Intelligent Systems Programming

Lecture 12: **Linear Programming III**

Duality

Today’s Program

- [10:00-11:05]
  - The dual problem
  - The duality theorem
- [11:15-12:00]
  - Relationship between primal and dual problems
  - Economic interpretation of dual variables

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Lower Bounds from Primal Problem

Maximize $$\sum_{j=1}^{n} c_j x_j$$

Subject to $$\sum_{j=1}^{n} a_{ij} x_j \leq b_i \qquad (i = 1, \ldots, m)$$

$$x_j \geq 0 \qquad (j = 1, \ldots, n)$$

- Every feasible solution to $x_1, x_2, \ldots, x_n$ defines a lower bound of z^*

Lower Bound Example

$$
\begin{array}{ll}
\text{Maximize} & 4x_1 + x_2 + 5x_3 + 3x_4 \\
\text{Subject to} & x_1 - x_2 - x_3 + 3x_4 \leq 1 \\
& 5x_1 + x_2 + 3x_3 + 8x_4 \leq 55 \\
& -x_1 + 2x_2 + 3x_3 - 5x_4 \leq 3 \\
& x_1, x_2, x_3, x_4 \geq 0
\end{array}
$$

- $x_1 = 1, x_2 = 0, x_3 = 1, x_4 = 0$ is a feasible solution
  Thus: $z^* \geq 4\times1 + 5\times1 = 9$
- Is there a smart way to define upper bounds also?

Upper Bound Example I

Maximize $\quad 4x_1 + x_2 + 5x_3 + 3x_4$

Subject to

$$
\begin{aligned}
x_1 - x_2 - x_3 + 3x_4 &\leq 1 \\
5x_1 + x_2 + 3x_3 + 8x_4 &\leq 55 \\
-x_1 + 2x_2 + 3x_3 - 5x_4 &\leq 3 \\
x_1, x_2, x_3, x_4 &\geq 0
\end{aligned}
$$

Multiply 2. constraint by 5/3:

$$\frac{5}{3}(5x_1 + x_2 + 3x_3 + 8x_4 \leq 55) \quad = \quad \frac{25}{3}x_1 + \frac{5}{3}x_2 + 5x_3 + \frac{40}{3}x_4 \leq \frac{275}{3}$$

Upper Bound Example I

Maximize $\quad 4x_1 + x_2 + 5x_3 + 3x_4$

Subject to

$$\begin{aligned} x_1 - x_2 - x_3 + 3x_4 &\leq 1 \\ 5x_1 + x_2 + 3x_3 + 8x_4 &\leq 55 \\ -x_1 + 2x_2 + 3x_3 - 5x_4 &\leq 3 \\ x_1, x_2, x_3, x_4 &\geq 0 \end{aligned}$$

Multiply 2. constraint by 5/3:

$$\frac{5}{3}(5x_1 + x_2 + 3x_3 + 8x_4 \leq 55) \quad = \quad \frac{25}{3}x_1 + \frac{5}{3}x_2 + 5x_3 + \frac{40}{3}x_4 \leq \frac{275}{3}$$

We have:

$$\begin{array}{ccccccccc} 4x_1 & + & x_2 & + & 5x_3 & + & 3x_4 & \leq & \\ \leq & & \leq & & \leq & & \leq & & \\ \frac{25}{3}x_1 & + & \frac{5}{3}x_2 & + & 5x_3 & + & \frac{40}{3}x_4 & \leq & \frac{275}{3} \end{array}$$

Thus:

$$z^* \leq \frac{275}{3} = 91.6$$

Upper Bound Example II

Maximize $4x_1 + x_2 + 5x_3 + 3x_4$

Subject to

$$\begin{aligned} x_1 - x_2 - x_3 + 3x_4 &\le 1 \\ 5x_1 + x_2 + 3x_3 + 8x_4 &\le 55 \\ -x_1 + 2x_2 + 3x_3 - 5x_4 &\le 3 \\ x_1, x_2, x_3, x_4 &\ge 0 \end{aligned}$$

Sum 2. and 3. constraint:

$$(5-1)x_1 + (1+2)x_2 + (3+3)x_3 + (8-5)x_4 \le (55+3) \;=\; 4x_1 + 3x_2 + 6x_3 + 3x_4 \le 58$$

We have:

$$\begin{array}{ccccccccc} 4x_1 & + & x_2 & + & 5x_3 & + & 3x_4 & \le & \\ \le & & \le & & \le & & \le & & \\ 4x_1 & + & 3x_2 & + & 6x_3 & + & 3x_4 & \le & 58 \end{array}$$

Thus:

$z^* \le 58$

The Dual Problem of Example

- **Conclusion**: any linear combination of the constraints with larger coefficients will do.
- Dual = LP to find minimum upper bound!!

