MTH 9875 The Volatility Surface: Fall 2017

Lecture 13: Rough volatility

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Outline of Lecture 13

- The time series of historical volatility
  - Scaling properties
- The RFSV model
- The Rough Bergomi model
- The Rough Heston model
- Forecasting realized variance
- The time series of variance swaps
- Relating historical and implied

The time series of realized variance

- We would like to study the time series of instantaneous variance v_t but of course cannot because v_t is latent.
- On the other hand, integrated variance $\frac{1}{\delta}\int_t^{t+\delta} v_s\,ds$ may (in principle) be estimated arbitrarily accurately given enough price data.
  - In practice, market microstructure noise makes estimation harder at very high frequency.
  - Sophisticated estimators of integrated variance have been developed to adjust for market microstructure noise. See Gatheral and Oomen [9] (for example) for details of these.

- The Oxford-Man Institute of Quantitative Finance makes historical realized variance (RV) estimates freely available at http://realized.oxford-man.ox.ac.uk (http://realized.oxford-man.ox.ac.uk). These estimates are updated daily.
  - Each day, for 21 different indices, all trades and quotes are used to estimate realized (or integrated) variance over the trading day from open to close.
- Using daily RV estimates as proxies for instantaneous variance, we may investigate the time series properties of v_t empirically.

First update and save the latest Oxford-Man data:

In [1]:
```
download.file(url="http://realized.oxford-man.ox.ac.uk/media/1366/oxfordmanrealizedvolatilityindices.zip", destfile="oxfordRvData.zip")
unzip(zipfile="oxfordRvData.zip")
```

There are many different estimates of realized variance, all of them very similar. We will use the realized kernel estimates denoted by ".rk".

In [2]:
```
library(xts)
library(xtable)

rv.data <- read.csv("OxfordManRealizedVolatilityIndices.csv")
colnums <- which(sapply(rv.data, function(x) grep(".rk",x))>0)
col.names <- names(colnums)

rv1 <- rv.data[,colnums]
index.names <- rv1[2,]

datesRaw <- rv.data[-(1:2),1]
dates <- strptime(datesRaw,"%Y%m%d")

rv.list <- NULL
index.names <- as.matrix(index.names)

n <- length(index.names)
for (i in 1:n){
    tmp.krv1 <- xts(rv1[-(1:2),i],order.by=dates)
    rv.list[[i]] <- tmp.krv1[(tmp.krv1!="")&(tmp.krv1!="0")]
}

names(rv.list)<- index.names

save(rv.list, file="oxfordRV.rData")
```

```
Loading required package: zoo

Attaching package: ‘zoo’

The following objects are masked from ‘package:base’:

    as.Date, as.Date.numeric
```

Let's plot SPX realized variance.

In [3]:
```
# Load Oxford-Man KRV data
load("oxfordRV.rData")
names(rv.list)

spx.rk <- rv.list[["SPX2.rk"]]
```

'SPX2.rk' 'FTSE2.rk' 'N2252.rk' 'GDAXI2.rk' 'RUT2.rk' 'AORD2.rk' 'DJI2.rk' 'IXIC2.rk' 'FCHI2.rk' 'HSI2.rk' 'KS11.rk' 'AEX.rk' 'SSMI.rk' 'IBEX2.rk' 'NSEI.rk' 'MXX.rk' 'BVSP.rk' 'GSPTSE.rk' 'STOXX50E.rk' 'FTSTI.rk' 'FTSEMIB.rk'

In [4]:
```
library(repr)
```

In [5]:
```
options(repr.plot.width=14,repr.plot.height=8)
```

In [6]:
```
plot(spx.rk, main="SPX realized variance",plot=NULL)
lines(spx.rk,col="red")
```

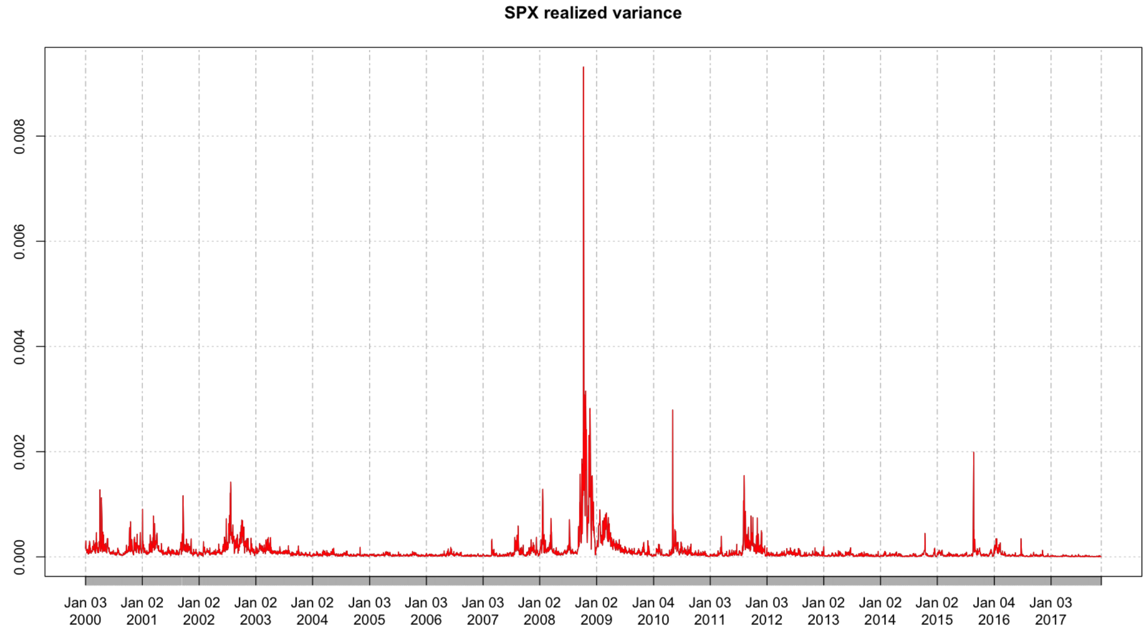


Figure 1: Oxford-Man KRV estimates of SPX realized variance from January 2000 to the current date.

In [7]:
```
print(head(spx.rk))
print(tail(spx.rk))
```

```
           [,1]
2000-01-03 "0.000160726642338866"
2000-01-04 "0.000264396469319473"
2000-01-05 "0.000304650302935347"
2000-01-06 "0.000148582063339039"
2000-01-07 "0.000123266970191763"
2000-01-10 "0.000130693391920629"
           [,1]
2017-11-15 "1.73779387282484E-05"
2017-11-16 "6.58115235088268E-06"
2017-11-17 "7.01827122946616E-06"
2017-11-20 "4.1453844078278E-06"
2017-11-21 "3.44015357109983E-06"
2017-11-22 "3.23374577964428E-06"
```

Scaling of the volatility process

For $q \geq 0$, we define the qth sample moment of differences of log-volatility at a given lag Δ.($\langle \cdot \rangle$ denotes the sample average):

$$m(q, \Delta) = \langle |\log \sigma_{t+\Delta} - \log \sigma_t|^q \rangle$$

For example

$$m(2, \Delta) = \langle (\log \sigma_{t+\Delta} - \log \sigma_t)^2 \rangle$$

is just the sample variance of differences in log-volatility at the lag Δ.

Scaling of $m(q, \Delta)$ with lag Δ

In [9]:

```
sig <- sqrt(as.numeric(spx.rk))

mq.del.Raw <- function(q,lag){mean(abs(diff(log(sig),lag=lag))^q)}
mq.del <- function(x,q){sapply(x,function(x){mq.del.Raw(q,x)})}

# Plot mq.del(1:100,q) for various q

x <- 1:100

mycol <- rainbow(5)

ylab <- expression(paste(log," ",m(q,Delta)))
xlab <- expression(paste(log, " ", Delta))

qVec <- c(.5,1,1.5,2,3)
zeta.q <- numeric(5)
q <- qVec[1]
```

In [10]:

```
options(repr.plot.height=7, repr.plot.width=10)
```

In [11]:
```
plot(log(x),log(mq.del(x,q)),pch=20,cex=.5,
          ylab=ylab, xlab=xlab,ylim=c(-3,-.5))
fit.lm <- lm(log(mq.del(x,q)) ~ log(x))
abline(fit.lm, col=mycol[1],lwd=2)
zeta.q[1] <- coef(fit.lm)[2]

for (i in 2:5){
    q <- qVec[i]
    points(log(x),log(mq.del(x,q)),pch=20,cex=.5)
    fit.lm <- lm(log(mq.del(x,q)) ~ log(x))
    abline(fit.lm, col=mycol[i],lwd=2)
    zeta.q[i] <- coef(fit.lm)[2]
    }
 legend("bottomright", c("q = 0.5","q = 1.0","q = 1.5","q = 2.0","q = 3.
0"),inset=0.05, lty=1, col = mycol)

print(zeta.q)
```

```
[1] 0.06952471 0.13598982 0.19935973 0.25970759 0.37180920
```

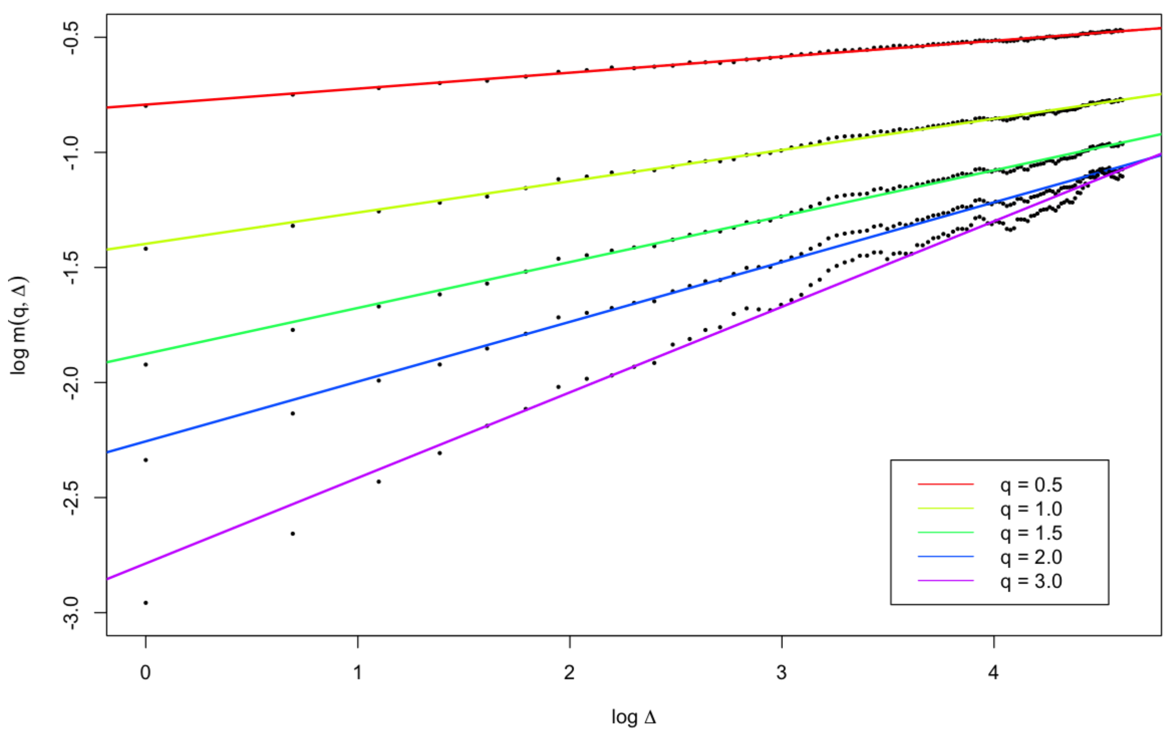


Figure 2: $\log m(q, \Delta)$ as a function of $\log \Delta$, SPX.

Monofractal scaling result

- From the above log-log plot, we see that for each q, $m(q, \Delta) \propto \Delta^{\zeta_q}$.
- How does ζ_q scale with q?

Scaling of ζ_q with q

In [12]:
```
plot(qVec,zeta.q,xlab="q",ylab=expression(zeta[q]),pch=20,col="blue",cex
=2)
fit.lm <- lm(zeta.q[1:4] ~ qVec[1:4]+0)
abline(fit.lm, col="red",lwd=2)
(h.est <- coef(fit.lm)[1])
```

qVec[1:4]: 0.131894259705048

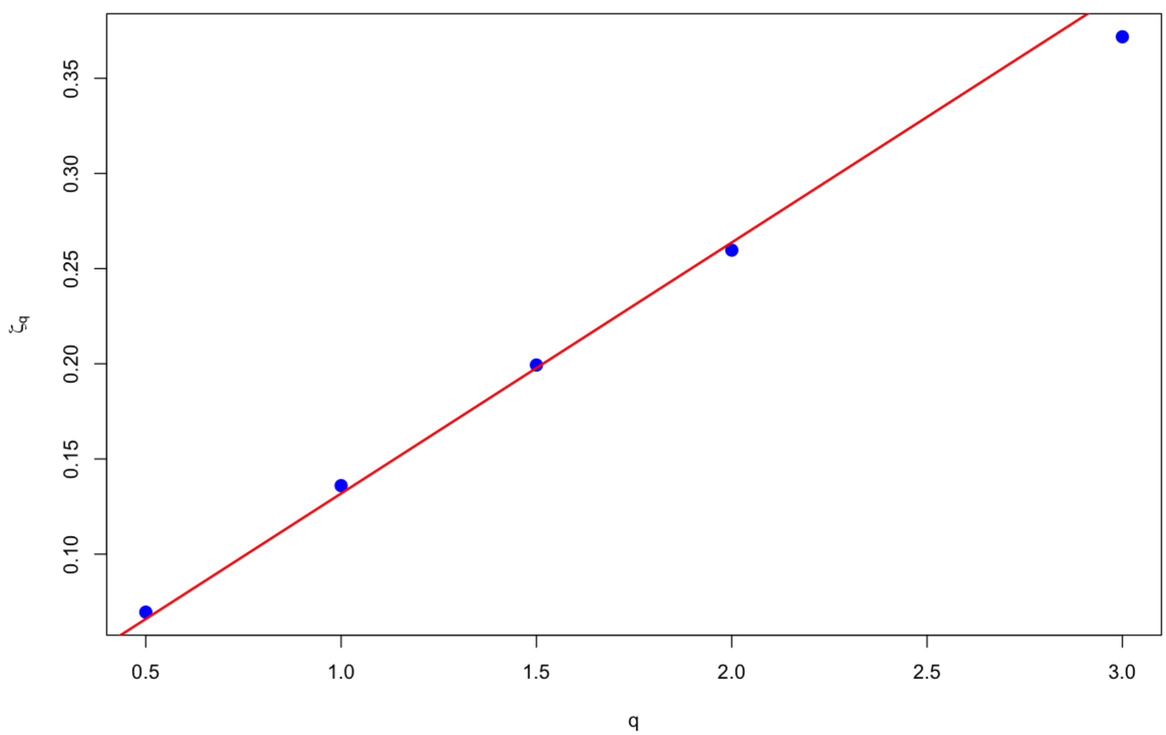


Figure 3: Scaling of ζ_q with q.

