CENTER FOR
ECONOMIC
BEHAVIOR &
INEQUALITY

1. Recap of stationary equilibrium

Lectures at IIES

Jeppe Druedahl

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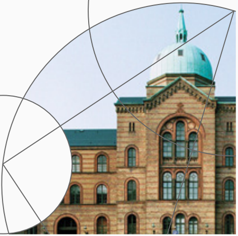


Introduction

- **Ultimate learning goal:** *Analyze heterogeneous agent models using sequence-space methods*
- **Stationary equilibrium** + *dynamic effects of shocks and policies*
  1. Long-run structural transformations
  2. Short-run business cycle fluctuations
- **Lectures:**
  1. Recap of consumption-saving and stationary equilibrium
  2. Transitional dynamics in sequence space
  3. Aggregate risk, linearized dynamics and analytical analysis
  4. Open-Economy HANK + HANK with search-and-matching (SAM)
- **Exercises:**
  1. Using the numerical methods in practice
  2. In Python using my *GEModelTools* package

Macroeconomic models with heterogeneous agents

- **Models:**
  1. **HANC:** Heterogeneous Agent *Neo-Classical* model
     (Aiyagari-Bewley-Hugget-Imrohoroglu, Standard Incomplete Markets)
  2. **HANK:** Heterogeneous Agent *New Keynesian* model
     (i.e. include price and wage setting frictions)
- **History:**
  1. Heathcote et al. (2009), »Quantitative Macroeconomics with Heterogeneous Households«
  2. Kaplan and Violante (2018), »Microeconomic Heterogeneity and Macroeconomic Shocks«
  3. Cherrier et al. (2023), »Household Heterogeneity in Macroeconomic Models: A Historical Perspective«

Recap: Consumption-saving

1. Permanent income hypothesis (PIH)
2. MPC (marginal propensity to consume)
3. Liquidity/borrowing constraints
4. Euler-equation
5. Natural borrowing constraint
6. Buffer-stock target

Recap: Dynamic programming

1. Bellman equation (state and control variables, continuation value)
2. Value and policy functions
3. Beginning- and end-of-period value functions
4. Discrete grids and linear interpolation
5. Discrete transition probabilities from stochastic processes (Tauchen, Rouwenhorst, Gauss-Hermite quadrature)
6. Value function iteration (VFI)
7. Endogenous grid point method (EGM)
8. Monte carlo simulation (stochastic)
9. Histogram simulation (deterministic)

Recap: Stationary equilibrium

1. Neoclassical firm (Ramsey)
2. Market clearing
3. Ex ante and ex post heterogeneity
4. Incomplete vs. complete markets (partial insurance)
5. Root-finding

Consumption-saving

Generations of models

1. Permanent income hypothesis (Friedman, 1957)
   or life-cycle model (Modigliani and Brumburg, 1954)
2. Buffer-stock consumption model
   (Deaton, 1991, 1992; Carroll, 1992, 1997)
3. Multiple-asset buffer-stock consumption models
   (e.g. Kaplan and Violante (2014))

Consumption-saving

$$v_0 = \max_{\{c_t\}_{t=0}^{T-1}} \sum_{t=0}^{T-1} \beta^t u(c_t)$$

$$\text{s.t.}$$

$$a_t = (1+r)a_{t-1} + wz_t - c_t$$

$$a_{T-1} \geq 0$$

- **Variables:**
  Consumption: c_t
  Productivity: z_t
  End-of-period savings: a_t (*no debt at death*)
- **Parameters:**
  Discount factor: β
  Wage: w
  Interest rate: r (define $R \equiv 1 + r$ as interest factor)

It is a *static* problem

$$
\begin{aligned}
v_0 &= \max_{\{c_t\}_{t=0}^{T-1}} \sum_{t=0}^{T-1} \beta^t u(c_t) \\
&\text{s.t.} \\
a_t &= (1+r)a_{t-1} + wz_t - c_t \\
a_{T-1} &\geq 0
\end{aligned}
$$

- **It is a *static* problem:**
  1. **Information:** z_t is known for all t at $t = 0$
  2. **Target:** Discounted utility, $\sum_{t=0}^{T-1} \beta^t u(c_t)$
  3. **Behavior:** Choose $c_0, c_1, \ldots, c_{T-1}$ *simultaneously*
  4. **Solution:** Sequence of consumption *choices* $c_0^*, c_1^*, \ldots, c_{T-1}^*$

IBC

- **Substitution** implies *Intertemporal Budget Constraint* (IBC)

$$\begin{aligned} a_{T-1} &= Ra_{T-2} + wz_{T-1} - c_{T-1} \\ &= R^2 a_{T-3} + Rwz_{T-2} - Rc_{T-2} + wz_{T-1} - c_{T-1} \\ &= R^T a_{-1} + \sum_{t=0}^{T-1} R^{T-1-t}(wz_t - c_t) \end{aligned}$$

- Use **terminal condition** $a_{T-1} = 0$ (equality due utility max.)

$$R^{-(T-1)} a_{T-1} = 0 \quad \Leftrightarrow \quad s_0 + h_0 - \sum_{t=0}^{T-1} R^{-t} c_t = 0$$

where $s_0 \equiv Ra_{-1}$ (after-interest assets)
and $h_0 \equiv \sum_{t=0}^{T-1} R^{-t} wz_t$ (human capital)

FOC and Euler-equation

$$\mathcal{L} = \sum_{t=0}^{T-1} \beta^t u(c_t) + \lambda \left[\sum_{t=0}^{T-1} R^{-t} c_t - s_0 - h_0 \right]$$

- **First order conditions**:

$$\forall t : 0 = \beta^t u'(c_t) - \lambda (1+r)^{-t} \Leftrightarrow u'(c_t) = -\lambda \left(\beta R \right)^{-t}$$

- **Euler-equation** for $k \in \{1, 2, \dots\}$:

$$\frac{u'(c_t)}{u'(c_{t+k})} = \frac{-\lambda \left(\beta R \right)^{-t}}{-\lambda \left(\beta R \right)^{-(t+k)}} = \left(\beta R \right)^k$$

Consumption choice

- **CRRA:** $u(c_t) = \frac{c_t^{1-\sigma}}{1-\sigma}$ imply Euler-equation

$$\frac{c_0^{-\sigma}}{c_t^{-\sigma}} = (\beta R)^t \Leftrightarrow c_t = (\beta R)^{\frac{t}{\sigma}} c_0$$

- Insert **Euler** into **IBC** to get consumption choice

$$\begin{aligned} \sum_{t=0}^{T-1} R^{-t} (\beta R)^{t/\sigma} c_0 &= s_0 + h_0 \Leftrightarrow \\ c_0^* &= \frac{1-(\beta R)^{1/\sigma} R^{-1}}{1-\left((\beta R)^{1/\sigma} R^{-1}\right)^T} (s_0 + h_0) \end{aligned}$$

Infinite horizon

- **Infinite horizon** for $(\beta R)^{1/\sigma} R^{-1} < 1$: Let $T \to \infty$ to get

$$c_0^* = \left(1 - \frac{(\beta R)^{1/\sigma}}{R}\right)(s_0 + h_0)$$

- **Interesting properties** are e.g.:
  1. Interest rate sensitivity: $\frac{\partial c_0}{\partial r}$
  2. MPC of permanent income change: $\frac{\partial c_0}{\partial w}$
  3. MPC of future income: $\frac{\partial c_0}{\partial z_t}$
  4. MPC of windfall income: $\frac{\partial c_0}{\partial s_0}$
     Small when $\beta R \approx 1$ and $1 - R^{-1} \approx r \Rightarrow \frac{\partial c_0}{\partial s_0} \approx r$
- **No borrowing constraints or uncertainty**
- **Other simplifications:** No age life-cycle, bequests etc.

