MTH 9875 The Volatility Surface: Fall 2017

Lecture 5: Local and implied volatility under stochastic volatility

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Outline of lecture 5

- Approximating implied volatilities given local volatilities
  - An exact but implicit expression
  - The most-likely-path
- Local volatility in the Heston model
- Implied volatility in the Heston model
  - Term structure of ATM variance and ATM skew
- The SVI parameterization
- Comparison between empirical SPX volatility surface and Heston fit

Our objective

- Our objective is to approximate the shape of the implied volatility surface for any given stochastic volatility model, the Heston model in particular.
- In Lecture 2, we showed how to compute local volatilities from implied volatilities:

(1)

$$v_\ell(k,T) = \frac{\frac{\partial w}{\partial T}}{\left(1-\frac{1}{2}\frac{k}{w}\frac{\partial w}{\partial k}\right)^2 - \frac{1}{4}\left(\frac{1}{4}+\frac{1}{w}\right)\left(\frac{\partial w}{\partial k}\right)^2 + \frac{1}{2}\frac{\partial^2 w}{\partial k^2}}$$

- Recall that given a model, we may approximate the conditional expectation of v_t to get v_ℓ using

(2)

$$v_\ell(k,T) = \mathbb{E}[v_T | x_T = k].$$

- An obvious approach would then be to compute local volatility in the stochastic volatility model, then invert (1) to get implied volatility.
  - We showed in Lecture 2 how to invert (1) for small times to expiration.
  - However, nobody has yet figured out how to do this in general!

- Instead, by exploiting the insight (originally due to Bruno Dupire) that implied volatility is related to the breakeven level of a delta-hedging strategy, we derive a general path-integral representation of Black-Scholes implied variance.

Delta hedging under stochastic volatility

Assume the stock price S_t satisfies the SDE

$$\frac{dS_t}{S_t} = \mu_t dt + \sigma_t\, dZ_t$$

where the volatility σ_t may be random. Then path-by-path, for any suitably smooth function $f(S_t, t)$,

$$\begin{aligned} & f(S_T,T) - f(S_0,0) \\ &= \int_0^T df \\ &= \int_0^T \left\{ \frac{\partial f}{\partial S_t}\, dS_t + \frac{\partial f}{\partial t}\, dt + \frac{\sigma_t^2}{2}\, S_t^2\, \frac{\partial^2 f}{\partial S_t^2}\, dt \right\} \end{aligned}$$

In particular (with $f(\cdot) = C_{BS}(\cdot)$), the value of a call option is given by:

$$\begin{aligned} & C(S_0,K,T) \\ &= \mathbb{E}\left[(S_T-K)^+\right] \\ &= \mathbb{E}\left[C_{BS}\left(S_T,K,\bar\sigma(T),0\right)\right] \\ &= C_{BS}\left(S_0,K,\bar\sigma(0),T\right) \\ &\quad + \mathbb{E}\left[\int_0^T \left\{ \frac{\partial C_{BS}}{\partial S_t} dS_t + \frac{\partial C_{BS}}{\partial t} dt + \frac{1}{2}\sigma_t^2 S_t^2\, \frac{\partial^2 C_{BS}}{\partial S_t^2} dt \right\}\right] \end{aligned}$$

where $\bar\sigma(t),\ t \in [0,T]$ is some bounded positive function of t.

The breakeven computation

Now let

$$\bar\sigma(t)^2 = \frac{1}{T-t} \int_t^T v(s)\, ds$$

for some deterministic (bounded and non-negative) forward variance function $v(\cdot)$.

Then $C_{BS}\left(S_t, K, \bar{\sigma}(t), T-t\right)$ satisfies the Black-Scholes equation (assuming zero interest rates and dividends):

$$\frac{\partial C_{BS}}{\partial t} = -\frac{1}{2}\, v(t)\, S_t^2\, \frac{\partial^2 C_{BS}}{\partial S_t^2}.$$

Substituting for the time derivative $\frac{\partial C_{BS}}{\partial t}$, and using that $\mathbb{E}[dS_t] = 0$, we obtain:

$$\begin{aligned} & C\left(S_0, K, T\right) \\ &= C_{\rm BS}\left(S_0, K, \bar{\sigma}(0), T\right) \\ &\quad + \mathbb{E}\left[\int_0^T \frac{1}{2}\left\{\sigma_t^2 - v(t)\right\} S_t^2\, \frac{\partial^2 C_{BS}}{\partial S_t^2} dt\right]. \end{aligned}$$

In particular, if we choose $\bar{\sigma}(0) = \sigma_{\rm BS}(k, T)$, we have $C\left(S_0, K, T\right) = C_{\rm BS}\left(S_0, K, \bar{\sigma}(0), T\right)$ and

(3)

$$\mathbb{E}\left[\int_0^T \frac{1}{2}\left\{\sigma_t^2 - v(t)\right\} S_t^2\, \frac{\partial^2 C_{BS}}{\partial S_t^2} dt\right] = 0.$$

Representation of implied volatility

We then have the following lemma (from [Guyon and Henry-Labordère][6]):

> **Lemma 1 (Guyon and Henry-Labordère)**
>
> (4)
>
> $$\sigma_{BS}^2(K, T) = \frac{1}{T}\int_0^T v(t)\, dt$$
>
> if and only if
>
> $$\int_0^T \mathbb{E}\left[\frac{1}{2}\left\{\sigma_t^2 - v(t)\right\} S_t^2\, \frac{\partial^2 C_{BS}^v}{\partial S_t^2}\right]\, dt = 0.$$
>
> where C_{BS}^v represents the Black-Scholes formula computed with $v(t)$ as the forward variance function.

σ_{BS} as the breakeven implied volatility

- In words,

$$\int_0^T \mathbb{E}\left[\frac{1}{2}\left\{\sigma_t^2 - v(t)\right\} S_t^2\, \frac{\partial^2 C_{BS}^v}{\partial S_t^2}\right]\, dt$$

  gives the expected realized profit on the purchase of a European call or put option at the market price, delta-hedged using the deterministic forward variance function $v(t)$ when the actual realized volatility is σ_t.

Black-Scholes gamma

For fixed K and T, define the Black-Scholes gamma

$$\Gamma_{\text{BS}}(S_t,t) := \frac{\partial^2}{\partial S_t^2}\, C_{\text{BS}}(S_t,K,\bar{\sigma}(t),T-t).$$

Then, in terms of Γ_{BS}, the breakeven condition becomes

$$\mathbb{E}\left[\int_0^T \frac{1}{2}\left\{\sigma_t^2 - v(t)\right\} S_t^2\, \Gamma_{\text{BS}}(S_t,t)\,dt\right] = 0.$$

Black-Scholes forward implied variance

Now choose $v(t)$ to be the *Black-Scholes forward implied variance* function

(5)

$$v(t) = \frac{\mathbb{E}\left[\sigma_t^2\, S_t^2\, \Gamma_{BS}(S_t,t)\right]}{\mathbb{E}\left[S_t^2\, \Gamma_{BS}(S_t,t)\right]}.$$

Then

$$\mathbb{E}\left[S_t^2\, \Gamma_{BS}(S_t,t)\right] v\,(t) = \mathbb{E}\left[\sigma_t^2\, S_t^2\, \Gamma_{BS}(S_t,t)\right]$$

and the breakeven condition (4) is satisfied.

σ_{BS}^2 as a path integral

It then follows from Lemma 1 (5) that

Path integral representation of Black-Scholes implied volatility

(6)

$$\sigma_{BS}(K,T)^2 = \frac{1}{T}\int_0^T \frac{\mathbb{E}\left[\sigma_t^2 S_t^2 \Gamma_{BS}(S_t,t)\right]}{\mathbb{E}\left[S_t^2 \Gamma_{BS}(S_t,t)\right]}dt$$

- Equation (6) expresses implied variance as the time-integral of expected instantaneous variance σ_t^2 under some probability measure.
- (6) is *exact* but implicit.

Interpretation

- To compute the Black-Scholes implied volatility of an option, we need to average the possible realized volatilities over all possible scenarios.
- Each such scenario is weighted by the gamma of the option.
  - The profitability of the delta-hedger in any time interval is directly proportional to the gamma and the difference between “expected instantaneous variance” (or local variance) and realized instantaneous variance.
- Only paths that start at S_0 and end at the strike price K need be included in the average.
  - At inception of the delta-hedge, the stock price can only be S_0.
  - At expiration, gamma is precisely zero if $S_T \neq K$. Gamma explodes at $S_T = K$.

An elegant recasting of (6)

Following Roger Lee, we may rewrite (6) more elegantly as

(7)

$$\sigma_{BS}(K,T)^2 = \bar{\sigma}(0)^2 = \frac{1}{T}\int_0^T \mathbb{E}^{G_t}[\sigma_t^2]\,dt$$

thus interpreting the definition (5) of $v(t)$ as the expectation of σ_t^2 with respect to the probability measure $\mathbb{G}_t$ defined, relative to the pricing measure $\mathbb{Q}$, by the Radon-Nikodym derivative

$$\frac{d\mathbb{G}_t}{d\mathbb{Q}} := \frac{S_t^2\,\Gamma_{BS}(S_t,\bar{\sigma}(t))}{\mathbb{E}\left[S_t^2\,\Gamma_{BS}(S_t,\bar{\sigma}(t))\right]}.$$

Remark

Equations (5) and (6) are implicit because the gamma $\Gamma_{BS}(S_t,t)$ of the option depends on all the forward implied variances $v\,(t)$.

Special Case: Black-Scholes

Suppose $\sigma_t = \sigma(t)$, a function of t only. Then

$$v\,(t) = \frac{\mathbb{E}\left[\sigma(t)^2 S_t^2 \Gamma_{BS}(S_t)\right]}{\mathbb{E}\left[S_t^2 \Gamma_{BS}(S_t)\right]} = \sigma(t)^2.$$

The forward implied variance $v(t)$ and the forward variance $\sigma(t)^2$ coincide.

As expected,in this case, $v(t)$ has no dependence on the strike K or the option expiration T.

Understanding the integral representation (6)

In order to get better intuition for the path integral representation (6), first recall how to compute a risk-neutral expectation:

$$\mathbb{E}^{\mathbb{Q}}\left[f\left(S_t\right)\right]=\int dS_t\, p\left(S_t,t;S_0\right) f\left(S_t\right).$$

We get the risk-neutral pdf of the stock price at time t by taking the second derivative of the market price of European options with respect to strike price.

$$p\left(S_t,t;S_0\right)=\left.\frac{\partial^2 C\left(S_0,K,t\right)}{\partial K^2}\right|_{K=S_t}.$$

Then from equation (7) we have

(8)

$$\begin{aligned} v(t) &= \mathbb{E}^{G_t}\left[\sigma_t^2\right]\\ &= \mathbb{E}^{\mathbb{Q}}\left[\sigma_t^2 \frac{d\mathbb{G}_t}{d\mathbb{Q}}\right]\\ &= \int dS_t\, q\left(S_t;S_0,K,T\right)\mathbb{E}^{\mathbb{Q}}\left[\sigma_t^2\middle|S_t\right]\\ &= \int dS_t\, q\left(S_t;S_0,K,T\right) v_\ell\left(S_t,t\right)\end{aligned}$$

where we further define

$$q\left(S_t,t;S_0,K,T\right):=\frac{p\left(S_t,t;S_0\right)\ S_t^2\ \Gamma_{BS}(S_t,t)}{\mathbb{E}\left[S_t^2\ \Gamma_{BS}(S_t,t)\right]}$$

and $v_\ell\left(S_t,t\right)=\mathbb{E}^{\mathbb{Q}}\left[\sigma_t^2\middle|S_t\right]$ is the local variance.

Brownian Bridge-like density

- $q\left(S_t,t;S_0,K,T\right)$ looks like a Brownian Bridge density for the stock price:
  - $p\left(S_t,t;S_0\right)$ has a delta-function peak at S_0 at time 0 and
  - $\Gamma_{BS}(S_t,t)$ has a delta-function peak at K at expiration T.