Primal of Example

Maximize

$$4x_1 + x_2 + 5x_3 + 3x_4$$

Subject to

$$\begin{aligned} x_1 - x_2 - x_3 + 3x_4 &\le 1 \\ 5x_1 + x_2 + 3x_3 + 8x_4 &\le 55 \\ -x_1 + 2x_2 + 3x_3 - 5x_4 &\le 3 \\ x_1, x_2, x_3, x_4 &\ge 0 \end{aligned}$$

Dual of Example

Minimize

$$y_1 + 55y_2 + 3y_3$$

Subject to

$$\begin{aligned} y_1 + 5y_2 - y_3 &\ge 4 \\ -y_1 + y_2 + 2y_3 &\ge 1 \\ -y_1 + 3y_2 + 3y_3 &\ge 5 \\ 3y_1 + 8y_2 - 5y_3 &\ge 3 \\ y_1, y_2, y_3 &\ge 0 \end{aligned}$$

The General Dual Problem

Primal Problem

Maximize

$$\sum_{j=1}^{n} c_j x_j$$

Subject to

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i \quad (i = 1, \dots, m)$$

$$x_j \geq 0 \quad (j = 1, \dots, n)$$

Dual Problem

Minimize

$$\sum_{i=1}^{m} b_i y_i$$

Subject to

$$\sum_{i=1}^{m} a_{ij} y_i \geq c_j \quad (j = 1, \dots, n)$$

$$y_i \geq 0 \quad (i = 1, \dots, m)$$

Duals are Upper Bounds (succinct proof)

If $x_1,\ldots,x_n$ is primal feasible and $y_1,\ldots,y_m$ is dual feasible then:

$$\sum_{j=1}^{n} c_j x_j \le \sum_{i=1}^{m} b_i y_i$$

Proof

$$\sum_{j=1}^{n} c_j x_j \le \sum_{j=1}^{n}\left(\sum_{i=1}^{m} a_{ij} y_i\right) x_j = \sum_{i=1}^{m}\left(\sum_{j=1}^{n} a_{ij} x_j\right) y_i \le \sum_{i=1}^{m} b_i y_i$$

$$\sum_{i=1}^{m} a_{ij} y_i \ge c_j$$

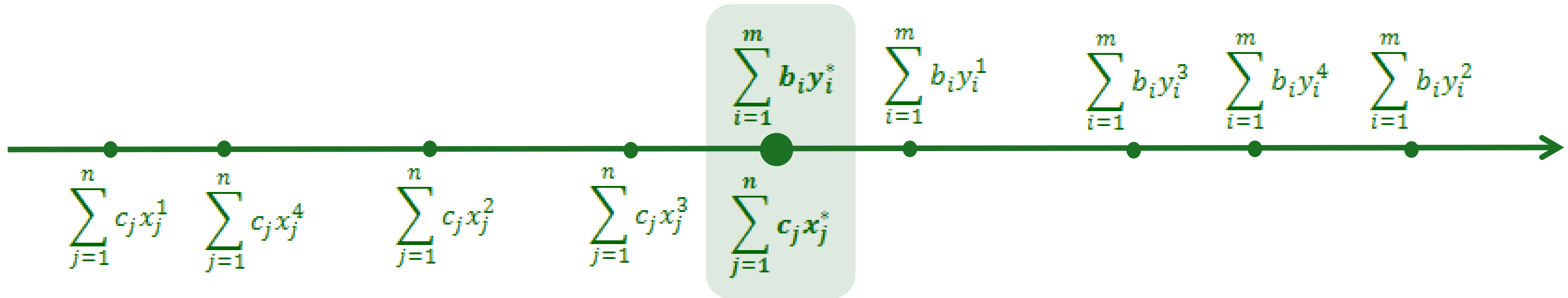
(from dual feasibility)

math

$$\sum_{j=1}^{n} a_{ij} x_j \le b_i$$

(from primal feasibility)

Useful Consequences



- If you stumble over a primal and a dual feasible solution with same objective value, then
- No primal solution with higher objective value and no dual solution with lower objective value can be found, thus
- The primal and dual solutions are optimal!

