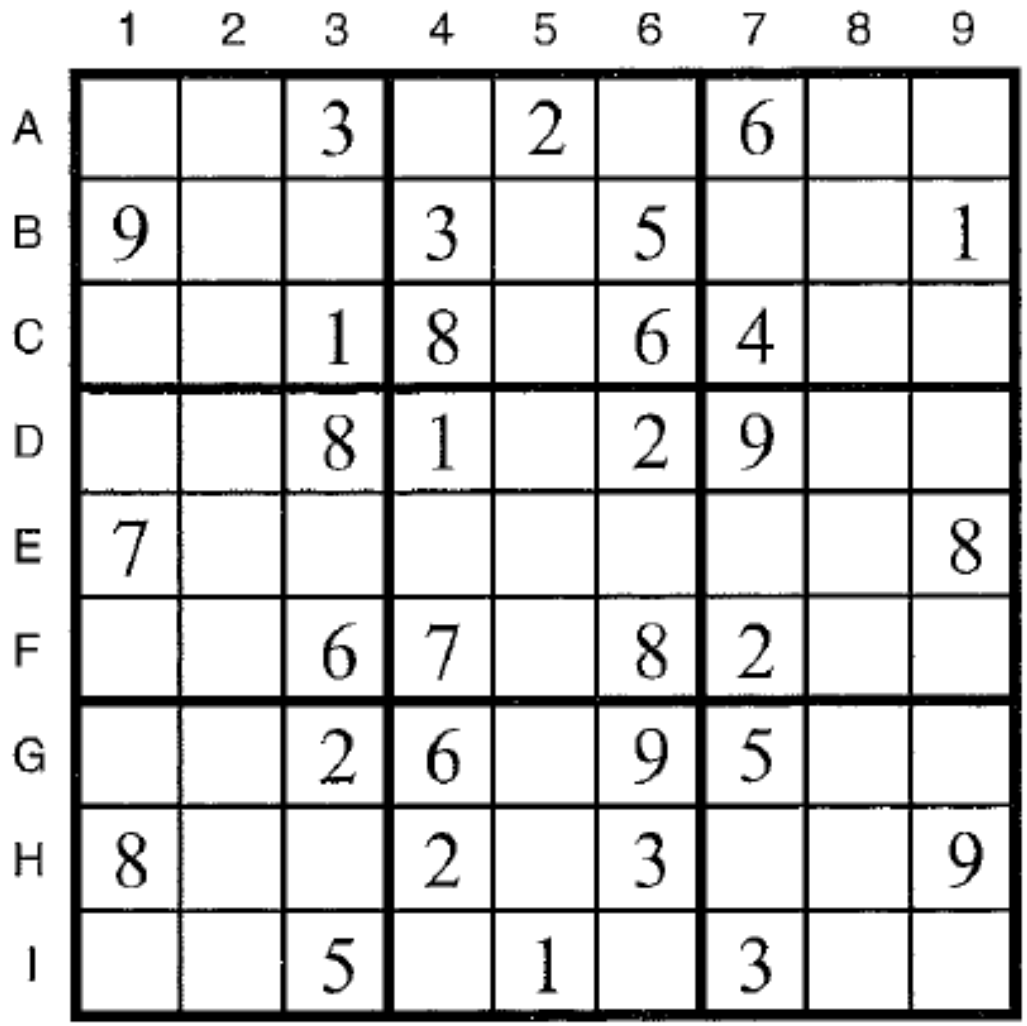
Intelligent Systems Programming

Lecture 7: **Constraint Programming I**

Today's Program

- [10:00-10:55]
  - Constraint Satisfaction Problems (CSPs)
  - Constraint Propagation
- [11:05-12:00]
  - Backtracking
  - Forward Checking
  - Maintaining arc consistency (MAC) algorithm

Constraint Satisfaction Problems (CSPs)



	1	2	3	4	5	6	7	8	9
A			3		2		6		
B	9			3		5			1
C			1	8		6	4		
D			8	1		2	9		
E	7								8
F			6	7		8	2		
G			2	6		9	5		
H	8			2		3			9
I			5		1		3		

Constraint Satisfaction Problems (CSPs)

Definition of CSPs

A CSP is a triple $\langle X,D,C \rangle$, where:

$X = \{X_1,\ldots,X_n\}$ is a finite set of variables.

$D = \{D_1,\ldots,D_n\}$ is a set of domains of possible values for each variable, where $D_i = \{v_1,\ldots,v_{ki}\}$

$C = \{C_1,\ldots,C_m\}$ is a set of constraints, where $C_i = \langle$scope, relation$\rangle$

e.g. $X_1 \in \{A,B\}$, $X_2 \in \{A,B\}$

Implicit constraint representation: $\langle (X_1,X_2), X_1 \neq X_2 \rangle$

Explicit constraint representation: $\langle (X_1,X_2), [(A,B), (B,A)] \rangle$

Typical short notation: $X_1 \neq X_2$

CSP Solutions

- Partial Assignment : Values assigned to only some of the variables.
- Complete Assignment : Each variable has a value assigned.
- Consistent Assignment : Each constraint, where all variables in its scope are assigned, is satisfied.
- Solution : Complete consistent assignment.