For convenience in what follows, we now rewrite equation (8) in terms of $x_t := \log\left(S_t/S_0\right)$:

(9)

$$v(t)=\int dx_t\, q\left(x_t,t;x_T,T\right) v_\ell\left(x_t,t\right)$$

Download some code

In [38]:
```
download.file(url="http://mfe.baruch.cuny.edu/wp-content/uploads/2014/09/MTH98
unzip(zipfile="rFiles4.zip")

source("BlackScholes.R")
source("Heston2.R")
source("plotIvols.R")
source("svi.R")
source("sviVolSurface.R")
```

The quasi-Brownian Bridge density

In [39]:
```
p <- function(St,t,S0,sigma){
    x <- log(St/S0)
    sig <- sigma*sqrt(t)
    return(dnorm(x/sig-sig/2)/sig/St)
   }

gammaBS <- function(S0,K,T,sigma){
    x <- log(S0/K)
    sig <- sigma*sqrt(T)
    d1 <- x/sig+sig/2
    return(dnorm(d1)/sig/S0)
   }

q <- function(St,t,S0,K,T,sigma){p(St,t,S0,sigma)*gammaBS(St,K,T-t,sigma)*St^2

q1 <- function(x,t){q(exp(x),t,S0=1,K=1.3,T=1,sigma=.2)}
q1Vec <- function(k,t){sapply(t,function(t){q1(k,t)})} # Vectorized version fo
```

In [40]:
```
library(repr)
options(repr.plot.height=5)
```

In [41]:
```
# Setup the plot
x <- seq(-.4,.4,0.01) # Vector of log-strikes
t <- seq(0.04,0.96,0.02) # Vector of times
z <- q1Vec(x,t) # Array of volatilities

# Add colors
nbcol <- 100
color <- rainbow(nbcol,start=.4,end=.8)
nrz <- nrow(z)
ncz <- ncol(z)
# Compute the z-value at the facet centres
zfacet <- z[-1, -1] + z[-1, -ncz] + z[-nrz, -1] + z[-nrz, -ncz]
# Recode facet z-values into color indices
facetcol <- cut(zfacet, nbcol)
```

In [42]:
```
options(repr.plot.width=14,repr.plot.height=10)
```

In [43]:
```
# Generate 3D plot
persp(x, t, z, col=color[facetcol], phi=50, theta=30,
    r=1/sqrt(3)*20,d=5,expand=.4,ltheta=-135,lphi=20,ticktype="detailed",
    shade=.5,border=NA,xlab="Log-spot x",ylab="Time t",box=T, axes=c(T,T,F),z
```

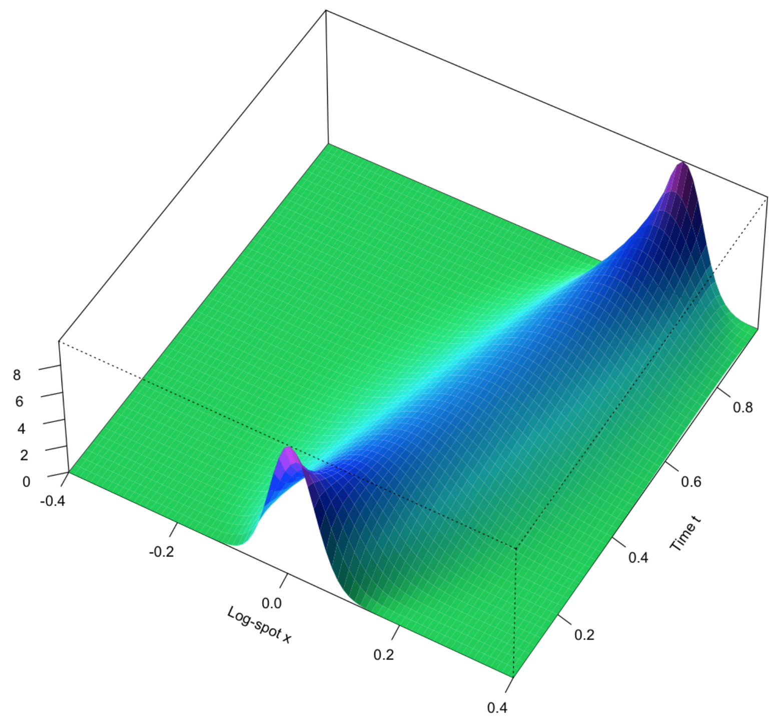


Figure 1: Graph of the density of x_t conditional on $x_T = \log(K)$ for a 1 year European option, strike 1.3 with current stock price $= 1$ and 20% volatility.

The most-likely-path approximation

- Figure 1 shows what $q\,(x_t, t; x_T, T)$ looks like in the case of a 1 year European option struck at 1.3 with a flat 20% volatility.
    - We see that $q\,(x_t, t; x_T, T)$ peaks on a line (which we will denote by $\tilde{x}_t$ and call the *most-likely-path*) joining the stock price today with the strike price at expiration.
    - Moreover, the density looks roughly symmetric around the peak. This suggests an expansion around the peak $\tilde{x}_t$ (at which the derivative of $q\,(x_t, t; x_t, T)$ with respect to x_t is zero).

This motivates us to write:

(10)

$$q\,(x_t, t; x_T, T) \approx q(\tilde{x}_t, t; x_T, T) + \frac{1}{2}(x_t - \tilde{x}_t)^2 \left.\frac{\partial^2 q}{\partial x_t^2}\right|_{x_t=\tilde{x}_t}$$

In practice, the local variance $v_\ell(x_t, t)$ is typically not so far from linear in x_t in the region where $q(x_t, t; x_T, T)$ is significant so we may further write

(11)
$$v_\ell(x_t, t) \approx v_\ell(\tilde{x}_t, t) + (x_t - \tilde{x}_t) \left.\frac{\partial v_\ell}{\partial x_t}\right|_{x_t=\tilde{x}_t}$$

Substituting (10) and (11) into the integrand in equation (9) gives

$$v(t) \approx v_\ell(\tilde{x}_t, t)$$

and we may rewrite equation (6) as

(12)
$$\sigma_{BS}(K, T)^2 \approx \frac{1}{T} \int_0^T v_\ell(\tilde{x}_t)\, dt$$

The MLP approximation in words

In words, equation (12) says that

> The Black-Scholes implied variance of an option with strike K is given approximately by the integral from valuation date ($t = 0$) to the expiration date ($t = T$) of the local variances along the path $\tilde{x}_t$ that maximizes the Brownian Bridge density $q(x_t, t; x_T, T)$.

- Note that in practice, it's not trivial to compute the most likely path $\tilde{x}_t$.
  - [Adil Reghai][7] describes an efficient fixed-point algorithm to do this.

A fixed point algorithm for finding the most likely path

For each log-strike $k := \log(K/S_0)$, we approximate the most likely path $\tilde{x}_t$ as

$$\tilde{x}_t = \frac{w_t}{w_T}\, k$$

with

$$w_t = \int_0^t ds\, v_\ell(\tilde{x}_s)$$

Obviously, this definition is circular. We solve by starting with the straight line

$$\tilde{x}_t = \frac{t}{T}\, k$$

as our initial guess and iterating until the path doesn't change. This iteration is extremely fast in practice.

Problems with the MLP approximation

- The MLP approximation should work well if local variance is roughly linear in the neighborhood of the peak of $q(\cdot)$.
  - Conversely, if there is substantial curvature, the approximation does not work well.
    - For example, at-the-money if there is curvature and little or no skew (as in FX).
  - The MLP approximation is too rough for longer dates when the density $q(\cdot)$ is not so highly peaked.

- It's not clear that the peak of the density is the appropriate most-likely-path. For example, Reghai actually approximates the conditional expectation:

$$\tilde{S}_t := \mathbb{E}\left[S_t | S_T = K\right]$$

- [Guyon and Henry-Labordère][5] show how to correct for curvature (it is rather slow and complicated).

Variational MLP

More recently, [Gatheral and Wang][4] came up with another version of MLP where the most-likely-path is now uniquely defined.

Define

$$f(x(t),t) = \partial_t \log \sigma(x,t)|_{x=x(t)}$$

Then

$$\sigma_{\rm BS}(k,T) \approx \frac{\frac{1}{T}\int_0^T \sigma(\tilde{x}(t),t)\,\exp\left\{\int_0^t f(\tilde{x}(s),s)\,ds\right\}\,dt}{\sqrt{\frac{1}{T}\int_0^T \exp\left\{2\int_0^t f(\tilde{x}(s),s)\,ds\right\}\,dt}}.$$

The most-likely-path $\tilde{x}(t)$ given (implicitly again) by

(13)

$$\tilde{x}(t) = k\,\frac{\int_0^t du\,\sigma(\tilde{x}(u),u)\,\exp\left\{\int_0^u f(\tilde{x}(s),s)\,ds\right\}}{\int_0^T du\,\sigma(\tilde{x}(u),u)\,\exp\left\{\int_0^u f(\tilde{x}(s),s)\,ds\right\}}.$$

Local variance in the Heston model

Recall the Heston SDE:

(14)

$$\begin{aligned} dx_t &= -\frac{v_t}{2}dt + \sqrt{v_t}\,dZ_t \\ dv_t &= -\lambda\,(v_t - \bar{v})dt + \rho\,\eta\,\sqrt{v_t}\,dZ_t + \sqrt{1-\rho^2}\,\eta\,\sqrt{v_t}\,dZ_t^{\perp} \end{aligned}$$

where dZ_t and $dZ_t^{\perp}$ are orthogonal.

Eliminating $\sqrt{v_t} dZ_t$, we get

(15)

$$dv_t = -\lambda\,(v_t - \bar{v})\,dt + \rho\,\eta\,\left(dx_t + \frac{1}{2}\,v_t\,dt\right) + \sqrt{1-\rho^2}\,\eta\,\sqrt{v_t}\,dZ_t^{\perp}$$

Our strategy will be to compute local variances in the Heston model and then integrate local variance from valuation date to expiration date to approximate the BS implied variance using equation (12).

Expected variance

First, consider the unconditional expectation $\hat{v}_s$ of the instantaneous variance at time s. Solving equation (15) gives

$$\hat{v}(s) \;=\; (v_0 - \bar{v})\,e^{-\lambda\,s} \;+\; \bar{v}.$$

Then define the expected total variance to time t through the relation

$$\hat{w}(t) \;:=\; \int_0^t \hat{v}(s)ds \;=\; (v_0 - \bar{v})\left\{\frac{1-e^{-\lambda\,t}}{\lambda}\right\} \;+\; \bar{v}\,t.$$

Finally, let $u_t := \mathbb{E}[v_t\,|x_T] = v_\ell(x_T, T)$ be the expectation of the instantaneous variance at time t conditional on the final value x_T of x.

An ansatz

According to Wikipedia:

> "... an ansatz is an educated guess that is verified later by its results."

Without loss of generality, assume $x_0 = 0$. Then, the ansatz is that

$$\mathbb{E}[x_s\,|x_T] = x_T\,\frac{\hat{w}(s)}{\hat{w}(T)}$$

where $\hat{w}(t) := \int_0^t ds\,\hat{v}(s)$ is the expected total variance to time t.

To see that this ansatz is plausible, note that

$$\mathbb{E}\,(x_s) = \mathbb{E}(x_T)\,\frac{\hat{w}(s)}{\hat{w}(T)} = -\frac{\hat{w}(T)}{2}\,\frac{\hat{w}(s)}{\hat{w}(T)} = -\frac{\hat{w}(s)}{2}.$$

If the process for x_t were a conventional Brownian Bridge process, the ansatz would be true; in this case, it is only approximately true.

Approximate computation of local variance

Assuming the ansatz to be correct, we may take the conditional expectation of (15) to get:

(16)

$$du(t) \;=\; -\lambda (u(t) - \bar{v}) dt \;+\; \frac{\rho\eta}{2} u(t) dt \;+\; \rho\eta \frac{x_T}{\hat{w}(T)} d\hat{w}(t) \\ +\; \sqrt{1-\rho^2}\,\eta \sqrt{v_t}\; \mathbb{E}[dZ_t^{\perp} | x_T]$$

where $u(t) = \mathbb{E}[v_t | x_T]$.

