Baby-MAKRO*

DISCLAIMER: WORK-IN-PROGRESS ⇒ BEWARE OF ERRORS!

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Abstract

This note outlines a simplified, »baby«, version of the MAKRO model used by the Danish Ministry of Finance. The model is for a small open economy with a fixed exchange and overlapping generations. The model have perfect foresight, but is full of imperfections due to e.g. frictions in the labor market and adjustment costs. The model is written and solved in terms of a series of ordered blocks. This clarifies the model dynamics and make it easier to solve for fluctuations around the steady state using a numerical equation system solver. Online code is provided for solving the model in Python.

The model is designed so undergraduate students can work with it, and analyze potential extensions in their thesis work. The model structure is similar to state-of-the-art heterogeneous agent models (see this course) and the model is thus relevant for further academic studies. The similarity to the grown-up MAKRO modelmake it relevant for potential future job tasks and the public debate.

The note concludes with a status report.

Online code: github.com/NumEconCopenhagen/BabyMAKRO

MAKRO: See online documentation

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1 Overview

We consider a *small open economy* with a *fixed exchange rate* and *overlapping generations*. Time is discrete, $t \in \{0, 1, \dots\}$ and the frequency is annual.

Households live for up to # periods, and their age is denoted by a. The age dependent mortality is $\zeta_a \in (0, 1)$ and the population and demographic structure is constant. Households exogenously search for jobs and supply labor, receives inheritances and choose consumption and savings to get utility from consumption and bequests.

The *foreign economy* provides a fixed nominal rate of return, sell import goods at fixed prices, and have a demand curve for the domestic export good.

The *production* in the economy is layered as follows:

1. *Production firms* rent *capital* and *labor* to produce the domestic output good.
2. *Repacking firms* combine imported goods with the domestic output good to produce a consumption good, an investment good, and an export good.
3. Capital is rented from a *capital agency*, which purchases the investment good to accumulate capital subject to adjustment costs.
4. Labor is rented from a *labor agency*, which post vacancies a search-and-match labor market to purchase labor from the households.

All firms are price takers. Prices are thus flexible, except for wages which is determined by ad hoc bargaining. All *goods markets clear* and the matching process is determined by a *matching function*.

There is *perfect foresight* in the economy. I.e. the value of all current and future variables are known. This is a strong assumption, and in many ways the model should be considered a first order approximation to a full model with both idiosyncratic and aggregate risk. It can be relevant to introduce model elements, which proxy for the effects of risks. Utility-of-wealth can e.g. proxy for the precautionary saving motive.

1.1 Equilibrium path

The *equilibrium path* in the economy is a set of paths for all variables, which satisfies all accounting identities, optimal firm and household behavior in terms of first order conditions, and implies market clearing. When all variables are constant over time, the equilibrium path is a *steady state*.

In terms of math, the model is just an *equation system* stacking the accounting identifies, first order conditions and market clearing conditions. If the economy is initially out of steady state, we solve for the equilibrium path by truncating the equation system to T periods. The assumption is that the economy has settled down to the steady state well before period T, and we can assume variables from period T onward are at their steady state value. The economy can be out of steady state both because lagged *endogenous* variables are initially not at their steady state values and/or because the *exogenous* variables are not at their steady state values. We talk of an *impulse response* when the economy starts at the steady state, but some exogenous variables *temporarily* deviate from the steady state following some converging auto-regressive process.[1]

We simplify the model and the resulting equation system by writing it in terms of a *ordered series of block*. We start from a set of *exogenous* variables (e.g. variables determined in the foreign economy) and a set of *unknown* variables. Each block then takes in the path of some variables, return the path of other variables, and imply *targets*, which must be zero if the model equations are satisfied. Each block can use the unknown variables and output variables of previous blocks as input variables. In the end we collect all the targets. The number of unknown variables must equal the number of target variables.

To solve the model, we must first find the steady state. As explained in Section 2, this can be done by manually choosing values for a selection of the endogenous variables and the deriving the rest from closed form expressions or solving sub-systems with a numerical equation system solver. Next, we solve for the equilibrium path again using a numerical equation system solver.

The block structure and ordering is not unique. If a different set of unknowns is chosen, a different ordering of blocks must also be chosen. If an additional variable

[1] This is also called an MIT shock. A shock in a model with perfect foresight is to some degree a contradiction in terms. The assumption is that even though the agents experiences a shock, they expect that there will never be a shock again. Accounting for expecting of future shocks is much more complicated. Some realism can be mimicked by studying impulse response to shocks about the future, which when the future comes never materialized as a new opposite signed shock negates it. Multiple shocks arriving sequentially can also be studied.

is considered an *unknown*, an additional equation must be considered a target instead of being used to calculate a output. In the limit, all variables can be considered as unknowns and all equations as targets. This is inefficient as the number of variables can be very large.[2]

1.2 On CES technology

The assumption of CES technology is used repeatedly in the model. It is therefore beneficial to recap it briefly. Consider a firm producing good X using good X_i and X_j with a CES technology. Input prices are P_i and P_j and the output price is P. The firm is a price taker in all markets. The *profit maximization* problem of the firm is

$$\max_{X_i,X_j} PX - P_iX_i - P_jX_j \text{ s.t. } X = \Gamma\left(\mu_i^{\frac{1}{\sigma}}X_i^{\frac{\sigma-1}{\sigma}} + \mu_j^{\frac{1}{\sigma}}X_j^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}}, \ \ \mu_i+\mu_j=1,\ \mu_i,\sigma,\Gamma>0,\ \sigma\neq 1 \tag{1}$$

The generic *first order condition* is

$$\begin{aligned}
0 &= P\mu_i^{\frac{1}{\sigma}}X_i^{\frac{\sigma-1}{\sigma}-1}\Gamma\left(\mu_i^{\frac{1}{\sigma}}X_i^{\frac{\sigma-1}{\sigma}} + \mu_j^{\frac{1}{\sigma}}X_j^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}-1} - P_i \Leftrightarrow \\
P_i &= P\mu_i^{\frac{1}{\sigma}}X_i^{-\frac{1}{\sigma}}\Gamma\left(\mu_i^{\frac{1}{\sigma}}X_i^{\frac{\sigma-1}{\sigma}} + \mu_j^{\frac{1}{\sigma}}X_j^{\frac{\sigma-1}{\sigma}}\right)^{\frac{1}{\sigma-1}} \Leftrightarrow \\
P_i &= P\mu_i^{\frac{1}{\sigma}}X_i^{-\frac{1}{\sigma}}\Gamma^{\frac{\sigma-1}{\sigma}}X^{\frac{1}{\sigma}} \Leftrightarrow \\
X_i &= \mu_i\left(\frac{P}{P_i}\right)^{\sigma}\Gamma^{\sigma-1}X.
\end{aligned} \tag{2}$$