Initial liquidity/borrowing constraint

- Implied period 0 **savings** are:

$$a_0 = Ra_{-1} + wz_0 - c_0$$

- **Borrowing constraint:** $a_0 \geq -w \cdot b$
- **Maximum consumption:** $\overline{c}_0 = Ra_{-1} + wz_0 + wb$
- **Optimal consumption:** Constrained or unconstrained.

$$c_0^* = \min\left\{\overline{c}_0, \left(1 - \frac{(\beta R)^{1/\sigma}}{R}\right)(s_0 + h_0)\right\}$$

- **Empirical realism.** Incl. high MPC of constrained.
- **Technical issue:** Borrowing constraints further in the future complicates the analytical solution considerably.

Uncertainty and always borrowing constraint

$$v_0(z_0, a_{-1}) = \max_{\{c_t\}_{t=0}^{\infty}} \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t u(c_t) \right]$$

s.t.

$$a_t = (1+r)a_{t-1} + wz_t - c_t$$

$$z_{t+1} \sim \mathcal{Z}(z_t)$$

$$a_t \geq -wb$$

$$\lim_{t \to \infty} (1+r)^{-t} a_t \geq 0 \quad \text{[No-Ponzi game]}$$

- **Stochastic income** from 1st order Markov-process, $\mathcal{Z}$
- ***A true dynamic problem:***
    1. **Information:** z_t is revealed period-by-period
    2. **Target:** *Expected* discounted utility, $\mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t u(c_t) \right]$
    3. **Behavior:** Choose c_t *sequentially* as information is revealed
    4. **Solution:** Sequence of consumption *functions*, $c_t^*(z_t, a_{t-1})$

IBC

- **Substitution** still implies:

$$R^{-(T-1)}a_{T-1} = 0 \quad \Leftrightarrow \quad s_0 + h_0 - \sum_{t=0}^{T-1} R^{-t} c_t = 0$$

- **What if $T \to \infty$?** We must have $\lim_{T\to\infty} R^{-(T-1)}a_{T-1} = 0$
  1. $\lim_{T\to\infty} R^{-(T-1)}a_{T-1} > 0$: Consumption can be increased
  2. $\lim_{T\to\infty} R^{-(T-1)}a_{T-1} < 0$: Violates No-Ponzi game condition
- For $T \to \infty$ we have the **IBC**:

$$\sum_{t=0}^{\infty} R^{-t} c_t = Ra_{-1} + \sum_{t=0}^{\infty} R^{-t} w z_t$$

Natural borrowing constraint

- Denote **minimum possible productivity** by $\underline{z}$
- **Consumption must be non-negative** $\Rightarrow$
  interest payments must be less than minimum income

$$c_t \geq 0 \Rightarrow r(-a_t) \leq w\underline{z} \Leftrightarrow a_t \geq -\frac{w\underline{z}}{r}$$

  If debt was larger it would in the worst case ($\forall z_t = \underline{z}$) grow without bound even with zero consumption ($\forall c_t = 0$)

$$\begin{aligned}
a_0 &= -\frac{w\underline{z}}{r} - \Delta \\
a_1 &= (1+r)a_0 + w\underline{z} = a_0 - (1+r)\Delta \\
a_2 &= (1+r)a_1 + w\underline{z} = a_0 - (1+r)^2\Delta \\
&\vdots
\end{aligned}$$

- **Natural borrowing constraint:** $a_t \geq \underline{a} = -w \min\left\{b, \frac{\underline{z}}{r}\right\}$

Euler-equation from variation argument

- **Case I:** If $u'(c_t) > \beta R \mathbb{E}_t [u'(c_{t+1})]$:
  Increase c_t by marginal $\Delta > 0$, and lower c_{t+1} by $R\Delta$
  1. **Feasible:** Yes, if $a_t > \underline{a}$
  2. **Utility change:** $u'(c_t) + \beta(-R)\mathbb{E}_t [u'(c_{t+1})] > 0$

- **Case II:** If $u'(c_t) < \beta R \mathbb{E}_t [u'(c_{t+1})]$:
  Lower c_t by marginal $\Delta > 0$, and increase c_{t+1} by $R\Delta$
  1. **Feasible:** Yes (always)
  2. **Utility change:** $u'(c_t) + \beta R \mathbb{E}_t [u'(c_{t+1})] > 0$

- **Conclusion:** By contradiction
  1. **Constrained:** $a_t = \underline{a}$ and $u'(c_t) \geq \beta R \mathbb{E}_t [u'(c_{t+1})]$, or
  2. **Unconstrained:** $a_t > \underline{a}$ and $u'(c_t) = \beta R \mathbb{E}_t [u'(c_{t+1})]$

Special case I: Quadratic utility

- **Quadratic utility:** $u(c_t) = -\frac{1}{2}(\overline{c} - c)^2$ with $\beta R = 1$ and »large« $\overline{c}$
- **Euler-equation:** *Consumption = expected future consumptio,n*

$$(\overline{c} - c_t) = \mathbb{E}_t\left[(\overline{c} - c_{t+k})\right] \Leftrightarrow c_t = \mathbb{E}_t\left[c_{t+k}\right]$$

- Use **IBC** in expectation to get **consumption function:**

$$\sum_{t=0}^{\infty} R^{-t}\mathbb{E}_0\left[c_t\right] = Ra_{-1} + \sum_{t=0}^{\infty} R^{-t} w\mathbb{E}_0\left[z_t\right] \Rightarrow$$

$$c^*(z_t, a_{t-1}) = c_0 = ra_{-1} + \frac{r}{R}\sum_{t=0}^{T} R^{-t} w\mathbb{E}_0\left[z_t\right]$$

where we formally disregard the borrowing constraint

- **Certainty equivalence:** *Only expected income matter.*

Special case II: CARA utility

- **CARA utility:** $u(c_t) = -\frac{1}{\alpha} e^{-\alpha c}$
- **Productivity is absolute random walk:**

$$\begin{aligned} z_t &= z_{t-1} + \psi_t \\ \psi_t &\sim \mathcal{N}(0, \sigma_\psi^2) \end{aligned}$$

- **Consumption function** (see proof)**:**

$$c^*(a_{t-1}, z_t) = r a_{t-1} + w z_t - \frac{\log{(\beta R)^{\frac{1}{\alpha}}} + \alpha \frac{\sigma_\psi^2}{2}}{r^2}$$

where we formally disregard the borrowing constraint

- **Precautionary saving:** $\sigma_\psi^2 \uparrow$ implies $c_t^* \downarrow$ for given z_t and a_{t-1}
$\Rightarrow$ *accumulation of buffer-stock*

Further resources

1. Lecture notes by Christopher Carroll
2. Lecture notes by Pierre-Olivier Gourinchas
3. The Economics of Consumption, Jappelli and Pistaferri (2017)
4. »Liquidity constraints and precautionary saving«
   Carroll, Holm, Kimball (JET, 2021)
5. »Theoretical Foundations of Buffer Stock Saving«
   Carroll (QE, forthcomming)

Dynamic solution: Bellman's Principle of Optimality

- **In words:** *An optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision.* (See Bellman, 1957, Chap. III.3.)
- **In math:**
  1. **Value function, v_t:** Defined *recursively* from

$$v_t(z_t, a_{t-1}) = \max_{c_t} u(c_t) + \beta \mathbb{E}_t[v_{t+1}(z_{t+1}, a_t)]$$
$$\text{s.t. } a_t = (1+r)a_{t-1} + wz_t - c_t \geq \underline{a}$$

  with $v_T(\bullet) = 0$.

