Baby-MAKRO[*]

DISCLAIMER: WORK-IN-PROGRESS ⇒ BEWARE OF ERRORS!

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Abstract

This note outlines a simplified »baby« version of the MAKRO model used by the Danish Ministry of Finance. The model is for a small open economy with a fixed exchange and overlapping generations. The model has perfect foresight, but is full of imperfections due to e.g. frictions in the labor market and adjustment costs. The model is written and solved in terms of a series of ordered blocks. This clarifies the model dynamics and makes it easier to solve for fluctuations around the steady state using a numerical equation system solver. Online code is provided for solving the model in Python.

The model is designed so undergraduate students can work with it, and analyze potential extensions in their thesis work. The model structure is similar to state-of-the-art heterogeneous agent models (see this course) and the model is thus relevant for further academic studies. The similarity to the grown-up MAKRO model makes it relevant for potential future job tasks and the public debate.

The note concludes with a status report on the continuing development of the model.

Online code: github.com/NumEconCopenhagen/BabyMAKRO

MAKRO: See online documentation

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1 Overview

We consider a *small open economy* with a *fixed exchange rate* and *overlapping generations*. Time is discrete, $t \in \{0, 1, \ldots\}$ and the frequency is annual.

Households live for up to # periods, and their age is denoted by a. The age dependent mortality is $\zeta_a \in (0, 1)$ and the population and demographic structure is constant. Households exogenously search for jobs and supply labor, receive inheritances and choose consumption and savings to get utility from consumption and bequests.

The *foreign economy* provides a fixed nominal rate of return, sells import goods at fixed prices, and have a demand curve for the domestic export good.

The *production* in the economy is layered as follows:

1. *Production firms* rent *capital* and *labor* to produce the domestic output good.
2. *Repacking firms* combine imported goods with the domestic output good to produce a consumption good, an investment good, and an export good.
3. Capital is rented from a *capital agency*, which purchases the investment good to accumulate capital subject to adjustment costs.
4. Labor is rented from a *labor agency*, which posts vacancies for a search-and-match labor market to purchase labor from the households.

All firms mentioned above are price takers. The domestic good is bought by intermediaries producing differentiated goods and who are subject to price adjustment costs. Wages are determined by ad hoc bargaining. All *goods markets clear* and the matching process is determined by a *matching function*.

There is *perfect foresight* in the economy. I.e. the value of all current and future variables are known. This is a strong assumption, and in many ways the model should be considered a first order approximation to a full model with both idiosyncratic and aggregate risk. It can be relevant to introduce model elements, which proxy for the effects of risks. Utility-of-wealth can e.g. proxy for a precautionary saving motive.

1.1 Equilibrium path

The *equilibrium path* in the economy is a set of paths for all variables, which satisfies all accounting identities, optimal firm and household behavior in terms of first order conditions, and implies market clearing. When all variables are constant over time, the equilibrium path is a *steady state*.

In terms of math, the model is just an *equation system* stacking the accounting identifies, first order conditions and market clearing conditions. If the economy is initially out of steady state, we solve for the equilibrium path by truncating the equation system to T periods. The assumption is that the economy has settled down to the steady state well before period T, and we can assume variables from period T onward are at their steady state value. The economy can be out of steady state both because lagged *endogenous* variables are initially not at their steady state values and/or because the *exogenous* variables are not at their steady state values. We talk of an *impulse response* when the economy starts at the steady state, but some exogenous variables *temporarily* deviate from the steady state following some converging auto-regressive process.[1]

We simplify the model and the resulting equation system by writing it in terms of a *ordered series of block*. We start from a set of *exogenous* variables (e.g. variables determined in the foreign economy) and a set of *unknown* variables. Each block then takes in the path of some variables, return the path of other variables, and imply *targets*, which must be zero if the model equations are satisfied. Each block can use the unknown variables and output variables of previous blocks as input variables. In the end we collect all the targets. The *number of unknown variables* must equal the *number of target variables*.

To solve the model, we must first find the steady state. As explained in Section 3, this can be done by manually choosing values for a selection of the endogenous variables and the deriving the rest from closed form expressions or solving sub-systems with a numerical equation system solver. Next, we solve for the equilibrium path again using a numerical equation system solver.

The block structure and ordering is *not* unique. If a different set of unknowns is chosen, a different ordering of blocks must also be chosen. If an additional variable

[1] This is also called an MIT shock. A shock in a model with perfect foresight is to some degree a contradiction in terms. The assumption is that even though the agents experiences a shock, they expect that there will never be a shock again. Accounting for expecting of future shocks is much more complicated. Some realism can be mimicked by studying impulse response to shocks about the future, which when the future comes never materialized as a new opposite signed shock negates it. Multiple shocks arriving sequentially can also be studied.

is considered an *unknown*, an additional equation must be considered a target instead of being used to calculate a output. In the limit, all variables can be considered as unknowns and all equations as targets. This is inefficient as the number of variables can be very large.[2]

1.2 On CES technology

The assumption of CES technology is used repeatedly in the model. It is therefore beneficial to recap it briefly. Consider a firm producing good X using good X_i and X_j with a CES technology. Input prices are P_i and P_j and the output price is P. The firm is a price taker in all markets. The *profit maximization* problem of the firm is

$$\max_{X_i,X_j} PX - P_iX_i - P_jX_j \text{ s.t. } X = \Gamma\left(\mu_i^{\frac{1}{\sigma}} X_i^{\frac{\sigma-1}{\sigma}} + \mu_j^{\frac{1}{\sigma}} X_j^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}},\ \ \mu_i + \mu_j = 1,\ \mu_i, \sigma, \Gamma > 0,\ \sigma \neq 1 \tag{1}$$

The generic *first order condition*

$$\begin{aligned}
0 &= P\mu_i^{\frac{1}{\sigma}} X_i^{\frac{\sigma-1}{\sigma}-1}\Gamma\left(\mu_i^{\frac{1}{\sigma}} X_i^{\frac{\sigma-1}{\sigma}} + \mu_j^{\frac{1}{\sigma}} X_j^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}-1} - P_i \Leftrightarrow \\
P_i &= P\mu_i^{\frac{1}{\sigma}} X_i^{-\frac{1}{\sigma}}\Gamma\left(\mu_i^{\frac{1}{\sigma}} X_i^{\frac{\sigma-1}{\sigma}} + \mu_j^{\frac{1}{\sigma}} X_j^{\frac{\sigma-1}{\sigma}}\right)^{\frac{1}{\sigma-1}} \Leftrightarrow \\
P_i &= P\mu_i^{\frac{1}{\sigma}} X_i^{-\frac{1}{\sigma}}\Gamma^{\frac{\sigma-1}{\sigma}} X^{\frac{1}{\sigma}} \Leftrightarrow \\
X_i &= \mu_i\left(\frac{P}{P_i}\right)^{\sigma}\Gamma^{\sigma-1}X.
\end{aligned} \tag{2}$$

As the production technology has constant return-to-scale, there are infinitely many solutions to the FOCs. They all satisfy that inputs are used in proportion as follows

$$\frac{X_i}{X_j} = \frac{\mu_i}{\mu_j}\left(\frac{P_j}{P_i}\right)^{\sigma} \tag{3}$$

[2] Modeling systems such as GAMS can combined with a state-of-the-art solver such as CONOOPT automatically analyze the structure of the equation system and thereby ripe the benefits we get from manually ordering the blocks.

Assuming *free entry*, and thus *zero profits*, the output price is uniquely determined from the input prices as

$$\begin{aligned}
0 &= PX - P_i X_i - P_j X_j \Leftrightarrow \\
P &= \frac{P_i X_i + P_j X_j}{X} \\
&= \mu_i \left(\frac{P}{P_i}\right)^{\sigma} \Gamma^{\sigma-1} P_i + \mu_j \left(\frac{P}{P_j}\right)^{\sigma} \Gamma^{\sigma-1} P_j \Leftrightarrow \\
(\Gamma P)^{1-\sigma} &= \mu_i P_i^{1-\sigma} + \mu_j P_j^{1-\sigma} \Leftrightarrow \\
P &= \frac{1}{\Gamma} \left(\mu_i P_i^{1-\sigma} + \mu_j P_j^{1-\sigma}\right)^{\frac{1}{1-\sigma}} .
\end{aligned} \tag{4}$$

2 Blocks

2.1 Exogenous variables

The *exogenous* variables are:

1. Γ_t, technology
2. G_t, public spending
3. χ_t, *foreign* demand shifter (»market size«)
4. $P_t^{M,C}$, import price of *private consumption* component good
5. $P_t^{M,G}$, import price of *public consumption* component good
6. $P_t^{M,I}$, import price of *investment* component good
7. $P_t^{M,X}$, import price of *export* component good
8. P_t^F, *foreign* price level
9. r_t^{hh}, *foreign* interest rate

2.2 Unknown variables

The chosen *unknown* variables are:

1. A_t^q, inheritance flow (T unknowns)
2. A_t^{death}, wealth of households at $a = \# - 1$ (T unknowns)
3. K_t, capital (T unknowns)
4. L_t, labor supply (T unknowns)
5. r_t^K, rental price for capital (T unknowns)
6. P_t^Y, price of domestic output (T unknowns)

The total number of unknowns thus is $6 \times T$.

2.3 Demographics

The age-specific number of household is N_a, which we normalize to 1 at age $a = 0$. The mortality rate at period a is ζ_a, the potential life-span is # and retirement age is $\#_{work} < \#$.