As the production technology has constant return-to-scale, there are infinitely many solutions to the FOCs. They all satisfy that inputs are used in proportion as follows

$$\frac{X_i}{X_j} = \frac{\mu_i}{\mu_j}\left(\frac{P_j}{P_i}\right)^{\sigma} \tag{3}$$

[2] Modeling systems such as GAMS can combined with a state-of-the-art solver such as CONOOPT automatically analyze the structure of the equation system and thereby ripe the benefits we get from manually ordering the blocks.

Assuming *free entry*, and thus *zero profits*, the output price is uniquely determined from the input prices as

$$
\begin{aligned}
0 &= PX - P_i X_i - P_j X_j \Leftrightarrow \\
P &= \frac{P_i X_i + P_j X_j}{X} \\
&= \mu_i \left(\frac{P}{P_i}\right)^{\sigma} \Gamma^{\sigma-1} P_i + \mu_j \left(\frac{P}{P_j}\right)^{\sigma} \Gamma^{\sigma-1} P_j \Leftrightarrow \\
(\Gamma P)^{1-\sigma} &= \mu_i P_i^{1-\sigma} + \mu_j P_j^{1-\sigma} \Leftrightarrow \\
P &= \frac{1}{\Gamma}\left(\mu_i P_i^{1-\sigma} + \mu_j P_j^{1-\sigma}\right)^{\frac{1}{1-\sigma}}.
\end{aligned}
\tag{4}
$$

Blocks

1.3 Exogenous variables

The *exogenous* variables are:

1. $P_t^{M,C}$, import price of *private consumption* component good
2. $P_t^{M,G}$, import price of *public consumption* component good
3. $P_t^{M,I}$, import price of *investment* component good
4. $P_t^{M,X}$, import price of *export* component good
5. P_t^{F} , *foreign* price level
6. χ_t, *foreign* demand shifter
7. G_t, public spending
8. Γ_t, technology

1.4 Unknown variables

The chosen *unknown* variables are:

1. A_t^{death}, wealth of households at $a = \# - 1$ (T unknowns)

2. A_t^q, inheritance flow (T unknowns)

3. L_t, labor supply (T unknowns)

4. K_t, capital (T unknowns)

5. r_t^K, rental price for capital (T unknowns)

6. P_t^Y, price set in Phillips-curve (T unknowns)

7. W_t, wage (T unknowns)

The total number of unknowns thus is $7 \times T$.

1.5 Demographics

The age-specific number of household is N_a, which we normalize to 1 at age $a = 0$. The mortality rate at period a is ζ_a, the potential life-span is # and retirement age is $\#_{work} < \#$.

The demographic structure and population is then given by

$$N_a = \begin{cases} 1 & \text{if } a = 0 \\ (1 - \zeta_{a-1})N_{a-1} & \text{if } a > 0 \end{cases}$$

$$\zeta_a = \begin{cases} 0 & \text{if } a < \#_{work} \\ \left(\frac{a+1-\#_{work}}{\#-\#_{work}}\right)^{\varsigma} & \text{if } a < \# - 1 \\ 1 & \text{if } a = \# - 1 \end{cases}$$

$$N = \sum_{a=0}^{\#-1} N_a$$

$$N^{work} = \sum_{a=0}^{\#_{\text{work}}-1} N_a$$

1.6 Block I. Households - search behavior and matching

Households search for a job and supply labor exogenously. The age-specific job-separation probability is $\delta_a^L \in (0, 1)$. All unemployed search for a job. As an initial condition, we have $L_{-1,t-1} = 0$.

The quantity of searchers is

$$S_{a,t} = \begin{cases} 1 & \text{if } a = 0 \\ (1-\zeta_{a-1})\left[(N_{a-1} - L_{a-1,t-1}) + \delta_a^L L_{a-1,t-1}\right] & \text{if } a < \#_{work} \\ 0 & \text{if } a \geq \#_{work} \end{cases}$$
$$S_t = \sum_a S_{a,t}.$$

The quantity of households with a job *before matching* is

$$\underline{L}_{a,t} = \begin{cases} 0 & \text{if } a = 0 \\ (1-\zeta_{a-1})(1-\delta_a^L)L_{a-1,t-1} & \text{if } a < \#_{work} \\ 0 & \text{if } a \geq \#_{work} \end{cases}$$
$$\underline{L}_t = \sum_a \underline{L}_{a,t}.$$

The aggregate job-separation rate incl. mortality is

$$\delta_t^L = \frac{L_{t-1} - \underline{L}_t}{L_{t-1}}.$$

The quantity of vacancies is v_t and the number of matches, $\mathcal{M}_t$, is given by the *matching function*

$$\mathcal{M}_t = \frac{S_t v_t}{\left(S_t^{\frac{1}{\sigma^m}} + (\nu v_t)^{\frac{1}{\sigma^m}}\right)^{\sigma^m}},$$

where ν is the efficiency of vacancies.