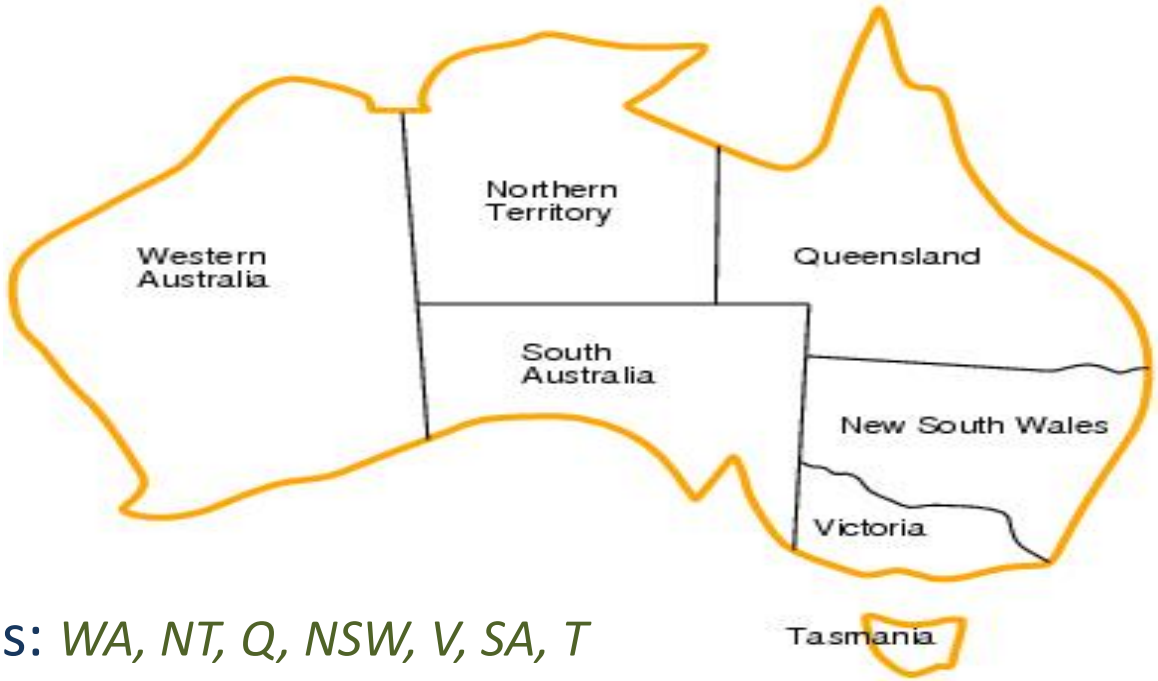
Types of Constraints

Different Arity

- Unary constraints involve a single variable.
  - e.g. $X \neq 12$
- Binary constraints involve pairs of variables.
  - e.g. $X > Y$, $P = MIB \implies C = Black$
- Global constraints involve arbitrary number of variables.
  - e.g. *AllDifferent*

Today we assume constraints to be unary or binary!

CSP Example: Map Coloring

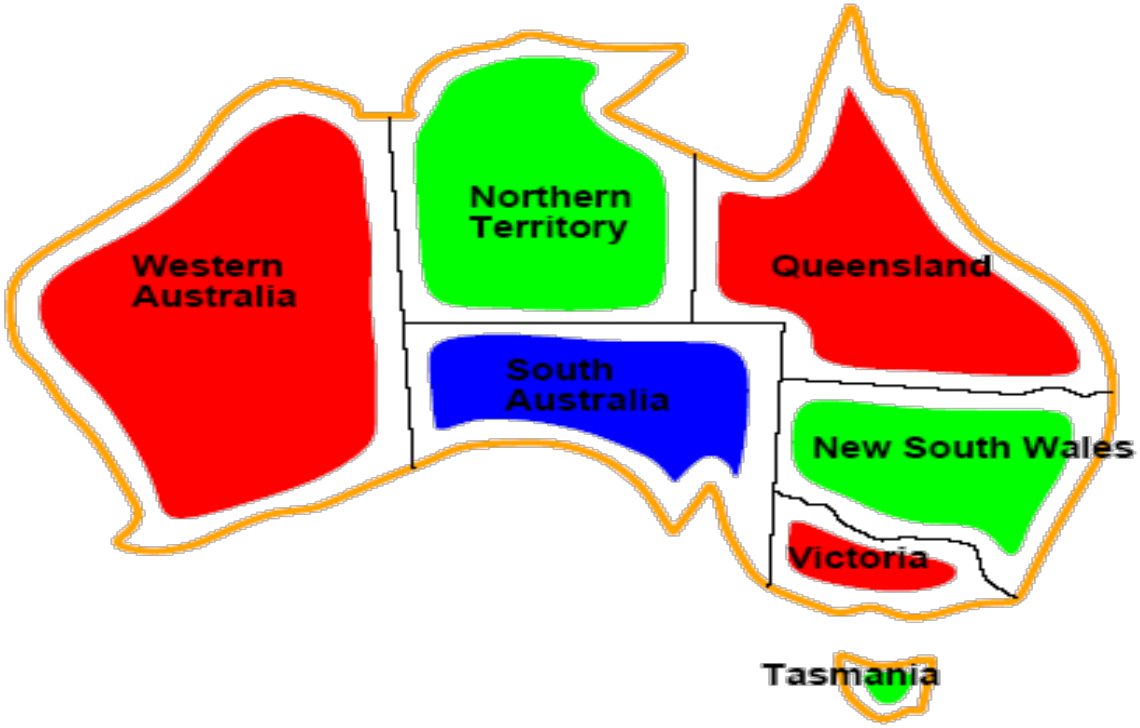


- Variables: *WA, NT, Q, NSW, V, SA, T*
- Domains: D_i = {*red, green, blue*}
- Constraints: adjacent regions must have different colors.