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The Duality Theorem

The Duality Theorem

If the primal problem has an optimal solution $(x_1^*, x_2^*, \dots, x_n^*)$ then the dual problem has an optimal solution $(y_1^*, y_2^*, \dots, y_m^*)$ such that

$$\sum_{j=1}^{n} c_j x_j^* = \sum_{i=1}^{m} b_i y_i^*$$

Proof 1/5

Proof

- **Claim**

  For the optimal dictionary, $(y_1^*, y_2^*, \dots, y_m^*)$ is the negative *z*-coefficients of slack variables.

  Thus, if we write

$$z = z^* + \sum_{k=1}^{n+m} \bar{c}_k \, x_k$$

  The claim is

$$y_i^* = -\bar{c}_{n+i} \quad (i = 1, \dots, m)$$

Proof 2/5

- **Example**: Optimal dictionary from lecture 10:

$$
\begin{aligned}
x_3 &= 1 + x_2 - 3x_4 - 2x_6 \\
x_1 &= 2 - 2x_2 - 2x_4 + x_6 \\
x_5 &= 1 + 5x_2 + 2x_4 \\
z &= 13 - 3x_2 - x_4 - x_6
\end{aligned}
$$

- Slack variables x_4, x_5, and x_6
- We have $\bar{c}_4 = -1$, $\bar{c}_5 = 0$, and $\bar{c}_6 = -1$
- Thus $y_1^* = 1$, $y_2^* = 0$, and $y_3^* = 1$

Proof 3/5

1) Set $y_i^* = -\bar{c}_{n+i} \quad (i = 1, \ldots, m)$
2) Show that $(y_1^*, y_2^*, \ldots, y_m^*)$ is optimal and

$$\sum_{j=1}^{n} c_j x_j^* = \sum_{i=1}^{m} b_i y_i^*$$

Proof 3/5

1) Set $y_i^* = -\bar{c}_{n+i} \quad (i = 1, \dots, m)$

2) Show that $(y_1^*, y_2^*, \dots, y_m^*)$ is optimal and

$$\sum_{j=1}^{n} c_j x_j^* = \sum_{i=1}^{m} b_i y_i^*$$

We only need to show that $(y_1^*, y_2^*, \dots, y_m^*)$ is feasible (**A**) and

$$\sum_{j=1}^{n} c_j x_j^* = \sum_{i=1}^{m} b_i y_i^* \quad \textbf{(B)}$$

Why?

Proof 4/5

- By substituting $y_i^* = -\bar{c}_{n+i} \quad (i = 1, \dots, m)$ we get:

$$z = \sum_{j=1}^{n} c_j x_j = z^* + \sum_{k=1}^{n+m} \bar{c}_k\, x_k = z^* + \sum_{j=1}^{n} \bar{c}_j x_j - \sum_{i=1}^{m} y_i^* \underbrace{\left(b_i - \sum_{j=1}^{n} a_{ij} x_j \right)}_{\text{Def. of slack vars.}}$$

which may be rewritten to

$$\sum_{j=1}^{n} c_j x_j = \left(z^* - \sum_{i=1}^{m} b_i y_i^* \right) + \sum_{j=1}^{n} \left(\bar{c}_j + \sum_{i=1}^{m} a_{ij} y_i^* \right) x_j$$

Proof 5/5

- Since

$$\sum_{j=1}^{n} c_j x_j = \left(z^* - \sum_{i=1}^{m} b_i y_i^* \right) + \sum_{j=1}^{n} \left(\bar{c}_j + \sum_{i=1}^{m} a_{ij} y_i^* \right) x_j$$

holds for all values of x_j, we have:

A:

$$c_j = \bar{c}_j + \sum_{i=1}^{m} a_{ij} y_i^* \qquad (j = 1, \dots, n)$$

Thus, since $\bar{c}_k \leq 0 \;\; (k = 1, \dots, n+m)$

$$\sum_{i=1}^{m} a_{ij} y_i^* \geq c_j \quad (j = 1, \dots, n)$$

and

$$y_i^* \geq 0 \quad (i = 1, \dots, m)$$

Proof 5/5

- Since

$$\sum_{j=1}^{n} c_j x_j = \left(z^* - \sum_{i=1}^{m} b_i y_i^*\right) + \sum_{j=1}^{n}\left(\bar{c}_j + \sum_{i=1}^{m} a_{ij} y_i^*\right) x_j$$

holds for all values of x_j, we have:

