Ax-Grothendieck in lean

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June 10, 2022

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Polynomial maps

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Polynomial maps on a field K are regular endomorphisms on K^n, i.e. n polynomials in $K[x_1, \dots, x_n]$.

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Code

```
def poly_map (K : Type*) [comm_semiring K] (n : ℕ) : Type* :=
fin n → mv_polynomial (fin n) K

def eval : poly_map K n → (fin n → K) → (fin n → K) :=
λ ps as k, mv_polynomial.eval as (ps k)
```

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- Amazing fact: Lefschetz also says that if we show it for all ACF_p for large p then it is also true for ACF_0.
- Good news: We can easily show it for algebraic closures of $\mathbb{F}_p$ for any p.

Locally finite fields

Definition (Locally finite fields)

Let K be a field of characteristic p a prime. Then the following are equivalent definitions for K being a *locally finite field*:

1. The minimal subfield generated by any finite subset of K is finite.
2. $\mathbb{F}_p \to K$ is algebraic.
3. K embeds into an algebraic closure of $\mathbb{F}_p$.

Theorem

Locally finite fields satisfy Ax-Grothendieck.

Proof.

$$\mathbb{F}_p \hookrightarrow \mathbb{F}_p(\texttt{coeffs}) \hookrightarrow K$$

$$\downarrow_{\mathbb{F}_p(\texttt{coeffs})} \text{ respects injectivity}$$

$\square$

Theorem (Lefschetz principle)
Let ϕ be a sentence in the language of rings. Then the following are equivalent:

1. *Some model of* ACF_0 *satisfies* ϕ. *(If you like* $\mathbb{C} \vDash \phi$.*)*
2. $\mathsf{ACF}_0 \vDash \phi$
3. *There exists* $n \in \mathbb{N}$ *such that for any prime* p *greater than* n, $\mathsf{ACF}_p \vDash \phi$
4. *There exists* $n \in \mathbb{N}$ *such that for any prime* p *greater than* n, *some model of* ACF_p *satisfies* ϕ.

(1. $\leftrightarrow$ *2.) and (3.* $\leftrightarrow$ *4.) are due to the theories* ACF_p *being complete for any* p *(0 or prime); (2.* $\leftrightarrow$ *3.) due to compactness theorem.*

Proof overview

An overview of the proof:

$$
\begin{tikzcd}
\text{(algebraic) } \chi_p \text{ Ax-G} \arrow[d, "\text{soundness}"] & & \\
\text{(model th.) } \chi_p \text{ Ax-G} \arrow[d, "\mathsf{ACF}_p \text{ is complete}"] & & \\
\text{(model th.) } \chi_p \text{ Ax-G} \arrow[r, "\text{completeness}"'] \arrow[d, "\text{Lefschetz } \chi\text{-change}"] & \text{(algebraic) } \chi_p \text{ Ax-G} \arrow[dr] & \\
\text{(model th.) } \chi_0 \text{ Ax-G} \arrow[r, "\text{completeness}"'] & \text{(algebraic) } \chi_0 \text{ Ax-G} \arrow[r, "\text{case on } \chi"] & \text{(algebraic) Ax-G}
\end{tikzcd}
$$