CENTER FOR
ECONOMIC
BEHAVIOR &
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2. Transitional dynamics in sequence space

Lectures at IIES

Jeppe Druedahl

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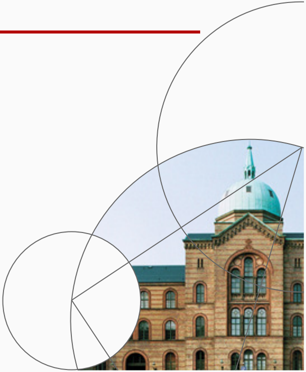

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Introduction

Introduction

- **Last time:** *Stationary equilibrium*
- **Today:** *Transition path*
- **Model:** Heterogeneous Agent Neo-Classical (HANC) model
- **Literature:**
    1. Auclert et. al. (2021), »Using the Sequence-Space Jacobian to Solve and Estimate Heterogeneous-Agent Models«
    2. Documentation for GEModelTools
       (except stuff on *linearized solution* and *simulation*)
    3. Kirkby (2017)

Ramsey

Ramsey: Summary

- **Simplified form:**

$$u'(C_t^{hh}) = \beta(1 + F_K(\Gamma_t, K_t, 1) - \delta)u'(C_{t+1}^{hh})$$
$$K_t = (1-\delta)K_{t-1} + F(\Gamma_t, K_{t-1}, 1) - C_t^{hh}$$

- **Production function:** $\Gamma_t K_{t-1}^{\alpha} L_t^{1-\alpha}$
- **Utility function:** $\frac{\left(C_t^{hh}\right)^{1-\sigma}}{1-\sigma}$
- **Steady state:**

$$K_{ss} = \left(\frac{\left(\frac{1}{\beta} - 1 + \delta\right)}{\Gamma_{ss}\alpha}\right)^{\frac{1}{\alpha-1}}$$
$$C_{ss}^{hh} = (1-\delta)K_{ss} + \Gamma_{ss}K_{ss}^{\alpha} - K_{ss}$$

Ramsey: As an equation system

$$
\begin{bmatrix}
r_t^K - \alpha \Gamma_t K_t^{\alpha-1} L_t^{1-\alpha} \\
w_t - (1-\alpha)\Gamma_t K_t^{\alpha} L_t^{-\alpha} \\
r_t - (r_t^K - \delta) \\
A_t - K_t \\
C_t^{hh,-\sigma} - \beta(1+r_{t+1}) C_{t+1}^{hh,-\sigma} \\
L_t^{hh} - 1 \\
A_t^{hh} - \left((1+r_t)A_{t-1}^{hh} + w_t L_t^{hh} - C_t^{hh}\right) \\
A_t - A_t^{hh} \\
L_t - L_t^{hh} \\
\forall t \in \{0,1,\dots\}, \text{given } K_{-1}
\end{bmatrix} = \mathbf{0}
$$

Remember: Perfect foresight

Truncated, reduced vector form

$$\boldsymbol{H}(\boldsymbol{K}, \boldsymbol{L}, \boldsymbol{\Gamma}, K_{-1}) = \begin{bmatrix} A_t - A_t^{hh} \\ L_t - L_t^{hh} \\ \forall t \in \{0, 1, \ldots, T-1\} \end{bmatrix} = \mathbf{0}$$

where $\boldsymbol{X} = (X_0, X_1, \ldots, X_{T-1})$, $A_{-1}^{hh} = K_{-1}$ and

$$\begin{aligned}
r_t^K &= \alpha \Gamma_t (K_{t-1}/L_t)^{\alpha-1} \\
w_t &= (1-\alpha)\Gamma_t (K_{t-1}/L_t)^{\alpha} \\
A_t &= K_t \\
r_t &= r_t^K - \delta \\
C_t^{hh} &= \left(\beta(1+r_{t+1})\right)^{-\sigma} C_{t+1}^{hh} \ \text{(backwards, } C_T^{hh} = C_{ss}\text{)} \\
L_t^{hh} &= 1 \\
A_t^{hh} &= (1+r_t)A_{t-1}^{hh} + w_t L_t^{hh} - C_t^{hh} \ \text{(forwards, } A_{-1}^{hh} \text{ known)}
\end{aligned}$$

Truncation: $T < \infty$ fine when $\Gamma_t = \Gamma_{ss}$ for all $t > \underline{t}$ with $\underline{t} \ll T$

Further reduced

$$\boldsymbol{H}(\boldsymbol{K}, \boldsymbol{\Gamma}, K_{-1}) = \begin{bmatrix} A_t - A_t^{hh} \\ \forall t \in \{0, 1, \ldots, T-1\} \end{bmatrix} = \mathbf{0}$$

where $\boldsymbol{X} = (X_0, X_1, \ldots, X_{T-1})$, $A_{-1}^{hh} = K_{-1}$ and

$$\begin{aligned} L_t = L_t^{hh} = 1 & \\ r_t^K &= \alpha \Gamma_t (K_{t-1}/L_t)^{\alpha-1} \\ w_t &= (1-\alpha)\Gamma_t (K_{t-1}/L_t)^{\alpha} \\ A_t &= K_t \\ r_t &= r_t^K - \delta \\ C_t^{hh} &= (\beta(1+r_{t+1}))^{-\sigma} C_{t+1}^{hh} \text{ (backwards, } C_T^{hh} = C_{ss}\text{)} \\ A_t^{hh} &= (1+r_t)A_{t-1}^{hh} + w_t L_t^{hh} - C_t^{hh} \text{ (forwards, } A_{-1}^{hh} \text{ known)} \end{aligned}$$