A:

$$c_j = \bar{c}_j + \sum_{i=1}^{m} a_{ij} y_i^* \qquad (j = 1, \dots, n)$$

Thus, since $\bar{c}_k \leq 0 \ \ (k = 1, \dots, n + m)$

$$\sum_{i=1}^{m} a_{ij} y_i^* \geq c_j \quad (j = 1, \dots, n)$$

and

$$y_i^* \geq 0 \quad (i = 1, \dots, m)$$

B:

$$z^* = \sum_{i=1}^{m} b_i y_i^*$$

Thus

$$\sum_{j=1}^{n} c_j x_j^* = \sum_{i=1}^{m} b_i y_i^*$$

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Dual of Dual = Primal

Dual		Rewritten Dual
Minimize $\sum_{i=1}^{m} b_i y_i$		Maximize $\sum_{i=1}^{m} (-b_i) y_i$
Subject to $\sum_{i=1}^{m} a_{ij} y_i \geq c_j \quad j = (1, \dots, n)$ $y_i \geq 0 \quad i = (1, \dots, m)$	=	Subject to $\sum_{i=1}^{m} (-a_{ij}) y_i \leq (-c_j) \quad j = (1, \dots, n)$ $y_i \geq 0 \quad i = (1, \dots, m)$

Dual of Dual = Primal

Rewritten Dual

Maximize

$$\sum_{i=1}^{m}(-b_i)y_i$$

Subject to

$$\sum_{i=1}^{m}(-a_{ij})y_i \leq (-c_j) \quad j = (1, \dots, n)$$

$$y_i \geq 0 \qquad i = (1, \dots, m)$$

Dual of Dual

Minimize

$$\sum_{j=1}^{n}(-c_j)x_j$$

Subject to

$$\sum_{j=1}^{n}(-a_{ij})x_j \geq (-b_i) \quad i = (1, \dots, m)$$

$$x_j \geq 0 \qquad j = (1, \dots, n)$$

Dual of Dual = Primal

Dual of Dual

Minimize

$$\sum_{j=1}^{n}(-c_j)x_j$$

Subject to

$$\sum_{j=1}^{n}(-a_{ij})x_j \geq (-b_i) \quad i = (1, \dots, m)$$

$$x_j \geq 0 \qquad j = (1, \dots, n)$$

=

Primal

Maximize

$$\sum_{j=1}^{n} c_j x_j$$

Subject to

$$\sum_{j=1}^{n} a_{ij}x_j \leq b_i \qquad i = (1, \dots, m)$$

$$x_j \geq 0 \qquad j = (1, \dots, n)$$

Corollary to Duality Theorem

$$\boxed{\text{Primal has optimal sol.} \Leftrightarrow \text{Dual has optimal sol.}}$$

Proof

$\Rightarrow$: follows directly from duality theorem.

$\Leftarrow$: dual has optimal sol. $\Rightarrow$ (by duality theorem)

dual of dual has optimal sol. $\Rightarrow$

primal has optimal solution.

Unboundedness and Infeasibility

- From

$$\sum_{j=1}^{n} c_j x_j \leq \sum_{i=1}^{m} b_i y_i$$

It follows:

1. Primal unbounded $\Rightarrow$ Dual infeasible
2. Dual unbounded $\Rightarrow$ Primal infeasible

But we can have

- Primal unbounded and Dual unbounded? or
- Primal infeasible and Dual infeasible?

Example: Dual and Primal Infeasible

Primal

Maximize

$$2x_1 - x_2$$

Subject to

$$\begin{aligned} x_1 - x_2 &\leq 1 \\ -x_1 + x_2 &\leq -2 \end{aligned}$$

$$x_1, x_2 \geq 0$$

Dual

Minimize

$$y_1 - 2y_2$$

Subject to

$$\begin{aligned} y_1 - y_2 &\geq 2 \\ -y_1 + y_2 &\geq -1 \end{aligned}$$

$$y_1, y_2 \geq 0$$

Primal-Dual Combinations

		Dual		
		Optimal	Infeasible	Unbounded
Primal	Optimal			
	Infeasible			
	Unbounded			

Practical usage of Dual Problem

- Since the number of simplex iterations $< 3m$ in practice, change to dual if more constraints than variables!
- The optimal primal solution is equal to the dual solution of the optimal dual dictionary

Checking Solutions

- Assume that for an LP problem
  1. ***X**** is claimed to be an optimal primal solution
  2. ***Y**** is claimed to be the corresponding optimal dual solution

- To verify this don't run simplex just:
  1. Check that ***X**** is feasible - Proves feasibility
  2. Check that ***Y**** is dual feasible
  3. Check that $\sum_{j=1}^{n} c_j x_j^* = \sum_{i=1}^{m} b_i y_i^*$

  Proves optimality why?