If the dependence of $dZ_t^{\perp}$ on x_T is weak or if $\sqrt{1-\rho^2}$ is very small, we may drop the last term to get

$$du(t) \;\approx\; -\lambda' (u(t) - \bar{v}') dt \;+\; \rho\eta \frac{x_T}{\hat{w}(T)} \hat{v}(t) dt$$

with $\lambda' = \lambda - \rho\eta/2$, $\bar{v}' = \bar{v}\lambda/\lambda'$. The solution to this equation is

(17)

$$v_\ell(x_T, T) = u(T) \approx \hat{v}'(T) \;+\; \rho\eta\, \frac{x_T}{\hat{w}(T)} \int_0^T \hat{v}(s)\, e^{-\lambda' (T-s)} ds$$

with $\hat{v}'(s) := \left(v - \bar{v}'\right) e^{-\lambda' s} + \bar{v}'$.

The accuracy of our Heston local volatility approximation

- Equation (17) gives us an approximate but surprisingly accurate formula for local variance within the Heston model (an extremely accurate approximation when $\rho = \pm 1$) .
  - Local variance is approximately linear in $x = \log(F/K)$.
- To get (17), we made two approximations: the ansatz and dropping the last term in (16).

- For reasonable parameters, equation (17) gives good intuition for the functional form of local variance.
  - When $\rho = \pm 1$, it is almost exact.
  - Equation (17) is in fact exact to first order in η whether or not the ansatz holds or $\sqrt{1-\rho^2}$ is small.

A special case of Heston: The Heston-Nandi model

From equation (17), if $\rho = \pm 1$ and if $v = \bar{v}$, Heston local variance should be well approximated by

(18)

$$
\begin{aligned}
v_\ell(x_T,T) &= \left[\hat{v}'_T - \eta\,\frac{x_T}{w_T}\int_0^T \hat{v}_s\, e^{-\lambda'(T-s)} ds\right]^+ \\
&= \left[(v-\bar{v}')e^{-\lambda' T} + \bar{v}' - \eta\, x_T\,\left\{\frac{1-e^{-\lambda' T}}{\lambda' T}\right\}\right]^+
\end{aligned}
$$

with $\lambda' = \lambda + \frac{\eta}{2}$, $\bar{v}' = \bar{v}\frac{\lambda}{\lambda'}$. The whole expression must be bounded below by zero - all stock prices above the critical stock price at which the local variance reaches zero are unattainable.

Heston-Nandi (HN) parameters

As throughout [The Volatility Surface][2], we will adopt the following Heston parameters:

$$
\begin{aligned}
v &= 0.04 \\
\bar{v} &= 0.04 \\
\lambda &= 10 \\
\eta &= 1 \\
\rho &= -1.
\end{aligned}
$$

Monte-Carlo vs approximate Heston-Nandi local variance smile

Adapting the local variance estimation code from Problem 4 of HW4, we get the following plot:

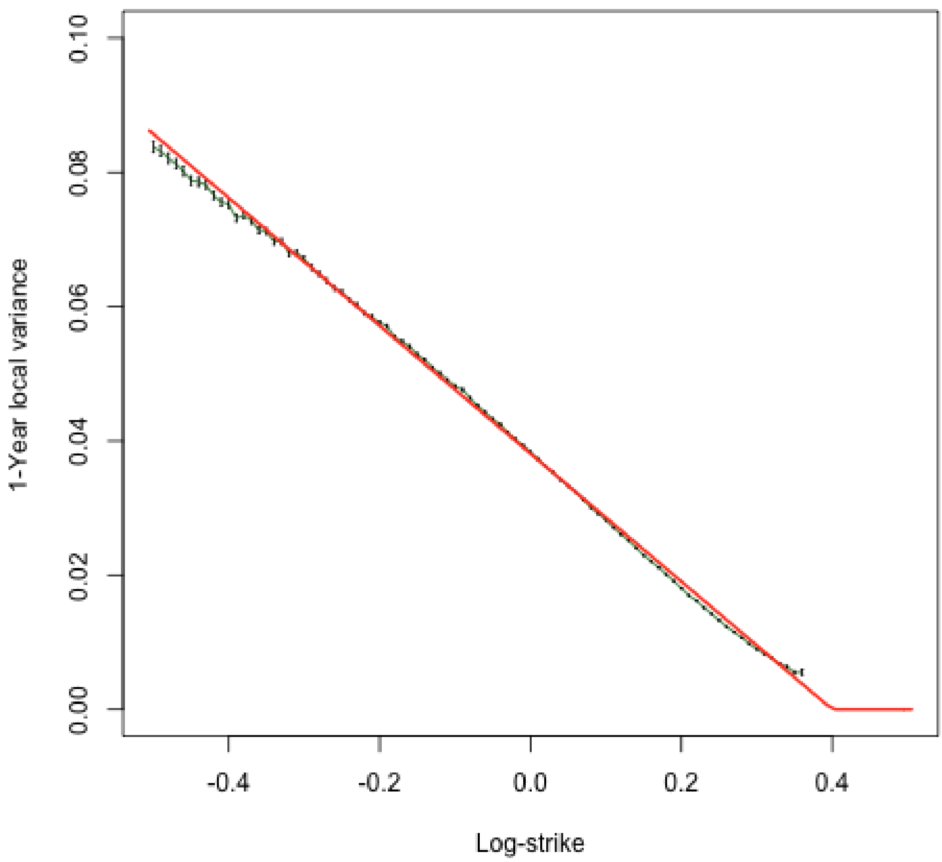


Figure 2: Local variance from Monte Carlo simulation (5mm paths) in dark green; the approximate local variance formula (18) in red.

Implied volatility in the Heston model

- Having computed approximate local variance in the Heston model, we now apply the MLP appproximation to get implied variance.
  - We need to integrate the Heston local variance along the most likely stock price path joining the initial stock price to the strike price at expiration (the one which maximizes the Brownian Bridge probability density).

Then, the Black-Scholes implied variance is given by

(19)

$$\sigma_{BS}(K,T)^2 \approx \frac{1}{T} \int_0^T v_\ell(\tilde{x}_t, t)\, dt$$

where $\{\tilde{x}_t\}$ is the most likely path (as defined earlier).

Recall that the Brownian Bridge density $q\,(x_t, t; x_T, T)$ is roughly symmetric and peaked around $\tilde{x}_t$, so $\mathbb{E}\,[x_t - \tilde{x}_t\,|x_T] \approx 0$. Applying the ansatz once again, we obtain

$$\tilde{x}_t = \mathbb{E}\left[\tilde{x}_t \left| x_T\right.\right] = \mathbb{E}\left[\tilde{x}_t - x_t \left| x_T\right.\right] + \mathbb{E}\left[x_t \left| x_T\right.\right] \approx \frac{\hat{w}(t)}{\hat{w}(T)} x_T$$

We substitute this expression back into equations (17) and (19) to get

(20)

$$\begin{aligned}\sigma_{BS}(K,T)^2 &\approx \frac{1}{T}\int_0^T u_t(\tilde{x}_t)dt \\ &\approx \frac{1}{T}\int_0^T \hat{v}'(t)dt + \rho\eta\,\frac{x_T}{\hat{w}(T)}\,\frac{1}{T}\int_0^T dt\int_0^t \hat{v}(s)\,e^{-\lambda'(t-s)}ds\end{aligned}$$

The term structure of implied volatility in the Heston model

The at-the-money term structure of BS implied variance in the Heston model is obtained by setting $x_T = 0$ in equation (20). Performing the integration explicitly gives

$$\begin{aligned}\sigma_{BS}(K,T)^2\Big|_{K=F_T} &\approx \frac{1}{T}\int_0^T \hat{v}'(t)dt \\ &= \frac{1}{T}\int_0^T \left[\left(v - \bar{v}'\right)e^{-\lambda' t} + \bar{v}'\right]dt \\ &= \left(v - \bar{v}'\right)\frac{1 - e^{-\lambda' T}}{\lambda' T} + \bar{v}'.\end{aligned}$$

We see that in the Heston model, the at-the-money Black-Scholes implied variance

$$\sigma_{BS}(K,T)^2\Big|_{K=F_T} \rightarrow v$$

as $T \rightarrow 0$ and as $T \rightarrow \infty$, the at-the-money Black-Scholes implied variance reverts to $\bar{v}'$.

The implied volatility skew in the Heston model

- By inspection of (19) the implied variance skew in the Heston model is approximately linear in the correlation ρ and the volatility of volatility η.
- When $v_0 = \bar{v}$, $\hat{v}(s) = \bar{v}$ and $\hat{w}(t) = \bar{v}\,t$.
  - The most likely path $\tilde{x}_t \approx \frac{t}{T} x_T$ is exactly a straight line in log-space between the initial stock price on valuation date and the strike price at expiration.

Performing the integrations in equation (20) explicitly, we get

(21)

$$\begin{aligned}\sigma_{BS}(K,T)^2 &\approx \frac{\hat{w}'(T)}{T} + \rho\eta\frac{x_T}{T^2}\int_0^T dt\,\frac{1}{T}\int_0^t e^{-\lambda'(t-s)}ds \\ &= \frac{\hat{w}'(T)}{T} + \rho\eta\,\frac{x_T}{\lambda' T}\left\{1 - \frac{\left(1 - e^{-\lambda' T}\right)}{\lambda' T}\right\}\end{aligned}$$

where

$$\hat{w}'(T) = \int_0^T \hat{v}'(t)dt.$$

Qualitative features of the ATM skew

- From (21), we see that the implied variance skew $\frac{\partial}{\partial x_t}\sigma_{BS}(K,T)^2$ is *independent* of the level of instantaneous variance v or long-term mean variance $\bar{v}$.
  - This remains approximately true even when $v \neq \bar{v}$.

```
- <span>It follows that we now have a fast way of calibrating the Heston
model to observed implied volatility skews. </span>

    - <span>Just two expirations would in principle allow us to determine
$\lambda'$ and the product $\rho
\eta$. </span>

    - <span> We can then fit the term structure of volatility to determine
the
long term mean variance $\bar v$ and the instantaneous variance
$v_0$.</span>


- <span> The order $\eta^2$ curvature of the smile (not discussed here)
would allow us to
determine $\rho$ and $\eta$ separately.</span>
    - Also, equation [(21)](#eq21:hestonsmile) generates ATM volatilities
that are too high.  We need the $O(\eta^2)$ (convexity) term to correct
this.


- <span> We note that as we increase either the correlation $\rho$ or the
volatility of volatility $\eta$, the skew increases.</span>
```

The Bergomi-Guyon expansion

- As we will see in more detail in Lecture 8, the [Bergomi and Guyon][1] (BG) implied volatility expansion allows us to compute Heston implied volatility up to order η^2.
  - Smile curvature and the correction to the ATM volatility are computable up to this order.
  - We will be able to check explicitly that the BG skew is consistent with equation (21) to first order in η.

Asymptotic analysis

- The very short-dated skew is independent of λ and T:

$$
\begin{aligned}
\frac{\partial}{\partial x_t}\sigma_{BS}(K,T)^2 &= \rho\eta\,\frac{1}{\lambda' T}\left\{1-\frac{\left(1-e^{-\lambda' T}\right)}{\lambda' T}\right\} \\
&\to \frac{\rho\eta}{2} \text{ as } T\to 0.
\end{aligned}
$$

- The long-dated skew is inversely proportional to T:

$$
\begin{aligned}
\frac{\partial}{\partial x_t}\sigma_{BS}(K,T)^2 &= \rho\eta\,\frac{1}{\lambda' T}\left\{1-\frac{\left(1-e^{-\lambda' T}\right)}{\lambda' T}\right\} \\
&\sim \frac{\rho\eta}{\lambda' T} \text{ as } T\to \infty.
\end{aligned}
$$

- Finally, increasing η causes the curvature of the implied volatility skew (related to the kurtosis of the risk-neutral density) to increase but we haven't shown that here.