Solution method

1. **Set truncation** T
2. **Find Jacobian around steady state** $\boldsymbol{H_K}$
   by *numerical differentiation*
3. **Solve** $\boldsymbol{H}(\boldsymbol{K}, \boldsymbol{\Gamma}, K_{-1})$ in $\boldsymbol{K}$ for given $\boldsymbol{\Gamma}$ and K_{-1} with a quasi-Newton solver such as Broyden's method

- **Notebook:** *Ramsey.ipynb*

Jacobian

$$\boldsymbol{H_K} = \begin{bmatrix} \frac{\partial(A_0 - A_0^{hh})}{\partial K_0} & \frac{\partial(A_0 - A_0^{hh})}{\partial K_1} & \cdots \\ \frac{\partial(A_1 - A_1^{hh})}{\partial K_0} & \ddots & \ddots \\ \vdots & \ddots & \ddots \end{bmatrix}$$

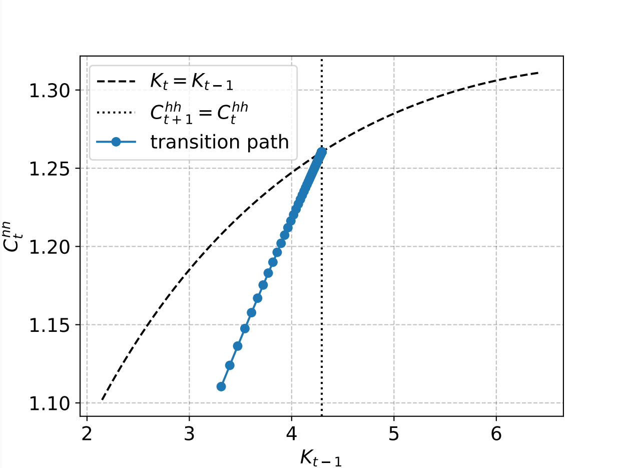
- **Column** s**:** Dynamic effect of change in capital in period s
- **Decomposition:**

$$\boldsymbol{H_K} = \boldsymbol{I} - \left(\mathcal{J}^{A^{hh},r}\mathcal{J}^{r,K} + \mathcal{J}^{A^{hh},r}\mathcal{J}^{w,K}\right)$$

1. Mechanic effect: $\frac{\partial \boldsymbol{A}}{\partial \boldsymbol{K}} = \boldsymbol{I}$
2. Pricing through firms: $\mathcal{J}^{r,K}$ and $\mathcal{J}^{w,K}$
3. Consumption-saving through households: $\mathcal{J}^{A^{hh},r}$ and $\mathcal{J}^{A^{hh},w}$

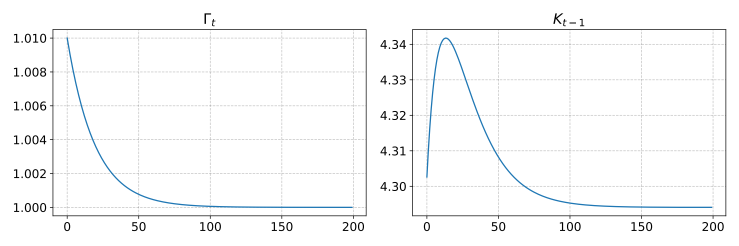
Example 1: Initially low capital

Initially away from steady state: $K_{-1} = 0.75 K_{ss}$



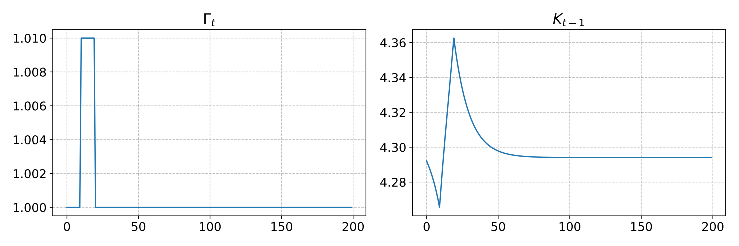
Example 2: Technology shock

Technology shock: $\Gamma_t = 0.01\Gamma_{ss}0.95^t$ (exogenous, deterministic)



Example 3: Future technology shock

Technology shock: $\Gamma_t = \begin{cases} 1.01\Gamma_{ss} & \text{if } t \in [10, 20) \\ \Gamma_{ss} & \text{else} \end{cases}$ (exogenous, deterministic)



Transition path

Heterogeneous households

- **Utility maximization** for household i:

$$v_0(\beta_i, z_{it}, a_{it-1}) = \max_{\{c_{it}\}_{t=0}^{\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta_i^t u(c_{it})$$

s.t.

$$\ell_{it} = z_{it}$$

$$a_{it} = (1 + r_t)a_{it-1} + w_t \ell_{it} - c_{it} + \Pi_t$$

$$\log z_{it+1} = \rho_z \log z_{it} + \psi_{it+1}, \;\; \psi_{it} \sim \mathcal{N}(\mu_\psi, \sigma_\psi), \;\; \mathbb{E}[z_{it}] = 1$$

$$a_{it} \geq 0$$

Distributions and aggregates

- **Policy functions:** Aggregate prices are hidden as inputs, i.e.

$$x_t^*(\beta_i, z_{it}, a_{it-1}) = x^*(\beta_i, z_{it}, a_{it-1}, \{r_\tau, w_\tau\}_{\tau \geq t}) \text{ for } x \in \{a, \ell, c\}$$

- **Distributions** (vector of probabilities):