e.g. <(*WA*,*NT*), *WA* ≠ *NT*>

<(*WA*,*NT*) , [(*red, green*), (*red, blue*), (*green, red*), …]>

CSP Example: Map Coloring

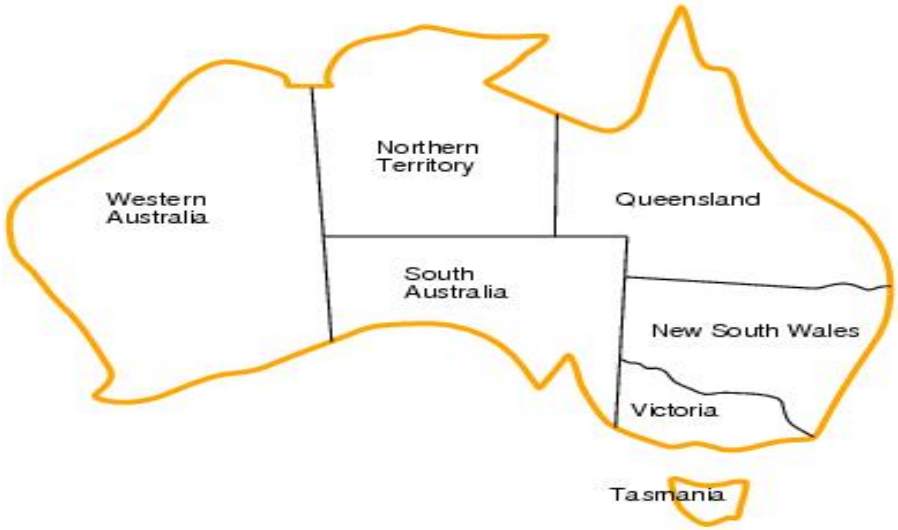


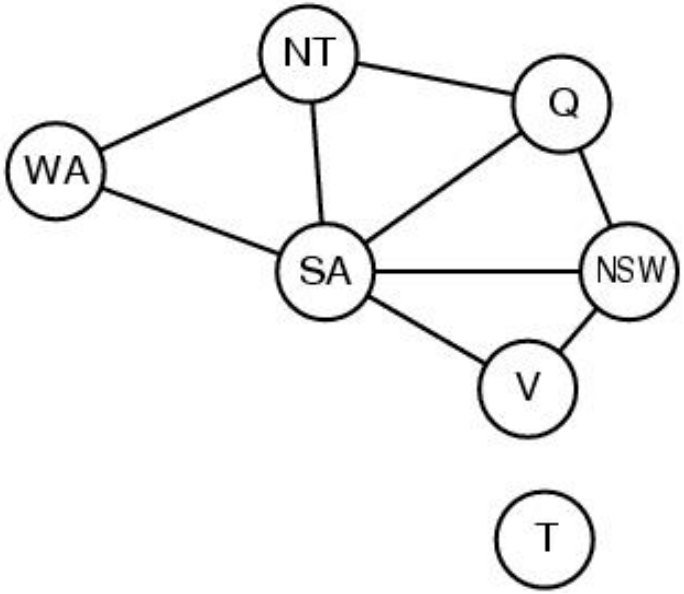
- Solutions are assignments satisfying all constraints, e.g.

  {*WA=red, NT=green, Q=red, NSW=green, V=red, SA=blue, T=green*}

Constraint Graph Representation of CSP

- Nodes are variables
- Edges are binary constraints





Constraint Propagation (Rule Inference)

CSPs are solved combining search and constraint propagation

Constraint Propagation

- Node consistency : for every value v_i of variable X, all unary constraints of X are satisfied.

  Example

  $SA \in \{red, green, blue\}, SA \neq \{green\}$

  $SA \in \{red, blue\}$

- Arc consistency: for every $X \rightarrow Y$ arc, every value v_i in X has a support value u_j in Y.

Arc Consistency Examples

1) $SA \in \{red, green, blue\}$, $NT \in \{blue\}$, $SA \neq NT$

$SA \rightarrow NT$ arc-consistent: $SA \in \{red, green\}$

2) $X,Y \in \{0,1,...,9\}$, $Y=X^2$

$X \rightarrow Y$ arc-consistent : $X \in \{0,1,2,3\}$, $Y \in \{0,1,...,9\}$

$Y \rightarrow X$ arc-consistent: $X \in \{0,1,...,9\}$, $Y \in \{0,1,4,9\}$

3) $SA \in \{red, green, blue\}$, $WA \in \{red, green, blue\}$, $SA \neq WA$

$SA \rightarrow WA$ / $WA \rightarrow SA$ arc-consistent: can we prune any values?

Arc Consistency Algorithm AC-3

function AC-3(csp) **returns** false if an inconsistency is found and true otherwise
  inputs: csp, a binary CSP with components (X, D, C)
  local variables: $queue$, a queue of arcs, initially all the arcs in csp

  while $queue$ is not empty **do**
    $(X_i, X_j) \leftarrow$ REMOVE-FIRST($queue$)
    if REVISE(csp, X_i, X_j) **then**
      if size of $D_i = 0$ **then return** $false$
      for each X_k **in** X_i.NEIGHBORS - $\{X_j\}$ **do**
        add (X_k, X_i) to $queue$
  return $true$

Obs: two arcs for each binary constraint!

---

function REVISE(csp, X_i, X_j) **returns** true iff we revise the domain of X_i
  $revised \leftarrow false$
  for each x **in** D_i **do**
    if no value y in D_j allows (x,y) to satisfy the constraint between X_i and X_j **then**
      delete x from D_i
      $revised \leftarrow true$
  return $revised$

Complexity of AC-3

Assume

- n variables,
- at most d values in domains
- c binary constraints

Observations

- An arc can at most be added d times
- An arc can be revised in d^2
- Thus, worst case runtime is $O(cd^3)$

CSP Solving

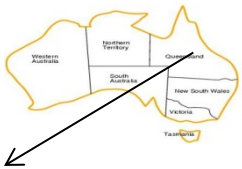
Search in CSP

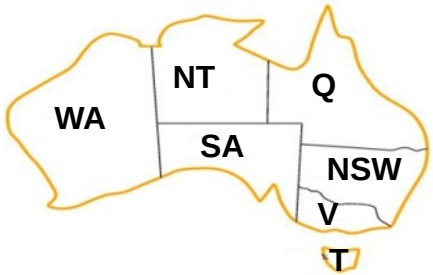
- Inference is not enough
- Apply depth-first search:
  - State: Partial assignment
  - Action: *var* = *value*
- **Complexity**
  - Branching factor b at the top level is $\boldsymbol{nd}$
  - $b = (n-l)d$ at depth l, hence $n!d^n$ leaves
  - But only d^n complete assignments?!

Backtracking Search

- **Insight**:
  - If we assign the first *k* variables to values, it does not matter in what order we did it in.
  - Thus, after choosing which variable to assign in a node, do not change it.
- **Backtracking Algorithm:**
  - Choose values for one variable at the time.
  - Backtrack when a variable has no legal values left.

Backtracking Example





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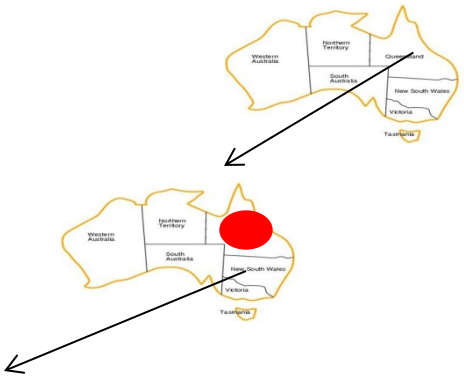

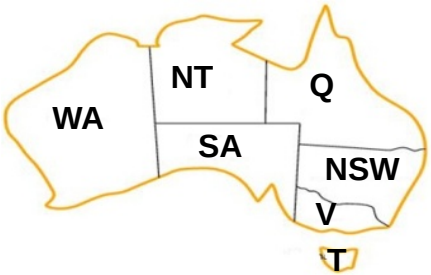

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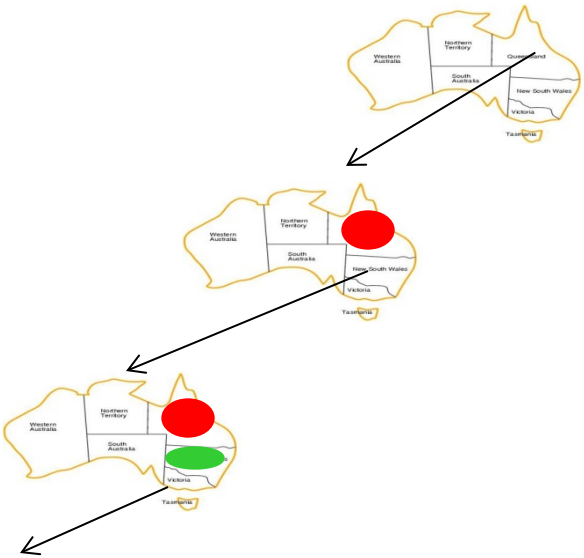