Dynamics of the volatility skew under local volatility

Recall formula (1.10) from TVS for local volatility in terms of implied volatility:

$$
v_\ell = \frac{\frac{\partial w}{\partial T}}{1-\frac{k}{w}\frac{\partial w}{\partial k}+\frac{1}{4}\left(-\frac{1}{4}-\frac{1}{w}+\frac{k^2}{w^2}\right)\left(\frac{\partial w}{\partial k}\right)^2+\frac{1}{2}\frac{\partial^2 w}{\partial k^2}}.
$$

Differentiating with respect to k and considering only the leading term in $\frac{\partial w}{\partial k}$ (which is small for large T) , we find

$$
\frac{\partial v_\ell}{\partial k} \approx \frac{\partial}{\partial T}\frac{\partial w}{\partial k} + \frac{1}{w}\frac{\partial w}{\partial T}\frac{\partial w}{\partial k}.
$$

That is, the local variance skew $\frac{\partial v_\ell}{\partial k}$ decays with the BS implied total variance skew $\frac{\partial w}{\partial k}$.

Dynamics of the volatility skew under local volatility

- To get the forward volatility surface in a local volatility model, we integrate over the local volatilities from the (forward) valuation date to the expiration of the option along the most likely path.
  - The forward volatility surface will be substantially flatter than today's because the forward local volatility skews are all flatter.

- Contrast this with a stochastic volatility model where implied volatility skews are approximately time-homogeneous.
  - Local volatility models imply that future BS implied volatility surfaces will be flat (relative to today's).
  - Stochastic volatility models imply that future BS implied volatility surfaces will look like today's.

The SVI parameterization

In the SVI ("Stochastic Volatility Inspired") parameterization, for each expiration, we write

(22)

$$\sigma^2_{BS}(k) = a + b\,\left\{\rho\,(k-m) + \sqrt{(k-m)^2+\sigma^2}\right\}$$

where the coefficients a, b, ρ, σ and m depend on the expiration.

- SVI is just a *parameterization*, there are no underlying dynamics associated with it.

- In [Gatheral and Jacquier][3], we show that it is possible to guarantee no static arbitrage when fitting SVI.
  - The functional form (22) is fitted to all expirations simultaneously subject to the constraint that there should be no calendar spread or butterfly arbitrage.

- Total implied variance may then be interpolated between slices to give a smooth surface.
  - Here we use Stineman monotonic spline interpolation.

The SPX implied volatility surface

Do implied volatilities produced by the Heston model look like implied volatilities in the market?

To get a sense of what an actual implied volatility surface looks like, Figure 2 shows the surface resulting from an SVI fit to observed implied variance as a function of k for each expiration on September 15, 2005, the day before the September *triple witching day*.

A triple witching day is a day on which both index option contracts and index futures contracts expire.

Figure 3.2: 3D plot of volatility surface

In [44]:
```
download.file(url="http://mfe.baruch.cuny.edu/wp-content/uploads/2014/09/spxO
unzip(zipfile="spxOptData050915.rData.zip")

load("spxOptData050915.rData")

head(spxOptData)
```

Expiry	Texp	Strike	Bid	Ask	Fwd	CallMid
20050917	0.003832991	500	NA	4.614143	1227.82	NA
20050917	0.003832991	600	NA	3.717240	1227.82	NA
20050917	0.003832991	650	NA	3.322663	1227.82	NA
20050917	0.003832991	700	NA	2.956414	1227.82	NA
20050917	0.003832991	750	NA	2.614263	1227.82	NA
20050917	0.003832991	800	NA	2.292761	1227.82	NA

Here is a pre-cooked SVI fit:

In [45]:
```
texp <- sort(unique(spxOptData$Texp))

svidata <- c(
 c(-0.0001449630, 0.0092965440, 0.0196713280, -0.2941176470,-0.0054273230),
 c(-0.000832134 , 0.024439766 , 0.069869455 , -0.299975308 , 0.02648364  ),
 c(-0.0008676750, 0.0282906450, 0.0873835580, -0.2892204290, 0.0592703000),
 c(-0.0000591593, 0.0331790820, 0.0812872370, -0.3014043240, 0.0652549210),
 c(0.0011431940 , 0.0462796440, 0.1040682980, -0.3530782140, 0.0942000770),
 c(0.0022640980 , 0.0562604150, 0.1305339330, -0.4387409470, 0.1111230690),
 c(0.0040335530 , 0.0733707550, 0.1707947600, -0.4968970370, 0.1496609160),
 c(0.0034526910 , 0.0917230540, 0.2236814130, -0.4942213210, 0.1854128490))

sviMatrix <- as.data.frame(t(array(svidata,dim=c(5,8))))
colnames(sviMatrix)<-c("a","b","sig","rho","m")
```

In [46]:
```
sviMatrix
```

a	b	sig	rho	m
-0.0001449630	0.009296544	0.01967133	-0.2941176	-0.005427323
-0.0008321340	0.024439766	0.06986945	-0.2999753	0.026483640
-0.0008676750	0.028290645	0.08738356	-0.2892204	0.059270300
-0.0000591593	0.033179082	0.08128724	-0.3014043	0.065254921
0.0011431940	0.046279644	0.10406830	-0.3530782	0.094200077
0.0022640980	0.056260415	0.13053393	-0.4387409	0.111123069
0.0040335530	0.073370755	0.17079476	-0.4968970	0.149660916
0.0034526910	0.091723054	0.22368141	-0.4942213	0.185412849

Now build a 3D plot of the SVI fit to the SPX volatility surface.

In [47]:
```
volTVS <- function(k,t){sqrt(sviW(sviMatrix,texp,k,t)/t)}

# Setup the plot
k <- seq(-.5,.5,0.01) # Vector of log-strikes
t <- seq(0.04,1.74,0.02) # Vector of times
z <- t(volTVS(k,t)) # Array of volatilities

# Add colors
nbcol <- 100
color <- rainbow(nbcol,start=.3,end=.4)
nrz <- nrow(z)
ncz <- ncol(z)

# Compute the z-value at the facet centres
zfacet <- z[-1, -1] + z[-1, -ncz] + z[-nrz, -1] + z[-nrz, -ncz]
# Recode facet z-values into color indices
facetcol <- cut(zfacet, nbcol)
```

In [54]:
```
options(repr.plot.width=10,repr.plot.height=7)
```

Finally, generate the plot:

In [57]:
```
persp(k, t, z, col=color[facetcol], phi=30, theta=30,
    r=1/sqrt(3)*20,d=5,expand=.5,ltheta=-135,lphi=20,ticktype="detailed",
    shade=.8,border=NA,xlab="Log-strike k",ylab="Expiration t",zlab="Implied v
```

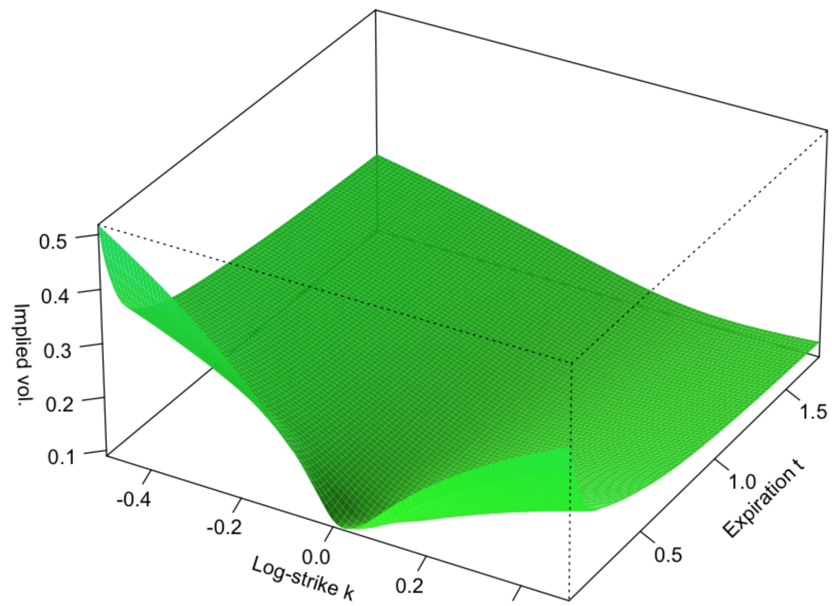
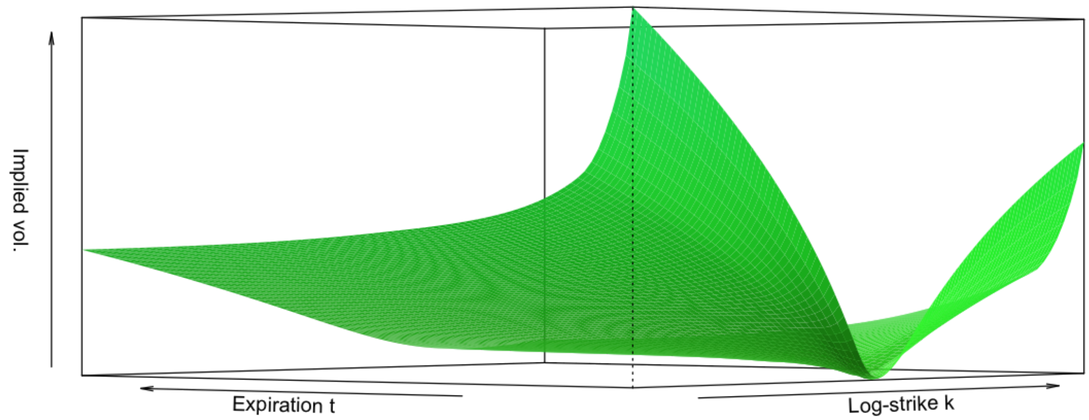


Figure 3: Graph of the SPX implied volatility surface as of the close on September 15, 2005, the day before triple witching.

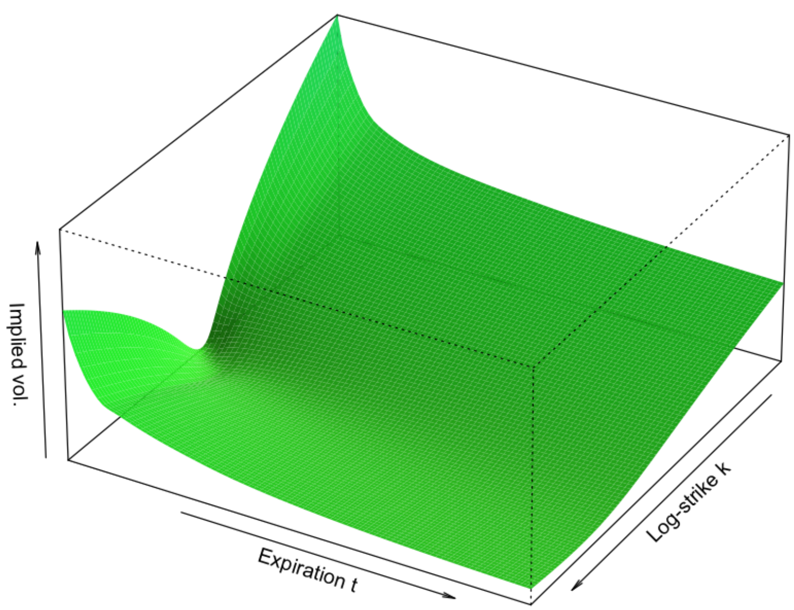
Now wrap into a simple function and experiment...