  1. Beginning-of-period: $\underline{\boldsymbol{D}}_t$ over β_i, z_{it-1} and a_{it-1}
  2. Productivity transition: $\boldsymbol{D}_t = \Pi_z' \underline{\boldsymbol{D}}_t$ over β_i, z_{it} and a_{it-1}
  3. Savings transition: $\underline{\boldsymbol{D}}_{t+1} = \Lambda_t' \boldsymbol{D}_t$ where again

$$\Lambda_t = \Lambda\left(\{r_\tau, w_\tau\}_{\tau \geq t}\right)$$

- **Aggregate consumption and savings:**

$$\begin{aligned} X_t^{hh} &= \int x_t^*(\beta_i, z_{it}, a_{it-1}) d\boldsymbol{D}_t \text{ for } x \in \{a, \ell, c\} \\ &= X^{hh}\left(\{r_\tau, w_\tau\}_{\tau \geq t}, \underline{\boldsymbol{D}}_0\right) \\ &= \boldsymbol{x}_t^{*\prime} \boldsymbol{D}_t \end{aligned}$$

Equation system

The model can be written as an **equation system**

$$\begin{bmatrix} r_t^K - F_K(K_{t-1}, L_t) \\ w_t - F_L(K_{t-1}, L_t) \\ r_t - (r_t^K - \delta) \\ A_t - K_t \\ \boldsymbol{D}_t - \Pi_z \underline{\boldsymbol{D}}_t \\ \underline{\boldsymbol{D}}_{t+1} - \Lambda_t \boldsymbol{D}_t \\ A_t - \boldsymbol{a}_t^{*\prime} \boldsymbol{D}_t \\ L_t - \ell_t^{*\prime} \boldsymbol{D}_t \\ \forall t \in \{0, 1, \dots\}, \text{given } \underline{\boldsymbol{D}}_0 \end{bmatrix} = \mathbf{0}$$

where $\{\Gamma_t\}_{t \geq 0}$ is a given technology path and $K_{-1} = \int a_{t-1} d\underline{\boldsymbol{D}}_0$

Transition path - close to verbal definition

For a given $\underline{\boldsymbol{D}}_0$ and a path $\{\Gamma_t\}$

1. Quantities $\{K_t\}$ and $\{L_t\}$,
2. prices $\{r_t\}$ and $\{w_t\}$,
3. the distributions $\{\boldsymbol{D}_t\}$ over β_i, z_t and a_{t-1}
4. and the policy functions $\{a_t^*\}$, $\{\ell_t^*\}$ and $\{c_t^*\}$

are such that in all periods

1. Firms maximize profits (prices)
2. Household maximize expected utility (policy functions)
3. $\boldsymbol{D}_t$ is implied by simulating the household problem forwards from $\underline{\boldsymbol{D}}_0$
4. Mutual fund balance sheet is satisfied
5. The capital market clears
6. The labor market clears
7. The goods market clears

What are we finding

- **Underlying assumption:** No aggregate uncertainty
- **»Shock«, Γ:** A fully unexpected non-recurrent event $\equiv$ *MIT shock*
- **Transition path, $\boldsymbol{K}$:** Non-linear perfect foresight response to
    1. Initial distribution, $\underline{\boldsymbol{D}}_0 \neq \boldsymbol{D}_{ss}$, *or to*
    2. Shock, $\Gamma_t \neq \Gamma_{ss}$ for some t (i.e. impulse-response)

Truncated, reduced vector form

$$\boldsymbol{H}(\boldsymbol{K}, \boldsymbol{L}, \boldsymbol{\Gamma}, \underline{\boldsymbol{D}}_0) = \begin{bmatrix} A_t - A_t^{hh} \\ L_t - L_t^{hh} \\ \forall t \in \{0, 1, \dots, T-1\} \end{bmatrix} = \mathbf{0}$$

where $\boldsymbol{X} = (X_0, X_1, \dots, X_{T-1})$, $K_{-1} = \int a_{t-1} d\underline{\boldsymbol{D}}_0$ and

$$\begin{aligned} r_t^K &= \alpha \Gamma_t (K_{t-1}/L_t)^{\alpha-1} \\ w_t &= (1-\alpha)\Gamma_t (K_{t-1}/L_t)^{\alpha} \\ A_t &= K_t \\ \boldsymbol{D}_t &= \Pi_z' \underline{\boldsymbol{D}}_t \\ \underline{\boldsymbol{D}}_{t+1} &= \Lambda_t' \boldsymbol{D}_t \\ A_t^{hh} &= \boldsymbol{a}_t^{*\prime} \boldsymbol{D}_t \\ L_t^{hh} &= \boldsymbol{\ell}_t^{*\prime} \boldsymbol{D}_t \\ & \forall t \in \{0, 1, \dots, T-1\} \end{aligned}$$

Truncation: $T < \infty$ fine when $\Gamma_t = \Gamma_{ss}$ for all $t > \underline{t}$ with $\underline{t} \ll T$

Further reduction

$$\boldsymbol{H}(\boldsymbol{K}, \boldsymbol{\Gamma}, \underline{\boldsymbol{D}}_0) = \begin{bmatrix} A_t - A_t^{hh} \\ \forall t \in \{0, 1, \ldots, T-1\} \end{bmatrix} = \mathbf{0}$$

where $\boldsymbol{X} = (X_0, X_1, \ldots, X_{T-1})$, $K_{-1} = \int a_{t-1} d\underline{\boldsymbol{D}}_0$ and

$$\begin{aligned} L_t &= 1 \\ A_t &= K_t \\ r_t^K &= \alpha \Gamma_t (K_{t-1}/L_t)^{\alpha-1} \\ w_t &= (1-\alpha)\Gamma_t (K_{t-1}/L_t)^{\alpha} \\ \boldsymbol{D}_t &= \Pi_z' \underline{\boldsymbol{D}}_t \\ \underline{\boldsymbol{D}}_{t+1} &= \Lambda_t' \boldsymbol{D}_t \\ A_t^{hh} &= \boldsymbol{a}_t^{*\prime} \boldsymbol{D}_t \\ &\forall t \in \{0, 1, \ldots, T-1\} \end{aligned}$$

Truncation: $T < \infty$ fine when $\Gamma_t = \Gamma_{ss}$ for all $t > \underline{t}$ with $\underline{t} \ll T$

Use Broyden's method?