The demographic structure and population is then given by

$$N_a = \begin{cases} 1 & \text{if } a = 0 \\ (1 - \zeta_{a-1}) N_{a-1} & \text{if } a > 0 \end{cases} \tag{5}$$

$$\zeta_a = \begin{cases} 0 & \text{if } a < \#_{\text{work}} \\ \left(\frac{a+1-\#_{\text{work}}}{\# - \#_{\text{work}}} \right)^{\zeta} & \text{if } a < \# - 1 \\ 1 & \text{if } a = \# - 1 \end{cases} \tag{6}$$

$$N = \sum_{a=0}^{\#-1} N_a \tag{7}$$

$$N_{\text{work}} = \sum_{a=0}^{\#_{\text{work}}-1} N_a. \tag{8}$$

2.4 Block I. Repacking firms - prices

The output good, Y_t, can be used for either private consumption, C_t, public consumption, G_t, investment, I_t, or exports, X_t. For each use the output good must be repacked with imported goods. This is done by repacking firms with a CES production technology, where they take the output price and the import prices as given..

Using the results with CES technology derived in sub-section 1.2 with $\Gamma = 1$, we write the **block** for the pricing part of this as:

1. **Inputs:** $\{P_t^Y\}$, $\left\{P_t^{M,\bullet}\right\}$ for $\bullet \in \{C, G, I, X\}$
2. **Output:** $\{P_t^\bullet\}$ for $\bullet \in \{C, G, I, X\}$

$$P_t^\bullet = \left(\mu^{M,\bullet}\left(P_t^{M,\bullet}\right)^{1-\sigma^\bullet} + \left(1-\mu^{M,\bullet}\right)\left(P_t^Y\right)^{1-\sigma^\bullet}\right)^{\frac{1}{1-\sigma^\bullet}}. \tag{9}$$

2.5 Block II. Wage determination

The wage, W_t, is determined by an unmodeled bargaining mechanism, and it is set such that it is increasing in the labor demand.

- **Inputs:** $\{L_t\}$
- **Outputs:** $\{W_t\}$

$$W_t = W_{ss}\left(\frac{L_t}{L_{ss}}\right)^{\epsilon_w}. \tag{10}$$

2.6 Block III. Households - search behavior and matching

Households search for a job and supply labor exogenously. The age-specific job-separation probability is $\delta_a^L \in (0, 1)$. All unemployed search for a job. As an initial condition, we have $L_{-1,t-1} = 0$.

The quantity of searchers is

$$S_{a,t} = \begin{cases} 1 & \text{if } a = 0 \\ (1-\zeta_{a-1})\left[(N_{a-1} - L_{a-1,t-1}) + \delta_a^L L_{a-1,t-1}\right] & \text{if } a < \#_{\text{work}} \\ 0 & \text{if } a \geq \#_{\text{work}} \end{cases}$$

$$S_t = \sum_a S_{a,t}.$$

The quantity of households with a job *before matching* is

$$\underline{L}_{a,t} = \begin{cases} 0 & \text{if } a = 0 \\ (1-\zeta_{a-1})(1-\delta_a^L)L_{a-1,t-1} & \text{if } a < \#_{\text{work}} \\ 0 & \text{if } a \geq \#_{\text{work}} \end{cases}$$

$$\underline{L}_t = \sum_a \underline{L}_{a,t}.$$

The aggregate job-separation rate incl. mortality is

$$\delta_t^L = \frac{L_{t-1} - \underline{L}_t}{L_{t-1}}.$$

The quantity of vacancies is v_t and the number of matches, $\mathcal{M}_t$, is given by the *matching function*

$$\mathcal{M}_t = \frac{S_t v_t}{\left(S_t^{\frac{1}{\sigma^m}} + v_t^{\frac{1}{\sigma^m}}\right)^{\sigma^m}}.$$

The job-filling rate, m_t^v, and the job-finding rate, m_t^s, are thus

$$m_t^v = \frac{\mathcal{M}_t}{v_t}$$

$$m_t^s = \frac{\mathcal{M}_t}{S_t}.$$

The number of employed therefore is

$$L_{a,t} = \underline{L}_{a,t} + m_t^s S_{a,t}.$$

This implies that the number of unemployed is

$$U_{a,t} = \begin{cases} N_a - L_{a,t} & \text{if } a < \#_{\text{work}} \\ 0 & \text{if } a \geq \#_{\text{work}} \end{cases}.$$

In equilibrium, the number of matches must equal the number of new hires, i.e.

$$\mathcal{M}_t = L_t - \underline{L}_t.$$

It is assumed that the households accumulate human capital as a function of their working experience, wich is partly cohort specific and partly steady state driven

$$\begin{aligned} H_{a,t} &= 1 + \rho_1 \cdot x_{a,t} - \rho_2 \cdot x_{a,t}^2 \\ x_{a,t} &= \begin{cases} 0 & \text{if } a = 0 \\ x_{a-1,t-1} + \left(\frac{L_{a-1,t-1}}{N_{a-1}}\right)^{\Phi} \left(\frac{L_{a-1,ss}}{N_{a-1}}\right)^{1-\Phi} & \text{else} \end{cases} \end{aligned}$$

The amount of human capital affects the *effective* employment in cohort a such that

$$L_{a,t}^H = L_{a,t} H_{a,t}$$

We write the **block** in terms of inputs, and outputs as:

- **Inputs:** $\{L_t\}$

- **Outputs:** $\{S_{a,t}\}$, $\{S_t\}$, $\{\delta_t^L\}$, $\{\mathcal{M}\}$,$\{v_t\}$, $\{m_t^v\}$, $\{m_t^s\}$, $\{L_{a,t}\}$, $\{L_{a,t}^H\}$,$\{U_{a,t}\}$,$\{U_t\}$,$\{L_t^H\}$,$\{H_t\}$

$$S_{a,t} = \begin{cases} 1 & \textbf{if } a = 0 \\ (1-\zeta_{a-1})\left[(N_{a-1} - L_{a-1,t-1}) + \delta_a^L L_{a-1,t-1}\right] & \textbf{if } a < \#_{\textbf{work}} \\ 0 & \textbf{if } a \geq \#_{\textbf{work}} \end{cases} \tag{11}$$

$$\underline{L}_{a,t} = \begin{cases} 0 & \textbf{if } a = 0 \\ (1-\zeta_{a-1})(1-\delta_a^L)L_{a-1,t-1} & \textbf{if } a < \#_{\textbf{work}} \\ 0 & \textbf{if } a \geq \#_{\textbf{work}} \end{cases} \tag{12}$$

$$x_{a,t} = \begin{cases} 0 & \textbf{if } a = 0 \\ x_{a-1,t-1} + \left(\frac{L_{a-1,t-1}}{N_{a-1}}\right)^{\Phi} \left(\frac{L_{a-1,ss}}{N_{a-1}}\right)^{1-\Phi} & \textbf{else} \end{cases} \tag{13}$$

$$H_{a,t} = 1 + \rho_1 \cdot x_{a,t} - \rho_2 \cdot x_{a,t}^2 \tag{14}$$

$$S_t = \sum_{a=0}^{\#-1} S_{a,t} \tag{15}$$

$$\underline{L}_t = \sum_{a=0}^{\#-1} \underline{L}_{a,t} \tag{16}$$

$$\delta_t^L = \frac{L_{t-1} - \underline{L}_t}{L_{t-1}} \tag{17}$$

$$\mathcal{M}_t = L_t - \underline{L}_t \tag{18}$$

$$m_t^s = \frac{\mathcal{M}_t}{S_t} \tag{19}$$

$$v_t = \left(\frac{\mathcal{M}_t^{\frac{1}{\sigma^m}}}{1 - \left(\frac{\mathcal{M}_t}{S_t}\right)^{\frac{1}{\sigma^m}}} \right)^{\sigma^m} \tag{20}$$

$$m_t^v = \frac{\mathcal{M}_t}{v_t} \tag{21}$$

$$L_{a,t} = \underline{L}_{a,t} + m_t^s S_{a,t} \tag{22}$$

$$L_{a,t}^H = L_{a,t} H_{a,t} \tag{23}$$

$$U_{a,t} = \begin{cases} N_a - L_{a,t} & \textbf{if } a < \#_{\textbf{work}} \\ 0 & \textbf{if } a \geq \#_{\textbf{work}} \end{cases} \tag{24}$$

$$U_t = \sum_{a=0}^{\#-1} U_{a,t}. \tag{25}$$

$$L_t^H = \sum_{a=0}^{\#-1} L_{a,t}^H. \tag{26}$$

$$H_t = \frac{L_t^H}{L_t} \tag{27}$$

For $t = 0$, the variable $L_{a-1,t-1}$ is pre-determined.

2.7 Block IV. Labor agency

The labor agency firms post vacancies, v_t, to hire effective labor L_t^H. The labor cost of posting each vacancy is κ^L in units of labor. The firms can therefore rent out $\ell_t = L_t^H - \kappa^L v_t$ units of labor to the production firms at the rental price r_t^ℓ.

Effective labor follows the law-of-motion $L_t^H = \left(1 - \delta_t^L\right) L_{t-1}^H + m_t^v H_t v_t$, and the wage, W_t, is determined by the bargaining mechanism. Matching occurs according to the matching function, and the firms take the separation rate, δ_t^L, and the vacancy filling rate, m_t^v as given. Since lagged employment, L_{t-1}^H, is pre-determined, we consider L_t^H to be the choice value and derive the required number of vacancies, v_t, and the implied labor for rent, ℓ_t.

The labor agency problem is then:

$$V_0^{\text{labor}}(L_{t-1}) = \max_{\{L_t^H\}} \sum_{t=0}^{\infty} \left(1 + r^{\text{firm}}\right)^{-t} \left[r_t^{\ell} \ell_t - W_t L_t^H\right]$$

$$\text{s.t.}$$

$$v_t = \frac{L_t - \left(1 - \delta_t^L\right) L_{t-1}}{H_t m_t^v}$$

$$\ell_t = L_t^H - \kappa^L v_t.$$

Using the FOC to L_t from the labor agency problem, we write the **block** as:

- **Inputs:** $\{W_t\}$, $\{m_t^v\}$, $\{\delta_t^L\}$, $\{L_t\}$, $\{v_t\}$, $\{\underline{H}_t\}$
- **Outputs:** $\{r_t^{\ell}\}$, $\{\ell_t\}$

$$r_t^{\ell} = \frac{1}{1 - \frac{\kappa^L}{H_t m_t^v}} \left[W - r_{t+1}^{\ell} \frac{1 - \delta_{t+1}^L}{1 + r^{\text{firm}}} \frac{\kappa^L}{H_{t+1} m_{t+1}^v}\right] \tag{28}$$

$$\ell_t = L_t^H - \kappa^L v_t. \tag{29}$$

The variable L_{-1}^H is pre-determined.