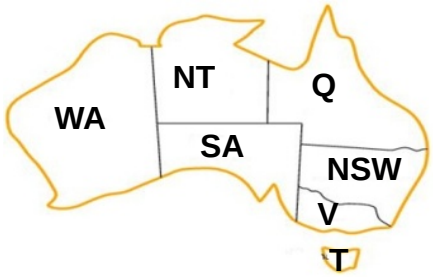

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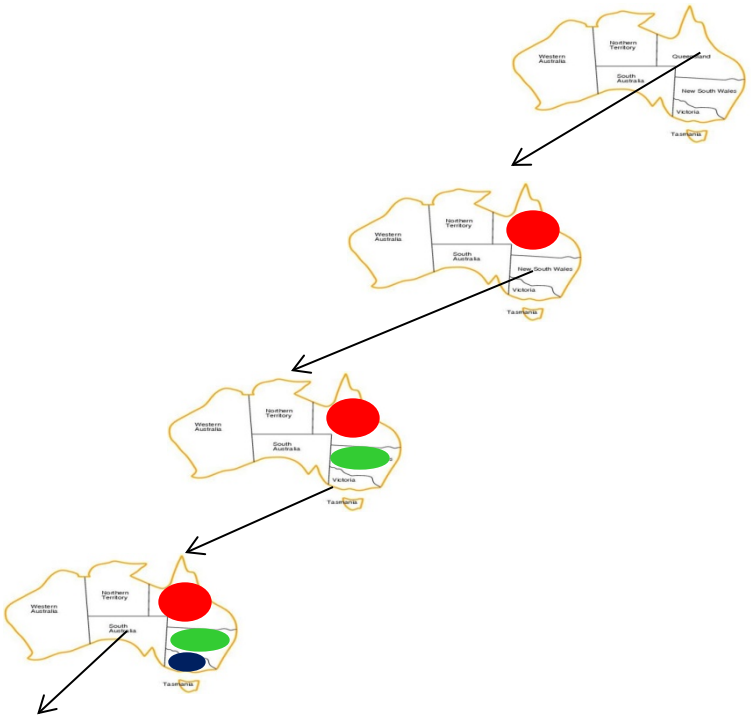

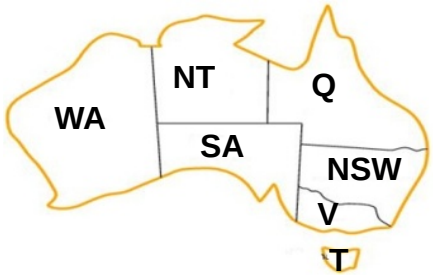

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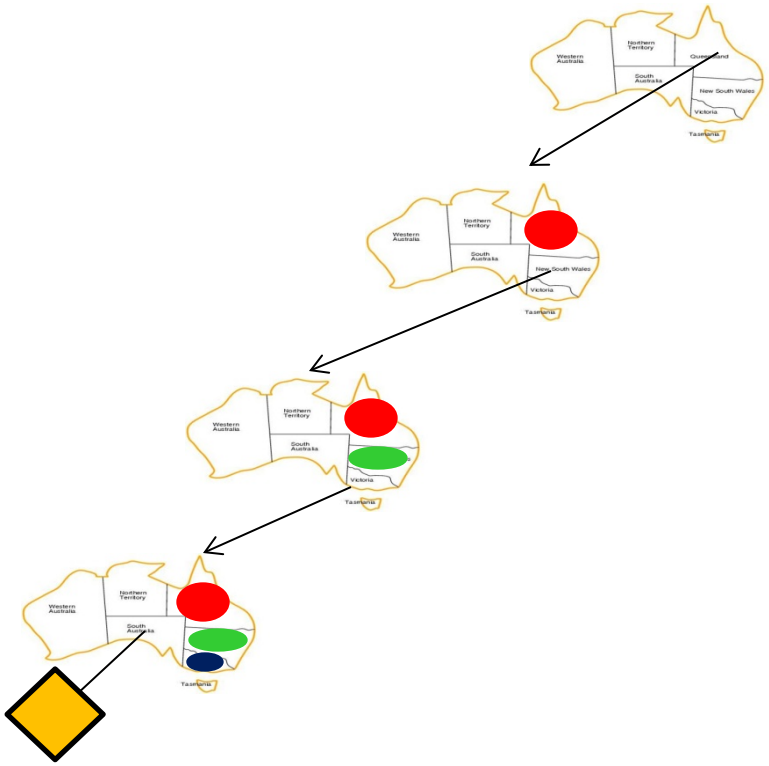

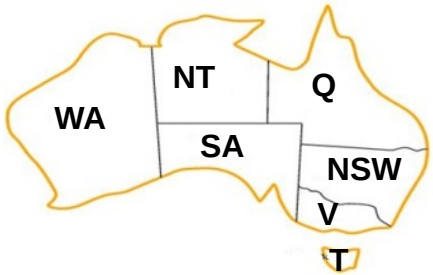

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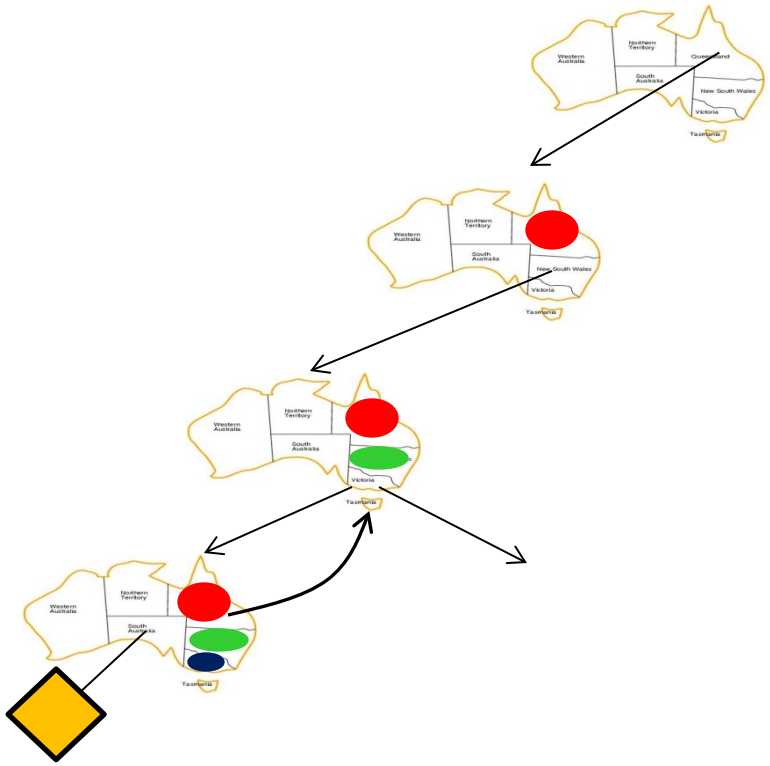