In [58]:
```
# Wrap into simple function
plotVols <- function(phi,theta){
    persp(k, t, z, col=color[facetcol], phi=phi, theta=theta,
        r=1/sqrt(3)*20,d=5,expand=.5,ltheta=-135,lphi=20,ticktype="simple",
        shade=.8,border=NA,xlab="Log-strike k",ylab="Expiration t",zlab="Impl:
}
```

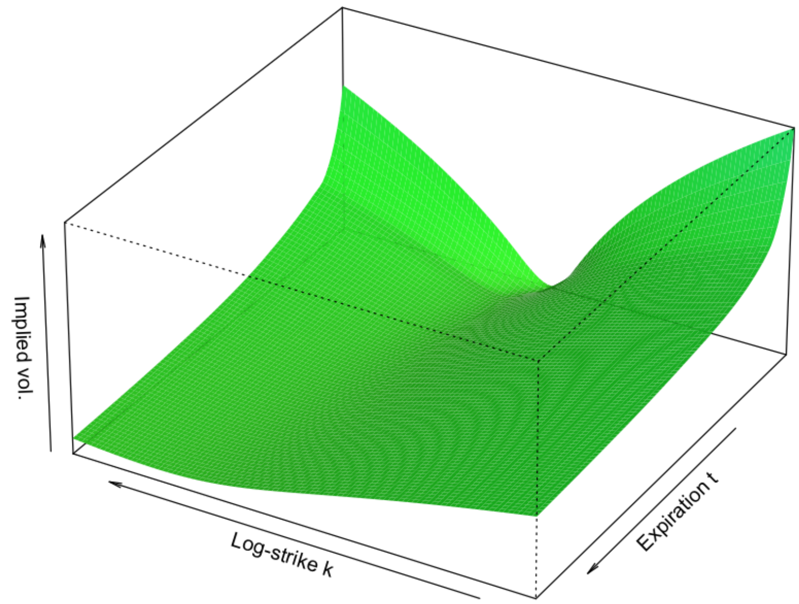
In [59]:
```
plotVols(.1,-50)
```



In [60]:
```
plotVols(30,120)
```



In [61]:
```
plotVols(30,210)
```



SVI fits to individual expirations

In [62]:
```
res <- plotIvols(spxOptData,sviMatrix)
```

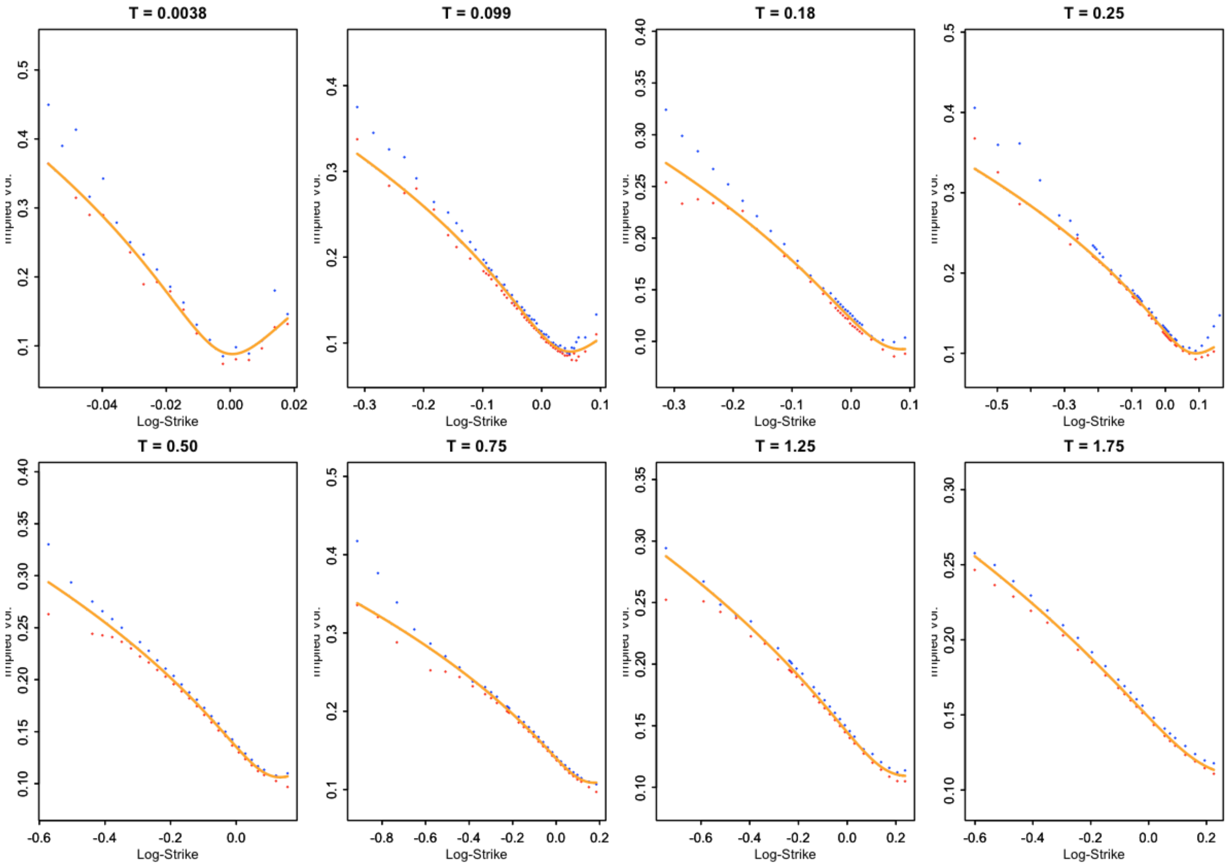


Figure 4: Plots of the SVI fits to SPX implied volatilities as of September 15, 2005. Bid vols are in red; offer vols in blue; the orange line is the SVI fit.

Empirical ATM levels and skews

From the SVI fit, we impute the at-the-money forward variance levels and skews listed in Table 1. (Recall that by at-the-money skew, we mean $\frac{\partial}{\partial k}\sigma_{BS}(k,T)^2\Big|$ where k is the log-strike).

Expiration	Time to expiry	ATM Variance	ATM Skew
Sep-05	1 day	0.0078	-0.0683
Oct-05	1 month	0.0121	-0.1623
Nov-05	2 months	0.0149	-0.1372
Dec-05	3 months	0.0161	-0.1221
Mar-06	6 months	0.0183	-0.0946
Jun-06	9 months	0.0195	-0.0815
Dec-06	15 months	0.0209	-0.0679
Jun-07	21 months	0.0220	-0.0595

Table 1: At-the-money SPX variance levels and skews as of the close on September 15, 2005, the day before expiration.

Code to generate Table 1.

In [63]:
```
sviVarianceATM <- function(sviParams){

    a <- sviParams$a
    b <- sviParams$b
    rho <- sviParams$rho
    m <- sviParams$m
    sig <- sviParams$sig

    var <-a + b*(-m*rho+sqrt(m^2+sig^2))
    return(var)

    }

sviSkewATM <- function(sviParams){

    b <- sviParams$b
    rho <- sviParams$rho
    m <- sviParams$m
    sig <- sviParams$sig

    skew <- b*(rho-m/sqrt(m^2+sig^2))
    return(skew)

    }

nexp <- length(texp)
atmVars <- numeric(nexp)
atmSkews <- numeric(nexp)

for (i in 1:nexp){

    atmVars[i] <- sviVarianceATM(sviMatrix[i,])/texp[i]
    atmSkews[i] <- sviSkewATM(sviMatrix[i,])/texp[i]

    }

atmVols <- sqrt(atmVars)
```

In [64]:
```
data.frame(atmVars,atmSkews)
```

atmVars	atmSkews
0.007802062	-0.06828599
0.012055005	-0.16226925
0.014853978	-0.13723756
0.016079491	-0.12210809
0.018315711	-0.09457975
0.019531042	-0.08151867
0.020945102	-0.06792718
0.022044728	-0.05946294

Term structure of ATM skew

A function to draw the approximate term structure of Heston skew:

In [65]:
```
skewHeston <- function(skewParams,texp){

    rhoeta <- skewParams[1]
    lam <- skewParams[2]
    lt <- lam*texp
    skew <- rhoeta/lt*(1-(1-exp(-lt))/lt)
    return(skew)

    }
```

Now fit this skew function to the empirical points:

In [66]:
```
skewObjective1 <- function(skewParams){

    skewModel <- skewHeston(skewParams,texp[-1])
    skewEmpirical <- atmSkews[-1]
    obj <-sum((skewModel-skewEmpirical)^2)
    return(obj*1000000)

    }

res1 <- optim(c(-.7*.39,1.15),skewObjective1)
fit1 <- res1$par
```

Fit again dropping the first three points:

In [67]:

```
skewObjective2 <- function(skewParams){

    skewModel <- skewHeston(skewParams,texp[-(1:3)])
    skewEmpirical <- atmSkews[-(1:3)]
    obj <-sum((skewModel-skewEmpirical)^2)
    return(obj*1000000)

    }

res2 <- optim(c(-.7*.39,1.15),skewObjective2)
fit2 <- res2$par
```

Now fit a power law:

In [68]:

```
skewObjectivePL <- function(alpha){

    skewModel <- atmSkews[5]*(texp[5]/texp[-1])^alpha
    skewEmpirical <- atmSkews[-1]
    obj <-sum((skewModel-skewEmpirical)^2)
    return(obj*1000000)

    }

resPL <- optim(.4,skewObjectivePL)
(fitPL <- resPL$par)
```

```
Warning message:
In optim(0.4, skewObjectivePL): one-dimensional optimization by Nelder-Me
ad is unreliable:
use "Brent" or optimize() directly
```

0.34220703125

Finally, draw Figure 5.

In [69]:
```
plot(texp,atmSkews,col="blue",ylim=c(-.17,-.05),pch=20,cex=2,xlab="Time to exp
curve(skewHeston(fit1,x),from=0,to=2,col="red",add=T)
curve(skewHeston(fit2,x),from=0,to=2,col="orange",add=T)
curve(atmSkews[5]*(texp[5]/x)^fitPL,from=0,to=2,col="green4",lwd=2, add=T)
```

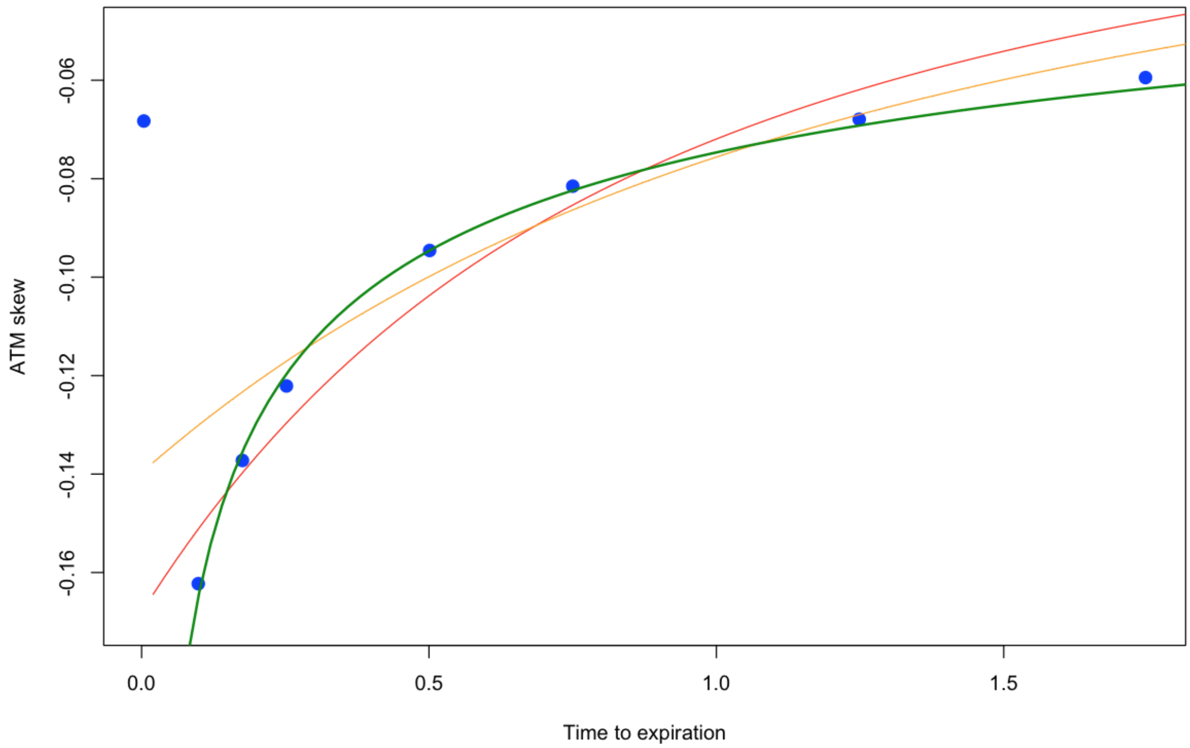


Figure 5: SPX ATM skew vs time to expiry. Blue points are empirical skews; the red line is a fit of (23) to all empirical skews except the first; the dark green fit excludes the first 3 data points; the orange line is a power-law fit with exponent 0.34.

