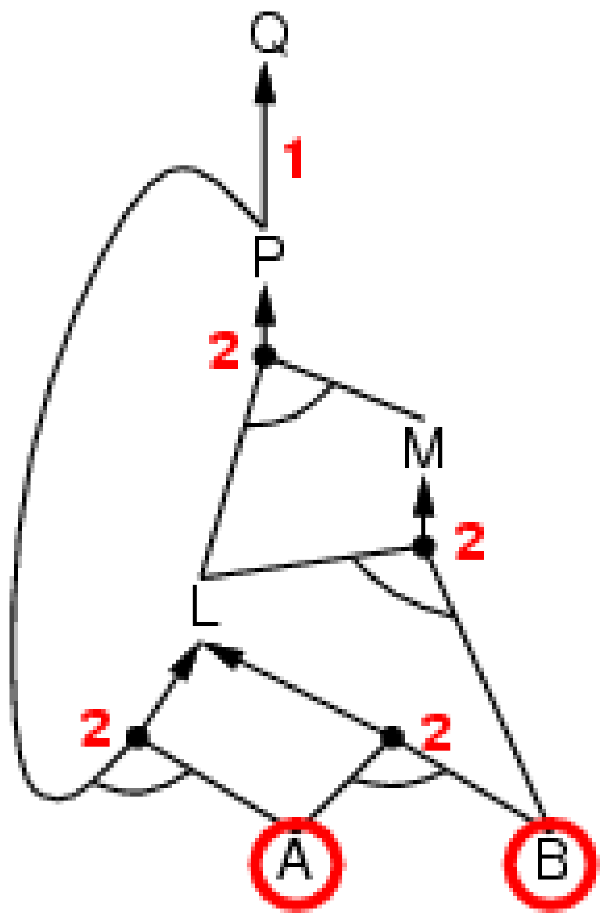
A

B

$KB \vDash Q$?



Forward chaining example



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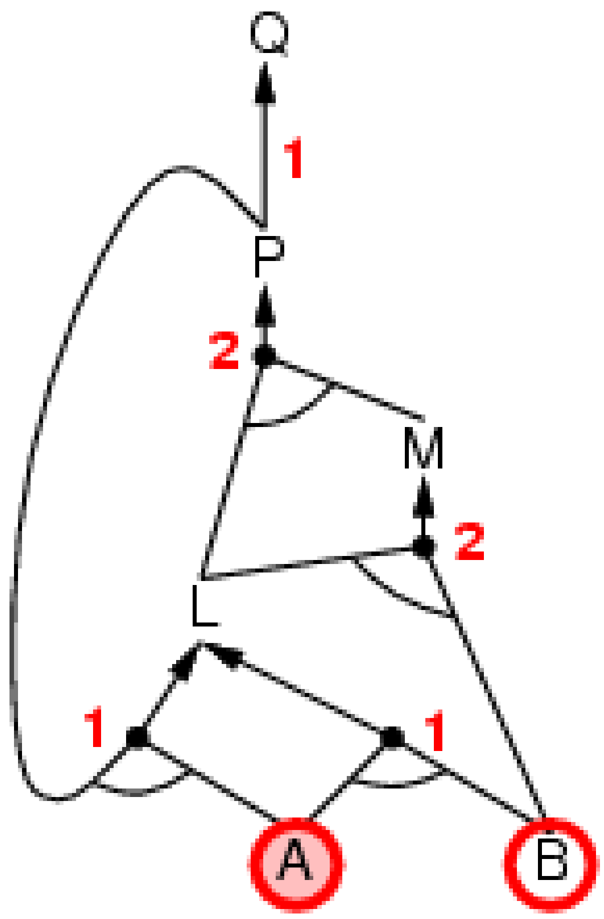