We find the monofractal scaling relationship

$$\zeta_q = q\,H$$

with $H \approx 0.13$.

- Note however that H does vary over time, in a narrow range, as we will see later.
- Note also that our estimate of H is biased high because we proxied instantaneous variance v_t with its average over each day $\frac{1}{T}\int_0^T v_t\,dt$, where T is one trading day.
  - On the other hand, the time series of realized variance is noisy and this causes our estimate of H to be biased low.

- This scaling property as $\Delta \to 0$ is equivalent to H-Hölder continuity of paths of the volatility.
  - Since $H \ll 1/2$, *volatility is rough*!

Estimated H for all indices

We now repeat this analysis for all 21 indices in the Oxford-Man dataset.

In [13]:
```
n <- length(rv.list)
h <- numeric(n) # H is estimated as half of the slope
nu <- numeric(n)

for (i in 1:n){ # Run all the regressions
  v <- rv.list[[i]]
  sig <- sqrt(as.numeric(v))

  x <- 1:100
  dlsig2 <- function(lag){mean((diff(log(sig),lag=lag))^2)}
  dlsig2Vec <- function(x){sapply(x,dlsig2)}

  fit.lm <- lm(log(dlsig2Vec(x)) ~ log(x))

  nu[i] <- sqrt(exp(coef(fit.lm)[1]))
  h[i] <- coef(fit.lm)[2]/2

}
```

In [14]:
```
(OxfordH <- data.frame(names(rv.list),h.est=h,nu.est=nu))
```

names.rv.list.	h.est	nu.est
SPX2.rk	0.12985380	0.3237351
FTSE2.rk	0.14070020	0.2669710
N2252.rk	0.10996875	0.3271248
GDAXI2.rk	0.14618433	0.2766664
RUT2.rk	0.11742913	0.3305939
AORD2.rk	0.08179224	0.3583750
DJI2.rk	0.12795934	0.3177994
IXIC2.rk	0.12351297	0.2993373
FCHI2.rk	0.12758347	0.2925884
HSI2.rk	0.09870516	0.2815690
KS11.rk	0.11836042	0.2803263
AEX.rk	0.14190035	0.2914356
SSMI.rk	0.17662100	0.2209443
IBEX2.rk	0.12395749	0.2834995
NSEI.rk	0.10808951	0.3217038
MXX.rk	0.08937631	0.3233862
BVSP.rk	0.10738326	0.3124622
GSPTSE.rk	0.11783238	0.3047810
STOXX50E.rk	0.11631641	0.3388185
FTSTI.rk	0.12710425	0.2289036
FTSEMIB.rk	0.13215250	0.2952422

In [15]:
```
save(OxfordH,file="OxfordH.rData")
```

Distributions of $(\log \sigma_{t+\Delta} - \log \sigma_t)$ for various lags Δ

Having established these beautiful scaling results for the moments, how do the histograms look?

In [16]:
```
plotScaling <- function(j,scaleFactor){
  v <- as.numeric(rv.list[[j]])
  x <- 1:100

  xDel <- function(x,lag){diff(x,lag=lag)}
  sd1 <- sd(xDel(log(v),1))
  sdl <- function(lag){sd(xDel(log(v),lag))}

  h <- OxfordH$h.est[j]

  plotLag <- function(lag){
    y <- xDel(log(v),lag)
    hist(y,breaks=100,freq=F,main=paste("Lag =",lag,"Days"),xlab=NA)# Ve
ry long tailed!
    curve(dnorm(x,mean=mean(y),sd=sd(y)),add=T,col="red",lwd=2)
    curve(dnorm(x,mean=0,sd=sd1*lag^h),add=T,lty=2,lwd=2,col="blue")
#    lines(density(y)$x,density(y)$y,col="dark green",lty=3,lwd=3)
  }

  (lags <- scaleFactor^(0:3))
  print(names(rv.list)[j])
  par(mfrow=c(2,2))
  par(mar=c(3,2,1,3))
  for (i in 1:4){plotLag(lags[i])}
  par(mfrow=c(1,1))
}
```

In [17]:
```
options(repr.plot.height=5, repr.plot.width=10)
```

In [18]:
```
plotScaling(1,5)
```

```
[1] "SPX2.rk"
```

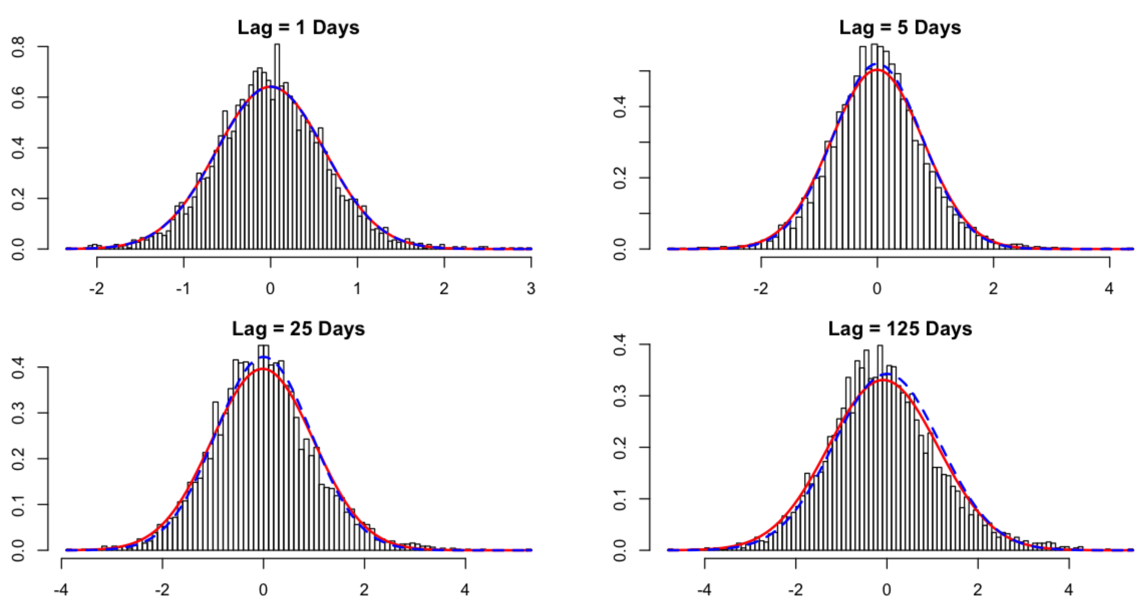


Figure 4: Histograms of $(\log \sigma_{t+\Delta} - \log \sigma_t)$ for various lags Δ; normal fit in red; $\Delta = 1$ normal fit scaled by $\Delta^{0.14}$ in blue.

Universality?

- [Gatheral, Jaisson and Rosenbaum][8]</sup> compute daily realized variance estimates over one hour windows for DAX and Bund futures contracts, finding similar scaling relationships.
- We have also checked that Gold and Crude Oil futures scale similarly.
  - Although the increments $(\log \sigma_{t+\Delta} - \log \sigma_t)$ seem to be fatter tailed than Gaussian.
- [Bennedsen et al.][5]</sup>, estimate volatility time series for more than five thousand individual US equities, finding rough volatility in every case.

A microstructural explanation: A Hawkes model of price formation

- Why might rough volatility be universal?
- [Jaisson and Rosenbaum][9]</sup> show that rough volatility can be obtained as a scaling limit of a simple model of price dynamics in terms of Hawkes processes.
- Remarkably, [El Euch and Rosenbaum][7]</sup> were able to compute the characteristic function of the resulting *rough Heston* model.

A Hawkes model of price formation

[Jaisson and Rosenbaum][11]</sup> considered a generalization of a simple model of price dynamics in terms of Hawkes processes due to Bacry and Muzy with the following properties:

- Reflecting the high degree of endogeneity of the market, the L^1 norm of the kernel matrix is close to one (nearly unstable).
- No drift in the price process imposes a relationship between buy and sell kernels.
- Liquidity asymmetry: The average impact of a sell order is greater than the impact of a buy order.
- Splitting of metaorders motivates power-law decay of the Hawkes kernels $\varphi(\tau) \sim \tau^{-(1+\alpha)}$ (empirically $\alpha \approx 0.6$).

The scaling limit of the price model

They also showed that after a suitable rescaling in time and space, the long term limit of the Hawkes price model is given by the following Rough Heston dynamics:

$$\frac{dS_t}{S_t} = \sqrt{v_t}\, dZ_t$$
$$v_t = v_0 + \frac{\lambda}{\Gamma(\alpha)} \int_0^t \frac{\theta - v_s}{(t-s)^{1-\alpha}}\, ds + \frac{\lambda\, \nu}{\Gamma(\alpha)} \int_0^t \frac{\sqrt{v_s}\, dW_s}{(t-s)^{1-\alpha}}$$

with

$$d\langle Z, W\rangle_t = \rho\, dt.$$

- The correlation ρ is related to a liquidity asymmetry parameter.

The characteristic function

Define the fractional integral and differential operators:

$$I^{1-\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f(s)}{(t-s)^{\alpha}}\, ds; \quad D^{\alpha} f(t) = \frac{d}{dt}\, I^{1-\alpha} f(t).$$

Remarkably, [El Euch and Rosenbaum][6]</sup> have computed the following expression for the characteristic function of the Rough Heston model:

$$\phi_t(u) = \exp\left\{ \int_0^t g(u,s)\, ds + \frac{v_0}{\lambda\, \nu}\, I^{1-\alpha} g(u,t) \right\}.$$

Here $g(u, \cdot)$ solves the fractional Riccati equation

$$D^{\alpha} g(u,s) = -\frac{\lambda\, \theta}{2}\, u\,(u+i) + \lambda\,(i\,\rho\,\nu\,u - 1)\, g(u,s) + \frac{\lambda\,\nu^2}{2\,\theta}\, g^2(u,s).$$

- This is a fractional version of the conventional Heston Riccati equation.

A natural model of realized volatility

- Distributions of differences in the log of realized volatility are close to Gaussian.
  - This motivates us to model σ_t as a lognormal random variable.
- Moreover, the scaling property of variance of RV differences suggests the model:

(1)

$$\log \sigma_{t+\Delta} - \log \sigma_t = \nu\, \left(W^H_{t+\Delta} - W^H_t \right)$$

where W^H is fractional Brownian motion.

- In [Gatheral, Jaisson and Rosenbaum][9]</sup>, we refer to a stationary version of (1) as the RFSV (for Rough Fractional Stochastic Volatility) model.

Fractional Brownian motion (fBm)

- *Fractional Brownian motion* (fBm) $\{W^H_t; t \in \mathbb{R}\}$ is the unique Gaussian process with mean zero and autocovariance function
$$\mathbb{E}\left[W^H_t\,W^H_s\right]=\frac{1}{2}\left\{|t|^{2\,H}+|s|^{2\,H}-|t-s|^{2\,H}\right\}$$
where $H \in (0,1)$ is called the *Hurst index* or parameter.
  - In particular, when $H = 1/2$, fBm is just Brownian motion.
  - If $H > 1/2$, increments are positively correlated.% so the process is trending.
  - If $H < 1/2$, increments are negatively correlated.% so the process is reverting.

Representations of fBm

There are infinitely many possible representations of fBm in terms of Brownian motion. For example, with $\gamma=\frac{1}{2}-H$,

> **Mandelbrot-Van Ness**
>
> $$W^H_t = C_H\,\left\{\int_{-\infty}^t \frac{dW_s}{(t-s)^\gamma} - \int_{-\infty}^0 \frac{dW_s}{(-s)^\gamma}\right\}.$$

where the choice

$$C_H = \sqrt{\frac{2\,H\,\Gamma(3/2-H)}{\Gamma(H+1/2)\,\Gamma(2-2\,H)}}$$

ensures that

$$\mathbb{E}\left[W^H_t\,W^H_s\right]=\frac{1}{2}\left\{t^{2H}+s^{2H}-|t-s|^{2H}\right\}.$$

Efficient estimation of H

- So far, we just used simple regression to estimate H.
- When H is small, as we find empirically, out of all the estimators that we tested, the ACF estimator adopted by [Bennedsen et al.][5]</sup> is the most efficient.

Heuristic derivation of the ACF estimator

Once again, the covariance structure of fBm is given by

$$\mathbb{E}\left[W_t^H\, W_s^H\right] = \frac{1}{2}\left\{t^{2H} + s^{2H} - |t-s|^{2H}\right\}.$$

Up to a multiplicative factor, our model is

$$y_t = \log v_t = W_t^H.$$

Then $\mathrm{var}[y_t] = t^{2H}$. and

$$\mathrm{cov}[y_t, y_{t+\Delta}] = \frac{1}{2}\left\{t^{2H} + (t+\Delta)^{2H} - \Delta^{2H}\right\}$$

Dividing one by the other gives

$$\rho(\Delta) = \frac{1}{2}\left\{1 + \left(1 + \frac{\Delta}{t}\right)^{2H} - \left(\frac{\Delta}{t}\right)^{2H}\right\}$$

Thus, for Δ/t sufficiently small,

$$1 - \rho(\Delta) = \frac{1}{2}\left(\frac{\Delta}{t}\right)^{2H} + O\left(\frac{\Delta}{t}\right).$$

- Note in particular that we expect the ACF estimator to work best when $H \ll \frac{1}{2}$.
- Also, when $H = \frac{1}{2}$, we have $\rho(\Delta) = 1$ as we would expect for Brownian motion.

The ACF estimator

Taking logs of each side, we obtain

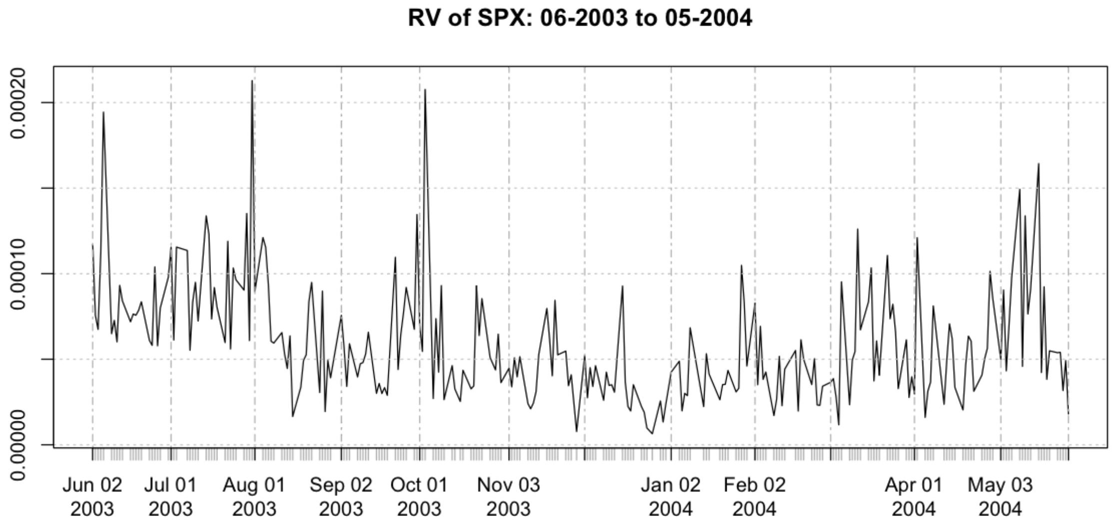
$$\log(1 - \rho(\Delta)) = a + 2\,H\ \log\Delta.$$

- Thus H can be estimated efficiently by regression.