  2. **Policy function, c_t^*:** Is the same as

$$c_t^*(z_t, a_{t-1}) = \arg\max_{c_t} u(c_t) + \beta \mathbb{E}_t[v_{t+1}(z_{t+1}, a_t)]$$
$$\text{s.t. } a_t = (1+r)a_{t-1} + wz_t - c_t \geq \underline{a}$$

Vocabulary

$$v_t(z_t, a_{t-1}) = \max_{c_t} u(c_t) + \beta \mathbb{E}_t[v_{t+1}(z_{t+1}, a_t)]$$

$$\text{s.t. } a_t = (1+r)a_{t-1} + wz_t - c_t \geq \underline{a}$$

1. **State variables:** z_t and a_{t-1}
2. **Control variable:** c_t
3. **Continuation value:** $\beta \mathbb{E}_t[v_{t+1}(z_{t+1}, a_t)]$
4. **Parameters:** r, w, and stuff in $u(\bullet)$

Note: Straightforward to extend to more goods, more assets or other states, more complex uncertainty, bounded rationality etc.

Infinite horizon: $T \to \infty$?

$$v_t(z_t, a_{t-1}) = \max_{c_t} u(c_t) + \beta \mathbb{E}_t[v_{t+1}(z_{t+1}, a_t)]$$

$$\text{s.t. } a_t = (1+r)a_{t-1} + wz_t - c_t \geq \underline{a}$$

- **Contraction mapping result:** *If β is low enough (strong enough impatience) then the value and policy functions converge to $v(z_t, a_{t-1})$ and $c^*(z_t, a_{t-1})$ for large enough T*
- **Maximum upper limit for β:** $\frac{1}{1+r}$
- **In practice:**
  1. Make arbitrary initial guess (e.g. $v_{t+1} = 0$)
  2. Solve backwards until value and policy functions does not change anymore (given some tolerance)

Dynamic programming

Recursive problem: Timing of shocks

- **Realization of shocks:** *First in the period before choices are made*
- **Beginning-of-period value function** (before realization)**:**

$$\underline{v}_t(z_{t-1}, a_{t-1}) = \mathbb{E}\left[v_t(z_t, a_{t-1}) \,|\, z_{t-1}, a_{t-1}\right]$$
$$z_t \sim \mathcal{Z}(z_{t-1})$$

- **End-of-period value function** (after realization)**:**

$$v_t(z_t, a_{t-1}) = \max_{c_t} u(c_t) + \beta \underline{v}_{t+1}(z_t, a_t)$$
$$\text{s.t.}$$
$$a_t = (1+r)a_{t-1} + wz_t - c_t$$
$$a_t \geq \underline{a}$$

- **FOC:** $u'(c_t) = \beta \underline{v}_{a,t+1}(z_t, a_t)$
- **Envelope condition:** $\underline{v}_{a,t}(z_{t-1}, a_{t-1}) = \mathbb{E}\left[(1+r)u'(c_t)\right]$

Discretization and linear interpolation

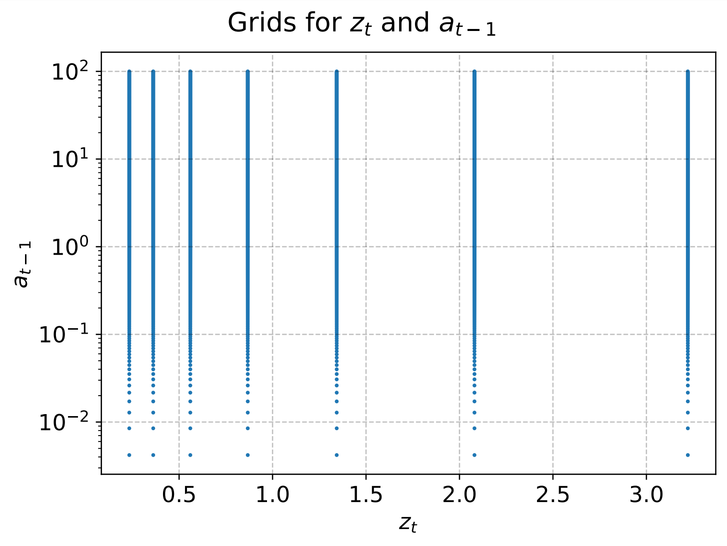
- **Discretization:** All state variables belong to discrete sets $\equiv$ *grids*,

$$
\begin{aligned}
z_t &\in \mathcal{G}_z = \{z^0, z^1, \dots, z^{\#_z - 1}\} \\
a_t &\in \mathcal{G}_a = \{a^0, a^1, \dots, a^{\#_a - 1}\} \\
a^0 &= \underline{a}
\end{aligned}
$$

- **Transition probabilities:** $\pi_{i_{z-}, i_z} = \Pr[z_t = z^{i_z} \mid z_{t-1} = z^{i_{z-}}]$
- **Linear interpolation** (function approximation):
  1. Assume $\underline{v}_{t+1}$ is known on $\mathcal{G}_z \times \mathcal{G}_a$ (tensor product)
  2. Evaluate $\underline{v}_{t+1}(z^{i_z}, a)$ for arbitrary a by

$$
\begin{aligned}
\underline{\breve{v}}_{t+1}(z^{i_z}, a) &= \underline{v}_{t+1}(z^{i_z}, a^{\iota}) + \omega(a - a^{\iota}) \\
\omega &\equiv \frac{v_{t+1}(z^{i_z}, a^{\iota+1}) - v_{t+1}(z^{i_z}, a^{\iota})}{a^{\iota+1} - a^{\iota}} \\
\iota &\equiv \text{largest } i_a \in \{0, 1, \dots, \#_a - 2\} \text{ such that } a^{i_a} \leq a
\end{aligned}
$$