1. Guess $\boldsymbol{K}^0$ and set $i = 0$
2. Calculate the steady state Jacobian $\boldsymbol{H}_{\boldsymbol{K},ss} = \boldsymbol{H}_{\boldsymbol{K}}(\boldsymbol{K}_{ss}, \boldsymbol{\Gamma}_{ss}, K_{ss})$
3. Calculate $\boldsymbol{H}^i = \boldsymbol{H}(\boldsymbol{\Gamma}, \boldsymbol{K}^i, K_{-1})$
4. Update Jacobian by
$$\boldsymbol{H}_{\boldsymbol{K}}^i = \begin{cases} \boldsymbol{H}_{\boldsymbol{K},ss} & \text{if } i = 0 \\ \boldsymbol{H}_{\boldsymbol{K}}^{i-1} + \frac{(\boldsymbol{H}^i - \boldsymbol{H}^{i-1}) - \boldsymbol{H}_{\boldsymbol{K}}^{i-1}(\boldsymbol{K}^i - \boldsymbol{K}^{i-1})}{\left|\boldsymbol{K}^i - \boldsymbol{K}^{i-1}\right|_2} \left(\boldsymbol{K}^i - \boldsymbol{K}^{i-1}\right)' & \text{if } i > 0 \end{cases}$$
5. Stop if $\left|\boldsymbol{H}^i\right|_\infty$ below tolerance
6. Update guess by $\boldsymbol{K}^{i+1} = \boldsymbol{K}^i - \left(\boldsymbol{H}_{\boldsymbol{K}}^i\right)^{-1} \boldsymbol{H}^i$
7. Increment i and return to step 3

Note: *We find the fully non-linear solution*

Much more stable than relaxation (esp. with many variables)

Bottleneck: How do we find the Jacobian?

1. **Naive approach:** For each $s \in \{0, 1, ..., T-1\}$ do
   1.1 Set $K_t = K_{ss} + \mathbf{1}\{t = s\} \cdot \Delta, \qquad \Delta = 10^{-4}$
   1.2 Find $\boldsymbol{r}$ and $\boldsymbol{w}$
   1.3 Solve household problem backwards along transition path
   1.4 Simulate households forward along transition path
   1.5 Calculate $\frac{\partial H_t}{\partial K_s} = \frac{K_t - A_t^{hh}}{\Delta}$ for all t

   Bottleneck: We need T^2 solution steps and simulation steps!

2. **Fake news algorithm:** From household Jacobian to full Jacobian

$$\boldsymbol{H_K} = \boldsymbol{I} - \left(\mathcal{J}^{A^{hh},r}\mathcal{J}^{r,K} + \mathcal{J}^{A^{hh},w}\mathcal{J}^{w,K}\right)$$

   $\mathcal{J}^{r,K}, \mathcal{J}^{w,K}$: Fast from the onset - *only involve aggregates*
   $\mathcal{J}^{A^{hh},r}, \mathcal{J}^{A^{hh},w}$: Only requires T solution steps and simulation steps!

   $\Rightarrow$ *detailed discussed later today*

Full block structure

- **Shocks** are $\boldsymbol{Z} = \boldsymbol{\Gamma}$ and **unknowns** are $\boldsymbol{U} = \left[\begin{array}{cc} \boldsymbol{K} & \boldsymbol{L} \end{array}\right]'$
- **Ordered blocks:**
  1. Production firm: $\boldsymbol{\Gamma}, \boldsymbol{K}, \boldsymbol{L}, K_{-1} \rightarrow \boldsymbol{r^K}, \boldsymbol{w}$
  2. Mutual fund: $\boldsymbol{K}, \boldsymbol{r^K} \rightarrow \boldsymbol{A}, \boldsymbol{r}$
  3. Households: $\boldsymbol{r}, \boldsymbol{w}, \underline{\boldsymbol{D}}_0 \rightarrow \boldsymbol{A^{hh}}, \boldsymbol{L^{hh}}$
  4. Market clearing: $\boldsymbol{A}, \boldsymbol{L}, \boldsymbol{A^{hh}}, \boldsymbol{L^{hh}} \rightarrow \boldsymbol{A} - \boldsymbol{A^{hh}}, \boldsymbol{L} - \boldsymbol{L^{hh}}$
- **Jacobian:**

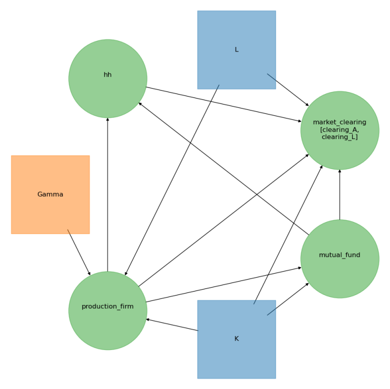
$$\boldsymbol{H_U} = \left[\begin{array}{cc} \boldsymbol{H_K} & \boldsymbol{H_L} \end{array}\right]$$

$$\boldsymbol{H_K} = \left[\begin{array}{c} \mathcal{J}^{A,K} - \left(\mathcal{J}^{A^{hh},r}\mathcal{J}^{r,r^K}\mathcal{J}^{r^K,K} + \mathcal{J}^{A^{hh},w}\mathcal{J}^{w,K}\right) \\ \boldsymbol{0} \end{array}\right]$$

$$\boldsymbol{H_L} = \left[\begin{array}{c} \mathcal{J}^{A^{hh},r}\mathcal{J}^{r,r^K}\mathcal{J}^{r^K,L} + \mathcal{J}^{A^{hh},w}\mathcal{J}^{w,L} \\ \boldsymbol{I} \end{array}\right]$$

DAG: Directed Acyclical Growth

- **Orange square:** Shocks (exogenous)
- **Purple square:** Unknowns (endogenous)
- **Green circles:** Blocks (with variables and targets inside)



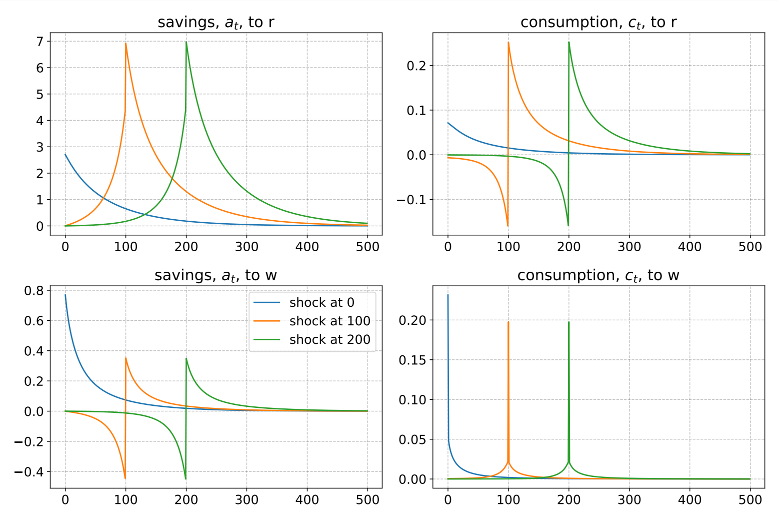
Interpreting the household Jacobians

- **Jacobian of consumption wrt. wage:** *What happens to consumption in period t when the wage (and thus income) increases in period s?*

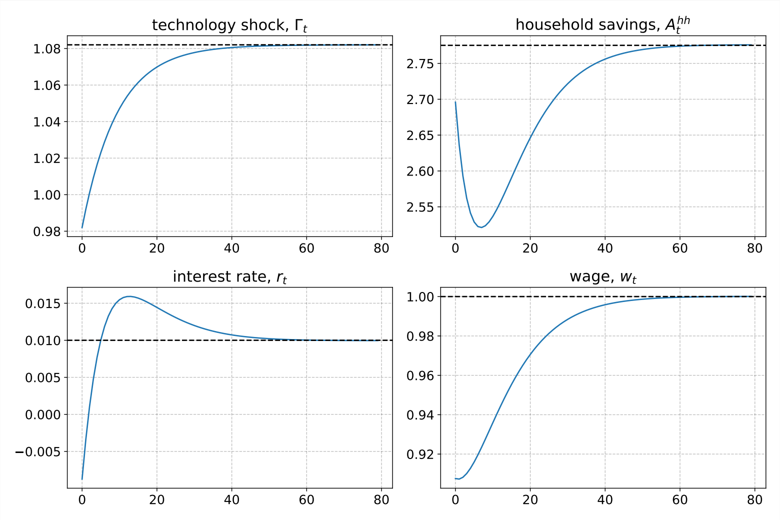
$$\mathcal{J}^{C^{hh},w} = \begin{bmatrix} \frac{\partial C_0^{hh}}{\partial w_0} & \frac{\partial C_0^{hh}}{\partial w_1} & \cdots \\ \frac{\partial C_1^{hh}}{\partial w_0} & \ddots & \ddots \\ \vdots & \ddots & \ddots \end{bmatrix}$$