In [36]:
```
h.acf <- function(path){
    y.acf <- acf(path,plot=F)
    log.del <- log(y.acf$lag[-1])
    log.lhs <- log(1-y.acf$acf[-1])
    fit.lm <- lm(log.lhs ~ log.del)
    return(fit.lm$coef[2]/2)
        }
```

An example

In [39]:
```
yPath <- spx.rk["2003-06-01::2004-05-31"]
plot(yPath,main="RV of SPX: 06-2003 to 05-2004")
```



In [40]:
```
h.acf(as.numeric(yPath))
```

log.del: 0.0601516764681327

Time series of H using ACF

- We now draw the time series of H using the ACF estimator.

In [41]:
```
h.acf.i <- function(series)function(del)function(i){
    rk.path <- as.numeric(series[(i-del):i])
    h.acf(rk.path)
    }
```

In [42]:
```
h.acf.i(spx.rk)(252)(1234)
```

log.del: 0.0877924997525288

In [43]:
```
h.acf.series <- function(series)function(del){
    require(xts)
    n <- length(series)
    res <- sapply((1+del):n,h.acf.i(series)(del))
    return(xts(res,order.by=index(series[(1+del):length(series)]),tzone
= Sys.getenv("TZ")))
}
```

Compare the two estimates of H

In [44]:
```
n.spx <- length(spx.rk)
h.spx.acf <- as.numeric(h.acf.series(spx.rk)(n.spx-1))
h.spx.regression <- OxfordH$h.est[1]
nu.spx.regression <- OxfordH$nu.est[1]
data.frame(h.spx.acf,h.spx.regression)
```

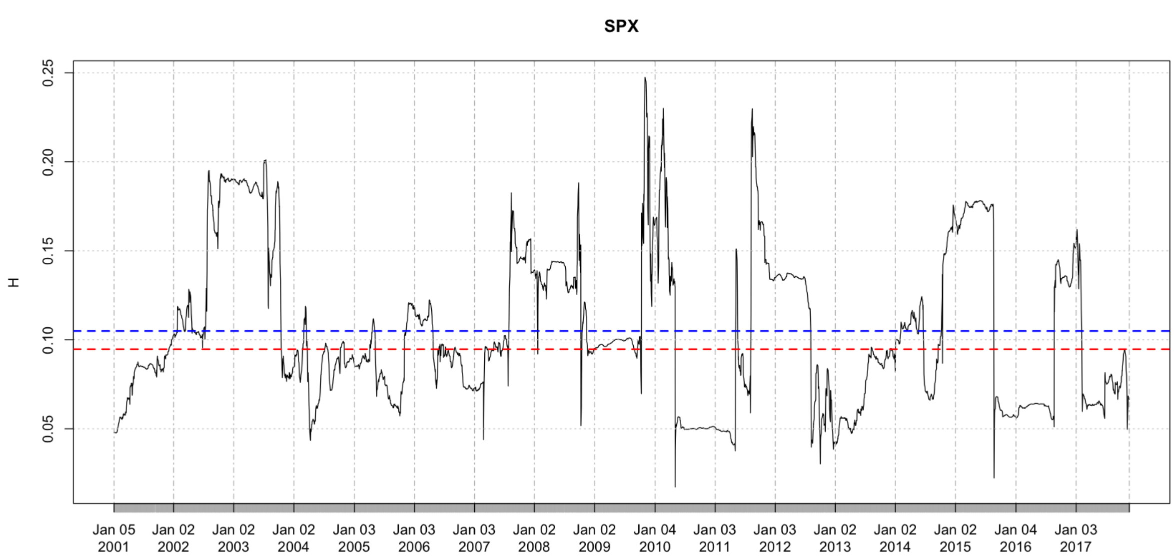
h.spx.acf	h.spx.regression
0.09802246	0.1298538

- Looking again at the log-log plots of $m_q(\Delta)$ against Δ, we note that the points don't quite lie on a straight line.
- A more careful analysis that takes account of the bias due to averaging and the noisiness of the time series of realized variance gives us an estimate of H more consistent with the ACF estimate.

Time series of H for SPX

In [45]:
```
h.spx.252 <-  h.acf.series(spx.rk)(252)
```

In [46]:
```
options(repr.plot.width=14,repr.plot.height=7)
plot(h.spx.252,main="SPX",ylab="H")
abline(h=median(h.spx.252),lty=2,col="red",lwd=2)
abline(h=mean(h.spx.252),lty=2,col="blue",lwd=2)
```



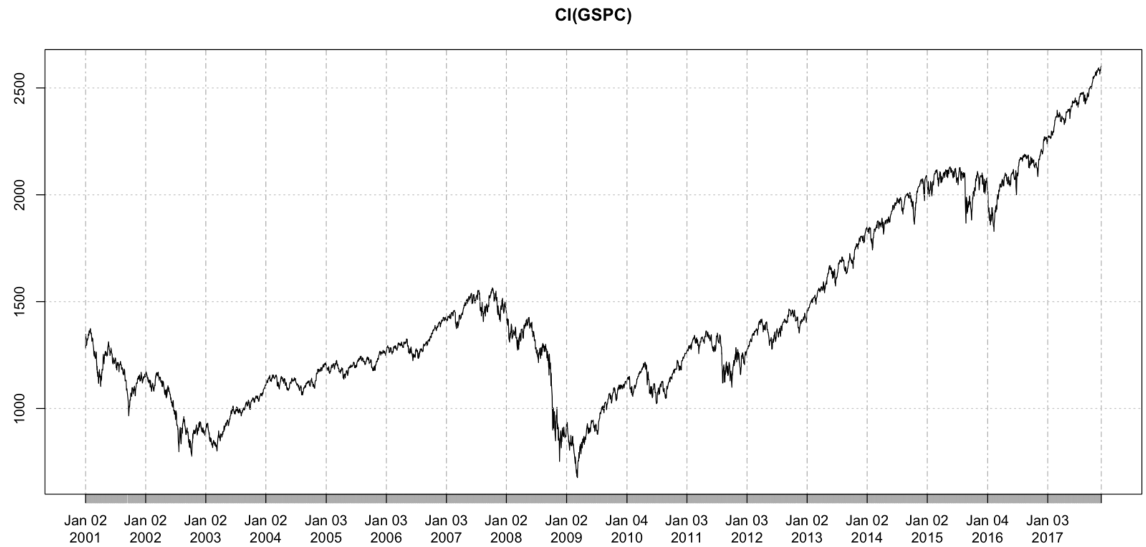
Line up time series of H with SPX

- First we use `quantmod` to download SPX data.

In [47]:
```
getSymbols('^GSPC',from="2001-01-01")
```

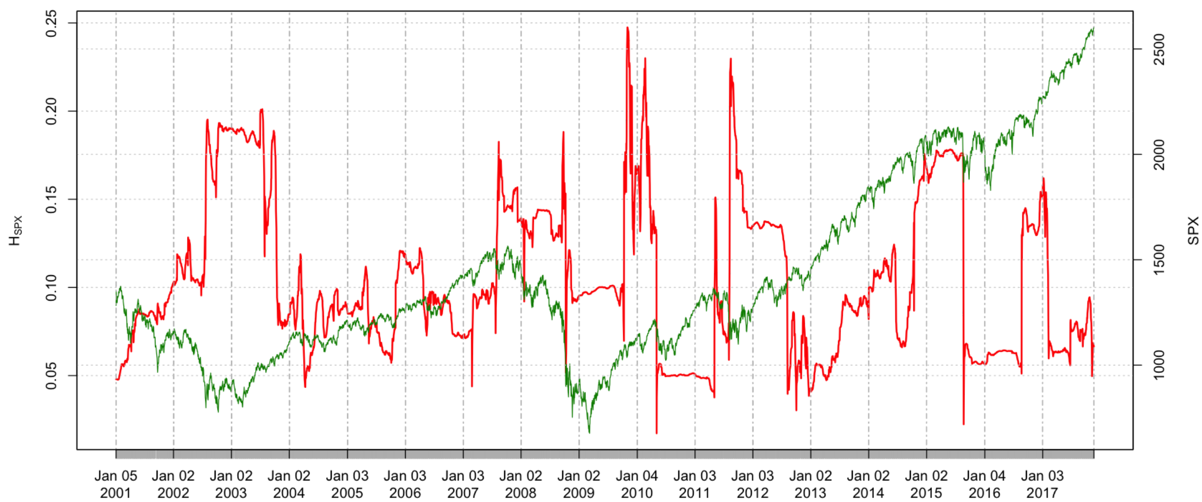
'GSPC'

In [48]:
```
plot(Cl(GSPC))
```



Superimpose the two time series

In [49]:
```
z <- Cl(GSPC)
par(mar = c(5, 4, 4, 4) + 0.3)  # Leave space for z axis
plot(h.spx.252,type="n",ylab=expression(H[SPX]),main=NULL)
lines(h.spx.252,col="red",lwd=2)
par(new = TRUE)
plot(z, type = "l", axes = FALSE, bty = "n", xlab = "", ylab = "",col="g
reen4",main=NULL)
axis(side=4, at = pretty(range(z)))
mtext("SPX", side=4, line=3)
```



Observations

- H tends to spike when the market is under stress.
  - And seems close to zero when the market is calm.
- Note the following peaks
  - The Greek debt crisis in late 2011.
  - The Brexit vote in 2015. In this case H rises with uncertainty then collapses.
- When the market crashes, H rises. But often H rises without the market crashing.
- In particular, H of the volatility time series seems to be a meaningful and relevant statistic.

Comte and Renault: FSV model

[Comte and Renault][6]</sup> were perhaps the first to model volatility using fractional Brownian motion. In their fractional stochastic volatility (FSV) model,

$$\frac{dS_t}{S_t} = \sigma_t\,dZ_t$$
$$d\log\sigma_t = -\alpha\,(\log\sigma_t - \theta)\,dt + \gamma\,d\hat{W}^H_t$$

with

$$\hat{W}^H_t = \int_0^t \frac{(t-s)^{H-1/2}}{\Gamma(H+1/2)}\,dW_s, \quad 1/2 \le H < 1$$

and $\mathbb{E}\left[dW_t\,dZ_t\right] = \rho\,dt$.

- The FSV model is a generalization of the Hull-White stochastic volatility model.

RFSV and FSV

- The model (1):
$$\log\sigma_{t+\Delta} - \log\sigma_t = \nu\,\left(W^H_{t+\Delta} - W^H_t\right)$$
is not stationary.
- Stationarity is desirable both for mathematical tractability and also to ensure reasonableness of the model at very large times.
- The RFSV model (the stationary version of (1) is formally identical to the FSV model. Except that
  - $H < 1/2$ in RFSV vs $H > 1/2$ in FSV.
  - $\alpha\,T \gg 1$ in RFSV vs $\alpha\,T \sim 1$ in FSV, where T is a typical timescale of interest.

Heuristic derivation of autocorrelation function

We assume that $\sigma_t = \bar{\sigma}_t\,e^{\nu\,W^H_t}$. Then

$$\begin{aligned}&\mathrm{cov}\left[\sigma_t,\sigma_{t+\Delta}\right]\\ &= \bar{\sigma}_t\,\bar{\sigma}_{t+\Delta}\,\left[\exp\left\{\frac{\nu^2}{2}\left(t^{2H} + (t+\Delta)^{2H} - \Delta^{2H}\right)\right\} - 1\right]\\ &\sim \bar{\sigma}_t\,\bar{\sigma}_{t+\Delta}\,\exp\left\{\frac{\nu^2}{2}\left(t^{2H} + (t+\Delta)^{2H} - \Delta^{2H}\right)\right\} \text{ as } t\to\infty.\end{aligned}$$

Similarly, $\mathrm{var}\left[\sigma_t\right] \sim \bar{\sigma}_t^2\,\exp\left\{\nu^2\,t^{2H}\right\}$ as $t \to \infty$. Thus

$$\rho(\Delta) = \frac{\mathrm{cov}\left[\sigma_t,\sigma_{t+\Delta}\right]}{\sqrt{\mathrm{var}\left[\sigma_t\right]\,\mathrm{var}\left[\sigma_{t+\Delta}\right]}} \sim \exp\left\{-\frac{\nu^2}{2}\,\Delta^{2H}\right\}.$$

FSV and long memory

- Why did [Comte and Renault][6]</sup> choose $H > 1/2$?
    - Because it has been a widely-accepted stylized fact that the volatility time series exhibits long memory.
- In this technical sense, *long memory* means that the autocorrelation function of volatility decays as a power-law.

- One of the influential papers that established this was [Andersen, Bollerslev, Diebold and Ebens][1] </sup> which estimated the degree d of fractional integration from daily realized variance data for the 30 DJIA stocks.
    - Using the GPH (Geweke-Porter-Hudak) estimator, they found d around 0.35 which implies that the ACF $\rho(\tau) \sim \tau^{2\,d-1} = \tau^{-0.3}$ as $\tau \to \infty$.

- But every statistical estimator assumes the validity of some underlying model!

- In the RFSV model,

$$\rho(\Delta) \sim \exp\left\{-\frac{\nu^2}{2}\,\Delta^{2\,H}\right\}.$$

- Using the same GPH estimator on the Oxford-Man RV data we find $d = 0.48$ which according to their test would indicate extreme long memory. But our model (1) is different from that of [Andersen, Bollerslev, Diebold and Ebens][1]</sup>; it does not have long memory.