The job-filling rate, m_t^v, and the job-finding rate, m_t^s, are thus

$$m_t^v = \frac{\mathcal{M}_t}{v_t}$$
$$m_t^s = \frac{\mathcal{M}_t}{S_t}.$$

The number of employed therefore is

$$L_{a,t} = \underline{L}_{a,t} + m_t^s S_{a,t}.$$

This implies that the number of unemployed is

$$U_{a,t} = \begin{cases} N_a - L_{a,t} & \text{if } a < \#_{work} \\ 0 & \text{if } a \geq \#_{\text{work}} \end{cases}$$

In equilibrium, the number of matches must equal the number of new hires, i.e.

$$\mathcal{M}_t = L_t - \underline{L}_t.$$

We write the **block** in terms of inputs, and outputs as:

- **Inputs:** $\{L_t\}$
- **Outputs:** $\{S_{a,t}\}$, $\{S_t\}$, $\{\delta_t^L\}$, $\{\mathcal{M}_t\}$,$\{v_t\}$, $\{m_t^v\}$, $\{m_t^s\}$, $\{L_{a,t}\}$, $\{U_{a,t}\}$, $\{U_t\}$

$$S_{a,t} = \begin{cases} 1 & \textbf{if } a = 0 \\ (1-\zeta_{a-1})\left[(N_{a-1} - L_{a-1,t-1}) + \delta^L_a L_{a-1,t-1}\right] & \textbf{if } a < \#_{work} \\ 0 & \textbf{if } a \geq \#_{work} \end{cases} \tag{5}$$

$$\underline{L}_{a,t} = \begin{cases} 0 & \textbf{if } a = 0 \\ (1-\zeta_{a-1})(1-\delta^L_a) L_{a-1,t-1} & \textbf{if } a < \#_{work} \\ 0 & \textbf{if } a \geq \#_{work} \end{cases} \tag{6}$$

$$S_t = \sum_{a=0}^{\#-1} S_{a,t}. \tag{7}$$

$$\underline{L}_t = \sum_{a=0}^{\#-1} \underline{L}_{a,t}. \tag{8}$$

$$\delta^L_t = \frac{L_{t-1} - \underline{L}_t}{L_{t-1}} \tag{9}$$

$$\mathcal{M}_t = L_t - \underline{L}_t \tag{10}$$

$$m^s_t = \frac{\mathcal{M}_t}{S_t}$$

$$v_t = \frac{1}{\nu} \left(\frac{(m^s_t)^{\frac{1}{\sigma^m}} S_t^{\frac{1}{\sigma^m}}}{1 - (m^s_t)^{\frac{1}{\sigma^m}}} \right)^{\sigma^m} \tag{11}$$

$$m^v_t = \frac{\mathcal{M}_t}{v_t} \tag{12}$$

$$L_{a,t} = \underline{L}_{a,t} + m^s_t S_{a,t}. \tag{13}$$

$$U_{a,t} = \begin{cases} N_a - L_{a,t} & \textbf{if } a < \#_{\textbf{work}} \\ 0 & \textbf{if } a \geq \#_{\textbf{work}} \end{cases} \tag{14}$$

$$U_t = \sum_{a=0}^{\#-1} U_{a,t}. \tag{15}$$

For $t = 0$, the variable $L_{a-1,t-1}$ is pre-determined.

1.7 Block II. Labor agency

The labor agency firms post vacancies, v_t, to hire labor L_t. The cost of each vacancy is κ^L in units of labor. The firms can therefore rent out $\ell_t = L_t - \kappa^L v_t$ units of labor to the production firms at the rental price r_t^ℓ.

Labor follows the law-of-motion $L_t = \left(1-\delta_t^L\right) L_{t-1} + m_t^v v_t$, and the wage, W_t, is determined by bargaining. Matching occurs according to the matching function, and the firms take the separation rate, δ_t^L, and the vacancy filling rate, m_t^v as given. Since lagged employment, L_{t-1}, is pre-determined, we consider L_t to be the choice value and derive the required number of vacancies, v_t, and the implied labor for rent, ℓ_t.

The labor agency problem then is:

$$
\begin{aligned}
V_0(L_{t-1}) = \max_{\{L_t\}} \sum_{t=0}^{\infty} \left(1+r^{\text{firm}}\right)^{-t} &\left[r_t^\ell \ell_t - W_t L_t\right] \\
\text{s.t.}& \\
v_t &= \frac{L_t - \left(1-\delta_t^L\right) L_{t-1}}{m_t^v} \\
\ell_t &= L_t - \kappa^L v_t.
\end{aligned}
$$

Using the FOC to L_t from the labor agency problem, we write the **block** as:

- **Inputs:** $\{W_t\}$, $\{m_t^v\}$, $\{\delta_t^L\}$, $\{L_t\}$, $\{v_t\}$
- **Outputs:** $\{r_t^\ell\}$, $\{\ell_t\}$

$$
r_t^\ell = \frac{1}{1-\frac{\kappa^L}{m_t^v}} \left[W_t - r_{t+1}^\ell \frac{1-\delta_{t+1}^L}{1+r^{\text{firm}}} \frac{\kappa^L}{m_{t+1}^v}\right] \tag{16}
$$

$$
\ell_t = L_t - \kappa^L v_t. \tag{17}
$$

The variable L_{-1} is pre-determined.

1.8 Block III. Production firms

The production firms use capital, K_{t-1}, and labor, ℓ_t, to produce output, Y_t, with a CES technology. The rental price of capital is r_t^K and the rental price of labor is r_t^ℓ. The firms are price takers in all markets and free entry implies zero profits.

Using the results with CES technology derived in sub-section 1.2, we write the **block** as:

- **Inputs:** $\{K_t\}$, $\{\ell_t\}$, $\{r_t^K\}$, $\{r_t^\ell\}$
- **Outputs:** $\{Y_t\}$, $\left\{P_t^{Y,0}\right\}$

$$Y_t = \Gamma\left(\left(\mu^K\right)^{\frac{1}{\sigma^Y}} K_{t-1}^{\frac{\sigma^Y-1}{\sigma^Y}} + \left(1-\mu^K\right)^{\frac{1}{\sigma^Y}} \ell_t^{\frac{\sigma^Y-1}{\sigma^Y}}\right)^{\frac{\sigma^Y}{\sigma^Y-1}} \tag{18}$$

$$P_t^{Y,0} = \frac{1}{\Gamma}\left(\mu^K\left(r_t^K\right)^{1-\sigma^Y} + \left(1-\mu^K\right)\left(r_t^\ell\right)^{1-\sigma^Y}\right)^{\frac{1}{1-\sigma^Y}}. \tag{19}$$

- **Targets:**

$$\frac{K_{t-1}}{\ell_t} = \frac{\mu_K}{1-\mu_K}\left(\frac{r_t^\ell}{r_t^K}\right)^{\sigma^Y}. \tag{20}$$

The variable K_{-1} is pre-determined.