2.8 Block V. Production firms

The production firms use capital, K_{t-1}, and labor, ℓ_t, to produce output, Y_t, with a CES technology. The rental price of capital is r_t^K and the rental price of labor is r_t^{ℓ}. The marginal cost of output is denoted $P_t^{Y,0}$.

Using the results with CES technology derived in sub-section 1.2, we write the **block** as:

- **Inputs:** $\{\Gamma_t\}$, $\{K_t\}$, $\{\ell_t\}$, $\{r_t^K\}$, $\{r_t^{\ell}\}$

- **Outputs:** $\{Y_t\}$, $\left\{P_t^{Y,0}\right\}$

$$Y_t = \Gamma_t \left(\left(\mu^K\right)^{\frac{1}{\sigma^Y}} K_{t-1}^{\frac{\sigma^Y-1}{\sigma^Y}} + \left(1-\mu^K\right)^{\frac{1}{\sigma^Y}} \ell_t^{\frac{\sigma^Y-1}{\sigma^Y}} \right)^{\frac{\sigma^Y}{\sigma^Y-1}} \tag{30}$$

$$P_t^{Y,0} = \frac{1}{\Gamma_t} \left(\mu^K \left(r_t^K\right)^{1-\sigma^Y} + \left(1-\mu^K\right) \left(r_t^\ell\right)^{1-\sigma^Y} \right)^{\frac{1}{1-\sigma^Y}}. \tag{31}$$

Targets:

$$\frac{K_{t-1}}{\ell_t} = \frac{\mu_K}{1-\mu_K} \left(\frac{r_t^\ell}{r_t^K}\right)^{\sigma^Y}. \tag{32}$$

The variable K_{-1} is pre-determined.

2.9 Block VI. Phillips Curve

The output price, P_t^Y, is a mark-up over the marginal cost, $P_t^{Y,0}$, to capture monopolistic behavior by the firms. In addition, the final prices are sticky, and the firms pay a quadratic adjustment cost, g_t, to change them. The adjustment costs are applied to changes in inflation such that if the inflation in period t deviates from the inflation in period $t-1$, then the firm faces positive price adjustment costs. The adjustment costs are proportional to $P_t^Y$and $Y_t$implying that a higher price and output level increases the adjustment costs. The demand for each good is y_t.

$$\begin{aligned}
V_t^{\text{intermediary}} &= \max_{\{p_t^Y\}} \left(p_t^Y - P_t^{Y,0}\right) y_t - \vartheta_t + \frac{1}{1+r^{\text{firm}}} V_{t+1}^{\text{intermediary}} \\
&\text{s.t.} \\
\vartheta_t &= \frac{\gamma}{2} \left[\frac{p_t^Y / p_{t-1}^Y}{p_{t-1}^Y / p_{t-2}^Y} - 1 \right]^2 P_t^Y Y_t \\
y_t &= \left(\frac{p_t^Y}{P_t^Y}\right)^{-\sigma_D} Y_t.
\end{aligned}$$

Using the FOC and symmetry across firms, we get

- **Inputs:** $\{Y_t\}$, $\left\{P_t^{Y,0}\right\}$

- **Outputs:** $\{P_t^Y\}$

$$\begin{aligned}P_t^Y =&(1+\theta)P_t^{Y,0} - \eta\left(\frac{P_t^Y/P_{t-1}^Y}{P_{t-1}^Y/P_{t-2}^Y} - 1\right)\frac{P_t^Y/P_{t-1}^Y}{P_{t-1}^Y/P_{t-2}^Y}P_t^Y \\ &+ \frac{2\eta}{1+r^{\mathbf{firm}}}\frac{Y_{t+1}}{Y_t}\left(\frac{P_{t+1}^Y/P_t^Y}{P_t^Y/P_{t-1}^Y} - 1\right)\frac{P_{t+1}^Y/P_t^Y}{P_t^Y/P_{t-1}^Y}P_{t+1}^Y\end{aligned}$$

$$\begin{aligned}\theta &\equiv \frac{1}{\sigma_D - 1} \\ \eta &\equiv \theta\gamma.\end{aligned}$$

Targets:

$$\begin{aligned}P_t^Y =& (1+\theta)P_t^{Y,0} - \eta\left(\frac{P_t^Y/P_{t-1}^Y}{P_{t-1}^Y/P_{t-2}^Y} - 1\right)\frac{P_t^Y/P_{t-1}^Y}{P_{t-1}^Y/P_{t-2}^Y}P_t^Y \\ &+ \frac{2\eta}{1+r^{\text{firm}}}\frac{Y_{t+1}}{Y_t}\left(\frac{P_{t+1}^Y/P_t^Y}{P_t^Y/P_{t-1}^Y} - 1\right)\frac{P_{t+1}^Y/P_t^Y}{P_t^Y/P_{t-1}^Y}P_{t+1}^Y.\end{aligned} \tag{33}$$

2.10 Block VII. Foreign economy

The foreign economy has so-called Armington demand of the domestic export good. We write the **block** as:

1. **Inputs:** $\{P_t^F\}$,$\{\chi_t\}$, $\{P_t^X\}$
2. **Outputs:** $\{X_t\}$

$$X_t = \gamma^X X_{t-1} + (1-\gamma^X)\chi_t\left(\frac{P_t^X}{P_t^F}\right)^{-\sigma^F}. \tag{34}$$

2.11 Block VIII. Capital agency

The capital agency firm buys investment goods, I_t, at price, P_t^I, to accumulate capital, K_t, which it rents out to production at the rental rate r_{t+1}^K in the following period. The investment decision is subject to convex adjustment costs Ψ in terms of wasted investment goods, such that effective investment is ι_t. Future profits are discounted with r^{firm}. The capital agency takes prices as given, and its problem thus is:

$$
\begin{aligned}
V_0^{\text{capital}}\left(K_{t-1}\right) = \max_{\{K_t\}} \sum_{t=0}^{\infty} \left(1 + r^{\text{firm}}\right)^{-t} & \left[r_t^K K_{t-1} - P_t^I \left(\iota_t + \Psi(\iota_t, K_{t-1})\right)\right] \\
\text{s.t.} & \\
I_t &= \iota_t + \Psi(\iota_t, K_{t-1}) \\
K_t &= (1 - \delta^K) K_{t-1} + \iota_t.
\end{aligned}
$$

We choose the functional form

$$
\Psi(\iota_t, K_{t-1}) = \frac{\Psi_0}{2} \left(\frac{\iota_t}{K_{t-1}} - \delta^K\right)^2 K_{t-1},
$$

implying

$$
\begin{aligned}
\Psi_\iota(\iota_t, K_{t-1}) &= \Psi_0 \left(\frac{\iota_t}{K_{t-1}} - \delta^K\right) \\
\Psi_K(\iota_t, K_{t-1}) &= \frac{\Psi_0}{2} \left(\frac{\iota_t}{K_{t-1}} - \delta^K\right)^2 - \Psi_0 \left(\frac{\iota_t}{K_{t-1}} - \delta^K\right) \frac{\iota_t}{K_{t-1}}.
\end{aligned}
$$

We write the **block** as

- **Inputs:** $\{r_t^K\}$, $\{P_t^I\}$, $\{K_t\}$
- **Outputs:** $\{\iota_t\}, \{I_t\}$

$$
\iota_t = K_t - \left(1 - \delta^K\right) K_{t-1} \tag{35}
$$

$$
I_t = \iota_t + \Psi(\iota_t, K_{t-1}). \tag{36}
$$

- **Targets**:

$$
\begin{aligned}
0 = & - P_t^I \left(1 + \Psi_\iota\left(\iota_t, K_{t-1}\right)\right) \\
& + \left(1 + r^{\text{firm}}\right)^{-1} \left[r_{t+1}^k + P_{t+1}^I (1 - \delta^K)(1 + \Psi_\iota(\iota_{t+1}, K_t)) - P_{t+1}^I \Psi_K\left(\iota_{t+1}, K_t\right)\right].
\end{aligned} \tag{37}
$$

The variable K_{-1} is pre-determined.

2.12 Block IX. Government

The government budget is given by

$$B_t = (1+r^B)B_{t-1} + P_t^G G_t + (1-\tau_t)W_U W_{ss} U_t + (1-\tau_t)W_R W_{ss}(N_t - N_t^{work}) - \tau_t W_t L_t^H, \tag{38}$$

where r^B is the interest rate on government debt determined in the foreign economy. We assume the government gradually adjusts taxes to get back to steady state debt.

- **Inputs:** $\{P_t^G\}$, $\{G_t\}$, $\{W_t\}$, $\{L_t^H\}$, $\{N_t\}$, $\{N_t^{work}\}$
- **Outputs:** $\{\tau_t\}$, $\{B_t\}$, $\{\tilde{B}_t\}$

$$\begin{aligned} \tilde{B}_t &= (1+r^B)B_{t-1} + P_t^G G_t + (1-\tau_{ss})W_U W_{ss} U_t \\ &\quad + (1-\tau_{ss})W_R W_{ss}(N_t - N_{\text{work}}) - \tau_{ss} W_t L_t^H \end{aligned} \tag{39}$$

$$\tau_t = \tau_{ss} + \varepsilon_B \frac{\tilde{B}_t - B_{ss}}{W_t L_t^H + W_U W_{ss} U_t + W_R W_{ss}(N_t - N_{\text{work}})} \tag{40}$$

$$\begin{aligned} B_t &= (1+r^B)B_{t-1} + P_t^G G_t + (1-\tau_t)W_U W_{ss} U_t \\ &\quad + (1-\tau_t)W_R W_{ss}(N_t - N_{\text{work}}) - \tau_t W_t L_t^H. \end{aligned} \tag{41}$$

The variable B_{-1} is pre-determined.

