Baby-MAKRO*

DISCLAIMER: WORK-IN-PROGRESS ⇒ BEWARE OF ERRORS!

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Abstract

This note outlines a simplified »baby« version of the MAKRO model used by the Danish Ministry of Finance. The model is for a small open economy with a fixed exchange and overlapping generations. The model have perfect foresight, but is full of imperfections due to e.g. frictions in the labor market and adjustment costs. The model is written and solved in terms of a series of ordered blocks. This clarifies the model dynamics and make it easier to solve for fluctuations around the steady state using a numerical equation system solver. Online code is provided for solving the model in Python.

The model is designed so undergraduate students can work with it, and analyze potential extensions in their thesis work. The model structure is similar to state-of-the-art heterogeneous agent models (see this course) and the model is thus relevant for further academic studies. The similarity to the grown-up MAKRO model make it relevant for potential future job tasks and the public debate.

The note concludes with a status report on the continuing development of the model.

Online code: github.com/NumEconCopenhagen/BabyMAKRO

MAKRO: See online documentation

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1 Overview

We consider a *small open economy* with a *fixed exchange rate* and *overlapping generations*. Time is discrete, $t \in \{0, 1, \dots\}$ and the frequency is annual.

Households live for up to # periods, and their age is denoted by a. The age dependent mortality is $\zeta_a \in (0, 1)$ and the population and demographic structure is constant. Households exogenously search for jobs and supply labor, receive inheritances and choose consumption and savings to get utility from consumption and bequests.

The *foreign economy* provides a fixed nominal rate of return, sell import goods at fixed prices, and have a demand curve for the domestic export good.

The *production* in the economy is layered as follows:

1. *Production firms* rent *capital* and *labor* to produce the domestic output good.
2. *Repacking firms* combine imported goods with the domestic output good to produce a consumption good, an investment good, and an export good.
3. Capital is rented from a *capital agency*, which purchases the investment good to accumulate capital subject to adjustment costs.
4. Labor is rented from a *labor agency*, which post vacancies a search-and-match labor market to purchase labor from the households.

All firms are price takers. Prices are thus flexible, except for wages which is determined by ad hoc bargaining. All *goods markets clear* and the matching process is determined by a *matching function*.

There is *perfect foresight* in the economy. I.e. the value of all current and future variables are known. This is a strong assumption, and in many ways the model should be considered a first order approximation to a full model with both idiosyncratic and aggregate risk. It can be relevant to introduce model elements, which proxy for the effects of risks. Utility-of-wealth can e.g. proxy for a precautionary saving motive.

1.1 Equilibrium path

The *equilibrium path* in the economy is a set of paths for all variables, which satisfies all accounting identities, optimal firm and household behavior in terms of first order conditions, and implies market clearing. When all variables are constant over time, the equilibrium path is a *steady state*.

In terms of math, the model is just an *equation system* stacking the accounting identifies, first order conditions and market clearing conditions. If the economy is initially out of steady state, we solve for the equilibrium path by truncating the equation system to T periods. The assumption is that the economy has settled down to the steady state well before period T, and we can assume variables from period T onward are at their steady state value. The economy can be out of steady state both because lagged *endogenous* variables are initially not at their steady state values and/or because the *exogenous* variables are not at their steady state values. We talk of an *impulse response* when the economy starts at the steady state, but some exogenous variables *temporarily* deviate from the steady state following some converging auto-regressive process.[1]

We simplify the model and the resulting equation system by writing it in terms of a *ordered series of block*. We start from a set of *exogenous* variables (e.g. variables determined in the foreign economy) and a set of *unknown* variables. Each block then takes in the path of some variables, return the path of other variables, and imply *targets*, which must be zero if the model equations are satisfied. Each block can use the unknown variables and output variables of previous blocks as input variables. In the end we collect all the targets. The *number of unknown variables* must equal the *number of target variables*.

To solve the model, we must first find the steady state. As explained in Section 3, this can be done by manually choosing values for a selection of the endogenous variables and the deriving the rest from closed form expressions or solving sub-systems with a numerical equation system solver. Next, we solve for the equilibrium path again using a numerical equation system solver.

The block structure and ordering is *not* unique. If a different set of unknowns is chosen, a different ordering of blocks must also be chosen. If an additional variable

[1] This is also called an MIT shock. A shock in a model with perfect foresight is to some degree a contradiction in terms. The assumption is that even though the agents experiences a shock, they expect that there will never be a shock again. Accounting for expecting of future shocks is much more complicated. Some realism can be mimicked by studying impulse response to shocks about the future, which when the future comes never materialized as a new opposite signed shock negates it. Multiple shocks arriving sequentially can also be studied.

is considered an *unknown*, an additional equation must be considered a target instead of being used to calculate a output. In the limit, all variables can be considered as unknowns and all equations as targets. This is inefficient as the number of variables can be very large.[2]

1.2 On CES technology

The assumption of CES technology is used repeatedly in the model. It is therefore beneficial to recap it briefly. Consider a firm producing good X using good X_i and X_j with a CES technology. Input prices are P_i and P_j and the output price is P. The firm is a price taker in all markets. The *profit maximization* problem of the firm is

$$\max_{X_i,X_j} PX - P_iX_i - P_jX_j \text{ s.t. } X = \Gamma\left(\mu_i^{\frac{1}{\sigma}}X_i^{\frac{\sigma-1}{\sigma}} + \mu_j^{\frac{1}{\sigma}}X_j^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}}, \;\; \mu_i+\mu_j = 1,\, \mu_i,\sigma,\Gamma > 0,\, \sigma \neq 1 \tag{1}$$

The generic *first order condition*

$$\begin{aligned}
0 &= P\mu_i^{\frac{1}{\sigma}}X_i^{\frac{\sigma-1}{\sigma}-1}\Gamma\left(\mu_i^{\frac{1}{\sigma}}X_i^{\frac{\sigma-1}{\sigma}} + \mu_j^{\frac{1}{\sigma}}X_j^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}-1} - P_i \Leftrightarrow \\
P_i &= P\mu_i^{\frac{1}{\sigma}}X_i^{-\frac{1}{\sigma}}\Gamma\left(\mu_i^{\frac{1}{\sigma}}X_i^{\frac{\sigma-1}{\sigma}} + \mu_j^{\frac{1}{\sigma}}X_j^{\frac{\sigma-1}{\sigma}}\right)^{\frac{1}{\sigma-1}} \Leftrightarrow \\
P_i &= P\mu_i^{\frac{1}{\sigma}}X_i^{-\frac{1}{\sigma}}\Gamma^{\frac{\sigma-1}{\sigma}}X^{\frac{1}{\sigma}} \Leftrightarrow \\
X_i &= \mu_i\left(\frac{P}{P_i}\right)^{\sigma}\Gamma^{\sigma-1}X.
\end{aligned} \tag{2}$$

As the production technology has constant return-to-scale, there are infinitely many solutions to the FOCs. They all satisfy that inputs are used in proportion as follows

$$\frac{X_i}{X_j} = \frac{\mu_i}{\mu_j}\left(\frac{P_j}{P_i}\right)^{\sigma} \tag{3}$$

[2] Modeling systems such as GAMS can combined with a state-of-the-art solver such as CONOOPT automatically analyze the structure of the equation system and thereby ripe the benefits we get from manually ordering the blocks.

Assuming *free entry*, and thus *zero profits*, the output price is uniquely determined from the input prices as

$$\begin{aligned}
0 &= PX - P_i X_i - P_j X_j \Leftrightarrow \\
P &= \frac{P_i X_i + P_j X_j}{X} \\
&= \mu_i \left(\frac{P}{P_i}\right)^{\sigma} \Gamma^{\sigma-1} P_i + \mu_j \left(\frac{P}{P_j}\right)^{\sigma} \Gamma^{\sigma-1} P_j \Leftrightarrow \\
(\Gamma P)^{1-\sigma} &= \mu_i P_i^{1-\sigma} + \mu_j P_j^{1-\sigma} \Leftrightarrow \\
P &= \frac{1}{\Gamma} \left(\mu_i P_i^{1-\sigma} + \mu_j P_j^{1-\sigma}\right)^{\frac{1}{1-\sigma}} . \qquad (4)
\end{aligned}$$

2 Blocks

2.1 Exogenous variables

The *exogenous* variables are:

1. Γ_t, technology
2. G_t, public spending
3. χ_t, *foreign* demand shifter (»market size«)
4. $P_t^{M,C}$, import price of *private consumption* component good
5. $P_t^{M,G}$, import price of *public consumption* component good
6. $P_t^{M,I}$, import price of *investment* component good
7. $P_t^{M,X}$, import price of *export* component good
8. P_t^{F} , *foreign* price level
9. r_t^{hh}, *foreign* interest rate

2.2 Unknown variables

The chosen *unknown* variables are:

1. A_t^{death}, wealth of households at $a = \# - 1$ (T unknowns)
2. A_t^q, inheritance flow (T unknowns)
3. L_t, labor supply (T unknowns)
4. K_t, capital (T unknowns)
5. r_t^K, rental price for capital (T unknowns)
6. P_t^Y, price of domestic output (T unknowns)
7. W_t, wage (T unknowns)

The total number of unknowns thus is $7 \times T$.

2.3 Demographics

The age-specific number of household is N_a, which we normalize to 1 at age $a = 0$. The mortality rate at period a is ζ_a, the potential life-span is # and retirement age is $\#_{work} < \#$.