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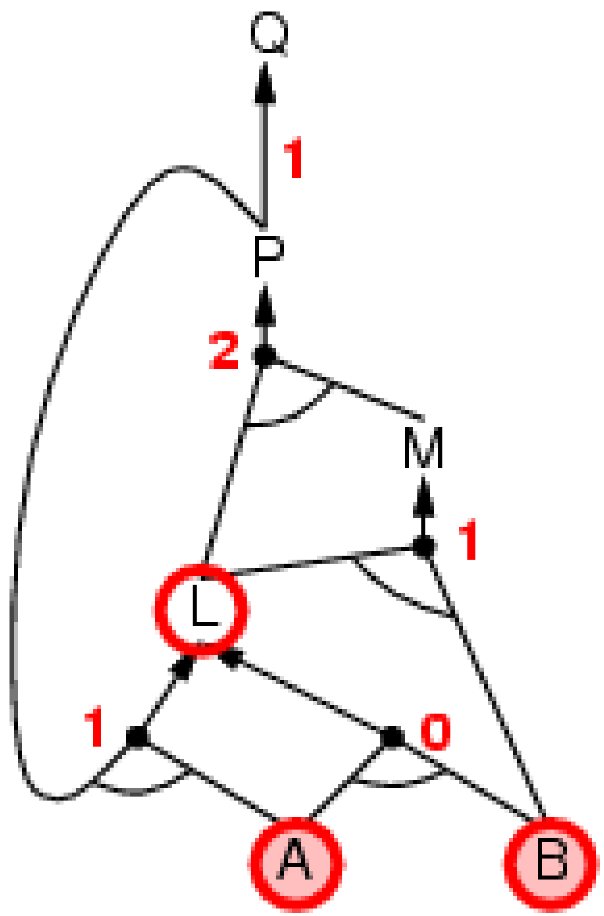
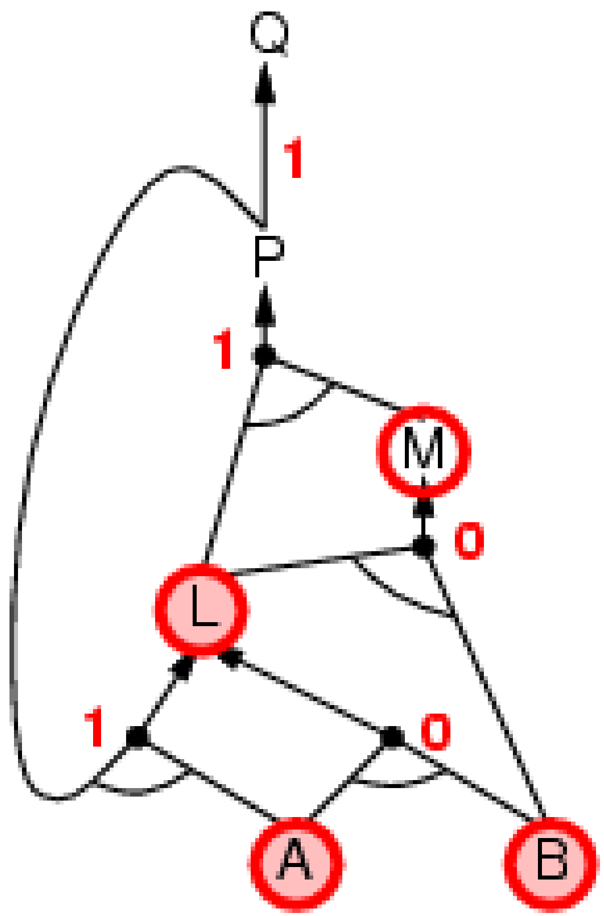
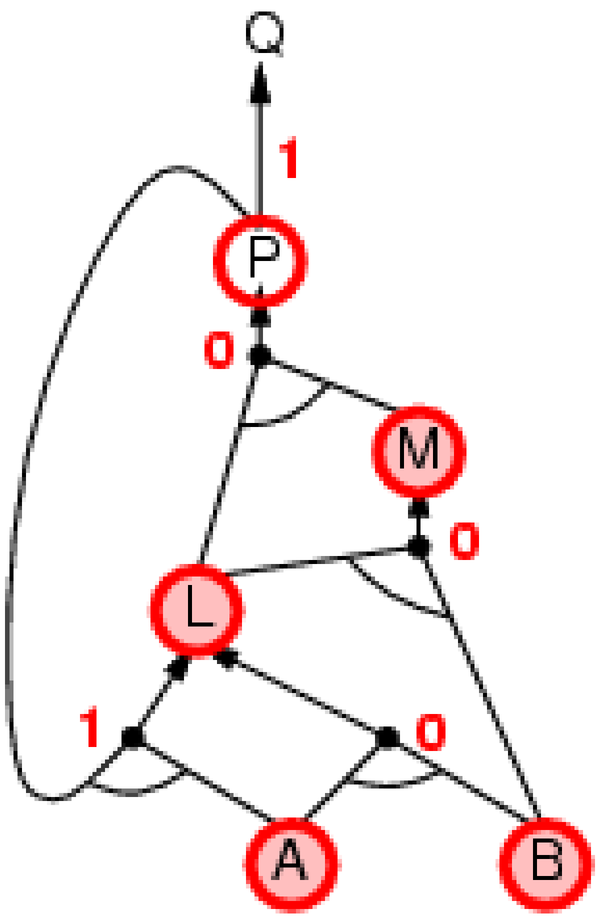
Forward chaining example