2.13 Block X. Households - consumption -saving

The model has two types of households. A share λ of households is hands-to-mouth and a share $1-\lambda$ of households is unconstrained (Ricardian). All households have four sources of income:

1. Post-tax labor income, $(1-\tau_t)\, W_t \frac{H_{a,t} L_{a,t}^H}{N_a}$
2. Post-tax unemployment benefits, $(1-\tau_t)\, W^U W_{ss} \frac{U_{a,t}}{N_a}$
3. Post-tax retirement benefits, $(1-\tau_t)\, W^R W_{ss} \frac{N_a - (L_{a,t} + U_{a,t})}{N_a}$
4. Equally divided inheritance, $\frac{A_t^q}{N}$

The post-tax labor income depends on the effective employment, $L_{a,t}^{H}$, and thus the income depends on the amount of human capital. The age specific income is

$$\mathrm{inc}_{a,t} = (1-\tau_t)\, W_t \frac{L_{a,t}^{H}}{N_a} + (1-\tau_t)\, W^U W_{ss} \frac{U_{a,t}}{N_a} + (1-\tau_t)\, W^R W_{ss} 1_{\{a \geq \#_{\mathrm{work}}\}} + \frac{A_t^q}{N}.$$

The price of consumption goods is P_t^C. Consumption is $C_{a,t}^{\bullet}$ and end-of-period nominal savings is $A_{a,t}^{\bullet}$, where $\bullet \in \{\mathrm{HtM}, R\}$. The behavior of *surviving* hands-to-mouth households is

$$\begin{aligned} C_{a,t}^{\mathrm{HtM}} &= \frac{\mathrm{inc}_{a,t}}{P_t^C} \\ A_{a,t}^{\mathrm{HtM}} &= 0. \end{aligned}$$

The Ricardian households making their first decision in period t_0 solve the problem

$$\begin{aligned} V_{t_0} = \max_{\left\{C_{a,t}^R\right\}_{a=0}^{\#-1}} & \sum_{a=0}^{\#-1} \left(\Pi_{j=1}^{a} \beta (1-\zeta_{j-1})\right) \left[\frac{\left(C_{a,t}^R\right)^{1-\sigma}}{1-\sigma} + \zeta_a \mu^{A^q} \frac{\left(\frac{A_{a,t}^R}{P_t^C}\right)^{1-\sigma}}{1-\sigma} \right] \\ \text{s.t.} & \\ t &= t_0 + a \\ A_{-1,t}^R &= 0 \\ A_{a,t}^R &= (1+r_t^{hh}) A_{a-1,t-1}^R + \mathrm{inc}_{a,t} - P_t^C C_{a,t}^R. \end{aligned}$$

Aggregation implies

$$\begin{aligned} C_{a,t} &= \lambda C_{a,t}^{\mathrm{HtM}} + (1-\lambda) C_{a,t}^R \\ A_{a,t} &= \lambda A_{a,t}^{\mathrm{HtM}} + (1-\lambda) A_{a,t}^R, \end{aligned}$$

and

$$\begin{aligned} C_t &= \sum_{a=0}^{\#-1} N_a C_{a,t} \\ A_t &= \sum_{a=0}^{\#-1} N_a C_{a,t}. \end{aligned}$$

Bequests are

$$A_t^q = \left(1 + r^{hh}\right) \sum_{a=0}^{\#-1} \zeta_a N_a A_{a,t}.$$

Using the FOC for we can write the **block** as:

1. **Inputs:** $\{L_{a,t}^H\}$, $\{U_{a,t}\}$, $\{P_t^C\}$, $\{W_t\}$, $\{\tau_t\}$, $\{A_t^q\}$, $\left\{A_{\#-1,t}^R\right\}$,$\{r_t^{hh}\}$

2. **Outputs:** $\{A_{a,t}^{\text{HtM}}\}$, $\{A_{a,t}^R\}$,$\{A_{a,t}\}$, $\{A_t\}$, $\{C_{a,t}^{\text{HtM}}\}$,$\{C_{a,t}^R\}$, $\{C_{a,t}\}$, $\{C_t\}$, $\{\text{inc}_{a,t}\}$, $\{\text{inc}_t\}$, $\{\pi_t^{hh}\}$

   Calculate

$$\pi_t^{hh} = \frac{P_t^C}{P_{t-1}^C} - 1 \tag{42}$$

$$\text{inc}_{a,t} = (1-\tau_t)\, W_t \frac{L_{a,t}^H}{N_a} + (1-\tau_t)\, W^U W_{ss} \frac{U_{a,t}}{N_a} + (1-\tau_t)\, W^R W_{ss} 1_{\{a \geq \#_{\text{work}}\}} + \frac{A_t^q}{N} \tag{43}$$

$$C_{a,t}^{\text{HtM}} = \frac{\text{inc}_{a,t}}{P_t^C} \tag{44}$$

$$A_{a,t}^{\text{HtM}} = 0. \tag{45}$$

   For each birth cohort $t_0 \in \{-\# + 1, -\# + 2, \ldots, T-1\}$ iterate backwards from $a = \# - 1$ with $t = t_0 + a$, but skipping steps where $t < 0$ or $t > T - 1$:

$$C_{a,t}^R = \begin{cases} \left(\zeta_a \mu^{A^q} \left(\frac{A_{\#-1,t}^R}{P_t^C}\right)^{-\sigma}\right)^{-\frac{1}{\sigma}} & \text{if } a = \# - 1 \\ \left(\beta(1-\zeta_a)\frac{1+r_{ss}^{hh}}{1+\pi_{ss}^{hh}}\left(C_{a+1,ss}^{R,}\right)^{-\sigma} + \zeta_a \mu^{A^q}\left(\frac{A_{a,ss}^R}{P_{ss}^C}\right)^{-\sigma}\right)^{-\frac{1}{\sigma}} & \text{elif } t = T-1 \\ \left(\beta(1-\zeta_a)\frac{1+r_{t+1}^{hh}}{1+\pi_{t+1}^{hh}}\left(C_{a+1,t+1}^{R,}\right)^{-\sigma} + \zeta_a \mu^{A^q}\left(\frac{A_{a,t}^R}{P_t^C}\right)^{-\sigma}\right)^{-\frac{1}{\sigma}} & \text{else} \end{cases}$$

$$A_{a-1,t-1}^R = \frac{A_{a,t}^R - \text{inc}_{a,t} + P_t^C C_{a,t}^R}{1 + r_t^{hh}}.$$

Aggregates

$$C_{a,t} = \lambda C_{a,t}^{\text{HtM}} + (1-\lambda) C_{a,t}^{R} \tag{46}$$

$$A_{a,t} = \lambda A_{a,t}^{\text{HtM}} + (1-\lambda) A_{a,t}^{R} \tag{47}$$

$$\text{inc}_t = \sum_{a=0}^{\#-1} N_a \text{inc}_{a,t} \tag{48}$$

$$C_t = \sum_{a=0}^{\#-1} N_a C_{a,t} \tag{49}$$

$$A_t = \sum_{a=0}^{\#-1} N_a A_{a,t}. \tag{50}$$

Targets:

$$0 = A_t^q - \left(1 + r_t^{hh}\right) \sum_{a=0}^{\#-1} \zeta_a N_a A_{a,t-1} \tag{51}$$

$$0 = \sum_{t_0=-\#+1}^{-1} \left(A_{-t_0-1,-1}^{R} - A_{-t_0-1,ss}\right) + \sum_{t_0=0}^{T-1-\#+1} \left(A_{-1,t_0}^{R} - 0.0\right) \tag{52}$$

For $t = 0$, we have that the variable $A_{a,t-1}^{R}$ is pre-determined.

2.14 Block XI. Repacking firms - components

The repacking firms were described in sub-section 2.4. Using additional results from sub-section 1.2 on CES technology with $\Gamma = 1$, we write the **block** as:

1. **Inputs:** $\{P_t^Y\}$, $\left\{P_t^{M,\bullet}\right\}$, $\{P_t^{\bullet}\}$, $\{\bullet_t\}$ for $\bullet \in \{C, G, I, X\}$

2. **Output:** $\{\bullet_t^M\}$, $\{\bullet_t^Y\}$ for $\bullet \in \{C, G, I, X\}$

$$\bullet_t^M = \mu^{M,\bullet} \left(\frac{P_t^{\bullet}}{P_t^{M,\bullet}}\right)^{\sigma^{\bullet}} \bullet_t \tag{53}$$

$$\bullet_t^Y = \left(1 - \mu^{M,\bullet}\right) \left(\frac{P_t^{\bullet}}{P_t^{Y}}\right)^{\sigma^{\bullet}} \bullet_t \,. \tag{54}$$

2.15 Block XII. Goods market clearing

The production of the domestic output good must match the above output goods used by the repacking firms. Imports, M_t, are the sum of the imports used by the repacking firms.

We write the **block** as:

- **Inputs:** $\{\bullet_t^Y\}$, $\{\bullet_t^M\}$ for $\bullet \in \{C, G, I, X\}$
- **Outputs:** $\{M_t\}$

$$M_t = \sum_{\bullet \in \{C,G,I,X\}} \bullet_t^M. \tag{55}$$

- **Targets:**

$$Y_t = \sum_{\bullet \in \{C,G,I,X\}} \bullet_t^Y. \tag{56}$$

2.16 Total number of targets

We have $6 \times T$ targets in Equation (32), (33), (37), (51), (52), and (56), which is equal to number of unknowns.