Term structure of ATM skew: Observations

- From the pattern of the points, we would suspect that a simple functional form should be able to fit.
- The red and orange lines show the result of fitting

  (23)
  $$\rho\,\eta\,\frac{1}{\lambda' T}\left\{1-\frac{\left(1-e^{-\lambda' T}\right)}{\lambda' T}\right\}$$

  to the observed skews.

- The green power-law fit is much better!
  - The Heston model appears to be mis-specified.
  - And, we will see that all SV models generate the same-shaped surface.

- We can improve the fit quality by adding jumps or more stochastic volatility factors as in the Bergomi or DMR models.

Term structure of ATM variance

A function to approximate Heston ATM variance term structure:

In [70]:
```
varHeston <- function(varParams,texp){

    v <- varParams[1]
    vbar <- varParams[2]
    lam <- varParams[3]
    lt <- lam*texp
    var <- vbar + (v-vbar)*(1-exp(-lt))/lt
    return(var)

    }
```

Fit empirically ATM variances dropping the first point:

In [71]:
```
varObjective <- function(varParams){

    varModel <- varHeston(varParams,texp[-1])
    varEmpirical <- atmVars[-1]
    obj <-sum((varModel-varEmpirical)^2)
    return(obj*1000000)

    }

(res <- optim(c(0.04,0.04,1.15),varObjective))
fit <- res$par
```

```
$par
[1] 0.008855222 0.022875872 6.268353214

$value
[1] 0.8863727

$counts
function gradient
     282       NA

$convergence
[1] 0

$message
NULL
```

Finally, draw Figure 3.5 from the book.

In [72]:
```
plot(texp,atmVars,col="blue",xlab="Time to expiration",pch=20,cex=2,ylab="ATM
curve(varHeston(fit,x),from=0,to=2,col="red",n=500,add=T)
```

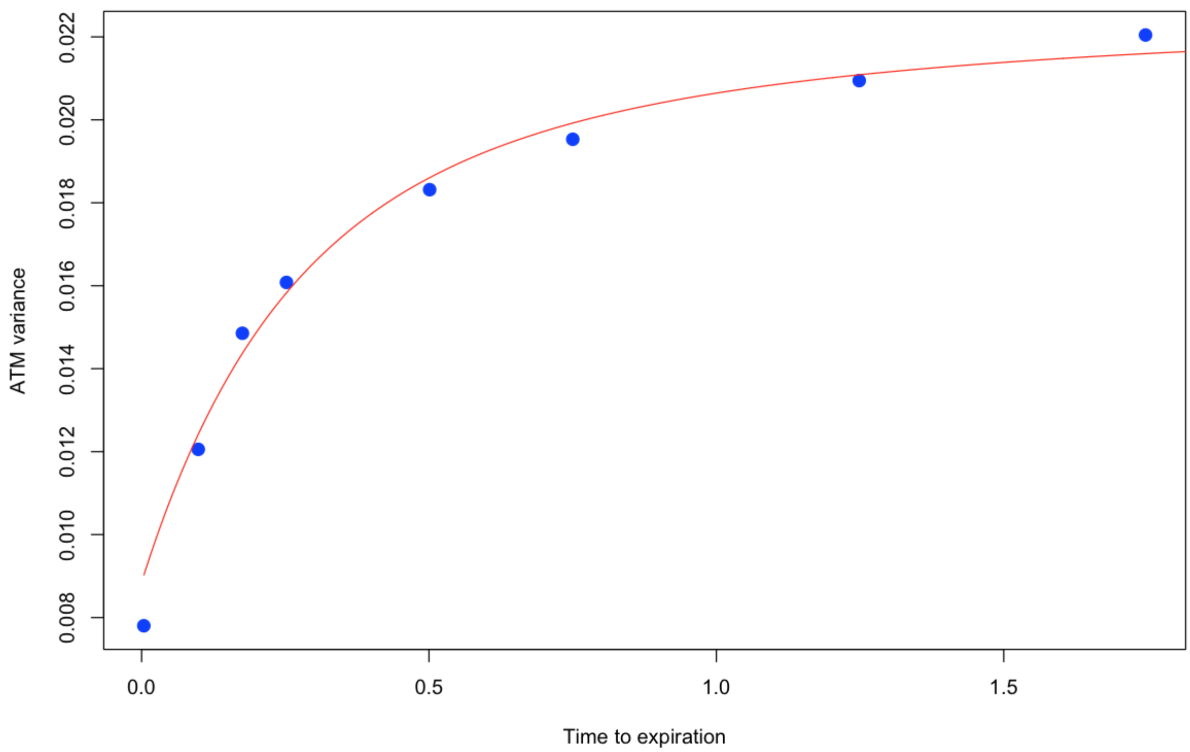


Figure 6: Blue points are empirical SPX ATM variances; the red line is a fit of the approximate ATM variance formula (20).

Observations on ATM volatility term structure

- In Figure 6, we see that on this particular date, our simple formula:
$$\sigma_{BS}(K,T)^2\big|_{K=F_T} \approx \left(v - \bar{v}'\right)\,\frac{1}{\lambda' T}\left\{1 - \frac{\left(1 - e^{-\lambda' T}\right)}{\lambda' T}\right\} + \bar{v}'$$
fits the data pretty well.
  - This is not always the case; in general, the term structure of volatility can be quite intricate at the short-end.

SPX volatility smiles on 15-Sep-2011

- It so happens that 16-Sep-2011 was also a triple witching day.
- Let’s take a look at the volatility surface as of 3PM the previous day (September 15, 2011).
  - Note that volatility is much higher than in 2005 and that the term structure of volatility is now downward-sloping.

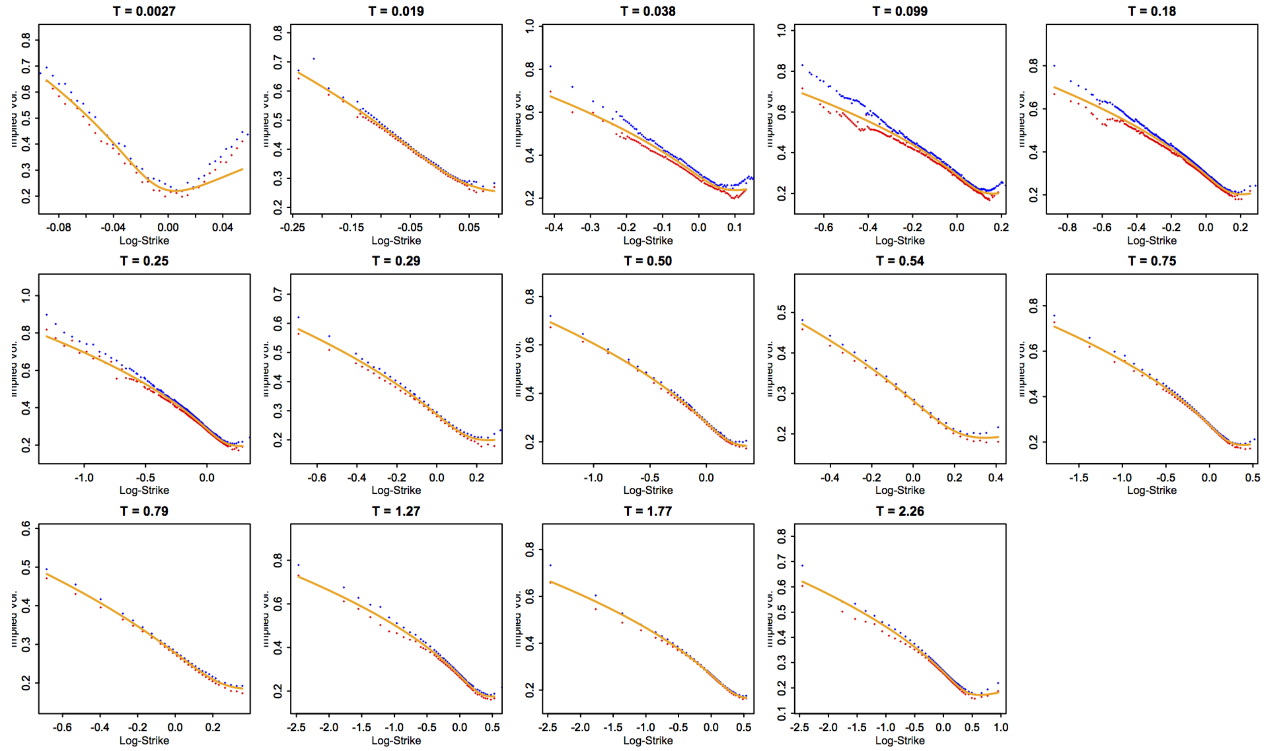


Figure 7: SPX implied volatilities as of September 15, 2011. Bid vols are in red; offer vols in blue; the orange line is the SVI fit.

Empirical ATM levels and skews

Expiration	Time to expiry	ATM Variance	ATM Skew
Sep-11	1 day	0.0494	-0.7343
Oct-11	1 month	0.0831	-0.5164
Nov-11	2 months	0.0841	-0.4439
Dec-11	3 months	0.0825	-0.3741
Mar-12	6 months	0.0786	-0.2525
Jun-12	9 months	0.0761	-0.2122
Dec-12	15 months	0.0728	-0.1658
Jun-13	21 months	0.0699	-0.1339
Dec-13	27 months	0.0692	-0.1185

Table 2: At-the-money SPX variance levels and variance skews as of 3pm on September 15, 2011, the day before expiration.

Term structure of ATM skew on 15-Sep-2011

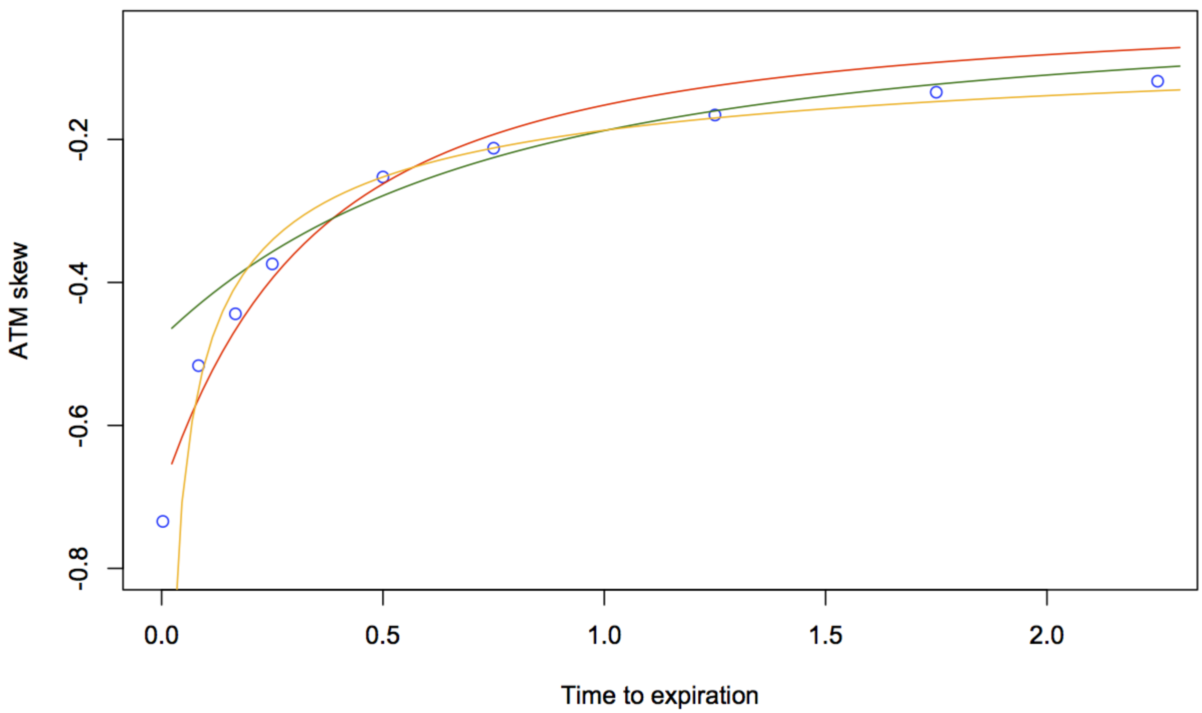


Figure 8: SPX ATM skew vs time to expiry. Blue points are empirical skews; the red line is a fit of (23) to all empirical skews; the dark green fit excludes the first 3 data points; the orange line is a power-law fit.