- **Columns:** The full dynamic response to a shock in period s

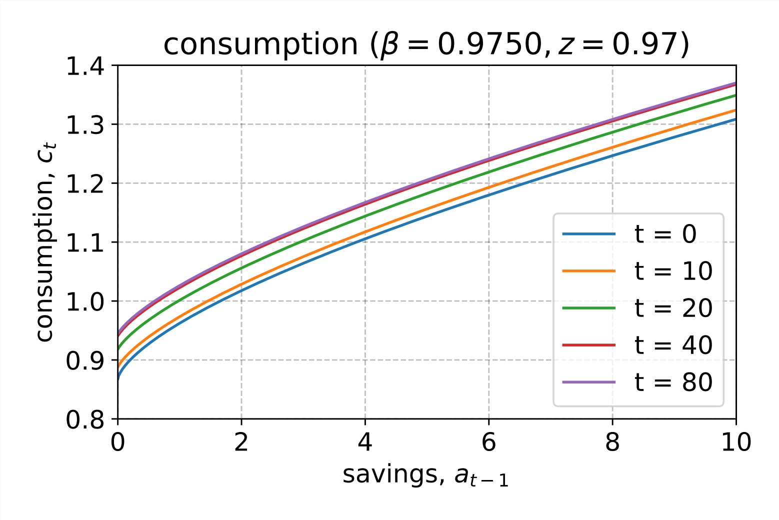
Household Jacobians



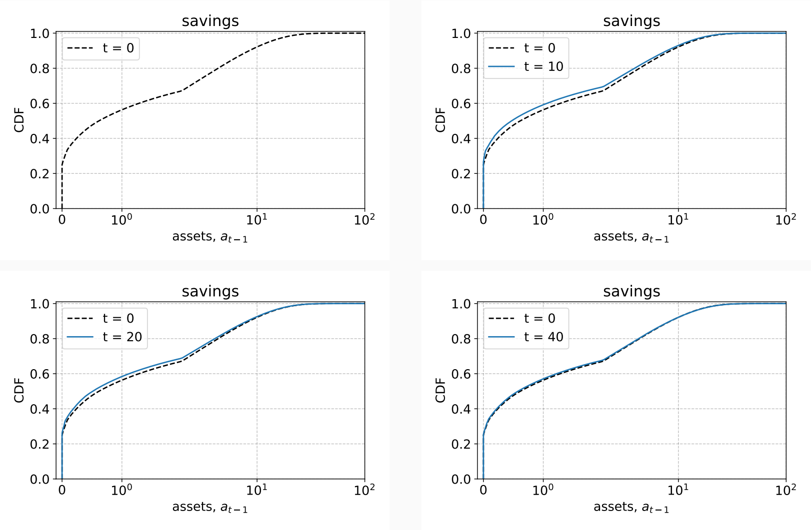
Transition path to technology shock



Consumption functions along transition path



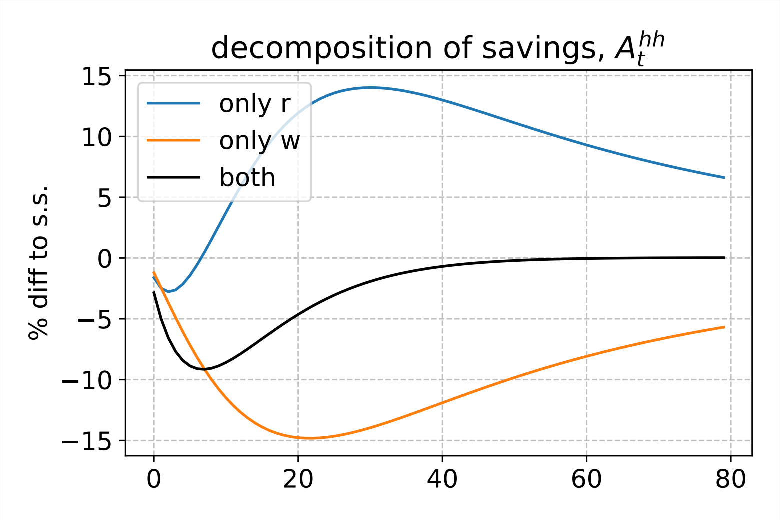
Distributions along transition path



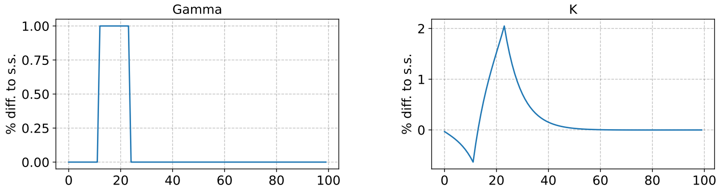
Decomposition of GE response

- **GE transition path:** $\boldsymbol{r}^*$ and $\boldsymbol{w}^*$
- **PE response of each:**
  1. Set $(\boldsymbol{r}, \boldsymbol{w}) \in \{(\boldsymbol{r}^*, \boldsymbol{w}_{ss}), (\boldsymbol{r}_{ss}, \boldsymbol{w}^*)\}$
  2. Solve household problem backwards along transition path
  3. Simulate households forward along transition path
  4. Calculate outcomes of interest
- **Additionally:** We can vary the initial distribution, $\underline{\boldsymbol{D}}_0$, to find the response of sub-groups

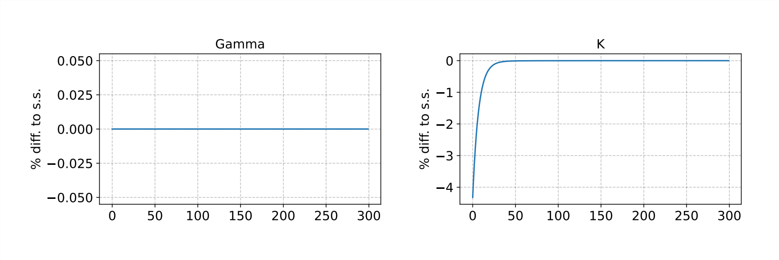
Decomposition of savings



More shocks: Future technology shock



More shocks: 5% less capital



Distribution: *Proportional reduction of savings for everybody*

DAGs

General model class I

1. Time is discrete (index t).
2. There is a continuum of households (index i, when needed).
3. There is *perfect foresight* wrt. all aggregate variables, $\boldsymbol{X}$, indexed by $\mathcal{N}$, $\boldsymbol{X} = \{\boldsymbol{X}_t\}_{t=0}^{\infty} = \{\boldsymbol{X}^j\}_{j\in\mathcal{N}} = \{X_t^j\}_{t=0,j\in\mathcal{N}}^{\infty}$, where $\mathcal{N} = \mathcal{Z} \cup \mathcal{U} \cup \mathcal{O}$, and $\mathcal{Z}$ are *exogenous shocks,* $\mathcal{U}$ are *unknowns*, $\mathcal{O}$ are outputs, and $\mathcal{H} \in \mathcal{O}$ are *targets.*
4. The model structure is described in terms of a set of *blocks* indexed by $\mathcal{B}$, where each block has inputs, $\mathcal{I}_b \subset \mathcal{N}$, and outputs, $\mathcal{O}_b \subset \mathcal{O}$, and there exists functions $h^o(\{\boldsymbol{X}^i\}_{i\in\mathcal{I}_b})$ for all $o \in \mathcal{O}_b$.
5. The blocks are *ordered* such that (i) each output is *unique* to a block, (ii) the first block only have shocks and unknowns as inputs, and (iii) later blocks only additionally take outputs of previous blocks as inputs. This implies the blocks can be structured as a *directed acyclical graph* (DAG).

General model class II

6. The number of targets are equal to the number of unknowns, and an *equilibrium* implies $\boldsymbol{X}^{o} = 0$ for all $o \in \mathcal{H}$. Equivalently, the model can be summarized by an *target equation system* from the unknowns and shocks to the targets,

$$\boldsymbol{H}(\boldsymbol{U}, \boldsymbol{Z}) = \boldsymbol{0},$$

and an *auxiliary model equation* to infer all variables

$$\boldsymbol{X} = \boldsymbol{M}(\boldsymbol{U}, \boldsymbol{Z}).$$

A *steady state* satisfy

$$\boldsymbol{H}(\boldsymbol{U}_{ss}, \boldsymbol{Z}_{ss}) = 0 \text{ and } \boldsymbol{X}_{ss} = \boldsymbol{M}(\boldsymbol{U}_{ss}, \boldsymbol{Z}_{ss})$$

General model class III

7. The *discretized household block* can be written recursively as

$$
\begin{aligned}
\boldsymbol{v}_t &= v(\underline{\boldsymbol{v}}_{t+1}, \boldsymbol{X}_t^{hh}) \\
\underline{\boldsymbol{v}}_t &= \Pi(\boldsymbol{X}_t^{hh})\boldsymbol{v}_t \\
\boldsymbol{D}_t &= \Pi(\boldsymbol{X}_t^{hh})'\underline{\boldsymbol{D}}_t \\
\underline{\boldsymbol{D}}_{t+1} &= \Lambda(\underline{\boldsymbol{v}}_{t+1}, \boldsymbol{X}_t^{hh})'\boldsymbol{D}_t \\
\boldsymbol{a}_t^* &= \boldsymbol{a}^*(\underline{\boldsymbol{v}}_{t+1}, \boldsymbol{X}_t^{hh}) \\
\boldsymbol{Y}_t^{hh} &= \boldsymbol{y}(\underline{\boldsymbol{v}}_{t+1}, \boldsymbol{X}_t^{hh})'\boldsymbol{D}_t \\
&\underline{\boldsymbol{D}}_0 \text{ is given,} \\
&\boldsymbol{X}_t^{hh} = \{\boldsymbol{X}_t^i\}_{i \in \mathcal{I}_{hh}}, \boldsymbol{Y}_t^{hh} = \{\boldsymbol{X}_t^o\}_{o \in \mathcal{O}_{hh}},
\end{aligned}
$$

where $\boldsymbol{Y}_t$ is aggregated outputs with $\boldsymbol{y}(\underline{\boldsymbol{v}}_{t+1}, \boldsymbol{X}_t^{hh})$ as individual level measures (savings, consumption labor supply etc.).

8. Given the sequence of shocks, $\boldsymbol{Z}$, there exists a *truncation period*, T, such all variables return to steady state beforehand.

Fake News Algorithm

Fake news algorithm

- **Household block:**

$$\boldsymbol{Y}^{hh} = hh(\boldsymbol{X}^{hh})$$

- **Goal:** Fast computation of

$$\mathcal{J}^{hh} = \frac{dhh(\boldsymbol{X}_{ss}^{hh})}{d\boldsymbol{X}^{hh}}$$

- **Naive approach:** Requires T^2 solution and simulation steps
- **Next slides:** *Sketch of much faster approach*
  (with $\Pi_t = \Pi_{ss}$ for notational simplicity)

Forward looking behavior

- **Notation:** $\bullet_t^{s,i}$ when there in period s is a shock to variable i
- **Time to shock:** Sufficient statistic for value and policy functions

$$\underline{\boldsymbol{v}}_t^{s,i} = \begin{cases} \underline{\boldsymbol{v}}_{ss} & \text{for } t > s \\ \underline{\boldsymbol{v}}_{T-1-(s-t)}^{T-1,i} & \text{for } t \leq s \end{cases} \text{ and } \boldsymbol{v}_t^{s,i} = \begin{cases} \boldsymbol{v}_{ss} & \text{for } t > s \\ \boldsymbol{v}_{T-1-(s-t)}^{T-1,i} & \text{for } t \leq s \end{cases}$$

$$\boldsymbol{y}_t^{s,i} = \begin{cases} \boldsymbol{y}_{ss} & t > s \\ \boldsymbol{y}_{T-1-(s-t)}^{T-1,i} & t \leq s \end{cases} \text{ and } \Lambda_t^{s,i} = \begin{cases} \Lambda_{ss} & t > s \\ \Lambda_{T-1-(s-t)}^{T-1,i} & t \leq s \end{cases}$$

- **Computation:** Only a single backward iteration required!
- **Note:** This is not an approximation

The first steps forward

- **Effect on output variable o in period 0:**

$$\mathcal{Y}_{0,s}^{o,i} \equiv \frac{dY_0^{o,s,i}}{dx} = \frac{\left(d\boldsymbol{y}_0^{o,s,i}\right)'}{dx} \Pi_{ss}' \underline{\boldsymbol{D}}_{ss}$$

- **Effect on beginning-of-period distribution in period 1:**

$$\underline{\mathcal{D}}_{1,s}^{i} \equiv \frac{d\underline{\boldsymbol{D}}_1^{s,i}}{dx} = \frac{\left(d\Lambda_0^{s,i}\right)'}{dx} \Pi_{ss}' \underline{\boldsymbol{D}}_{ss}$$

- **Expectation vector:** $\mathcal{E}_t^o \equiv \left(\Pi_{ss}\Lambda_{ss}\right)^t \Pi_{ss}\boldsymbol{y}_{ss}^o$,
- **Computational cost:**
  1. The cost of computing $\mathcal{Y}_{0,s}^{o,i}$ and $\underline{\mathcal{D}}_{1,s}^{i}$ for $s \in \{0, 1, \ldots, T-1\}$ are similar to a full forward simulation for T periods.
  2. The cost of computing $\mathcal{E}_s^o$ is negligible in comparison and can be done recursively, $\mathcal{E}_t^o = \Pi_{ss}\Lambda_{ss}\mathcal{E}_{t-1}^o$ with $\mathcal{E}_0^o = \Pi_{ss}\boldsymbol{y}_{ss}^o$.