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Economic Significance of Duals

- Consider an LP in standard form:

Maximize $\sum_{j=1}^{n} c_j x_j$

Subject to $\sum_{j=1}^{n} a_{ij} x_j \leq b_i \quad (i = 1, \ldots, m)$

$$x_j \geq 0 \quad (j = 1, \ldots, n)$$

- How much does objective of optimal solution increase per extra b_i unit?

Economic Significance of Duals

- Consider an LP in standard form:

Maximize $\sum_{j=1}^{n} c_j x_j$

Subject to $\sum_{j=1}^{n} a_{ij} x_j \leq b_i \quad (i = 1, \ldots, m)$

$$x_j \geq 0 \quad (j = 1, \ldots, n)$$

- How much does objective of optimal solution increase per extra b_i unit?
- Answer: y_i^* since $z^* = \sum_{i=1}^{m} b_i y_i^*$

Theorem

If an LP problem on standard form has a non-degenerate basic optimal solution, then there is a positive ε with the following property: if $|t_i| \leq \varepsilon$ for all $i = 1, \dots, m$, then the problem

$$\text{Maximize} \quad \sum_{j=1}^{n} c_j x_j$$

$$\text{Subject to} \quad \sum_{j=1}^{n} a_{ij} x_j \leq b_i + t_i \quad (i = 1, \dots, m)$$

