Center for Economic Behavior & Inequality

3. Aggregate risk and analytical analysis

IIES lectures

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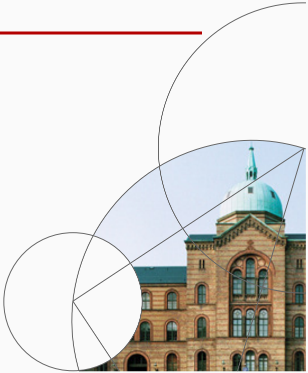


Introduction

Introduction

- **Previously:** Non-linear transition path and perfect foresight
- **Today:**
  1. Linearized Impulse Response Function (IRF)
  2. Linearized simulation with aggregate risk
- **Literature:**
  1. Auclert et. al. (2021), »Using the Sequence-Space Jacobian to Solve and Estimate Heterogeneous-Agent Models«
  2. Auclert et. al. (2023), »The Intertemporal Keynesian Cross«
  3. Boppart et. al. (2018), »Exploiting MIT shocks in heterogeneous-agent economies: The impulse response as a numerical derivative«
  4. Documentation for GEModelTools

IRFs and simulation

Reminder of model class

- **Unknowns:** $\boldsymbol{U}$
- **Shock:** Z
- **Additional variables:** $\boldsymbol{X}$
- **Target equation system:**

$$\boldsymbol{H}(\boldsymbol{U}, \boldsymbol{Z}) = \boldsymbol{0}$$

- **Auxiliary model equations**

$$\boldsymbol{X} = \boldsymbol{M}(\boldsymbol{U}, \boldsymbol{Z})$$

Linearized IRFs

- **Today:** Just consider the *first order solution*
  1. Solve for IRFs for unknowns

$$\boldsymbol{H}(\boldsymbol{U},\boldsymbol{Z})=0 \Rightarrow \boldsymbol{H_U}d\boldsymbol{U}+\boldsymbol{H_Z}d\boldsymbol{Z}=0 \Leftrightarrow d\boldsymbol{U}=\underbrace{-\boldsymbol{H_U^{-1}}\boldsymbol{H_Z}}_{\equiv\boldsymbol{G_U}}d\boldsymbol{Z}$$

  2. Derive all other IRFs for

$$\begin{aligned}\boldsymbol{X}=\boldsymbol{M}(\boldsymbol{U},\boldsymbol{Z}) \Rightarrow d\boldsymbol{X} &= M_{\boldsymbol{U}}d\boldsymbol{U}+M_{\boldsymbol{Z}}d\boldsymbol{Z}\\ &= \underbrace{(-M_{\boldsymbol{U}}\boldsymbol{H_U^{-1}}\boldsymbol{H_Z}+M_{\boldsymbol{Z}})}_{\equiv\boldsymbol{G}}d\boldsymbol{Z}\end{aligned}$$

- **Computation:** Same for $\boldsymbol{Z}$ as for $\boldsymbol{U}$
- **Limitations:**
  1. Imprecise for *large* shocks
  2. Imprecise in models with *aggregate non-linearities*
     (direct in aggregate equations or through micro-behavior)

Aggregate risk

- **Aggregate stochastic variables:** $\boldsymbol{Z}$ follow some known process
- **Observation:** Linearization of aggregate variables imply *certainty equivalence* with respect to these
- **Insight:** *The IRF from an MIT shock is* equivalent *to the IRF in a model with aggregate risk, which is linearized in the aggregate variables* (Boppart et. al., 2018)

Comparisons

- **State-space approach with linearization**: Ahn et al. (2018); Bayer and Luetticke (2020); Bhandari et al. (2023); Bilal (2023)
  Con:
  1. Harder to implement in my view
  2. Valuable to be able to interpret Jacobians

  Pro**:**
  1. More similar to standard approaches for RBC and NK models
  2. Easier path to 2nd and higher order approximations
- **Global solution:** The distribution of households is a state variable for each household ⇒ *explosion in complexity*
  1. Original: Krusell and Smith (1997, 1998); Algan et al. (2014);
  2. Deep learning: Fernández-Villaverde et al. (2021); Maliar et al. (2021); Han et al. (2021); Kase et al. (2022); Azinovic et al. (2022); Gu et al. (2023); Chen et al. (2023)
- **Discrete aggregate risk:** Lin and Peruffo (2023)

Basic linearized simulation

- **Shocks:** Write the shocks as an $MA(\infty)$ with coefficients $d\boldsymbol{Z}_s$ for $s \in \{0, 1, \dots\}$ driven by the innovation $\boldsymbol{\epsilon}_t$.
- **Linearized simulation** (with truncation)**:**
  1. Draw time series of innovations, $\tilde{\boldsymbol{\epsilon}}_t$
  2. Calculate the time series of shocks as $d\tilde{\boldsymbol{Z}}_t = \sum_{s=0}^{t} d\boldsymbol{Z}_s \tilde{\boldsymbol{\epsilon}}_{t-s}$
  3. Calculate the time series of other aggregate variables as

     $$d\tilde{\boldsymbol{X}}_t = \sum_{s=0}^{t} d\boldsymbol{X}_s \tilde{\boldsymbol{\epsilon}}_{t-s}$$

     where $d\boldsymbol{X}_s$ is the IRF to a unit-shock after s periods
- **Intuition:** Sum of first order effects from all previous shocks
- **Equivalence:**
  1. Same result if we linearize all aggregated equations and write the model in MA(∞) form
  2. The state space form can also be recovered (not needed)