Grids



Deriving transition probabilities

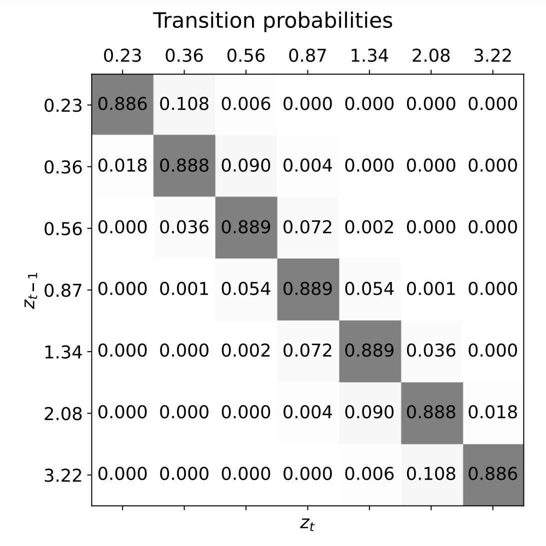
- **Specification:** Assume

$$z_t = \tilde{z}_t \xi_t, \;\; \log \xi_t \sim \mathcal{N}(\mu_\xi, \sigma_\xi)$$
$$\log \tilde{z}_{t+1} = \rho_z \log \tilde{z}_t + \psi_{t+1}, \;\; \psi_{t+1} \sim \mathcal{N}(\mu_\psi, \sigma_\psi)$$

  where μ_ξ and μ_ψ ensures $\mathbb{E}[\xi_t] = 1$, $\mathbb{E}[\tilde{z}_t] = 1$ and $\mathbb{E}[z_t] = 1$

- **Discretization of $\tilde{z}_t$:** Derive $\mathcal{G}_{\tilde{z}}$ and $\pi_{i_{\tilde{z}-}, i_{\tilde{z}}}$ given ρ_z and σ_ψ (e.g. using Tauchen (1986) or Rouwenhorst (1995))
- **Discretization of ξ_t:** Derive $\mathcal{G}_\xi$ and $\pi_{i_{\xi-}, i_\xi}$ given σ_ξ (e.g. using Gauss-Hermite quadrature)
- **Combined:** Derive $\mathcal{G}_z = \mathcal{G}_{\tilde{z}} \times \mathcal{G}_\xi$ (tensor product) and use independence of $\tilde{z}_t$ and ξ_t to get transition probabilities π_{i_{z-}, i_z} (kronecker product)

Transition probability matrix



Value function iteration (VFI)

- **Beginning-of-period value function:**

$$\underline{v}_t(z^{i_{z-}}, a^{i_{a-}}) = \sum_{i_z=0}^{\#_z-1} \pi_{i_{z-},i_z} v_t(z^{i_z}, a^{i_{a-}})$$

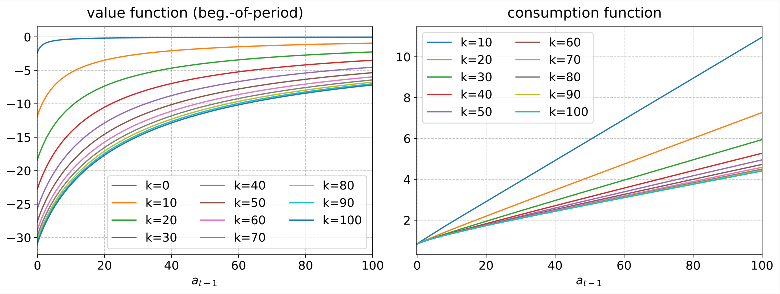
- **End-of-period value-of-choice:**

$$v_t(z^{i_z}, a^{i_{a-}}) = \max_{c_t} v_t(z^{i_z}, a^{i_{a-}} | c_t)$$
$$\text{with } c_t \in [0, (1+r)a^{i_{a-}} + wz^{i_z} + \underline{a}]$$

$$v_t(z^{i_z}, a^{i_{a-}} | c_t) = u(c_t) + \check{\underline{v}}_{t+1}(z^{i_z}, a_t)$$
$$\text{with } a_t = (1+r)a^{i_{a-}} + wz^{i_z} - c_t$$

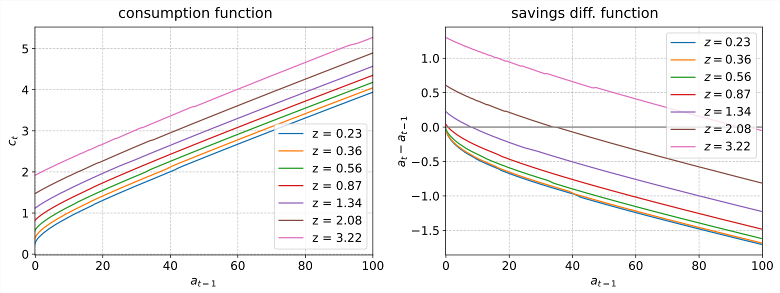
- **Inner loop:** For each grid point in $\mathcal{G}_z \times \mathcal{G}_a$ find $c_t^*(z_t, a_{t-1})$ and therefore $v_t(z_t, a_{t-1})$ with a *numerical optimizer*
- **Outer loop:** Backwards from $t = T-1$ (with $\underline{v}_T = 0$, or known)

Convergence ($t = T - 1 - k$)



with $z_t = 0.87$

Converged policy functions



Precautionary saving:

1. Consumption lower than without risk (same slope for $a_{t-1} \to \infty$)
2. Especially at low savings ($\to$ concave function in a_{t-1})

Buffer-stock target: $a_t = a_{t-1}$ for constant income *realizations*

Endogenous grid-point method (EGM)

- **Cash-on-hand:** $m_t \equiv (1+r)a_{t-1} + wz_t \Rightarrow a_t + c_t = m_t$
- **Solution step:** For each grid point in $\mathcal{G}_z \times \mathcal{G}_a$
  1. Find consumption by inverting FOC
  $$\tilde{c}_t(z^{i_z}, a^{i_a}) = u'^{-1}\left(\beta \underline{v}_{a,t+1}(z^{i_z}, a^{i_a})\right)$$
  2. Calculate endogenous grid: $\tilde{m}_t(z^{i_z}, a^{i_a}) = a^{i_a} + \tilde{c}_t(z^{i_z}, a^{i_a})$
  3. Calculate exogenous grid: $m_t(z^{i_z}, a^{i_{a-}}) = (1+r)a^{i_{a-}} + wz^{i_z}$
  4. Interpolate $\tilde{m}_t \to \tilde{c}_t$ at $m(z^{i_z}, a^{i_{a-}})$ to get $a_t^{\bullet}(z^{i_z}, a^{i_{a-}})$
  5. Enforce constraint by $a_t^*(z^{i_z}, a^{i_{a-}}) = \max\{a_t^{\bullet}(z^{i_z}, a^{i_{a-}}), \underline{a}\}$
  6. Consumption is $c_t^*(z^{i_z}, a^{i_{a-}}) = m_t(z^{i_z}, a^{i_{a-}}) - a_t^*(z^{i_z}, a^{i_{a-}})$
- **Expectation step:** For each grid point in $\mathcal{G}_z \times \mathcal{G}_a$

$$\underline{v}_{a,t}(z^{i_{z-}}, a^{i_{a-}}) = \sum_{i_z=0}^{\#_z-1} \pi_{i_{z-},i_z}(1+r)u'\left(c_t^*(z^{i_z}, a^{i_{a-}})\right)$$