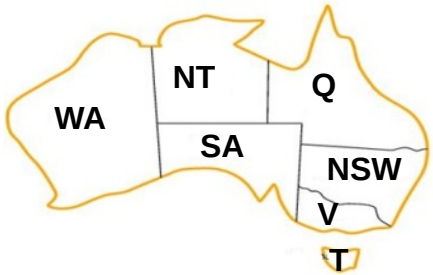

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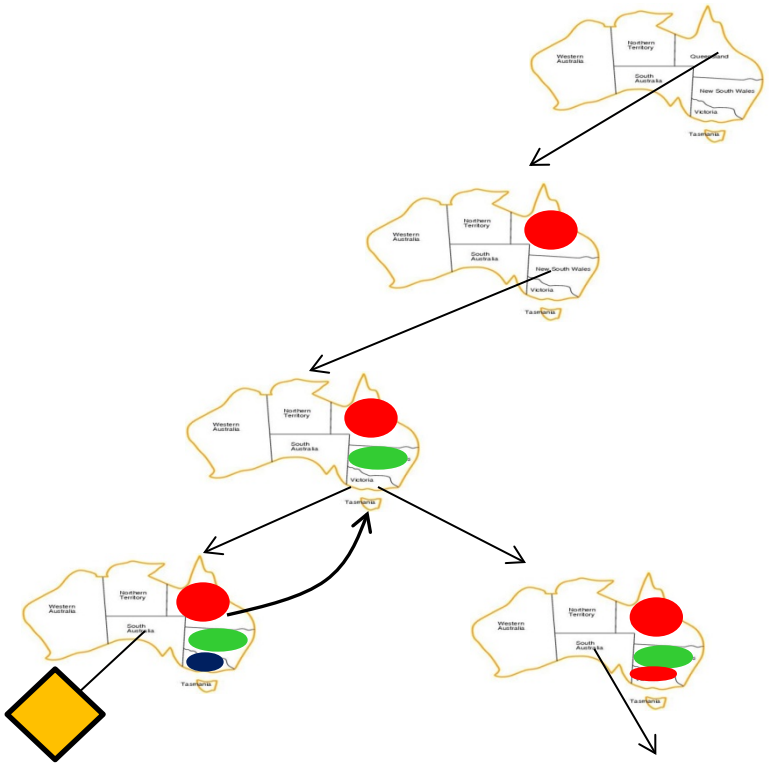

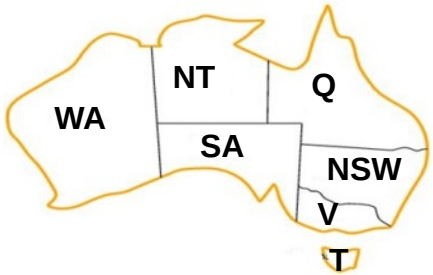

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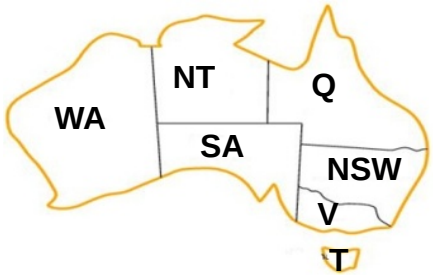

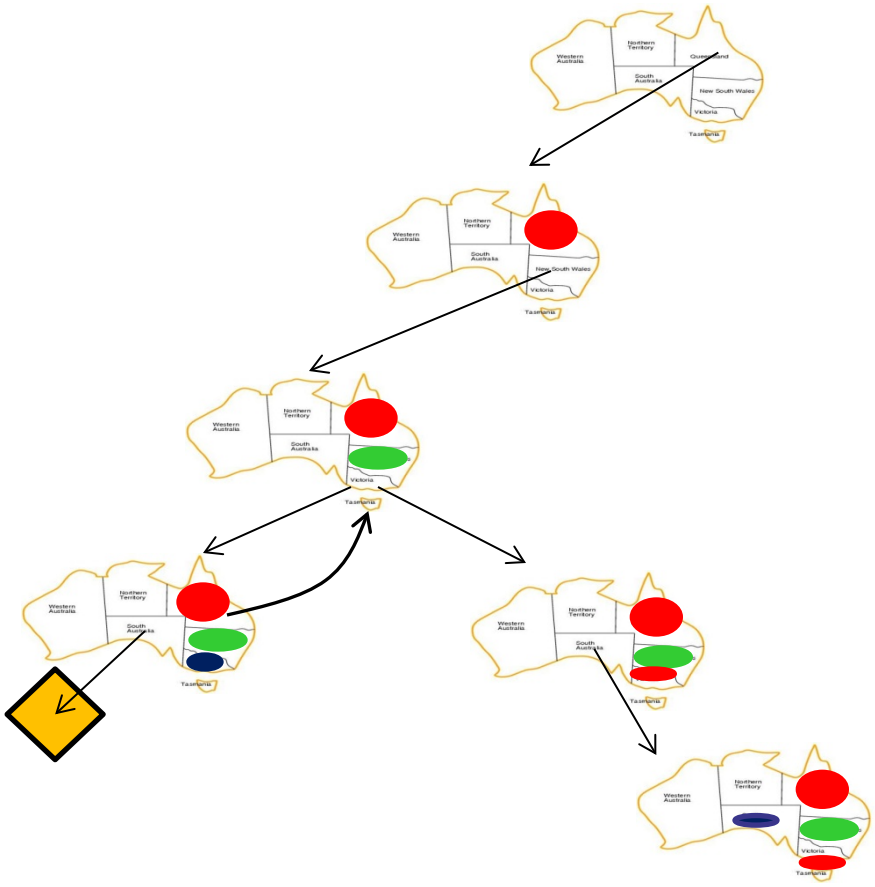

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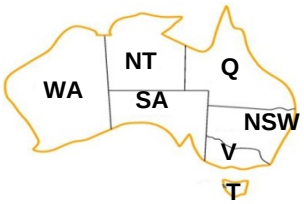
Backtracking Algorithm

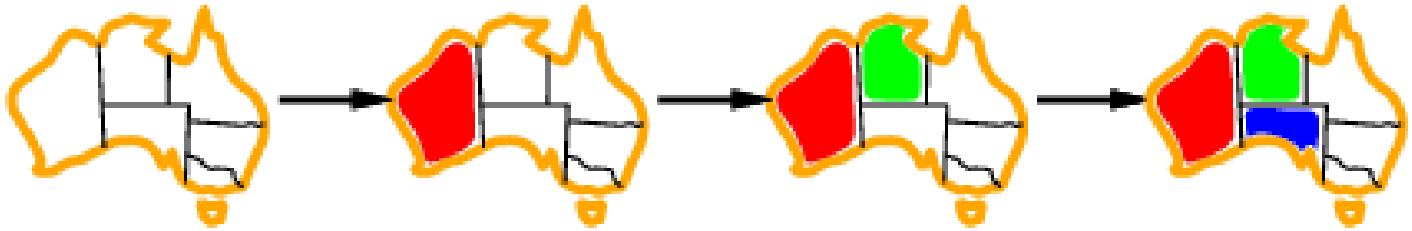
```
function BACKTRACKING-SEARCH(csp) returns a solution, or failure
  return BACKTRACK({ }, csp)

function BACKTRACK(assignment, csp) returns a solution, or failure
  if assignment is complete then return assignment
  var ← SELECT-UNASSIGNED-VARIABLE(csp)
  for each value in ORDER-DOMAIN-VALUES(var, assignment, csp) do
    if value is consistent with assignment then
      add {var = value} to assignment
      inferences ← INFERENCE(csp, var, value)
      if inferences ≠ failure then
        add inferences to assignment
        result ← BACKTRACK(assignment, csp)
        if result ≠ failure then
          return result
    remove {var = value} and inferences from assignment
  return failure
```

Improving Backtracking Efficiency

- General-purpose methods can give huge gains in speed:
  - Which variable should be assigned next? (SELECT-UNASSIGNED-VARIABLE)
  - In what order should its values be tried? (ORDER-DOMAIN-VALUES)
  - What inferences can be performed (INFERENCE)

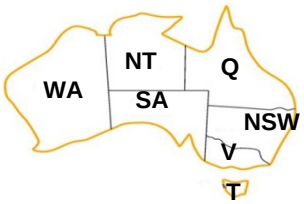
Selecting Variable

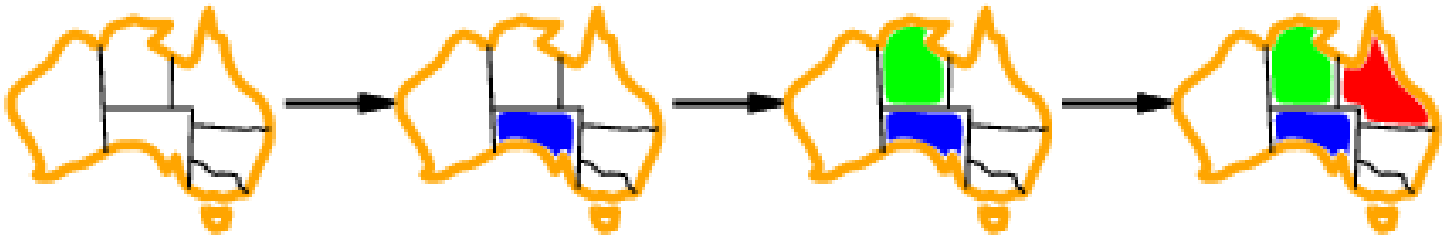




- Minimum remaining values (MRV)
- Rule: choose variable with the **fewest legal values**

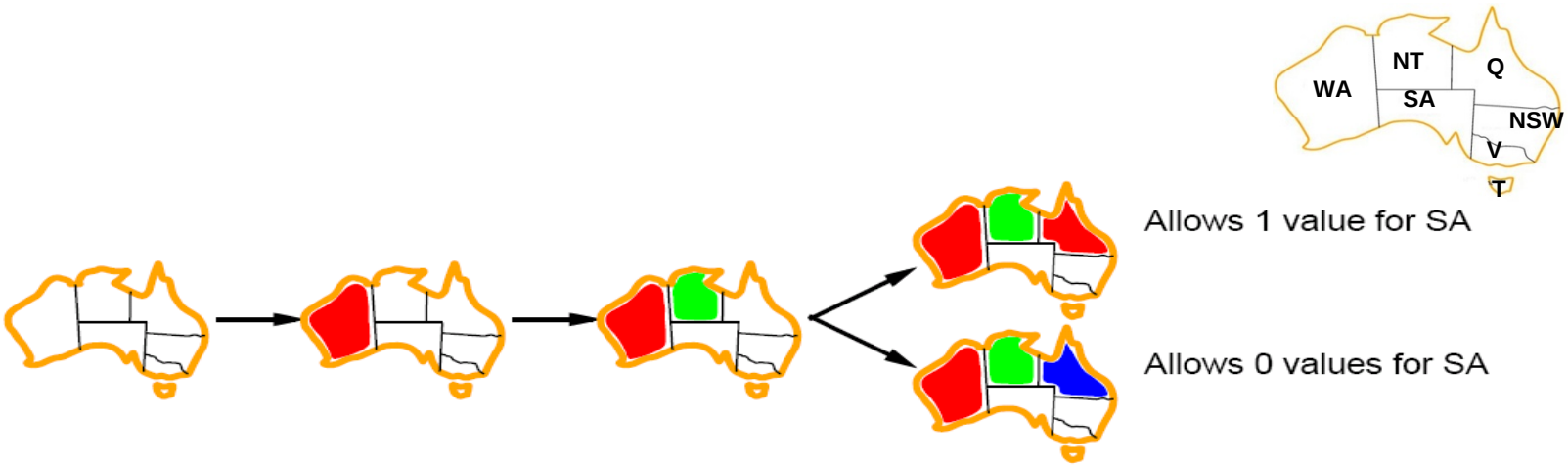
Selecting Variable





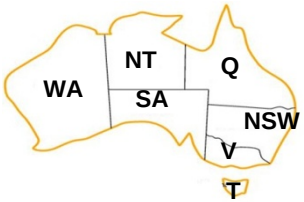
- Degree heuristic
- Rule: select variable that is involved in the largest number of constraints on other unassigned variables.
- Degree heuristic is very useful as a tie breaker

Selecting Value



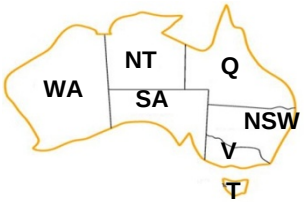
- Least constraining value heuristic
- Rule: given a variable choose the least constraining value i.e., the one that leaves the maximum flexibility for subsequent variable assignments.

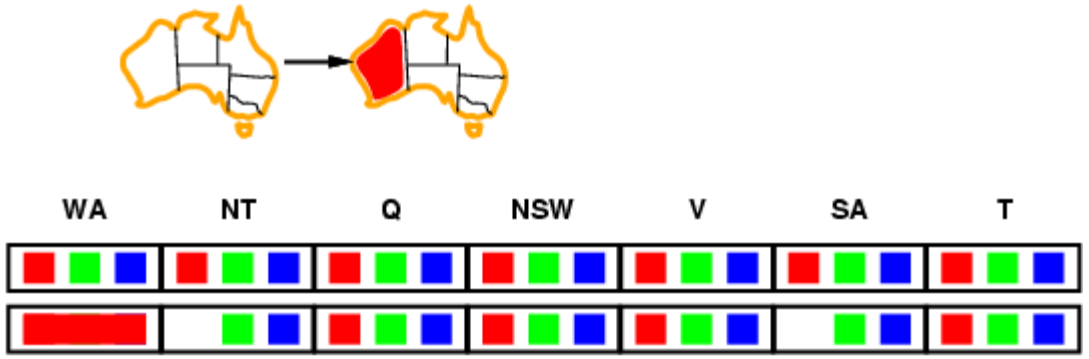
Forward Checking



- Forward checking: Whenever a value v is assigned to a variable X_i, make all variables **consistent** with **this** assignment.
- Terminates search when any variable has no legal values.

Forward Checking





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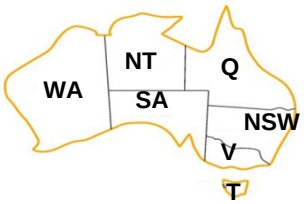

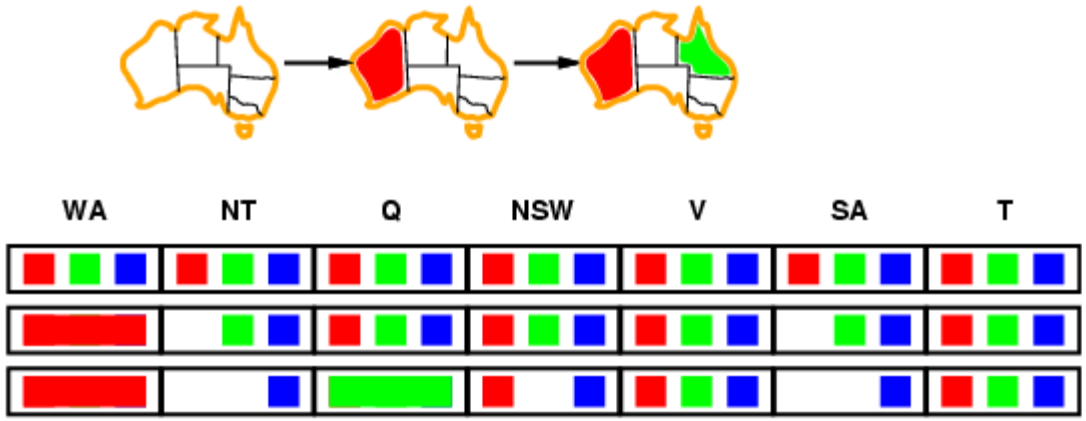

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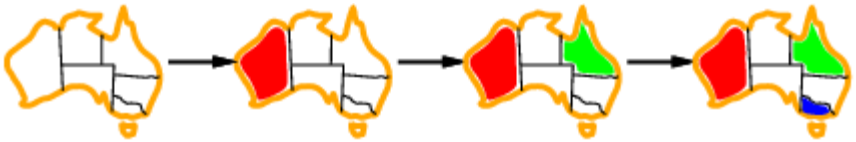
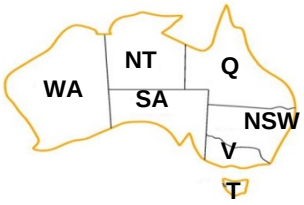




- Forward checking: Whenever a value v is assigned to a variable X_i, make all variables **consistent** with **this** assignment.
- Terminates search when any variable has no legal values.

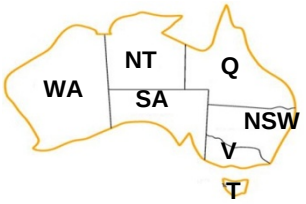
Forward Checking Algorithm

function FORWARD-CHECKING-SEARCH(*csp*) **returns** a solution or failure
 return RECURSIVE-FORWARD-CHECKING({},*csp*)

function RECURSIVE-FORWARD-CHECKING(*assignment*,*csp*) **returns** a solution or failure
 if *assigment* is complete **then return** *assigment*
  var ←SELECT-UNASSIGNED-VARIABLE(VARIABLES[*csp*],*assigment*,*csp*)
  for each *value* **in** ORDER-DOMAIN-VALUES(*var, assigment, csp*) **do**
   if *value* is consistent with *assigment* **do**
    add{*var=value*} to *assigment*
    ***inferences* ← remove all domain values of remaining variables inconsistent with {var = value}**
    if *inferences≠failure* **then**
     add *inferences* to *assigment* and update *csp*
     result←RECURSIVE-FORWARD-CHECKING(*assigment, csp*)
     if *result≠failure* **then**
      result result
   remove {*var=value*} and *inferences* from *assigment* and *csp*
 return *failure*