3 Steady state

We fix a number of variables:

1. The nominal wage, $W_{ss} = 1$
2. The inflation, $\pi_{ss}^{hh} = 0$
3. The job-finding rate, $m_{ss}^s = 0.75$
4. The job-filling rate, $m_{ss}^v = 0.75$
5. The government debt, $B_{ss} = 0$
6. The foreign interest rate, $r_{ss}^{hh} = 0.04$

We allow for the adjustment of the exogenous variables and other parameters to fit with this. We can then find the steady state as follows:

1. Price normalization:

$$P_{ss}^{Y} = P_{ss}^{F} = P_{ss}^{M,\bullet} = 1, \bullet \in \{C, G, I, X\}.$$

2. The pricing behavior of repacking firms then implies

$$P_{ss}^{\bullet} = 1, \bullet \in \{C, G, I, X\}.$$

3. The exogenous labor supply and search-and-matching imply

$$S_{a,ss} = \begin{cases} 1 & \text{if } a = 0 \\ (1-\zeta_{a-1})\left[(N_{a-1} - L_{a-1,ss}) + \delta_a^L L_{a-1,ss}\right] & \text{if } a < \#_{\text{work}} \\ 0 & \text{else} \end{cases}$$

$$\underline{L}_{a,ss} = \begin{cases} 0 & \text{if } a = 0 \\ (1-\zeta_{a-1})(1-\delta_a^L)L_{a-1,ss} & \text{if } a < \#_{\text{work}} \\ 0 & \text{else} \end{cases}$$

$$x_{a,ss} = \begin{cases} 0 & \text{if } a = 0 \\ x_{a-1,ss} + \frac{L_{a-1,ss}}{N_{a-1}} & \text{else} \end{cases}$$

$$H_{a,ss} = 1 + \rho_1 \cdot x_{a,ss} - \rho_2 \cdot x_{a,ss}^2$$

$$L_{a,ss} = \underline{L}_{a,ss} + m_{ss}^{s} S_{a,ss}$$

$$L_{a,ss}^{H} = L_{a,ss} H_{a,ss}$$

$$
\begin{aligned}
U_{a,ss} &= \begin{cases} N_a - L_{a,ss} & \text{if } a < \#_{\text{work}} \\ 0 & \text{else} \end{cases} \\
L_{ss} &= \sum_a L_{a,ss} \\
L^H_{ss} &= \sum_a L^H_{a,ss} \\
H_{ss} &= \frac{L^H_{ss}}{L_{ss}} \\
S_{ss} &= \sum_a S_{a,ss} \\
U_{ss} &= \sum_a U_{a,ss} \\
\delta^L_{ss} &= \frac{L_{ss} - \underline{L}_{ss}}{L_{ss}} \\
\mathcal{M}_{ss} &= \delta^L_{ss} L_{ss} \\
v_{ss} &= \frac{\mathcal{M}_{ss}}{m^v_{ss}} \\
\nu &= \frac{1}{v_{ss}} \left(\frac{\left(m^s_{ss}\right)^{\frac{1}{\sigma^m}} S_{ss}^{\frac{1}{\sigma^m}}}{1 - \left(m^s_{ss}\right)^{\frac{1}{\sigma^m}}} \right)^{\sigma^m}
\end{aligned}
$$

4. Capital agency behavior implies

$$
r^K_{ss} = r^{\text{firm}} + \delta^K.
$$

5. Labor agency behavior implies

$$
\begin{aligned}
r^\ell_{ss} &= \frac{W_{ss}}{1 - \left(1 + \frac{1-\delta^L_{ss}}{1+r^{firm}}\right) \frac{\kappa^L}{H_{ss} m^v_{ss}}} \\
\ell_{ss} &= L^H_{ss} - \kappa^L v_{ss}.
\end{aligned}
$$

6. Set the prices using the Phillips-curve

$$
\begin{aligned}
P^{Y,0}_{ss} &= \frac{P^Y_{ss}}{1+\theta} \\
\Gamma_{ss} &= \frac{1}{P^{Y,0}_{ss}} \left(\mu^K \left(r^K_{ss}\right)^{1-\sigma^Y} + \left(1-\mu^K\right) \left(r^\ell_{ss}\right)^{1-\sigma^Y} \right).
\end{aligned}
$$

7. Determine the capital and output from the production firm

$$
\begin{aligned}
K_{ss} &= \frac{\mu_K}{1-\mu_K}\left(\frac{r^{\ell}_{ss}}{r^{K}_{ss}}\right)^{\sigma^Y} \ell_{ss} \\
Y_{ss} &= \Gamma_{ss}\left(\left(\mu^K\right)^{\frac{1}{\sigma^Y}} K_{ss}^{\frac{\sigma^Y-1}{\sigma^Y}} + \left(1-\mu^K\right)^{\frac{1}{\sigma^Y}} \ell_{ss}^{\frac{\sigma^Y-1}{\sigma^Y}}\right)^{\frac{\sigma^Y}{\sigma^Y-1}}.
\end{aligned}
$$

8. From capital accumulation equations

$$
\iota_{ss} = I_{ss} = \delta^K K_{ss}.
$$

9. Exogenously, set the government spending as a share of the output and find the tax rate

$$
\begin{aligned}
G_{ss} &= G^{share} Y_{ss} \\
\tau_{ss} &= \frac{r_B B_{ss} + P^G_{ss} G_{ss} + W_U U_{ss} + W_R (N_{ss} - N^{work}_{ss})}{W_{ss} L^H_{ss} + W_U U_{ss} + W_R (N_{ss} - N^{work}_{ss})}.
\end{aligned}
$$

10. Find the age-specific income for the households and use this to define the consumption for the hands-to-mouth households.
Guess on A^q_{ss} and check $\left(1+r^{hh}_{ss}\right)\sum_{a=0}^{\#-1}\zeta_a N_a A_{a,ss} = A^q_{ss}$. Find the consumption

of Ricardian households and their assets backwards.

$$\begin{aligned}
\text{inc}_{a,ss} &= (1-\tau_{ss})\, W_{ss} \frac{L^H_{a,ss}}{N_a} + (1-\tau_{ss})\, W_U W_{ss} \frac{U_{a,ss}}{N_a} + (1-\tau_{ss})\, W_R W_{ss} 1_{\{a \geq \#_{\text{work}}\}} + \frac{A^q_{ss}}{N} \\
C^{\text{HtM}}_{a,ss} &= \frac{\text{inc}_{a,ss}}{P^C_{ss}} \\
A^{\text{HtM}}_{a,ss} &= 0 \\
C^R_{a,ss} &= \begin{cases} \left(\zeta_a \mu^{A^q} \left(\frac{A^R_{\#-1,ss}}{P^C_{ss}} \right)^{-\sigma} \right)^{-\frac{1}{\sigma}} & \text{if } a = \# - 1 \\ \left(\beta (1-\zeta_a) \frac{1+r^{hh}_{ss}}{1+\pi^{hh}_{ss}} \left(C^R_{a+1,ss} \right)^{-\sigma} + \zeta_a \mu^{A^q} \left(\frac{A^R_{a,ss}}{P^C_t} \right)^{-\sigma} \right)^{-\frac{1}{\sigma}} & \text{else} \end{cases} \\
A^R_{a-1,ss} &= \frac{A^R_{a,ss} - \text{inc}_{a,ss} + P^C_{ss} C^R_{a,ss}}{1 + r^{hh}_{ss}} \\
C_{a,ss} &= \lambda C^{\text{HtM}}_{a,ss} + (1-\lambda) C^R_{a,ss} \\
A_{a,ss} &= \lambda A^{\text{HtM}}_{a,ss} + (1-\lambda) A^R_{a,ss} \\
A^q_{ss} &= \left(1 + r^{hh}_{ss}\right) \sum_{a=0}^{\#-1} \zeta_a N_a A_{a,ss}.
\end{aligned}$$

11. Determine package components for consumption and investment

$$\begin{aligned}
\bullet^M_{ss} &= \mu^{M,\bullet} \left(\frac{P^\bullet_{ss}}{P^{M,\bullet}_{ss}} \right)^{\sigma^\bullet} \bullet_{ss}, \text{ for } \bullet \in \{C, G, I\} \\
\bullet^Y_{ss} &= (1 - \mu^{M,\bullet}) \left(\frac{P^\bullet_{ss}}{P^{M,\bullet}_{ss}} \right)^{\sigma^\bullet} \bullet_{ss}, \text{ for } \bullet \in \{C, G, I\}.
\end{aligned}$$

12. Determine χ_{ss} to get market clearing

$$\begin{aligned}
X^Y_{ss} &= Y_{ss} - \left(C^Y_{ss} + G^Y_{ss} + I^Y_{ss} \right) \\
\chi_{ss} = X_{ss} &= \frac{X^Y_{ss}}{(1-\mu^{M,X}) \Gamma^{\sigma^X - 1}} \\
X^M_{ss} &= \mu^{M,X} \Gamma^{\sigma^X - 1} \left(\frac{P^X_{ss}}{P^{M,X}_{ss}} \right)^{\sigma^X} X_{ss} \\
M_{ss} &= C^M_{ss} + G^M_{ss} + I^M_{ss} + X^M_{ss}.
\end{aligned}$$

3.1 Data

See 0b - steady state - data.ipynb.

The model's steady state is compared to data from Statistics Denmark which is pulled in 0b - steady state - data.ipynb. The ratios of M_t, X_t, C_t, G_t and I_t relative to Y_t from the national accounts data are used for the comparison with the model's steady state. The employment and unemployment rates, and the capital stock relative to output are also compared to the model. The steady state ratios from the data are assumed to be given by the historical averages. Data for income and wealth over the life-cycle is used to calibrate the steady state life-cycle income profile and wealth-income ratio using estimates from the regression:

$$\text{inc}_{a,t} = \text{birth year}_a + \rho_1 \text{age}_{a,t} + \rho_2 \text{age}^2_{a,t} + \epsilon_{a,t}$$

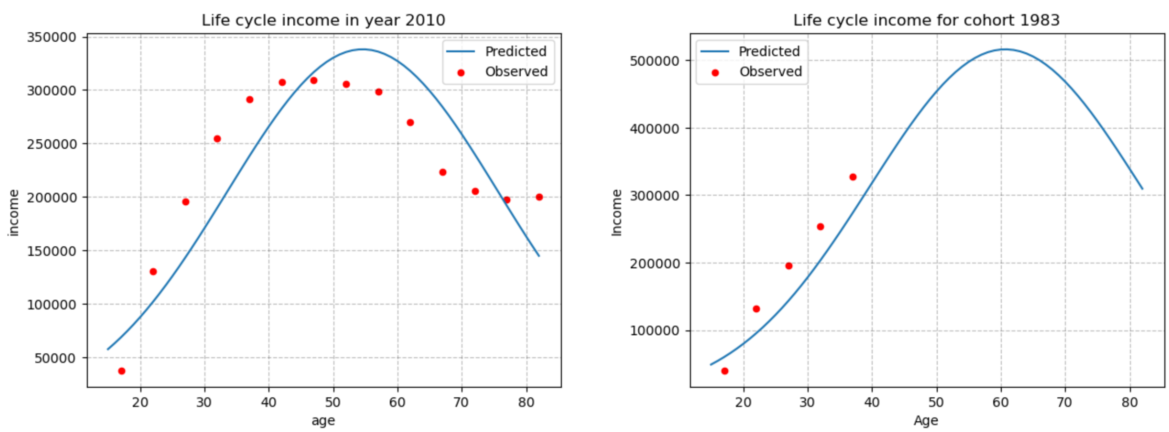

Figure 1. Data Points and Estimates used for Income Callibration

3.2 Households

See 0c - steady state - households.ipynb.

The household's employment status and consumption-saving behaviour in steady state is derived and plotted in 0c - steady state - households.ipynb. The resulting household income and wealth income ratio is compared to the historical data (shown in figure 2).

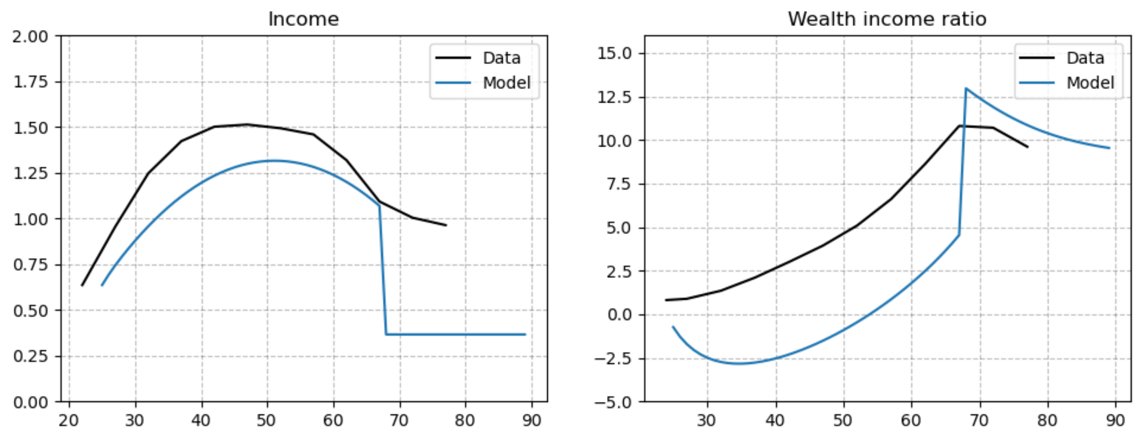

Figure 2. Icome and Savings Comparison

The notebook then compares model outcomes when certain paramterers are changed, as an intuitive check of the model's behaviour:

- A_{ss}^{q}: If the households receive less bequest, their income are lower and they will consume less.
- μ^{A^q} : If the households have a lower preference for bequest, they will save less and consume more.
- W_{ss} : If the households receive a higher wage, they will have more income which implies more consumption.
- τ_{ss} : If the tax rate is lower, the households receive more after-tax wage. Therefore, this is similar to increasing the wage.
- W_U: If the households receive a higher unemployment benefit, they will have more income before retirement; thus, they can consume and save more. However, this effect is much smaller than the effect from a higher wage, as the number of unemployed is much smaller than the number of employed.
- W_R: If the households receive more in retirement benefits, they will have more income once they retire. This implies that they save less, and they consume more throughout their life.
- β: If beta decreases, the households prefer to consume more today relative to tomorrow. As such, they save less and they consume more in the first part of their lifetime.

- r_{hh} : If the interest rate decreases, it is relatively more expensive to save than consume. Therefore, the households will save less and consume more initially.

- λ: If the share of HtM households decreases, the aggregation of households will smooth consumption more.

3.3 Aggregates

3.4 Aggregates

See 0d - steady state - aggregate.ipynb

The model's steady state is computed and the agregate ratios are compared to those found in the data:

Table 1. Comparison of Aggregate Steady State Ratios

	Model	Data
M/Y	0.46	0.46
X/Y	0.60	0.52
C/Y	0.36	0.47
G/Y	0.25	0.25
I/Y	0.26	0.22
K/Y	2.56	5.44
Employment to population ratio	0.72	0.74
Unemployment rate	1.67	3.93

4 Shocks

In this section we consider different shocks to exogenous variables and study the dynamics of the model compared to empiricle impulse responses. We use data from the report »*Matching af impuls responser og øvrige kortsigtsmomenter: MAKRO ift. empirien*« by DREAM. The report contains a foreign demand shock, a government spending shock and a foreign interest rate shock. We shock these exogenous variables in our model to best match the empiricle shocks, and compare the dynamics in the other variables to the empiricle impulse responses.

4.1 Foreign demand shock

See 1b - IRF export market.ipynb.

Initial effect: Initially, there is a shock to the foreign demand shifter, $\chi \uparrow$, the foreign prices,$P^F \uparrow$, and the import prices,$P^{M,\bullet} \uparrow$ for $\bullet \in \{C, G, I, X\}$and the foreign interest rate, $r^{HH} \uparrow$. These are chosen exogenously to match the empirical shocks.

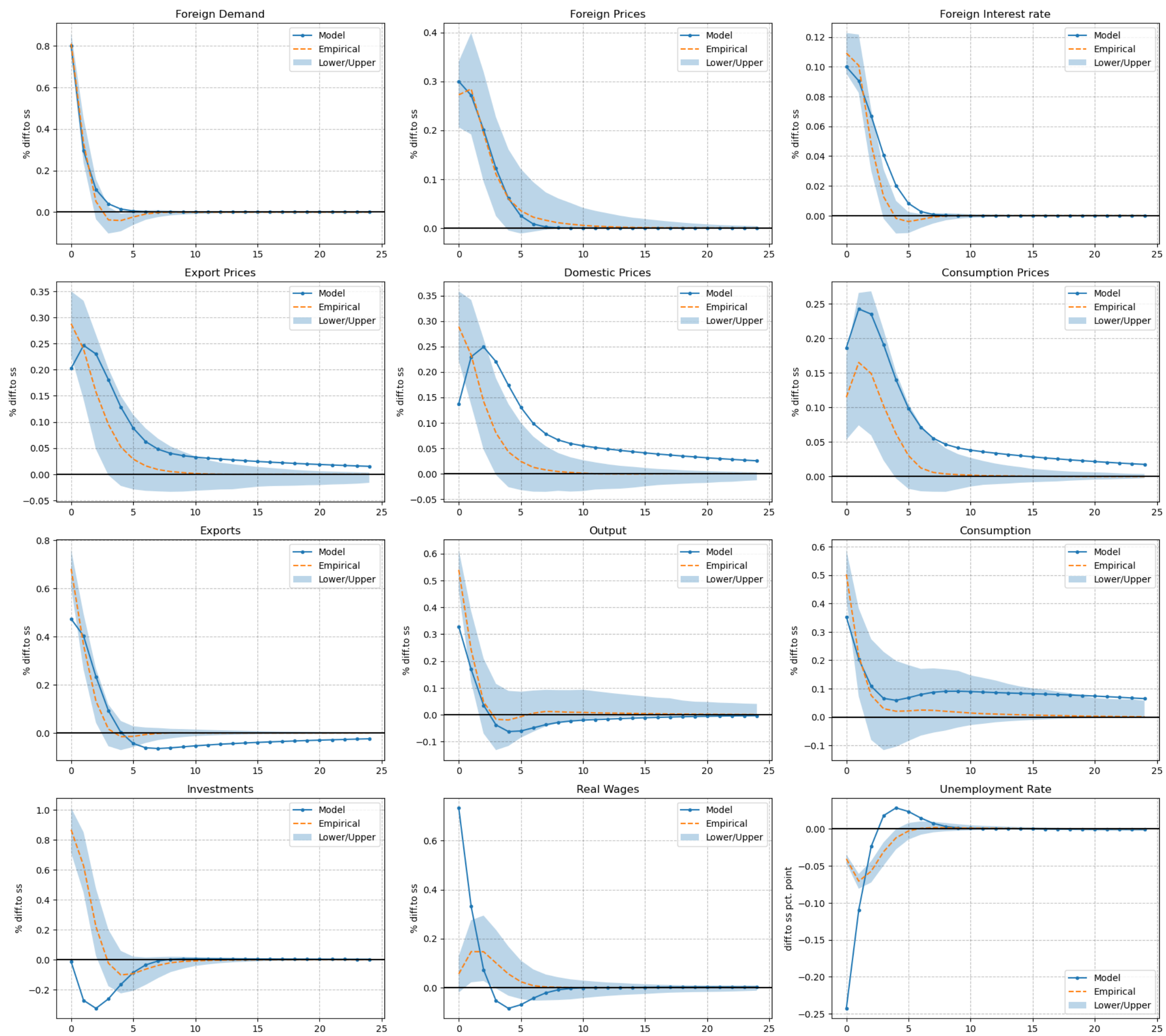

Figure 3. Foreign demand shock

As seen above the model does reasonably well matching the empirical response in the national account variables with the exception of investemnts. Prices also follow a simmilar path as the empirical IRFs, but are more sticky and have a slower convergence back towards steady state. The labor market differs from the empirical response, and shows much larger initial fluctuations. Real wages rise more than expected and unemployment falls more than expected. The shock to the foreign demand affects the domestic economy through the three exogenous channels: the foreign demand, the foreign prices and the foreign interest rate. We look at the effects from the channels one by one.