The demographic structure and population is then given by

$$N_a = \begin{cases} 1 & \text{if } a = 0 \\ (1 - \zeta_{a-1})N_{a-1} & \text{if } a > 0 \end{cases} \tag{5}$$

$$\zeta_a = \begin{cases} 0 & \text{if } a < \#_{\text{work}} \\ \left(\frac{a+1-\#_{\text{work}}}{\#-\#_{\text{work}}}\right)^{\varsigma} & \text{if } a < \# - 1 \\ 1 & \text{if } a = \# - 1 \end{cases} \tag{6}$$

$$N = \sum_{a=0}^{\#-1} N_a \tag{7}$$

$$N_{\text{work}} = \sum_{a=0}^{\#_{\text{work}}-1} N_a. \tag{8}$$

2.4 Block I. Households - search behavior and matching

Households search for a job and supply labor exogenously. The age-specific job-separation probability is $\delta_a^L \in (0,1)$. All unemployed search for a job. As an initial condition, we have $L_{-1,t-1} = 0$.

The quantity of searchers is

$$S_{a,t} = \begin{cases} 1 & \text{if } a = 0 \\ (1-\zeta_{a-1})\left[(N_{a-1} - L_{a-1,t-1}) + \delta_a^L L_{a-1,t-1}\right] & \text{if } a < \#_{\text{work}} \\ 0 & \text{if } a \geq \#_{\text{work}} \end{cases}$$
$$S_t = \sum_a S_{a,t}.$$

The quantity of households with a job *before matching* is

$$\underline{L}_{a,t} = \begin{cases} 0 & \text{if } a = 0 \\ (1-\zeta_{a-1})(1-\delta_a^L)L_{a-1,t-1} & \text{if } a < \#_{\text{work}} \\ 0 & \text{if } a \geq \#_{\text{work}} \end{cases}$$
$$\underline{L}_t = \sum_a \underline{L}_{a,t}.$$

The aggregate job-separation rate incl. mortality is

$$\delta_t^L = \frac{L_{t-1} - \underline{L}_t}{L_{t-1}}.$$

The quantity of vacancies is v_t and the number of matches, $\mathcal{M}_t$, is given by the *matching function*

$$\mathcal{M}_t = \frac{S_t v_t}{\left(S_t^{\frac{1}{\sigma^m}} + (\nu v_t)^{\frac{1}{\sigma^m}}\right)^{\sigma^m}},$$

where ν is the efficiency of vacancies.

The job-filling rate, m_t^v, and the job-finding rate, m_t^s, are thus

$$m_t^v = \frac{\mathcal{M}_t}{v_t}$$
$$m_t^s = \frac{\mathcal{M}_t}{S_t}.$$

The number of employed therefore is

$$L_{a,t} = \underline{L}_{a,t} + m_t^s S_{a,t}.$$

This implies that the number of unemployed is

$$U_{a,t} = \begin{cases} N_a - L_{a,t} & \text{if } a < \#_{\text{work}} \\ 0 & \text{if } a \geq \#_{\text{work}} \end{cases}.$$

In equilibrium, the number of matches must equal the number of new hires, i.e.

$$\mathcal{M}_t = L_t - \underline{L}_t.$$

We write the **block** in terms of inputs, and outputs as:

- **Inputs:** $\{L_t\}$

- **Outputs:** $\{S_{a,t}\}$, $\{S_t\}$, $\left\{\delta_t^L\right\}$, $\{\mathcal{M}_t\}$,$\{v_t\}$, $\{m_t^v\}$, $\{m_t^s\}$, $\{L_{a,t}\}$, $\{U_{a,t}\}$, $\{U_t\}$

$$S_{a,t} = \begin{cases} 1 & \textbf{if } a = 0 \\ (1-\zeta_{a-1})\left[(N_{a-1} - L_{a-1,t-1}) + \delta_a^L L_{a-1,t-1}\right] & \textbf{if } a < \#_{\textbf{work}} \\ 0 & \textbf{if } a \geq \#_{\textbf{work}} \end{cases} \tag{9}$$

$$\underline{L}_{a,t} = \begin{cases} 0 & \textbf{if } a = 0 \\ (1-\zeta_{a-1})(1-\delta_a^L)L_{a-1,t-1} & \textbf{if } a < \#_{\textbf{work}} \\ 0 & \textbf{if } a \geq \#_{\textbf{work}} \end{cases} \tag{10}$$

$$S_t = \sum_{a=0}^{\#-1} S_{a,t} \tag{11}$$

$$\underline{L}_t = \sum_{a=0}^{\#-1} \underline{L}_{a,t} \tag{12}$$

$$\delta_t^L = \frac{L_{t-1} - \underline{L}_t}{L_{t-1}} \tag{13}$$

$$\mathcal{M}_t = L_t - \underline{L}_t \tag{14}$$

$$m_t^s = \frac{\mathcal{M}_t}{S_t}$$

$$v_t = \frac{1}{\nu}\left(\frac{(m_t^s)^{\frac{1}{\sigma^m}}\, S_t^{\frac{1}{\sigma^m}}}{1-(m_t^s)^{\frac{1}{\sigma^m}}}\right)^{\sigma^m} \tag{15}$$

$$m_t^v = \frac{\mathcal{M}_t}{v_t} \tag{16}$$

$$L_{a,t} = \underline{L}_{a,t} + m_t^s S_{a,t} \tag{17}$$

$$U_{a,t} = \begin{cases} N_a - L_{a,t} & \textbf{if } a < \#_{\textbf{work}} \\ 0 & \textbf{if } a \geq \#_{\textbf{work}} \end{cases} \tag{18}$$

$$U_t = \sum_{a=0}^{\#-1} U_{a,t}. \tag{19}$$

For $t = 0$, the variable $L_{a-1,t-1}$ is pre-determined.

2.5 Block II. Labor agency

The labor agency firms post vacancies, v_t, to hire labor L_t. The cost of each vacancy is κ^L in units of labor. The firms can therefore rent out $\ell_t = L_t - \kappa^L v_t$ units of labor to the production firms at the rental price r_t^ℓ.

Labor follows the law-of-motion $L_t = \left(1 - \delta_t^L\right) L_{t-1} + m_t^v v_t$, and the wage, W_t, is determined by bargaining. Matching occurs according to the matching function, and the firms take the separation rate, δ_t^L, and the vacancy filling rate, m_t^v as given. Since lagged employment, L_{t-1}, is pre-determined, we consider L_t to be the choice value and derive the required number of vacancies, v_t, and the implied labor for rent, ℓ_t.

The labor agency problem then is:

$$V_0^{\text{labor}}(L_{t-1}) = \max_{\{L_t\}} \sum_{t=0}^{\infty} \left(1 + r^{\text{firm}}\right)^{-t} \left[r_t^\ell \ell_t - W_t L_t\right]$$

$$\text{s.t.}$$

$$v_t = \frac{L_t - \left(1 - \delta_t^L\right) L_{t-1}}{m_t^v}$$

$$\ell_t = L_t - \kappa^L v_t.$$

Using the FOC to L_t from the labor agency problem, we write the **block** as:

- **Inputs:** $\{W_t\}$, $\{m_t^v\}$, $\{\delta_t^L\}$, $\{L_t\}$, $\{v_t\}$
- **Outputs:** $\{r_t^\ell\}$, $\{\ell_t\}$

$$r_t^\ell = \frac{1}{1 - \frac{\kappa^L}{m_t^v}} \left[W_t - r_{t+1}^\ell \frac{1 - \delta_{t+1}^L}{1 + r^{\text{firm}}} \frac{\kappa^L}{m_{t+1}^v}\right] \tag{20}$$

$$\ell_t = L_t - \kappa^L v_t. \tag{21}$$

The variable L_{-1} is pre-determined.

2.6 Block III. Production firms

The production firms use capital, K_{t-1}, and labor, ℓ_t, to produce output, Y_t, with a CES technology. The rental price of capital is r_t^K and the rental price of labor is r_t^ℓ. The firms are price takers in all markets and free entry implies zero profits. The output price is denoted $P_t^{Y,0}$.

Using the results with CES technology derived in sub-section 1.2, we write the **block** as:

- **Inputs:** $\{K_t\}, \{\ell_t\}, \{r_t^K\}, \{r_t^\ell\}$
- **Outputs:** $\{Y_t\}, \left\{P_t^{Y,0}\right\}$

$$Y_t = \Gamma\left(\left(\mu^K\right)^{\frac{1}{\sigma^Y}} K_{t-1}^{\frac{\sigma^Y-1}{\sigma^Y}} + \left(1-\mu^K\right)^{\frac{1}{\sigma^Y}} \ell_t^{\frac{\sigma^Y-1}{\sigma^Y}}\right)^{\frac{\sigma^Y}{\sigma^Y-1}} \tag{22}$$

$$P_t^{Y,0} = \frac{1}{\Gamma}\left(\mu^K\left(r_t^K\right)^{1-\sigma^Y} + \left(1-\mu^K\right)\left(r_t^\ell\right)^{1-\sigma^Y}\right)^{\frac{1}{1-\sigma^Y}}. \tag{23}$$

Targets:

$$\frac{K_{t-1}}{\ell_t} = \frac{\mu_K}{1-\mu_K}\left(\frac{r_t^\ell}{r_t^K}\right)^{\sigma^Y}. \tag{24}$$

The variable K_{-1} is pre-determined.