$$x_j \geq 0 \quad (j = 1, \dots, n)$$

has an optimal solution and its optimal value equals

$z^* + \sum_{i=1}^{m} y_i^* t_i$

Example

Standard Form

Maximize

$$x_2$$

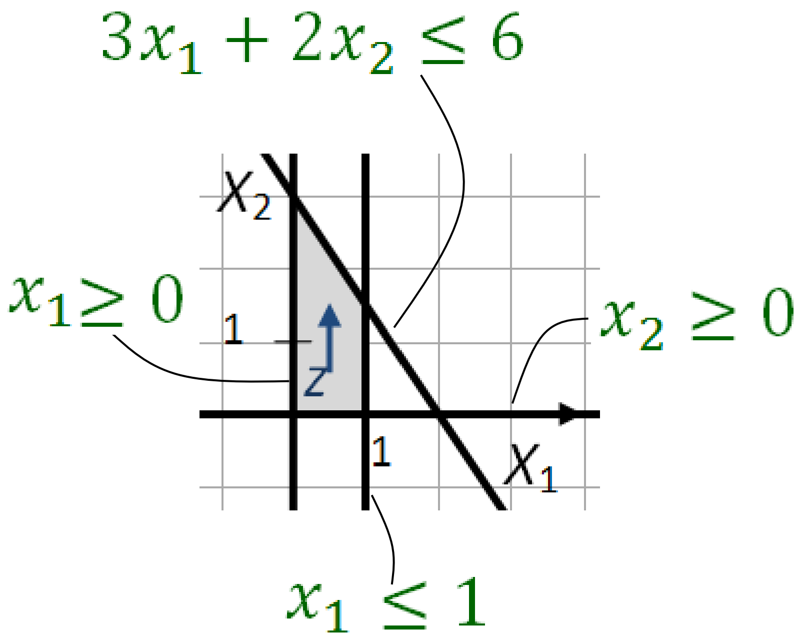
Subject to

$$3x_1 + 2x_2 \leq 6$$

$$x_1 \leq 1$$

$$x_1, x_2 \geq 0$$

Geometric Interpretation



Example

Slack Form

Maximize

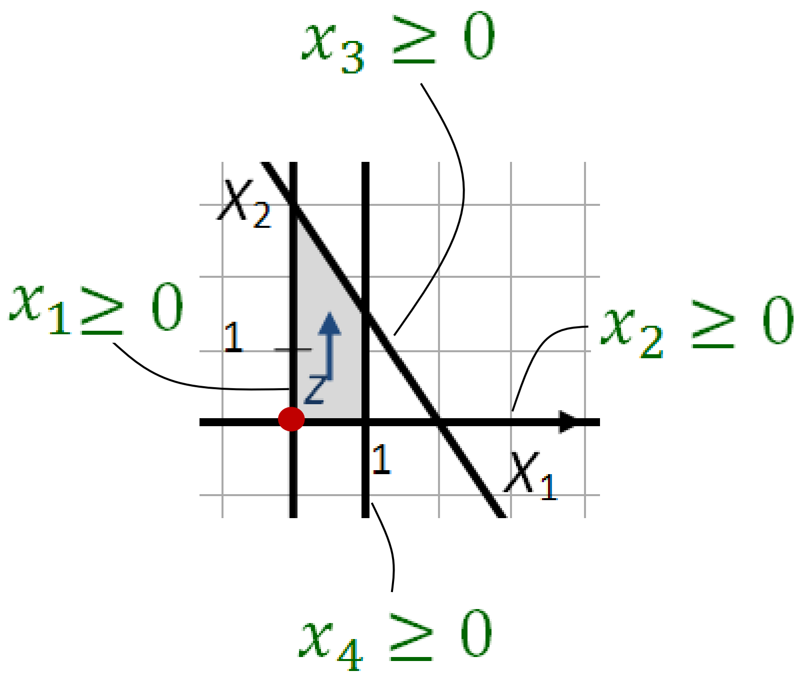
$$z$$

Subject to

$$\begin{aligned} x_3 &= 6 - 3x_1 - 2x_2 \\ x_4 &= 1 - x_1 \\ z &= x_2 \end{aligned}$$

$$x_1, x_2, x_3, x_4 \geq 0$$

Geometric Interpretation



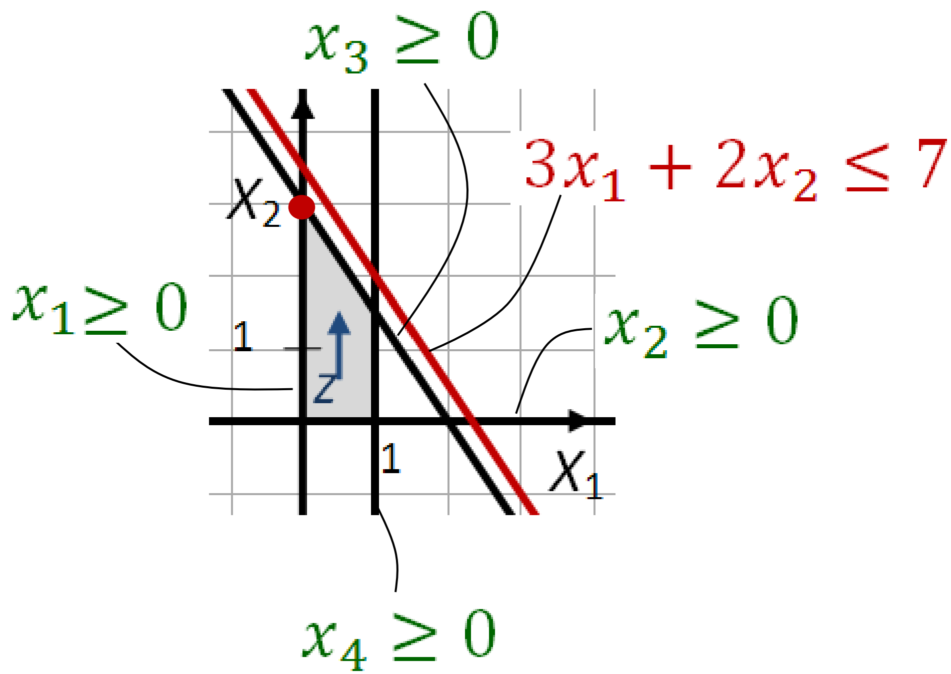
Example

Optimal Dictionary

$$x_2 = 3 - \frac{3}{2}x_1 - \frac{1}{2}x_3$$
$$x_4 = 1 - x_1$$
$$z = 3 - \frac{3}{2}x_1 - \frac{1}{2}x_3$$

$$y_1 = \frac{1}{2}$$
$$y_2 = 0$$

Geometric Interpretation



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