Generalized linearized simulation

- **Generality:** Extend the model with auxiliary variables (incl. distributional moments) to calculate additional IRFs and simulations
- **Full distribution:**
  1. The IRF for grid point i_g in a policy function can be calculated as

$$da_{i_g,s}^{*}=\sum_{s'=s}^{T-1}\sum_{X^{hh}\in\boldsymbol{X}^{hh}}\frac{\partial a_{i_g}^{*}}{\partial X_{s'-s}^{hh}}dX_{s'}^{hh}.$$

     where $\partial\boldsymbol{a}_{i_g}^{*}/\partial X_{k}^{hh}$ is the derivative to a k-period ahead shock to input X^{hh} (calculated in fake news algorithm)
  2. The policy function can there be simulated as

$$\boldsymbol{a}_{i_g,t}^{*}=\sum_{s=0}^{t}da_{i_g,s}^{*}\tilde{\boldsymbol{\epsilon}}_{t-s}$$

  3. Distribution can then be simulated forwards

Calculating moments

- **Calculating moments such as var(dC_t):**
  1. From the simulation, or
  2. From the IRFs,

$$\text{var}(dC_t) = \sum_i \sigma_i^2 \sum_{s=0}^{T} \left(dC_s^i\right)^2$$

where dC_s^i is the IRF to a unit-shock to i after s periods and σ_i is the standard deviation of shock i

- **Also work for covariances, correlations etc.**
- **For a new set of parameters?**
  1. Only shock processes change $\Rightarrow$ same Jacobians
  2. Only need to re-compute Jacobian of aggregate variables?
  3. Also need to re-compute Jacobian of household problem?
  4. Also need to find stationary equilibrium again?
- **Speed-up:** Use Fast Fourier Transform (FFT), see SSJ

More moments: Covariances

$$\text{cov}(dC_t, dC_{t+1}) = \sum_i \sigma_i^2 \sum_{s=0}^{T} dC_s^i dC_{s+1}^i$$

$$\text{cov}(dC_t, dY_t) = \sum_i \sigma_i^2 \sum_{s=0}^{T} dC_s^i dY_s^i$$

Sticky wages

Households

- **Household problem:**

$$
\begin{aligned}
v_t(z_t, a_{t-1}) &= \max_{c_t} \frac{c_t^{1-\sigma}}{1-\sigma} - \varphi \frac{\ell_t^{1+\nu}}{1+\nu} + \beta \mathbb{E}_t \left[v_{t+1}(z_{t+1}, a_t) \right] \\
\text{s.t. } a_t + c_t &= (1 + r_t^a) a_{t-1} + (1 - \tau_t) w_t \ell_t z_t + \chi_t \\
\log z_{t+1} &= \rho_z \log z_t + \psi_{t+1} \ \ , \psi_t \sim \mathcal{N}(\mu_\psi, \sigma_\psi), \ \mathbb{E}[z_t] = 1 \\
a_t &\geq 0
\end{aligned}
$$

- **Active decisions:** Consumption-saving, c_t (and a_t)
- **Union decision:** Labor supply, ℓ_t
- **Consumption function:** $C_t^{hh} = C^{hh}\left(\{r_s^a, \tau_s, w_s, \ell_s, \chi_s\}_{s \geq t}\right)$

Firms

- **Production and profits:**

$$Y_t = \Gamma_t L_t$$
$$\Pi_t = P_t Y_t - W_t L_t$$

- **First order condition:**

$$\frac{\partial \Pi_t}{\partial L_t} = 0 \Leftrightarrow P_t \Gamma_t - W_t = 0 \Leftrightarrow w_t \equiv W_t / P_t = \Gamma_t$$

Zero profits: $\Pi_t = 0$

- **Wage and price inflation:**

$$\pi_t^w \equiv W_t / W_{t-1} - 1$$
$$\pi_t \equiv \frac{P_t}{P_{t-1}} - 1 = \frac{W_t / \Gamma_t}{W_{t-1} / \Gamma_{t-1}} - 1 = \frac{1 + \pi_t^w}{\Gamma_t / \Gamma_{t-1}} - 1$$

Union

- Everybody works the same:

$$\ell_t = L_t^{hh}$$

- Unspecified *wage adjustment costs* imply a **New Keynesian Wage (Phillips) Curve** (NKWPC or NKWC)

$$\pi_t^w = \kappa \left(\varphi \left(L_t^{hh} \right)^{\nu} - \frac{1}{\mu} \left(1 - \tau_t \right) w_t \left(C_t^{hh} \right)^{\text{-}\sigma} \right) + \beta \pi_{t+1}^w$$

Government

- **Spending:** G_t
- **Tax bill:** T_t

$$T_t = \int \tau_t w_t \ell_t z_t d\boldsymbol{D}_t = \tau_t \Gamma_t L_t = \tau_t Y_t$$

- If: **One-period bonds:**

$$B_t = (1 + r_t^b) B_{t-1} + G_t + \chi_t - T_t$$

- If: **Long-term bonds:** Geometrically declining payment stream of $1, \delta, \delta^2, \ldots$ for $\delta \in [0, 1]$. The bond price is q_t.

$$q_t (B_t - \delta B_{t-1}) = B_{t-1} + G_t + \chi_t - T_t$$

- Potential **tax-rule:**

$$\tau_t = \tau_{ss} + \omega q_{ss} \frac{B_{t-1} - B_{ss}}{Y_{ss}}$$

Central bank

- Standard **Taylor rule:**

$$1 + i_t = (1 + i_{t-1})^{\rho_i} \left((1 + r_{ss}) (1 + \pi_t)^{\phi_\pi} \right)^{1-\rho_i}$$

Alternative: Real rate rule

$$1 + i_t = (1 + r_{ss})(1 + \pi_{t+1})$$

Indeterminacy: Consider limit or assume future tightening

- **Fisher-equation:**

$$1 + r_t = \frac{1 + i_t}{1 + \pi_{t+1}}$$

Arbitrage

1. One-period *real* bond, $q_t = 1$:

$$t > 0: \; r_t^b = r_t^a = r_{t-1}$$
$$r_0^b = r_0^a = 1 + r_{ss}$$

2. One-period *nominal* bond, $q_t = 1$:

$$t > 0: \; r_t^b = r_t^a = r_{t-1}$$
$$t > 0: \; r_0^b = r_0^a = (1 + r_{ss})(1 + \pi_{ss}) / (1 + \pi_0)$$