1.9 Block IV. Phillips Curve

The final price, P_t^Y, is a mark-up over the marginal cost price, $P_t^{Y,0}$, to capture monopolitic behavior by the firms. The demand for each good is y_t. The final prices are sticky, and the firms pay a quadratic adjustment cost, g_t, to change them.

$$V_t = \max_{\{p_t^Y\}} \left(p_t^Y - P_t^{Y,0}\right) y_t - g_t + \frac{1}{1+r^{\text{firm}}} V_{t+1}$$

s.t.

$$g_t = \frac{\gamma}{2}\left[\frac{p_t^Y/p_{t-1}^Y}{p_{t-1}^Y/p_{t-2}^Y} - 1\right]^2 P_t^Y Y_t$$

$$y_t = \left(\frac{p_t^Y}{P_t^Y}\right)^{-\sigma_D} Y_t$$

Using the first order condition we get

- **Inputs:** $\{Y_t\}, \left\{P_t^{Y,0}\right\}$
- **Outputs:** $\left\{P_t^Y\right\}$

$$
\begin{aligned}
p_t^Y &= \frac{\sigma_D}{\sigma_D - 1} P_t^{Y,0} - \frac{1}{\sigma_D - 1} \frac{p_t^Y}{y_t} \left(\frac{\partial g_t}{\partial p_t^Y} + \frac{1}{1 + r^{\text{firm}}} \frac{\partial g_{t+1}}{\partial p_t^Y} \right) \\
p_t^Y &= \frac{\sigma_D}{\sigma_D - 1} P_t^{Y,0} - \frac{1}{\sigma_D - 1} \frac{p_t^Y}{y_t} \left(\gamma \left[\frac{p_t^Y / p_{t-1}^Y}{p_{t-1}^Y / p_{t-2}^Y} - 1 \right] \left(\frac{1 / p_{t-1}^Y}{p_{t-1}^Y / p_{t-2}^Y} \right) P_t^Y Y_t + \frac{1}{1 + r^{\text{firm}}} \gamma \left[\frac{p_{t+1}^Y / p_t^Y}{p_t^Y / p_{t-1}^Y} - \right. \right. \\
P_t^Y &= \frac{\sigma_D}{\sigma_D - 1} P_t^{Y,0} - \frac{1}{\sigma_D - 1} \left(\gamma \left[\frac{P_t^Y / P_{t-1}^Y}{P_{t-1}^Y / P_{t-2}^Y} - 1 \right] \left(\frac{P_t^Y / P_{t-1}^Y}{P_{t-1}^Y / P_{t-2}^Y} \right) P_t^Y + \frac{2}{1 + r^{\text{firm}}} \gamma \left[\frac{P_{t+1}^Y / P_t^Y}{P_t^Y / P_{t-1}^Y} - 1 \right] \left(\right. \right. \\
P_t^Y &= (1 + \theta) P_t^{Y,0} - \eta \left(\frac{P_t^Y / P_{t-1}^Y}{P_{t-1}^Y / P_{t-2}^Y} - 1 \right) \frac{P_t^Y / P_{t-1}^Y}{P_{t-1}^Y / P_{t-2}^Y} P_t^Y + \frac{2}{1 + r^{\text{firm}}} \eta \frac{Y_{t+1}}{Y_t} \left(\frac{P_{t+1}^Y / P_t^Y}{P_t^Y / P_{t-1}^Y} - 1 \right) \frac{P_{t+1}^Y / P_t^Y}{P_t^Y / P_{t-1}^Y}
\end{aligned}
$$

where$\theta = \dfrac{1}{\sigma_D - 1} \vee \eta = \theta\gamma$

- **Targets:**

$$
P_t^Y = (1 + \theta) P_t^{Y,0} - \eta \left(\frac{P_t^Y / P_{t-1}^Y}{P_{t-1}^Y / P_{t-2}^Y} - 1 \right) \frac{P_t^Y / P_{t-1}^Y}{P_{t-1}^Y / P_{t-2}^Y} P_t^Y + \frac{2}{1 + r^{\text{firm}}} \eta \frac{Y_{t+1}}{Y_t} \left(\frac{P_{t+1}^Y / P_t^Y}{P_t^Y / P_{t-1}^Y} - 1 \right) \frac{P_{t+1}^Y / P_t^Y}{P_t^Y / P_{t-1}^Y} P_{t+1}^Y \tag{21}
$$

The variable K_{-1} is pre-determined.

1.10 Block V. Bargaining

The target wage, W_t^*, in the bargaining is specified ad hoc as a weighted average of the profit the firms get from a marginal unit of labor, $\overline{W}_t$, and the worker's outside option,$\underline{W}_t$. The bargained wage, W_t, is again a weighted average of the past wage and the current target wage to create a real rigidity.

- **Inputs:** $\{W_t\}$, $\{Y_t\}$, $\{\ell_t\}$

- **Outputs:** $\{\overline{W}_t, \underline{W}_t, W_t^*\}$

$$\overline{W}_t = P_t^Y \left((1-\mu_K)\, \Gamma^{\sigma^Y - 1} \frac{Y_t}{\ell_t} \right)^{\frac{1}{\sigma^Y}} \tag{22}$$

$$\underline{W}_t = W^U \tag{23}$$

$$W_t^* = \psi \overline{W}_t + (1-\psi)\underline{W}_t \tag{24}$$

- **Targets:**

$$W_t = \gamma^w W_{t-1} + (1-\gamma^w) W_t^* \tag{25}$$

1.11 Block VI. Repacking firms - prices

The output good, Y_t, can be used for either private consumption, C_t, public consumption, G_t, investment, I_t, or exports, X_t. For each use the output good must be repacked with imported goods. This is done by repacking firms with a CES production technology.

