Baby-MAKRO*

DISCLAIMER: WORK-IN-PROGRESS ⇒ BEWARE OF ERRORS!

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Abstract

This note outlines a simplified »baby« version of the MAKRO model used by the Danish Ministry of Finance. The model is for a small open economy with a fixed exchange and overlapping generations. The model has perfect foresight, but is full of imperfections due to e.g. frictions in the labor market and adjustment costs. The model is written and solved in terms of a series of ordered blocks. This clarifies the model dynamics and makes it easier to solve for fluctuations around the steady state using a numerical equation system solver. Online code is provided for solving the model in Python.

The model is designed so undergraduate students can work with it, and analyze potential extensions in their thesis work. The model structure is similar to state-of-the-art heterogeneous agent models (see this course) and the model is thus relevant for further academic studies. The similarity to the grown-up MAKRO model makes it relevant for potential future job tasks and the public debate.

The note concludes with a status report on the continuing development of the model.

Online code: github.com/NumEconCopenhagen/BabyMAKRO

MAKRO: See online documentation

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1 Overview

We consider a *small open economy* with a *fixed exchange rate* and *overlapping generations*. Time is discrete, $t \in \{0, 1, \dots\}$ and the frequency is annual.

Households live for up to # periods, and their age is denoted by a. The age dependent mortality is $\zeta_a \in (0, 1)$ and the population and demographic structure is constant. Households exogenously search for jobs and supply labor, receive inheritances and choose consumption and savings to get utility from consumption and bequests.

The *foreign economy* provides a fixed nominal rate of return, sells import goods at fixed prices, and have a demand curve for the domestic export good.

The *production* in the economy is layered as follows:

1. *Production firms* rent *capital* and *labor* to produce the domestic output good.
2. *Repacking firms* combine imported goods with the domestic output good to produce a consumption good, an investment good, and an export good.
3. Capital is rented from a *capital agency*, which purchases the investment good to accumulate capital subject to adjustment costs.
4. Labor is rented from a *labor agency*, which posts vacancies for a search-and-match labor market to purchase labor from the households.

All firms mentioned above are price takers. The domestic good is bought by intermediaries producing differentiated goods and who are subject to price adjustment costs. Wages are determined by ad hoc bargaining. All *goods markets clear* and the matching process is determined by a *matching function*.

There is *perfect foresight* in the economy. I.e. the value of all current and future variables are known. This is a strong assumption, and in many ways the model should be considered a first order approximation to a full model with both idiosyncratic and aggregate risk. It can be relevant to introduce model elements, which proxy for the effects of risks. Utility-of-wealth can e.g. proxy for a precautionary saving motive.

1.1 Equilibrium path

The *equilibrium path* in the economy is a set of paths for all variables, which satisfies all accounting identities, optimal firm and household behavior in terms of first order conditions, and implies market clearing. When all variables are constant over time, the equilibrium path is a *steady state*.

In terms of math, the model is just an *equation system* stacking the accounting identifies, first order conditions and market clearing conditions. If the economy is initially out of steady state, we solve for the equilibrium path by truncating the equation system to T periods. The assumption is that the economy has settled down to the steady state well before period T, and we can assume variables from period T onward are at their steady state value. The economy can be out of steady state both because lagged *endogenous* variables are initially not at their steady state values and/or because the *exogenous* variables are not at their steady state values. We talk of an *impulse response* when the economy starts at the steady state, but some exogenous variables *temporarily* deviate from the steady state following some converging auto-regressive process.[1]

We simplify the model and the resulting equation system by writing it in terms of a *ordered series of block*. We start from a set of *exogenous* variables (e.g. variables determined in the foreign economy) and a set of *unknown* variables. Each block then takes in the path of some variables, return the path of other variables, and imply *targets*, which must be zero if the model equations are satisfied. Each block can use the unknown variables and output variables of previous blocks as input variables. In the end we collect all the targets. The *number of unknown variables* must equal the *number of target variables*.

To solve the model, we must first find the steady state. As explained in Section 3, this can be done by manually choosing values for a selection of the endogenous variables and the deriving the rest from closed form expressions or solving sub-systems with a numerical equation system solver. Next, we solve for the equilibrium path again using a numerical equation system solver.

The block structure and ordering is *not* unique. If a different set of unknowns is chosen, a different ordering of blocks must also be chosen. If an additional variable

[1] This is also called an MIT shock. A shock in a model with perfect foresight is to some degree a contradiction in terms. The assumption is that even though the agents experiences a shock, they expect that there will never be a shock again. Accounting for expecting of future shocks is much more complicated. Some realism can be mimicked by studying impulse response to shocks about the future, which when the future comes never materialized as a new opposite signed shock negates it. Multiple shocks arriving sequentially can also be studied.

is considered an *unknown*, an additional equation must be considered a target instead of being used to calculate a output. In the limit, all variables can be considered as unknowns and all equations as targets. This is inefficient as the number of variables can be very large.[2]

1.2 On CES technology

The assumption of CES technology is used repeatedly in the model. It is therefore beneficial to recap it briefly. Consider a firm producing good X using good X_i and X_j with a CES technology. Input prices are P_i and P_j and the output price is P. The firm is a price taker in all markets. The *profit maximization* problem of the firm is

$$\max_{X_i,X_j} PX - P_iX_i - P_jX_j \text{ s.t. } X = \Gamma\left(\mu_i^{\frac{1}{\sigma}}X_i^{\frac{\sigma-1}{\sigma}} + \mu_j^{\frac{1}{\sigma}}X_j^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}}, \;\; \mu_i + \mu_j = 1,\, \mu_i, \sigma, \Gamma > 0,\, \sigma \neq 1 \tag{1}$$

The generic *first order condition*

$$\begin{aligned}
0 &= P\mu_i^{\frac{1}{\sigma}}X_i^{\frac{\sigma-1}{\sigma}-1}\Gamma\left(\mu_i^{\frac{1}{\sigma}}X_i^{\frac{\sigma-1}{\sigma}} + \mu_j^{\frac{1}{\sigma}}X_j^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}-1} - P_i \Leftrightarrow \\
P_i &= P\mu_i^{\frac{1}{\sigma}}X_i^{-\frac{1}{\sigma}}\Gamma\left(\mu_i^{\frac{1}{\sigma}}X_i^{\frac{\sigma-1}{\sigma}} + \mu_j^{\frac{1}{\sigma}}X_j^{\frac{\sigma-1}{\sigma}}\right)^{\frac{1}{\sigma-1}} \Leftrightarrow \\
P_i &= P\mu_i^{\frac{1}{\sigma}}X_i^{-\frac{1}{\sigma}}\Gamma^{\frac{\sigma-1}{\sigma}}X^{\frac{1}{\sigma}} \Leftrightarrow \\
X_i &= \mu_i\left(\frac{P}{P_i}\right)^{\sigma}\Gamma^{\sigma-1}X.
\end{aligned} \tag{2}$$

As the production technology has constant return-to-scale, there are infinitely many solutions to the FOCs. They all satisfy that inputs are used in proportion as follows

$$\frac{X_i}{X_j} = \frac{\mu_i}{\mu_j}\left(\frac{P_j}{P_i}\right)^{\sigma} \tag{3}$$

[2] Modeling systems such as GAMS can combined with a state-of-the-art solver such as CONOOPT automatically analyze the structure of the equation system and thereby ripe the benefits we get from manually ordering the blocks.

Assuming *free entry*, and thus *zero profits*, the output price is uniquely determined from the input prices as

$$
\begin{aligned}
0 &= PX - P_iX_i - P_jX_j \Leftrightarrow \\
P &= \frac{P_iX_i + P_jX_j}{X} \\
&= \mu_i \left(\frac{P}{P_i}\right)^{\sigma} \Gamma^{\sigma-1} P_i + \mu_j \left(\frac{P}{P_j}\right)^{\sigma} \Gamma^{\sigma-1} P_j \Leftrightarrow \\
(\Gamma P)^{1-\sigma} &= \mu_i P_i^{1-\sigma} + \mu_j P_j^{1-\sigma} \Leftrightarrow \\
P &= \frac{1}{\Gamma}\left(\mu_i P_i^{1-\sigma} + \mu_j P_j^{1-\sigma}\right)^{\frac{1}{1-\sigma}}. \qquad (4)
\end{aligned}
$$

2 Blocks

2.1 Exogenous variables

The *exogenous* variables are:

1. Γ_t, technology
2. G_t, public spending
3. χ_t, *foreign* demand shifter (»market size«)
4. $P_t^{M,C}$, import price of *private consumption* component good
5. $P_t^{M,G}$, import price of *public consumption* component good
6. $P_t^{M,I}$, import price of *investment* component good
7. $P_t^{M,X}$, import price of *export* component good
8. P_t^F, *foreign* price level
9. r_t^{hh}, *foreign* interest rate

2.2 Unknown variables

The chosen *unknown* variables are:

1. A_t^q, inheritance flow (T unknowns)
2. A_t^{death}, wealth of households at $a = \# - 1$ (T unknowns)
3. K_t, capital (T unknowns)
4. L_t, labor supply (T unknowns)
5. r_t^K, rental price for capital (T unknowns)
6. P_t^Y, price of domestic output (T unknowns)

The total number of unknowns thus is $6 \times T$.

2.3 Demographics

The age-specific number of household is N_a, which we normalize to 1 at age $a = 0$. The mortality rate at period a is ζ_a, the potential life-span is # and retirement age is $\#_{work} < \#$.

The demographic structure and population is then given by

$$N_a = \begin{cases} 1 & \text{if } a = 0 \\ (1 - \zeta_{a-1}) N_{a-1} & \text{if } a > 0 \end{cases} \tag{5}$$

$$\zeta_a = \begin{cases} 0 & \text{if } a < \#_{\text{work}} \\ \left(\frac{a+1-\#_{\text{work}}}{\#-\#_{\text{work}}} \right)^{\zeta} & \text{if } a < \# - 1 \\ 1 & \text{if } a = \# - 1 \end{cases} \tag{6}$$

$$N = \sum_{a=0}^{\#-1} N_a \tag{7}$$

$$N_{\text{work}} = \sum_{a=0}^{\#_{\text{work}}-1} N_a. \tag{8}$$

2.4 Block I. Repacking firms - prices

The output good, Y_t, can be used for either private consumption, C_t, public consumption, G_t, investment, I_t, or exports, X_t. For each use the output good must be repacked with imported goods. This is done by repacking firms with a CES production technology, where they take the output price and the import prices as given..

Using the results with CES technology derived in sub-section 1.2 with $\Gamma = 1$, we write the **block** for the pricing part of this as:

1. **Inputs:** $\{P_t^Y\}$, $\left\{P_t^{M,\bullet}\right\}$ for $\bullet \in \{C, G, I, X\}$
2. **Output:** $\{P_t^\bullet\}$ for $\bullet \in \{C, G, I, X\}$

$$P_t^\bullet = \left(\mu^{M,\bullet}\left(P_t^{M,\bullet}\right)^{1-\sigma^\bullet} + \left(1-\mu^{M,\bullet}\right)\left(P_t^Y\right)^{1-\sigma^\bullet}\right)^{\frac{1}{1-\sigma^\bullet}}. \tag{9}$$

2.5 Block II. Wage determination

The wage, W_t, is determined by an unmodeled bargaining mechanism, and it is set such that it is increasing in the labor demand.

- **Inputs:** $\{L_t\}$
- **Outputs:** $\{W_t\}$

$$W_t = W_{ss}\left(\frac{L_t}{L_{ss}}\right)^{\epsilon_w}. \tag{10}$$

2.6 Block III. Households - search behavior and matching

Households search for a job and supply labor exogenously. The age-specific job-separation probability is $\delta_a^L \in (0, 1)$. All unemployed search for a job. As an initial condition, we have $L_{-1,t-1} = 0$.

The quantity of searchers is

$$
\begin{aligned}
S_{a,t} &= \begin{cases} 1 & \text{if } a = 0 \\ (1-\zeta_{a-1})\left[(N_{a-1} - L_{a-1,t-1}) + \delta_a^L L_{a-1,t-1}\right] & \text{if } a < \#_{\text{work}} \\ 0 & \text{if } a \geq \#_{\text{work}} \end{cases} \\
S_t &= \sum_a S_{a,t}.
\end{aligned}
$$

The quantity of households with a job *before matching* is

$$
\begin{aligned}
\underline{L}_{a,t} &= \begin{cases} 0 & \text{if } a = 0 \\ (1-\zeta_{a-1})(1-\delta_a^L)L_{a-1,t-1} & \text{if } a < \#_{\text{work}} \\ 0 & \text{if } a \geq \#_{\text{work}} \end{cases} \\
\underline{L}_t &= \sum_a \underline{L}_{a,t}.
\end{aligned}
$$

The aggregate job-separation rate incl. mortality is

$$
\delta_t^L = \frac{L_{t-1} - \underline{L}_t}{L_{t-1}}.
$$

The quantity of vacancies is v_t and the number of matches, $\mathcal{M}_t$, is given by the *matching function*

$$
\mathcal{M}_t = \frac{S_t v_t}{\left(S_t^{\frac{1}{\sigma^m}} + v_t^{\frac{1}{\sigma^m}}\right)^{\sigma^m}}.
$$

The job-filling rate, m_t^v, and the job-finding rate, m_t^s, are thus

$$
\begin{aligned}
m_t^v &= \frac{\mathcal{M}_t}{v_t} \\
m_t^s &= \frac{\mathcal{M}_t}{S_t}.
\end{aligned}
$$

The number of employed therefore is

$$
L_{a,t} = \underline{L}_{a,t} + m_t^s S_{a,t}.
$$

This implies that the number of unemployed is

$$U_{a,t} = \begin{cases} N_a - L_{a,t} & \text{if } a < \#_{\text{work}} \\ 0 & \text{if } a \geq \#_{\text{work}} \end{cases}.$$

In equilibrium, the number of matches must equal the number of new hires, i.e.

$$\mathcal{M}_t = L_t - \underline{L}_t.$$

It is assumed that the households accumulate human capital as a function of their age

$$H_a = \rho_1 \cdot a - \rho_2 \cdot a^2.$$

The amount of human capital affects the *effective* number of searchers and therefore the *effective* employment in cohort a such that

$$L^H_{a,t} = \underline{L}^H_{a,t} + m^s_t S^H_{a,t} H_a.$$

The effective quantity of searchers is

$$S^H_{a,t} = \begin{cases} 1 & \text{if } a = 0 \\ (1 - \zeta_{a-1}) \left[(N^H_{a-1} - L^H_{a-1,t-1}) + \delta^L_a L^H_{a-1,t-1} \right] & \text{if } a < \#_{\text{work}} \\ 0 & \text{if } a \geq \#_{\text{work}} \end{cases}$$

where

$$N^H_a = \begin{cases} 1 & \text{if } a = 0 \\ (1 - \zeta_{a-1}) N^H_{a-1} H_a & \text{if } a > 0 \end{cases}$$

The effective quantity of households with a job before mathcing is

$$\underline{L}^H_{a,t} = \begin{cases} 0 & \text{if } a = 0 \\ (1 - \zeta_{a-1})(1 - \delta^L_a) L^H_{a-1,t-1} & \text{if } a < \#_{\text{work}} \\ 0 & \text{if } a \geq \#_{\text{work}} \end{cases}$$

We write the **block** in terms of inputs, and outputs as:

- **Inputs:** $\{L_t\}$
- **Outputs:** $\{S_{a,t}\}$,$\{S^H_{a,t}\}$, $\{S_t\}$, $\{\delta^L_t\}$, $\{\mathcal{M}\}$,$\{v_t\}$, $\{m^v_t\}$, $\{m^s_t\}$, $\{L_{a,t}\}$,$\{L^H_{a,t}\}$,$\{U_{a,t}\}$,$\{U_t\}$

$$S_{a,t} = \begin{cases} 1 & \textbf{if } a = 0 \\ (1-\zeta_{a-1})\left[(N_{a-1} - L_{a-1,t-1}) + \delta^L_a L_{a-1,t-1}\right] & \textbf{if } a < \#_{\textbf{work}} \\ 0 & \textbf{if } a \geq \#_{\textbf{work}} \end{cases} \tag{11}$$

$$S^H_{a,t} = \begin{cases} 1 & \textbf{if } a = 0 \\ (1-\zeta_{a-1})\left[(N^H_{a-1} - L^H_{a-1,t-1}) + \delta^L_a L^H_{a-1,t-1}\right] & \textbf{if } a < \#_{\textbf{work}} \\ 0 & \textbf{if } a \geq \#_{\textbf{work}} \end{cases} \tag{12}$$

$$\underline{L}_{a,t} = \begin{cases} 0 & \textbf{if } a = 0 \\ (1-\zeta_{a-1})(1-\delta^L_a)L_{a-1,t-1} & \textbf{if } a < \#_{\textbf{work}} \\ 0 & \textbf{if } a \geq \#_{\textbf{work}} \end{cases} \tag{13}$$

$$\underline{L}^H_{a,t} = \begin{cases} 0 & \textbf{if } a = 0 \\ (1-\zeta_{a-1})(1-\delta^L_a)L^H_{a-1,t-1} & \textbf{if } a < \#_{\textbf{work}} \\ 0 & \textbf{if } a \geq \#_{\textbf{work}} \end{cases} \tag{14}$$

$$S_t = \sum_{a=0}^{\#-1} S_{a,t} \tag{15}$$

$$\underline{L}_t = \sum_{a=0}^{\#-1} \underline{L}_{a,t} \tag{16}$$

$$\delta_t^L = \frac{L_{t-1} - \underline{L}_t}{L_{t-1}} \tag{17}$$

$$\mathcal{M}_t = L_t - \underline{L}_t \tag{18}$$

$$m_t^s = \frac{\mathcal{M}_t}{S_t} \tag{19}$$

$$v_t = \left(\frac{\mathcal{M}_t^{\frac{1}{\sigma^m}}}{1 - \left(\frac{\mathcal{M}_t}{S_t}\right)^{\frac{1}{\sigma^m}}} \right)^{\sigma^m} \tag{20}$$

$$m_t^v = \frac{\mathcal{M}_t}{v_t} \tag{21}$$

$$L_{a,t} = \underline{L}_{a,t} + m_t^s S_{a,t} \tag{22}$$

$$L_{a,t}^H = \underline{L}_{a,t}^H + m_t^s S_{a,t}^H H_a \tag{23}$$

$$U_{a,t} = \begin{cases} N_a - L_{a,t} & \textbf{if } a < \#_{\textbf{work}} \\ 0 & \textbf{if } a \geq \#_{\textbf{work}} \end{cases} \tag{24}$$

$$U_t = \sum_{a=0}^{\#-1} U_{a,t}. \tag{25}$$

For $t = 0$, the variable $L_{a-1,t-1}$ is pre-determined.

2.7 Block IV. Labor agency

The labor agency firms post vacancies, v_t, to hire labor L_t. The labor cost of posting each vacancy is κ^L in units of labor. The firms can therefore rent out $\ell_t = L_t - \kappa^L v_t$ units of labor to the production firms at the rental price r_t^ℓ.

Labor follows the law-of-motion $L_t = \left(1 - \delta_t^L\right) L_{t-1} + m_t^v v_t$, and the wage, W_t, is determined by the bargaining mechanism. Matching occurs according to the matching function, and the firms take the separation rate, δ_t^L, and the vacancy filling rate, m_t^v as given. Since lagged employment, L_{t-1}, is pre-determined, we consider L_t to be the choice value and derive the required number of vacancies, v_t, and the implied labor for rent, ℓ_t.

The labor agency problem is then:

$$V_0^{\text{labor}}(L_{t-1}) = \max_{\{L_t\}} \sum_{t=0}^{\infty} \left(1 + r^{\text{firm}}\right)^{-t} \left[r_t^{\ell} \ell_t - W_t L_t\right]$$

$$\text{s.t.}$$

$$v_t = \frac{L_t - \left(1 - \delta_t^L\right) L_{t-1}}{m_t^v}$$

$$\ell_t = L_t - \kappa^L v_t.$$

Using the FOC to L_t from the labor agency problem, we write the **block** as:

- **Inputs:** $\{W_t\}$, $\{m_t^v\}$, $\{\delta_t^L\}$, $\{L_t\}$, $\{v_t\}$
- **Outputs:** $\{r_t^{\ell}\}$, $\{\ell_t\}$

$$r_t^{\ell} = \frac{1}{1 - \frac{\kappa^L}{m_t^v}} \left[W_t - r_{t+1}^{\ell} \frac{1 - \delta_{t+1}^L}{1 + r^{\text{firm}}} \frac{\kappa^L}{m_{t+1}^v}\right] \tag{26}$$

$$\ell_t = L_t - \kappa^L v_t. \tag{27}$$

The variable L_{-1} is pre-determined.

2.8 Block V. Production firms

The production firms use capital, K_{t-1}, and labor, ℓ_t, to produce output, Y_t, with a CES technology. The rental price of capital is r_t^K and the rental price of labor is r_t^{ℓ}. The marginal cost of output is denoted $P_t^{Y,0}$.

Using the results with CES technology derived in sub-section 1.2, we write the **block** as:

- **Inputs:** $\{\Gamma_t\}$, $\{K_t\}$, $\{\ell_t\}$, $\{r_t^K\}$, $\{r_t^{\ell}\}$

- **Outputs:** $\{Y_t\}$, $\left\{P_t^{Y,0}\right\}$

$$Y_t = \Gamma_t \left(\left(\mu^K\right)^{\frac{1}{\sigma^Y}} K_{t-1}^{\frac{\sigma^Y-1}{\sigma^Y}} + \left(1-\mu^K\right)^{\frac{1}{\sigma^Y}} \ell_t^{\frac{\sigma^Y-1}{\sigma^Y}} \right)^{\frac{\sigma^Y}{\sigma^Y-1}} \tag{28}$$

$$P_t^{Y,0} = \frac{1}{\Gamma_t} \left(\mu^K \left(r_t^K\right)^{1-\sigma^Y} + \left(1-\mu^K\right) \left(r_t^\ell\right)^{1-\sigma^Y} \right)^{\frac{1}{1-\sigma^Y}}. \tag{29}$$

Targets:

$$\frac{K_{t-1}}{\ell_t} = \frac{\mu_K}{1-\mu_K} \left(\frac{r_t^\ell}{r_t^K}\right)^{\sigma^Y}. \tag{30}$$

The variable K_{-1} is pre-determined.

2.9 Block VI. Phillips Curve

The output price, P_t^Y, is a mark-up over the marginal cost, $P_t^{Y,0}$, to capture monopolistic behavior by the firms. In addition, the final prices are sticky, and the firms pay a quadratic adjustment cost, g_t, to change them. The adjustment costs are applied to changes in inflation such that if the inflation in period t deviates from the inflation in period $t-1$, then the firm faces positive price adjustment costs. The adjustment costs are proportional to $P_t^Y$and $Y_t$implying that a higher price and output level increases the adjustment costs. The demand for each good is y_t.

$$\begin{aligned} V_t^{\text{intermediary}} &= \max_{\{p_t^Y\}} \left(p_t^Y - P_t^{Y,0}\right) y_t - \vartheta_t + \frac{1}{1+r^{\text{firm}}} V_{t+1}^{\text{intermediary}} \\ &\text{s.t.} \\ \vartheta_t &= \frac{\gamma}{2} \left[\frac{p_t^Y / p_{t-1}^Y}{p_{t-1}^Y / p_{t-2}^Y} - 1 \right]^2 P_t^Y Y_t \\ y_t &= \left(\frac{p_t^Y}{P_t^Y}\right)^{-\sigma_D} Y_t. \end{aligned}$$

Using the FOC and symmetry across firms, we get

- **Inputs:** $\{Y_t\}$, $\left\{P_t^{Y,0}\right\}$

- **Outputs:** $\left\{P_t^Y\right\}$

$$P_t^Y = (1+\theta)P_t^{Y,0} - \eta\left(\frac{P_t^Y/P_{t-1}^Y}{P_{t-1}^Y/P_{t-2}^Y} - 1\right)\frac{P_t^Y/P_{t-1}^Y}{P_{t-1}^Y/P_{t-2}^Y}P_t^Y + \frac{2\eta}{1+r^{\textbf{firm}}}\frac{Y_{t+1}}{Y_t}\left(\frac{P_{t+1}^Y/P_t^Y}{P_t^Y/P_{t-1}^Y} - 1\right)\frac{P_{t+1}^Y/P_t^Y}{P_t^Y/P_{t-1}^Y}P_{t+1}^Y$$

$$\theta \equiv \frac{1}{\sigma_D - 1}$$
$$\eta \equiv \theta\gamma.$$

Targets:

$$P_t^Y = (1+\theta)P_t^{Y,0} - \eta\left(\frac{P_t^Y/P_{t-1}^Y}{P_{t-1}^Y/P_{t-2}^Y} - 1\right)\frac{P_t^Y/P_{t-1}^Y}{P_{t-1}^Y/P_{t-2}^Y}P_t^Y + \frac{2\eta}{1+r^{\text{firm}}}\frac{Y_{t+1}}{Y_t}\left(\frac{P_{t+1}^Y/P_t^Y}{P_t^Y/P_{t-1}^Y} - 1\right)\frac{P_{t+1}^Y/P_t^Y}{P_t^Y/P_{t-1}^Y}P_{t+1}^Y. \tag{31}$$

2.10 Block VII. Foreign economy

The foreign economy has so-called Armington demand of the domestic export good. We write the **block** as:

1. **Inputs:** $\left\{P_t^F\right\}$,$\{\chi_t\}$, $\left\{P_t^X\right\}$
2. **Outputs:** $\{X_t\}$

$$X_t = \gamma^X X_{t-1} + (1-\gamma^X)\chi_t\left(\frac{P_t^X}{P_t^F}\right)^{-\sigma^F}. \tag{32}$$

2.11 Block VIII. Capital agency

The capital agency firm buys investment goods, I_t, at price, P_t^I, to accumulate capital, K_t, which it rents out to production at the rental rate r_{t+1}^K in the following period. The investment decision is subject to convex adjustment costs Ψ in terms of wasted investment goods, such that effective investment is ι_t. Future profits are discounted with r^{firm}. The capital agency takes prices as given, and its problem thus is:

$$
\begin{aligned}
V_0^{\text{capital}}\left(K_{t-1}\right) = \max_{\{K_t\}} \sum_{t=0}^{\infty} \left(1+r^{\text{firm}}\right)^{-t} & \left[r_t^K K_{t-1} - P_t^I \left(\iota_t + \Psi(\iota_t, K_{t-1})\right)\right] \\
\text{s.t.} & \\
I_t &= \iota_t + \Psi(\iota_t, K_{t-1}) \\
K_t &= (1-\delta^K)K_{t-1} + \iota_t.
\end{aligned}
$$

We choose the functional form

$$
\Psi(\iota_t, K_{t-1}) = \frac{\Psi_0}{2}\left(\frac{\iota_t}{K_{t-1}} - \delta^K\right)^2 K_{t-1},
$$

implying

$$
\begin{aligned}
\Psi_\iota(\iota_t, K_{t-1}) &= \Psi_0\left(\frac{\iota_t}{K_{t-1}} - \delta^K\right) \\
\Psi_K(\iota_t, K_{t-1}) &= \frac{\Psi_0}{2}\left(\frac{\iota_t}{K_{t-1}} - \delta^K\right)^2 - \Psi_0\left(\frac{\iota_t}{K_{t-1}} - \delta^K\right)\frac{\iota_t}{K_{t-1}}.
\end{aligned}
$$

We write the **block** as

- **Inputs:** $\{r_t^K\}$, $\{P_t^I\}$, $\{K_t\}$
- **Outputs:** $\{\iota_t\}, \{I_t\}$

$$
\iota_t = K_t - \left(1-\delta^K\right)K_{t-1} \tag{33}
$$

$$
I_t = \iota_t + \Psi(\iota_t, K_{t-1}). \tag{34}
$$

- **Targets**:

$$
\begin{aligned}
0 = &- P_t^I\left(1 + \Psi_\iota\left(\iota_t, K_{t-1}\right)\right) \\
&+ \left(1+r^{\text{firm}}\right)^{-1}\left[r_{t+1}^k + P_{t+1}^I(1-\delta^K)(1+\Psi_\iota(\iota_{t+1}, K_t)) - P_{t+1}^I \Psi_K\left(\iota_{t+1}, K_t\right)\right].
\end{aligned} \tag{35}
$$

The variable K_{-1} is pre-determined.

2.12 Block IX. Government

The government budget is given by

$$B_t = (1+r^B)B_{t-1} + P_t^G G_t + (1-\tau_t)W_U W_{ss} U_t + (1-\tau_t)W_R W_{ss}(N_t - N_t^{work}) - \tau_t W_t L_t, \tag{36}$$

where r^B is the interest rate on government debt determined in the foreign economy. We assume the government gradually adjusts taxes to get back to steady state debt.

- **Inputs:** $\{P_t^G\}$, $\{G_t\}$, $\{W_t\}$, $\{L_t\}$, $\{N_t\}$, $\{N_t^{work}\}$
- **Outputs:** $\{\tau_t\}$, $\{B_t\}$, $\{\tilde{B}_t\}$

$$\begin{aligned}\tilde{B}_t &= (1+r^B)B_{t-1} + P_t^G G_t + (1-\tau_{ss})W_U W_{ss} U_t \\ &\quad + (1-\tau_{ss})W_R W_{ss}(N_t - N_{\text{work}}) - \tau_{ss} W_t L_t\end{aligned} \tag{37}$$

$$\tau_t = \tau_{ss} + \varepsilon_B \frac{\tilde{B}_t - B_{ss}}{W_t L_t + W_U W_{ss} U_t + W_R W_{ss}(N_t - N_{\text{work}})} \tag{38}$$

$$\begin{aligned}B_t &= (1+r^B)B_{t-1} + P_t^G G_t + (1-\tau_t)W_U W_{ss} U_t \\ &\quad + (1-\tau_t)W_R W_{ss}(N_t - N_{\text{work}}) - \tau_t W_t L_t.\end{aligned} \tag{39}$$

The variable B_{-1} is pre-determined.

2.13 Block X. Households - consumption -saving

The model has two types of households. A share λ of households is hands-to-mouth and a share $1-\lambda$ of households is unconstrained (Ricardian). All households have four sources of income:

1. Post-tax labor income, $(1-\tau_t)\, W_t \frac{L_{a,t}^H}{N_a}$
2. Post-tax unemployment benefits, $(1-\tau_t)\, W^U W_{ss} \frac{U_{a,t}}{N_a}$
3. Post-tax retirement benefits, $(1-\tau_t)\, W^R W_{ss} \frac{N_a - (L_{a,t} + U_{a,t})}{N_a}$
4. Equally divided inheritance, $\frac{A_t^q}{N}$

The post-tax labor income depends on the effective employment, $L^H_{a,t}$, and thus the income depends on the amount of human capital. The age specific income is

$$\mathrm{inc}_{a,t} = (1-\tau_t)\, W_t \frac{L^H_{a,t}}{N_a} + (1-\tau_t)\, W^U W_{ss} \frac{U_{a,t}}{N_a} + (1-\tau_t)\, W^R W_{ss} 1_{\{a \geq \#_{\mathrm{work}}\}} + \frac{A^q_t}{N}.$$

The price of consumption goods is P^C_t. Consumption is $C^\bullet_{a,t}$ and end-of-period nominal savings is $A^\bullet_{a,t}$, where $\bullet \in \{\mathrm{HtM}, R\}$. The behavior of *surviving* hands-to-mouth households is

$$\begin{aligned} C^{\mathrm{HtM}}_{a,t} &= \frac{\mathrm{inc}_{a,t}}{P^C_t} \\ A^{\mathrm{HtM}}_{a,t} &= 0. \end{aligned}$$

The Ricardian households making their first decision in period t_0 solve the problem

$$\begin{aligned} V_{t_0} = & \max_{\left\{C^R_{a,t}\right\}_{a=0}^{\#-1}} \sum_{a=0}^{\#-1} \left(\Pi_{j=1}^{a} \beta(1-\zeta_{j-1})\right) \left[\frac{\left(C^R_{a,t}\right)^{1-\sigma}}{1-\sigma} + \zeta_a \mu^{A^q} \frac{\left(\frac{A^R_{a,t}}{P^C_t}\right)^{1-\sigma}}{1-\sigma} \right] \\ & \text{s.t.} \\ t &= t_0 + a \\ A^R_{-1,t} &= 0 \\ A^R_{a,t} &= (1+r^{hh}_t) A^R_{a-1,t-1} + \mathrm{inc}_{a,t} - P^C_t C^R_{a,t}. \end{aligned}$$

Aggregation implies

$$\begin{aligned} C_{a,t} &= \lambda C^{\mathrm{HtM}}_{a,t} + (1-\lambda) C^R_{a,t} \\ A_{a,t} &= \lambda A^{\mathrm{HtM}}_{a,t} + (1-\lambda) A^R_{a,t}, \end{aligned}$$

and

$$\begin{aligned} C_t &= \sum_{a=0}^{\#-1} N_a C_{a,t} \\ A_t &= \sum_{a=0}^{\#-1} N_a C_{a,t}. \end{aligned}$$

Bequests are

$$A_t^q = \left(1 + r^{hh}\right) \sum_{a=0}^{\#-1} \zeta_a N_a A_{a,t}.$$

Using the FOC for we can write the **block** as:

1. **Inputs:** $\{L_{a,t}\}$, $\{U_{a,t}\}$, $\{P_t^C\}$, $\{W_t\}$, $\{\tau_t\}$, $\{A_t^q\}$, $\left\{A_{\#-1,t}^R\right\}$,$\{r_t^{hh}\}$

2. **Outputs:** $\{A_{a,t}^{\text{HtM}}\}$, $\{A_{a,t}^R\}$,$\{A_{a,t}\}$, $\{A_t\}$, $\{C_{a,t}^{\text{HtM}}\}$,$\{C_{a,t}^R\}$, $\{C_{a,t}\}$, $\{C_t\}$, $\{\text{inc}_{a,t}\}$, $\{\text{inc}_t\}$, $\{\pi_t^{hh}\}$

   Calculate

$$\pi_t^{hh} = \frac{P_t^C}{P_{t-1}^C} - 1 \tag{40}$$

$$\text{inc}_{a,t} = (1-\tau_t) W_t \frac{L_{a,t}}{N_a} + (1-\tau_t) W^U W_{ss} \frac{U_{a,t}}{N_a} + (1-\tau_t) W^R W_{ss} 1_{\{a \geq \#_{\text{work}}\}} + \frac{A_t^q}{N} \tag{41}$$

$$C_{a,t}^{\text{HtM}} = \frac{\text{inc}_{a,t}}{P_t^C} \tag{42}$$

$$A_{a,t}^{\text{HtM}} = 0. \tag{43}$$

   For each birth cohort $t_0 \in \{-\# + 1, -\# + 2, \ldots, T-1\}$ iterate backwards from $a = \# - 1$ with $t = t_0 + a$, but skipping steps where $t < 0$ or $t > T - 1$:

$$C_{a,t}^R = \begin{cases} \left(\zeta_a \mu^{A^q} \left(\frac{A_{\#-1,t}^R}{P_t^C}\right)^{-\sigma}\right)^{-\frac{1}{\sigma}} & \text{if } a = \# - 1 \\ \left(\beta(1-\zeta_a) \frac{1+r_{ss}^{hh}}{1+\pi_{ss}^{hh}} \left(C_{a+1,ss}^{R,}\right)^{-\sigma} + \zeta_a \mu^{A^q} \left(\frac{A_{a,ss}^R}{P_{ss}^C}\right)^{-\sigma}\right)^{-\frac{1}{\sigma}} & \text{elif } t = T - 1 \\ \left(\beta(1-\zeta_a) \frac{1+r_{t+1}^{hh}}{1+\pi_{t+1}^{hh}} \left(C_{a+1,t+1}^{R,}\right)^{-\sigma} + \zeta_a \mu^{A^q} \left(\frac{A_{a,t}^R}{P_t^C}\right)^{-\sigma}\right)^{-\frac{1}{\sigma}} & \text{else} \end{cases}$$

$$A_{a-1,t-1}^R = \frac{A_{a,t}^R - \text{inc}_{a,t} + P_t^C C_{a,t}^R}{1 + r_t^{hh}}.$$

Aggregates

$$C_{a,t} = \lambda C^{\text{HtM}}_{a,t} + (1-\lambda) C^{R}_{a,t} \tag{44}$$

$$A_{a,t} = \lambda A^{\text{HtM}}_{a,t} + (1-\lambda) A^{R}_{a,t} \tag{45}$$

$$\text{inc}_t = \sum_{a=0}^{\#-1} N_a \text{inc}_{a,t} \tag{46}$$

$$C_t = \sum_{a=0}^{\#-1} N_a C_{a,t} \tag{47}$$

$$A_t = \sum_{a=0}^{\#-1} N_a A_{a,t}. \tag{48}$$

Targets:

$$0 = A^{q}_{t} - \left(1 + r^{hh}_{t}\right) \sum_{a=0}^{\#-1} \zeta_a N_a A_{a,t-1} \tag{49}$$

$$0 = \sum_{t_0=-\#+1}^{-1} \left(A^{R}_{-t_0-1,-1} - A_{-t_0-1,ss}\right) + \sum_{t_0=0}^{T-1-\#+1} \left(A^{R}_{-1,t_0} - 0.0\right) \tag{50}$$

For $t = 0$, we have that the variable $A^{R}_{a,t-1}$ is pre-determined.

2.14 Block XI. Repacking firms - components

The repacking firms were described in sub-section 2.4. Using additional results from sub-section 1.2 on CES technology with $\Gamma = 1$, we write the **block** as:

1. **Inputs:** $\{P^{Y}_{t}\}$, $\left\{P^{M,\bullet}_{t}\right\}$, $\{P^{\bullet}_{t}\}$, $\{\bullet_t\}$ for $\bullet \in \{C, G, I, X\}$
2. **Output:** $\{\bullet^{M}_{t}\}$, $\{\bullet^{Y}_{t}\}$ for $\bullet \in \{C, G, I, X\}$

$$\bullet^{M}_{t} = \mu^{M,\bullet} \left(\frac{P^{\bullet}_{t}}{P^{M,\bullet}_{t}}\right)^{\sigma^{\bullet}} \bullet_t \tag{51}$$

$$\bullet^{Y}_{t} = \left(1 - \mu^{M,\bullet}\right) \left(\frac{P^{\bullet}_{t}}{P^{Y}_{t}}\right)^{\sigma^{\bullet}} \bullet_t\,. \tag{52}$$

2.15 Block XII. Goods market clearing

The production of the domestic output good must match the above output goods used by the repacking firms. Imports, M_t, are the sum of the imports used by the repacking firms.

We write the **block** as:

- **Inputs:** $\{\bullet_t^Y\}$, $\{\bullet_t^M\}$ for $\bullet \in \{C, G, I, X\}$
- **Outputs:** $\{M_t\}$

$$M_t = \sum_{\bullet \in \{C,G,I,X\}} \bullet_t^M. \tag{53}$$

- **Targets:**

$$Y_t = \sum_{\bullet \in \{C,G,I,X\}} \bullet_t^Y. \tag{54}$$

2.16 Total number of targets

We have $6 \times T$ targets in Equation (30), (31), (35), (49), (50), and (54), which is equal to number of unknowns.

3 Steady state

We fix a number of variables:

1. The nominal wage, $W_{ss} = 1$
2. The inflation, $\pi_{ss}^{hh} = 0$
3. The job-finding rate, $m_{ss}^s = 0.75$
4. The job-filling rate, $m_{ss}^v = 0.75$
5. The government debt, $B_{ss} = 0$
6. The foreign interest rate, $r_{ss}^{hh} = 0.04$

We allow for the adjustment of the exogenous variables and other parameters to fit with this. We can then find the steady state as follows:

1. Price normalization:

$$P_{ss}^{Y} = P_{ss}^{F} = P_{ss}^{M,\bullet} = 1,\ \bullet \in \{C, G, I, X\}.$$

2. The pricing behavior of repacking firms then implies

$$P_{ss}^{\bullet} = 1,\ \bullet \in \{C, G, I, X\}.$$

3. The exogenous labor supply and search-and-matching imply

$$S_{a,ss} = \begin{cases} 1 & \text{if } a = 0 \\ (1-\zeta_{a-1})\left[(N_{a-1} - L_{a-1,ss}) + \delta_a^L L_{a-1,ss}\right] & \text{if } a < \#_{\text{work}} \\ 0 & \text{else} \end{cases}$$

$$S_{a,ss}^{H} = \begin{cases} 1 & \text{if } a = 0 \\ (1-\zeta_{a-1})\left[(N_{a-1}^{H} - L_{a-1,ss}^{H}) + \delta_a^L L_{a-1,ss}^{H}\right] & \text{if } a < \#_{\text{work}} \\ 0 & \text{if } a \geq \#_{\text{work}} \end{cases}$$

$$\underline{L}_{a,ss} = \begin{cases} 0 & \text{if } a = 0 \\ (1-\zeta_{a-1})(1-\delta_a^L)L_{a-1,ss} & \text{if } a < \#_{\text{work}} \\ 0 & \text{else} \end{cases}$$

$$\underline{L}_{a,ss}^{H} = \begin{cases} 0 & \text{if } a = 0 \\ (1-\zeta_{a-1})(1-\delta_a^L)L_{a-1,ss}^{H} & \text{if } a < \#_{\text{work}} \\ 0 & \text{if } a \geq \#_{\text{work}} \end{cases}$$

$$L_{a,ss} = \underline{L}_{a,ss} + m_{ss}^{s} S_{a,ss}$$
$$L_{a,ss}^{H} = \underline{L}_{a,ss}^{H} + m_{ss}^{s} S_{a,ss}^{H} H_a$$

$$U_{a,ss} = \begin{cases} N_a - L_{a,ss} & \text{if } a < \#_{\text{work}} \\ 0 & \text{else} \end{cases}$$

$$L_{ss} = \sum_a L_{a,ss}$$

$$S_{ss} = \sum_a S_{a,ss}$$

$$U_{ss} = \sum_a U_{a,ss}$$

$$\delta^L_{ss} = \frac{L_{ss} - \underline{L}_{ss}}{L_{ss}}$$

$$\mathcal{M}_{ss} = \delta^L_{ss} L_{ss}$$

$$v_{ss} = \frac{\mathcal{M}_{ss}}{m^v_{ss}}$$

$$\nu = \frac{1}{v_{ss}} \left(\frac{(m^s_{ss})^{\frac{1}{\sigma^m}} \, S_{ss}^{\frac{1}{\sigma^m}}}{1 - (m^s_{ss})^{\frac{1}{\sigma^m}}} \right)^{\sigma^m}$$

4. Capital agency behavior implies

$$r^K_{ss} = r^{\text{firm}} + \delta^K.$$

5. Labor agency behavior implies

$$r^\ell_{ss} = \frac{W_{ss}}{1 - \left(1 + \frac{1-\delta^L_{ss}}{1+r^{firm}}\right) \frac{\kappa^L}{m^v_{ss}}}$$

$$\ell_{ss} = L_{ss} - \kappa^L v_{ss}.$$

6. Set the prices using the Phillips-curve

$$P^{Y,0}_{ss} = \frac{P^Y_{ss}}{1+\theta}$$

$$\Gamma_{ss} = \frac{1}{P^{Y,0}_{ss}} \left(\mu^K \left(r^K_{ss}\right)^{1-\sigma^Y} + \left(1-\mu^K\right) \left(r^\ell_{ss}\right)^{1-\sigma^Y} \right).$$

7. Determine the capital and output from the production firm

$$
\begin{aligned}
K_{ss} &= \frac{\mu_K}{1-\mu_K}\left(\frac{r^{\ell}_{ss}}{r^{K}_{ss}}\right)^{\sigma^Y}\ell_{ss} \\
Y_{ss} &= \Gamma_{ss}\left(\left(\mu^K\right)^{\frac{1}{\sigma^Y}}K_{ss}^{\frac{\sigma^Y-1}{\sigma^Y}}+\left(1-\mu^K\right)^{\frac{1}{\sigma^Y}}\ell_{ss}^{\frac{\sigma^Y-1}{\sigma^Y}}\right)^{\frac{\sigma^Y}{\sigma^Y-1}}.
\end{aligned}
$$

8. From capital accumulation equations

$$
\iota_{ss} = I_{ss} = \delta^K K_{ss}.
$$

9. Exogenously, set the government spending as a share of the output and find the tax rate

$$
\begin{aligned}
G_{ss} &= G^{share}Y_{ss} \\
\tau_{ss} &= \frac{r_B B_{ss} + P^G_{ss}G_{ss} + W_U U_{ss} + W_R(N_{ss} - N^{work}_{ss})}{W_{ss}L_{ss} + W_U U_{ss} + W_R(N_{ss} - N^{work}_{ss})}.
\end{aligned}
$$

10. Find the age-specific income for the households and use this to define the consumption for the hands-to-mouth households.
Guess on A^q_{ss} and check $\left(1+r^{hh}_{ss}\right)\sum_{a=0}^{\#-1}\zeta_a N_a A_{a,ss} = A^q_{ss}$. Find the consumption

of Ricardian households and their assets backwards.

$$
\begin{aligned}
\text{inc}_{a,ss} &= (1-\tau_{ss})\, W_{ss} \frac{L_{a,ss}}{N_a} + (1-\tau_{ss})\, W_U W_{ss} \frac{U_{a,ss}}{N_a} + (1-\tau_{ss})\, W_R W_{ss} 1_{\{a \geq \#_{\text{work}}\}} + \frac{A^q_{ss}}{N} \\
C^{\text{HtM}}_{a,ss} &= \frac{\text{inc}_{a,ss}}{P^C_{ss}} \\
A^{\text{HtM}}_{a,ss} &= 0 \\
C^R_{a,ss} &= \begin{cases} \left(\zeta_a \mu^{A^q} \left(\frac{A^R_{\#-1,ss}}{P^C_{ss}} \right)^{-\sigma} \right)^{-\frac{1}{\sigma}} & \text{if } a = \# - 1 \\ \left(\beta (1-\zeta_a) \frac{1+r^{hh}_{ss}}{1+\pi^{hh}_{ss}} \left(C^R_{a+1,ss} \right)^{-\sigma} + \zeta_a \mu^{A^q} \left(\frac{A^R_{a,ss}}{P^C_t} \right)^{-\sigma} \right)^{-\frac{1}{\sigma}} & \text{else} \end{cases} \\
A^R_{a-1,ss} &= \frac{A^R_{a,ss} - \text{inc}_{a,ss} + P^C_{ss} C^R_{a,ss}}{1 + r^{hh}_{ss}} \\
C_{a,ss} &= \lambda C^{\text{HtM}}_{a,ss} + (1-\lambda) C^R_{a,ss} \\
A_{a,ss} &= \lambda A^{\text{HtM}}_{a,ss} + (1-\lambda) A^R_{a,ss} \\
A^q_{ss} &= \left(1 + r^{hh}_{ss} \right) \sum_{a=0}^{\#-1} \zeta_a N_a A_{a,ss}.
\end{aligned}
$$

11. Determine package components for consumption and investment

$$
\begin{aligned}
\bullet^M_{ss} &= \mu^{M,\bullet} \left(\frac{P^\bullet_{ss}}{P^{M,\bullet}_{ss}} \right)^{\sigma^\bullet} \bullet_{ss}, \text{ for } \bullet \in \{C, G, I\} \\
\bullet^Y_{ss} &= (1-\mu^{M,\bullet}) \left(\frac{P^\bullet_{ss}}{P^{M,\bullet}_{ss}} \right)^{\sigma^\bullet} \bullet_{ss}, \text{ for } \bullet \in \{C, G, I\}.
\end{aligned}
$$

12. Determine χ_{ss} to get market clearing

$$
\begin{aligned}
X^Y_{ss} &= Y_{ss} - \left(C^Y_{ss} + G^Y_{ss} + I^Y_{ss} \right) \\
\chi_{ss} = X_{ss} &= \frac{X^Y_{ss}}{(1-\mu^{M,X}) \Gamma^{\sigma^X - 1}} \\
X^M_{ss} &= \mu^{M,X} \Gamma^{\sigma^X - 1} \left(\frac{P^X_{ss}}{P^{M,X}_{ss}} \right)^{\sigma^X} X_{ss} \\
M_{ss} &= C^M_{ss} + G^M_{ss} + I^M_{ss} + X^M_{ss}.
\end{aligned}
$$

3.1 Data

See 0b - steady state - data.ipynb.

1. Download data: Data from Statistics Denmark is downloaded through DstApi to calibrate the steady state.

2. National account: Data for the national accounts used to calculate the historical average ratios of M_t, X_t, C_t, G_t and I_t relative to Y_t.

3. Capital, employment and pouplation: Data for capital, employment and population used to calculate the historical averages of capital and employment rate.

4. Unemployment rate: Data for the unemployment rate used to calculate the historical average.

5. Steady state aggreate: Historical averages of the variables above.

6. Income and wealth over life-cycle: Data for historical averages of income and wealth over the life-cycle. The regression below is perfomed to seperate time and life-cycle effects on income. The data and estimates are then used to calibrate the steady state life-cycle income profile and wealth-income ratio.

$$\text{inc}_{a,t} = \text{birth year}_a + \rho_1 \text{age}_{a,t} + \rho_2 \text{age}^2_{a,t} + \epsilon_{a,t}$$

3.2 Households

See 0c - steady state - households.ipynb.

1. Setup: An instance of the baseline model is created named »model«. The parameters, steady state and solution namespaces are set.

2. Mortality and population: First, a function for creating an interactive plot is defined, where it is possible to adjust the power of the mortality rate ζ_a. Then, figures showing the mortality rate and the population are plotted. The average life expectancy is increasing in the power of ζ_a, as this implies a lower mortality rate for all $a < \#$. In the extreme case where the power is approaching infinity, everybody dies in the last period, and in the opposite case where the power is 0, everybody dies in the first period after retirement.

Reset the demographic structure (to 4), such that it will not affect the other results.

3. Labor supply: Using the job-finding rate, it is possible to find the household search behavior in steady state. A function for an interactive figure is defined, where it is possible to adjust the separation rate, δ_a^L. Figures for the separation rate, the number of searchers and the employment/unemployment levels are plotted. The separation rate is constant throughout the work life period, and a higher separation rate implies more workers loss their job; thus, they become unemployed and search for a new job. In the extreme case where the separation rate is 1, all workers loss their job at the end; all workers search for a new job. As such, the employment level will be equal to the job-finding rate, m^s.
   Reset the demographic structure and household search ($\delta_a^L = 0.1$), such that it will not affect the other results.

4. Consumption-saving: Using the required inputs, its is possible to find the household consumption and savings behavior. The age-specific income, consumption and savings are plotted for the Ricardian and HtM households together with the aggregations. The income and saving profiles is compared to historical data. The Ricardian households smooth their consumption, while the HtM consume all their income in every period. The aggregated consumption and savings are a weighted average determined by the share of each type, λ.

5. Varying central inputs: A function for comparing to models is defined, which plots the income, consumption and savings for varying inputs:
   - A_{ss}^q: If the households receive less bequest, their income are lower and they will consume less.
   - μ^{A^q} : If the households have a lower preference for bequest, they will save less and consume more.
   - W_{ss} : If the households receive a higher wage, they will have more income which implies more consumption.
   - τ_{ss} : If the tax rate is lower, the households receive more after-tax wage. Therefore, this is similar to increasing the wage.
   - W_U: If the households receive a higher unemployment benefit, they will have more income before retirement; thus, they can consume and save more.

However, this effect is much smaller than the effect from a higher wage, as the number of unemployed is much smaller than the number of employed.

- W_R: If the households receive more in retirement benefits, they will have more income once they retire. This implies that they save less, and they consume more throughout their life.
- β: If beta decreases, the households prefer to consume more today relative to tomorrow. As such, they save less and they consume more in the first part of their lifetime.
- r_{hh} : If the interest rate decreases, it is relatively more expensive to save than consume. Therefore, the households will save less and consume more initially.
- λ: If the share of HtM households decreases, the aggregation of households will smooth consumption more.

6. Test household blocks: The solution variables are set to their steady state value, and the model is set to be in steady state. The search and matching block and the household consumption behavior are evaluated, and it is checked that all variables remain at their steady state levels.

3.3 Aggregates

See 0d - steady state - aggregate.ipynb

1. Setup: An instance of the baseline model is created named »model«.
2. Find steady steady: The steady state values are found and printed. In addition, steady state are compared to historical averages from the data.
3. Speed and error tests: The exogenous and unknown steady state variables are specified. The blocks are evaluated using python evaluation, and it is verified that all targets are zero. Next, the blocks are evaluated using numba evaluation. The initial evaluation is slower when using numba, but subsequent evaluations are much faster than python evaluations.

4 Shocks

In this section, we consider a number of shocks to central exogenous variables and analyze what happens to all other variables and the convergence back to steady state.

4.1 Government spending shock

See 1 - shock - government spending.ipynb.

- **Effect on Government** (see 2.12)

  Initial effect: The government increases real consumption, $G \uparrow$, which implies an increase in the demand for goods. The firms increase their production to satisfy the excess demand, $Y \uparrow$, and as such they have to employ more workers, $L \uparrow$. In isolation, this improves the government budget due to less payments related to unemployment benefits and a higher tax revenue. However, the overall effect on the government budget is ambigious as goverment consumption increases as well. Overall, the debt decreases, $B \downarrow$, which allows the government to lower its tax rate, $\tau \downarrow$. The price increases, $P^G \uparrow$, following higher output demand and higher marginal costs. However, as prices are sticky, it does not fully adjust.
  Transition path: Government consumption decreases as the shock fades. The debt grows due to a decline in the production and the employment, and the fact that prices continue to increase due to the adjustment costs. The government raises the tax rate as a consequence, which further decreases the excess demand. Eventually, the debt is gradually reduced once the tax rate has grown sufficiently large.

- **Effect on Production firms, Phillips-curve and Investments** (see 2.8,2.9,2.11)

  Initial effect: The firms increases production to meet the higher demand. Since capital is fixed, the production firms can only produce more by hiring more labor, $\ell \uparrow$. This makes the capital more productive, $r^K \uparrow$, but also drives up the marginal cost price, $P^{Y,0} \uparrow$. As the output price is a mark-up of the marginal cost, the output prices increase, $P^Y \uparrow$. However, this increase is limited due to the stickyness of the output prices. A higher rent on capital implies that the capital agency has an incentive to accumulate more capital, $K \uparrow$, which it achieve throguh more investment $I \uparrow$.

Transition path: The excess demand stemming from government consumption shock declines, such that the firms lower their production, $Y \downarrow$. As such, less labor is needed, which makes capital less productive. Consequencely, the capital agency invests in less capital, and the capital stock declines. A lower stock of capital drives up the return on capital, such that this process eventually reverses.

- **Effect on Labor agency, Labor search and match and Wage** (see 2.6, 2.7, 2.5)

  Initial effect: The increase in demand for ℓ implies that the labor agency hires more labor, $L \uparrow$. More hired labor implies more matches, $\mathcal{M} \uparrow$, due to more job vancancies, $v \uparrow$, which further implies an increase in the job-finding rate, $m^s \uparrow$. However, the job-filling rate decreases, $m^v \downarrow$. The wage increases as it is increasing in the labor demand, $W \uparrow$.
  Transition path: The employment decreases from its initial spike, as the demand for labor decreases. This implies that the amount of searchers and the job-filling rate increase. As the employment rate is higher, both the amount of matches and vacancies decrease to a level, which is lower than their steady state levels. The wage decreases, as the excess demand for labor decreases.

- **Effect on Consumption** (see 2.13)

  Initial effect: The higher real wage and the higher employment rate imply that the real income increases, $\text{inc}^{real} \uparrow$. As such, the HtM households increase their consumption, $C^{HtM} \uparrow$. The Ricardian households forsee that the government will raise the tax rate in future periods and in order to smooth their consumption, they choose to save up, $A \uparrow$. This implies that they can afford less consumption, $C^R \downarrow$. Overall, total consumption increases, $C \uparrow$. The price on consumption increases, $P^C \uparrow$, following the higher output prices, which implies inflation and thereby a lower real interest rate, $r^{real} \downarrow$.

  Transition path: The real income decreases following the declining real wage and the increase in the tax rate, thus the consumption of the HtM households decreases. Likewise, the consumption of the Ricardian households decreases, which is futher enlarged by the fact that the real interest rate increases, making savings more attractive. Once, the price on consumption starts to decrease, so does the real interest rates, and the Ricardian households begin using their savings to increase consumption. The aggregated consumption decreases for a few periods, and once the government lowers its tax rate again, the real income increases and so does the consumption.

- **Effect on Foreign economy** (see 2.10)

  Initial effect: The domestic prices have increased relative to the foreign price. As such, export decreases, $X \downarrow$, and import increases, $M \uparrow$.
  Transition path: As the excess domestic demand declines, so does the domestic prices. This implies that import decreases and export increases.

- **Further comments:**

  The effect of tax rate adjustment speed: The effect of the government spending shock is affected by how fast the government adjusts the tax rate, which is captured by the parameter ε_B. In 2 - sensitivity - technology.ipynb., a sensitivity analysis is conducted to analysis the effect of adjustment time. The basaline case is compared to two different case; one where the adjustment rate is higher, and one where it is lower. If ε_B decreases, the govenrment initially decreases the tax rate less. Hence, it receives less tax payments, and the government debt is higher than the baseline case. Likewise, the government is also slower to adjust the tax rate once the debt starts growing, and as a consequence hereof the initial shock has long term effect on the government budget. The slower adjustment of the tax rate also affects the households as the initial effect on their income is smaller, but the future effect is much more persistent. If ε_B increases, it has the opposite effect.

4.2 Productivity shock

See 1 - shock - technology.ipynb.

- **Effect on Production firms, Phillips-curve and Investments** (see 2.8,2.9,2.11)

  Initial effect: The productivity increases due to the technology shock, $\Gamma \uparrow$, such that the production increases, $Y \uparrow$, marginal cost prices decrease, $P^{Y,0} \downarrow$, and output prices decrease, $P^Y \downarrow$. However, the sticky nature of the output prices, implies that this price reduction is much smaller initially. As a consequence, the gain of increasing production is much smaller for the firms, and they choose to demand less labor, $\ell \downarrow$. Less labor makes capital less productive, such that the firms decrease the stock of capital, $K \downarrow$, which they achieve by investing less, $I \downarrow$.

Transition path: The productivity declines as the shock fades. As the output prices slowly adjust, the higher productivity leads to higher real returns on labor and capital. Hence, the firms increase production by hiring labor and investing in capital. As, the labor force and the capital stock increase, the marginal products decrease, which leads to a reduction of both inputs and a reduction in the production.

- **Effect on Labor agency, Labor search & match and Wage** (see 2.6, 2.7, 2.5)

  Initial effect: The labor agency hires less labor, $L \downarrow$, following the decrease in demand for ℓ. Less labor implies less matches, $\mathcal{M} \downarrow$, due to less vacancies, $v \downarrow$, which further implies a lower job-finding rate, $m^s \uparrow$. However, the job-filling rate increases, $m^v \uparrow$. The wage is decreasing in the lower labor demand, $W \downarrow$.
  Transition path: The demand for labor increases, thus the labor agency hires more employees. This implies more matches, a higher job-filling rate, more vancancies and a higher wage.

- **Effect on Government** (see 2.12)

  Initial effect: The government debt is negatively affected by the the decreasing price on government consumption, but positively affected by increasing payments to unemployment benefits. The overall effect is a higher government debt, $B \uparrow$. Therefore, the government to raise the tax rate, $\tau \uparrow$.

  Transition path: The debt continue to rise due to higher interest payments and more expensive government consumption. However, as the unemployment benefit payouts decrease due to an incrasing employment rate, and as the tax revenue increases following the higher tax rate, the government debt decreases. Due to the high employment rates, the government debt becomes negative, which allows the government to lower its tax rate.

- **Effect on Consumption** (see 2.13)

  Initial effect: The real income drops, $\text{inc}^{real} \downarrow$, following the drop in the wage, which implies lower consumption for the HtM households, $C^{HtM} \downarrow$. The Ricardian households, however, smooth their consumption. They know that their future income will increase as the tax rate will decrease and the wage will increase. Thus, they initially borrow, $A \downarrow$, to increase consumption, $C^R \uparrow$. The effect on aggregated consumption is dominated by the drop in consumption from the HtM households and thus $C \downarrow$. The price on consumption decreases, $P^C \downarrow$, following the lower output prices, which implies deflation and thereby a higher real interest rate, $r^{real} \uparrow$.

Transition path: The real income increases due to a higher wage rate and a lower tax rate, thereby increasing the consumption of the HtM households. The Ricardian households increase consumption as well, which is reinforced by a falling real interest rate. As such, they substitute towards consumption by borrowing even more. Once they start paying their debt, the consumption decreases to a level beneath its steady state. This is caused by increased interest rate payments following their higher debt.

- **Effect on Foreign economy** (see 2.10)

  Initial effect: The domestic firm is relatively more productive than the foreign firms, hence the export increases, $X \uparrow$, and import decreases, $M \downarrow$.
  Transition path: As the excess domestic productiveity declines, so does the export.

- **Further comments:**

  The effect of price stickiness: The effects of the productivity shock is affected by the stickiness of the output prices, which is captured by the parameter η (determined by γ). The case where $\eta = \gamma = 0$ corresponds to fully flexible prices. In 2 - sensitivity - technology.ipynb., a sensitivity analysis is conducted, where the case of fully flexible prices and sticky prices is examined. If prices are fully flexible, the real marginal product of capital and labor immidiately increase, such that the production firms hire more labor and invest in capital. As a consequence of more labor, the real wage increases and the government debt decreases, which allows the government to lower the tax rate. This implies an unanimous increase in the real income of the households, such that they increase consumption immidiately. As such, the signs of these effects are opposite to the case of the baseline case with sticky prices, where the households were worse of. If η increases, the effects are qualitatively similar to the baseline model.

4.3 Export shock

See 3 - shock decomposition - export.ipynb.

- **Effect on Foreign economy, Prices and Goods market clearing** (see 2.10,2.4,2.15)

  Initial effect: Initially, there is a shock to the foreign demand shifter, $\chi \uparrow$, the foreign prices,$P^F \uparrow$, and the import prices,$P^{M,\bullet} \uparrow$ for $\bullet \in \{C, G, I, X\}$. These shocks are here decomposed into two channels: The foreign demand shock channel containing the shock to the foreign demand shifter, and the price shock channel containing the shock to foreign and import prices.

  Examining the foreign demand shock channel, a higher foreign demand implies more goods are exported, $X \uparrow$, and additionally an increasing price of export, $P^X \uparrow$. As a consequence, the total demand and the output prices increase, $Y \uparrow$ and $P^Y \uparrow$. The higher output prices affect prices on consumption, investment and government consumption, such that these increase, $P^\bullet \uparrow$ for $\bullet \in \{C, G, I\}$. As the import prices are unaffected in this channel, imports becomes relatively cheaper compared to the domestic production, such that import increases, $M \uparrow$.
  Examining the price shock channel, higher foreign prices increase the foreign demand for domestic goods. As such, it is qualitatively similar to an increase in the demand shifter, and the total demand as well as domestic prices increase, $Y \uparrow$ and$P^\bullet \uparrow$ for $\bullet \in \{Y, C, G, I\}$. However, the higher import prices make import less attractive, $M \downarrow$, via direct effects from domestic demand $\bullet_t^M \downarrow$ for $\bullet \in \{C, G, I, X\}$.

  Overall, the combination of the two channels suggests that export, total demand, and all prices unanimously increase, whereas the effect on import is ambiguous. However, the model suggests that the overall effect on import is positive, as the import prices are exogenous and thus independent of the foreign prices, P^F.
  Transition path: Taking a starting point in the foreign demand shock channel, the foreign demand shifter decreases as per assumption. A lower foreign demand implies less exported goods. The domestic prices increase even further which is caused by the stickiness of the output prices and as a consequence the total demand drops. The import declines by a smaller rate as the domestic prices are now even higher relative to the import prices. Hereafter, prices declines as the foreign demand shock decreases implying that the import declines as well.
  Next, consider the price shock channel, where both foreign prices and import prices decrease as per assumption. Again, a lower foreign demand stemming from lower foreign prices implies less exported goods, and the domestic prices increase even further due to stickiness of the output prices. The import grows due to the declining import prices. As the import prices declines faster than the

domestic prices, the import grows above its steady state value and export drops below its steady state value.

- **Effect on Production firms, Phillips-curve and Investments** (see 2.8,2.9,2.11)

  Initial effect: Through both channels, the production firms increase production, $Y \uparrow$, to meet the increased demand stemming from the more export. As capital is fixed for the production firms, they increase the labor, $\ell \uparrow$, to produce more. The real marginal product of capital is increasing in labor $\frac{r^K}{P^Y} \uparrow$ as well as the real marginal product of labor, $\frac{r^\ell}{P^Y} \uparrow$. The effects on capital and investment are different through each of the channels. Taking a starting point in the foreign demand shock channel, the dominating effect is that firms want more capital, as the real marginal product has increased, $K \uparrow$, which is achived by investing more, $I \uparrow$. However through the price shock channel, the effect from higher import prices is a negative, domestic demand shock. The firms foresee that the lower future demand implies a lower future production. Hence, less investments are made, $I \downarrow$, such that there is a drop in the capital, $K \downarrow$. Overall, the effect from higher real marginal product of capital seems to dominate, such that investments and capital initially increase.

  Transition path: The production decreases as the excess demand diminishes, which implies that the labor demand decreases. Likewise, the capital declines due to a low real marginal product of capital, but eventually a higher domestic demand implies a higher real marginal product of capital, implying more capital. This low level of capital also implies that the production and labor demand drops below their steady state levels. The effects stemming from each channel are qualitatively similar, even though the quantitative effects on capital is much greater through the price shock channel.

- **Effect on Labor agency, Labor search & match and Wage** (see 2.6, 2.7, 2.5)

  Initial effect: The initial effect on the demand for labor from each shock channel is qualitatively the same. The demand for labor increases as a consequence of higher foreign demand, implying that the labor agency hires more labor, $L \uparrow$. More hired labor implies more matches, $\mathcal{M} \uparrow$, as job vacancies increase, $v \uparrow$, which further implies an increase in the job-finding rate, $m^s \uparrow$. However, the job-filling rate decreases, $m^v \uparrow$. The wage as well as the real wage is increasing in the labor demand, $W \uparrow$.

Transition path: As the production declines, so does the labor. Less labor implies more searchers, less vacancies and less matches. The wage is also declining due to less labor demand.

- **Effect on Government** (see 2.12)

  Initial effect: The initial effects on the government budget are qualitatively the same through each of the two channels. A higher foreign demand implies a higher price on government consumption, hence increasing the government expenditures. On the other hand, a higher demand initially implies that more people becomes employed. Hence, the government receives more in tax revenue and pays less in unemployment benefits, which suggests lower government expenditures. The overall effect seems to reduce the expenditures of the government, $B \downarrow$, which allows the government to lower its tax rate, $\tau \downarrow$.

  Transition path: After the initial reduction of government debt, it starts to increase due to an increase in the unemployment as described in »Labor agency, Labor search & match and Wage«, a lower tax rate, and the fact that the price on government consumption remains high. However, the growth in the debt is limited due to a decrease in the interest payments following the initial drop in debt. The debt continues to increase as the foreign demand diminshes.

- **Effect on Consumption** (see2.13)

  Initial effect: Both shocks increase the foreign demand which implies more labor, a lower tax rate and thereby a higher real income for the households, $\text{inc}^{\text{real}} \uparrow$. Therefore, the HtM increases consumption, $C^{HtM} \uparrow$. The Ricardian household smooth consumption by initially increasing their savings, $A \uparrow$, as they foresee that the tax rate will increase in the future.

  The price shock channel implies less import and thereby less consumption import. Additionally, the real income of the household decreases as the effects on the labor supply is significantly smaller combined with the fact that the real unemployment and retirement benefits decrease. Both of these effects suggest lower consumption, $C \downarrow$. The initial price shock implies future deflation, and thereby an increase in the real interest rate, $\text{inc}^{\text{real}} \uparrow$. As such, Ricardian households substitute towards savings, $A \uparrow$, which also implies they consume less than the HtM households initially.

The combined effect of both channels suggests an increase in both the real interest rate and savings. The effect on consumption and real income is ambiguous, but the overall effect is a reduction in both.

Transition path: Taking a starting point in the foreign demand shock, the real income continues to rise due to a lower tax rate. This allows for more consumption as well as more savings. As the savings increase so does the related interest payments, which over time allows the Ricardian households to have consumption above their real income. Once, the tax rate starts converging towards its steady state level, so does the real income and thereby the consumption.

Turning to the price shock channel, the real interest decreases from its initial high level, which induces the Ricardian households to save up less than initially, thus consuming more. The higher interest payment allows the Ricardian household to increase consumption even though the real income continues to decline due to an increase in the tax rate. Eventually, the real interest rate has decreased too much for the interest payment to be sufficient to cover the excess consumption (consumption above the real wage), thus the Ricardian households start to use their savings to cover their consumption. Once, the real income converges, so will the consumption and the savings.

4.4 Interest rate shock

See 1 - shock - interest rate.ipynb.

- **Effect on Consumption** (see2.13)

  Initial effect: Initially, the foreign interest increases, $r^{hh} \uparrow$. This has two effects on the Ricardian households' savings - namely an income and a substitution effect. The income effect is an increase return on savings already held by the households. As such, they can reach the same savings level even if they decrease future savings. Overall, this income effect implies that the consumption would initially increase. The substitution effect is that the interest rate for future savings are higher. This makes it more attractive to build up savings, which implies an initial drop in consumption. Overall, the substitution effect seems to be dominant as consumption for the Ricardian households initially decreases, $C^R \downarrow$, and savings increases, $A \uparrow$. The real income decreases whch is cause by a production and thereby less labor demand. As such, the HtM households can afford less consumption, $C^{HtM} \downarrow$.

Aggregated consumption decreases, $C \downarrow$, which drives down the price on consumption $P^C \downarrow$. There is an initial increase in bequests, as these are affected by the interest rate $Aq \uparrow$.
Transition path: The foreign interest rate decreases as per assumption. After the initial drop, the Ricardians' consumption increases as they need to save less due to higher interest payments. Higher demand for consumption drives up prices, but also implies higher output levels and thereby higher employment levels. Higher employment has two positive effects for consumption; higher wage income compared to unemployment benefits and a lower tax rate (even though this effect is small). These two effects also increase the HtM households' consumption, so overall the aggregated consumption continues to grow. Eventually, the decline in interest payment implies that the consumption as well as the savings converge back to its steady state level.

- **Effect on Production firms, Phillips-curve and Investments** (see 2.8,2.9,2.11)

  Initial effect: The production firms decrease production, $Y \downarrow$, to meet the lower demand stemming from the less consumption. As the capital is fixed, they must fire workers, $\ell \downarrow$, which drives down return on capital, $r^K \downarrow$. This implies a lower marginal cost price, $P^{Y,0} \downarrow$, and a lower output price, $P^Y \downarrow$s. The output price is sticky, thus the decrease is relatively smaller. A lower output price implies that the price on investment also decreases, $P^I \downarrow$. The investments increase, $I \uparrow$, as the firms foresee a higher real return on capital in the future stemming from higher future demand for consumption. More investments implies an increase in the capital, $K \uparrow$.
  Transition path: The production firms increase production due to the higher demand stemming from the more consumption. As such, more workers are hired, which drives up the return on capital as well as the marginal cost and output prices. These converge toward their steady state level as the excess demand diminishes.

- **Effect on Labor agency, Labor search & match and Wage** (see 2.6, 2.7, 2.5)

  Initial effect: The decrease in demand for ℓ implies that the labor agency fires labor, $L \downarrow$. Less hired labor implies less matches, $\mathcal{M} \downarrow$, which further implies an decrease in the job-finding rate, $m^s \downarrow$, and the vacancies, $v \downarrow$. However, the job-filling rate increase, $m^v \uparrow$. The real wage is held constant, thus the nominal wage follows the consumer prices.

Transition path: Higher output production implies a higher demand for workers. Therefore, more labor is hired which implies more matches, a higher job-finding rate, more vacancies, less searches and a lower job-filling rate. Once, the excess demand diminishes so does employment rate and all variables converge towards their steady state levels.

- **Effect on Government** (see 2.12)

  Initial effect: The government debt increase when the employment rates decrease as the government receives less in tax revenue and pays more in unemployment benefits. On the other hand, the nominal expenditures decrease as the price on government consumption decreases. Overall, the initial effect on debt seems to be slightly positive, such that the government raises the tax rate.

  Transition path: The increasing employment rate implies a declining debt, which is not counteracted by the increasing prices. As such, the debt becomes negative, and the government receives increasing interest rate payments. The government reacts by lowering the tax rate, which increases the demand for consumption even further; thus, raising the employment rates further. Eventually, the excess demand diminishes, such that both the government and the tax rate increase towards their steady state levels.

- **Effect on Foreign economy** (see 2.10)

  Initial effect: The price on exports decreases initially, which is caused by the lower output prices. As the foreign prices are unaffected, the foreign demand for domestic goods increases, such that more goods are exported, $X \uparrow$. It also implies that imported goods are relatively more expensive compared to domestic goods, thus less goods are imported, $M \downarrow$.

  Transition path: As the domestic demand increases, so does the domestic prices, which reduces the competitiveness of the domestic firms. Therefore, less goods are exported and more goods are imported. The export and import normalizes once the excess domestic demand vanishes.

5 Status report

Status: The described model is implemented in Python. Results are still not understood in detail.

Tasks:

1. Clarify steady state calibration.
2. Perform impulse-response matching.
3. Improve detailed accounting of main shocks.
4. Add technology growth, population growth, and trend inflation.
5. Add financial flows accounts wrt. to the foreign economy.
6. Add more government with taxes and spending.
7. Add endogenous labor supply.
8. Add multiple sectors and input-output structure.
9. Make the model quarterly.

Computational improvements: Improve calculation of Jacobian (e.g.. automatic differentiation)[3].

[3] Investigate what is done efficiently in MAKRO (GAMS+CONOPT)

A Derivations

A.1 Capital agency

$$
\begin{aligned}
V\left(K_{t-1}\right) = \max_{\{K_t\}} \sum_{t=0}^{\infty} \frac{1}{\left(1+r^{\text{firm}}\right)^t} & \left[r_t^K K_{t-1} - P_t^I(\iota_t + \Psi(\iota_t, K_{t-1}))\right] \\
\text{s.t.} & \\
I_t &= \iota_t + \Psi(\iota_t, K_{t-1}) \\
K_t &= (1-\delta^K)K_{t-1} + \iota_t \\
\text{with } \Psi(\iota_t, K_{t-1}) &= \frac{\Psi_0}{2}\left(\frac{\iota_t}{K_{t-1}} - \delta^K\right)^2 K_{t-1}
\end{aligned}
$$

Inserting constraints:

$$
V\left(K_{t-1}\right) = \max_{\{K_t\}} \sum_{t=0}^{\infty} \frac{1}{\left(1+r^{\text{firm}}\right)^t} \left[r_t^K K_{t-1} - P_t^I(K_t - (1-\delta^K)K_{t-1} + \Psi(\iota_t, K_{t-1}))\right]
$$

First order condition:

$$
\begin{aligned}
\frac{\partial V(K_{t-1})}{\partial K_t} = & \frac{1}{(1+r^{\text{firm}})^t}\left(-P_t^I\left(1+\Psi_\iota(\iota_t, K_{t-1})\right)\right) \\
& + \frac{1}{(1+r^{\text{firm}})^{t+1}}\left(r_{t+1}^K - P_{t+1}^I\left(-(1-\delta^K) + \Psi_\iota(\iota_{t+1}, K_t)(1-\delta^K) + \Psi_K(\iota_{t+1}, K_t)\right)\right) = 0
\end{aligned}
$$

From which it follows:

$$
\begin{aligned}
& -P_t^I\left(1+\Psi_\iota(\iota_t, K_{t-1})\right) \\
+ \frac{1}{(1+r^{\text{firm}})} & \left(r_{t+1}^K - P_{t+1}^I\left(-(1-\delta^K) - \Psi_\iota(\iota_{t+1}, K_t)(1-\delta^K) + \Psi_K(\iota_{t+1}, K_t)\right)\right) = 0
\end{aligned}
$$

A.2 Labor agency

$$
\begin{aligned}
V(L_{t-1}) = \max_{\{L_t\}} \sum_{t=0}^{\infty} & \frac{1}{(1+r^{\text{firm}})^t} \left[r_t^{\ell} \ell_t - W_t L_t \right] \\
\text{s.t.} & \\
\ell_t &= L_t - \kappa^L v_t \\
L_t &= \left(1 - \delta_t^L\right) L_{t-1} + m_t^v v_t
\end{aligned}
$$

Inserting constraints:

$$
V(L_{t-1}) = \max_{\{L_t\}} \sum_{t=0}^{\infty} \frac{1}{(1+r^{\text{firm}})^t} \left[r_t^{\ell} \left(L_t - \kappa^L \frac{L_t - (1-\delta_t^L) L_{t-1}}{m_t^v} \right) - W_t L_t \right]
$$

First order condition:

$$
\frac{\partial V(L_{t-1})}{\partial L_t} = \frac{1}{(1+r^{\text{firm}})^t} \left(r_t^{\ell} \left(1 - \frac{\kappa^L}{m_t^v} \right) - W_t \right) + \frac{1}{(1+r^{\text{firm}})^{t+1}} \left(r_{t+1}^{\ell} \frac{\kappa^L}{m_{t+1}^v} (1 - \delta_{t+1}^L) \right) = 0
$$

From which it follows:

$$
\begin{aligned}
\left(r_t^{\ell} \left(1 - \frac{\kappa^L}{m_t^v} \right) - W_t \right) + \frac{1}{(1+r^{\text{firm}})} \left(r_{t+1}^{\ell} \frac{\kappa^L}{m_{t+1}^v} (1 - \delta_{t+1}^L) \right) &= 0 \\
\frac{1}{1 - \frac{\kappa^L}{m_t^v}} \left[W_t - \frac{1}{(1+r^{\text{firm}})} \left(r_{t+1}^{\ell} \frac{\kappa^L}{m_{t+1}^v} (1 - \delta_{t+1}^L) \right) \right] &= r_t^{\ell}
\end{aligned}
$$

A.3 Phillips Curve

$$
\begin{aligned}
p_t^Y &= \frac{\sigma_D}{\sigma_D - 1} P_t^{Y,0} - \frac{1}{\sigma_D - 1} \frac{p_t^Y}{y_t} \left(\frac{\partial g_t}{\partial p_t^Y} + \frac{1}{1 + r^{\text{firm}}} \frac{\partial g_{t+1}}{\partial p_t^Y} \right) \\
p_t^Y &= \frac{\sigma_D}{\sigma_D - 1} P_t^{Y,0} - \frac{1}{\sigma_D - 1} \frac{p_t^Y}{y_t} \left(\gamma \left[\frac{p_t^Y / p_{t-1}^Y}{p_{t-1}^Y / p_{t-2}^Y} - 1 \right] \left(\frac{1 / p_{t-1}^Y}{p_{t-1}^Y / p_{t-2}^Y} \right) P_t^Y Y_t \right. \\
&\quad \left. + \frac{1}{1 + r^{\text{firm}}} \gamma \left[\frac{p_{t+1}^Y / p_t^Y}{p_t^Y / p_{t-1}^Y} - 1 \right] 2 \left(\frac{p_{t+1}^Y / p_{t-1}^Y}{p_{t-1}^Y / p_{t-2}^Y} \right) P_{t+1}^Y Y_{t+1} \right) \\
P_t^Y &= \frac{\sigma_D}{\sigma_D - 1} P_t^{Y,0} - \frac{1}{\sigma_D - 1} \left(\gamma \left[\frac{P_t^Y / P_{t-1}^Y}{P_{t-1}^Y / P_{t-2}^Y} - 1 \right] \left(\frac{P_t^Y / P_{t-1}^Y}{P_{t-1}^Y / P_{t-2}^Y} \right) P_t^Y \right. \\
&\quad \left. + \frac{2}{1 + r^{\text{firm}}} \gamma \left[\frac{P_{t+1}^Y / P_t^Y}{P_t^Y / P_{t-1}^Y} - 1 \right] \left(\frac{P_{t+1}^Y / P_{t-1}^Y}{P_{t-1}^Y / P_{t-2}^Y} \right) P_{t+1}^Y \frac{Y_{t+1}}{Y_t} \right) \\
P_t^Y &= (1 + \theta) P_t^{Y,0} - \eta \left(\frac{P_t^Y / P_{t-1}^Y}{P_{t-1}^Y / P_{t-2}^Y} - 1 \right) \frac{P_t^Y / P_{t-1}^Y}{P_{t-1}^Y / P_{t-2}^Y} P_t^Y \\
&\quad + \frac{2}{1 + r^{\text{firm}}} \eta \frac{Y_{t+1}}{Y_t} \left(\frac{P_{t+1}^Y / P_t^Y}{P_t^Y / P_{t-1}^Y} - 1 \right) \frac{P_{t+1}^Y / P_t^Y}{P_t^Y / P_{t-1}^Y} P_{t+1}^Y
\end{aligned}
$$

where $\theta = \frac{1}{\sigma_D - 1}$.