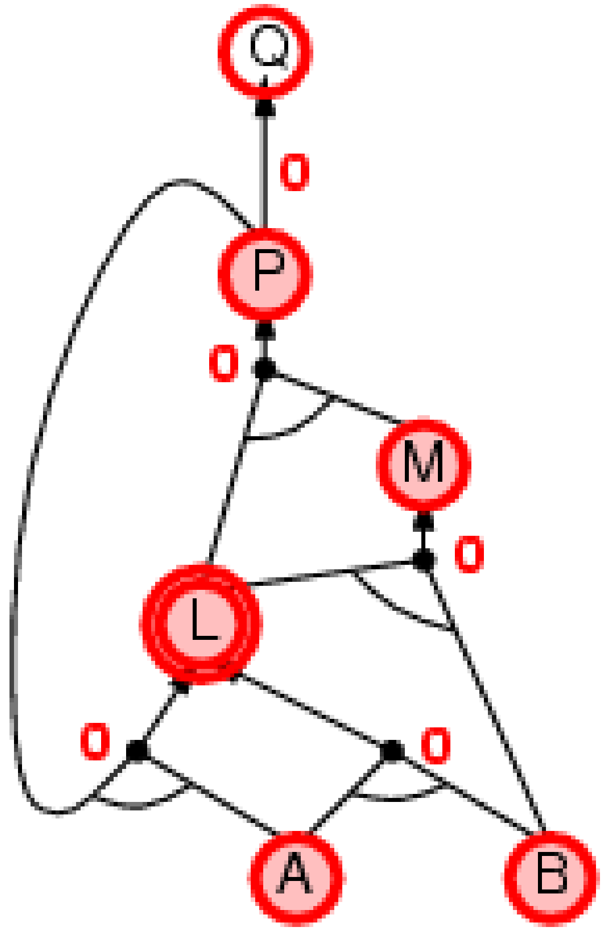
Forward chaining example



Forward chaining example



Forward chaining example



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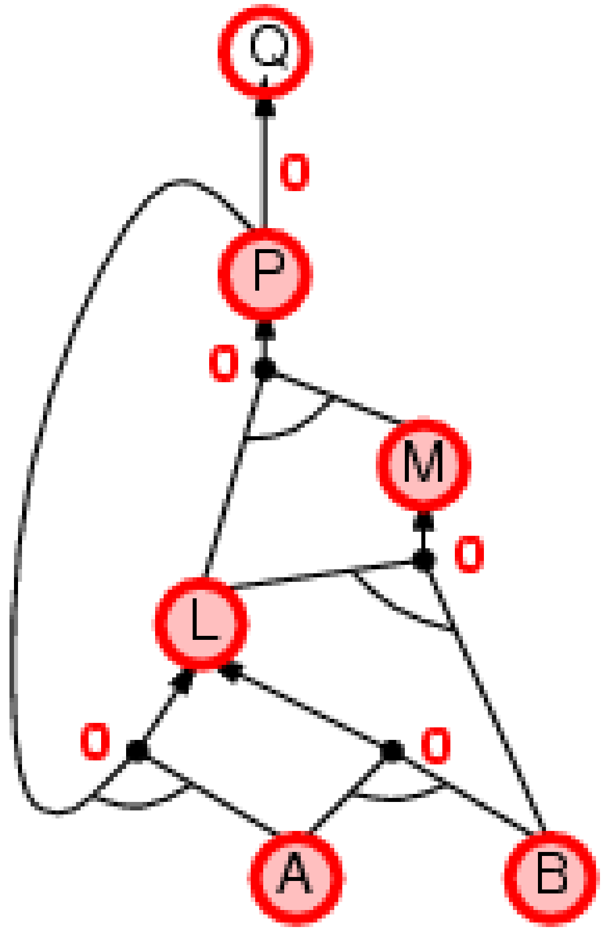