Examining the foreign demand shock channel, a higher foreign demand implies more goods are exported, $X \uparrow$, and additionally an increasing price of export, $P^X \uparrow$. As a consequence, the total demand and the output prices increase, $Y \uparrow$ and $P^Y \uparrow$. The higher output prices affect prices on consumption, investment and government consumption, such that these increase, $P^\bullet \uparrow$ for $\bullet \in \{C, G, I\}$. As the import prices are unaffected in this channel, imports becomes relatively cheaper compared to the domestic production, such that import increases, $M \uparrow$. The higher demand, $Y \uparrow$ leads to a higher labor demand increasing wages and decreasing unemployment, $W \uparrow$ and $U \downarrow$, thus increasing real income, consumption and savings. The higher employment increases the real marginal product of capital inducing more investments.

Examining the price shock channel, higher foreign prices increase the foreign demand for domestic goods. As such, it is qualitatively similar to an increase in the demand shifter, and the total demand as well as domestic prices increase, $Y \uparrow$ and$P^\bullet \uparrow$ for $\bullet \in \{Y, C, G, I\}$. However, the higher import prices make import less attractive, $M \downarrow$, via direct effects from domestic demand $\bullet_t^M \downarrow$ for $\bullet \in \{C, G, I, X\}$. Again, the higher demand, $Y \uparrow$ increases wages, employment, income, consumption and savings. However, the effect from higher import prices is a negative, domestic demand shock. The firms foresee that the lower future demand implies a lower future production. Hence, less investments are made, $I \downarrow$, such that there is a drop in the capital, $K \downarrow$.

The interest rate channel affects the ricardian households who face an income effect and substitution effect. The substitution effect dominates in this model leading to lower domestic demand and higher savings. The lower domestic demand among ricardian houseolds leads to lower consumption, $C \downarrow$ decreasing output $Y \downarrow$ and demand for labour $U \uparrow$. These effects are dominated by the foreign demand and price channels, so total consumption, output and savings grow, while unemployment falls. The interest rate channel has a positive effect on investments. The lower demand for labor lowers the marginal product of capital, but the lower demand decreases prices, including prices on investments. This leads to an increase in investments, but the positive effects from this channel and the foreign demand channel are dominated by the foreign price channel, so investments fall in the model. In the data investments initially increase substantialy as a result of the higher demand.

4.2 Government spending shock

See 1C - IRF government spending.ipynb.

Initial effect: The government increases real consumption, $G \uparrow$, which implies an increase in the demand for goods. The shock to $G \uparrow$ is chosen exogenously to match the empirical shocks.

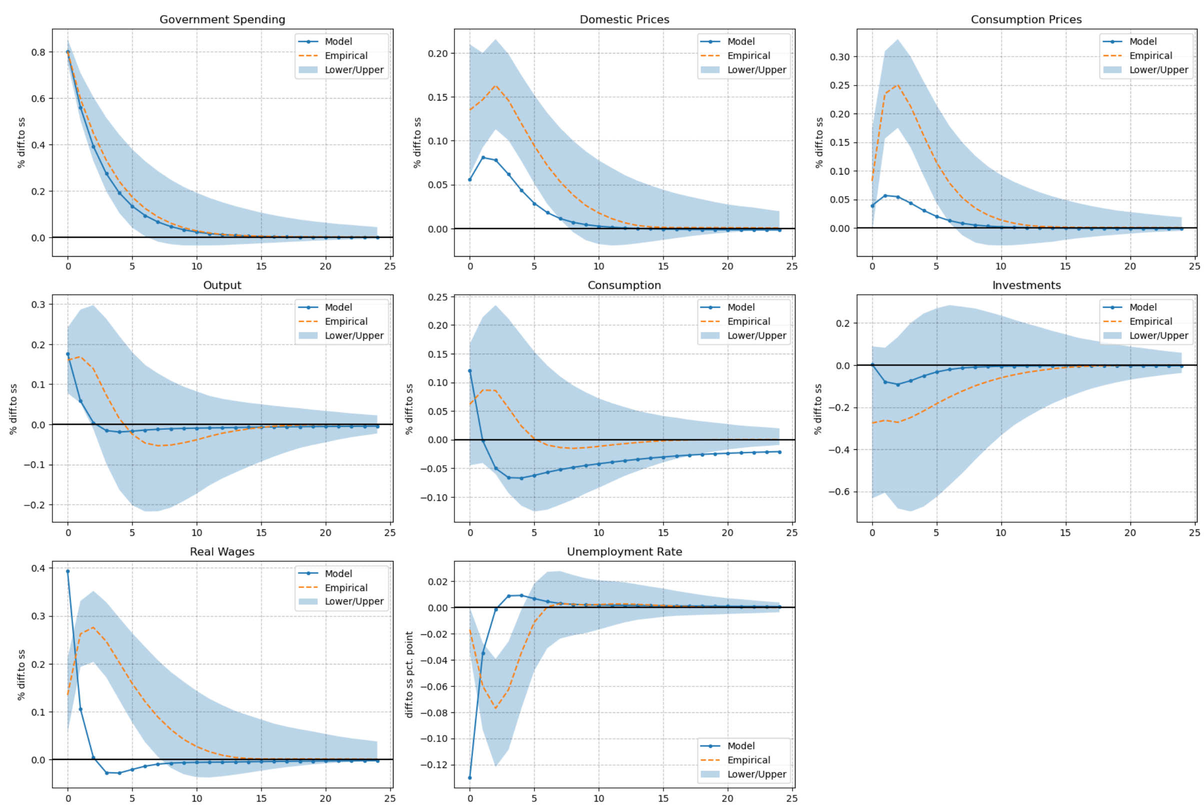

Figure 4. Government spending shock

The model struggles to match the empirical response in prices following higher government spending, with the model yielding to low inflation. The model mathces the national acount identities and labor market levels better, but the responses are to swift and decay faster than seen empiricaly.

The government increases real consumption, $G \uparrow$, which implies an increase in the demand for goods. The firms increases production to meet the higher demand. Since capital is fixed, the production firms can only produce more by hiring more labor, $\ell \uparrow$. This increases wages and drives up the marginal cost price, $P^{Y,0} \uparrow$. As the output price is a mark-up of the marginal cost, the output prices increase, $P^{Y} \uparrow$. However, this increase is limited due to the stickyness of the output prices, and real wages increase. The higher real wage and the higher employment rate imply that the real income increases, $\text{inc}^{real} \uparrow$. As such, the HtM households increase their consumption, $C^{HtM} \uparrow$. The Ricardian households forsee that the government will raise the tax rate in future periods and in order to smooth their consumption, they choose to save up, $A \uparrow$. This implies that they can afford less consumption, $C^{R} \downarrow$. Overall, total consumption increases, $C \uparrow$. The price on consumption increases, $P^{C} \uparrow$, following the higher output prices, which implies inflation and thereby a lower real interest rate, $r^{real} \downarrow$. This leads to fewer investments, implying a crowding out effect on investments.

4.3 Foreign interest rate shock

See 1d - IRF interest rate.ipynb.

Initial effect: Initially, there is a shock to the foreign interest rate, $r^{HH} \uparrow$, which decreases the foreign demand, $\chi \downarrow$ the foreign prices,$P^{F} \downarrow$, and the import prices,$P^{M,\bullet} \downarrow$ for $\bullet \in \{C, G, I, X\}$. These are chosen exogenously to match the empirical shocks.