3. Long-term *(real)* bonds:

$$\frac{1 + \delta q_{t+1}}{q_t} = 1 + r_t$$

$$1 + r_t^b = 1 + r_t^a = \frac{1 + \delta q_t}{q_{t-1}} = \begin{cases} \frac{1+\delta q_0}{q_{-1}} & \text{if } t = 0 \\ 1 + r_{t-1} & \text{else} \end{cases}$$

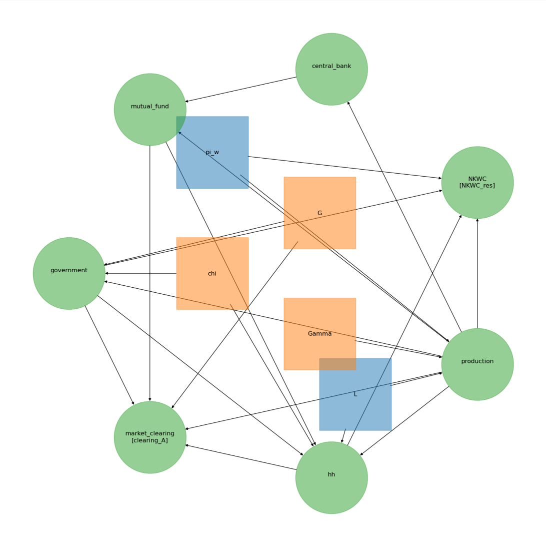
Market clearing

1. Asset market: $q_t B_t = A_t^{hh}$
2. Labor market: $L_t = L_t^{hh}$
3. Goods market: $Y_t = C_t^{hh} + G_t$

Equation system

$$
\boldsymbol{H}(\boldsymbol{\pi^w}, \boldsymbol{L}, \boldsymbol{G}, \chi, \boldsymbol{\Gamma}) = \begin{bmatrix}
w_t - \Gamma_t \\
1 + \pi_t - \frac{1+\pi_t^w}{\Gamma_t / \Gamma_{t-1}} \\
Y_t - \Gamma_t L_t \\
1 + i_t - (1 + i_{t-1})^{\rho_i} \left((1 + r_{ss}) (1 + \pi_t)^{\phi_\pi} \right)^{1-\rho_i} \\
1 + r_t - \frac{1+i_t}{1+\pi_{t+1}} \\
\frac{1+\delta q_{t+1}}{q_t} - (1 + r_t) \\
1 + r_t^a - \frac{1+\delta q_t}{q_{t-1}} \\
\tau_t - \left[\tau_{ss} + \omega q_{ss} \frac{B_{t-1} - B_{ss}}{Y_{ss}} \right] \\
q_t(B_t - \delta B_{t-1}) - [B_{t-1} + G_t + \chi_t - \tau_t Y_t] \\
q_t B_t - A_t^{hh} \\
\pi_t^w - \left[\kappa \left(\varphi \left(L_t^{hh} \right)^{\nu} - \frac{1}{\mu} (1 - \tau_t) w_t \left(C_t^{hh} \right)^{-\sigma} \right) + \beta \pi_{t+1}^W \right]
\end{bmatrix} = \boldsymbol{0}
$$

DAG



central_bank
mutual_fund
pi_w
NKWC
[NKWC_res]
G
government
chi
Gamma
production
L
market_clearing
[clearing_A]
hh

Analytical insights

Simpler consumption function

- **Assumptions:**
  1. One-period real bond
  2. No lump-sum transfers, $\chi_t = 0$
  3. Real rate rule: $r_t = r_{ss}$
  4. Fiscal policy in terms of dG_t and dT_t satisfying IBC

$$\sum_{t=0}^{\infty} (1 + r_{ss})^{-t} (dG_t - dT_t) = 0$$

- **Tax-bill:** $T_t = \tau_t w_t \int \ell_t z_t d\boldsymbol{D}_t = \tau_t \Gamma_t L_t = \tau_t L_t$
- **Household income:** $(1 - \tau_t) w_t \ell_t z_t = (Y_t - T_t) z_t$
- **Consumption function:** Simplifies to

$$C_t^{hh} = C^{hh}(\{Y_s - T_s\}_{s \geq 0}) = C^{hh}(\boldsymbol{Y} - \boldsymbol{T})$$

Two-equation version in $\boldsymbol{Y}$ and $\boldsymbol{r}$

$$\boldsymbol{Y} = C^{hh}(\boldsymbol{r}, \boldsymbol{Y} - \boldsymbol{T}) + \boldsymbol{G}$$
$$\boldsymbol{r} = \mathcal{R}(\boldsymbol{Y}; \boldsymbol{G}, \boldsymbol{T})$$

- **First equation:** Good market clearing
- **Second equation:**
  1. **Government:** $\boldsymbol{T}, \boldsymbol{Y} \rightarrow \boldsymbol{\tau}$
  2. **Resource constraint:** $\boldsymbol{G}, \boldsymbol{Y} \rightarrow \boldsymbol{C}$
  3. **Firm behavior I:** $\boldsymbol{\Gamma}, \boldsymbol{Y} \rightarrow \boldsymbol{L}, \boldsymbol{w}$
  4. **NKWC:** $\boldsymbol{L}, \boldsymbol{w}, \boldsymbol{\tau} \rightarrow \boldsymbol{\pi}^{\boldsymbol{w}}$
  5. **Firm behavior II:** $\boldsymbol{\pi}^{\boldsymbol{w}} \rightarrow \boldsymbol{\pi}$
  6. **Central bank:** $\boldsymbol{\pi} \rightarrow \boldsymbol{i}$
  7. **Fisher:** $\boldsymbol{i}, \boldsymbol{\pi} \rightarrow \boldsymbol{r}$
- **Heterogeneity does not enter** $\mathcal{R}(\boldsymbol{Y}; \boldsymbol{G}, \boldsymbol{T})$
- **Real rate rule:** Inflation is a side-show