2.7 Block IV. Phillips Curve

The final output price, P_t^Y, is a mark-up over the marginal cost price, $P_t^{Y,0}$, to capture monopolistic behavior by the firms. The demand for each good is y_t. The final prices are sticky, and the firms pay a quadratic adjustment cost, g_t, to change them.

$$\begin{aligned}
V_t^{\text{intermediary}} &= \max_{\{p_t^Y\}} \left(p_t^Y - P_t^{Y,0}\right) y_t - g_t + \frac{1}{1+r^{\text{firm}}} V_{t+1}^{\text{intermediary}} \\
&\text{s.t.} \\
g_t &= \frac{\gamma}{2}\left[\frac{p_t^Y/p_{t-1}^Y}{p_{t-1}^Y/p_{t-2}^Y} - 1\right]^2 P_t^Y Y_t \\
y_t &= \left(\frac{p_t^Y}{P_t^Y}\right)^{-\sigma_D} Y_t
\end{aligned}$$

Using the FOC and symmetry across firms, we get

- **Inputs:** $\{Y_t\}, \left\{P_t^{Y,0}\right\}$

- **Outputs:** $\left\{P_t^Y\right\}$

$$
\begin{aligned}
P_t^Y =& (1+\theta)P_t^{Y,0} - \eta\left(\frac{P_t^Y/P_{t-1}^Y}{P_{t-1}^Y/P_{t-2}^Y}-1\right)\frac{P_t^Y/P_{t-1}^Y}{P_{t-1}^Y/P_{t-2}^Y}P_t^Y \\
&+ \frac{2\eta}{1+r^{\textbf{firm}}}\frac{Y_{t+1}}{Y_t}\left(\frac{P_{t+1}^Y/P_t^Y}{P_t^Y/P_{t-1}^Y}-1\right)\frac{P_{t+1}^Y/P_t^Y}{P_t^Y/P_{t-1}^Y}P_{t+1}^Y
\end{aligned}
$$

$$
\begin{aligned}
\theta &\equiv \frac{1}{\sigma_D - 1} \\
\eta &\equiv \theta\gamma.
\end{aligned}
$$

Targets:

$$
\begin{aligned}
P_t^Y =& (1+\theta)P_t^{Y,0} - \eta\left(\frac{P_t^Y/P_{t-1}^Y}{P_{t-1}^Y/P_{t-2}^Y}-1\right)\frac{P_t^Y/P_{t-1}^Y}{P_{t-1}^Y/P_{t-2}^Y}P_t^Y \\
&+ \frac{2\eta}{1+r^{\text{firm}}}\frac{Y_{t+1}}{Y_t}\left(\frac{P_{t+1}^Y/P_t^Y}{P_t^Y/P_{t-1}^Y}-1\right)\frac{P_{t+1}^Y/P_t^Y}{P_t^Y/P_{t-1}^Y}P_{t+1}^Y.
\end{aligned} \tag{25}
$$

2.8 Block V. Bargaining

The target wage, W_t^*, in the bargaining is specified ad hoc as a weighted average of the profit the firms get from a marginal unit of labor, $\overline{W}_t$, and the worker's outside option,$\underline{W}_t$. The bargained wage, W_t, is again a weighted average of the past wage and the current target wage to create a real rigidity.

- **Inputs:** $\{W_t\}$, $\{Y_t\}$, $\{\ell_t\}$

- **Outputs:** $\{\overline{W}_t, \underline{W}_t, W_t^*\}$

$$
\overline{W}_t = P_t^Y\left((1-\mu_K)\,\Gamma^{\sigma^Y-1}\frac{Y_t}{\ell_t}\right)^{\frac{1}{\sigma^Y}} \tag{26}
$$

$$
\underline{W}_t = W^U \tag{27}
$$

$$
W_t^* = \psi\overline{W}_t + (1-\psi)\underline{W}_t. \tag{28}
$$

- **Targets:**

$$W_t = \gamma^w W_{t-1} + (1-\gamma^w) W_t^*. \tag{29}$$

2.9 Block VI. Repacking firms - prices

The output good, Y_t, can be used for either private consumption, C_t, public consumption, G_t, investment, I_t, or exports, X_t. For each use the output good must be repacked with imported goods. This is done by repacking firms with a CES production technology.

Using the results with CES technology derived in sub-section 1.2 with $\Gamma = 1$, we write the **block** for the pricing part of this as:

1. **Inputs:** $\{P_t^Y\}$, $\left\{P_t^{M,\bullet}\right\}$ for $\bullet \in \{C, G, I, X\}$
2. **Output:** $\{P_t^\bullet\}$ for $\bullet \in \{C, G, I, X\}$

$$P_t^\bullet = \left(\mu^{M,\bullet}\left(P_t^{M,\bullet}\right)^{1-\sigma^\bullet} + \left(1-\mu^{M,\bullet}\right)\left(P_t^Y\right)^{1-\sigma^\bullet}\right)^{\frac{1}{1-\sigma^\bullet}}. \tag{30}$$

2.10 Block VII. Foreign economy

The foreign economy has so-called Armington demand of the domestic export good. We write the **block** as:

1. **Inputs:** $\{P_t^F\}$,$\{\chi_t\}$, $\{P_t^X\}$
2. **Outputs:** $\{X_t\}$

$$X_t = \gamma^X X_{t-1} + (1-\gamma^X)\chi_t\left(\frac{P_t^X}{P_t^F}\right)^{-\sigma^F}. \tag{31}$$

2.11 Block VIII. Capital agency

The capital agency firm buys investment goods, I_t, at price, P_t^I, to accumulate capital, K_t, which it rents out to production at the rental rate r_{t+1}^K in the following period. The investment decision is subject to convex adjustment costs in terms of wasted investment goods, such that effective investment is ι_t. Future profits are discounted with r^{firm}. The capital agency takes prices as given, and its problem thus is:

$$
\begin{aligned}
V_0^{\text{capital}}\left(K_{t-1}\right) = \max_{\{K_t\}} \sum_{t=0}^{\infty} \left(1 + r^{\text{firm}}\right)^{-t} & \left[r_t^K K_{t-1} - P_t^I \left(\iota_t + \Psi(\iota_t, K_{t-1})\right)\right] \\
\text{s.t.} & \\
I_t &= \iota_t + \Psi(\iota_t, K_{t-1}) \\
K_t &= (1 - \delta^K) K_{t-1} + \iota_t.
\end{aligned}
$$

We choose the functional form

$$
\Psi(\iota_t, K_{t-1}) = \frac{\Psi_0}{2} \left(\frac{\iota_t}{K_{t-1}} - \delta^K\right)^2 K_{t-1},
$$

implying

$$
\begin{aligned}
\Psi_\iota(\iota_t, K_{t-1}) &= \Psi_0 \left(\frac{\iota_t}{K_{t-1}} - \delta^K\right) \\
\Psi_K(\iota_t, K_{t-1}) &= \frac{\Psi_0}{2} \left(\frac{\iota_t}{K_{t-1}} - \delta^K\right)^2 - \Psi_0 \left(\frac{\iota_t}{K_{t-1}} - \delta^K\right) \frac{\iota_t}{K_{t-1}}.
\end{aligned}
$$

We write the **block** as

- **Inputs:** $\left\{r_t^K\right\}$, $\left\{P_t^I\right\}$, $\{K_t\}$
- **Outputs:** $\{\iota_t\}, \{I_t\}$

$$
\begin{aligned}
\iota_t &= K_t - \left(1 - \delta^K\right) K_{t-1} && (32) \\
I_t &= \iota_t + \Psi(\iota_t, K_{t-1}). && (33)
\end{aligned}
$$

- **Targets**:

$$
\begin{aligned}
0 = & - P_t^I \left(1 + \Psi_\iota\left(\iota_t, K_{t-1}\right)\right) && (34) \\
& + \left(1 + r^{\text{firm}}\right)^{-1} \left[r_{t+1}^k + P_{t+1}^I (1 - \delta^K)(1 + \Psi_\iota(\iota_{t+1}, K_t)) - P_{t+1}^I \Psi_K\left(\iota_{t+1}, K_t\right)\right].
\end{aligned}
$$

The variable K_{-1} is pre-determined.

2.12 Block IX. Government

The government budget is given by

$$B_t = (1 + r^B)B_{t-1} + P_t^G G_t + (1 - \tau_t)W_U W_{ss} U_t + (1 - \tau_t)W_R W_{ss}(N_t - N_t^{work}) - \tau_t W_t L_t \tag{35}$$

where r^B is the interest rate on government debt determined in the foreign economy. We assume the government has the exogenous tax rate τ_{ss} for t_B years and then begins to adjust taxes to get back to steady state debt. For Δ_B years this is done gradually, and thereafter it is done fully.