Using the results with CES technology derived in sub-section 1.2 with $\Gamma = 1$, we write the **block** for the pricing part of this as:

1. **Inputs:** $\{P_t^Y\}$, $\left\{P_t^{M,\bullet}\right\}$ for $\bullet \in \{C, G, I, X\}$
2. **Output:** $\{P_t^\bullet\}$ for $\bullet \in \{C, G, I, X\}$

$$P_t^\bullet = \left(\mu^{M,\bullet} \left(P_t^{M,\bullet} \right)^{1-\sigma^\bullet} + \left(1 - \mu^{M,\bullet}\right) \left(P_t^Y \right)^{1-\sigma^\bullet} \right)^{\frac{1}{1-\sigma^\bullet}}. \tag{26}$$

1.12 Block VII. Foreign economy

The foreign economy has so-called Armington demand of the domestic export good. We write the **block** as:

1. **Inputs:** $\{P_t^F\}$,$\{\chi_t\}$, $\{P_t^X\}$
2. **Outputs:** $\{X_t\}$

$$X_t = \chi_t \left(\frac{P_t^X}{P_t^F} \right)^{-\sigma^F} \tag{27}$$

1.13 Block VIII. Capital agency

The capital agency firm buys investment goods, I_t, at price, P_t^I, to accumulate capital, K_t, which it rents out to production at the rental rate r_{t+1}^K in the following period. The investment decision is subject to convex adjustment costs in terms of wasted investment goods, such that effective investment is ι_t. Future profits are discounted with r^{firm}. The capital agency takes prices as given, and its problem thus is:

$$
\begin{aligned}
V_0\left(K_{t-1}\right) &= \max_{\{K_t\}} \sum_{t=0}^{\infty} \left(1+r^{\text{firm}}\right)^{-t} \left[r_t^K K_{t-1} - P_t^I\left(\iota_t + \Psi(\iota_t, K_{t-1})\right)\right] \\
&\text{s.t.} \\
I_t &= \iota_t + \Psi(\iota_t, K_{t-1}) \\
K_t &= (1-\delta^K)K_{t-1} + \iota_t.
\end{aligned}
$$

We choose

$$
\begin{aligned}
\Psi(\iota_t, K_{t-1}) &= \frac{\Psi_0}{2}\left(\frac{\iota_t}{K_{t-1}} - \delta^K\right)^2 K_{t-1} \\
\Psi_\iota(\iota_t, K_{t-1}) &= \Psi_0\left(\frac{\iota_t}{K_{t-1}} - \delta^K\right) \\
\Psi_K(\iota_t, K_{t-1}) &= \frac{\Psi_0}{2}\left(\frac{\iota_t}{K_{t-1}} - \delta^K\right)^2 - \Psi_0\left(\frac{\iota_t}{K_{t-1}} - \delta^K\right)\frac{\iota_t}{K_{t-1}}
\end{aligned}
$$

We write the **block** as

- **Inputs:** $\{r_t^K\}$, $\{P_t^I\}$, $\{K_t\}$
- **Outputs:** $\{\iota_t\}, \{I_t\}$

$$
\iota_t = K_t - \left(1-\delta^K\right)K_{t-1} \tag{28}
$$

$$
I_t = \iota_t + \Psi(\iota_t, K_{t-1}) \tag{29}
$$

- **Targets:**

$$
\begin{aligned}
0 = &- P_t^I\left(1 + \Psi_\iota\left(\iota_t, K_{t-1}\right)\right) \\
&+ \left(1+r^{\text{firm}}\right)^{-1}\left[r_{t+1}^k + P_{t+1}^I(1-\delta^K)(1+\Psi_\iota(\iota_{t+1}, K_t)) - P_{t+1}^I\Psi_K\left(\iota_{t+1}, K_t\right)\right]
\end{aligned} \tag{30}
$$

The variable K_{-1} is pre-determined.

1.14 Block IX. Government

The government budget is given by

$$B_t = (1+r^B)B_{t-1} + P_t^G G_t + (1-\tau_t)W_U W_{ss} U_t + (1-\tau_t)W_R W_{ss}(N_t - N_t^{work}) - \tau_t W_t L_t \tag{31}$$

where r^B is the interest rate on government debt determined in the foreign economy. We assume the government has the exogenous tax rate τ_{ss} for t_B years and then begins to adjust taxes to get back to steady state debt. For Δ_B years this is done gradually, and thereafter it is done fully.

- **Inputs:** $\{P_t^G\}$, $\{G_t\}$, $\{W_t\}$, $\{L_t\}$, $\{N_t\}$, $\{N_t^{work}\}$
- **Outputs:** $\{\tau_t\}$, $\{\tilde{\tau}_t\}$, $\{B_t\}$,$\{\tilde{B}_t\}$

$$\begin{aligned}
\tilde{B}_t &= (1+r^B)B_{t-1} + P_t^G G_t + (1-\tau_{ss})W_U W_{ss} U_t + (1-\tau_{ss})W_R W_{ss}(N_t - N_t^{work}) - \tau_{ss} W_t L_t \\
\tilde{\tau}_t &= \tau_{ss} + \varepsilon_B \frac{\tilde{B}_t - B_{ss}}{W_t L_t + W_U W_{ss} U_t + W_R W_{ss}(N_t - N_t^{work})} \\
\tau_t &= \begin{cases} \tau_{ss} & \text{if } t < t_B \\ (1-\omega_t)\tau_{ss} + \omega_t \tilde{\tau}_t & \text{if } t \in [t_B, t_B + \Delta_B] \\ \tilde{\tau}_t & \text{if } t > t_B + \Delta_B \end{cases} && (32) \\
\omega_t &= 3\left(\frac{t-t_B}{\Delta_B}\right)^2 - 2\left(\frac{t-t_B}{\Delta_B}\right)^3 \in (0,1) \\
B_t &= (1+r^B)B_{t-1} + P_t^G G_t + (1-\tau_t)W_U W_{ss} U_t + (1-\tau_t)W_R W_{ss}(N_t - N_t^{work}) - \tau_t W_t L_t && (33)
\end{aligned}$$

The variable B_{-1} is pre-determined.