Term structure of ATM variance on 15-Sep-2011

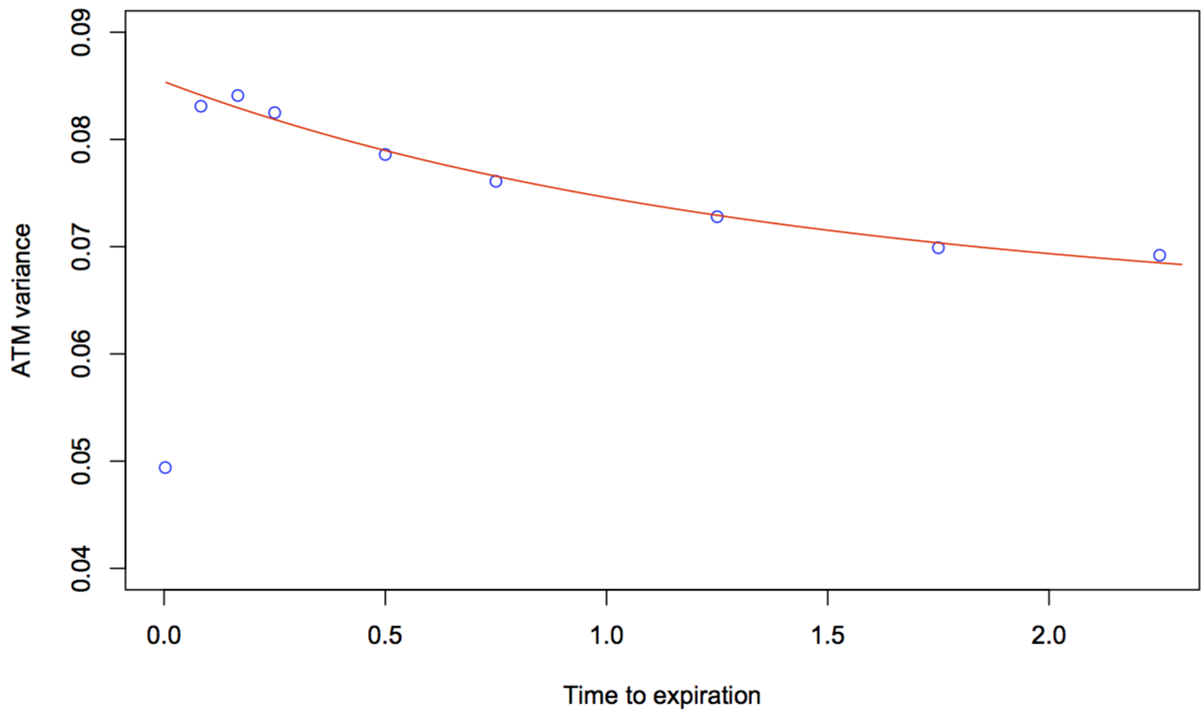


Figure 9: Blue points are empirical SPX ATM variances; the red line is a fit of the approximate ATM variance formula (20).

More remarks

- So, sometimes it’s possible to fit the term structure of at-the-money volatility with a stochastic volatility model, but it’s never possible to fit the term structure of the volatility skew for short expirations.
    - That’s one reason why practitioners prefer local volatility models: a stochastic volatility model with time-homogeneous parameters cannot fit market prices!

- Perhaps an extended stochastic volatility model with correlated jumps in stock price and volatility or more volatility factors might fit better?

- But how would traders choose their input parameters?
    - How would the SPX index book trader choose his volatility of volatility parameter?
    - Worse, what about the correlation between jumps in stock price and jumps in volatility?

Heston fit to the SPX implied volatility surface

v	-0.0174
$\bar{v}$	-0.0354
η	-0.3877

ρ	-0.7165
λ	-1.3253

Table 3: Heston fit to the SPX surface as of the close on September 15, 2005.

- The fitted Heston parameters are not so different from the BCC parameters:
  - This is not usual and in general fitted Heston parameters move slowly over time.
  - For example, the fitted η increases as the general volatility level increases.

Comparison of Heston fit to empirical surface: 3D plot

First a function to return Heston implied vols for a vector of log-strikes k and expirations t.

In [73]:
```
paramsHeston050915 <- list(
    v= 0.0174,
    vbar=0.0354,
    eta=0.3877,
    rho=-0.7165,
    lambda=1.3253)

k <- seq(-.5,.5,0.01) # Vector of log-strikes
t <- seq(0.04,1.74,0.02) # Vector of times

vol1 <- function(k,t){bsvol(phiHeston(paramsHeston050915),k,t)}
vol2 <- function(k,t){sapply(k,function(k){vol1(k,t)})}
vol <- function(k,t){sapply(t,function(t){vol2(k,t)})}
```

Now built the Heston volatility surface:

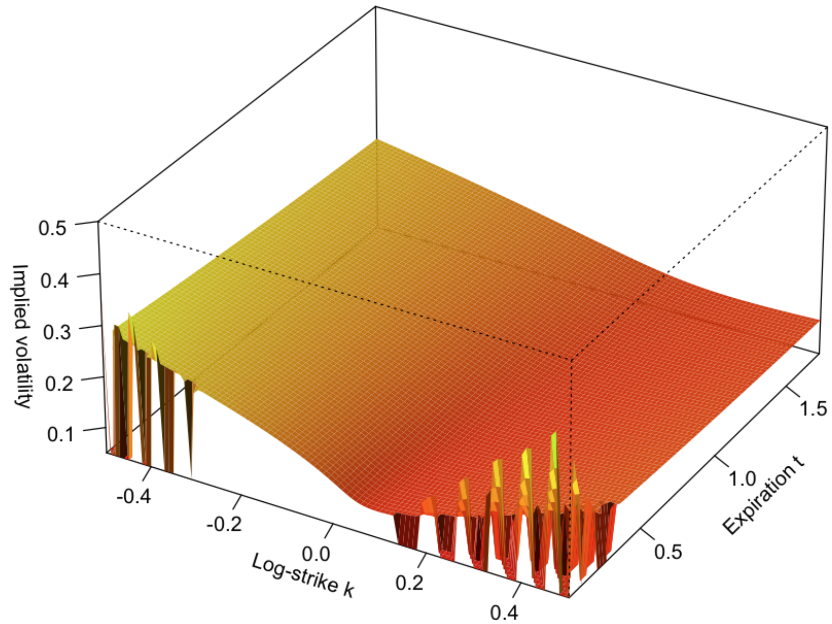
In [74]:
```
zvol <- vol(k,t)

# First the analytical formula:
z <- (zvol>.05)*zvol+(zvol <= 0.05)*.05

# Add colors
nbcol <- 100
color <- rainbow(nbcol,start=.0,end=.2)
nrz <- nrow(z)
ncz <- ncol(z)
# Compute the z-value at the facet centres
zfacet <- z[-1, -1] + z[-1, -ncz] + z[-nrz, -1] + z[-nrz, -ncz]
# Recode facet z-values into color indices
facetcol <- cut(zfacet, nbcol)
```

Plot the Heston surface:

In [75]:
```
persp(k, t, z, col=color[facetcol], phi=30, theta=30,
    r=1/sqrt(3)*20,d=5,expand=.5,ltheta=-135,lphi=20,ticktype="detailed",
    shade=.5,border=NA,xlab="Log-strike k",ylab="Expiration t",zlab="Implied v
```



Next build the empirical surface:

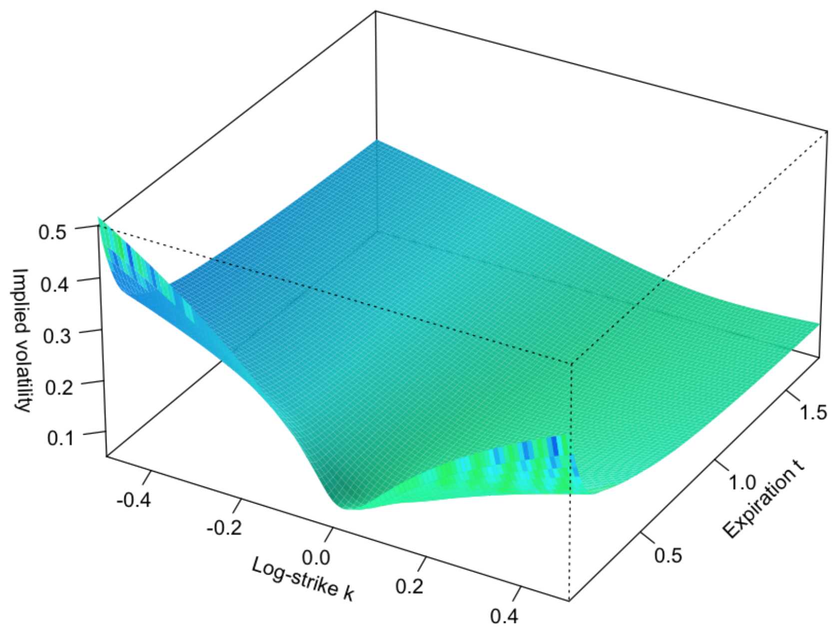
In [76]:
```
z2 <- t(volTVS(k,t)) # Array of empirical volatilities

# Add colors
nbcol <- 100
color2 <- rainbow(nbcol,start=.4,end=.6)
nrz <- nrow(z)
ncz <- ncol(z)
# Compute the z-value at the facet centres
zfacet <- z[-1, -1] + z[-1, -ncz] + z[-nrz, -1] + z[-nrz, -ncz]
# Recode facet z-values into color indices
facetcol <- cut(zfacet, nbcol)
```

and plot it:

In [77]:
```
persp(k, t, z2, col=color2[facetcol], phi=30, theta=30,
    r=1/sqrt(3)*20,d=5,expand=.5,ltheta=-135,lphi=20,ticktype="detailed",
    shade=.5,border=NA,xlab="Log-strike k",ylab="Expiration t",zlab="Implied v
```

```
Warning message:
In persp.default(k, t, z2, col = color2[facetcol], phi = 30, theta = 30,
 : surface extends beyond the box
```



Now plot them together with the empirical surface shifted up by two vol points (to make the comparison easier):

In [78]:
```
persp(k, t, z, col=color[facetcol], phi=30, theta=30,
    r=1/sqrt(3)*20,d=5,expand=.5,ltheta=-135,lphi=20,ticktype="detailed",
    shade=.5,border=NA,xlab="Log-strike k",ylab="Expiration t",zlab="Implied v
par(new=T)
persp(k, t, z2+.02, col=color2[facetcol], phi=30, theta=30,
    r=1/sqrt(3)*20,d=5,expand=.5,ltheta=-135,lphi=20,ticktype="detailed",
    shade=.5,border=NA,xlab="Log-strike k",ylab="Expiration t",zlab="Implied v
```

```
Warning message:
In persp.default(k, t, z2 + 0.02, col = color2[facetcol], phi = 30, : sur
face extends beyond the box
```

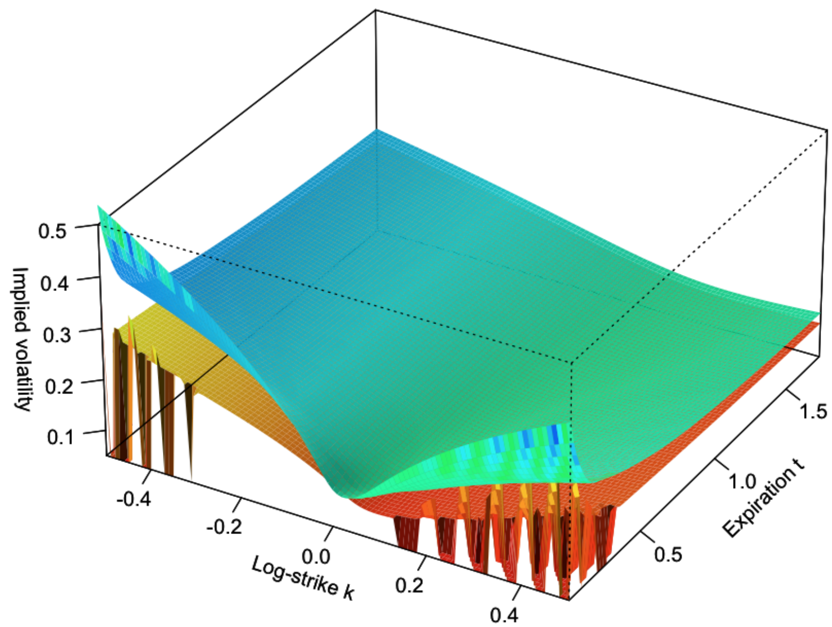


Figure 9: Upper surface is empirical, lower surface is Heston fit.