Numerical simulation

Numerical Monte Carlo simulation

- **Initial distribution:** Draw $z_{i,-1}$ and $a_{i,-1}$ for $i \in \{0, 1, \ldots, N-1\}$
- **Simulation:** Forwards in time from $t = 0$ and in each time period
  1. Draw z_{it} given transition probabilities
  2. Use linear interpolation to evaluate

$$
\begin{aligned}
c_{it} &= \breve{c}_t^*(z_{it}, a_{it-1}) \\
a_{it} &= (1+r)a_{it-1} + wz_{it} - c_{it}
\end{aligned}
$$

- **Review:**
  - **Pro:** Simple to implement
  - **Con:** Computationally costly and introduces randomness

Numerical histogram simulation

- **Initial distribution:** Choose $\underline{\boldsymbol{D}}_0(z_{-1}, a_{-1})$, which is defined on $\mathcal{G}_z \times \mathcal{G}_a$ and sum to $1 \equiv$ *histogram*
- **Simulation:** Forwards in time from $t = 0$ and in each time period
  1. **Distribute stochastic mass:** For each i_z and i_{a-} calculate $\boldsymbol{D}_t(z^{i_z}, a^{i_{a-}}) = \sum_{i_{z-}=0}^{\#_z-1} \pi_{i_{z-},i_z} \underline{\boldsymbol{D}}_t(z^{i_{z-}}, a^{i_{a-}})$
  2. **Initial zero mass:** Set $\underline{\boldsymbol{D}}_{t+1}(z^{i_z}, a^{i_a}) = 0$ for all i_z and i_a
  3. **Distribute endogenous mass:** For each i_z and i_{a-} do
     - 3.1 Find $\iota \equiv$ largest $i_a \in \{0, 1, \dots, \#_a - 2\}$ such that $a^{i_a} \leq a_t^*(z^{i_z}, a^{i_{a-}})$
     - 3.2 Calculate $\omega = \frac{a^{\iota+1} - a^*(z^{i_z}, a^{i_{a-}})}{a^{\iota+1} - a^{\iota}} \in [0, 1]$
     - 3.3 Increment $\underline{\boldsymbol{D}}_{t+1}(z^{i_z}, a^{\iota})$ with $\omega \boldsymbol{D}_t(z^{i_z}, a^{i_{a-}})$
     - 3.4 Increment $\underline{\boldsymbol{D}}_{t+1}(z^{i_z}, a^{\iota+1})$ with $(1-\omega)\boldsymbol{D}_t(z^{i_z}, a^{i_{a-}})$
- **Review:**
  1. **Pro:** Computationally efficient and no randomness
  2. **Con:** Introduces a non-continuous distribution

Small example

- **Grids:** $\mathcal{G}_z = \{\underline{z}, \overline{z}\}$ and $\mathcal{G}_a = \{0, 1\}$
- **Transition matrix:** $\pi_{0,0} = \pi_{1,1} = 0.5$
- **Policy function:**
  - Low income: $a^*(\underline{z}, 0) = a^*(\underline{z}, 1) = 0$
  - High income: Let $a^*(\overline{z}, 0) = 0.5$ and $a^*(\overline{z}, 1) = 1$
- **Initial distribution**: $\underline{\boldsymbol{D}}_0(z_{it}, a_{it-1}) = \begin{cases} 1 & \text{if } z_{it} = \underline{z} \text{ and } a_{it} = 0 \\ 0 & \text{else} \end{cases}$
- **Task:** Calculate by hand the transitions to

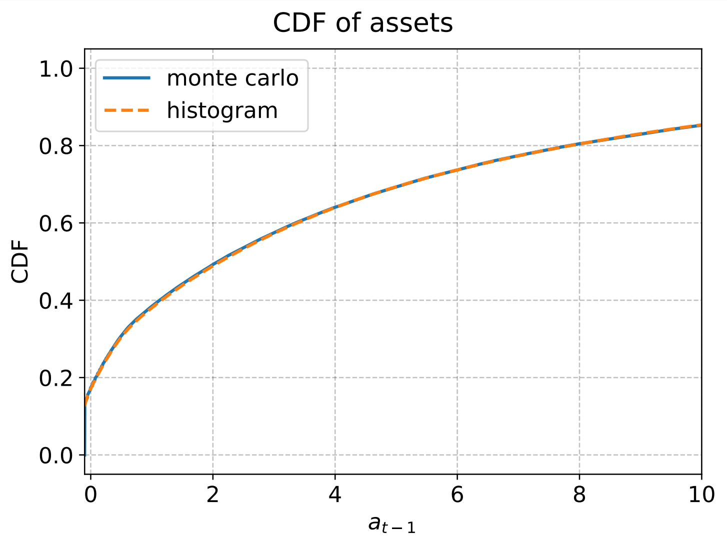
$$\boldsymbol{D}_0, \, \underline{\boldsymbol{D}}_1, \, \boldsymbol{D}_1, \ldots$$

See simple simple_histogram_simulation.xlsx

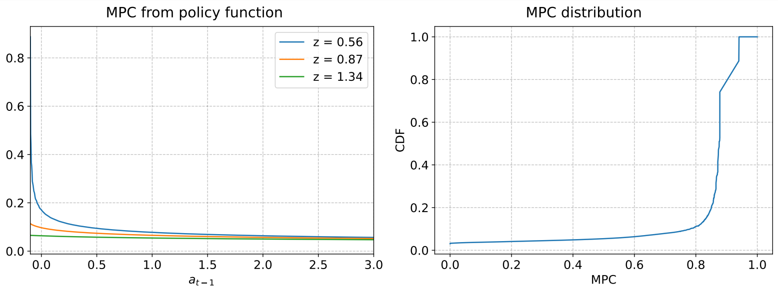
Infinite horizon: $T \to \infty$?

- **Initial guess:** Can be arbitrary.
  1. Everyone in one grid point, or
  2. Ergodic distribution of z_{it} and everyone has zero savings,
- **Convergence:** Simulate forward until the distribution does not change anymore (given some tolerance)

Converged CDF of savings



MPCs



MPC from policy function
z = 0.56
z = 0.87
z = 1.34
0.0
0.2
0.4
0.6
0.8
0.0
0.5
1.0
1.5
2.0
2.5
3.0
a_{t-1}
MPC distribution
CDF
0.0
0.2
0.4
0.6
0.8
1.0
MPC

Side-note: Matrix formulation

- The histogram method can be written in **matrix form**:

$$\boldsymbol{D}_t = \Pi_z' \underline{\boldsymbol{D}}_t$$
$$\underline{\boldsymbol{D}}_{t+1} = \Lambda_t' \boldsymbol{D}_t$$

  where

  $\underline{\boldsymbol{D}}_t$ is vector of length $\#_z \times \#_a$

  $\boldsymbol{D}_t$ is vector of length $\#_z \times \#_a$

  Π_z' is derived from the π_{i_{z-},i_z}'s

  Λ_t' is derived from the ι's and ω's

- **Note:** Example shown in notebook
- **Further details:** Young (2010), Tan (2020), Ocampo and Robinson (2022)

Ramsey

Ramsey: Firms

- **Production function:** $Y_t = F(\Gamma_t, K_{t-1}, L_t)$ [note timing of capital] where Γ_t is technology
- **Profits:** $\Pi_t = Y_t - w_t L_t - r_t^K K_{t-1}$
- **Profit maximization:** $\max_{K_{t-1},L_t} \Pi_t$
  1. Rental rate: $\frac{\partial \Pi_t}{\partial K_{t-1}} = 0 \Leftrightarrow r_t^K = F_K(\Gamma_t, K_{t-1}, L_t)$
  2. Real wage: $\frac{\partial \Pi_t}{\partial L_t} = 0 \Leftrightarrow w_t = F_L(\Gamma_t, K_{t-1}, L_t)$

  Zero profits: $\Pi_t = 0 \Rightarrow$

  $Y_t = w_t L_t + r_t^K K_{t-1}$ [functional income distribution]

Ramsey: Zero-profit mutual fund

- **Owns all capital**
- **Capital depreciate** with rate $\delta \in (0,1)$,

$$K_t = (1-\delta)K_{t-1} + I_t$$

- **Deposits** (from households), A_{t-1}: The rate of return is

$$r_t = r_t^K - \delta$$

- **Balance sheet:**

$$A_{t-1} = K_{t-1}$$

- **Utility maximization:**

$$v_0(A_{-1}^{hh}) = \max_{\{C_t^{hh}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(C_t^{hh})$$

s.t.

$$A_t^{hh} = (1+r_t)A_{t-1}^{hh} + w_t L_t^{hh} - C_t^{hh}$$

Exogenous labor supply: $L_t^{hh} = 1$

- **Euler-equation** (implied by Lagrangian):

$$u'(C_t^{hh}) = \beta(1+r_{t+1})u'(C_{t+1}^{hh})$$

Ramsey: Market Clearing

- **Capital market:** $K_t = A_t = A_t^{hh}$
- **Labor market:** $L_t = L_t^{hh} = 1$
- **Goods market:** $Y_t = C_t^{hh} + I_t$
- **Walras:** Capital and labor market clears $\Rightarrow$ goods market clears

$$
\begin{aligned}
C_t^{hh} + I_t &= \left[(1+r_t)A_{t-1}^{hh} + w_t L_t^{hh} - A_t^{hh})\right] + (K_t - (1-\delta)K_{t-1}) \\
&= [(1+r_t)K_{t-1} + w_t L_t - K_t)] + (K_t - (1-\delta)K_{t-1}) \\
&= r_t^K K_{t-1} + w_t L_t \\
&= Y_t
\end{aligned}
$$

Ramsey: Summary

- **Simplified form:**

$$u'(C_t^{hh}) = \beta(1 + F_K(\Gamma_t, K_t, 1) - \delta)u'(C_{t+1}^{hh})$$
$$K_t = (1-\delta)K_{t-1} + F(\Gamma_t, K_{t-1}, 1) - C_t^{hh}$$

- **Extended form:**

$$\begin{aligned}
r_t^K &= F_K(\Gamma_t, K_{t-1}, L_t) \\
w_t &= F_L(\Gamma_t, K_{t-1}, L_t) \\
r_t &= r_t^K - \delta \\
A_t &= K_t \\
A_t^{hh} &= (1 + r_t)A_{t-1}^{hh} + w_t L_t^{hh} - C_t^{hh} \\
u'(C_t^{hh}) &= \beta(1 + r_{t+1})u'(C_{t+1}^{hh}) \\
A_t &= A_t^{hh} \\
L_t &= L_t^{hh}
\end{aligned}$$

Ramsey: As an equation system

$$\begin{bmatrix} r_t^K - F_K(\Gamma_t, K_{t-1}, L_t) \\ w_t - F_L(\Gamma_t, K_{t-1}, L_t) \\ r_t - (r_t^K - \delta) \\ A_t - K_t \\ u'(C_t^{hh}) - \beta(1 + r_{t+1})u'(C_{t+1}^{hh}) \\ L_t^{hh} - 1 \\ A_t^{hh} - \left((1 + r_t)A_{t-1}^{hh} + w_t L_t^{hh} - C_t^{hh}\right) \\ A_t - A_t^{hh} \\ L_t - L_t^{hh} \\ \forall t \in \{0, 1, \dots\}, \text{given } K_{-1} \end{bmatrix} = \mathbf{0}$$

Note I: There is *perfect foresight*.

Note II: This is the so-called *sequence-space* formulation.

Ramsey: Steady state

- **Euler-equation** can be solved for K_{ss}:

$$u'(C_{ss}) = \beta(1 + F_K(\Gamma_{ss}, K_{ss}, 1) - \delta)u'(C_{ss}) \Leftrightarrow$$
$$F_K(\Gamma_{ss}, K_{ss}, 1) = \frac{1}{\beta} - 1 + \delta$$

- **Accumulation equation** then implies C_{ss}:

$$K_{ss} = (1-\delta)K_{ss} + F(\Gamma_{ss}, K_{ss}, 1) - C_{ss} \Leftrightarrow$$
$$C_{ss} = (1-\delta)K_{ss} + F(\Gamma_{ss}, K_{ss}, 1) - K_{ss}$$

HANC

HANC model overview

- **Model blocks:**
  1. **Firms:** Rent capital from mutual fund and hire labor from the households, produce with given technology, and sell output goods
  2. **Zero-profit mutual funds:** Own capital and rent it to firms, take deposits and pay return to household
  3. **Households:** Face idiosyncratic productivity shocks, supplies labor exogenously and makes consumption-saving decisions
  4. **Markets:** Perfect competition in labor, goods and capital markets
- **Add-on to Ramsey-Cass-Koopman:** *Heterogeneous households*
- **Other names:**
  1. The Aiyagari-model
  2. The Aiyagari-Bewley-Hugget-Imrohoroglu-model
  3. The Standard Incomplete Markets (SIM) model

Heterogeneous households

- **Utility maximization** for household *i*:

$$v_0(\beta_i, z_{it}, a_{it-1}) = \max_{\{c_{it}\}_{t=0}^{\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta_i^t u(c_{it})$$

$$\text{s.t.}$$

$$\begin{aligned}
\ell_{it} &= z_{it} \\
a_{it} &= (1+r_t)a_{it-1} + w_t \ell_{it} - c_{it} + \Pi_t \\
\log z_{it+1} &= \rho_z \log z_{it} + \psi_{it+1}, \;\; \psi_{it} \sim \mathcal{N}(\mu_\psi, \sigma_\psi), \;\; \mathbb{E}[z_{it}] = 1 \\
a_{it} &\geq 0
\end{aligned}$$

- **Where are there heterogeneity?**
  1. *Ex ante* due to different preferences, β_i
  2. *Ex post* due to stochastic productivity, z_{it}
- **Incomplete markets due to borrowing constraint**
  (fancy words: partial self-insurrance, lack of Arrow-Debreu securities)

Recursive formulation

- **Value function** (at decision)

$$
\begin{aligned}
v_t(\beta_i, z_{it}, a_{it-1}) &= \max_{c_t} u(c_t) + \beta \underline{v}_{t+1}(\beta_i, z_{it}, a_{it}) \\
&\text{s.t.} \\
\ell_{it} &= z_{it} \\
a_{it} &= (1 + r_t) a_{it-1} + w_t \ell_{it} - c_{it} + \Pi_t \\
\log z_{it+1} &= \rho_z \log z_{it} + \psi_{it+1} \\
a_{it} &\geq 0
\end{aligned}
$$