Main result

- **Result:** Tedious algebra imply the Jacobian can be constructed from the known objects as

$$\mathcal{F}_{t,s}^{,i,o} \equiv \begin{cases} \mathcal{Y}_{0,s}^{o,i} & t = 0 \\ \left(\mathcal{E}_{t-1}^{o}\right)' \underline{\mathcal{D}}_{1,s}^{i} & t \geq 1 \end{cases}$$

$$\mathcal{J}_{t,s}^{hh,i,o} = \sum_{k=0}^{\min\{t,s\}} \mathcal{F}_{t-k,s-k}^{i,o}$$

- **Intuition:** ???
- **Mathematically:** Use the chain-rule over and over again
- **Note:** Use linearity and that we start from steady state

Chain-rule unfolding $t = 0$

$$\mathcal{J}^{hh,i,o}_{0,s} = \mathcal{F}^{i,o}_{0,s} = \mathcal{Y}^{o,i}_{0,s} = \underbrace{\frac{dY_0^{o,s,i}}{dx}}_{\text{change in policy}}$$

Chain-rule unfolding $t = 1$

$$\mathcal{J}^{hh,i,o}_{1,0} = \mathcal{F}^{i,o}_{1,0} = (\mathcal{E}^{o}_{0})' \, \underline{\mathcal{D}}^{i}_{1,0} = \underbrace{(\boldsymbol{y}^{o}_{ss})' \, \Pi'_{ss} \frac{d\underline{\boldsymbol{D}}^{0,i}_{1}}{dx}}_{\text{change in distribution}}$$

$$s \geq 1: \; \mathcal{J}^{hh,i,o}_{1,s} = \mathcal{F}^{i,o}_{1,s} + \mathcal{F}^{i,o}_{0,s-1} = \underbrace{(\boldsymbol{y}^{o}_{ss})' \, \Pi'_{ss} \frac{d\underline{\boldsymbol{D}}^{s,i}_{1}}{dx}}_{\text{change in distribution}} + \underbrace{\frac{dY^{o,s-1,i}_{0}}{dx}}_{\text{change in policy}}$$

Chain-rule unfolding $t = 2$

$$\mathcal{J}^{hh,i,o}_{2,0} = \mathcal{F}^{i,o}_{2,0} = \underbrace{(\boldsymbol{y}^o_{ss})' \, \Pi'_{ss}\Lambda'_{ss}\Pi'_{ss}\frac{d\underline{\boldsymbol{D}}^{0,i}_1}{dx}}_{\text{change in distribution}}$$

$$\mathcal{J}^{hh,i,o}_{2,1} = \mathcal{F}^{i,o}_{2,1} + \mathcal{F}^{i,o}_{1,0} = \underbrace{(\boldsymbol{y}^o_{ss})' \, \Pi'_{ss}\Lambda'_{ss}\Pi'_{ss}\frac{d\underline{\boldsymbol{D}}^{1,i}_1}{dx} + (\boldsymbol{y}^o_{ss})' \, \Pi'_{ss}\frac{d\underline{\boldsymbol{D}}^{0,i}_1}{dx}}_{\text{change in distribution}}$$

$$\begin{aligned} s \geq 2: \; \mathcal{J}^{hh,i,o}_{2,s} &= \mathcal{F}^{i,o}_{2,s} + \mathcal{F}^{i,o}_{1,s-1} + \mathcal{F}^{i,o}_{0,s-2} \\ &= \underbrace{(\boldsymbol{y}^o_{ss})' \, \Pi'_{ss}\Lambda'_{ss}\Pi'_{ss}\frac{d\underline{\boldsymbol{D}}^{s,i}_1}{dx} + (\boldsymbol{y}^o_{ss})' \, \Pi'_{ss}\frac{d\underline{\boldsymbol{D}}^{s-1,i}_1}{dx}}_{\text{change in distribution}} + \underbrace{\frac{dY^{o,s-2,i}_0}{dx}}_{\text{change in policy}} \end{aligned}$$

Bottlenecks

Bottlenecks

- **Small models:** *Finding the stationary equilibrium*
  - **Trick:** *(Modified) policy function iteration* (Howard improvement)
  - **Idea:** Multiple steps as once when finding the value function
    See e.g. Rendahl (2022) and Eslami and Phelan (2023)
- **Bigger models:** With many unknowns and targets both computing the Jacobian and solving the equation system can be costly
  $\Rightarrow$ *SSJ toolbox from Auclert et. al. (2021) has some methods for speeding this up not available in GEModelTools*

Summary

Summary

- **Today:**
  1. Transition path (with truncation)
  2. Jacobian around steady state
  3. Block-structure and Directed Acyclical Graph (DAG)
  4. Fake news algorithm
  5. Interpretation of household Jacobian
  6. Decomposition of GE dynamics
- **Afternoon:**
  1. Introduction to code
  2. Exercise on model with government
- **Next:** Aggregate risk, linearized dynamics and analytical analysis