Correlogram and test of scaling

In [19]:
```
v <- rv.list[[1]]  # Pick spx.rk
sig <- sqrt(as.numeric(v))

aclog <-acf(log(sig),lag=100,plot=F)
plot(aclog$lag[-1],aclog$acf[-1],pch=20,ylab=expression(rho(Delta)),xlab
=expression(paste(Delta," (days)")),log="xy",col="blue")
```

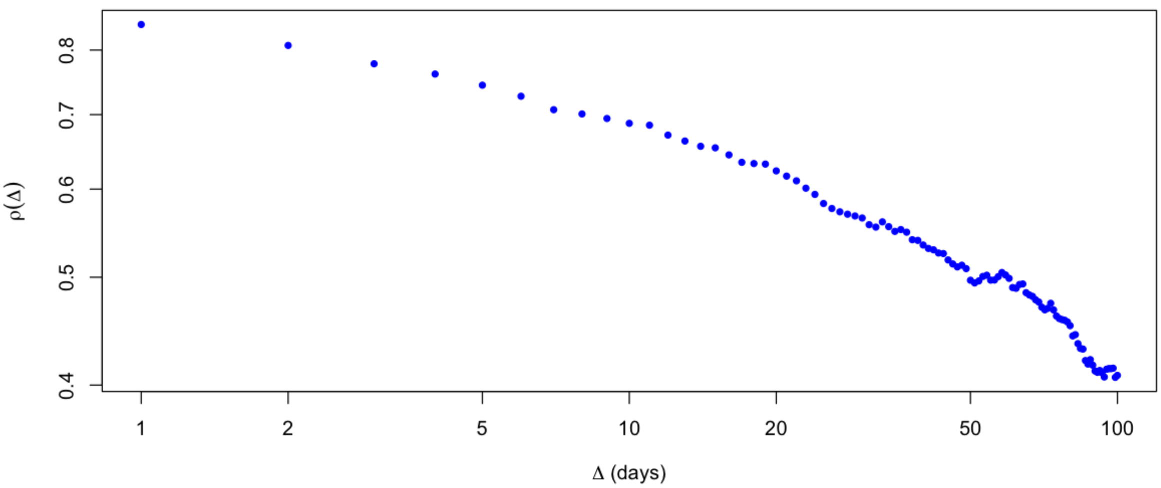


Figure 5: A correlogram of $\sigma_t = \sqrt{RV_t}$; it doesn't look linear!

In [20]:
```
esig2 <- mean(sig)^2
covdel <- acf(sig,lag.max=100,type="covariance",plot=F)$acf[-1]
x <- (1:100)^(2*h.est)
plot(x,log(covdel+esig2),pch=20,col="dark green",ylab=expression(phi(Del
ta)),xlab=expression(Delta^0.28 ))
abline(lm(log(covdel+esig2)~x),col="red",lwd=2)
```

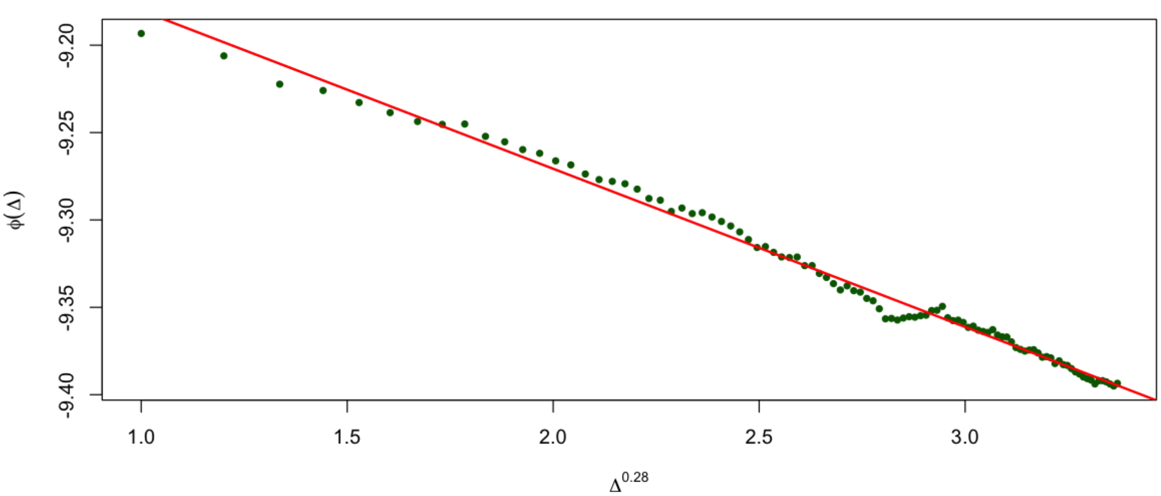


Figure 6: A plot of $\phi(\Delta) := \langle \log\left(\text{cov}(\sigma_{t+\Delta}, \sigma_t) + \langle \sigma_t \rangle^2\right) \rangle$ vs Δ^{2H} with $H \approx 0.14$. Clearly consistent with the scaling relationship $m(2, \Delta) \propto \Delta^{2H}$.

Model vs empirical autocorrelation functions

In [50]:
```
v <- rv.list[[1]]  # Pick spx.rk
sig <- sqrt(as.numeric(v))

aclog <-acf(log(sig),lag=100,plot=F)

y <- log(aclog$acf)
(h.spx <- OxfordH$h.est[1])
x <- aclog$lag^(2*h.spx)
fit.lm <- lm(y[-1]~x[-1])
a <- fit.lm$coef[1]
b <- fit.lm$coef[2]
```

0.129853795129515

In [51]:
```
plot(aclog$lag[-1],aclog$acf[-1],pch=20,ylab=expression(rho(Delta)),
     xlab=expression(paste(Delta," (days)")),log="xy",col="blue")
curve(exp(a+b*x^(2*h.est)),from=0.001,to=100,col="red",add=T,lwd=2,log=
"xy")
```

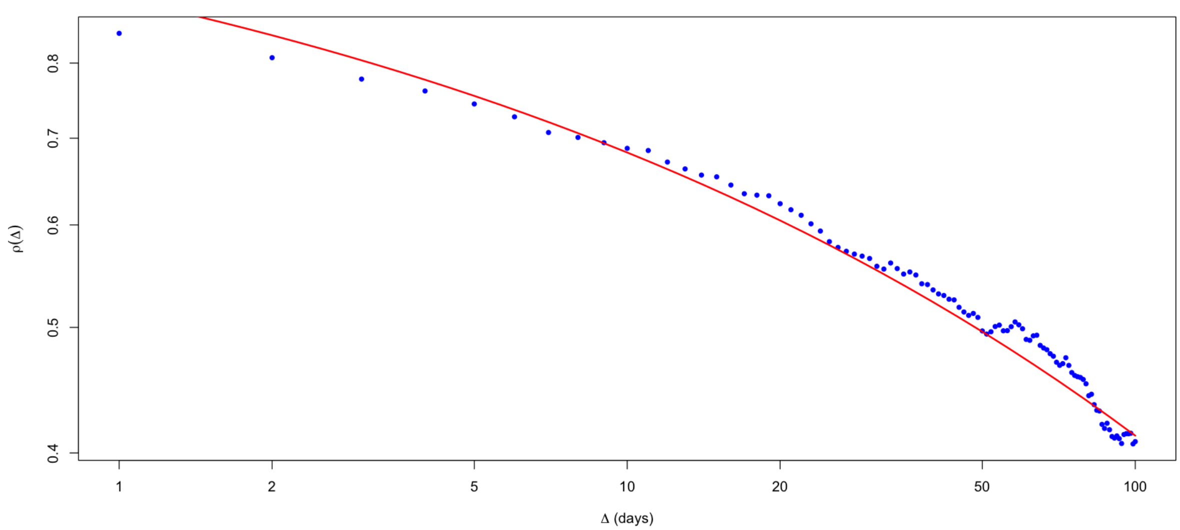


Figure 7: Here we superimpose the RFSV functional form $\rho(\Delta) \sim \exp\left\{-\frac{\nu^2}{2}\Delta^{2H}\right\}$ (in red) on the empirical curve (in blue).

Long memory of volatility may be spurious

- Looking at Figures 5, 6 and 7, there is no reason to suppose that volatility is long memory.
- Moreover, the RFSV model reproduces the observed autocorrelation function very closely.
- [Gatheral, Jaisson and Rosenbaum][6]</sup> further simulate volatility in the RFSV model and apply standard estimators to the simulated data.
  - Real data and simulated data generate very similar plots and similar estimates of the long memory parameter to those found in the prior literature.

- The RSFV model does not have the long memory property.
- Classical estimation procedures seem to identify spurious long memory of volatility.

Incompatibility of FSV with realized variance (RV) data}

- In Figure 8, we demonstrate graphically that long memory volatility models such as FSV with $H > 1/2$ are not compatible with the RV data.
- In the FSV model, the autocorrelation function $\rho(\Delta) \propto \Delta^{2\,H-2}$. Then, for long memory, we must have $1/2
- For $\Delta \gg 1/\alpha$, stationarity kicks in and $m(2,\Delta)$ tends to a constant as $\Delta \to \infty$.
- For $\Delta \ll 1/\alpha$, mean reversion is not significant and $m(2,\Delta) \propto \Delta^{2\,H}$.

RFSV vs FSV

- We can compute $m(2,\Delta)$ explicitly in both the FSV and RFSV models.
- The smallest possible value of H in FSV is $H = 1/2$. One empirical estimate in the literature says that $H \approx 0.53$ some time in 2008.

- Let's see how the theoretical estimates of $m(2,\Delta)$ compare with data.

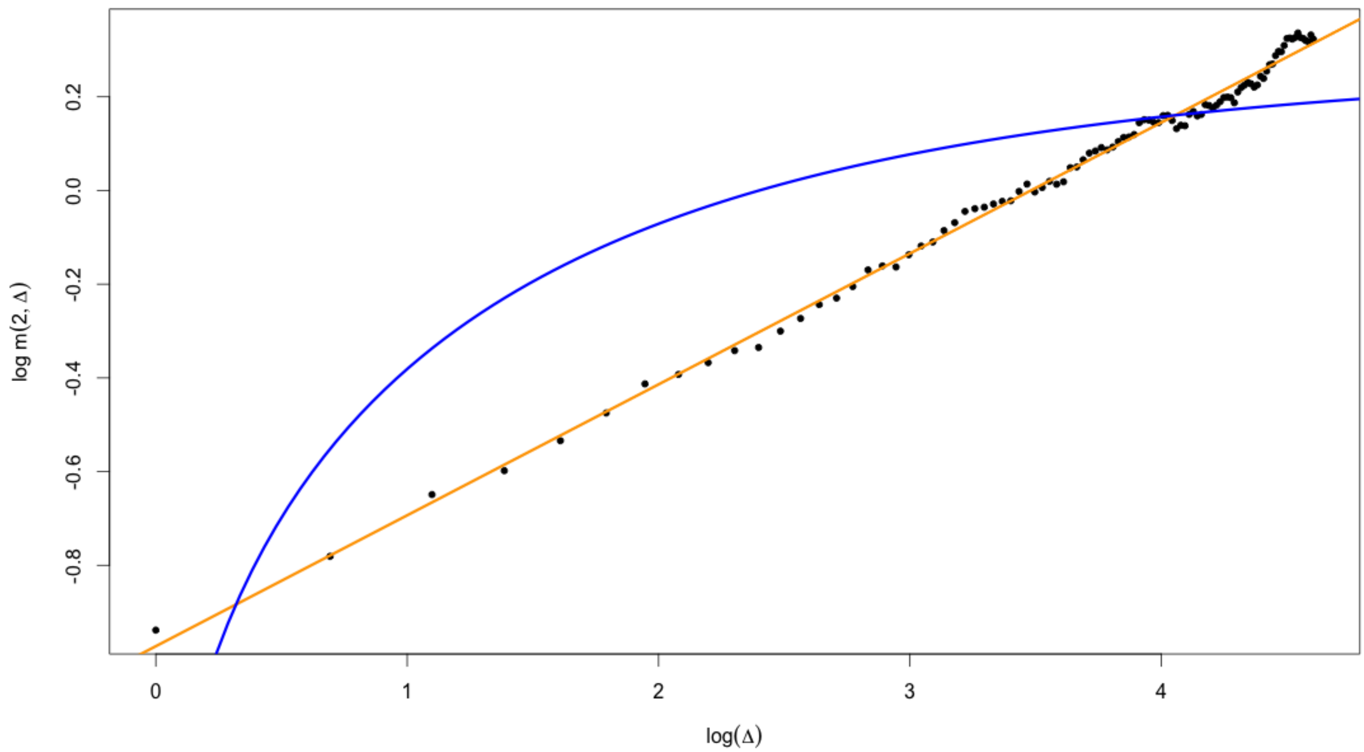


Figure 8: Black points are empirical estimates of $m(2,\Delta)$; the blue line is the FSV model with $\alpha=0.5$ and $H=0.53$; the orange line is the RFSV model with $\alpha=0$ and $H=0.14$.

Does simulated RSFV data look real?

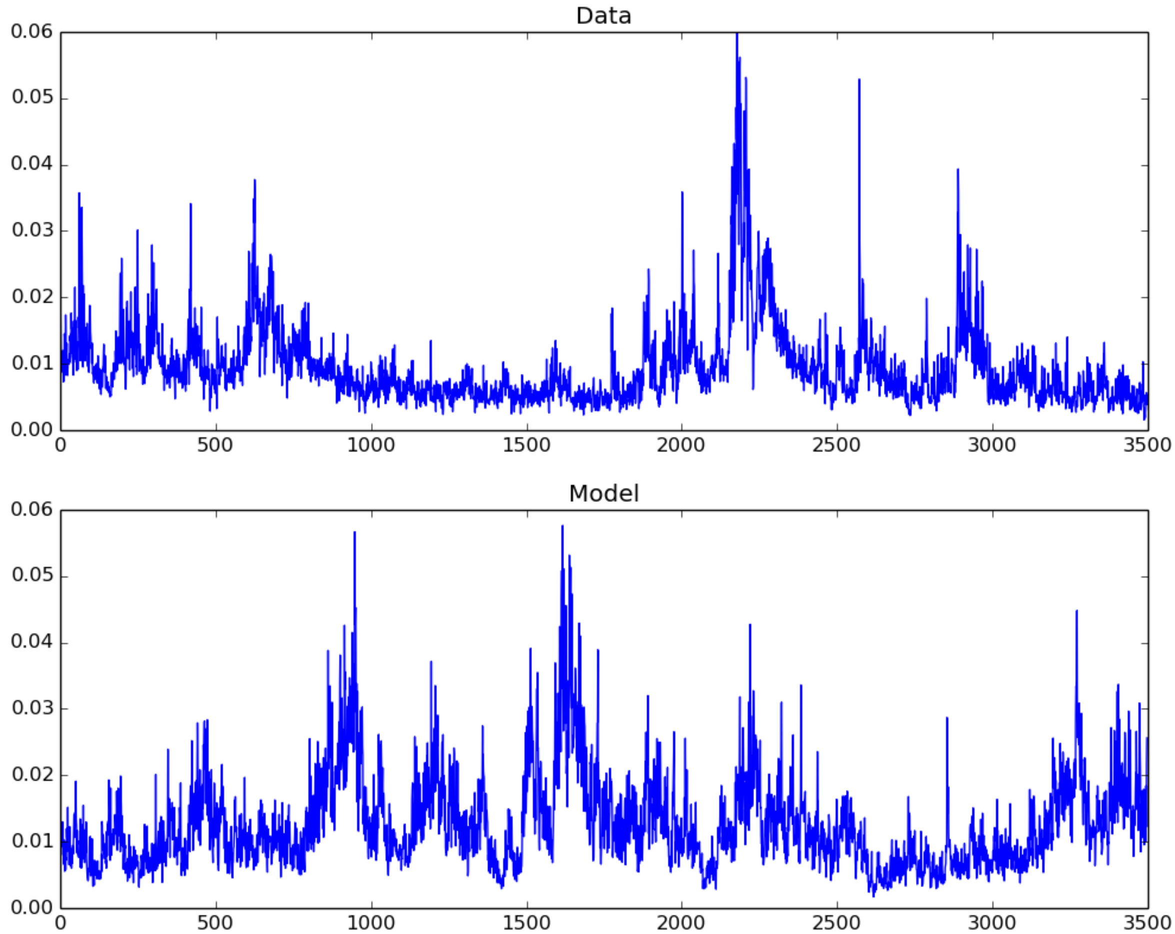


Figure 9: Volatility of SPX (above) and of the RFSV model (below).