Intertemporal Keynesian Cross

$$\boldsymbol{Y} = C^{hh}(\boldsymbol{Y} - \boldsymbol{T}) + \boldsymbol{G}$$

- **Total differentiation:**

$$dY_t = dG_t + \sum_{s=0}^{\infty} \frac{\partial C_t^{hh}}{\partial Z_s} (dY_t - dT_t)$$

IBC implies: $\sum_{t=0}^{\infty}(1+r_{ss})^{-t}\frac{\partial C_t^{hh}}{\partial Z_s} = (1+r_{ss})^{-s}$

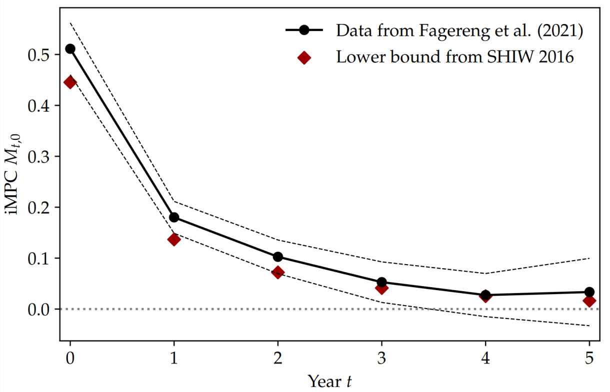
- **Intertemporal Keynesian Cross** in vector form

$$d\boldsymbol{Y} = d\boldsymbol{G} - d\boldsymbol{T} + \boldsymbol{M}d\boldsymbol{Y}$$

where $M_{t,s} = \frac{\partial C_t^{hh}}{\partial Z_s}$ encodes entire *complexity*

iMPCs in the data

Figure 1: iMPCs in the Norwegian and Italian data



Other columns: Druedahl et al. (2023) show in micro-data that consumption responds today to news about future income.

Perspective: Static Keynesian Cross

- **Old Keynesians:** Consumption only depend on current income

$$Y_t = G_t + C^{hh}(Y_t - T_t)$$

- **Total differentiate:**

$$\begin{aligned} dY_t &= dG_t + \frac{\partial C_t^{hh}}{\partial Z_t}(dY_t - dT_t) \\ &= dG_t + \text{mpc} \cdot (dY_t - dT_t) \end{aligned}$$

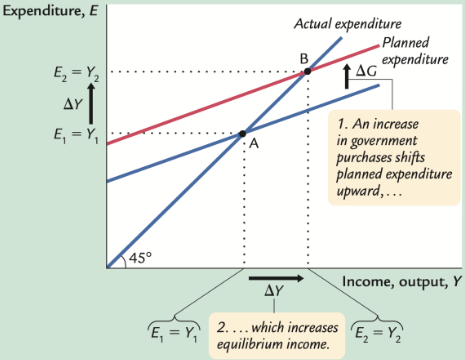
- **Solution**

$$dY_t = \frac{1}{1 - \text{mpc}} (dG_t - \text{mpc} dT_t)$$

from multiplier-process $1 + \text{mpc} + \text{mpc}^2 \cdots = \frac{1}{1-\text{mpc}}$

Static Keynesian Cross

figure 10-5



An Increase in Government Purchases in the Keynesian Cross An increase in government purchases of ΔG raises planned expenditure by that amount for any given level of income. The equilibrium moves from point A to point B, and income rises from Y_1 to Y_2. Note that the increase in income ΔY exceeds the increase in government purchases ΔG. Thus, fiscal policy has a multiplied effect on income.

Intertemporal solution technicalities

- **NPV-vector:** $\boldsymbol{q} = [1, (1+r_{ss})^{-1}, (1+r_{ss})^{-2}, \ldots]'$
- **Household IBC:** $\boldsymbol{q}'\boldsymbol{M} = \boldsymbol{q}'$ and $\boldsymbol{q}'(\boldsymbol{I} - \boldsymbol{M}) = \boldsymbol{0}$
- **Government IBC:** $\boldsymbol{q}'(d\boldsymbol{G} - d\boldsymbol{T}) = \boldsymbol{0}$
- **Problem:** $(\boldsymbol{I} - \boldsymbol{M})^{-1}$ cannot exist because

$$
\begin{aligned}
(\boldsymbol{I} - \boldsymbol{M})d\boldsymbol{Y} &= d\boldsymbol{G} - \boldsymbol{M}d\boldsymbol{T} \Leftrightarrow \\
\boldsymbol{q}'(\boldsymbol{I} - \boldsymbol{M})d\boldsymbol{Y} &= \boldsymbol{q}'(d\boldsymbol{G} - d\boldsymbol{T}) \Leftrightarrow \\
\boldsymbol{0} &= \boldsymbol{0}
\end{aligned}
$$

- **Problem:** If unique solution then on the form

$$
\begin{aligned}
d\boldsymbol{Y} &= \mathcal{M}(d\boldsymbol{G} - \boldsymbol{M}d\boldsymbol{T}) \\
\mathcal{M} &= (\boldsymbol{K}(\boldsymbol{I} - \boldsymbol{M}))^{-1}\boldsymbol{K}
\end{aligned}
$$

Indeterminancy: Still work-in-progress (Auclert et. al., 2023)