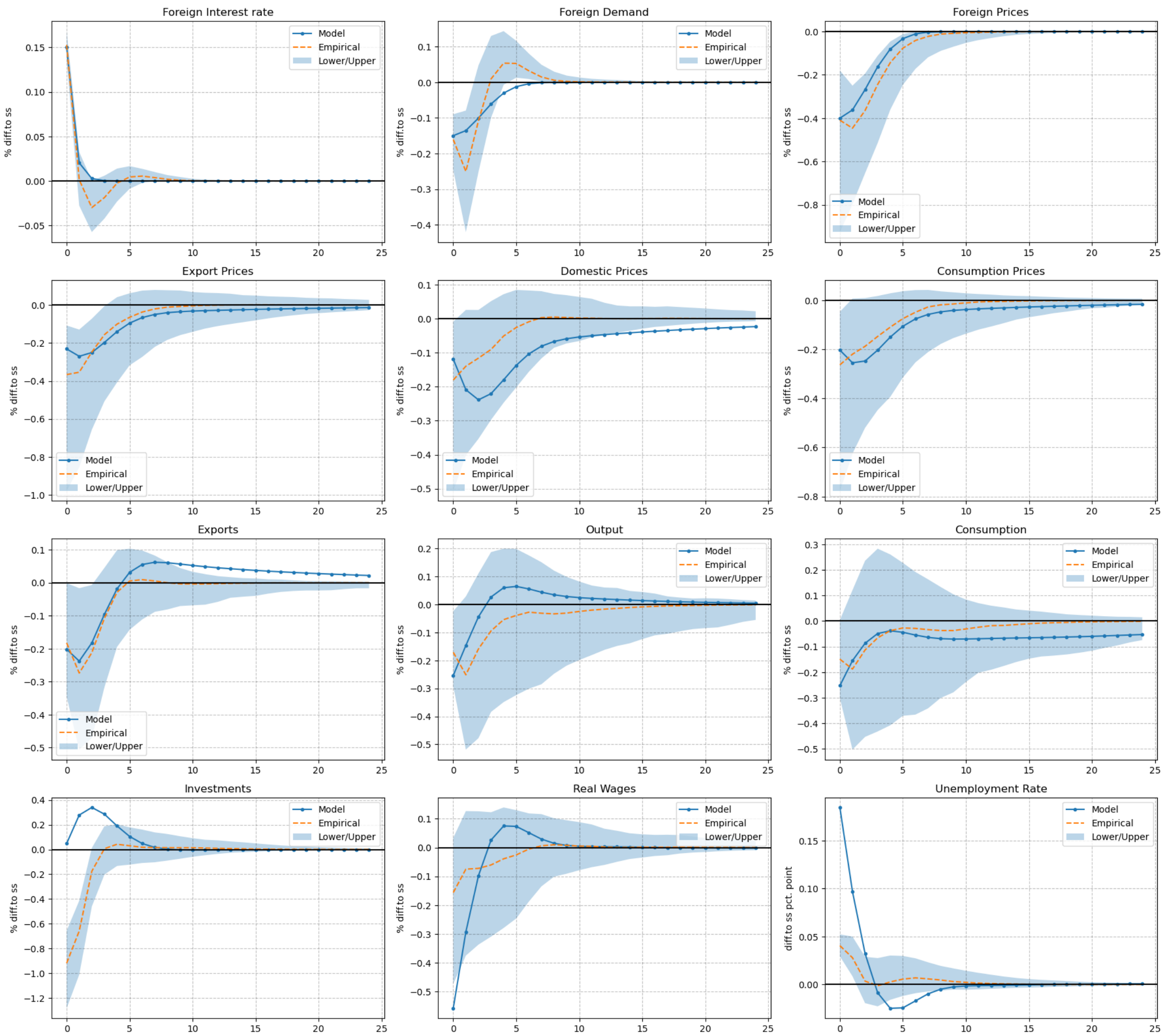

Figure 5. Foreign interest rate shock

As with the foreign demand shock, the model matches the empirical responses well, with the exception of investments, that move opposite of what the data suggests. The model is once again affected by a foreign demand channel, a foreign price channel, and a foreign interest rate channel. All the channels now decreases output, employment, wages and consumption. Investments are postive in the model due to lower investment prices, and firms expecting higher output in the future. In the data investments are significantly bellow steady state due to the low demand.

5 Status report

Status: The described model is implemented in Python. Results are still not understood in detail.

Tasks:

1. Clarify steady state calibration.
2. Perform impulse-response matching.
3. Improve detailed accounting of main shocks.
4. Add technology growth, population growth, and trend inflation.
5. Add financial flows accounts wrt. to the foreign economy.
6. Add more government with taxes and spending.
7. Add endogenous labor supply.
8. Add multiple sectors and input-output structure.
9. Make the model quarterly.

Computational improvements: Improve calculation of Jacobian (e.g.. automatic differentiation)[3].

[3] Investigate what is done efficiently in MAKRO (GAMS+CONOPT)

A Derivations

A.1 Capital agency

$$
\begin{aligned}
V\left(K_{t-1}\right) = \max_{\{K_t\}} \sum_{t=0}^{\infty} \frac{1}{\left(1+r^{\text{firm}}\right)^t} & \left[r_t^K K_{t-1} - P_t^I(\iota_t + \Psi(\iota_t, K_{t-1}))\right] \\
\text{s.t.} & \\
I_t &= \iota_t + \Psi(\iota_t, K_{t-1}) \\
K_t &= (1-\delta^K)K_{t-1} + \iota_t \\
\text{with } \Psi(\iota_t, K_{t-1}) &= \frac{\Psi_0}{2}\left(\frac{\iota_t}{K_{t-1}} - \delta^K\right)^2 K_{t-1}
\end{aligned}
$$

Inserting constraints:

$$
V\left(K_{t-1}\right) = \max_{\{K_t\}} \sum_{t=0}^{\infty} \frac{1}{\left(1+r^{\text{firm}}\right)^t} \left[r_t^K K_{t-1} - P_t^I(K_t - (1-\delta^K)K_{t-1} + \Psi(\iota_t, K_{t-1}))\right]
$$

First order condition:

$$
\begin{aligned}
\frac{\partial V(K_{t-1})}{\partial K_t} &= \frac{1}{(1+r^{\text{firm}})^t}\left(-P_t^I\left(1+\Psi_\iota(\iota_t, K_{t-1})\right)\right) \\
&+ \frac{1}{(1+r^{\text{firm}})^{t+1}}\left(r_{t+1}^K - P_{t+1}^I\left(-(1-\delta^K) + \Psi_\iota(\iota_{t+1}, K_t)(1-\delta^K) + \Psi_K(\iota_{t+1}, K_t)\right)\right) = 0
\end{aligned}
$$

From which it follows:

$$
\begin{aligned}
& -P_t^I\left(1+\Psi_\iota(\iota_t, K_{t-1})\right) \\
+ \frac{1}{(1+r^{\text{firm}})}\left(r_{t+1}^K - P_{t+1}^I\left(-(1-\delta^K) - \Psi_\iota(\iota_{t+1}, K_t)(1-\delta^K) + \Psi_K(\iota_{t+1}, K_t)\right)\right) &= 0
\end{aligned}
$$

A.2 Labor agency

$$\begin{aligned}
V(L^H_{t-1}) &= \max_{\{L_t\}} \sum_{t=0}^{\infty} \frac{1}{(1+r^{\text{firm}})^t} \left[r^\ell_t \ell_t - W_t L^H_t \right] \\
&\text{s.t.} \\
\ell_t &= L^H_t - \kappa^L v_t \\
L^H_t &= \left(1 - \delta^L_t\right) L^H_{t-1} + m^v_t H_t v_t
\end{aligned}$$

Inserting constraints:

$$V(L^H_{t-1}) = \max_{\{L_t\}} \sum_{t=0}^{\infty} \frac{1}{(1+r^{\text{firm}})^t} \left[r^\ell_t \left(L^H_t - \kappa^L \frac{L_t - (1-\delta^L_t) L_{t-1}}{m^v_t H_t} \right) - W_t L^H_t \right]$$

First order condition:

$$\frac{\partial V(L^H_{t-1})}{\partial L^H_t} = \frac{1}{(1+r^{\text{firm}})^t} \left(r^\ell_t \left(1 - \frac{\kappa^L}{m^v_t H_t} \right) - W_t \right) + \frac{1}{(1+r^{\text{firm}})^{t+1}} \left(r^\ell_{t+1} \frac{\kappa^L}{m^v_{t+1} H_{t+1}} (1 - \delta^L_{t+1}) \right) = 0$$

From which it follows:

$$\begin{aligned}
\left(r^\ell_t \left(1 - \frac{\kappa^L}{m^v_t H_t} \right) - W_t \right) + \frac{1}{(1+r^{\text{firm}})} \left(r^\ell_{t+1} \frac{\kappa^L}{m^v_{t+1} H_{t+1}} (1 - \delta^L_{t+1}) \right) &= 0 \\
\frac{1}{1 - \frac{\kappa^L}{m^v_t H_t}} \left[W_t - \frac{1}{(1+r^{\text{firm}})} \left(r^\ell_{t+1} \frac{\kappa^L}{m^v_{t+1} H_{t+1}} (1 - \delta^L_{t+1}) \right) \right] &= r^\ell_t
\end{aligned}$$

A.3 Phillips Curve

$$
\begin{aligned}
p_t^Y &= \frac{\sigma_D}{\sigma_D - 1} P_t^{Y,0} - \frac{1}{\sigma_D - 1} \frac{p_t^Y}{y_t} \left(\frac{\partial g_t}{\partial p_t^Y} + \frac{1}{1 + r^{\text{firm}}} \frac{\partial g_{t+1}}{\partial p_t^Y} \right) \\
p_t^Y &= \frac{\sigma_D}{\sigma_D - 1} P_t^{Y,0} - \frac{1}{\sigma_D - 1} \frac{p_t^Y}{y_t} \left(\gamma \left[\frac{p_t^Y / p_{t-1}^Y}{p_{t-1}^Y / p_{t-2}^Y} - 1 \right] \left(\frac{1 / p_{t-1}^Y}{p_{t-1}^Y / p_{t-2}^Y} \right) P_t^Y Y_t \right. \\
&\quad \left. + \frac{1}{1 + r^{\text{firm}}} \gamma \left[\frac{p_{t+1}^Y / p_t^Y}{p_t^Y / p_{t-1}^Y} - 1 \right] 2 \left(\frac{p_{t+1}^Y / p_{t-1}^Y}{p_{t-1}^Y / p_{t-2}^Y} \right) P_{t+1}^Y Y_{t+1} \right) \\
P_t^Y &= \frac{\sigma_D}{\sigma_D - 1} P_t^{Y,0} - \frac{1}{\sigma_D - 1} \left(\gamma \left[\frac{P_t^Y / P_{t-1}^Y}{P_{t-1}^Y / P_{t-2}^Y} - 1 \right] \left(\frac{P_t^Y / P_{t-1}^Y}{P_{t-1}^Y / P_{t-2}^Y} \right) P_t^Y \right. \\
&\quad \left. + \frac{2}{1 + r^{\text{firm}}} \gamma \left[\frac{P_{t+1}^Y / P_t^Y}{P_t^Y / P_{t-1}^Y} - 1 \right] \left(\frac{P_{t+1}^Y / P_{t-1}^Y}{P_{t-1}^Y / P_{t-2}^Y} \right) P_{t+1}^Y \frac{Y_{t+1}}{Y_t} \right) \\
P_t^Y &= (1 + \theta) P_t^{Y,0} - \eta \left(\frac{P_t^Y / P_{t-1}^Y}{P_{t-1}^Y / P_{t-2}^Y} - 1 \right) \frac{P_t^Y / P_{t-1}^Y}{P_{t-1}^Y / P_{t-2}^Y} P_t^Y \\
&\quad + \frac{2}{1 + r^{\text{firm}}} \eta \frac{Y_{t+1}}{Y_t} \left(\frac{P_{t+1}^Y / P_t^Y}{P_t^Y / P_{t-1}^Y} - 1 \right) \frac{P_{t+1}^Y / P_t^Y}{P_t^Y / P_{t-1}^Y} P_{t+1}^Y
\end{aligned}
$$

where $\theta = \frac{1}{\sigma_D - 1}$.