Remarks on the comparison

- The simulated and actual graphs look very alike.
- Persistent periods of high volatility alternate with low volatility periods.
- $H \sim 0.1$ generates very rough looking sample paths (compared with $H = 1/2$ for Brownian motion).
- Hence *rough volatility*.

- On closer inspection, we observe fractal-type behavior.
- The graph of volatility over a small time period looks like the same graph over a much longer time period.
- This feature of volatility has been investigated both empirically and theoretically in, for example, [Bacry and Muzy][2]</sup> .
- In particular, their Multifractal Random Walk (MRW) is related to a limiting case of the RSFV model as $H \to 0$.

Applications

- What is this rough volatility model good for?
- If we could change measure from $\mathbb{P}$ to $\mathbb{Q}$, we would be able to price options.
  - More on that later.
- Another obvious application is to volatility forecasting.

Forcasting fBm

- In the RFSV model (1), $\log \sigma_t \approx \nu\, W^H_t + C$ for some constant C.
- [Nuzman and Poor][12]</sup> show that $W^H_{t+\Delta}$ is conditionally Gaussian with conditional expectation

$$\mathbb{E}[W^H_{t+\Delta} | \mathcal{F}_t] = \frac{\cos(H\pi)}{\pi} \Delta^{H+1/2} \int_{-\infty}^t \frac{W^H_s}{(t-s+\Delta)(t-s)^{H+1/2}} ds$$

and conditional variance

$$\text{Var}[W^H_{t+\Delta} | \mathcal{F}_t] = \tilde{c}\, \Delta^{2H}.$$

where

$$\tilde{c} = \frac{\Gamma(3/2-H)}{\Gamma(H+1/2)\,\Gamma(2-2H)}.$$

A heuristic explanation of the formula

- The forecast formula comes from regressing $W^H_{t+\Delta}$ against the W^H_s with $s
- Let

$$\beta(u,\Delta) = \frac{\cos(H\pi)}{\pi} \Delta^{H+1/2} \frac{1}{(u+\Delta)\, u^{H+1/2}}.$$

  Then, for $t, \Delta > 0$ and $0
- In particular,

$$\int_0^\infty \beta(u,\Delta)\, du = 1.$$

- With $\beta(u,\Delta)$ thus defined and for $s
$$\mathbb{E}\left[W^H_s\,\left(W^H_{t+\Delta}-\int_{-\infty}^t\beta(t-u,\Delta)\,W^H_u\,du\right)\right]=0.$$
    - In other words, the $\beta(t-u,\Delta)$ are the normal regression coefficients.

The forecast formula

Using that W^H is a Gaussian random variable, we get that

Variance forecast formula

(3)
$$\mathbb{E}^{\mathbb{P}}\left[v_{t+\Delta}\,|\,\mathcal{F}_t\right]=\exp\left\{\mathbb{E}^{\mathbb{P}}\left[\log(v_{t+\Delta})|\,\mathcal{F}_t\right]+2\,\tilde{c}\,\nu^2\Delta^{2\,H}\right\}$$

where
$$\begin{aligned}&\mathbb{E}^{\mathbb{P}}\left[\log v_{t+\Delta}\,|\,\mathcal{F}_t\right]\\&=\frac{\cos(H\pi)}{\pi}\Delta^{H+1/2}\int_{-\infty}^t\frac{\log v_s}{(t-s+\Delta)(t-s)^{H+1/2}}\,ds.\end{aligned}$$

Discretization of the forecast formula

In [Gatheral, Jaisson, Rosenbaum][4]</sup>, we discretize the integral by taking mid-points as in
$$\mathbb{E}^{\mathbb{P}}\left[\log v_{t+\Delta}\,|\,\mathcal{F}_t\right]\approx\frac{1}{A}\sum_{j=0}^{L}\frac{\log v_{t-j}}{\left(j+\frac{1}{2}+\Delta\right)\,(j+\frac{1}{2})^{H+1/2}}.$$

where L is the maximum number of lags and the normalizing constant A is given by
$$A=\sum_{j=0}^{\infty}\frac{1}{\left(j+\frac{1}{2}+\Delta\right)\,(j+\frac{1}{2})^{H+1/2}}.$$

Inspired by [Bennedsen, Lunde and Pakkanen][4]</sup>, we approximate the first term in the sum more accurately as follows.

$$\mathbb{E}^{\mathbb{P}}\left[\log v_{t+\Delta} \,|\, \mathcal{F}_t\right] \approx \frac{1}{A}\left\{\frac{\log v_t}{(s^\star+\Delta)\,(s^\star)^{H+1/2}} + \sum_{j=1}^{L}\frac{\log v_{t-j}}{(j+\frac{1}{2}+\Delta)\,(j+\frac{1}{2})^{H+1/2}}\right\}$$

where $s^\star$ is chosen such that

$$\frac{1}{\gamma} = \int_0^1 \frac{ds}{s^{H+\frac{1}{2}}} = \frac{1}{{s^\star}^{H+\frac{1}{2}}} = \frac{1}{{s^\star}^{1-\gamma}}$$

where $\gamma = \frac{1}{2} - H$. Thus

$$s^\star = \gamma^{\frac{1}{1-\gamma}}.$$

Implement variance forecast in R

In [52]:

```
# Find all of the dates
dateIndex <- substr(as.character(index(spx.rk)),1,10) # Create index of dates

cTilde <- function(h){gamma(3/2-h)/(gamma(h+1/2)*gamma(2-2*h))} # Factor because we are computing conditional on \cF_t

# XTS compatible version of forecast
rv.forecast.XTS <- function(rvdata,h,date,nLags,delta,nu){
    gam <- 1/2-h
    j <- (1:nLags)-1
    cf <- 1/((j+1/2)^(h+1/2)*(j+1/2+delta)) # Lowest number should apply to latest date
    s.star <- gam^(1/(1-gam))
    cf[1] <- 1/(s.star^(h+1/2)*(s.star+delta))
    datepos <- which(dateIndex==date)
    ldata <- log(as.numeric(rvdata[datepos-j])) # Note that this object is ordered from earlier to later
    pick <- which(!is.na(ldata))
    norm <- sum(cf[pick])
    fcst <- cf[pick]%*%ldata[rev(pick)]/norm # Most recent dates get the highest weight
    return(exp(fcst+2*nu^2*cTilde(h)*delta^(2*h)))

    }
```

SPX actual vs forecast variance

- In order to forecast using (3), we need estimates of H and ν.
  - We use our estimates of H and ν from the regressions rather than from the ACF estimator.
  - The choice does not seem to make much difference.

In [54]:
```
var.forecast.spx <- function(h,nu)function(del){
    n <- length(spx.rk)
    nLags <- 200

    range <- nLags:(n-del)
    rv.predict <- sapply(dateIndex[range],function(d){rv.forecast.XTS(rv
data=spx.rk,h,d,nLags=nLags,delta=del,nu)})
    rv.actual <- spx.rk[range+del]
    return(list(rv.predict=rv.predict,rv.actual=rv.actual))
    }
```

- From experiment, we found that around 200 lags works best.

Scatter plot of delta days ahead predictions

In [56]:
```
del <- 1
vf <- var.forecast.spx(h=h.spx.regression,nu=nu.spx.regression)(del)
rv.predict <- vf$rv.predict
rv.actual <- vf$rv.actual
vol.predict <- sqrt(as.numeric(rv.predict))
vol.actual <- sqrt(as.numeric(rv.actual))
vol.actual <-  sqrt(as.numeric(rv.actual))
```

In [57]:
```
c(mean(vol.actual-vol.predict),sd(vol.actual-vol.predict))
```

-0.000411372312009911 0.00304734426814065

In [60]:
```
plot(vol.predict,vol.actual,col="blue",pch=20, ylab="Actual vol.", xlab=
"Predicted vol.")
abline(coef=c(0,1),col="red",lwd=3)
```

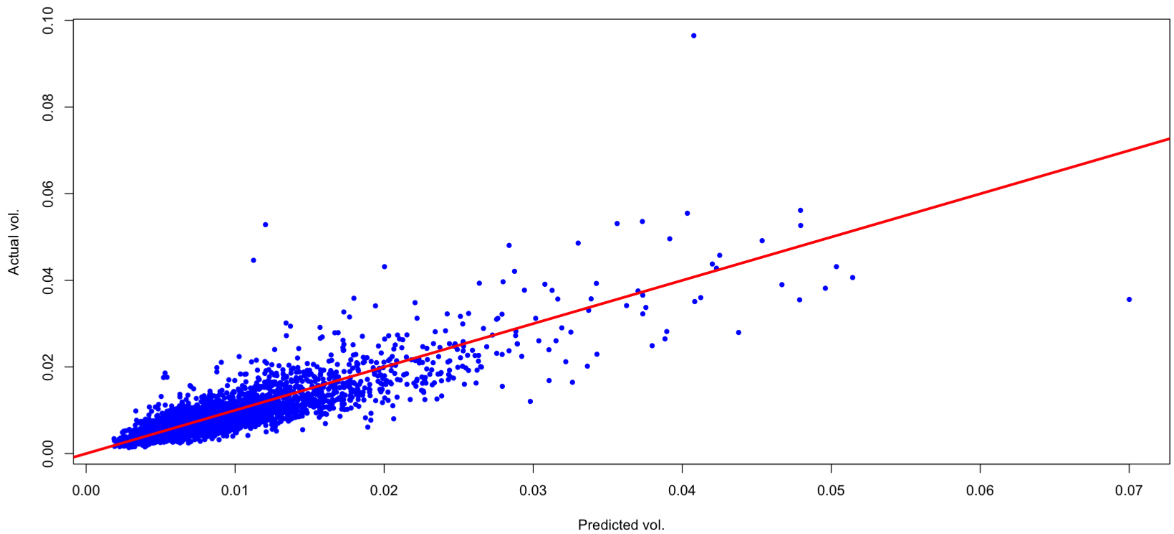


Figure 10: Actual vols vs predicted vols.

Superimpose actual and predicted vols

In [25]:
```
vol.actual <- sqrt(as.numeric(rv.actual)*252)
vol.predict <- sqrt(rv.predict*252)
plot(vol.actual, col="blue",type="l")
lines(vol.predict, col="red",type="l")
```

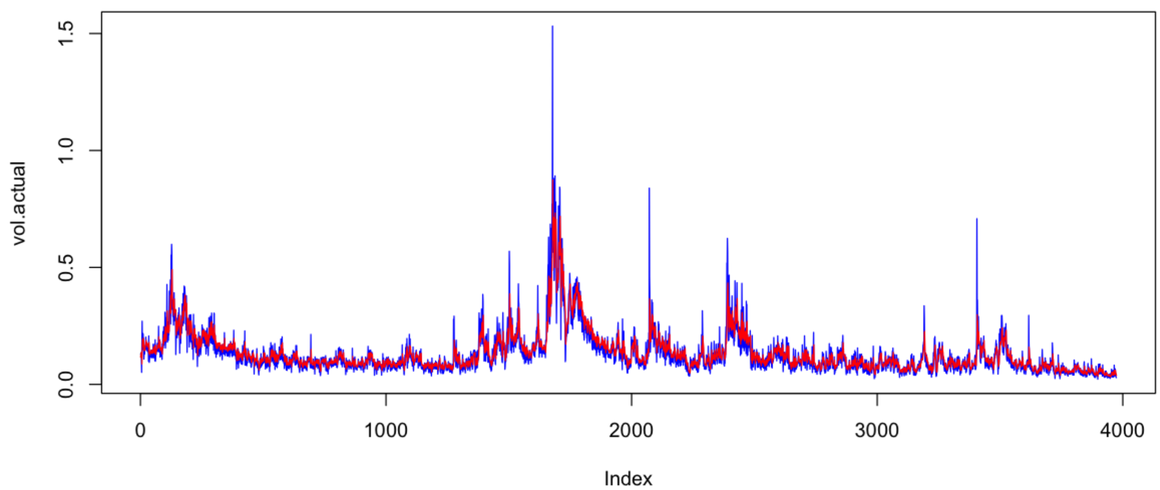
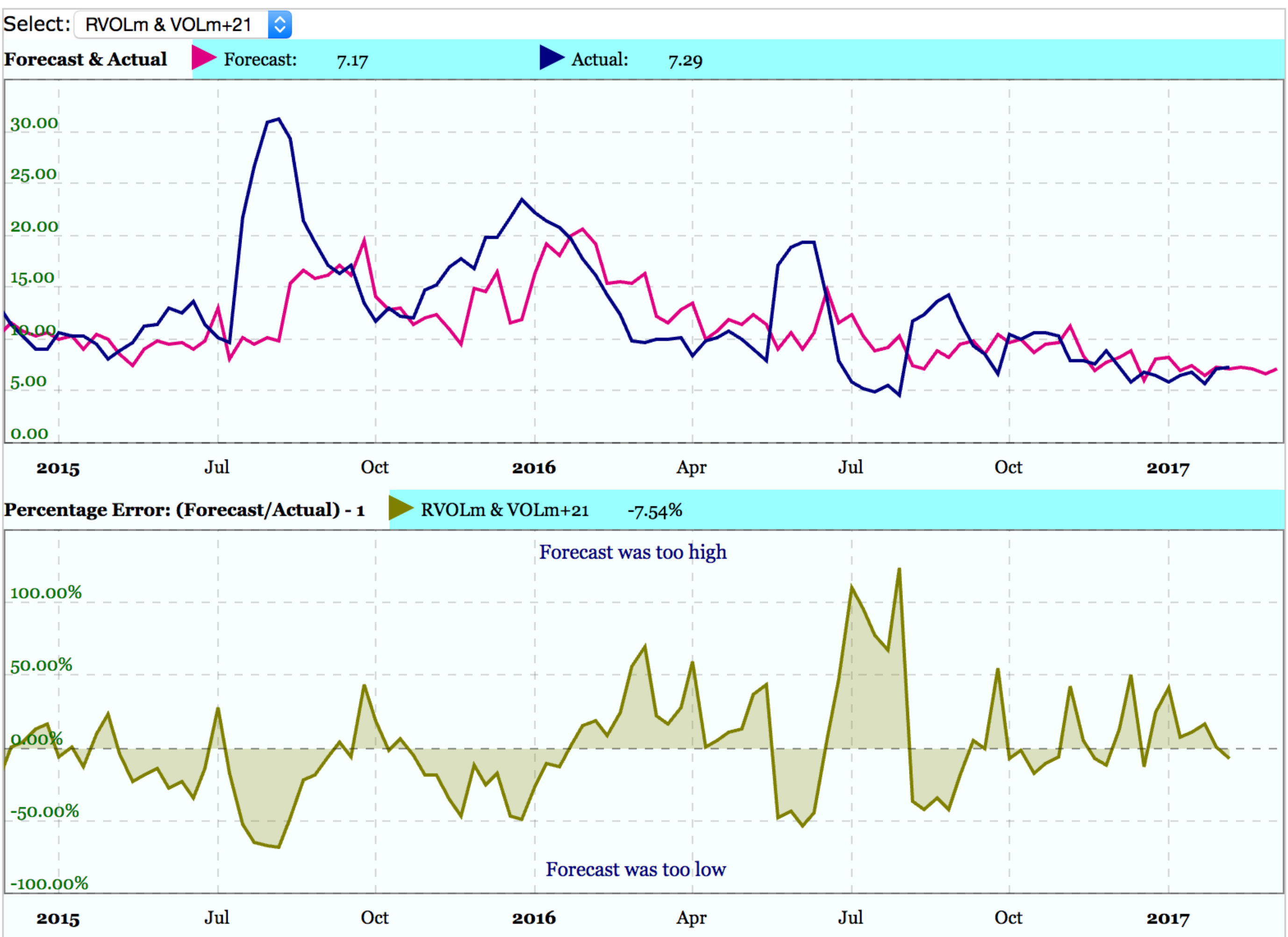