1.15 Block X. Households - consumption -saving

The model has two types of households. A share of λ of households is hands-to-mouth and a share of $1-\lambda$ households is unconstrained (Ricardian). All households have four sources of income:

1. Post-tax labor income, $(1-\tau_t)\, W_t \frac{L_{a,t}}{N_a}$

2. Post-tax unemployment benefits, $(1-\tau_t)\, W^U W_t \frac{U_{a,t}}{N_a}$

3. Post-tax retirement benefits, $(1-\tau_t)\, W^R W_t 1_{\{a \geq \#_{\text{work}}\}}$

4. Equally divided inheritance, $\frac{A_t^q}{N}$

The age specific income is

$$\text{inc}_{a,t} = (1-\tau_t)\, W_t \frac{L_{a,t}}{N_a} + (1-\tau_t)\, W^U W_{ss} \frac{U_{a,t}}{N_a} + (1-\tau_t)\, W^R W_{ss} 1_{\{a \geq \#_{\text{work}}\}} + \frac{A_t^q}{N}$$

The price of consumption goods is P_t^C.
Consumption is $C_{a,t}^{\bullet}$ and end-of-period nominal savings is $A_{a,t}^{\bullet}$, where $\bullet \in \{\text{HtM}, R\}$.
The behavior of *surviving* hands-to-mouth households are

$$\begin{aligned} C_{a,t}^{\text{HtM}} &= \frac{\text{inc}_{a,t}}{P_t^C} \\ A_{a,t}^{\text{HtM}} &0. \end{aligned}$$

The Ricardian households making their first decision in period t_0 solve the problem

$$\begin{aligned} V_{t_0} = & \max_{\left\{C_{a,t}^R, A_{a,t}^R\right\}_{a=0}^{\#-1}} \sum_{a=0}^{\#-1} \left(\Pi_{j=0}^{a-1} \beta(1-\zeta_j)\right) \left[\frac{\left(C_{a,t}^R\right)^{1-\sigma}}{1-\sigma} + \zeta_a \mu^{A^q} \frac{\left(\frac{A_{a,t}^R}{P_t^C}\right)^{1-\sigma}}{1-\sigma} \right] \\ & \text{s.t.} \\ t &= t_0 + a \\ A_{-1,t}^R &= 0 \\ A_{a,t}^R &= (1+r^{hh}) A_{a-1,t-1}^R + \text{inc}_{a,t} - P_t^C C_{a,t}^R. \end{aligned}$$

Aggregation implies

$$\begin{aligned} C_{a,t} &= \lambda C_{a,t}^{\text{HtM}} + (1-\lambda) C_{a,t}^R \\ A_{a,t} &= \lambda A_{a,t}^{\text{HtM}} + (1-\lambda) A_{a,t}^R \end{aligned}$$

and

$$
\begin{aligned}
C_t &= \sum_{a=0}^{\#-1} N_a C_{a,t} \\
A_t &= \sum_{a=0}^{\#-1} N_a C_{a,t}
\end{aligned}
$$

Bequests are

$$
A_t^q = \left(1 + r^{hh}\right) \sum_{a=0}^{\#-1} \zeta_a N_a A_{a,t-1}
$$

Using the FOC for we can write the **block** as:

1. **Inputs:** $\{L_{a,t}\}$, $\{U_{a,t}\}$, $\{P_t^C\}$, $\{W_t\}$, $\{\tau_t\}$, $\{A_t^q\}$, $\left\{A_{\#-1,t}^R\right\}$
2. **Outputs:** $\{A_{a,t}^{\text{HtM}}\}$, $\{A_{a,t}^R\}$,$\{A_{a,t}\}$, $\{A_t\}$, $\{C_{a,t}^{\text{HtM}}\}$,$\{C_{a,t}^R\}$, $\{C_{a,t}\}$, $\{C_t\}$, $\{\text{inc}_{a,t}\}$, $\{\text{inc}_t\}$, $\{\pi_t^{hh}\}$

   Calculate

$$
\pi_t^{hh} = \frac{P_t^C}{P_{t-1}^C} - 1 \tag{34}
$$

$$
\text{inc}_{a,t} = (1-\tau_t) W_t \frac{L_{a,t}}{N_a} + (1-\tau_t) W^U W_{ss} \frac{U_{a,t}}{N_a} + (1-\tau_t) W^R W_{ss} 1_{\{a \geq \#_{\text{work}}\}} + \frac{A_t^q}{N} \tag{35}
$$

$$
C_{a,t}^{\text{HtM}} = \frac{\text{inc}_{a,t}}{P_t^C} \tag{36}
$$

$$
A_{a,t}^{\text{HtM}} = 0 \tag{37}
$$

   For each birthcohort $t_0 \in \{-\# + 1, -\# + 2, \ldots, T-1\}$ iterate backwards from

$a = \# - 1$ with $t = t_0 + a$, but skipping steps where $t < 0$ or $t > T - 1$:

$$C^R_{a,t} = \begin{cases} \left(\zeta_a \mu^{A^q} \left(\frac{A^{\text{death}}_t}{P^C_t}\right)^{-\sigma}\right)^{-\frac{1}{\sigma}} & \text{if } a = \# - 1 \\ \left(\beta(1-\zeta_a)\frac{1+r^{hh}}{1+\pi^{hh}_{ss}}\left(C^{R,}_{a+1,ss}\right)^{-\sigma} + \zeta_a \mu^{A^q}\left(\frac{A^R_{a,ss}}{P^C_t}\right)^{-\sigma}\right)^{-\frac{1}{\sigma}} & \text{elif } t = T - 1 \\ \left(\beta(1-\zeta_a)\frac{1+r^{hh}}{1+\pi^{hh}_{t+1}}\left(C^{R,}_{a+1,t+1}\right)^{-\sigma} + \zeta_a \mu^{A^q}\left(\frac{A^R_{a,t}}{P^C_t}\right)^{-\sigma}\right)^{-\frac{1}{\sigma}} & \text{else} \end{cases}$$

$$A^R_{a-1,t-1} = \frac{A^R_{a,t} - \text{inc}_{a,t} + P^C_t C^R_{a,t}}{1 + r^{hh}}$$

Aggregates

$$C_{a,t} = \lambda C^{\text{HtM}}_{a,t} + (1-\lambda) C^R_{a,t} \tag{38}$$

$$A_{a,t} = \lambda A^{\text{HtM}}_{a,t} + (1-\lambda) A^R_{a,t} \tag{39}$$

$$\text{inc}_t = \sum_{a=0}^{\#-1} N_a \text{inc}_{a,t} \tag{40}$$

$$C_t = \sum_{a=0}^{\#-1} N_a C_{a,t} \tag{41}$$

$$A_t = \sum_{a=0}^{\#-1} N_a C_{a,t} \tag{42}$$

Targets

$$0 = A^q_t - \left(1 + r^{hh}\right) \sum_{a=0}^{\#-1} \zeta_a N_a A_{a,t-1} \tag{43}$$

$$0 = \sum_{t_0=-\#+1}^{-1} \left(A^R_{-t_0-1,-1} - A_{-t_0-1,ss}\right) + \sum_{t_0=0}^{T-1-\#+1} \left(A^R_{-1,t_0} - 0.0\right) \tag{44}$$

For $t = 0$, we have that the variable $A^R_{a,t-1}$ is pre-determined.