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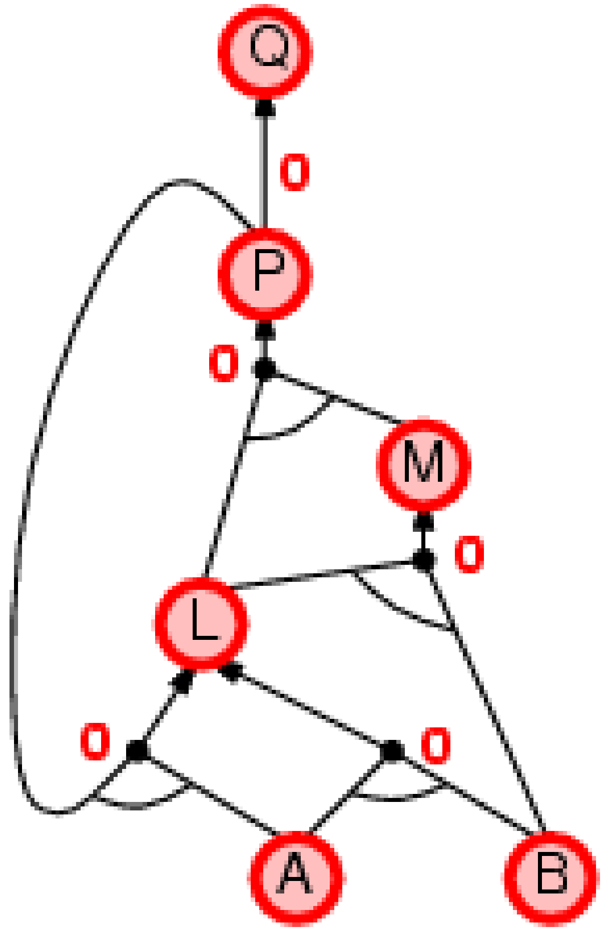
Forward chaining example

KB ⊨ Q

Yes!



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Properties of Forward Chaining

- **Sound**, yes since Modus Ponens is sound
- **Complete**, yes
- **Space and time**: $O(n)$, where n is the total number of clause literals

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Efficient SAT-Checking

Complete backtracking search algorithms

- DPLL algorithm (Davis, Putnam, Logemann, Loveland)

Incomplete local search algorithms

- `WalkSAT` algorithm

The DPLL Algorithm

Determine if a CNF sentence is satisfiable

Improvements over **truth table enumeration**:

1. **Early termination**

   A clause is true if any literal is true

   A sentence is false if any clause is false

2. **Pure symbol heuristic**

   Pure symbol: always appears with the same "sign" in all clauses

   e.g., In the three clauses $(A \lor \neg B)$, $(\neg B \lor \neg C)$, $(C \lor A)$, A and B are pure, C is impure

   Make a pure symbol literal true

3. **Unit propagation**

   Unit clause: only one literal in the clause

   The only literal in a unit clause must be true

   e.g., $(False \lor \neg B)$: B must be false

The DPLL Algorithm

function DPLL-SATISFIABLE?(s) **returns** *true* or *false*
  inputs: s, a sentence in propositional logic

  clauses ← the set of clauses in the CNF representation of s
  symbols ← a list of the proposition symbols in s
  return DPLL(*clauses*, *symbols*, { })

---

function DPLL(*clauses*, *symbols*, *model*) **returns** *true* or *false*

  if every clause in *clauses* is true in *model* **then return** *true*
  if some clause in *clauses* is false in *model* **then return** *false*
  P, *value* ← FIND-PURE-SYMBOL(*symbols*, *clauses*, *model*)
  if P is non-null **then return** DPLL(*clauses*, *symbols* – P, *model* ∪ {P=*value*})
  P, *value* ← FIND-UNIT-CLAUSE(*clauses*, *model*)
  if P is non-null **then return** DPLL(*clauses*, *symbols* – P, *model* ∪ {P=*value*})
  P ← FIRST(*symbols*); *rest* ← REST(*symbols*)
  return DPLL(*clauses*, *rest*, *model* ∪ {P=*true*}) **or**
      DPLL(*clauses*, *rest*, *model* ∪ {P=*false*}))

The `WalkSAT` algorithm

Determine if a CNF sentence is satisfiable

Algorithm

- Start with a random complete assignment
- In each iteration:
  - Pick random false clause
  - With probability *p*
    - flip random literal in clause
  - Else
    - flip literal that makes most clauses true