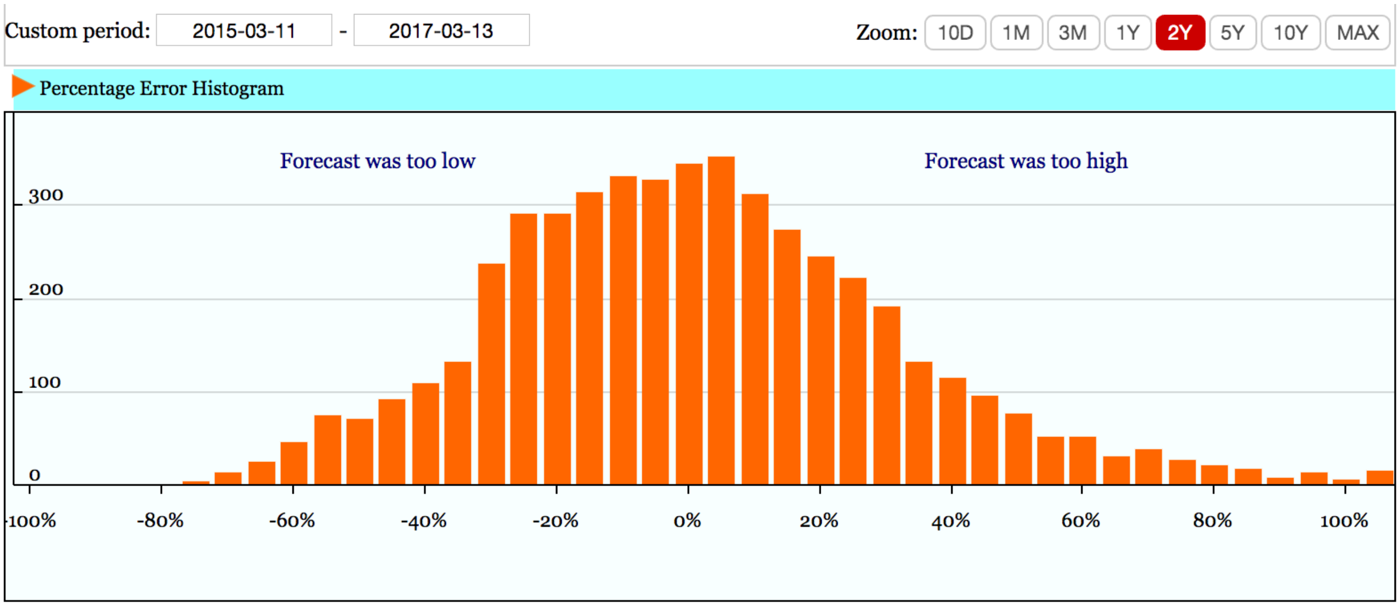
Figure 11: Actual volatilities in blue; predicted vols in red. Note that volatilities are in daily terms.

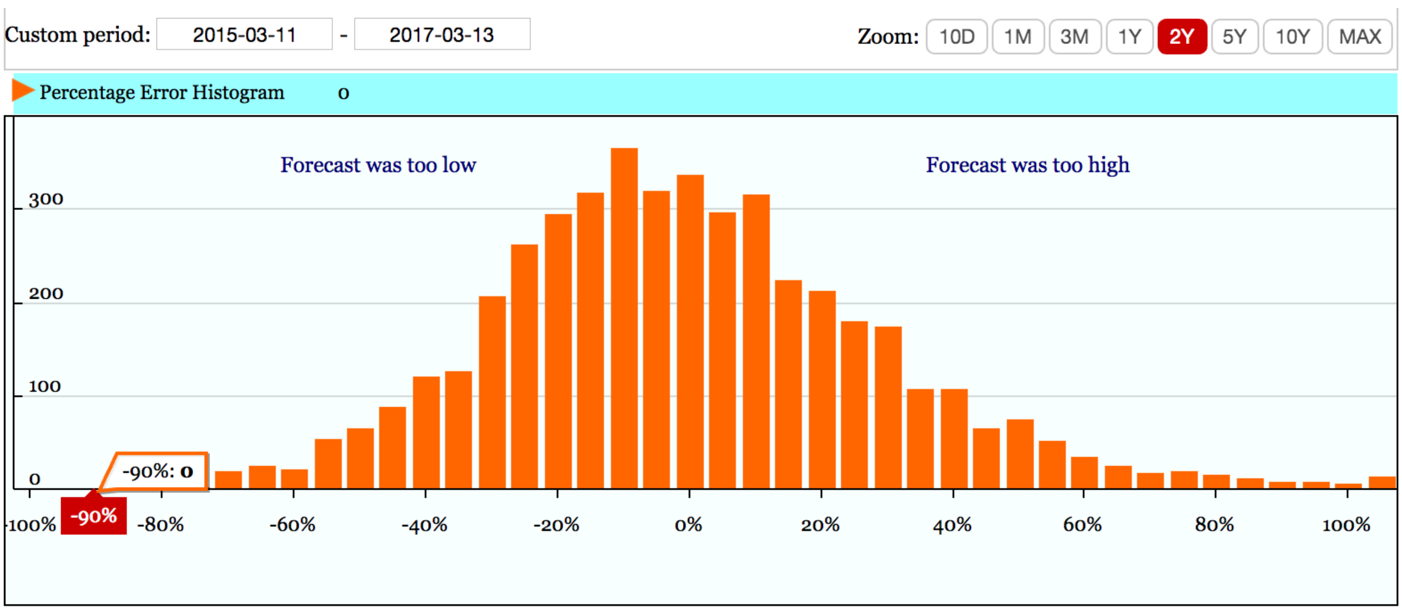
VolX

- The commercial company VolX (http://volx.us (http://volx.us)) has developed a number of RealVol Instruments and RealVol Indices based on realized volatility as defined by the RealVol Formulas.
    - Their business model is to license these indices to exchanges and information providers.
- They publish daily forecasts of RV using HARK (which is HAR-RV with Kalman filtering, and RVOL, an implementation of the Rough Volatility forecast.
- You can compare forecast versus actual volatility for the two estimators here: http://www.volx.us/volatilitycharts.shtml?2&SPY&PRED (http://www.volx.us/volatilitycharts.shtml?2&SPY&PRED).

VolX screenshots







Conditional and unconditional variances

- The HAR and rough volatility forecasts are both impressive.
  - Much superior to alternatives such as GARCH.
- However, HAR is a regression and rough volatility is a proper model.

- One practical consequence is that we can put error bars on our volatility forecasts.

So how good is the forecast?

Specifically, by how much is the variance of the future variance reduced by taking into account the whole history of the fBm?

- In practice of course, we only consider some finite history, 200 points say.
- We know this again from [Nuzman and Poor][10]</sup> who showed that the ratio of the conditional to the unconditional variance of the $\log v_t$ is

$$\tilde{c} = \frac{\Gamma(3/2 - H)}{\Gamma(H + 1/2)\,\Gamma(2 - 2H)}.$$

- We can compute this ratio empirically and compare with the model prediction.

Unconditional and conditional variance vs lag Δ

First we compute the time series of prediction errors.

In [64]:
```
log.vol.err <- function(del){
    vf <- var.forecast.spx(h=h.spx.regression,nu=nu.spx.regression)(del)
    rv.predict <- vf$rv.predict
    rv.actual <- vf$rv.actual
    vol.predict <- sqrt(as.numeric(rv.predict))
    vol.actual <-  sqrt(as.numeric(rv.actual))
    err <- log(vol.actual)-log(vol.predict)
    return(err)
    }
```

In [65]:
```
var.log.err <- function(del){
    var(log.vol.err(del))
}
```

In [66]:
```
var.log.err(10)
```

0.137335984832903

The following code takes about 2 minutes to run on my machine. You can run it by uncommenting the code.

In [70]:
```
del <- 1:100
#system.time(var.log.err.del <- sapply(del,var.log.err))
```

```
   user  system elapsed 
113.964   3.754 118.099 
```

In [72]:
```
#save(var.log.err.del ,file="varerr.rData")
load(file="varerr.rData")
```

Plot of conditional and unconditional variance

- The unconditional variance of differences in log-vol is given by
$$m(2,\Delta) = \langle (\log \sigma_{t+\Delta} - \log \sigma_t)^2 \rangle.$$
- The conditional variance is given by `var.log.err`(Δ).

In [73]:
```
plot(del,mq.del(del,2),pch=20,cex=1,ylab=expression(Variance),
     xlab=expression(Delta),col="blue",ylim=c(0,.35),
     main= "Unconditional and conditional variance")
curve(nu.spx.regression^2*x^(2*h.spx.regression),from=0,to=100,add=T,col
="red",lwd=2,n=1000)
points(del,var.log.err.del,col="green4",pch=20)
curve(cTilde(h.spx.regression)* nu.spx.regression^2*x^(2*h.spx.regressio
n),from=0,to=100,
      add=T,col="orange",lwd=2,n=1000)
```

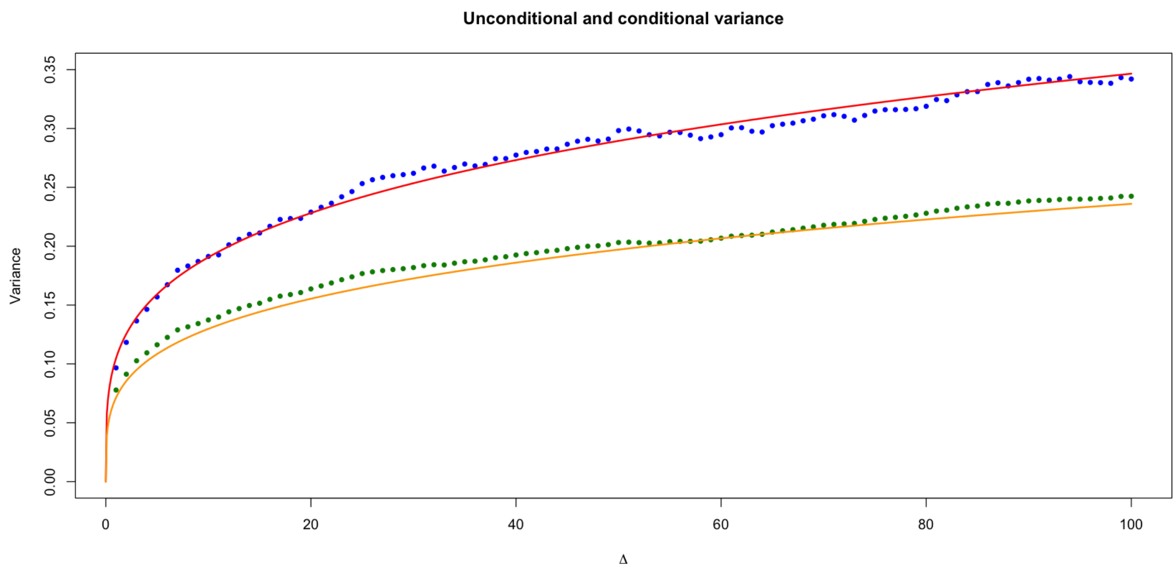


Figure 12. Actual unconditional variance in blue, rough volatility prediction in red; Actual conditional variance in green, rough volatility prediction in orange.

Amazing agreement between data and model

- We observe that the ratio of conditional to unconditional variance is more or less *exactly* as predicted by the model!