Wrap everything into another function

In [79]:
```
view <- function(phi,theta){

# Generate 3D plot of analytical solution
persp(k, t, z, col=color[facetcol], phi=phi, theta=theta,
    r=1/sqrt(3)*20,d=5,expand=.5,ltheta=-135,lphi=20,ticktype="detailed",
    shade=.5,border=NA,xlab="Log-strike k",ylab="Expiration t",zlab="Implied v

par(new=T) # To superimpose

persp(k, t, z2+.02, col=color2[facetcol], phi=phi, theta=theta,
    r=1/sqrt(3)*20,d=5,expand=.5,ltheta=-135,lphi=20,ticktype="detailed",
    shade=.5,border=NA,xlab="Log-strike k",ylab="Expiration t",zlab="Implied v
}
```

In [80]:
```
view(30,30)
```

```
Warning message:
In persp.default(k, t, z2 + 0.02, col = color2[facetcol], phi = phi, : su
rface extends beyond the box
```

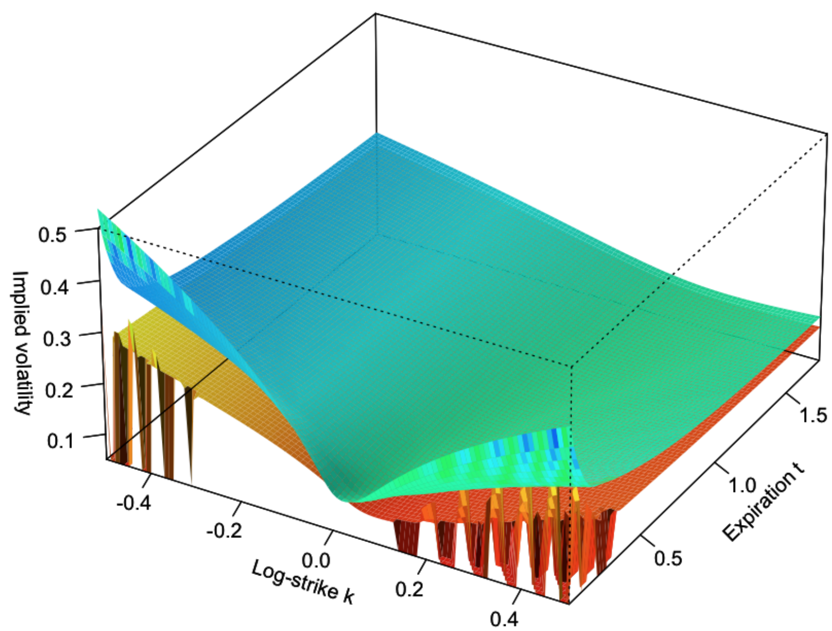
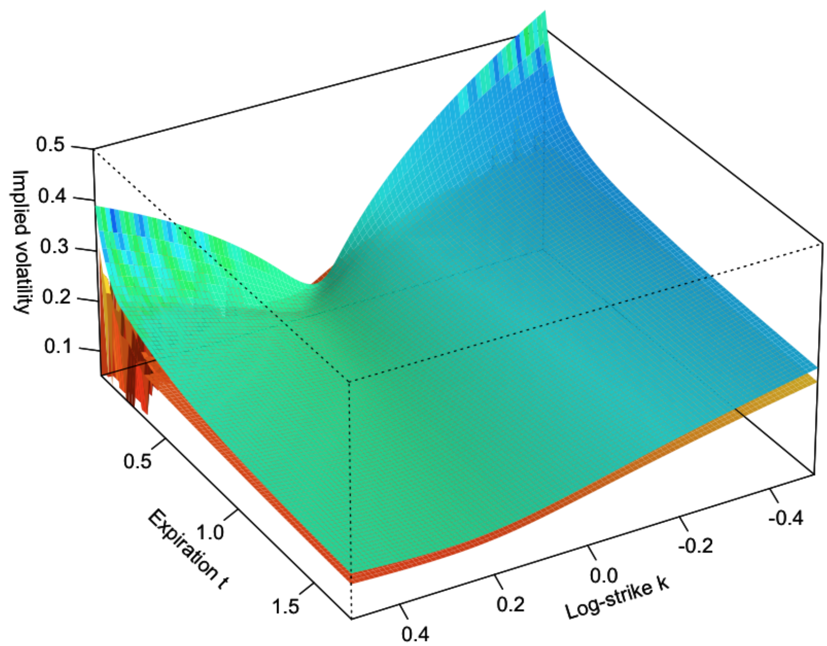


Figure 10: Upper surface is empirical, lower surface is Heston fit.

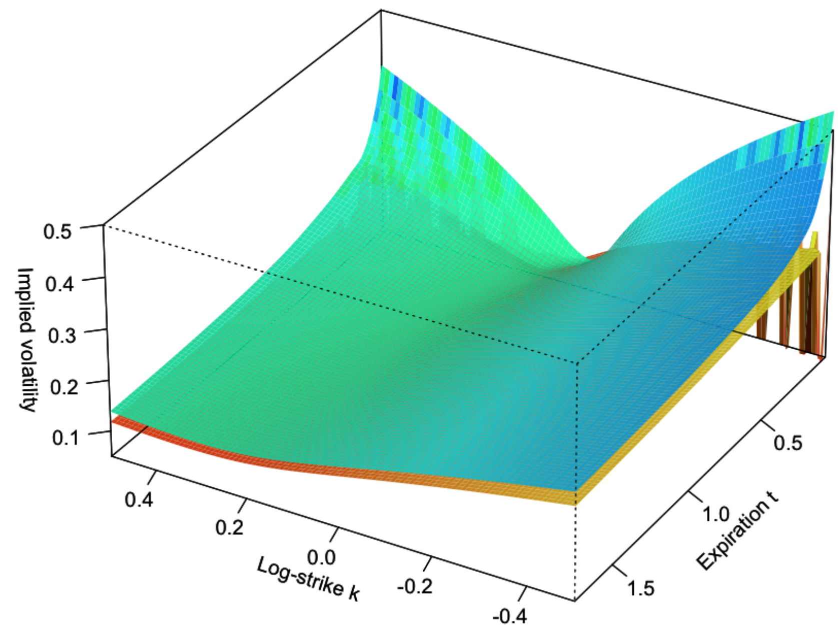
In [81]:
```
view(30,150)
```

```
Warning message:
In persp.default(k, t, z2 + 0.02, col = color2[facetcol], phi = phi, : su
rface extends beyond the box
```



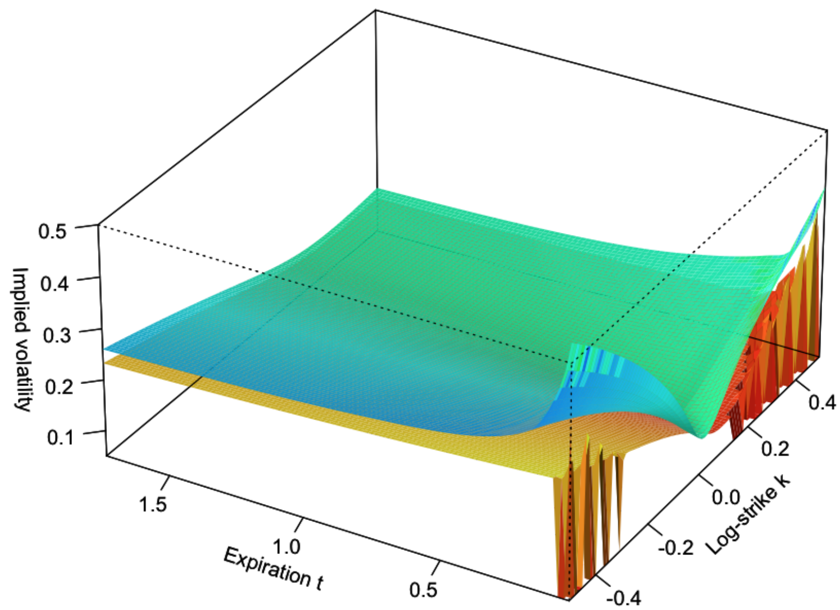
In [82]:
```
view(30,210)
```

```
Warning message:
In persp.default(k, t, z2 + 0.02, col = color2[facetcol], phi = phi, : su
rface extends beyond the box
```



In [83]:
```
view(30,300)
```

```
Warning message:
In persp.default(k, t, z2 + 0.02, col = color2[facetcol], phi = phi, : su
rface extends beyond the box
```



3D plot of Heston fit to the data: Remarks

- From the two views presented here, we can see that the Heston fit is pretty good for longer expirations but really not close for short expirations.
  - We see from Figure 9 that the smile generated by the Heston model is far too flat relative to the empirical implied volatility surface.
  - For longer expirations though, the fit isn't bad.

Final remarks on SV models and fitting the volatility surface

- It's quite clear from Figures 4 and 9 that the Heston model doesn't fit the observed implied volatility surface for short expirations
  - The fit is not bad for longer expirations.

- As we shall see later, all stochastic volatility models generate roughly the same shape of volatility surface.

- It follows that if we are looking for a model that fits options prices, we will need to look beyond conventional one-factor stochastic volatility models.

Summary of Lecture 5

- We showed how to approximate local volatility in the Heston model.
  - The approximation is accurate when $\rho = \pm 1$ (The Heston-Nandi model in particular) in which case we have a local volatility model that generates almost the same European option prices as the Heston model.

- The Black-Scholes implied variance of an option with strike K is given approximately by the integral from valuation date ($t = 0$) to the expiration date ($t = T$) of the local variances along the path $\tilde{x}_t$ that maximizes a Brownian Bridge-like density.

- We used this most-likely-path approximation to approximate implied volatility in the Heston model.
  - We were then able to approximate the term structures of ATM variance and ATM skew.

- We were then able to deduce that the Heston model generates a volatility surface that does not look like the empirically observed volatility surface.

References

1. ^ Lorenzo Bergomi and Julien Guyon, The smile in stochastic volatility models. *SSRN* (2011).
2. ^ Jim Gatheral, *The Volatility Surface: A Practitioner's Guide*, John Wiley and Sons, Hoboken, NJ (2006).
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4. ^ Jim Gatheral and Tai-Ho Wang, The heat-kernel most-likely-path approximation, *International Journal of Theoretical and Applied Finance* **15**(1) 1250001-1 – 1250001-18 (2012).
5. ^ Julien Guyon and Pierre Henry-Labordère, From spot volatilities to implied volatilities, *Risk Magazine* 79-84 (June 2011).
6. ^ Julien Guyon and Pierre Henry-Labordère, Nonlinear option pricing, *CRC Press, Boca Raton, FL* (2014).
7. ^ Adil Reghai, The hybrid most likely path, *Risk Magazine* (April 2006).

In [ ]: