Electromagnetic Theory Handbook for Objective-First Optimization

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1 Maxwell's equations

According to Eqs. 3.7 and 3.8 in [1], Maxwell's time-harmonic equations (E, H, J, and $M \propto e^{-i\omega t}$) are

$$-i\omega H = -\frac{1}{\mu}\nabla \times E - \frac{1}{\mu}M \tag{1}$$

$$-i\omega E = \frac{1}{\epsilon}\nabla \times H - \frac{1}{\epsilon}J \tag{2}$$

where M and J are the magnetic and electric current densities, respectively.

The wave equations are then,

$$\nabla \times \frac{1}{\mu}\nabla \times E - \omega^2 \epsilon E = i\omega J - \nabla \times \frac{1}{\mu}M \tag{3}$$

and

$$\nabla \times \frac{1}{\epsilon}\nabla \times H - \omega^2 \mu H = i\omega M + \nabla \times \frac{1}{\epsilon}J. \tag{4}$$

2 Maxwell's equations for a waveguide.

Assume that we have a uniform waveguide (not periodic, since periodic waveguides such as photonic crystal waveguides must be dealt with using Bloch's theorem). We want to find solutions of the form,

$$E(x, y, z, t) = E(x, y)e^{i\beta z - i\omega t}, \tag{5}$$

where β is the wave-vector in the direction of propagation (z).

The solution for a two-dimensional waveguide of non-magnetic material ($\mu = \mu_0$ everywhere) is messy[2], but the end result is

$$\left(\nabla \frac{1}{\epsilon_z} \nabla \cdot \epsilon - \nabla \times \nabla \times + \mu_0 \omega^2 \epsilon - \beta^2\right) E_t = 0, \tag{6}$$

where the transverse E-field components are $E_t = \hat{x} E_x + \hat{y} E_y$. We can then back-out the longitudinal component E_z using

$$\nabla \cdot \epsilon E = 0, \tag{7}$$

resulting in

$$E_z = \frac{i}{\beta \epsilon_z} \nabla \cdot \epsilon E_t. \tag{8}$$

If there is no variation in y (slab or one-dimensional waveguide), then we obtain

$$\left(\frac{\partial}{\partial x} \frac{1}{\epsilon_z} \frac{\partial}{\partial x} \epsilon_x + \mu_0 \omega^2 \epsilon_x - \beta^2\right) E_x = 0. \tag{9}$$

3 Perfectly matched layers

The upshot of ref. [3] is that a PML can be implemented by simply substituting partial derivatives in the following manner,

$$\frac{\delta}{\delta x} \rightarrow \frac{1}{1 + i \frac{\sigma_x(x)}{\omega}} \frac{\delta}{\delta x}, \tag{10}$$

where $\sigma_x(x) > 0$ in the PML and $\sigma_x = 0$ outside of it.

Further considerations include complex σ, Im $\sigma < 0$, to attenuate evanescent waves. Quadratic or cubic growth of σ to reduce numerical reflections arising from discretization error.

Generally, a half-wavelength thick PML layer is sufficient for acceptable attentuation.

References

[1] Allen Taflove, Susan C. Hagness, *Computational Electrodynamics, Third Edition* (Artech House, 2005).

[2] Jesse Lu, *2.5D Waveguide Equations.pdf* and *2.5D Waveguide Equations (simplified).pdf*, `https://github.com/JesseLu/misc/tree/master/scribbling`.

[3] Steven G. Johnson, *Notes on Perfectly Matched Layers (PMLs)* (2007).