1.16 Block XI. Repacking firms - components

The repacking firms were described in sub-section 1.11. Using additional results from sub-section 1.2 on CES technology with $\Gamma = 1$, we write the **block** as:

1. **Inputs:** $\{P_t^Y\}$, $\left\{P_t^{M,\bullet}\right\}$, $\{P_t^\bullet\}$, $\{\bullet_t\}$ for $\bullet \in \{C, G, I, X\}$

2. **Output:** $\{\bullet_t^M\}$, $\{\bullet_t^Y\}$ for $\bullet \in \{C, G, I, X\}$

$$\bullet_t^M = \mu^{M,\bullet} \left(\frac{P_t^\bullet}{P_t^{M,\bullet}} \right)^{\sigma^\bullet} \bullet_t \tag{45}$$

$$\bullet_t^Y = \left(1 - \mu^{M,\bullet}\right) \left(\frac{P_t^\bullet}{P_t^Y} \right)^{\sigma^\bullet} \bullet_t \tag{46}$$

1.17 Block XII. Goods market clearing

The production of the domestic output good must match the above output goods used by the repacking firms. Imports, M_t, are the sum of the imports used by the repacking firms.

We write the **block** as:

- **Inputs:** $\{\bullet_t^Y\}$, $\{\bullet_t^M\}$ for $\bullet \in \{C, G, I, X\}$
- **Outputs:** $\{M_t\}$

$$M_t = \sum_{\bullet \in \{C,G,I,X\}} \bullet_t^M \tag{47}$$

- **Targets:**

$$Y_t = \sum_{\bullet \in \{C,G,I,X\}} \bullet_t^Y \tag{48}$$

1.18 Total number of targets

We have $7 \times T$ targets in Equation (20), (21), (25), (30), (43), (44), and (48), which is equal to number of unknowns.

2 Steady state

We fix a number of variables:

1. The nominal wage,$W_{ss} = 1$

2. The inflation, $\pi_{ss}^{hh} = 0$

3. The job-finding rate, $m_{ss}^{s} = 0.75$

4. The job-filling rate, $m_{ss}^{v} = 0.75$

5. The government debt, $B_{ss} = 0$

We allow for the adjustment of the exogenous variables and other parameters to fit with this. We can then find the steady state as follows:

1. Price normalization:

$$P_{ss}^{Y} = P_{ss}^{F} = P_{ss}^{M,\bullet} = 1, \bullet \in \{C, G, I, X\}$$

2. The pricing behavior of repacking firms then implies

$$P_{ss}^{\bullet} = 1, \bullet \in \{C, G, I, X\}$$

3. The exogenous labor supply and search-and-matching imply

$$S_{a,ss} = \begin{cases} 1 & \text{if } a = 0 \\ (1-\zeta_{a-1})(N_{a-1} - L_{a-1,ss}) + \delta_a^L L_{a-1,ss} & \text{if } a < \#_{work} \\ 0 & \text{else} \end{cases}$$

$$\underline{L}_{a,ss} = \begin{cases} 0 & \text{if } a = 0 \\ (1-\zeta_{a-1})(1-\delta_a^L) L_{a-1,ss} & \text{if } a < \#_{work} \\ 0 & \text{else} \end{cases}$$

$$L_{a,ss} = \underline{L}_{a,ss} + m_{ss}^s S_{a,ss}$$

$$U_{a,ss} = \begin{cases} N_a - L_{a,ss} & \text{if } a < \#_{work} \\ 0 & \text{else} \end{cases}$$

$$L_{ss} = \sum_a L_{a,ss}$$

$$S_{ss} = \sum_a S_{a,ss}$$

$$U_{ss} = \sum_a U_{a,ss}$$

$$\delta_{ss}^L = \frac{L_{ss} - \underline{L}_{ss}}{L_{ss}}$$

$$\mathcal{M}_{ss} = \delta_{ss}^L L_{ss}$$

$$v_{ss} = \frac{\mathcal{M}_{ss}}{m_{ss}^v}$$

$$\nu = \frac{1}{v_{ss}} \left(\frac{(m_{ss}^s)^{\frac{1}{\sigma^m}} S_{ss}^{\frac{1}{\sigma^m}}}{1 - (m_{ss}^s)^{\frac{1}{\sigma^m}}} \right)^{\sigma^m}$$

4. Capital agency behavior implies

$$r_{ss}^K = r^{\text{firm}} + \delta^K$$

5. Labor agency behavior implies

$$
\begin{aligned}
r^{\ell}_{ss} &= \frac{W_{ss}}{1-\left(1-\frac{1-\delta^L_{ss}}{1+r^{firm}}\right)\frac{\kappa^L}{m^v_{ss}}} \\
\ell_{ss} &= L_{ss} - \kappa^L v_{ss}
\end{aligned}
$$

6. Set the prices using the Phillipps-curve

$$
\begin{aligned}
P^{Y,0}_{ss} &= \frac{P^Y_{ss}}{1+\theta} \\
\Gamma_{ss} &= \frac{1}{P^{Y,0}_{ss}}\left(\mu^K\left(r^K_{ss}\right)^{1-\sigma^Y} + \left(1-\mu^K\right)\left(r^{\ell}_{ss}\right)^{1-\sigma^Y}\right)
\end{aligned}
$$