Pricing under rough volatility

Following [Bayer, Friz and Gatheral][3]</sup>, the foregoing behavior suggest the following model for volatility under the real (or historical or physical) measure $\mathbb{P}$:

$$\log \sigma_u - \log \sigma_t = \nu \left(W^H_u - W^H_t \right), \quad u > t.$$

Let $\gamma = \frac{1}{2} - H$. We choose the Mandelbrot-Van Ness representation of fractional Brownian motion W^H as follows:

$$W^H_t = C_H \left\{ \int_{-\infty}^t \frac{dW_s^{\mathbb{P}}}{(t-s)^\gamma} - \int_{-\infty}^0 \frac{dW_s^{\mathbb{P}}}{(-s)^\gamma} \right\}$$

where the choice

$$C_H = \sqrt{\frac{2\,H\,\Gamma(3/2 - H)}{\Gamma(H + 1/2)\,\Gamma(2 - 2\,H)}}$$

ensures that

$$\mathbb{E}\left[W^H_t\,W^H_s\right] = \frac{1}{2}\left\{t^{2H} + s^{2H} - |t-s|^{2H}\right\}.$$

Then

$$\begin{aligned} &\log v_u - \log v_t \\ = &\quad 2\,\nu\,C_H \left\{ \int_t^u \frac{1}{(u-s)^\gamma}\,dW_s^{\mathbb{P}} \right. \\ &\left. + \int_{-\infty}^t \left[\frac{1}{(u-s)^\gamma} - \frac{1}{(t-s)^\gamma} \right] dW_s^{\mathbb{P}} \right\} \\ =: &\ 2\,\nu\,C_H\ \left[M_t(u) + Z_t(u) \right]. \end{aligned}$$

- Note that $\mathbb{E}^{\mathbb{P}}\left[M_t(u)|\,\mathcal{F}_t\right] = 0$ and $Z_t(u)$ is $\mathcal{F}_t$-measurable.
  - To price options, it would seem that we would need to know $\mathcal{F}_t$, the entire history of the Brownian motion W_s for $s

Pricing under $\mathbb{P}$

Let

$$\tilde{W}^{\mathbb{P}}_t(u) := \sqrt{2H} \int_t^u \frac{dW^{\mathbb{P}}_s}{(u-s)^\gamma}$$

With $\eta := 2\,\nu\, C_H/\sqrt{2H}$ we have $2\,\nu\, C_H\, M_t(u) = \eta\, \tilde{W}^{\mathbb{P}}_t(u)$ so denoting the stochastic exponential by $\mathcal{E}(\cdot)$, we may write

$$\begin{aligned} v_u &= v_t \exp\left\{ \eta \tilde{W}^{\mathbb{P}}_t(u) + 2\,\nu\, C_H\, Z_t(u) \right\} \\ &= \mathbb{E}^{\mathbb{P}}\left[v_u | \mathcal{F}_t\right] \mathcal{E}\left(\eta\, \tilde{W}^{\mathbb{P}}_t(u)\right). \end{aligned}$$

- The conditional distribution of v_u depends on $\mathcal{F}_t$ only through the variance forecasts $\mathbb{E}^{\mathbb{P}}\left[v_u | \mathcal{F}_t\right]$,
- To price options, one does not need to know $\mathcal{F}_t$, the entire history of the Brownian motion $W^{\mathbb{P}}_s$ for $s

Pricing under $\mathbb{Q}$

Our model under $\mathbb{P}$ reads:

(2)

$$v_u = \mathbb{E}^{\mathbb{P}}\left[v_u | \mathcal{F}_t\right] \mathcal{E}\left(\eta\, \tilde{W}^{\mathbb{P}}_t(u)\right).$$

Consider some general change of measure

$$dW^{\mathbb{P}}_s = dW^{\mathbb{Q}}_s + \lambda_s\, ds,$$

where $\{\lambda_s : s > t\}$ has a natural interpretation as the price of volatility risk. We may then rewrite (2) as

$$v_u = \mathbb{E}^{\mathbb{P}}\left[v_u | \mathcal{F}_t\right] \mathcal{E}\left(\eta\, \tilde{W}^{\mathbb{Q}}_t(u)\right) \exp\left\{ \eta\, \sqrt{2H} \int_t^u \frac{\lambda_s}{(u-s)^\gamma}\, ds \right\}.$$

- Although the conditional distribution of v_u under $\mathbb{P}$ is lognormal, it will not be lognormal in general under $\mathbb{Q}$.
  - The upward sloping smile in VIX options means λ_s cannot be deterministic in this picture.

The rough Bergomi (rBergomi) model

Let's nevertheless consider the simplest change of measure

$$dW_s^{\mathbb{P}} = dW_s^{\mathbb{Q}} + \lambda(s)\, ds,$$

where $\lambda(s)$ is a deterministic function of s. Then from (2), we would have

$$\begin{aligned} v_u &= \mathbb{E}^{\mathbb{P}}\left[v_u \,|\, \mathcal{F}_t\right] \mathcal{E}\left(\eta\, \tilde{W}_t^{\mathbb{Q}}(u)\right) \exp\left\{\eta \sqrt{2H} \int_t^u \frac{1}{(u-s)^{\gamma}}\, \lambda(s)\, ds\right\} \\ &= \xi_t(u)\, \mathcal{E}\left(\eta\, \tilde{W}_t^{\mathbb{Q}}(u)\right) \end{aligned}$$

where the forward variances $\xi_t(u) = \mathbb{E}^{\mathbb{Q}}\left[v_u \,|\, \mathcal{F}_t\right]$ are (at least in principle) tradable and observed in the market.

- $\xi_t(u)$ is the product of two terms:
- $\mathbb{E}^{\mathbb{P}}\left[v_u \,|\, \mathcal{F}_t\right]$ which depends on the historical path $\{W_s, s
- a term which depends on the price of risk $\lambda(s)$.

Features of the rough Bergomi model

- The rBergomi model is a non-Markovian generalization of the Bergomi model:
$$\mathbb{E}\left[v_u \,|\, \mathcal{F}_t\right] \neq \mathbb{E}[v_u | v_t].$$
  - The rBergomi model is Markovian in the (infinite-dimensional) state vector $\mathbb{E}^{\mathbb{Q}}\left[v_u \,|\, \mathcal{F}_t\right] = \xi_t(u)$.
- We have achieved our earlier aim of replacing the exponential kernels in the Bergomi model with a power-law kernel.
- We may therefore expect that the rBergomi model will generate a realistic term structure of ATM volatility skew.

Re-interpretation of the conventional Bergomi model

- A conventional n-factor Bergomi model is not self-consistent for an arbitrary choice of the initial forward variance curve $\xi_t(u)$.
  - $\xi_t(u) = \mathbb{E}\left[v_u \,|\, \mathcal{F}_t\right]$ should be consistent with the assumed dynamics.

- Viewed from the perspective of the fractional Bergomi model however:
  - The initial curve $\xi_t(u)$ reflects the history $\{W_s; s
  - The exponential kernels in the exponent of the conventional Bergomi model approximate more realistic power-law kernels.

- The conventional two-factor Bergomi model is then justified in practice as a tractable Markovian engineering approximation to a more realistic fractional Bergomi model.

The stock price process

- The observed anticorrelation between price moves and volatility moves may be modeled naturally by anticorrelating the Brownian motion W that drives the volatility process with the Brownian motion driving the price process.
- Thus
$$\frac{dS_t}{S_t} = \sqrt{v_t}\, dZ_t$$
with
$$dZ_t = \rho\, dW_t + \sqrt{1-\rho^2}\, dW_t^{\perp}$$
where ρ is the correlation between volatility moves and price moves.

Simulation of the rBergomi model

We simulate the rBergomi model as follows:

- Construct the joint covariance matrix for the Volterra process $\tilde{W}$ and the Brownian motion Z and compute its Cholesky decomposition.
- For each time, generate iid normal random vectors and multiply them by the lower-triangular matrix obtained by the Cholesky decomposition to get a $m \times 2\, n$ matrix of paths of $\tilde{W}$ and Z with the correct joint marginals.

- With these paths held in memory, we may evaluate the expectation under $\mathbb{Q}$ of any payoff of interest.
- This procedure is very slow! We need a faster computation.

Hybrid simulation of BSS processes

- The Rough Bergomi variance process is a special case of a Brownian Semistationary (BSS) process.
- [Bennedsen, Lunde and Pakkanen][4]</sup> show how to simulate such processes more efficiently.
- Very recently, [McCrickerd and Pakkanen][13] show how to massively increasing the efficiency of the hybrid scheme.
  - Moreover, they provide a sample Jupyter notebook!

- Their idea is roughly as follows:

$$\begin{aligned}\int_t^u \frac{dW_s}{(u-s)^\gamma} &= \sum_{k=1}^{n} \int_{t_{k+1}}^{t_k} \frac{dW_s}{(u-s)^\gamma} \\ &\approx \sum_{k=1}^{\kappa} \int_{t_{k+1}}^{t_k} \frac{dW_s}{(u-s)^\gamma} + \sum_{k=\kappa+1}^{n} \frac{1}{(u-s_k)^\gamma} \int_{t_{k+1}}^{t_k} dW_s \\ &= \sum_{k=1}^{\kappa} \int_{t_{k+1}}^{t_k} \frac{dW_s}{(u-s)^\gamma} + \sum_{k=\kappa+1}^{n} \frac{1}{(u-s_k)^\gamma} Z_k \sqrt{\frac{u-t}{n}}\end{aligned}$$

  where $t_k = u - \frac{k}{n}(u-t)$, the Z_k are iid $N(0,1)$ random variables and the s_k are such that

$$\int_{t_{k+1}}^{t_k} \frac{ds}{(u-s)^\gamma} = \frac{1}{(u-s_k)^\gamma}.$$

- The choice $\kappa = 1$ works well in practice.
- The choice $\kappa = 0$ corresponds to the Euler scheme which as expected performs poorly.

Guessing rBergomi model parameters

- The rBergomi model has only three parameters: H, η and ρ.
- If we had a fast simulation, we could just iterate on these parameters to find the best fit to observed option prices. But we don't.
- However, the model parameters H, η and ρ have very direct interpretations:
  - H controls the decay of ATM skew $\psi(\tau)$ for very short expirations.
  - The product $\rho\,\eta$ sets the level of the ATM skew for longer expirations.
    - Keeping $\rho\,\eta$ constant but decreasing ρ (so as to make it more negative) pushes the minimum of each smile towards higher strikes.
- So we can guess parameters in practice.
- As we will see, even without proper calibration (*i.e.* just guessing parameters), rBergomi model fits to the volatility surface are amazingly good.

SPX smiles in the rBergomi model

- In Figures 13 and 14, we show how well a rBergomi model simulation with guessed parameters fits the SPX option market as of August 14, 2013, one trading day before the third Friday expiration.

- Options set at the open of August 16, 2013 so only one trading day left.
  - rBergomi parameters were: $H = 0.05$, $\eta = 2.3$, $\rho = -0.9$.
  - Only three parameters to get a very good fit to the whole SPX volatility surface!

- Note in particular that the extreme short-dated smile is well reproduced by the rBergomi model.
  - There is no need to add jumps!

SPX smiles as of August 14, 2013

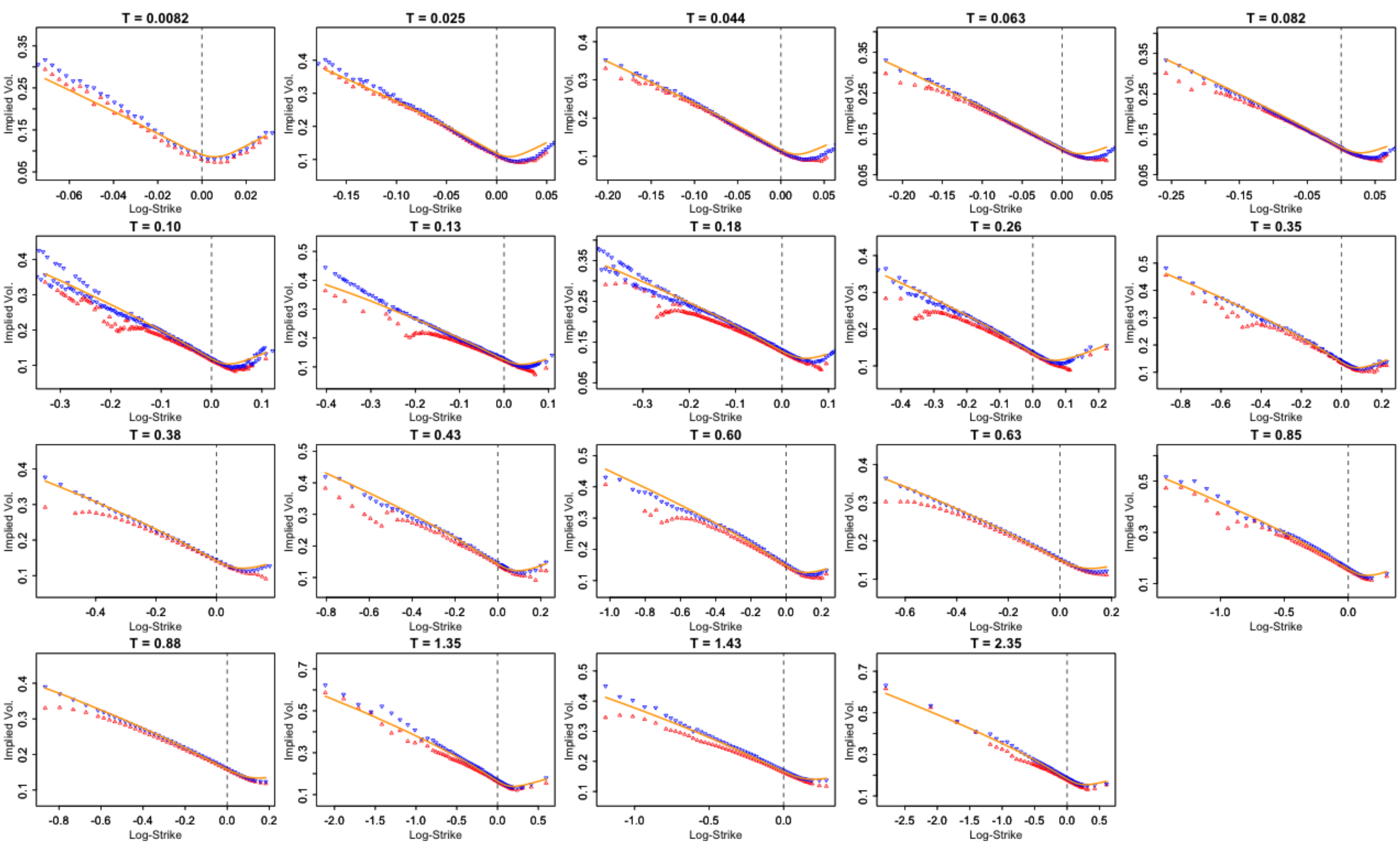


Figure 13: Red and blue points represent bid and offer SPX implied volatilities; orange smiles are from the rBergomi simulation.