- **Inputs:** $\{P_t^G\}$, $\{G_t\}$, $\{W_t\}$, $\{L_t\}$, $\{N_t\}$, $\{N_t^{work}\}$
- **Outputs:** $\{\tau_t\}$, $\{\tilde{\tau}_t\}$, $\{B_t\}$,$\{\tilde{B}_t\}$

$$\begin{aligned}
\tilde{B}_t &= (1 + r^B)B_{t-1} + P_t^G G_t + (1 - \tau_{ss})W_U W_{ss} U_t + (1 - \tau_{ss})W_R W_{ss}(N_t - N_{\text{work}}) - \tau_{ss} W_t L_t \\
\tilde{\tau}_t &= \tau_{ss} + \varepsilon_B \frac{\tilde{B}_t - B_{ss}}{W_t L_t + W_U W_{ss} U_t + W_R W_{ss}(N_t - N_{\text{work}})} \\
\tau_t &= \begin{cases} \tau_{ss} & \text{if } t < t_B \\ (1 - \omega_t)\tau_{ss} + \omega_t \tilde{\tau}_t & \text{if } t \in [t_{B,} t_B + \Delta_B] \\ \tilde{\tau}_t & \text{if } t > t_B + \Delta_B \end{cases} \qquad (36) \\
\omega_t &= 3\left(\frac{t - t_B}{\Delta_B}\right)^2 - 2\left(\frac{t - t_B}{\Delta_B}\right)^3 \in (0, 1) \\
B_t &= (1 + r^B)B_{t-1} + P_t^G G_t + (1 - \tau_t)W_U W_{ss} U_t + (1 - \tau_t)W_R W_{ss}(N_t - N_{\text{work}}) - \tau_t W_t L_t.
\end{aligned} \tag{37}$$

The variable B_{-1} is pre-determined.

2.13 Block X. Households - consumption -saving

The model has two types of households. A share of λ of households is hands-to-mouth and a share of $1 - \lambda$ households is unconstrained (Ricardian). All households have four sources of income:

1. Post-tax labor income, $(1 - \tau_t) W_t \frac{L_{a,t}}{N_a}$

2. Post-tax unemployment benefits, $(1-\tau_t)\, W^U W_{ss} \frac{U_{a,t}}{N_a}$

3. Post-tax retirement benefits, $(1-\tau_t)\, W^R W_{ss} \frac{N_a-(L_{a,t}+U_{a,t})}{N_a}$

4. Equally divided inheritance, $\frac{A_t^q}{N}$

The age specific income is

$$\text{inc}_{a,t} = (1-\tau_t)\, W_t \frac{L_{a,t}}{N_a} + (1-\tau_t)\, W^U W_{ss} \frac{U_{a,t}}{N_a} + (1-\tau_t)\, W^R W_{ss} 1_{\{a \geq \#_{\text{work}}\}} + \frac{A_t^q}{N}.$$

The price of consumption goods is P_t^C.
Consumption is $C_{a,t}^{\bullet}$ and end-of-period nominal savings is $A_{a,t}^{\bullet}$, where $\bullet \in \{\text{HtM}, R\}$.
The behavior of *surviving* hands-to-mouth households are

$$\begin{aligned} C_{a,t}^{\text{HtM}} &= \frac{\text{inc}_{a,t}}{P_t^C} \\ A_{a,t}^{\text{HtM}} & 0. \end{aligned}$$

The Ricardian households making their first decision in period t_0 solve the problem

$$\begin{aligned} V_{t_0} &= \max_{\left\{C_{a,t}^R, A_{a,t}^R\right\}_{a=0}^{\#-1}} \sum_{a=0}^{\#-1} \left(\Pi_{j=0}^{a-1} \beta(1-\zeta_j)\right) \left[\frac{\left(C_{a,t}^R\right)^{1-\sigma}}{1-\sigma} + \zeta_a \mu^{A^q} \frac{\left(\frac{A_{a,t}^R}{P_t^C}\right)^{1-\sigma}}{1-\sigma} \right] \\ & \text{s.t.} \\ t &= t_0 + a \\ A_{-1,t}^R &= 0 \\ A_{a,t}^R &= (1+r_t^{hh}) A_{a-1,t-1}^R + \text{inc}_{a,t} - P_t^C C_{a,t}^R. \end{aligned}$$

Aggregation implies

$$\begin{aligned} C_{a,t} &= \lambda C_{a,t}^{\text{HtM}} + (1-\lambda) C_{a,t}^R \\ A_{a,t} &= \lambda A_{a,t}^{\text{HtM}} + (1-\lambda) A_{a,t}^R, \end{aligned}$$

and

$$
\begin{aligned}
C_t &= \sum_{a=0}^{\#-1} N_a C_{a,t} \\
A_t &= \sum_{a=0}^{\#-1} N_a C_{a,t}.
\end{aligned}
$$

Bequests are

$$A_t^q = \left(1 + r^{hh}\right) \sum_{a=0}^{\#-1} \zeta_a N_a A_{a,t-1}.$$

Using the FOC for we can write the **block** as:

1. **Inputs:** $\{L_{a,t}\}$, $\{U_{a,t}\}$, $\{P_t^C\}$, $\{W_t\}$, $\{\tau_t\}$, $\{A_t^q\}$, $\left\{A_{\#-1,t}^R\right\}$,$\{r_t^{hh}\}$

2. **Outputs:** $\{A_{a,t}^{\text{HtM}}\}$, $\{A_{a,t}^R\}$,$\{A_{a,t}\}$, $\{A_t\}$, $\{C_{a,t}^{\text{HtM}}\}$,$\{C_{a,t}^R\}$, $\{C_{a,t}\}$, $\{C_t\}$, $\{\text{inc}_{a,t}\}$, $\{\text{inc}_t\}$, $\{\pi_t^{hh}\}$

   Calculate

$$\pi_t^{hh} = \frac{P_t^C}{P_{t-1}^C} - 1 \tag{38}$$

$$\text{inc}_{a,t} = (1-\tau_t)\, W_t \frac{L_{a,t}}{N_a} + (1-\tau_t)\, W^U W_{ss} \frac{U_{a,t}}{N_a} + (1-\tau_t)\, W^R W_{ss} 1_{\{a \geq \#_{\text{work}}\}} + \frac{A_t^q}{N} \tag{39}$$

$$C_{a,t}^{\text{HtM}} = \frac{\text{inc}_{a,t}}{P_t^C} \tag{40}$$

$$A_{a,t}^{\text{HtM}} = 0. \tag{41}$$

   For each birth cohort $t_0 \in \{-\# + 1, -\# + 2, \ldots, T - 1\}$ iterate backwards from

$a = \# - 1$ with $t = t_0 + a$, but skipping steps where $t < 0$ or $t > T - 1$:

$$C^R_{a,t} = \begin{cases} \left(\zeta_a \mu^{A^q} \left(\frac{A^R_{\#-1,t}}{P^C_t}\right)^{-\sigma}\right)^{-\frac{1}{\sigma}} & \text{if } a = \# - 1 \\ \left(\beta(1-\zeta_a)\frac{1+r^{hh}_{ss}}{1+\pi^{hh}_{ss}}\left(C^{R,}_{a+1,ss}\right)^{-\sigma} + \zeta_a \mu^{A^q}\left(\frac{A^R_{a,ss}}{P^C_t}\right)^{-\sigma}\right)^{-\frac{1}{\sigma}} & \text{elif } t = T - 1 \\ \left(\beta(1-\zeta_a)\frac{1+r^{hh}_{t+1}}{1+\pi^{hh}_{t+1}}\left(C^{R,}_{a+1,t+1}\right)^{-\sigma} + \zeta_a \mu^{A^q}\left(\frac{A^R_{a,t}}{P^C_t}\right)^{-\sigma}\right)^{-\frac{1}{\sigma}} & \text{else} \end{cases}$$

$$A^R_{a-1,t-1} = \frac{A^R_{a,t} - \mathrm{inc}_{a,t} + P^C_t C^R_{a,t}}{1 + r^{hh}}.$$

Aggregates

$$C_{a,t} = \lambda C^{\mathrm{HtM}}_{a,t} + (1-\lambda) C^R_{a,t} \tag{42}$$

$$A_{a,t} = \lambda A^{\mathrm{HtM}}_{a,t} + (1-\lambda) A^R_{a,t} \tag{43}$$

$$\mathrm{inc}_t = \sum_{a=0}^{\#-1} N_a \mathrm{inc}_{a,t} \tag{44}$$

$$C_t = \sum_{a=0}^{\#-1} N_a C_{a,t} \tag{45}$$

$$A_t = \sum_{a=0}^{\#-1} N_a C_{a,t}. \tag{46}$$

Targets:

$$0 = A^q_t - \left(1 + r^{hh}_t\right) \sum_{a=0}^{\#-1} \zeta_a N_a A_{a,t-1} \tag{47}$$

$$0 = \sum_{t_0=-\#+1}^{-1} \left(A^R_{-t_0-1,-1} - A_{-t_0-1,ss}\right) + \sum_{t_0=0}^{T-1-\#+1} \left(A^R_{-1,t_0} - 0.0\right). \tag{48}$$

For $t = 0$, we have that the variable $A^R_{a,t-1}$ is pre-determined.

2.14 Block XI. Repacking firms - components

The repacking firms were described in sub-section 2.9. Using additional results from sub-section 1.2 on CES technology with $\Gamma = 1$, we write the **block** as:

1. **Inputs:** $\{P_t^Y\}$, $\left\{P_t^{M,\bullet}\right\}$, $\{P_t^\bullet\}$, $\{\bullet_t\}$ for $\bullet \in \{C, G, I, X\}$
2. **Output:** $\{\bullet_t^M\}$, $\{\bullet_t^Y\}$ for $\bullet \in \{C, G, I, X\}$

$$\bullet_t^M = \mu^{M,\bullet} \left(\frac{P_t^\bullet}{P_t^{M,\bullet}} \right)^{\sigma^\bullet} \bullet_t \tag{49}$$

$$\bullet_t^Y = \left(1 - \mu^{M,\bullet}\right) \left(\frac{P_t^\bullet}{P_t^Y} \right)^{\sigma^\bullet} \bullet_t \, . \tag{50}$$

2.15 Block XII. Goods market clearing

The production of the domestic output good must match the above output goods used by the repacking firms. Imports, M_t, are the sum of the imports used by the repacking firms.

We write the **block** as:

- **Inputs:** $\{\bullet_t^Y\}$, $\{\bullet_t^M\}$ for $\bullet \in \{C, G, I, X\}$
- **Outputs:** $\{M_t\}$

$$M_t = \sum_{\bullet \in \{C,G,I,X\}} \bullet_t^M. \tag{51}$$

- **Targets:**

$$Y_t = \sum_{\bullet \in \{C,G,I,X\}} \bullet_t^Y. \tag{52}$$

2.16 Total number of targets

We have $7 \times T$ targets in Equation (24), (25), (29), (34), (47), (48), and (52), which is equal to number of unknowns.

3 Steady state

We fix a number of variables:

1. The nominal wage, $W_{ss} = 1$

2. The inflation, $\pi_{ss}^{hh} = 0$

3. The job-finding rate, $m_{ss}^{s} = 0.75$

4. The job-filling rate, $m_{ss}^{v} = 0.75$

5. The government debt, $B_{ss} = 0$

6. The foreign interest rate, $r_{ss}^{hh} = 0.04$

We allow for the adjustment of the exogenous variables and other parameters to fit with this. We can then find the steady state as follows:

1. Price normalization:

$$P_{ss}^{Y} = P_{ss}^{F} = P_{ss}^{M,\bullet} = 1, \bullet \in \{C, G, I, X\}.$$

2. The pricing behavior of repacking firms then implies

$$P_{ss}^{\bullet} = 1, \bullet \in \{C, G, I, X\}.$$

3. The exogenous labor supply and search-and-matching imply

$$
\begin{aligned}
S_{a,ss} &= \begin{cases} 1 & \text{if } a = 0 \\ (1-\zeta_{a-1})(N_{a-1} - L_{a-1,ss}) + \delta^L_a L_{a-1,ss} & \text{if } a < \#_{\text{work}} \\ 0 & \text{else} \end{cases} \\
\underline{L}_{a,ss} &= \begin{cases} 0 & \text{if } a = 0 \\ (1-\zeta_{a-1})(1-\delta^L_a) L_{a-1,ss} & \text{if } a < \#_{\text{work}} \\ 0 & \text{else} \end{cases} \\
L_{a,ss} &= \underline{L}_{a,ss} + m^s_{ss} S_{a,ss} \\
U_{a,ss} &= \begin{cases} N_a - L_{a,ss} & \text{if } a < \#_{\text{work}} \\ 0 & \text{else} \end{cases} \\
L_{ss} &= \sum_a L_{a,ss} \\
S_{ss} &= \sum_a S_{a,ss} \\
U_{ss} &= \sum_a U_{a,ss} \\
\delta^L_{ss} &= \frac{L_{ss} - \underline{L}_{ss}}{L_{ss}} \\
\mathcal{M}_{ss} &= \delta^L_{ss} L_{ss} \\
v_{ss} &= \frac{\mathcal{M}_{ss}}{m^v_{ss}} \\
\nu &= \frac{1}{v_{ss}} \left(\frac{(m^s_{ss})^{\frac{1}{\sigma^m}} S_{ss}^{\frac{1}{\sigma^m}}}{1 - (m^s_{ss})^{\frac{1}{\sigma^m}}} \right)^{\sigma^m} .
\end{aligned}
$$

4. Capital agency behavior implies

$$r^K_{ss} = r^{\text{firm}} + \delta^K.$$

5. Labor agency behavior implies

$$
\begin{aligned}
r^{\ell}_{ss} &= \frac{W_{ss}}{1-\left(1+\frac{1-\delta^{L}_{ss}}{1+r^{firm}}\right)\frac{\kappa^{L}}{m^{v}_{ss}}} \\
\ell_{ss} &= L_{ss}-\kappa^{L}v_{ss}.
\end{aligned}
$$

6. Set the prices using the Phillipps-curve

$$
\begin{aligned}
P^{Y,0}_{ss} &= \frac{P^{Y}_{ss}}{1+\theta} \\
\Gamma_{ss} &= \frac{1}{P^{Y,0}_{ss}}\left(\mu^{K}\left(r^{K}_{ss}\right)^{1-\sigma^{Y}}+\left(1-\mu^{K}\right)\left(r^{\ell}_{ss}\right)^{1-\sigma^{Y}}\right).
\end{aligned}
$$

7. Determine the capital and output from the production firm

$$
\begin{aligned}
K_{ss} &= \frac{\mu_{K}}{1-\mu_{K}}\left(\frac{r^{\ell}_{ss}}{r^{K}_{ss}}\right)^{\sigma^{Y}}\ell_{ss} \\
Y_{ss} &= \Gamma_{ss}\left(\left(\mu^{K}\right)^{\frac{1}{\sigma^{Y}}}K_{ss}^{\frac{\sigma^{Y}-1}{\sigma^{Y}}}+\left(1-\mu^{K}\right)^{\frac{1}{\sigma^{Y}}}\ell_{ss}^{\frac{\sigma^{Y}-1}{\sigma^{Y}}}\right)^{\frac{\sigma^{Y}}{\sigma^{Y}-1}},
\end{aligned}
$$

   idiosyncratic.

8. From capital accumulation equations

$$
\iota_{ss} = I_{ss} = \delta^{K}K_{ss}.
$$

9. Exogenously, set the government spending as a share of the output and find the tax rate

$$
\begin{aligned}
G_{ss} &= G^{share}Y_{ss} \\
\tau_{ss} &= \frac{r_{B}B_{ss}+P^{G}_{ss}G_{ss}+W_{U}U_{ss}+W_{R}(N_{ss}-N^{work}_{ss})}{W_{ss}L_{ss}+W_{U}U_{ss}+W_{R}(N_{ss}-N^{work}_{ss})}.
\end{aligned}
$$

10. Find the age-specific income for the households and use this to define the consumption for the hands-to-mouth households.
    Guess on A^{q}_{ss} and check $\left(1+r^{hh}_{ss}\right)\sum_{a=0}^{\#-1}\zeta_{a}N_{a}A_{a,ss} = A^{q}_{ss}$. Find the consumption

of Ricardian households and their assets backwards.

$$
\begin{aligned}
\text{inc}_{a,ss} &= (1-\tau_{ss})\, W_{ss}\frac{L_{a,ss}}{N_a} + (1-\tau_{ss})\, W_U W_{ss}\frac{U_{a,ss}}{N_a} + (1-\tau_{ss})\, W_R W_{ss} 1_{\{a\geq \#_{\text{work}}\}} + \frac{A^q_{ss}}{N} \\
C^{\text{HtM}}_{a,ss} &= \frac{\text{inc}_{a,ss}}{P^C_{ss}} \\
A^{\text{HtM}}_{a,ss} &= 0 \\
C^R_{a,ss} &= \begin{cases} \left(\zeta_a \mu^{A^q}\left(\frac{A^R_{\#-1,ss}}{P^C_{ss}}\right)^{-\sigma}\right)^{-\frac{1}{\sigma}} & \text{if } a = \#-1 \\ \left(\beta(1-\zeta_a)\frac{1+r^{hh}_{ss}}{1+\pi^{hh}_{ss}}\left(C^{R,}_{a+1,ss}\right)^{-\sigma} + \zeta_a\mu^{A^q}\left(\frac{A^R_{a,ss}}{P^C_t}\right)^{-\sigma}\right)^{-\frac{1}{\sigma}} & \text{else} \end{cases} \\
A^R_{a-1,ss} &= \frac{A^R_{a,ss} - \text{inc}_{a,ss} + P^C_{ss}C^R_{a,ss}}{1+r^{hh}_{ss}} \\
C_{a,ss} &= \lambda C^{\text{HtM}}_{a,ss} + (1-\lambda)C^R_{a,ss} \\
A_{a,ss} &= \lambda A^{\text{HtM}}_{a,ss} + (1-\lambda)A^R_{a,ss} \\
A^q_{ss} &= \left(1+r^{hh}_{ss}\right)\sum_{a=0}^{\#-1}\zeta_a N_a A_{a,ss}.
\end{aligned}
$$

11. Determine package components for consumption and investment

$$
\begin{aligned}
\bullet^M_{ss} &= \mu^{M,\bullet}\left(\frac{P^\bullet_{ss}}{P^{M,\bullet}_{ss}}\right)^{\sigma^\bullet}\bullet_{ss}, \text{ for } \bullet \in \{C,G,I\} \\
\bullet^Y_{ss} &= (1-\mu^{M,\bullet})\left(\frac{P^\bullet_{ss}}{P^{M,\bullet}_{ss}}\right)^{\sigma^\bullet}\bullet_{ss}, \text{ for } \bullet \in \{C,G,I\}.
\end{aligned}
$$

12. Determine χ_{ss} to get market clearing

$$
\begin{aligned}
X^Y_{ss} &= Y_{ss} - \left(C^Y_{ss} + G^Y_{ss} + I^Y_{ss}\right) \\
\chi_{ss} = X_{ss} &= \frac{X^Y_{ss}}{(1-\mu^{M,X})\Gamma^{\sigma^X-1}} \\
X^M_{ss} &= \mu^{M,X}\Gamma^{\sigma^X-1}\left(\frac{P^X_{ss}}{P^{M,X}_{ss}}\right)^{\sigma^X} X_{ss} \\
M_{ss} &= C^M_{ss} + G^M_{ss} + I^M_{ss} + X^M_{ss}.
\end{aligned}
$$

13. Let φ adjust to make bargaining fit

$$
\begin{aligned}
W^*_{ss} &= W_{ss} \\
\overline{W}_{ss} &= P^Y_{ss}\left((1-\mu_K)\,\Gamma_{ss}^{1-\sigma^Y}\frac{Y_{ss}}{\ell_{ss}}\right)^{\frac{1}{\sigma^Y}} \\
\underline{W}_{ss} &= W^U W^*_{ss} \\
\psi &= \frac{W^*_{ss}-\underline{W}_{ss}}{\overline{W}_{ss}-\underline{W}_{ss}}
\end{aligned}
$$

3.1 Aggregates

See 0 - steady state - aggregates.ipynb.

1. Setup: An instance of the baseline model is created and called model.

2. Find steady steady: The steady state values are found and printed for all blocks.

3. Speed and error tests: The exogenous and unknown steady state variables are specified. The blocks are evaluated using python evaluation, and it is verified that all targets are zero. Next, the blocks are evaluated using numba evaluation. The initial evaluation is slower when using numba, but additional evaluations are much faster than using python evaluation.

3.2 Households

See 0 - steady state - households.ipynb.

1. Setup: An instance of the baseline model is created and called model. The parameters, steady state and solution name-spaces are set.

2. Mortality and population: First, a function for creating an interactive plot is defined, where it is possible to adjust the power of ζ_a. Figures for the mortality rate and the population are plotted. The average life expectancy is increasing in the power of ζ_a, as this implies a lower mortality rate for all $a < \#$. In the extreme case where the power is approaching infinity, everybody dies in the last period, and in the opposite case where the power is 0, everybody dies in the first period after retirement.
   Reset the demographic structure, such that it will not affect the other results.

3. Labor supply: Using the job-finding rate, it is possible to find the household search behavior in steady state. A function for an interactive figure is defined, where it is possible to adjust the separation rate, δ_a^L. Figures for the separation rate, the number of searchers and the employment/unemployment levels are plotted. The separation rate is constant throughout the work life period, and a higher separation rate implies more workers loss their job; thus, they become unemployed and search for a new job. In the extreme case where the separation rate is 1, all workers loss their job at the end; thus, all workers search for a new job. As such, the employment level will be equal to the job-finding rate, m^s. Reset the demographic structure and household search, such that it will not affect the other results.

4. Consumption-saving: Using the required inputs, its is possible to find the household consumption and savings behavior. The age-specific income, consumption and savings are plotted for the Ricardian and HtM households together with the aggregations. The Ricardian households consumption smooth by accumulating savings, while the HtM consume all their income. The aggregated consumption and savings are a weighted average determined by the share of each type, λ.

5. Varying central inputs: A function for comparing to models is defined, which plots the income, consumption and savings for varying inputs:

   - If the households receive less bequest, their income are lower and they will consume and save less.
   - If the households have a lower preference for bequest, they will save less and consume more.
   - If the households receive more wage, they will have more income which implies more consumption and savings.
   - If the households receive more unemployment benefits, they will have more income before retirement; thus, they can consume and save more. This effect is much smaller than the effect from a higher wage, as the number of unemployed is much smaller than the number of employed.
   - If the households receive more in retirement benefits, they will have more income once they retire. This implies that they need less savings for retirement, and they can consume more throughout their life.

- If the tax rate is lower, the households receive more after-tax wage. Therefore, this is similar to increasing the wages.
- If beta decreases, the households prefer to consume more today relative to tomorrow. As such, they save less and they consume more in the first periods.
- If the interest rate decreases, it is relatively more expensive to save compared to consumption. Therefore, the households will save less and consume more initially.
- If the share of HtM households decreases, the aggregation of households will consumption smooth more by saving more.

6. Test household blocks: The solution variables are set to their steady state value, and the model is set to be in steady state. The search and match block and the household consumption behavior are evaluated, and it is checked that all variables remain at their steady state levels.

4 Shocks

In this section, we consider a number of shocks to central exogenous variables and analyze what happens to all other variables and the convergence back to steady state.

4.1 Government spending shock

- **Effect on the government** (see 2.12)

  Initial effect: The government increases real consumption, $G \uparrow$, which implies an increase in the government debt, $B \uparrow$. The tax-rate, τ, is fixed for t_b years, hence it does not change. A higher demand for government consumption is met by a higher price, $P^G \uparrow$, and a higher domestic production, $Y \uparrow$, and higher imports, $M \uparrow$.

  Transition path: Government consumption decreases as per assumption. The debt continue to grow due to interest payments and high real and nominal consumption. This is eventually reversed once the tax begin to

  as a consequences of the higher prices, which makes both consumption and prices converges towards their steady state levels. The budget continues to grow due to interest payment and the high consumption/prices until the new and higher tax-rate is implemented. It can be noted that the effects on the government budget stemming from labor supply and the wage are disregarded for now.

- **Effect on the production firms, the Phillips-curve and investments** (see 2.6,2.7,2.11)

  Initial effect: The firms increases production to meet the higher demand. Since capital is fixed, the production firms can only produce more by hiring more labor, $\ell \uparrow$. This makes the capital more productive, $r^K \uparrow$, and it also implies that the rent on capital must increase more than the rent on labor. The marginal cost price increases, $P^{Y,0} \uparrow$, following the increase in rents. As the output price is a mark-up of the marginal cost price, this increases as well, $P^Y \uparrow$. As the output prices are sticky, the increase should be relatively smaller. The effect on capital is ambiguous. On one hand, the rent on capital has increased, such that the capital agency has an incentive to accumulate more future capital, thus more investment $I \uparrow$ and capital $K \uparrow$. On the other hand, the price on investment has increased $P^I \uparrow$ (see Effect on prices below), hence less $I \downarrow$ and capital $K \downarrow$.

  Transition path: The total demand decreases, as it converges towards it steady state level. As such, the production firms needs less labor, such that capital is less productive. Both the marginal cost and output price decreases.

- **Effect on the Labor agency, Labor search and match and wage** (see 2.4, 2.5, 2.8)

  Initial effect: The increase in demand for ℓ implies that the labor agency hires more labor, $L \uparrow$. More hired labor implies more matches, $\mathcal{M} \uparrow$, which further implies an increase in the job-finding rate, $m^s \uparrow$, and the vacancies, $v \uparrow$. However, the job-filling rate decreases, $m^v \downarrow$.
  The effect on the wage is ambiguous. Wages should increase with higher output and output prices, but decrease as a consequence of more ℓ.
  Transition path: The amount of households with a job at the start of period 1 is high, thus $\underline{L} \uparrow$ and the number of searchers decreases $S \downarrow$. Hereafter, all variables converge towards their steady state level.

- **Effect on Prices** (see 2.9)

  Initial effect: Higher output prices imply higher prices on all goods through the repacking firms, $P_t^{\bullet} \uparrow$ for $\bullet \in \{C, I, X\}$
  Transition path: The total demand decreases as a consequence of higher prices,

which implies lower prices. As such, prices converge towards their steady state level.

- **Effect on Consumption and Exports** (see 2.10, 2.13)

  Initial effect: There is a direct effect on consumption and exports. As prices increases, they all decreases $\bullet \downarrow$ for $\bullet \in \{C, X\}$. It can be noted, that the higher total domestic consumption together with a lower export must be met by a higher import $M \uparrow$. For consumption, there is indirect effects as well.
  The income of the households is also affected through changes in labor supply and wages. The effect on wages is ambiguous, it is not clear whether income increases or decreases. However, it is clear that the Ricardian households decrease their consumption, as they see the higher tax rate in the future $C^R \downarrow$. Lower consumption and the fact the inflation is negative (higher real interest rate) imply more savings $A \uparrow$. There effect on bequest is neglectable as the households have chosen the savings beforehand.

  Transition path: Export converges smoothly towards it steady state level. After the initial drop in consumption, it increases towards its steady state level as prices converge. This is especially true for the hands-to-mouth households. By the 10th year, the new and higher tax rate is gradually implemented which leads to a large drop in consumption. The Ricardian households have foreseen this new tax rate, such that their consumption is more even throughout the period, but still with a decrease in consumption once the higher tax rate is implemented. They achieve the consumption smoothing by increasing their savings under the lower tax rate, and using this additional savings/income once the high tax rate is implemented. The bequests will not reach much before the new tax rate is implemented, because that a high proportion of the population made their savings decision before the shock to government consumption, and they will not/only partly be affected by a higher tax in the future. However, the higher tax rate has a larger effect on the later generation, which implies a lower income and less bequests.

4.2 Productivity shock - with constant real wage

- **Effect on the Production firms, the Phillips-curve and Investments** (see 2.6,2.7,2.11)

  Initial effect: The productivity increases due to the technology shock, $\Gamma \uparrow$, such that the production increases, $Y \uparrow$. and marginal cost prices decrease, $P^{Y,0} \downarrow$.The

output prices are a mark-up over the marginal cost prices and decrease as well, $P^Y \downarrow$. As the output prices are sticky, this decrease will be relatively smaller.
As capital is fixed for the production firms, they increase the labor, $\ell \uparrow$, to produce more. A higher productivity and more labor both implies an unanimous increase the real marginal product of capital $\frac{r^K}{P^Y} \uparrow$, but the nominal marginal product of capital , r^K, is not necessarily increasing as the output prices decrease. The effect on the real marginal product of labor is ambiguous, as it is increasing in the productivity but decreasing in the labor supply. The firms want more capital as the real marginal product has increased, $K \uparrow$, which is achieve by investing more, $I \uparrow$.
Transition path: The productivity decreases as per assumption. Due to the sticky nature of the output prices, these decrease for an additional year. The production increases to satisfy the increasing demand stemming from lower output prices. Again, the capital is fixed, and more labor is hired. A high level of productivity more labor both implies an increase the real marginal product of capital, and a high level of productivity and capital stock implies an increase in the real marginal product of labor. The firms want more capital as the real marginal product has increased, $K \uparrow$, which is achieve by investing more, $I \uparrow$.
After year 1, both the marginal cost prices and the output prices continuously grow towards their steady state level. The declining productivity implies decreasing real returns on both capital and labor. For the first 10 years, where the real return on capital is relatively high, the capital increases. Hereafter, capital continuously decreases. Production and labor continuously decreases, as the technology shock wears out.

- **Effect on the Labor agency, Labor search & match and wage** (see 2.4, 2.5, 2.8)

  Initial effect: The labor agency hires more labor, $L \uparrow$, following the increase in demand for ℓ. More labor implies more matches, $\mathcal{M} \uparrow$, which further implies an increase in the job-finding rate, $m^s \uparrow$, and the vacancies, $v \uparrow$. However, the job-filling rate decreases, $m^v \downarrow$.
  The real wage with respect to the output price is held constants. As such, the nominal wage follows the decrease in the output prices, $W \downarrow$. Here, W_{real} is defined in terms of the consumer prices, P^C. As these decrease less than the output prices, there is a negative effect, $W_{real} \downarrow$.
  Transition path: The amount of labor has a direct link to the demand for ℓ, and the transition paths between the two are similar. More employment means less

searchers, thus the amount of searchers is inverse proportional with the amount of labor. The vacancies, matches, and job-finding rate have a rapid increase in the first years as a consequence of the sudden growth in employment rates. Hereafter, they steadily decline.

- **Effect on Prices** (see 2.9)

  Initial effect: Lower output prices imply lower prices on all goods through the repacking firms, $P_t^{\bullet} \downarrow$ for $\bullet \in \{C, G, I, X\}$
  Transition path: The prices increase as the productivity parameter decreases. As such, prices converge towards their steady state level.

- **Effect on the Government** (see 2.12)

  Initial effect: The government debt decrease unambiguously due to two effect. The price on government consumption decreases, and a decrease in the unemployment benefits due to more employment. This allows the government to lower the tax rate after t_b years.

  Transition path: The debt continue to decline due to less interest payments, less nominal government consumption, and less unemployment benefit payouts due to a persistent higher employment rate, which allows the government to keeping the tax rate below its steady state level. Eventually, the unemployment rate and the price on government consumption converge to their steady state level. The government continues the lower tax rates as long as the debt is below its steady state, such that they both eventually converge to their steady state level.

- **Effect on Consumption and Exports** (see 2.10, 2.13)

  Initial effect: The domestic firm is relatively more productive than the foreign firms, hence the export increases, $X \uparrow$, and import decreases, $M \downarrow$. The effects on consumption can be decomposed into the effects on HtM consumption and Ricardian consumption. The HtM consumption follows the real income which increases due to the increased productivity, $\frac{\text{inc}}{P^C}, C^{HtM} \uparrow$. The Ricardian households, however, consumption smooth. They know that their future income will increase due to the lower tax rate, thus they initially borrows, $A \downarrow$, to increase consumption, $C^R \uparrow$. As both types of households increase their consumption, the total consumption increases as well, $C \uparrow$. There effect on bequest is neglectable as the households have chosen these beforehand.

Transition path: Again, there is an initial effect the first year due to the sticky nature of the output prices. As such, export increases further after one year, after which it smoothly declines towards its steady state level. Likewise, there is an increase in the consumption after one year. However, the total consumption also increases in year 2, which is driven by higher real income, such that the HtM households consume more relative to the decrease in the consumption of the Ricardian households. The consumption remains at a relatively high level due to the lower tax rate, but it eventually slowly decline towards its steady state levels. The savings are determined by the consumption smoothing of the Ricardian housholds. They borrow for the first periods, such that the savings decrease, as they know that the tax rate is decreasing. Once, the real income is sufficiently high, the Ricardian household starts to save more, such that the savings increase.

Further comments:

The initial of the productivity shock is affected by the stickiness of the output prices, which is captured by the parameter η. As such, it is interesting to see the effects of different values of η. Such sensitivity analysis is conducted in 2 - Technology shock - Sensitivity.ipynb.

- If η decreases, the initial level is less sticky. As such, the year 1 price adjustment effect disappears, such that the output price and the marginal cost price follow the same transition path. As a consequence, the full effect of the productivity affects the production and the real marginal products on capital and labor. Notably, the real marginal product on labor increases initially under fully flexible prices.
- If η increases, the initial level is more sticky. As such, the price adjustment continuous until year 2, but qualitatively the effects are similar to the baseline model.

4.3 Export shock

Shock to export ($\chi, P^F, P^{M,\bullet}$for $\bullet \in \{C, G, I, X\}$)

- **Effect on Foreign Economy and Prices** (see 2.10,2.9)

  Initial effect: Initially, there is a shock to the export productivity parameter, $\chi \uparrow$, the foreign prices,P^F $\uparrow$, and the import prices,$P^{M,\bullet}$ $\uparrow$ for $\bullet \in \{C, G, I, X\}$. The export increases through two shock channels, the export productivity channel and

the price channel. The shocks to the export productivity parameter and foreign prices directly increase the export, $X \uparrow$. However, the shock to the import prices has an indirect, opposite effect, as this drives up the export price, $P^X \uparrow$, through the repacking firms. The effect on the export price is smaller than the one on the foreign price, thus the overall effect on export is positive through both channels. This increase in export indirectly affects all prices stemming from higher total demand, such that these increase, $P^\bullet \uparrow$ for $\bullet \in \{C, G, I\}$. Additionally, there is a direct effect from higher import prices through the repacking firms. This is interesting as it implies, that the export demand shock stemming from the price channel comes with relatively higher prices compared to the export productivity channel. These two channels also affect the import in two opposite direction. The higher import prices imply less import whereas the higher export productivity implies more import. Overall, these two effect seems to almost cancel each other out, such that there only is a small positive effect.
Transition path: The export and the prices decrease as the shocks dissipate, such that they converge towards their steady state level.

- **Effect on the Production firms, the Phillips-curve and Investments** (see 2.6,2.7,2.11)

  Initial effect: The production firms increase production, $Y \uparrow$, to meet the increased demand stemming from the more export. To do this, they must hire more workers, $\ell \uparrow$, as the capital is fixed, which drives up return on capital, $r^K \uparrow$. This implies a higher marginal cost price, $P^{Y,0} \uparrow$, and a higher output price, $P^Y \uparrow$, because these are set as a mark-up over the marginal cost prices. The output price is sticky, thus the increase is relatively smaller. The effect on investments is affected by the increase return on capital relative to the price on investment. Overall, the effect is negative $I \downarrow$. However, the effect from each of the two shock channels is quite different. If we only considered the effect from the export productivity shock, the effect on investments would be positive. This is due to that the gain in the form of higher demand (and thereby higher return on capital) comes at a relatively small increase in the price on investments. This effect, however, is dominated by the price shock channel, where there is a relatively large increase in the price on investments. The overall negative effect on investments implies a decrease in the capital, $K \downarrow$.
  Transition path: The marginal cost price, the output price, the return on labor and the return on capital converges towards their steady state level once the shocks

dissipate. The investment increases from its initial drop, and it growth to become large than its steady state value. This is cause by the price on investments converge faster than the return on capital, such that it is profitable to invest. This cause the capital to also overshoot its steady state value, where after both variables converge towards steady state.

- **Effect on the Labor agency, Labor search & match and wage** (see 2.4, 2.5, 2.8)

  Initial effect: The increase in demand for ℓ implies that the labor agency hires more labor, $L \uparrow$. More hired labor implies more matches, $\mathcal{M} \uparrow$, which further implies an increase in the job-finding rate, $m^s \uparrow$, and the vacancies, $v \uparrow$. However, the job-filling rate decreases, $m^v \downarrow$.
  The effect on the wage is ambiguous. Wages should increase with higher output and output prices, but decrease as a consequence of more ℓ. Overall, there is a positive effect on the wage. As the return on workers is a markup over the wages, this increases as well, $r^\ell \uparrow$. The real wage, however, decreases due to the higher price on consumption.
  Transition path: The amount of households with a job at the start of period 1 is high, thus $\underline{L} \uparrow$ and the number of searchers decreases $S \downarrow$. Hereafter, all variables converge towards their steady state level.

- **Effect on the Government** (see 2.12)

  Initial effect: The effect on the government budget is ambiguous. On on hand, it increases as the price on government consumption increases. On the other hand, it decreases as more people becomes employed. Hence, the government receives more in tax revenue and pays less in unemployment benefits.Overall, the effect seems to be negative, such that the government lowers the tax rate.

  Transition path: The government budget adjusts, once the tax rate is lowered. Additionally, there is an effect stemming from the converges of the variables in the budget, i.e. the price on government consumption and the employment/unemployment rates.

- **Effect on Consumption** (see2.13)

  Initial effect: There are two opposite effects which affect the overall consumption level - higher price on consumption and higher income of the households. A

higher price on consumption, $P^C \uparrow$, affects the consumption through two channels. The direct effect is that higher prices implies a lower consumption. The indirect effect is that a higher initial price followed by a decrease in the price implies deflation, and thereby a higher real interest rate. The effect of a higher real interest rate is ambiguous and can be divided in an income effect and a substitution effect. A higher real interest rate implies that household gains a higher return on their saving. Therefore, they can obtain the same absolute return by saving less - implying more consumption today. The substitution effect is a higher return on their savings implies a greater alternative cost of consumption today - you could alternative invest a gain the real interest rate. Overall, the effect of a higher price on consumption is that consumption decreases.
The other, opposite effect is an increase in the income of the households due to higher wages and labor supply, which implies more consumption. Notably, the price on consumption increases more than the wages, such that the real wage decreases. Therefore, the overall initial effect on consumption is a decrease in the consumption level both for the Ricardian and hands-to-mouth consumers.

Transition path: After the initial drop, consumption increases towards steady state as prices stabilize. However, after 10 periods the new lower tax rate kicks in, which implies a higher income for the consumers. As such, the consumption increases to a level which is higher than its steady state level. Hereafter, consumption converges as the income converges towards steady state. As always, the Ricardian households foresee this increase in income, why they consumption smooth by lending initially. This implies the assets of the households decreases for the first periods, where after it follows .

4.4 Interest rate shock

Here we analyze a shock to the interest rate (r^{hh}):

- **Effect on Consumption** (see2.13)

  Initial effect: Initially, there is a shock to the interest rate, $r^{hh} \uparrow$. This has two effects on the Ricardian households' savings - namely an income and a substitution effect. The income effect is an increase return on savings already held by the households. As such, they can reach the same savings level even if they decrease future savings. Overall, this income effect implies that the consumption would initially increase. The substitution effect is that the interest rate for future savings are higher. This makes it more attractive to build up savings, which implies an initial drop in consumption. Overall, the substitution effect seems to be dominant as consumption for the Ricardian households initially decreases, $C^R \downarrow$, and savings increases, $A \uparrow$. A lower demand for consumption drives down the price on consumption $P^C \downarrow$, such that the Hands-to-Mouth households can afford more consumption, $C^{HtM} \uparrow$. The effect on total consumption is a decrease, $C \downarrow$. Notably, there is an initial increase in bequests, as these are affected by the interest rate $Aq \uparrow$.

Transition path: After the initial drop, the Ricardians' consumption increases as they save less. Higher demand drives up prices on demand, which in isolation should make consumption converge towards its steady state level. However, higher demand also implies higher output levels and thereby higher employment levels. This has two positive effects for consumption; higher income and less tax payments. These two effects also affects the Hands-to Mouth households' consumption, and they imply that the total consumption is above it steady state level for a long period. Eventually, the higher prices ensures that the consumption converge back to its steady state level. The effects on savings are similar, and these are above their steady state level as well.

- **Effect on the Production firms, the Phillips-curve and Investments** (see 2.6,2.7,2.11)

  Initial effect: The production firms decrease production, $Y \downarrow$, to meet the lower demand stemming from the less consumption. As the capital is fixed, they must fire workers, $\ell \downarrow$, which drives down return on capital, $r^K \downarrow$. This implies a lower marginal cost price, $P^{Y,0} \downarrow$, and a higher output price, $P^Y \downarrow$, because these are set as a mark-up over the marginal cost prices. The output price is sticky, thus the decrease is relatively smaller. A lower output price implies that the price on investment also decreases, $P^I \downarrow$. The investments are affected by the decrease in the return on capital relative to the decrease in the price on investment. Overall, the effect is positive $I \uparrow$. More investments implies an increase in the capital, $K \uparrow$.

  Transition path: The production firms increase production due to the higher demand stemming from the more consumption. As such, more workers are hired, which drives up the return on capital as well as the marginal cost and output prices. These converge toward their steady state level as the excess demand converges.

- **Effect on the Labor agency, Labor search & match and wage** (see 2.4, 2.5, 2.8)

  Initial effect: The decrease in demand for ℓ implies that the labor agency fires labor, $L \downarrow$. Less hired labor implies less matches, $\mathcal{M} \downarrow$, which further implies an decrease in the job-finding rate, $m^s \downarrow$, and the vacancies, $v \downarrow$. However, the job-filling rate increase, $m^v \uparrow$.
  The effect on the wage is ambiguous. Wages should decrease with higher output and output prices, but increase as a consequence of less ℓ. Overall, there is a negative effect on the wage. As the return on workers is a markup over the wages, this decreases as well, $r^\ell \downarrow$. The real wage, however, increases due to the lower price on consumption.
  Transition path: Higher output production implies more worker. Hence, the effects on the labor market is similar to the once for the production firms.

- **Effect on the Government** (see 2.12)

  Initial effect: The effect on the government budget is ambiguous. On on hand, it decreases as the price on government consumption decreases. On the other hand, it increases as more people becomes unemployed. Hence, the government receives less in tax revenue and pays more in unemployment benefits.Overall, the effect seems to be negative, such that the government lowers the tax rate.

  Transition path: The government budget adjusts, once the tax rate is lowered. Additionally, there is an effect stemming from the converges of the variables in the budget, i.e. the price on government consumption and the employment/unemployment rates.

- **Effect on Foreign Economy** (see 2.10)

  Initial effect: The price on exports decreases initially, which mechanically implies more export, $X \uparrow$, and less import, $M \downarrow$.

  Transition path: The export converges as the price on export converges.

5 Status report

Status: The described model is implemented in Python. Some results look weird, which could suggest a code error, a math error or weird assumptions or parameters. This should be checked and a baseline calibration established.

Economic extensions: Potential extensions include

1. **Extend household problem:** Add habit formation.
2. **Add more government with taxes and spending.**
3. **Add endogenous labor supply.**
4. **Add financial flows accounts wrt. to the foreign economy.**
5. **Add multiple sector and input-output structure.**
6. **Add technology growth, population growth, and trend inflation**

Computational improvements:

1. Simplify calculation of Jacobian with graph theory or automatic differentiation.
2. Speed-up calculation of Jacobian with parallelization.
3. Speed-up broyden-solver with sparse algebra.
4. Investigate what is done efficiently in MAKRO (GAMS+CONOPT)

A Derivations

A.1 Labor agency

The

A.2 Capital agency

TBA

A.3 Phillips Curve

$$p_t^Y = \frac{\sigma_D}{\sigma_D - 1} P_t^{Y,0} - \frac{1}{\sigma_D - 1} \frac{p_t^Y}{y_t} \left(\frac{\partial g_t}{\partial p_t^Y} + \frac{1}{1 + r^{\text{firm}}} \frac{\partial g_{t+1}}{\partial p_t^Y} \right)$$

$$p_t^Y = \frac{\sigma_D}{\sigma_D - 1} P_t^{Y,0} - \frac{1}{\sigma_D - 1} \frac{p_t^Y}{y_t} \left(\gamma \left[\frac{p_t^Y / p_{t-1}^Y}{p_{t-1}^Y / p_{t-2}^Y} - 1 \right] \left(\frac{1 / p_{t-1}^Y}{p_{t-1}^Y / p_{t-2}^Y} \right) P_t^Y Y_t + \frac{1}{1 + r^{\text{firm}}} \gamma \left[\frac{p_{t+1}^Y / p_t^Y}{p_t^Y / p_{t-1}^Y} - \right.\right.$$

$$P_t^Y = \frac{\sigma_D}{\sigma_D - 1} P_t^{Y,0} - \frac{1}{\sigma_D - 1} \left(\gamma \left[\frac{P_t^Y / P_{t-1}^Y}{P_{t-1}^Y / P_{t-2}^Y} - 1 \right] \left(\frac{P_t^Y / P_{t-1}^Y}{P_{t-1}^Y / P_{t-2}^Y} \right) P_t^Y + \frac{2}{1 + r^{\text{firm}}} \gamma \left[\frac{P_{t+1}^Y / P_t^Y}{P_t^Y / P_{t-1}^Y} - 1 \right] \left(\right.\right.$$

$$P_t^Y = (1 + \theta) P_t^{Y,0} - \eta \left(\frac{P_t^Y / P_{t-1}^Y}{P_{t-1}^Y / P_{t-2}^Y} - 1 \right) \frac{P_t^Y / P_{t-1}^Y}{P_{t-1}^Y / P_{t-2}^Y} P_t^Y + \frac{2}{1 + r^{\text{firm}}} \eta \frac{Y_{t+1}}{Y_t} \left(\frac{P_{t+1}^Y / P_t^Y}{P_t^Y / P_{t-1}^Y} - 1 \right) \frac{P_{t+1}^Y / P_t^Y}{P_t^Y / P_{t-1}^Y}$$

where$\theta = \frac{1}{\sigma_D - 1} \wedge$