Baby-MAKRO*

DISCLAIMER: WORK-IN-PROGRESS ⇒ BEWARE OF ERRORS!

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Abstract

This note outlines a simplified, »baby«, version of the MAKRO model used by the Danish Ministry of Finance. The model is for a small open economy with a fixed exchange and overlapping generations. The model have perfect foresight, but is full of imperfections due to e.g. frictions in the labor market and adjustment costs. The model is written and solved in terms of a series of ordered blocks. This clarifies the model dynamics and make it easier to solve for fluctuations around the steady state using a numerical equation system solver. Online code is provided for solving the model in Python.

The model is designed so undergraduate students can work with it, and analyze potential extensions in their thesis work. The model structure is similar to state-of-the-art heterogeneous agent models (see this course) and the model is thus relevant for further academic studies. The similarity to the grown-up MAKRO modelmake it relevant for potential future job tasks and the public debate.

The note concludes with a status report.

Online code: github.com/NumEconCopenhagen/BabyMAKRO

MAKRO: See online documentation

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1 Overview

We consider a *small open economy* with a *fixed exchange rate* and *overlapping generations*. Time is discrete, $t \in \{0, 1, \dots\}$ and the frequency is annual.

Households live for A periods, and their age is denoted by a. The population is constant. Households exogenously search for jobs and supply labor, receives inheritances and choose consumption and savings to get utility from consumption and bequests.

The *foreign economy* provides a fixed nominal rate of return, sell import goods at fixed prices, and have a demand curve for the domestic export good.

The *production* in the economy is layered as follows:

1. *Production firms* rent *capital* and *labor* to produce the domestic output good.
2. *Repacking firms* combine imported goods with the domestic output good to produce a consumption good, an investment good, and an export good.
3. Capital is rented from a *capital agency*, which purchases the investment good to accumulate capital subject to adjustment costs.
4. Labor is rented from a *labor agency*, which post vacancies a search-and-match labor market to purchase labor from the households.

All firms are price takers. Prices are thus flexible, except for wages which is determined by ad hoc bargaining. All *goods markets clear* and the matching process is determined by a *matching function*.

There is *perfect foresight* in the economy. I.e. the value of all current and future variables are known. This is a strong assumption, and in many ways the model should be considered a first order approximation to a full model with both idiosyncratic and aggregate risk. It can be relevant to introduce model elements, which proxy for the effects of risks. Utility-of-wealth can e.g. proxy for the precautionary saving motive.

1.1 Equilibrium path

The *equilibrium path* in the economy is a set of path for all variables, which satisfies all accounting identities, optimal firm and household behavior in terms of first order conditions, and implies market clearing. When all variables are constant over time, the equilibrium path is a *steady state*.

In terms of math, the model is just an *equation system* stacking the accounting identifies, first order conditions and market clearing conditions. If the economy is initially out of steady state, we solve for the equilibrium path by truncating the equation system to T periods. The assumption is that the economy has settled down to the steady state well before period T, and we can assume variables from period T onward are at their steady state value. The economy can be out of steady state both because *endogenous* variables initially are not at their steady state values and/or because the *exogenous* variables are out-of-steady state. We talk of an *impulse response* when the economy starts at the steady state, but some exogenous variables *temporarily* deviate from the steady state following some converging auto-regressive process.[1]

We simplify the model and the resulting equation system by writing it in terms of a *ordered series of block*. We start from a set of *exogenous* variables (e.g. variables determined in the foreign economy) and a set of *unknown* variables. Each block then takes in the path of some variables, return the path of other variables, and imply *targets*, which must be zero if the model equations are satisfied. Each block can use the unknown variables and output variables of previous blocks as input variables. In the end we collect all the targets. The number of unknown variables must equal the number of target variables. We then use a *numerical equation system solver* to first find the steady state and then equilibrium path.

The block structure and ordering is not unique. If a different set of unknowns is chosen, a different ordering of blocks must also be chosen. If an additional variable is considered an *unknown*, an additional equation must be considered a target instead of being used to calculate a output. In the limit, all variables can be considered as unknowns and all equations as targets. This is inefficient as the number of variables can be very large.[2]

[1] This is also called an MIT shock. A shock in a model with perfect foresight is to some degree a contradiction in terms. The assumption is that even though the agents experiences a shock, they expect that there will never be a shock again. Accounting for expecting of future shocks is much more complicated. Some realism can be mimicked by studying impulse response to shocks about the future, which when the future comes never materialized as a new opposite signed shock negates it. Multiple shocks arriving sequentially can also be studied.