Phase Transition

- Consider random 3-CNF sentences. e.g.,

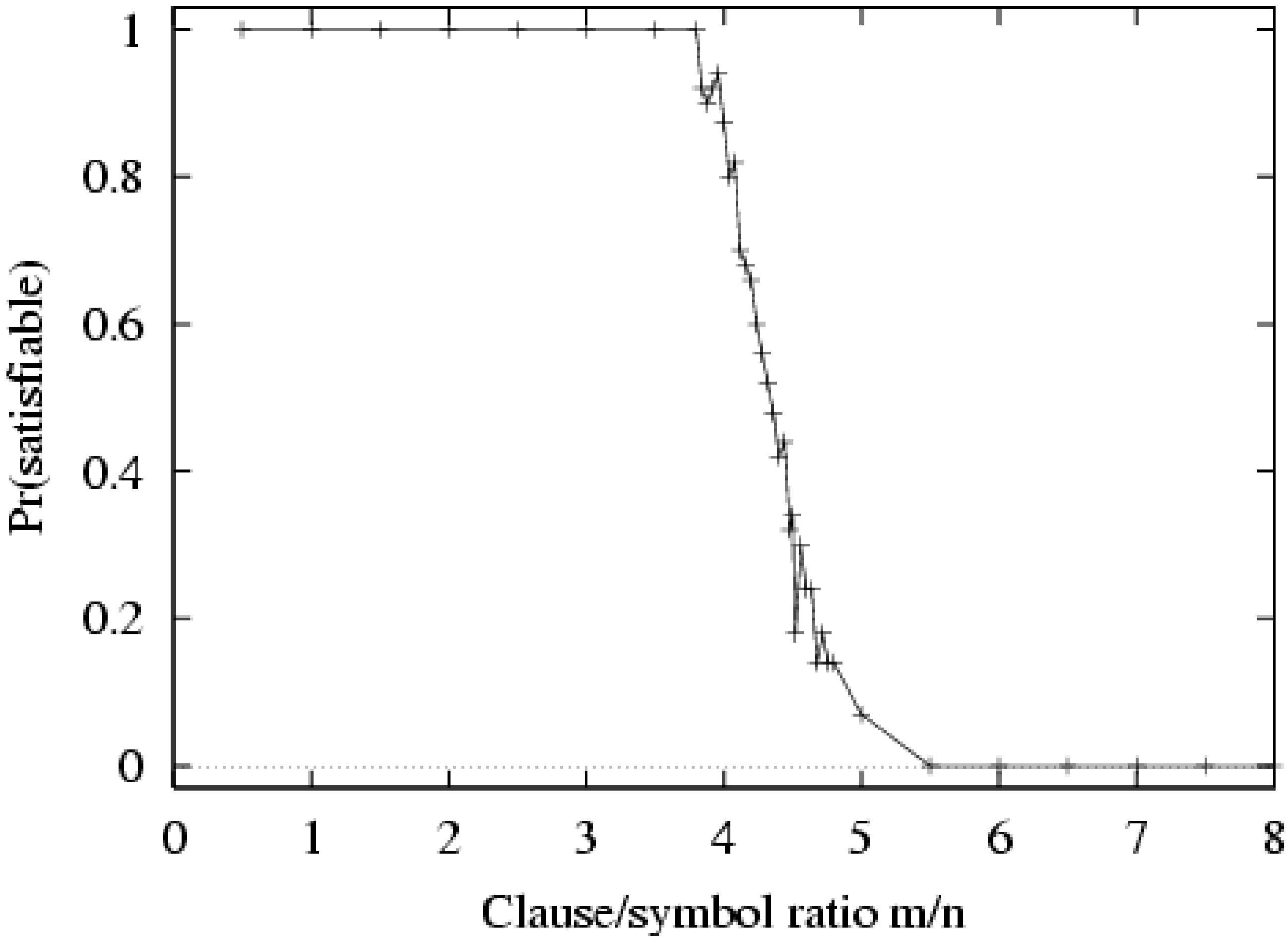
  $(\neg D \lor \neg B \lor C) \land (B \lor \neg A \lor \neg C) \land (\neg C \lor \neg B \lor E) \land (E \lor \neg D \lor B) \land (B \lor E \lor \neg C)$

  m = number of clauses

  n = number of symbols

  – Hard problems seem to cluster near $m/n = 4.3$! (phase transition)

Phase Transition



Phase Transition