Intermezzo: Response of consumption

$$
\begin{aligned}
d\boldsymbol{Y} &= d\boldsymbol{G} + \boldsymbol{M}(d\boldsymbol{Y} - d\boldsymbol{T}) \Leftrightarrow \\
d\boldsymbol{Y} - d\boldsymbol{G} &= \boldsymbol{M}(d\boldsymbol{G} - d\boldsymbol{T}) + \boldsymbol{M}(d\boldsymbol{Y} - d\boldsymbol{G}) \Leftrightarrow \\
d\boldsymbol{Y} - d\boldsymbol{G} &= \mathcal{M}\boldsymbol{M}(d\boldsymbol{G} - d\boldsymbol{T}) \Rightarrow \\
d\boldsymbol{C} &= \mathcal{M}\boldsymbol{M}(d\boldsymbol{G} - d\boldsymbol{T})
\end{aligned}
$$

Fiscal multipliers

$$d\boldsymbol{Y} = d\boldsymbol{G} + \underbrace{\mathcal{M}\boldsymbol{M}(d\boldsymbol{G} - d\boldsymbol{T})}_{d\boldsymbol{C}}$$

- **Balanced budget multiplier:**

$$d\boldsymbol{G} = d\boldsymbol{T} \Rightarrow d\boldsymbol{Y} = d\boldsymbol{G}, d\boldsymbol{C} = 0$$

  Note: Central that income and taxes affect household income in exactly the same way = no redistribution

- **Deficit multiplier:** $d\boldsymbol{G} \neq d\boldsymbol{T}$
  1. Larger effect of $d\boldsymbol{G}$ than $d\boldsymbol{T}$
  2. *Numerical results needed*

Fiscal multiplier

Impact-multiplier:

$$\frac{\partial Y_0}{\partial G_0}$$

Cummulative-multiplier:

$$\frac{\sum_{t=0}^{\infty}(1+r_{ss})^{-t}dY_t}{\sum_{t=0}^{\infty}(1+r_{ss})^{-t}dG_t}$$

Comparison with RA model

- **From lecture 1:** $\beta(1+r_{ss}) = 1$ implies

$$C_t = (1-\beta)\sum_{s=0}^{\infty}\beta^s Y_{t+s}^{hh} + r_{ss}a_{-1}$$

- The **iMPC-matrix** becomes

$$\boldsymbol{M}^{RA} = \begin{bmatrix} (1-\beta) & (1-\beta)\beta & (1-\beta)\beta^2 & \cdots \\ (1-\beta) & (1-\beta)\beta & (1-\beta)\beta^2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} = (1-\beta)\mathbf{1}\boldsymbol{q}'$$

- **Consumption response** is zero

$$\begin{aligned} d\boldsymbol{C}^{\boldsymbol{RA}} &= \mathcal{M}\boldsymbol{M}^{\boldsymbol{RA}}(d\boldsymbol{G} - d\boldsymbol{T}) \\ &= \mathcal{M}(1-\beta)\mathbf{1}\boldsymbol{q}'(d\boldsymbol{G} - d\boldsymbol{T}) \\ &= \mathbf{0} \Leftrightarrow d\boldsymbol{Y} = d\boldsymbol{G} \end{aligned}$$

Comparison with TA model

- **Hands-too-Mouth (HtM) households:** λ share have $C_t = Y_{t+s}^{hh}$

$$\boldsymbol{M}^{TA} = (1-\lambda)\boldsymbol{M}^{RA} + \lambda \boldsymbol{I}$$

- **Intertemporal Keynesian Cross** becomes

$$(\boldsymbol{I} - \boldsymbol{M}^{TA})d\boldsymbol{Y} = d\boldsymbol{G} - \boldsymbol{M}^{TA} d\boldsymbol{T}$$
$$(\boldsymbol{I} - \boldsymbol{M}^{RA})d\boldsymbol{Y} = \underbrace{\frac{1}{1-\lambda}\left[d\boldsymbol{G} - \lambda d\boldsymbol{T}\right]}_{d\tilde{\boldsymbol{G}}_t} - \boldsymbol{M}^{RA} d\boldsymbol{T}$$

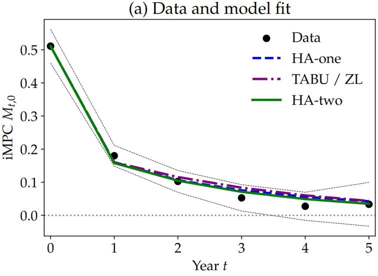
- **Same solution-form as RA:** $d\boldsymbol{Y} = d\tilde{\boldsymbol{G}}_t$

$$d\boldsymbol{Y} = d\tilde{\boldsymbol{G}}_t = d\boldsymbol{G}_t + \frac{\lambda}{1-\lambda}\left[d\boldsymbol{G} - dT\right]$$

- Still a cumulative multiplier of 1 (both for RA and HtM)

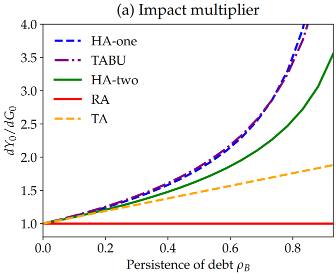
iMPCs in models

Figure 2: iMPCs in the Norwegian data and several models



Multipliers and debt-financing

Figure 5: Multipliers according to the IKC



Note. These figures assume a persistence of government spending equal to $\rho_G = 0.76$, and vary ρ_B in $dB_t = \rho_B(dB_{t-1} + dG_t)$. See section 7.1 for details on calibration choices.

Generalized IKC

- **Budget constraint** can be written with initial capital gain

$$a_t + c_t = (Y_t - T_t)z_t + \chi_t + \begin{cases} (1 + r_{t-1})a_{t-1} & \text{if } t > 0 \\ (1 + r_{ss} + \text{cap}_0)a_{t-1} & \text{if } t = 0 \end{cases}$$

1. Real bond: $\text{cap}_0 = 0$
2. Nominal bond:

$$\text{cap}_0 = \frac{(1 + r_{ss})(1 + \pi_{ss})}{1 + \pi_0} - (1 + r_{ss})$$

3. Long-term bond:

$$\text{cap}_0 = \frac{1 + \delta q_0}{q_{-1}} - (1 + r_{ss})$$

Generalized IKC

- **Consumption-function** $C_t^{hh} = C^{hh}(\boldsymbol{Y} - \boldsymbol{T}, \chi, \boldsymbol{r}, \mathbf{cap}_0)$ implies

$$d\boldsymbol{C} = \boldsymbol{M}(d\boldsymbol{Y} - d\boldsymbol{T}) + \boldsymbol{M}^{\chi} d\boldsymbol{\chi} + \boldsymbol{M}^{r} d\boldsymbol{r} + \boldsymbol{m}^{\text{cap}}\text{cap}_0$$

  where

$$\boldsymbol{M}^{\chi}_{t,s} = \left[\frac{\partial C_t^{hh}}{\partial \chi_s}\right], \boldsymbol{M}^{r}_{t,s} = \left[\frac{\partial C_t^{hh}}{\partial r_s}\right], \boldsymbol{m}^{\text{cap}}_t = \left[\frac{\partial C_t^{hh}}{\partial \text{cap}_0}\right]$$

- Why are $\boldsymbol{M}^{\chi}$ and $\boldsymbol{M}$ different?

Sticky prices

Overview

- **Households:**
  1. Differ by stochastic idiosyncratic productivity and savings
  2. Supply labor and choose consumption
  3. Subject to a borrowing constraint
- **Intermediary goods firms** (continuum)
  1. Produce differentiated goods with labor
  2. Set price under monopolistic competition
  3. Pay dividends to households
- **Final goods firms** (representative)
  1. Produce final good with intermediary goods
  2. Take price as given under perfect competition
- **Government:**
  1. Collect taxes from households
  2. Pays interest on government debt and choose public consumption
- **Central bank**: Set nominal interest rate

Final goods firms

- Intermediary goods indexed by $j \in [0, 1]$
- ***Static* problem for representative final good firm:**

$$\max_{y_{jt}\ \forall j} P_t Y_t - \int_0^1 p_{jt} y_{jt} dj \text{ s.t. } Y_t = \left(\int_0^1 y_{jt}^{\frac{1}{\mu}} dj \right)^{\mu}$$

for given output price, P_t, and input prices, p_{jt}

- **Demand curve** derived from FOC wrt. y_{jt}

$$\forall j : y_{jt} = \left(\frac{p_{jt}}{P_t} \right)^{-\frac{\mu}{\mu - 1}} Y_t$$

- **Note:** Zero profits (can be used to derive price index)

Derivation of demand curve

- **FOC wrt.** y_{jt}

$$
\begin{aligned}
0 &= P_t \mu \left(\int_0^1 y_{jt}^{\frac{1}{\mu}} \, dj \right)^{\mu - 1} \frac{1}{\mu} y_{jt}^{\frac{1}{\mu} - 1} - p_{jt} \Leftrightarrow \\
\frac{p_{jt}}{P_t} &= \left(\int_0^1 y_{jt}^{\frac{1}{\mu}} \, dj \right)^{\mu - 1} y_{jt}^{\frac{1-\mu}{\mu}} \Leftrightarrow \\
\left(\frac{p_{jt}}{P_t} \right)^{\frac{\mu}{\mu - 1}} &= \left(\int_0^1 y_{jt}^{\frac{1}{\mu}} \, dj \right)^{\mu} y_{jt}^{-1} \Leftrightarrow \\
y_{jt} &= \left(\frac{p_{jt}}{P_t} \right)^{-\frac{\mu}{\mu - 1}} Y_t
\end{aligned}
$$

Intermediary goods firms

- ***Dynamic* problem for intermediary goods firms:**

$$J_t(p_{jt-1}) = \max_{y_{jt},p_{jt},n_{jt}} \left\{ \frac{p_{jt}}{P_t} y_{jt} - w_t n_{jt} - \Omega(p_{jt}, p_{jt-1}) Y_t + \frac{J_{t+1}(p_{jt})}{1+r_{t+1}} \right\}$$

$$\text{s.t. } y_{jt} = Z_t n_{jt}, \;\; y_{jt} = \left(\frac{p_{jt}}{P_t}\right)^{-\frac{\mu}{\mu-1}} Y_t$$

$$\Omega(p_{jt}, p_{jt-1}) = \frac{\mu}{\mu-1}\frac{1}{2\kappa}\left[\log\left(\frac{p_{jt}}{p_{jt-1}}\right)\right]^2$$

- **Symmetry:** In equilibrium all firms set the same price, $p_{jt} = P_t$
- **NKPC** derived from FOC wrt. p_{jt} and envelope condition:

$$\log(1+\pi_t) = \kappa\left(\frac{w_t}{Z_t} - \frac{1}{\mu}\right) + \frac{Y_{t+1}}{Y_t}\frac{\log(1+\pi_{t+1})}{1+r_{t+1}}, \;\; \pi_t \equiv P_t/P_{t-1} - 1$$

- **Implied production:** $Y_t = y_{jt}$, $N_t = n_{jt}$ (from symmetry)
- **Implied dividends:** $d_t = Y_t - w_t N_t - \frac{\mu}{\mu-1}\frac{1}{2\kappa}\left[\log(1+\pi_t)\right]^2 Y_t$

Derivation of NKPC

- **FOC wrt. p_{jt}:**

$$0 = \left(1 - \frac{\mu}{\mu - 1}\right)\left(\frac{p_{jt}}{P_t}\right)^{-\frac{\mu}{\mu-1}} \frac{Y_t}{P_t} + \frac{\mu}{\mu - 1}\frac{w_t}{Z_t}\left(\frac{p_{jt}}{P_t}\right)^{-\frac{\mu}{\mu-1}} \frac{Y_t}{p_{jt}}$$

$$- \frac{\mu}{\mu - 1}\frac{1}{\kappa}\frac{\log\left(\frac{p_{jt}}{p_{jt-1}}\right)}{p_{jt}} Y_t + \frac{J'_{t+1}(p_{jt})}{1 + r_{t+1}}$$

- **Envelope condition:** $J'_{t+1}(p_{jt}) = \frac{\mu}{\mu-1}\frac{1}{\kappa}\frac{\log\left(\frac{p_{jt+1}}{p_{jt}}\right)}{p_{jt}} Y_{t+1}$
- **FOC + Envelope + Symmetry +** $\pi_t = P_t / P_{t-1} - 1$

$$0 = \left(1 - \frac{\mu}{\mu - 1}\right)\frac{Y_t}{P_t} + \frac{\mu}{\mu - 1}\frac{w_t}{Z_t}\frac{Y_t}{P_t}$$

$$+ \frac{\mu}{\mu - 1}\frac{1}{\kappa}\log\left(1 + \pi_t\right)\frac{Y_t}{P_t} + \frac{\frac{\mu}{\mu-1}\frac{1}{\kappa}\log\left(1 + \pi_{t+1}\right)\frac{Y_{t+1}}{P_t}}{1 + r_{t+1}}$$

Households

- **Household problem**: Distribution, $\boldsymbol{D}_t$, over z_t and a_{t-1}

$$v_t(z_t, a_{t-1}) = \max_{c_t} \frac{c_t^{1-\sigma}}{1-\sigma} - \varphi \frac{\ell_t^{1+\nu}}{1+\nu} + \beta \mathbb{E}\left[v_{t+1}(z_{t+1}, a_t) \,|\, z_t, a_t\right]$$

$$\text{s.t. } a_t = (1+r_t)a_{t-1} + (w_t \ell_t - \tau_t + d_t)z_t - c_t \geq 0$$

$$\log z_{t+1} = \rho_z \log z_t + \psi_{t+1} \ \ , \psi_t \sim \mathcal{N}(\mu_\psi, \sigma_\psi), \ \mathbb{E}[z_t] = 1$$

- **Dividends:** Distributed proportional to productivity (ad hoc)
- **Taxes:** Collected proportional to productivity (ad hoc)
- **Optimality conditions:**

$$\text{FOC wrt. } c_t : 0 = c_t^{-\sigma} - \beta \mathbb{E}_t\left[v_{a,t+1}(z_{t+1}, a_t)\right]$$

$$\text{FOC wrt. } \ell_t : 0 = w_t z_t \beta \mathbb{E}_t\left[v_{a,t+1}(z_{t+1}, a_t)\right] - \varphi \ell_t^\nu$$

$$\text{Envelope condition: } v_{a,t}(z_t, a_{t-1}) = (1+r_t)c_t^{-\sigma}$$

- **Effective labor-supply:** $n_t = z_t \ell_t$

EGM I

- **Beginning-of-period value function:**

$$\underline{v}_{a,t}(z_{t-1}, a_{t-1}) = \mathbb{E}_t \left[v_{a,t}(z_t, a_{t-1}) \right] = \mathbb{E}\left[(1+r_t) c_t^{-\sigma} \right]$$

- **Endogenous grid method:** Vary z_t and a_t to find

$$c_t = \left(\beta \underline{v}_{a,t+1}(z_t, a_t)\right)^{-\frac{1}{\sigma}}$$

$$\ell_t = \left(\frac{w_t z_t}{\varphi} c_t^{-\sigma} \right)^{\frac{1}{\nu}}$$

$$m_t = c_t + a_t - (w_t \ell_t - \tau_t + d_t) z_t$$

- **Consumption and labor supply:** Use linear interpolation to find

$$c^*(z_t, a_{t-1}) \text{ and } \ell^*(z_t, a_{t-1}) \text{ with } m_t = (1+r_t) a_{t-1}$$

- **Savings:** $a^*(z_t, a_{t-1}) = (1+r_t) a_{t-1} - c_t^* + (w_t \ell_t^* - \tau_t + d_t) z_t$

EGM II

- **Problem:** $a^*(z_t, a_{t-1}) < 0$ violate borrowing constraint
- **Refinement if** $a^*(z_t, a_{t-1}) < 0$ **by:**

  Find ℓ^* (and c^* and n^*) with *Newton solver* assuming $a^* = 0$

  1. Stop if $f(\ell^*) = \ell^* - \left(\frac{w_t z_t}{\varphi}\right)^{\frac{1}{\nu}} (c^*)^{-\frac{\sigma}{\nu}} <$ tol. where

     $$\begin{aligned} c^* &= (1 + r_t)a_{t-1} + (w_t \ell^* - \tau_t + d_t) z_t \\ n^* &= z_t \ell^* \end{aligned}$$

  2. Set

     $$\ell^* = \frac{f(\ell^*)}{f'(\ell^*)} = \frac{f(\ell^*)}{1 - \left(\frac{w_t z_t}{\varphi}\right)^{\frac{1}{\nu}} \left(-\frac{\sigma}{\nu}\right) (c^*)^{-\frac{\sigma}{\nu}} w_t z_t}$$

  3. Return to step 1

Government and central bank

- **Monetary policy:** Folow Taylor-rule:

$$i_t = i_t^* + \phi \pi_t$$

where i_t^* is a shock

- **Fisher relationship:**

$$r_t = (1 + i_{t-1})/(1 + \pi_t) - 1$$

- **Government:** Choose τ_t to keep debt constant and finance exogenous public consumption

$$\tau_t = r_t B_{ss} + G_t$$

Market clearing

1. Assets: $B_{ss} = \int a_t^*(z_t, a_{t-1}) d\boldsymbol{D}_t$
2. Labor: $N_t = \int n_t^*(z_t, a_{t-1}) d\boldsymbol{D}_t$ (in effective units)
3. Goods: $Y_t = \int c_t^*(z_t, a_{t-1}) d\boldsymbol{D}_t + G_t + \frac{\mu}{\mu-1} \frac{1}{2\kappa} \left[\log\left(1+\pi_t\right)\right]^2 Y_t$

As an equation system

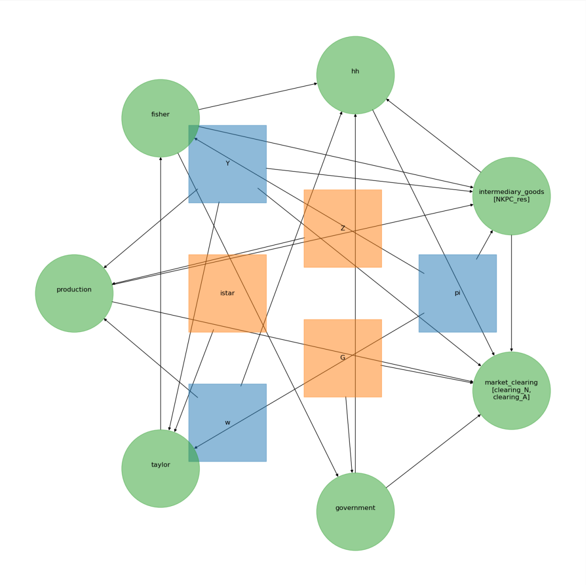
$$\boldsymbol{H}(\boldsymbol{\pi}, \boldsymbol{w}, \boldsymbol{Y}, \boldsymbol{i}^*, \boldsymbol{Z}, \underline{\boldsymbol{D}}_0) = \mathbf{0}$$

$$\begin{bmatrix} \log(1+\pi_t) - \left[\kappa\left(\frac{w_t}{Z_t} - \frac{1}{\mu}\right) + \frac{Y_{t+1}}{Y_t}\frac{\log(1+\pi_{t+1}}{1+r_{t+1}})\right] \\ N_t - \int n_t^*(z_t, a_{t-1}) d\boldsymbol{D}_t \\ B_{ss} - \int a_t^*(z_t, a_{t-1}) d\boldsymbol{D}_t \end{bmatrix} = \mathbf{0}$$

The rest of the model is given by

$$\boldsymbol{X} = \boldsymbol{M}(\boldsymbol{\pi}, \boldsymbol{w}, \boldsymbol{Y}, \boldsymbol{i}^*, \boldsymbol{Z})$$

As a DAG



hh
fisher
Y
intermediary_goods
[NKPC_res]
Z
production
istar
pi
G
market_clearing
[clearing_N,
clearing_A]
w
taylor
government

Steady state

- **Chosen:** B_{ss}, G_{ss}, r_{ss}
- **Analytically:**
  1. **Normalization:** $Z_{ss} = N_{ss} = 1$
  2. **Zero-inflation:** $\pi_{ss} = 0 \Rightarrow i_{ss} = i_{ss}^{*} = (1 + r_{ss})(1 + \pi_{ss}) - 1$
  3. **Firms:** $Y_{ss} = Z_{ss}N_{ss}$, $w_{ss} = \frac{Z_{ss}}{\mu}$ and $d_{ss} = Y_{ss} - w_{ss}N_{ss}$
  4. **Government:** $\tau_{ss} = r_{ss}B_{ss} + G_{ss}$
  5. **Assets:** $A_{ss} = B_{ss}$
- **Numerically:** Choose β and φ to get market clearing

Transmission mechanism to monetary policy shock

1. **Monetary policy shock:** $i_t^* \downarrow \Rightarrow i_t = i_t^* + \phi\pi_t \downarrow$
2. **Real interest rate:** $r_t = \frac{1+i_{t-1}}{1+\pi_t} \downarrow$
3. **Taxes:** $\tau_t = r_t B_{ss} \downarrow$
4. **Household consumption**, $C_t^{hh} \uparrow$, due to $r_t \downarrow$ and $\tau_t \downarrow$
5. **Firms production,** $Y_t \uparrow$, and **labor demand**, $N_t \uparrow$
6. **Inflation,** $\pi_t \uparrow$, and **wage**, $w_t \uparrow$ and **dividends**, $d_t \downarrow$
7. **Household labor supply**, $N_t^{hh} \uparrow$, due to $w_t \uparrow$ and $d_t \downarrow$, but dampened $\tau_t \downarrow$
8. **Nominal rate**, $i_t \uparrow$ due to $\pi_t \uparrow$ implying $r_t \uparrow$
9. **Household consumption**, $C_t^{hh} \uparrow$, due to $w_t \uparrow$ but dampened by $d_t \downarrow$ and $r_t \uparrow$

Exercise

Exercise

Use *HANK-sticky-wages* in sub-folder.

1. Compute *fiscal multipliers* varying:
   1.1 Bond maturity: δ
   1.2 Fiscal aggressiveness: ω
   1.3 Monetary aggressiveness: ϕ_π
2. Does the model match the evidence of intertemporal MPCs? What happens to the fiscal multiplier if the fit is improved?

Summary

Summary

- **Today:**
  1. Aggregate risk and linearized dynamics (IRF and simulation)
  2. Calculating aggregate moments (for calibration or estimation)
  3. HANK with sticky wages and/or prices
  4. Intertemporal Keynesian Cross
  5. Analysis of fiscal multipliers
- **Next:** Examples
  1. **I-HANK:**
     Auclert, et. al. (2021), »Exchange Rates and Monetary Policy with Heterogeneous Agents: Sizing up the Real Income Channel«
     Druedahl et al. (2022),
     »The Transmission of Foreign Demand Shocks«
  2. **HANK-SAM:**
     Broer et. al. (2023),
     »Fiscal stimulus policies according to HANK-SAM«