I. Supplemental Material

We only provide the formulation here. If you are interested in the detailed derivations, we recommend You can find all the relevant formula derivations in [1] and [2].

A. *Kinematics Function* $\mathbf{f}()$

Suppose the IMU sample interval is Δt, the kinematics function is:

$$\mathbf{f}(\mathbf{x}_i, \mathbf{u}_i, \mathbf{w}_i) = \Delta t \begin{bmatrix} \boldsymbol{\omega}_{m_i} - \mathbf{b}_{\boldsymbol{\omega}_i} - \mathbf{n}_{\boldsymbol{\omega}_i} \\ {}^G\mathbf{v}_{I_i} \\ \mathbf{0}_{3\times1} \\ {}^G\mathbf{R}_{I_i}(\mathbf{a}_{m_i} - \mathbf{b}_{\mathbf{a}_i - \mathbf{n}_{\mathbf{a}_i}}) + {}^G\mathbf{g}_i \\ \mathbf{n}_{\mathbf{b}\boldsymbol{\omega}_i} \\ \mathbf{n}_{\mathbf{ba}_i} \\ \mathbf{0}_{3\times1} \end{bmatrix}. \tag{1}$$

The system input and noise are:

$$\mathbf{u} \doteq \begin{bmatrix} \boldsymbol{\omega}_m^T & \mathbf{a}_m^T \end{bmatrix}^T, \mathbf{w} \doteq \begin{bmatrix} \mathbf{n}_{\boldsymbol{\omega}}^T & \mathbf{n}_{\mathbf{a}}^T & \mathbf{n}_{\mathbf{b}\boldsymbol{\omega}}^T & \mathbf{n}_{\mathbf{ba}}^T \end{bmatrix} \tag{2}$$

B. *Formulation of* $\mathbf{J}^n$

Recalling the definition of $\mathbf{J}^n$:

$$\mathbf{J}^n = \left(\left. \frac{\partial \left(\hat{\mathbf{x}}_k^n \boxplus \delta\mathbf{x}_k^n \right) \boxminus \hat{\mathbf{x}}_k}{\partial \mathbf{x}_k^n} \right|_{\delta\mathbf{x}_k^n = \mathbf{0}} \right)^{-1}. \tag{3}$$

Combining the definition of $\delta\mathbf{x}$ and $\mathbf{x}$, we have:

$$\mathbf{J}^n = \begin{bmatrix} \mathbf{A}({}^G\mathbf{R}_{I_k}^n)^T & \mathbf{0}_{3\times15} \\ \mathbf{0}_{15\times3} & \mathbf{I}_{15\times15} \end{bmatrix} \tag{4}$$

where $\mathbf{A}()$ is defined as:

$$\mathbf{A}(\mathbf{u}) = \mathbf{I} + \left(\frac{1 - \cos\|\mathbf{u}\|}{\|\mathbf{u}\|} \right) \frac{[\mathbf{u}]_\times}{\|\mathbf{u}\|} + \left(1 - \frac{\sin(\|\mathbf{u}\|)}{\mathbf{u}} \right) \frac{[\mathbf{u}]_\times^2}{\|\mathbf{u}\|^2} \tag{5}$$

C. *Derivation of Jacobian Matrix*

The Jacobian matrix is given by:

$$H_j^n = \frac{\partial \mathbf{z}_j(\hat{\mathbf{x}}_k \boxplus \delta\mathbf{x}_k^n, {}^{L_k}\mathbf{p}_j)}{\partial \delta\mathbf{x}_k^n} \tag{6}$$

and the observation function is:

$$\mathbf{z}_j(\hat{\mathbf{x}}_k \boxplus \delta\mathbf{x}_k^n, {}^L\mathbf{p}_j) = s_{j,i}^n \mathbf{D}_j{}^n \left({}^G\hat{\mathbf{p}}_j - {}^G\boldsymbol{\mu}_i \right) \tag{7}$$

The calculation of the derivative of $s_{j,i}^n\mathbf{D}_j^n$ w.r.t. $\delta\mathbf{x}_k^n$ is quite comprehensive and impractical. We assume $s_{j,i}^n\mathbf{D}_j^n$ as constant for each iteration and update its value after the iteration in the implementation. Let's pre-process the observation function:

$$\begin{aligned} \mathbf{z}_j(\hat{\mathbf{x}}_k \boxplus \delta\mathbf{x}_k^n, {}^L\mathbf{p}_j) = s_j\mathbf{D}_j^n \Big(& {}^G\hat{\mathbf{p}}_{I_k} + {}^G\delta\mathbf{p}_{I_k} - {}^G\boldsymbol{\mu}_j \\ & + {}^G\hat{\mathbf{R}}_{I_k} \mathrm{Exp}\left({}^G\delta\mathbf{r}_{I_k} \right) \left({}^I\mathbf{R}_{L_k} {}^L\mathbf{p}_j + {}^I\mathbf{p}_{L_k} \right) \Big). \end{aligned} \tag{8}$$

Due to the minute transformation between two LiDAR frames, we have:

$$\mathrm{Exp}\left({}^G\delta\mathbf{r}_{I_k} \right) \approx \mathbf{I} + \left[{}^G\delta\mathbf{r}_{I_k} \right]_\times. \tag{9}$$

