Solutions Lecture 12
Intelligent System Programming (ISP)

Exercise 1 (adapted from C83 3.9)

First Phase

Aux. problem

Maximize $-x_0$

Subject to

$x_1 - x_2 - x_0 \leq -1$
$-x_1 - x_2 - x_0 \leq -3$
$2x_1 + x_2 - x_0 \leq 4$

Pre-Initial dictionary

$x_3 = -1 - x_1 + x_2 + x_0$
$x_4 = -3 + x_1 + x_2 + x_0$ Pivot out most negative variable to get feasible dictionary
$x_5 = 4 - 2x_1 - x_2 + x_0$
$w = \qquad\qquad - x_0$

Initial dictionary: increase x_2 to 4

	Max increase of x_2
$x_3 = -1 - x_1 + x_2 + (3 - x_1 - x_2 + x_4)$	
$\quad = 2 - 2x_1 + x_4$	1
$x_0 = 3 - x_1 - x_2 + x_4$	3
$x_5 = 4 - 2x_1 - x_2 + (3 - x_1 - x_2 + x_4)$	
$\quad = 7 - 3x_1 - 2x_2 + x_4$	3.5
$w = -(3 - x_1 - x_2 + x_4)$	
$\quad = -3 + x_1 + x_2 - x_4$	

First dictionary: increase x_2 to 3 (we choose x_2 rather than x_1 since this makes x_0 non-basic)

$x_3 = 2 - 2x_1 + x_4$
$x_2 = 3 - x_1 + x_4 - x_0$
$x_5 = 7 - 3x_1 - 2(3 - x_1 + x_4 - x_0) + x_4$
$\quad = 1 - x_1 - x_4 + 2x_0$
$w = -3 + x_1 + (3 - x_1 + x_4 - x_0) - x_4$
$\quad = -x_0$ (not suprising)

First phase over, remove x_0 and express z in terms the other non-basic variables x_1 and x_4

Second Phase

Initial dictionary

	Max increase of x_4
$x_3 = 2 - 2x_1 + x_4$	inf
$x_2 = 3 - x_1 + x_4$	inf
$x_5 = 1 - x_1 - x_4$	1
$z = 3x_1 + x_2$	
$\quad = 3x_1 + (3 - x_1 + x_4)$	
$\quad = 3 + 2x_1 + x_4$	

Obs.: The text-book often chooses the variable in the z expression with largest coefficient. This would be x_1. But you are allowed to choose any variable with positive coefficient (in particular if this leads to simpler computations for you). In this case, we choose x_4.

First dictionary

	Max increase of x_1
$x_3 = 2 - 2x_1 + (1 - x_1 - x_5)$	
$\quad = 3 - 3x_1 - x_5$	1
$x_2 = 3 - x_1 + (1 - x_1 - x_5)$	
$\quad = 4 - 2x_1 - x_5$	2
$x_4 = 1 - x_1 - x_5$	1
$z = 3 + 2x_1 + (1 - x_1 - x_5)$	
$\quad = 4 + x_1 - x_5$	

Second dictionary (optimal)

$x_3 = 3 - 3(1 - x_4 - x_5) - x_5$
$\quad = 3x_4 + 2x_5$
$x_2 = 4 - 2(1 - x_4 - x_5) - x_5$
$\quad = 2 + 2x_4 + x_5$
$x_1 = 1 - x_4 - x_5$
$z = 4 + (1 - x_4 - x_5) - x_5$
$\quad = 5 - x_4 - 2x_5$ (error in solutions in C83!)

Exercise 2

There are many possible solutions. Below we show one of them

Standard Form

Maximize $x_1 + x_2$

Subject to

$x_1 + x_2 \leq 0$
$x_1 , x_2 \geq 0$

Slack Form (initial dictionary)

	Max increase of x_1
$x_3 = 0 - x_1 - x_2$	0
$z = x_1 + x_2$	

First dictionary (degenerate pivot since no change of basic solution)

$x_1 = 0 - x_2 - x_3$
$z = (0 - x_2 - x_3) + x_2$
$= 0 - x_3$

Exercise 3 (adopted from C83 3.2)

Initial dictionary

	Maximum increase of x_2
$x_5 = 0 + 2x_1 + 9x_2 - x_3 - 9x_4$	inf
$x_6 = 0 - 1/3x_1 - x_2 + 1/3x_3 + 2x_4$	0
$z = 2x_1 + 3x_2 - x_3 - 12x_4$	

1. dictionary

	Maximum increase of x_1
$x_5 = 0 + 2x_1 + 9(- 1/3x_1 + 1/3x_3 + 2x_4 - x_6) - x_3 - 9x_4$	0 (preferred due to tie breaking rule)
$= 0 - 1/3x_1 + 1/3x_3 + 2x_4 - x_6$	
$x_2 = 0 - 1/3x_1 + 1/3x_3 + 2x_4 - x_6$	0
$z = 2x_1 + 3(- 1/3x_1 + 1/3x_3 + 2x_4 - x_6) - x_3 - 12x_4$	
$= x_1 - 6x_4 - 3x_6$	

2. dictionary

	Maximum increase of x_4
$x_1 = 0 + 2x_3 + 9x_4 - x_5 - 9x_6$	inf
$x_2 = 0 - 1/3(2x_3 + 9x_4 - x_5 - 9x_6) + 1/3x_3 + 2x_4 - x_6$	
$\quad 0 - 1/3x_3 - x_4 + 1/3x_5 + 2x_6$	0
$z = (2x_3 + 9x_4 - x_5 - 9x_6) - 6x_4 - 3x_6$	
$\quad = 2x_3 + 3x_4 - x_5 - 12x_6$	

3. dictionary

	Maximum increase of x_3
$x_1 = 0 + 2x_3 + 9(-x_2 - 1/3x_3 + 1/3x_5 + 2x_6) - x_5 - 9x_6$	
$\quad = 0 - 9x_2 - x_3 + 2x_5 + 9x_6$	0 (preferred due to tie breaking rule)
$x_4 = 0 - x_2 - 1/3x_3 + 1/3x_5 + 2x_6$	0
$z = 2x_3 + 3(-x_2 - 1/3x_3 + 1/3x_5 + 2x_6) - x_5 - 12x_6$	
$\quad = -3x_2 + x_3 - 6x_6$	

4. dictionary

	Maximum increase of x_6
$x_3 = 0 - x_1 + 9x_2 + 2x_5 + 9x_6$	inf
$x_4 = 0 - x_2 - 1/3(-x_1 + 9x_2 + 2x_5 + 9x_6) + 1/3x_5 + 2x_6$	
$\quad = 0 + 1/3x_1 + 2x_2 - 1/3x_5 - x_6$	0
$z = -3x_2 + (-x_1 + 9x_2 + 2x_5 + 9x_6) - 6x_6$	
$\quad = -x_1 - 12x_2 + 2x_5 + 3x_6$	

5. dictionary

	Maximum increase of x_5
$x_3 = 0 - x_1 + 9x_2 + 2x_5 + 9(1/3x_1 + 2x_2 - x_4 - 1/3x_5)$	
$\quad = 0 + 2x_1 + 9x_2 - 9x_4 - x_5$	0 (preferred due to tie breaking rule)
$x_6 = 0 + 1/3x_1 + 2x_2 - x_4 - 1/3x_5$	0
$z = -x_1 - 12x_2 + 2x_5 + 3(1/3x_1 + 2x_2 - x_4 - 1/3x_5)$	
$\quad = -6x_2 - 3x_4 + x_5$	

6. dictionary (identical to initial dictionary, thus cycle found)

$x_5 = 0 + 2x_1 + 9x_2 - x_3 - 9x_4$
$x_6 = 0 + 1/3x_1 + 2x_2 - x_4 - 1/3(2x_1 + 9x_2 - x_3 - 9x_4)$
$= 0 - 1/3x_1 - x_2 + 1/3x_3 + 2x_4$
$z = - 6x_2 - 3x_4 + (2x_1 + 9x_2 - x_3 - 9x_4)$
$= 2x_1 + 3x_2 - x_3 - 12x_4$

Exercise 4

No, the simplex algorithm cannot in general find optimal solutions to such problems. The problem is that an optimal solution may only exist in the interior of the polyhedron defined by the constraints. For a linear objective this is never the case which simplex exploits by only exploring the corner points.