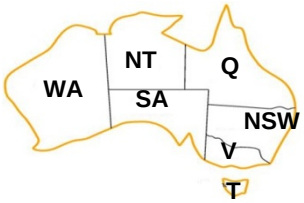
Forward Checking

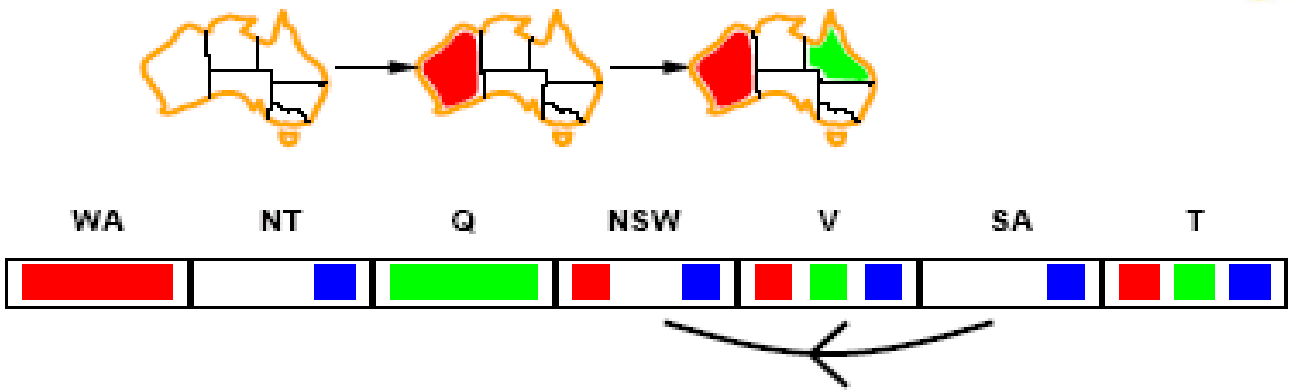
- **Forward checking** propagates information from assigned to unassigned variables, but doesn't provide early detection for all failures:



NT and SA cannot both be blue!

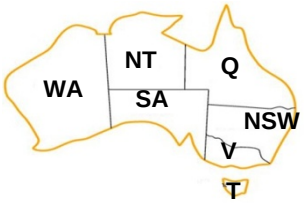
Arc Consistency

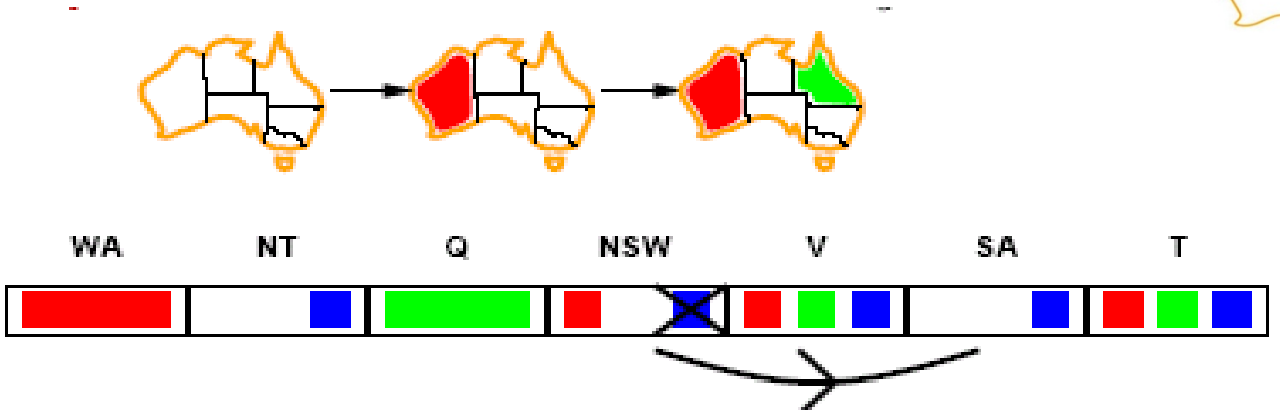




- $X \rightarrow Y$ is consistent iff for every value u_i of X there is some allowed v_j value in Y

Arc Consistency





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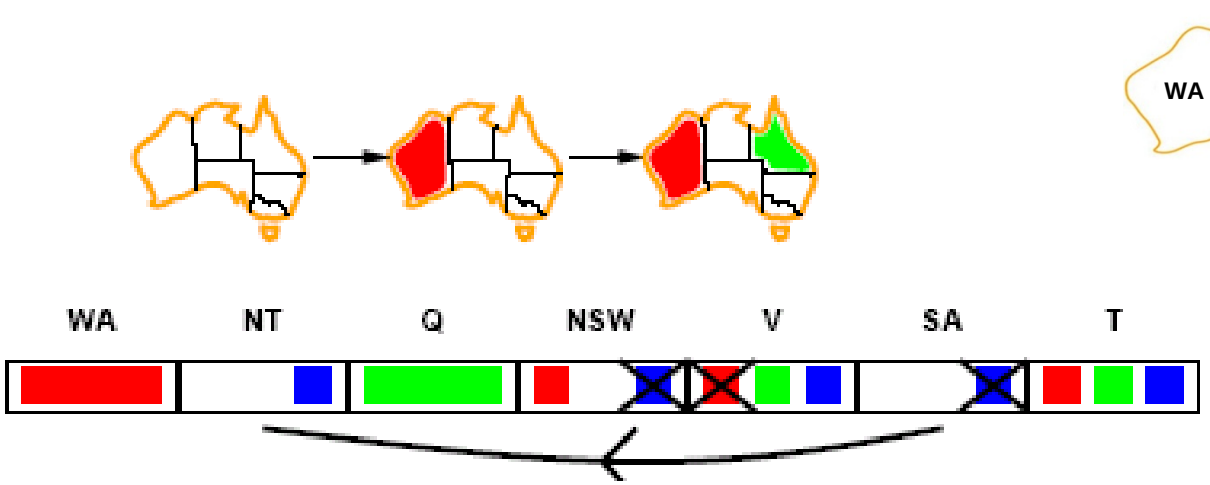
Arc Consistency



- $X \rightarrow Y$ is consistent iff for every value u_i of X there is some allowed v_j value in Y.
- Arc consistency detects failure earlier than FC

MAC

function MAC-SEARCH(*csp*) **returns** a solution or failure
  run **AC-3(csp)**
  return RECURSIVE-MAC({},*csp*)

function RECURSIVE-MAC(*assignment*,*csp*) **returns** a solution or failure
  if *assigment* is complete **then return** *assigment*
    var ←SELECT-UNASSIGNED-VARIABLE(VARIABLES[*csp*],*assigment*,*csp*)
    for each *value* **in** ORDER-DOMAIN-VALUES(*var, assigment, csp*) **do**
      if *value* is consisten with *assigment* **do**
        add{*var*=*value*} to *assigment*
        ***inferences* ←AC-3(csp)**
        if *inferences*≠*failure* **then**
          add *inferences* to *assigment* and update *csp*
          result←RECURSIVE-MAC(*assigment, csp*)
          if *result*≠*failure* **then**
            result result
        remove {*var*=*value*} and *inferences* from *assigment* and *csp*
  return *failure*