Combining (14) and (15) leads to:

$$\begin{aligned} \mathbf{z}_j(\hat{\mathbf{x}}_k \boxplus \delta\mathbf{x}_k^n, {}^L\mathbf{p}_j) = s_j\mathbf{D}_j^n \Big(& {}^G\hat{\mathbf{p}}_{I_k} + {}^G\delta\mathbf{p}_{I_k} - {}^G\boldsymbol{\mu}_j \\ & + {}^G\hat{\mathbf{R}}_{I_k} {}^I\mathbf{q}_j - {}^G\hat{\mathbf{R}}_{I_k} \left[{}^I\mathbf{q}_j \right]_\times {}^G\delta\mathbf{r}_{I_k} \Big), \end{aligned} \tag{10}$$

where ${}^I\mathbf{q}_j = {}^I\mathbf{R}_{L_k} {}^L\mathbf{p}_j + {}^I\mathbf{p}_{L_k}$. Recalling (12), it is easy to derive the Jacobian matrix $\mathbf{H}_j^n$ as following:

$$\mathbf{H}_j^n = \begin{bmatrix} -s_j\mathbf{D}_j^n \left[{}^I\mathbf{q}_j \right]_\times & s_j\mathbf{D}_j^n & \mathbf{0}_{3\times15} \end{bmatrix} \tag{11}$$

D. *Derivation of Jacobian Matrix*

The Jacobian matrix is given by:

$$H_j^n = \frac{\partial \mathbf{z}_j(\hat{\mathbf{x}}_k \boxplus \delta\mathbf{x}_k^n, {}^{L_k}\mathbf{p}_j)}{\partial \delta\mathbf{x}_k^n} \tag{12}$$

and the observation function is:

$$\mathbf{z}_j(\hat{\mathbf{x}}_k \boxplus \delta\mathbf{x}_k^n, {}^L\mathbf{p}_j) = s_{j,i}^n \mathbf{D}_j{}^n \left({}^G\hat{\mathbf{p}}_j - {}^G\boldsymbol{\mu}_i \right) \tag{13}$$

The calculation of the derivative of $s_{j,i}^n\mathbf{D}_j^n$ w.r.t. $\delta\mathbf{x}_k^n$ is quite comprehensive and impractical. We assume $s_{j,i}^n\mathbf{D}_j^n$ as constant for each iteration and update its value after the iteration in the implementation. Let's pre-process the observation function:

$$\begin{aligned} \mathbf{z}_j(\hat{\mathbf{x}}_k \boxplus \delta\mathbf{x}_k^n, {}^L\mathbf{p}_j) = s_j\mathbf{D}_j^n \Big(& {}^G\hat{\mathbf{p}}_{I_k} + {}^G\delta\mathbf{p}_{I_k} - {}^G\boldsymbol{\mu}_j \\ & + {}^G\hat{\mathbf{R}}_{I_k} \mathrm{Exp}\left({}^G\delta\mathbf{r}_{I_k} \right) \left({}^I\mathbf{R}_{L_k} {}^L\mathbf{p}_j + {}^I\mathbf{p}_{L_k} \right) \Big). \end{aligned} \tag{14}$$

Due to the minute transformation between two LiDAR frames, we have:

$$\mathrm{Exp}\left({}^G\delta\mathbf{r}_{I_k} \right) \approx \mathbf{I} + \left[{}^G\delta\mathbf{r}_{I_k} \right]_\times. \tag{15}$$

Combining (14) and (15) leads to:

$$\begin{aligned} \mathbf{z}_j(\hat{\mathbf{x}}_k \boxplus \delta\mathbf{x}_k^n, {}^L\mathbf{p}_j) = s_j\mathbf{D}_j^n \Big(& {}^G\hat{\mathbf{p}}_{I_k} + {}^G\delta\mathbf{p}_{I_k} - {}^G\boldsymbol{\mu}_j \\ & + {}^G\hat{\mathbf{R}}_{I_k} {}^I\mathbf{q}_j - {}^G\hat{\mathbf{R}}_{I_k} \left[{}^I\mathbf{q}_j \right]_\times {}^G\delta\mathbf{r}_{I_k} \Big), \end{aligned} \tag{16}$$

where ${}^I\mathbf{q}_j = {}^I\mathbf{R}_{L_k} {}^L\mathbf{p}_j + {}^I\mathbf{p}_{L_k}$. Recalling (12), it is easy to derive the Jacobian matrix $\mathbf{H}_j^n$ as following:

$$\mathbf{H}_j^n = \begin{bmatrix} -s_j\mathbf{D}_j^n \left[{}^I\mathbf{q}_j \right]_\times & s_j\mathbf{D}_j^n & \mathbf{0}_{3\times15} \end{bmatrix} \tag{17}$$

References

[1] J. Sola, "Quaternion kinematics for the error-state kalman filter," *arXiv preprint arXiv:1711.02508*, 2017.

[2] D. He, W. Xu, and F. Zhang, "Kalman filters on differentiable manifolds," *arXiv preprint arXiv:2102.03804*, 2021.