The one-month SPX smile as of August 14, 2013

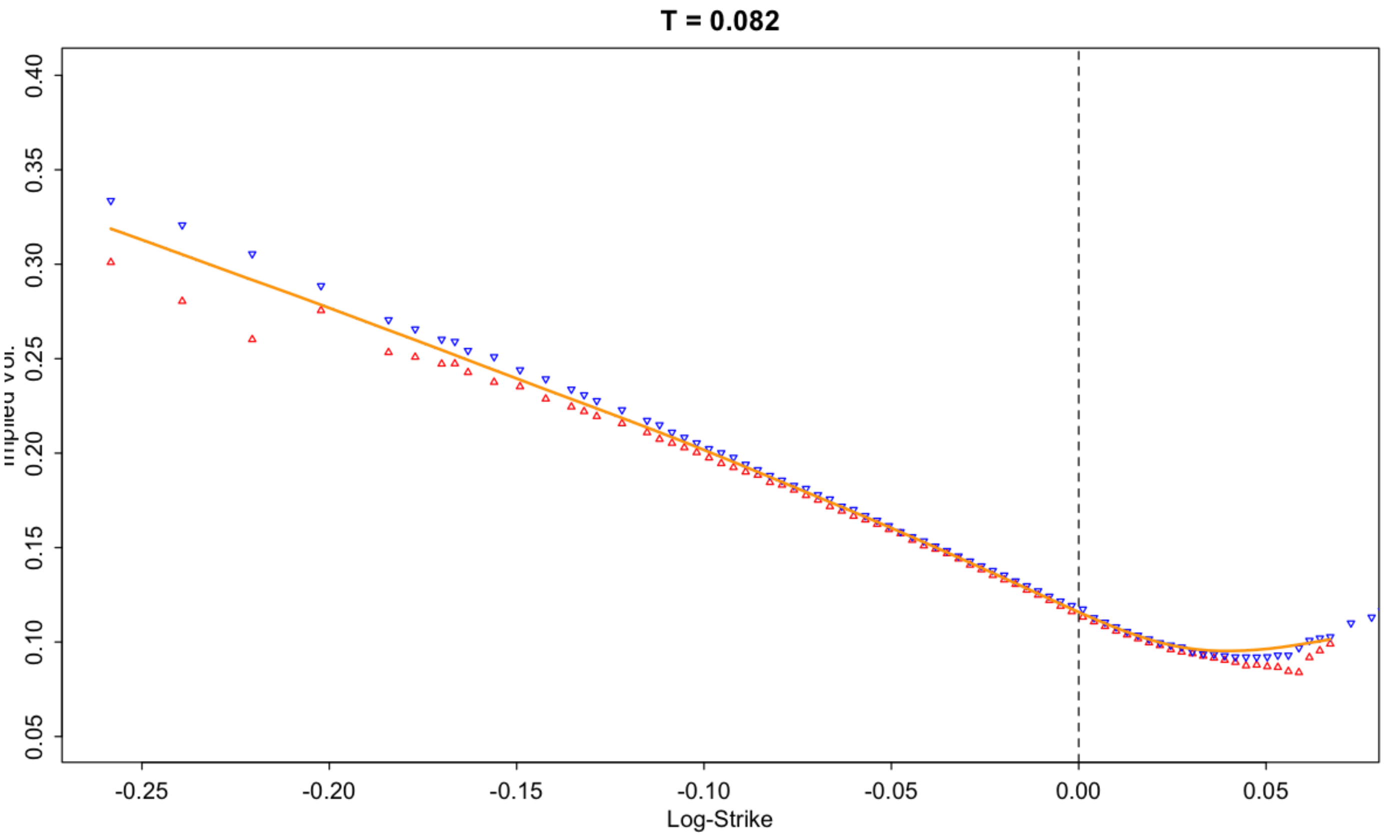


Figure 14: Red and blue points represent bid and offer SPX implied volatilities; orange smile is from the rBergomi simulation.

ATM volatilities and skews

In Figures 15 and 16, we see just how well the rBergomi model can match empirical skews and vols. Recall also that the parameters we used are just guesses!

Term structure of ATM skew as of August 14, 2013

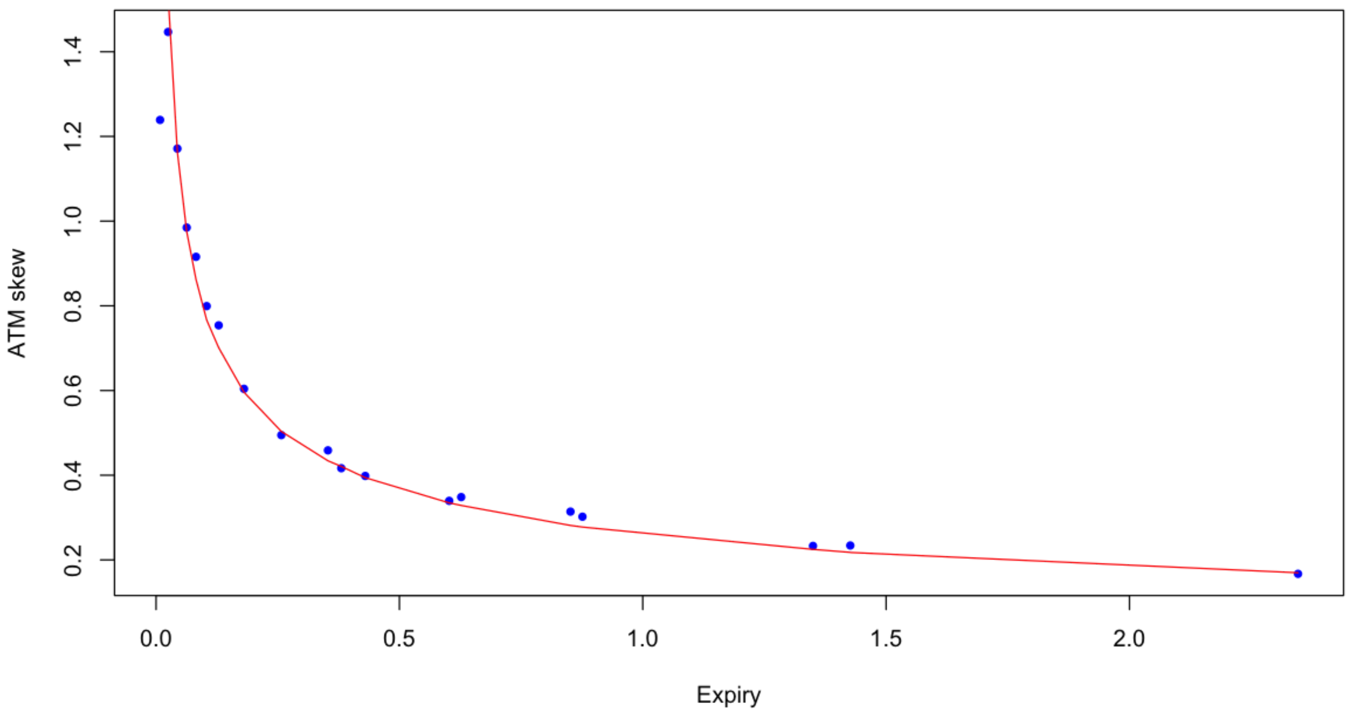


Figure 15: Blue points are empirical skews; the red line is from the rBergomi simulation.

Term structure of ATM vol as of August 14, 2013

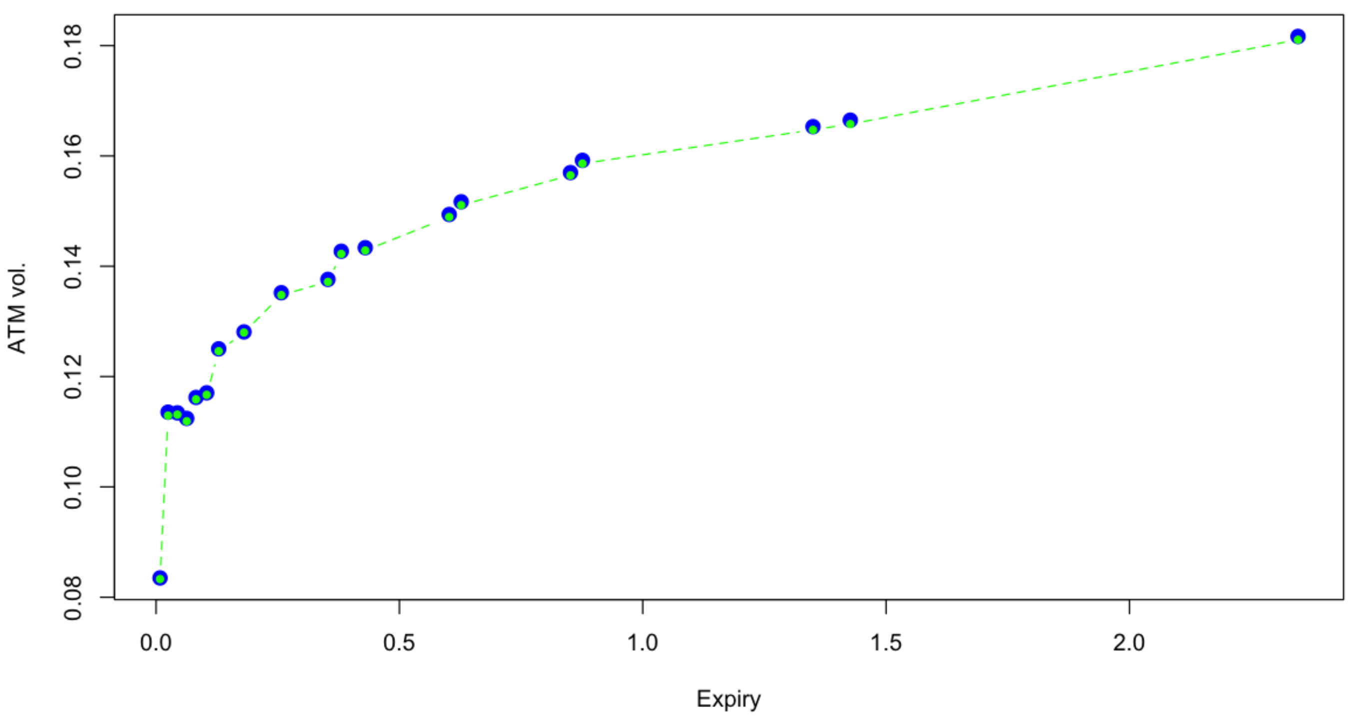


Figure 16: Blue points are empirical ATM volatilities; green points are from the rBergomi simulation. The two match very closely, as they should.

VIX options and futures under rough Bergomi

- The rough Bergomi model generates more or less flat VIX smiles.
  - Inconsistent with observed VIX smiles.

- Nevertheless, we can still try to impute the rough Bergomi parameters H and η by examining the term structure of VIX futures.

The distribution of VIX future payoffs

- Denote the terminal value of the VIX futures by $\sqrt{\zeta(T)}$. Then, by definition (see Chapter 11 of [The Volatility Surface][8]</sup> for more details),

$$\zeta(T)=\frac{1}{\Delta}\int_T^{T+\Delta}\mathbb{E}[v_u|\mathcal{F}_T]\,du.$$

  where Δ is one month.

- In the rough Bergomi model,

$$v_u=\xi_t(u)\,\mathcal{E}\left(\eta\sqrt{2H}\int_t^u\frac{dW_s}{(u-s)^\gamma}\right)$$

  with $\gamma=1/2-H$ so v_u is lognormal.

The lognormal approximation

- The VIX payoff and its square $\zeta(T)$ should be approximately lognormally distributed.
  - The quality of this approximation was confirmed by [Jacquier, Martini and Muguruza][10] </sup>.
  - In that case, the terminal distribution of $\zeta(T)$ is completely determined by $\mathbb{E}\left[\zeta(T)|\,\mathcal{F}_t\right]$ and $\mathrm{var}[\log\zeta(T)|\mathcal{F}_t]$.

- Obviously

$$\mathbb{E}\left[\zeta(T)|\,\mathcal{F}_t\right]=\frac{1}{\Delta}\int_T^{T+\Delta}\xi_t(u)\,du.$$

  - Recall that forward variances $\xi_t(u)$ may be estimated from variance swaps which can themselves be proxied by the log-strip (see Chapter 11 of [The Volatility Surface][8]</sup> again).
  - Alternatively they may be estimated from linear strips of VIX options.

Approximating the conditional variance of $\zeta(T)$

- To estimate the conditional variance of $\zeta(T)$, we approximate the arithmetic mean by the geometric mean as follows:

$$\zeta(T)\approx\exp\left\{\frac{1}{\Delta}\int_T^{T+\Delta}\mathbb{E}[\log v_u\,|\mathcal{F}_T]\,du\right\}.$$

- After some computation detailed in Appendix C of [Bayer, Friz and Gatheral][2]</sup>, we obtain that the variance of $\log VIX^2$ is given by

$$V_t(T) := \mathrm{var}[\log \zeta(T)|\mathcal{F}_t] \approx \eta^2\,(T-t)^{2\,H} f^H\left(\frac{\Delta}{T-t}\right)$$

where

$$f^H(\theta) = \frac{2H}{(H+1/2)^2}\,\frac{1}{\theta^2}\,\int_0^1\,\left[(1+\theta-x)^{1/2+H}-(1-x)^{1/2+H}\right]^2 dx.$$

The approximate fair value of VIX futures

- In [Bayer, Friz and Gatheral][2]</sup>, we chose to study the term structure of VVIX (the VIX of VIX).
- It is more natural to follow [Jacquier, Martini and Muguruza][9]</sup> and approximate the fair value of VIX futures.
- Under the lognormal approximtion, the fair value of the T-maturity VIX future is given by

$$\mathbb{E}\left[\sqrt{\zeta(T)}\middle|\mathcal{F}_t\right] = \sqrt{\mathbb{E}\left[\zeta(T)|\,\mathcal{F}_t\right]}\,\exp\left\{-\frac{1}{8}\,V_t(T)\right\}.$$

Results

- Both [Bayer, Friz and Gatheral][2] and [Jacquier, Martini and Muguruza][9]</sup> find that the volatility of volatility parameter η estimated by calibration to the SPX is roughly 20% greater than the estimate from VIX futures and options.
  - Arbitrage or model mis-specification?
- However, calibration to VIX futures is a very quick and practical way of fixing H.
- If H is fixed, calibration using the Hybrid BSS scheme becomes fast and efficient.
  - The expensive part is the simulation of W^H which is fixed if H is fixed.

Forecasting the variance swap curve

Finally, we forecast the whole variance swap curve using the variance forecasting formula (3).

In [26]:
```
library(stinepack)

xi1 <- function(date,nu,h,dt, tscale){ # dt=(u-t) is in units of years
  xi <- rv.forecast.XTS(spx.rk,h=h,date=date,nLags=500,delta=dt*tscale,n
u)
  return(xi)
}

# Forward variance curve (again the array tt should be in units of year
s)
xi <- function(date,tt,nu,h, tscale){sapply(tt,function(x){xi1(date,nu=n
u,h=h,x,tscale)})}


nu <- OxfordH$nu.est[1]
h <- OxfordH$h.est[1]

varSwapCurve <- function(date,bigT,nSteps,nu,h,tscale,onFactor){
  # Make vector of fwd variances
  tt <- seq(0,bigT,length.out=(nSteps+1))
  dt <- tt[2]
  xicurve <- xi(date,tt,nu,h,tscale)
  xicurve.mid <- (xicurve[1:nSteps]+xicurve[2:(nSteps+1)])/2
  int.xicurve <- cumsum(xicurve.mid)*dt
  varcurve <- int.xicurve/tt[-1]
  varcurve <- c(xicurve[1], varcurve)*onFactor*tscale #onFactor is to co
mpensate for overnight moves
  res <- data.frame(tt,sqrt(varcurve))
  names(res) <- c("texp","vsQuote")
  return(res)
}

varSwapForecast <- function(date,tau,nu,h,tscale,onFactor){
  vsc <- varSwapCurve(date,bigT=2.5,nSteps=100,nu=nu,h=h,tscale,onFactor
) # