Notes on Adjoint Solves using Maxwell

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Maxwell solves $(\nabla \times \mu^{-1} \nabla \times -\omega^2 \epsilon)E = i\omega J$, or

$$Ax = b, \tag{1}$$

where

- $\nabla \times \mu^{-1} \nabla \times -\omega^2 \epsilon \to A$,
- $E \to x$,
- $i\omega J \to b$.

In many cases, however, one requires the solution to the *adjoint* or conjugate-transpose of A, namely

$$A^\dagger y = d. \tag{2}$$

To do so, we introduce a diagonal symmetrization matrix, S, which has the property

$$SA = A^T S, \tag{3}$$

and is defined as $S = \text{diag}([s_x, s_y, s_z])$ where

$$s_x = s_x^{\text{dual}} s_y^{\text{prim}} s_z^{\text{prim}}, \tag{4a}$$
$$s_y = s_x^{\text{prim}} s_y^{\text{dual}} s_z^{\text{prim}}, \tag{4b}$$
$$s_z = s_x^{\text{prim}} s_y^{\text{prim}} s_z^{\text{dual}}. \tag{4c}$$

Using S, we perform the following operations on $Ax = b$,

$$SAx = Sb \tag{5a}$$
$$A^T Sx = Sb \tag{5b}$$
$$A^\dagger S^* x^* = S^* b^* \tag{5c}$$
$$A^\dagger y = d, \tag{5d}$$

from which we conclude that $y = S^* x^*$ and $d = S^* b^*$.

Therefore, in order to solve for $A^\dagger y = d$ given d using Maxwell, we simply compute

$$y = S^*(A^{-1}(S^{-1}d^*))^*. \tag{6}$$

Lastly, we may also consider the case where ω is an eigenfrequency and v is an eigenmode in the simulation domain, then

$$Av = 0, \quad v \neq 0. \tag{7}$$

We simply remark that in this case, a corresponding right eigenvector w may be found via

$$w = Sv \tag{8}$$

since

$$\begin{aligned} w^T A &= 0 && \text{(9a)} \\ v^T S A &= 0 && \text{(9b)} \\ v^T A^T S &= 0 && \text{(9c)} \\ SAv &= 0 && \text{(9d)} \\ Av &= 0. && \text{(9e)} \end{aligned}$$