- **Beginning-of-period value function** (before shock realization):

$$
\underline{v}_t(\beta_i, z_{it-1}, a_{it-1}) = \mathbb{E}\left[v_t(\beta_i, z_{it}, a_{it-1}) \,|\, \beta_i, z_{it-1}, a_{it-1} \right]
$$

Distributions and aggregates

- **Policy functions:** Aggregate prices are hidden as inputs, i.e.

$$x_t^*(\beta_i, z_{it}, a_{it-1}) = x^*(\beta_i, z_{it}, a_{it-1}, \{r_\tau, w_\tau\}_{\tau \geq t}) \text{ for } x \in \{a, \ell, c\}$$

- **Distributions** (vector of probabilities):
  1. Beginning-of-period: $\underline{\boldsymbol{D}}_t$ over β_i, z_{it-1} and a_{it-1}
  2. Productivity transition: $\boldsymbol{D}_t = \Pi_z' \underline{\boldsymbol{D}}_t$ over β_i, z_{it} and a_{it-1}
  3. Savings transition: $\underline{\boldsymbol{D}}_{t+1} = \Lambda_t' \boldsymbol{D}_t$ where again

$$\Lambda_t = \Lambda\left(\{r_\tau, w_\tau\}_{\tau \geq t}\right)$$

- **Aggregate consumption and savings:**

$$\begin{aligned} X_t^{hh} &= \int x_t^*(\beta_i, z_{it}, a_{it-1}) d\boldsymbol{D}_t \text{ for } x \in \{a, \ell, c\} \\ &= X^{hh}\left(\{r_\tau, w_\tau\}_{\tau \geq t}, \underline{\boldsymbol{D}}_0\right) \\ &= \boldsymbol{x}_t^{*\prime} \boldsymbol{D}_t \end{aligned}$$

Market clearing

- **Capital market:** $K_t = A_t = A_t^{hh} = \int a_t^*(\beta_i, z_{it}, a_{it-1}) d\boldsymbol{D}_t$
- **Labor market:** $L_t = L_t^{hh} = \int \ell_t^*(\beta_i, z_{it}, a_{it-1}) d\boldsymbol{D}_t = \int z_{it} d\boldsymbol{D}_t = 1$
- **Goods market:** $Y_t = C_t^{hh} + I_t$

- **Walras:** Capital and labor market clears $\Rightarrow$ goods market clears

$$
\begin{aligned}
C_t^{hh} + I_t &= \int c_{it}^* d\boldsymbol{D}_t + [K_t - (1-\delta)K_{t-1}] \\
&= \int \left[(1+r_t)a_{it-1} + w_t z_{it} - a_{it}\right] d\boldsymbol{D}_t \\
&= [(1+r_t)K_{t-1} + w_t L_t - K_t] + [K_t - (1-\delta)K_{t-1}] \\
&= r_t^K K_{t-1} + w_t L_t \\
&= Y_t
\end{aligned}
$$

Equation system

$$
\begin{bmatrix}
r_t^K - F_K(\Gamma_t, K_{t-1}, L_t) \\
w_t - F_L(\Gamma_t, K_{t-1}, L_t) \\
r_t - (r_t^K - \delta) \\
A_t - K_t \\
\boldsymbol{D}_t - \Pi_z' \underline{\boldsymbol{D}}_t \\
\underline{\boldsymbol{D}}_{t+1} - \Lambda_t' \boldsymbol{D}_t \\
A_t - \boldsymbol{a}_t^{*\prime} \boldsymbol{D}_t \\
L_t - \boldsymbol{\ell}_t^{*\prime} \boldsymbol{D}_t \\
\forall t \in \{0, 1, \dots\}, \text{given } \underline{\boldsymbol{D}}_0
\end{bmatrix} = \mathbf{0}
$$

where $K_{-1} = \int a_{it-1} d\underline{\boldsymbol{D}}_0$

1. **Perfect foresight** wrt. aggregate variables
2. **Stationary equilibrium:** Time-constant solution.
3. **Transition path:** Time-varying solution due to e.g. initial conditions or temporary deviations of exogenous variables.

Stationary equilibrium - equation system

The **stationary equilibrium** satisfies

$$\begin{bmatrix} r_{ss}^K - F_K(\Gamma_{ss}, K_{ss}, L_{ss}) \\ w_{ss} - F_L(\Gamma_{ss}, K_{ss}, L_{ss}) \\ r_{ss} - (r_{ss}^K - \delta) \\ A_{ss} - K_{ss} \\ \boldsymbol{D}_{ss} - \Pi_z' \underline{\boldsymbol{D}}_{ss} \\ \underline{\boldsymbol{D}}_{ss} - \Lambda_{ss}' \boldsymbol{D}_{ss} \\ A_{ss} - \boldsymbol{a}_{ss}^{*\prime} \boldsymbol{D}_{ss} \\ L_{ss} - \boldsymbol{\ell}_{ss}^{*\prime} \boldsymbol{D}_{ss} \end{bmatrix} = \mathbf{0}$$

Note I: Households still move around »inside« the distribution due to idiosyncratic shocks

Note II: Steady state for aggregates (quantities and prices) and the distribution as such

Stationary equilibrium - more verbal definition

For a given Γ_{ss}

1. Quantities K_{ss} and L_{ss},
2. prices r_{ss} and w_{ss} (always $\Pi_{ss} = 0$),
3. the distribution $\boldsymbol{D}_{ss}$ over β_i, z_{it} and a_{it-1}
4. and the policy functions a^*_{ss}, ℓ^*_{ss} and c^*_{ss}

are such that

1. Household maximize expected utility (policy functions)
2. Firms maximize profits (prices)
3. $\boldsymbol{D}_{ss}$ is the invariant distribution implied by the household problem
4. Mutual fund balance sheet is satisfied
5. The capital market clears
6. The labor market clears
7. The goods market clears

Direct implementation

Technology: $F(\Gamma, K, L) = \Gamma K^{\alpha} L^{1-\alpha}$

Root-finding problem in K_{ss} with the objective function:

1. Set $L_{ss} = 1$ (and $\Pi_{ss} = 0$)
2. Calculate $r_{ss} = \alpha \Gamma_{ss} (K_{ss})^{\alpha - 1} - \delta$ and $w_{ss} = (1-\alpha)\Gamma_{ss}(K_{ss})^{\alpha}$
3. Solve infinite horizon household problem *backwards*, i.e. find $\boldsymbol{a}_{ss}^{*}$
4. Simulate households *forwards* until convergence, i.e. find $\boldsymbol{D}_{ss}$
5. Return $K_{ss} - \boldsymbol{a}_{ss}^{*\prime} \boldsymbol{D}_{ss}$

Direct implementation (alternative)

Technology: $F(\Gamma, K, L) = \Gamma K^{\alpha} L^{1-\alpha}$

Root-finding problem in r_{ss} with the objective function:

1. Set $L_{ss} = 1$ (and $\Pi_{ss} = 0$)
2. Calculate $K_{ss} = \left(\frac{r_{ss}+\delta}{\alpha\Gamma_{ss}}\right)^{\frac{1}{\alpha-1}}$ and $w_{ss} = (1-\alpha)\Gamma_{ss}(K_{ss})^{\alpha}$
3. Solve infinite horizon household problem *backwards*, i.e. find $\boldsymbol{a}_{ss}^{*}$
4. Simulate households *forwards* until convergence, i.e. find $\boldsymbol{D}_{ss}$
5. Return $K_{ss} - \boldsymbol{a}_{ss}^{*\prime}\boldsymbol{D}_{ss}$

Indirect implementation

Technology: $F(K,L) = \Gamma K^{\alpha} L^{1-\alpha}$

Consider Γ_{ss} and δ as »free« parameters:

1. Choose r_{ss} and w_{ss}
2. Solve infinite horizon household problem *backwards*, i.e. find $\boldsymbol{a}_{ss}^{*}$
3. Simulate households *forwards* until convergence, i.e. find $\boldsymbol{D}_{ss}$
4. Set $K_{ss} = \boldsymbol{a}_{ss}^{*\prime}\boldsymbol{D}_{ss}$
5. Set $L_{ss} = 1$ (and $\Pi_{ss} = 0$)
6. Set $\Gamma_{ss} = \frac{w_{ss}}{(1-\alpha)(K_{ss})^{\alpha}}$
7. Set $r_{ss}^{K} = \alpha\Gamma_{ss}(K_{ss})^{\alpha-1}$
8. Set $\delta = r_{ss}^{k} - r_{ss}$

Calibration

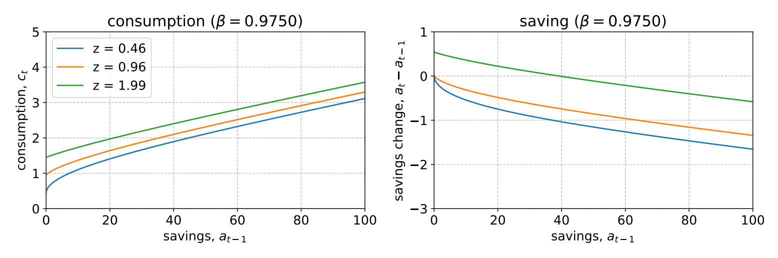
- **Preferences:** $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$
  1. Discount factors: $\beta \in \{0.965, 0.975, 0.985\}$ in equal pop. shares
  2. Relative risk aversion: $\sigma = 2$
- **Income:**
  1. AR(1): $\rho_z = 0.95$
  2. Std.: $\sigma_\psi = 0.30\sqrt{(1-\rho_z^2)}$
- **Technology:** $F(\Gamma, K, L) = \Gamma K^\alpha L^{1-\alpha}$
  1. Capital share: $\alpha = 0.36$
  2. TFP: $\Gamma_{ss} = 1.082$
  3. Depreciation: $\delta = 0.193$
- **Steady state:**
  1. Prices: $r_{ss} = 0.01$ and $w_{ss} = 1$
  2. Quantities: $K_{ss}/Y_{ss} = 1.776$

Consumption function

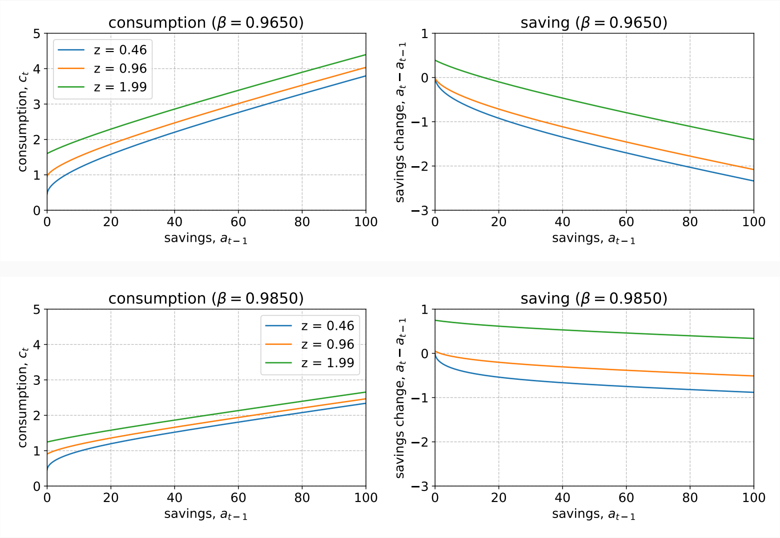
- **Euler-equation still necessary for $a_{it} > 0$:**

$$c_{it}^{-\sigma} = \beta_i(1 + r_{t+1})\mathbb{E}_t \left[c_{it+1}^{-\sigma}\right]$$

- **Precautionary saving:**
  1. Low consumption for low cash-on-hand → *buffer-stock target*
  2. Steep slope for low cash-on-hand → *high MPC*

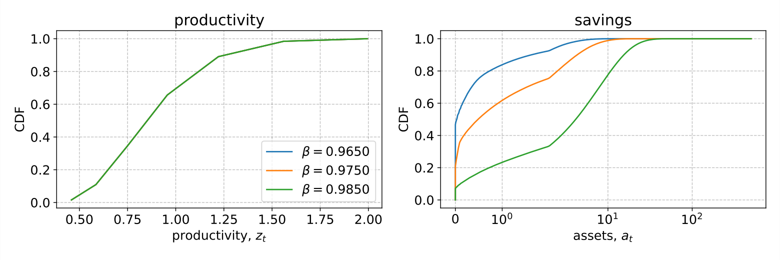


Low vs. high β_i



Distribution, D_t

- **Productivity:** Marginal distribution over only z_{it}
- **Savings:** Marginal distribution over a_{it} cond. on β_i



- **Drivers of wealth inequality:**
  1. Stochastic income
  2. Heterogeneous patience $\rightarrow$ savings behavior

Steady state interest rate

- **Representative agent / complete markets:**
  Derived from aggregate Euler-equation

$$C_t^{-\sigma} = \beta(1 + r_{t+1})C_{t+1}^{-\sigma} \Rightarrow C_{ss}^{-\sigma} = \beta(1 + r_{ss})C_{ss}^{-\sigma} \Leftrightarrow \beta = \frac{1}{1 + r_{ss}}$$

- **Heterogeneous agents:** *No such equation exists*
  1. Euler-equation replaced by asset market clearing condition
  2. Idiosyncratic income risk affects the steady state interest rate

σ_ψ	PE ($r_{ss} = 1\%$), A^{hh}	GE, r_{ss}	GE, A^{hh}
0.09	2.78	1.00%	2.78
0.14	7.39	0.12%	2.97
0.19	13.68	-1.11%	3.30

Partial Equilibrium: Same interest rate.

General Equilibrium: Capital+labor market clearing.

Summary

Summary

- **Recap:**
  1. Consumption-saving
  2. Dynamic programming
  3. Stationary equilibrium
- **Slightly new:**
  1. My notation
  2. Model formulation in sequence space
- **Next:** *Solution of transition path using sequence space Jacobian*