[2] Modeling systems such as GAMS can combined with a state-of-the-art solver such as CONOOPT automatically analyze the structure of the equation system and thereby ripe the benefits we get

1.2 On CES technology

The assumption of CES technology is used repeatedly in the model. It is therefore beneficial to recap it briefly. Consider a firm producing good X using good X_i and X_j with a CES technology. Input prices are P_i and P_j and the output price is P. The firm is a price taker in all markets. The *profit maximization* problem of the firm is

$$\max_{X_i, X_j} PX - P_i X_i - P_j X_j \text{ s.t. } X = \left(\mu_i^{\frac{1}{\sigma}} X_i^{\frac{\sigma-1}{\sigma}} + \mu_j^{\frac{1}{\sigma}} X_j^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}}, \;\; \mu_i + \mu_j = 1, \, \mu_i > 0, \, \sigma > 0 \tag{1}$$

The generic *first order condition* is

$$\begin{aligned}
0 &= P \mu_i^{\frac{1}{\sigma}} X_i^{\frac{\sigma-1}{\sigma}-1} \left(\mu_i^{\frac{1}{\sigma}} X_i^{\frac{\sigma-1}{\sigma}} + \mu_j^{\frac{1}{\sigma}} X_j^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}-1} - P_i \Leftrightarrow \\
P_i &= P \mu_i^{\frac{1}{\sigma}} X_i^{-\frac{1}{\sigma}} \left(\mu_i^{\frac{1}{\sigma}} X_i^{\frac{\sigma-1}{\sigma}} + \mu_j^{\frac{1}{\sigma}} X_j^{\frac{\sigma-1}{\sigma}} \right)^{\frac{1}{\sigma-1}} \Leftrightarrow \\
P_i &= P \mu_i^{\frac{1}{\sigma}} X_i^{-\frac{1}{\sigma}} Y^{\frac{1}{\sigma}} \Leftrightarrow \\
X_i &= \mu_i \left(\frac{P}{P_i} \right)^{\sigma} X.
\end{aligned} \tag{2}$$

As the production technology has constant return-to-scale, there are infinitely many solutions to the FOCs. They all satisfy that inputs are used in proportion as follows

$$\frac{X_i}{X_j} = \frac{\mu_i}{\mu_j} \left(\frac{P_j}{P_i} \right)^{\sigma} \tag{3}$$

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from manually ordering the blocks.

Assuming *free entry*, and thus *zero profits*, the output price is uniquely determined from the input prices as

$$
\begin{aligned}
0 &= PX - P_i X_i - P_j X_j \Leftrightarrow \\
P &= \frac{P_i X_i + P_j X_j}{X} \\
&= \mu_i \left(\frac{P}{P_i}\right)^{\sigma} P_i + \mu_j \left(\frac{P}{P_j}\right)^{\sigma} P_j \Leftrightarrow \\
P^{1-\sigma} &= \mu_i P_i^{1-\sigma} + \mu_j P_j^{1-\sigma} \Leftrightarrow \\
P &= \left(\mu_i P_i^{1-\sigma} + \mu_j P_j^{1-\sigma}\right)^{\frac{1}{1-\sigma}}. \qquad (4)
\end{aligned}
$$

2 Blocks

2.1 Exogenous variables

The *exogenous* variables are:

1. $P_t^{M,C}$, import price of *private consumption* component good
2. $P_t^{M,G}$, import price of *public consumption* component good
3. $P_t^{M,I}$, import price of *investment* component good
4. $P_t^{M,X}$, import price of *export* component good
5. P_t^{F} , *foreign* price level
6. χ_t, *foreign* demand shifter
7. G_t, public spending
8. $\tilde{\tau}_t$, tax rate (before adjustment)

2.2 Unknown variables

The chosen *unknown* variables are:

1. L_t, labor supply (T unknowns)

2. K_t, capital (T unknowns)

3. r_t^K, rental price for capital (T unknowns)

4. w_t, wage (T unknowns)

5. $B_t^{R,q}$, inheritance flow (T unknowns)

The total number of unknowns thus is $5T$.

2.3 Block I. Households - search behavior and matching

Households search for a job and supply labor exogenously. The age-specific job-separation probability is $\delta_a^L \in (0,1)$ and the retirement age is $A_R < A$. All unemployed search for a job. As an initial condition, we have $L_{-1,t-1} = 0$.
The quantity of searchers are

$$
\begin{aligned}
S_{a,t} &= \begin{cases} 1 & \text{if } a = 0 \\ \delta_a^L L_{a-1,t-1} & \text{if } a < A_R \\ 0 & \text{if } a \geq 0 \end{cases} \\
S_t &= \sum_a S_{a,t}.
\end{aligned}
$$

The quantity of households with a job *before matching* is

$$
\begin{aligned}
\underline{L}_{a,t} &= \begin{cases} 0 & \text{if } a = 0 \\ (1-\delta_a^L) L_{a-1,t-1} & \text{if } a < A_R \\ 0 & \text{if } a \geq 0 \end{cases} \\
\underline{L}_t &= \sum_a \underline{L}_{a,t}.
\end{aligned}
$$

The aggregate job-separation rate is

$$
\delta_t^L = \frac{\underline{L}_t - L_{t-1}}{L_{t-1}}.
$$

The quantity of vacancies is v_t and the number of matches, $\mathcal{M}_t$, is given by the *matching function*

$$\mathcal{M}_t = \frac{S_t v_t}{\left(S_t^{\frac{1}{\sigma^m}} + v_t^{\frac{1}{\sigma^m}}\right)^{\sigma^m}}.$$

The job-filling rate, m_t^v,and the job-finding rate, m_t^s, are thus

$$m_t^v = \frac{\mathcal{M}_t}{v_t}$$
$$m_t^s = \frac{\mathcal{M}_t}{S_t}.$$

The number of employed therefore is

$$L_{a,t} = \underline{L}_t + m_t^s S_{a,t}.$$

In equilibrium, the number of matches, must equal the number of new hires, i.e.

$$\mathcal{M}_t = L_t - \underline{L}_t.$$

We write the **block** in terms of inputs, and outputs as:

- **Inputs:** $\{L_t\}$
- **Outputs:** $\{S_{a,t}\}$, $\{S_t\}$, $\{\delta_t^L\}$, $\{\mathcal{M}_t\}$,$\{v_t\}$, $\{m_t^v\}$, $\{m_t^s\}$, $\{L_{a,t}\}$

$$
S_{a,t} = \begin{cases} 1 & \textbf{if } a = 0 \\ \delta^L_a L_{a-1,t-1} & \textbf{if } a < A_R \\ 0 & \textbf{if } a \geq 0 \end{cases} \tag{5}
$$

$$
\underline{L}_{a,t} = \begin{cases} 0 & \textbf{if } a = 0 \\ (1-\delta^L_a) L_{a-1,t-1} & \textbf{if } a < A_R \\ 0 & \textbf{if } a \geq 0 \end{cases} \tag{6}
$$

$$
S_t = \sum_a S_{a,t}. \tag{7}
$$

$$
\underline{L}_t = \sum_a \underline{L}_{a,t}. \tag{8}
$$

$$
\delta^L_t = \frac{L_{t-1} - \underline{L}_t}{L_{t-1}} \tag{9}
$$

$$
\mathcal{M}_t = L_t - \underline{L}_t \tag{10}
$$

$$
m^s_t = \frac{\mathcal{M}_t}{S_t}
$$

$$
v_t = \left(\frac{(m^s_t)^{\frac{1}{\sigma^m}} S_t^{\frac{1}{\sigma^m}}}{1 - (m^s_t)^{\frac{1}{\sigma^m}}} \right)^{\sigma^m} \tag{11}
$$

$$
m^v_t = \frac{\mathcal{M}_t}{v_t} \tag{12}
$$

$$
L_{a,t} = \underline{L}_t + m^s_t S_{a,t}. \tag{13}
$$

For $t = 0$, the variable $L_{a-1,t-1}$ is pre-determined.

2.4 Block II. Labor agency

The labor agency firms post vacancies, v_t, at the cost κ^L to hire labor, L_t, at the wage w_t, determined by bargaining, and renting it out to production firms at the rental price r^ℓ_t. Matching occurs according to a matching function, and the firms take the job-filling rate and prices as given.

The labor agency problem is:

$$
\begin{aligned}
V(L_{t-1}) = \max_{\{L_t\}} \sum_{t=0}^{\infty} \frac{1}{1+r^{\text{firm}}} & \left[r_t^\ell \ell_t - w_t L_t \right] \\
\ell_t &= L_t - \kappa^L v_t \\
L_t &= \left(1-\delta_t^L\right) L_{t-1} + m_t^v v_t.
\end{aligned}
$$

Using the FOC to L_t from the labor agency problem, we write the **block** as:

- **Inputs:** $\{w_t\}$, $\{m_t^v\}$, $\{\delta_t^L\}$, $\{L_t\}$, $\{v_t\}$
- **Outputs:** $\{r_t^\ell\}$, $\{\ell_t\}$

$$
r_t^\ell = \frac{w_t}{1 - \frac{\kappa^L}{m_t^v} + \frac{1-\delta_t^L}{1+r^{\text{firm}}} \frac{\kappa^L}{m_{t+1}^v}} \tag{14}
$$

$$
\ell_t = L_t - \kappa^L v_t. \tag{15}
$$

The variable L_{-1} is pre-determined.

2.5 Block III. Production firms

The production firms uses capital, K_{t-1}, and labor, ℓ_t, to produce output, Y_t, with a CES technology. The rental price of capital is r_t^K and the rental price of labor is r_t^ℓ. The firm is a price taker in all markets and free entry implies zero profits.

Using the results with CES technology derived in sub-section 1.2, we write the **block** as:

- **Inputs:** $\{K_t\}$, $\{\ell_t\}$, $\{r_t^K\}$, $\{r_t^\ell\}$
- **Outputs:** $\{Y_t\}$, $\{P_t^Y\}$

$$
Y_t = \left(\left(\mu^K\right)^{\frac{1}{\sigma^Y}} K_{t-1}^{\frac{\sigma^Y-1}{\sigma^Y}} + \left(1-\mu^K\right)^{\frac{1}{\sigma^Y}} \ell_t^{\frac{\sigma^Y-1}{\sigma^Y}} \right)^{\frac{\sigma^Y}{\sigma^Y-1}} \tag{16}
$$

$$
P_t^Y = \left(\mu^K \left(r_t^K\right)^{1-\sigma^Y} + \left(1-\mu^K\right) \left(r_t^\ell\right)^{1-\sigma^Y} \right)^{\frac{1}{1-\sigma^Y}}. \tag{17}
$$

- **Targets:**

$$\frac{K_{t-1}}{\ell_t} = \frac{\mu_K}{1-\mu_K}\left(\frac{r_t^{\ell}}{r_t^K}\right)^{\sigma^Y}. \tag{18}$$

The variable K_{-1} is pre-determined.

2.6 Block IV. Bargaining

The target wage in the bargaining is specified ad hoc as a weighted average of the marginal product of labor for production firm, MPL_t, and the worker outside option w^U. The bargained wage is a again a weighted average of the past wage and the current target wage to create a real rigidity.

- **Inputs:** $\{w_t\}$, $\{Y_t\}$, $\{\ell_t\}$
- **Outputs:** $\{w_t^*\}$

$$\mathbf{MPL}_t = \left((1-\mu_K)\frac{Y_t}{\ell_t}\right)^{\frac{1}{\sigma^Y}} \tag{19}$$

$$w_t^* = \psi\mathbf{MPL}_t + (1-\psi)w^U \tag{20}$$

- **Targets:**

$$w_t = \gamma^w w_{t-1} + (1-\gamma^w)w_t^* \tag{21}$$

2.7 Block V. Repacking firms - prices

The output good, Y_t, can be used for either private consumption, C_t, public consumption, G_t, investment, I_t, or exports, X_t. For each use the output good must be repacked with imported goods. This is done by repacking firms with a CES production technology.

Using the results with CES technology derived in sub-section 1.2, we write the **block** for the pricing part of this as:

1. **Inputs:** $\{P_t^Y\}$, $\left\{P_t^{M,\bullet}\right\}$ for $\bullet \in \{C, G, I, X\}$

2. **Output:** $\{P_t^\bullet\}$ for $\bullet \in \{C, G, I, X\}$

$$P_t^\bullet = \left(\mu^{M,\bullet}\left(P_t^{M,\bullet}\right)^{1-\sigma^\bullet} + \left(1-\mu^{M,\bullet}\right)\left(P_t^Y\right)^{1-\sigma^\bullet}\right)^{\frac{1}{1-\sigma^\bullet}}. \tag{22}$$

2.8 Block VI. Foreign economy

The foreign economy have so-called Armington demand the domestic export good. We write the **block** as:

1. **Inputs:** $\left\{P_t^F\right\}$,$\{\chi_t\}$, $\left\{P_t^Y\right\}$
2. **Outputs:** $\{X_t\}$

$$X_t = \chi_t \left(\frac{P_t^Y}{P_t^F}\right)^{-\sigma^F} \tag{23}$$

2.9 Block VII. Capital agency

The capital agency firm buys investment goods, I_t, at price P_t^I, to accumulate capital, K_t, which it rents out to production at the rental rate r_t^K in the following period. The investment decision is subject to convex adjustment costs in terms of wasted investment goods, such that effective investment is ι_t. Future profits are discounted with r^{firm}. The capital agency takes prices as given, and its problem thus is:

$$\begin{aligned}
V_0\left(K_{t-1}\right) &= \max_{\{K_t\}} \sum_{t=0}^{\infty} \frac{1}{\left(1+r^{\text{firm}}\right)^t}\left[r_t^K K_t - p_t^I(\iota_t + \Psi(\iota_t, K_{t-1}))\right] \\
&\text{s.t.} \\
I_t &= \iota_t + \Psi(\iota_t, K_{t-1}) \\
K_t &= (1-\delta^K)K_{t-1} + \iota_t.
\end{aligned}$$

We choose

$$\Psi(\iota_t, K_t) = \frac{\Psi_0}{2}\left(\frac{\iota_t}{K_t} - \delta\right)^2 K_t$$
$$\Psi_\iota(\iota_t, K_t) = \Psi_0\left(\frac{\iota_t}{K_t} - \delta\right)$$
$$\Psi_K(\iota_t, K_t) = \frac{\Psi_0}{2}\left(\frac{\iota_t}{K_t} - \delta\right)^2 - \Psi_0\left(\frac{\iota_t}{K_t} - \delta\right)\frac{\iota_t}{K_t}$$

We write the **block** as

- **Inputs:** $\{r_t^K\}$, $\{p_t^I\}$, $\{K_t\}$
- **Outputs:** $\{\iota_t\}, \{I_t\}$

$$\iota_t = K_t - \left(1 - \delta^K\right) K_{t-1} \tag{24}$$
$$I_t = \iota_t + \Psi(\iota_t, K_{t-1}) \tag{25}$$

- **Targets:**

$$0 = -P_t^I\left(1 + \Psi_\iota\left(\iota_t, K_{t-1}\right)\right) + \frac{1}{1 + r^{\text{firm}}}\left[r_{t+1}^k + (1-\delta)P_{t+1}^I(1 + \Psi_\iota(\iota_{t+1}, K_t)) - P_{t+1}^I \Psi_K\left(\iota_{t+1}, K_t\right)\right] \tag{26}$$

The variable K_{-1} is pre-determined.

2.10 BloV VIII. Government

The government budget is given by

$$B_t = (1 + r^B)B_{t-1} + P_t^G G_t - \tau_t w_t L_t \tag{27}$$

where r^B is the interest rate on government debt determined in the foreign economy. We assume the government have the exogenous tax rate $\tilde{\tau}_t$ for t_B and then begin to adjust taxes to get back to steady state debt. For Δ_B years this is done gradually, and thereafter it is done fully-

- **Inputs:** $\{\tilde{\tau}_t\}$, $\{P_t^G\}$, $\{w_t\}$, $\{L_t\}$

- **Outputs:** $\{\tau_t\}$, $\{\overline{\tau}_t\}$, $\{B_t\}$

$$\overline{\tau}_t = \tau_{ss}\left(B_{t-1}/B_{ss}\right)^{\varepsilon_B} \tag{28}$$

$$\tau_t = \begin{cases} \tilde{\tau}_t & \text{hvis } t < t_B \\ (1-\omega_t)\tilde{\tau}_t + \omega_t \overline{\tau}_t & \text{hvis } t \in [t_{B,} t_B + \Delta_B] \\ \overline{\tau}_t & \text{hvis } t > t_B + \Delta_B \end{cases} \tag{29}$$

$$\omega_t = 3\left(\frac{t-t_B}{\Delta_B}\right)^2 - 2\left(\frac{t-t_B}{\Delta_B}\right)^3$$

$$B_t = (1+r^B)B_{t-1} + P_t^G G_t - \tau_t w_t L_t \tag{30}$$

The variable B_{-1} is pre-determined.

2.11 Block IX. Households - consumption -saving

There is a share of λ hands-to-mouth household and a share of unconstrained $1-\lambda$ Ricardian households.

The Ricardian households earns income from labor, $w_t L_{a,t}$, pays taxes, $\tau_t w_t L_{a,t}$, gets an evenly divided inheritance, $\frac{(1-\lambda)B_t^{R,q}}{A}$, buys consumption goods at the price P_t^C and has end-of-period savings, $B_{a,t}$. It gets a fixed nominal return on savings given by r^{hh} . The Ricardian household making its first decision in period t_0 solves the problem

$$\begin{aligned}
V_{t_0} &= \max_{\left\{C_{a,t=t_0+a}\right\}_{a=0}^{A-1}} \sum_{a=0}^{A-1} \beta^a \left[\frac{\left(C_t^R\right)^{1-\sigma}}{1-\sigma} + \mathbf{1}_{a=A-1}\mu^B \frac{\left(B_{a,t}^R\right)^{1-\sigma}}{1-\sigma}\right] \\
&\text{s.t.} \\
t &= t_0 + a \\
B_{-1,t}^R &= 0 \\
B_{a,t}^R &= (1+r^{hh})B_{a-1,t-1}^R + (1-\tau_t)w_t L_{a,t} + \frac{(1-\lambda)B_t^{R,q}}{A} - P_t^C C_{a,t}^R.
\end{aligned}$$

The hands-too-mouth household consume all their income

$$C_{a,t}^{\text{HtM}} = (1-\tau_t)w_t L_{a,t} + \frac{(1-\lambda)B_t^{R,q}}{A}$$

Total consumption is

$$C_{a,t} = \lambda C_{a,t}^{\text{HtM}} + (1-\lambda) C_{a,t}^{R}$$

Using the FOC for C_t to the household problem, aggregation, and that inheritance inflows must match bequest outflows, $B_t^q = B_{A-1,t}$, let us write the **block** as:

1. **Inputs:** $\{L_{a,t}\}$, $\{P_t^C\}$, $\{w_t\}$, $\{\tau_t\}$, $\left\{B_t^{R,q}\right\}$
2. **Outputs:** $\{\pi_t^{hh}\}$, $\{C_{a,t}^{\text{HtM}}\}$, $\{C_{a,t}^{R}\}$, $\{B_{a,t}^{R}\}$, $\{C_t^R\}$, $\{B_t^R\}$, $\{C_t\}$

   Calculate

$$\pi_t^{hh} = \frac{p_t^C}{p_{t-1}^C} - 1 \tag{31}$$

$$B_{A-1,t}^{\text{R}} = B_t^{R,q} \tag{32}$$

$$C_{a,t}^{\text{HtM}} = (1-\tau_t) w_t L_{a,t} + \frac{(1-\lambda) B_t^{R,q}}{A} \tag{33}$$

   Use FOCs backwards

$$C_{a,t}^{R} = \begin{cases} \left(\mu^B \left(B_t^{R,q}\right)^{-\sigma}\right)^{-\frac{1}{\sigma}} & \text{if } a = A-1 \\ \left(\beta \frac{1+r_{hh}}{1+\pi_{ss}^{hh}} \left(C_{a+1,ss}^{R,}\right)^{-\sigma}\right)^{-\frac{1}{\sigma}} & \text{elif } t = T-1 \\ \left(\beta \frac{1+r_{hh}}{1+\pi_{t+1}^{hh}} \left(C_{a+1,t+1}^{R,}\right)^{-\sigma}\right)^{-\frac{1}{\sigma}} & \text{else} \end{cases}$$

   Calculate savings forwards

$$B_{a,t}^{R} = (1+r^{hh}) B_{a-1,t-1}^{R} + (1-\tau_t) w_t L_{a,t} + \frac{(1-\lambda) B_t^{R,q}}{A} - P_t^C C_{a,t}^{R} \tag{34}$$

   Calculate total consumption

$$C_{a,t} = \lambda C_{a,t}^{\text{HtM}} + (1-\lambda) C_{a,t}^{R} \tag{35}$$

$$C_t = \sum_a C_{a,t} \tag{36}$$

3. **Targets**

$$B_t^{R,q} = B_{a,t}^R \tag{37}$$

For $t = 0$, we have that the variable $B_{a-1,t-1}^R$ is pre-determined.

2.12 Block X. Repacking firms - components

The repacking firms were described in sub-section 2.7. Using additional results from sub-section 1.2 on CES technology, we write the **block** as:

1. **Inputs:** $\{P_t^Y\}$, $\left\{P_t^{M,\bullet}\right\}$, $\{P_t^\bullet\}$, $\{\bullet_t\}$ for $\bullet \in \{C, G, I, X\}$
2. **Output:** $\{\bullet_t^M\}$, $\{\bullet_t^Y\}$ for $\bullet \in \{C, G, I, X\}$

$$\bullet_t^M = \mu^{M,\bullet} \left(\frac{P_t^\bullet}{P_t^{M,\bullet}}\right)^{\sigma^\bullet} \bullet_t \tag{38}$$

$$\bullet_t^Y = \left(1 - \mu^{M,\bullet}\right) \left(\frac{P_t^\bullet}{P_t^Y}\right)^{\sigma^\bullet} \bullet_t \tag{39}$$

2.13 Block XI. Goods market clearing

The production of the domestic output good must match the about of the output good used by the repacking firms. Imports are used the sum of the imports of the repacking firms.

We write the **block** as:

- **Inputs:** $\{\bullet_t^Y\}$, $\{\bullet_t^X\}$ for $\bullet \in \{C, G, I, X\}$
- **Outputs:** $\{M_t\}$

$$M_t = \sum_{\bullet \in \{C,G,I,X\}} \bullet_t^M \tag{40}$$

- **Targets:**

$$Y_t = \sum_{\bullet \in \{C,G,I,X\}} \bullet_t^Y \tag{41}$$

2.14 Total number of targets

We have $5 \times T$ targets in Equation (18), (21), (26), (41), and and (37), which is equal to number of unknowns.

3 Steady state

We set $B_{ss} = 0$ and let G_{ss} be chosen freely.

We want the job-finding probability to be m_{ss}^{s} and adjust exogenous variables and other parameters to fit with this. We can then find the steady state as follows:

1. Price price normalization:

$$P_{ss}^{Y} = P_{ss}^{F} = P_{ss}^{M,\bullet} = 1, \bullet \in \{C, G, I, X\}$$

2. The pricing behavior of repacking firms then implies

$$\begin{aligned} P_{ss}^{\bullet} &= 1, \bullet \in \{C, G, I, X\} \\ \pi_{ss}^{hh} &= 0 \end{aligned}$$

3. The exogenous labor supply and search-and-matching implies

$$S_{a,ss} = \begin{cases} 1 & \text{if } a = 0 \\ \delta_a^L L_{a-1,ss} & \text{if } a < A_R \\ 0 & \text{else} \end{cases}$$

$$\underline{L}_{a,ss} = \begin{cases} 0 & \text{if } a = 0 \\ (1-\delta_a^L) L_{a-1,ss} & \text{if } a < A_R \\ 0 & \text{else} \end{cases}$$

$$\begin{aligned}
L_{a,ss} &= \underline{L}_{a,ss} + m_{ss}^s S_{a,ss} \\
L_{ss} &= \sum_a L_{a,ss} \\
S_{ss} &= \sum_a S_{a,ss} \\
\delta_{ss}^L &= \frac{L_{ss} - \underline{L}_t}{L_{ss}} \\
\mathcal{M}_{ss} &= \delta_{ss}^L L_{ss} \\
v_{ss} &= \frac{\mathcal{M}_{ss}}{\left(1 + S_{ss}^{\frac{1}{\sigma_m}}\right)^{\sigma_m}} \\
m_{ss}^v &= \frac{\mathcal{M}_{ss}}{v_{ss}}
\end{aligned}$$

4. Capital agency behavior implies

$$r_{ss}^K = r^{\text{firm}} + \delta^K$$

5. The rental price of labor is

$$r_t^\ell = \left(\frac{1 - \mu^K \left(r_t^K\right)^{1-\sigma^Y}}{1 - \mu^K} \right)^{\frac{1}{1-\sigma^Y}}$$

6. Labor for production and wages are

$$\ell_{ss} = L_{ss} - \kappa^L v_{ss}.$$

$$w_{ss} = r^{\ell}_{ss}\left(1 - \frac{\kappa^L}{m^v_{ss}} + \frac{1-\delta^L_{ss}}{1+r^{\text{firm}}}\frac{\kappa^L}{m^v_{ss}}\right)$$

7. Find the tax rate

$$\tau_{ss} = \frac{r_B B_{ss} + P^G_{ss} G_{ss}}{w_{ss} L_{ss}}$$

8. We guess on on B^q_{ss} and check $B_{A-1,t} = B^q_{ss}$ by

$$C_{a,t} = \begin{cases} \left(\mu^B \left(B^q_{ss}\right)^{-\sigma}\right)^{-\frac{1}{\sigma}} & \text{if } a = A-1 \\ \left(\beta \frac{1+r_{hh}}{1+\pi^{hh}_{ss}} C^{-\sigma}_{a+1,ss}\right)^{-\frac{1}{\sigma}} & \text{else} \end{cases}$$

$$B_{a,t} = (1+r^{hh})B_{a-1,t-1} + w_t L_{a,t} + \frac{B^q_t}{A} - P^C_t C_{a,t}$$

9. From production firm

$$K_{ss} = \frac{\mu_K}{1-\mu_K}\left(\frac{r^{\ell}_{ss}}{r^K_{ss}}\right)^{\sigma^Y} \ell_{ss}$$

10. From capital accumulation equations

$$\iota_{ss} = I_{ss} = \delta^K K_{ss}$$

11. Determine output

$$Y_{ss} = \left(\left(\mu^K\right)^{\frac{1}{\sigma^Y}} K_{ss}^{\frac{\sigma^Y-1}{\sigma^Y}} + \left(1-\mu^K\right)^{\frac{1}{\sigma^Y}} \ell_{ss}^{\frac{\sigma^Y-1}{\sigma^Y}}\right)$$

12. Determine package components for consumption and investment

$$\bullet^M_{ss} = \mu^{M,\bullet}\bullet_{ss}, \text{ for } \bullet \in \{C, I\}$$

$$\bullet^Y_{ss} = (1-\mu^{M,\bullet})\bullet_{ss}, \text{ for } \bullet \in \{C, I\}$$

13. Determine χ_{ss} to get market clearing

$$
\begin{aligned}
\chi_{ss} = X_{ss}^{Y} &= Y_{ss} - \left(C_{ss}^{Y} + I_{ss}^{Y}\right) \\
X_{ss} &= \frac{X_{ss}^{Y}}{1-\mu^{M,X}} \\
X_{ss}^{M} &= \mu^{M,X} X_{ss} \\
M_{ss} &= C_{ss}^{M} + I_{ss}^{M} + I_{ss}^{M}
\end{aligned}
$$

14. Let φ adjust to make bargaining fit

$$
\begin{aligned}
w_{ss}^{*} &= w_{ss} \\
\mathrm{MPL}_{ss} &= \left(\left(1-\mu^{K}\right)\frac{Y_{ss}}{\ell_{ss}}\right)^{\frac{1}{\sigma^{Y}}} \\
\varphi &= \frac{w_{ss}-w^{U}}{\mathrm{MPL}_{ss}-w^{U}}
\end{aligned}
$$

4 Status report

Status: The described model is implemented in Python. Some results look weird, which could suggest a code error, a math error or weird assumptions or parameters. This should be checked and a baseline calibration established.

Economic extensions: Potential extensions include

1. **Extend household problem:** Add gradual mortality and habit formation.
2. **Add hands-too-mouth households.**
3. **Add government with taxes and spending.**
4. **Make export demand more rigid.**
5. **Add endogenous labor supply.**
6. **Add financial flows accounts wrt. to the foreign economy.**
7. **Add multiple sector and input-output structure.**
8. **Add technology growth, population growth, and trend inflation**

Computational improvements:

1. Simplify calculation of Jacobian with graph theory or automatic differentiation.
2. Speed-up calculation of Jacobian with parallelization.
3. Speed-up broyden-solver with sparse algebra.
4. Investigate what is done efficiently in MAKRO (GAMS+CONOPT)