7. Determine the capital and output from the production firm

$$
\begin{aligned}
K_{ss} &= \frac{\mu_K}{1-\mu_K}\left(\frac{r^{\ell}_{ss}}{r^K_{ss}}\right)^{\sigma^Y}\ell_{ss} \\
Y_{ss} &= \Gamma_{ss}\left(\left(\mu^K\right)^{\frac{1}{\sigma^Y}} K_{ss}^{\frac{\sigma^Y-1}{\sigma^Y}} + \left(1-\mu^K\right)^{\frac{1}{\sigma^Y}} \ell_{ss}^{\frac{\sigma^Y-1}{\sigma^Y}}\right)^{\frac{\sigma^Y}{\sigma^Y-1}}
\end{aligned}
$$

8. From capital accumulation equations

$$
\iota_{ss} = I_{ss} = \delta^K K_{ss}
$$

9. Exogenously, set the government spending as a share of the output and find the tax rate

$$
\begin{aligned}
G_{ss} &= G^{share} Y_{ss} \\
\tau_{ss} &= \frac{r_B B_{ss} + P^G_{ss} G_{ss} + W_U U_{ss} + W_R\left(N_{ss} - N^{work}_{ss}\right)}{W_{ss} L_{ss} + W_U U_{ss} + W_R\left(N_{ss} - N^{work}_{ss}\right)}
\end{aligned}
$$

10. Find the age-specific income for the households and use this to define the consumption for the hands-to-mouth households.
Guess on A^q_{ss} and check $\left(1+r^{hh}\right)\sum_{a=0}^{\#-1}\zeta_a N_a A_{a,ss} = A^q_{ss}$. Find the Ricardian

housholds consumption and assets backwards.

$$
\begin{aligned}
\text{inc}_{a,ss} &= (1-\tau_{ss})\,W_{ss}\frac{L_{a,ss}}{N_a} + (1-\tau_{ss})\,W_U W_{ss}\frac{U_{a,ss}}{N_a} + (1-\tau_{ss})\,W_R W_{ss} 1_{\{a \geq \#_{\text{work}}\}} + \frac{A^q_{ss}}{N} \\
C^{\text{HtM}}_{a,ss} &= \frac{\text{inc}_{a,ss}}{P^C_{ss}} \\
A^{\text{HtM}}_{a,ss} &= 0 \\
C^R_{a,ss} &= \begin{cases} \left(\zeta_a \mu^{A^q}\left(\frac{A^R_{\#-1,ss}}{P^C_{ss}}\right)^{-\sigma}\right)^{-\frac{1}{\sigma}} & \text{if } a = \#-1 \\ \left(\beta(1-\zeta_a)\frac{1+r^{hh}}{1+\pi^{hh}_{ss}}\left(C^R_{a+1,ss}\right)^{-\sigma} + \zeta_a \mu^{A^q}\left(\frac{A^R_{a,ss}}{P^C_t}\right)^{-\sigma}\right)^{-\frac{1}{\sigma}} & \text{else} \end{cases} \\
A^R_{a-1,ss} &= \frac{A^R_{a,ss} - \text{inc}_{a,ss} + P^C_{ss} C^R_{a,ss}}{1+r^{hh}} \\
C_{a,ss} &= \lambda C^{\text{HtM}}_{a,ss} + (1-\lambda) C^R_{a,ss} \\
A_{a,ss} &= \lambda A^{\text{HtM}}_{a,ss} + (1-\lambda) A^R_{a,ss} \\
A^q_{ss} &= \left(1+r^{hh}\right)\sum_{a=0}^{\#-1} \zeta_a N_a A_{a,ss}
\end{aligned}
$$

11. Determine package components for consumption and investment

$$
\begin{aligned}
\bullet^M_{ss} &= \mu^{M,\bullet}\left(\frac{P^\bullet_{ss}}{P^{M,\bullet}_{ss}}\right)^{\sigma^\bullet} \bullet_{ss}, \text{ for } \bullet \in \{C,G,I\} \\
\bullet^Y_{ss} &= (1-\mu^{M,\bullet})\left(\frac{P^\bullet_{ss}}{P^{M,\bullet}_{ss}}\right)^{\sigma^\bullet} \bullet_{ss}, \text{ for } \bullet \in \{C,G,I\}
\end{aligned}
$$

12. Determine χ_{ss} to get market clearing

$$
\begin{aligned}
X^Y_{ss} &= Y_{ss} - \left(C^Y_{ss} + G^Y_{ss} + I^Y_{ss}\right) \\
\chi_{ss} = X_{ss} &= \frac{X^Y_{ss}}{(1-\mu^{M,X})\Gamma^{\sigma^X - 1}} \\
X^M_{ss} &= \mu^{M,X}\Gamma^{\sigma^X-1}\left(\frac{P^X_{ss}}{P^{M,X}_{ss}}\right)^{\sigma^X} X_{ss} \\
M_{ss} &= C^M_{ss} + G^M_{ss} + I^M_{ss} + X^M_{ss}
\end{aligned}
$$

13. Let φ adjust to make bargaining fit

$$
\begin{aligned}
W_{ss}^{*} &= W_{ss} \\
\overline{W}_{ss} &= P_{ss}^{Y}\left((1-\mu_K)\,\Gamma_{ss}^{1-\sigma^{Y}}\frac{Y_{ss}}{\ell_{ss}}\right)^{\frac{1}{\sigma^{Y}}} \\
\underline{W}_{ss} &= W^{U}W_{ss}^{*} \\
\psi &= \frac{W_{ss}^{*}-\underline{W}_{ss}}{\overline{W}_{ss}-\underline{W}_{ss}}
\end{aligned}
$$

3 Status report

Status: The described model is implemented in Python. Some results look weird, which could suggest a code error, a math error or weird assumptions or parameters. This should be checked and a baseline calibration established.

Economic extensions: Potential extensions include

1. **Extend household problem:** Add habit formation.
2. **Add more government with taxes and spending.**
3. **Add endogenous labor supply.**
4. **Add financial flows accounts wrt. to the foreign economy.**
5. **Add multiple sector and input-output structure.**
6. **Add technology growth, population growth, and trend inflation**

Computational improvements:

1. Simplify calculation of Jacobian with graph theory or automatic differentiation.
2. Speed-up calculation of Jacobian with parallelization.
3. Speed-up broyden-solver with sparse algebra.
4. Investigate what is done efficiently in MAKRO (GAMS+CONOPT)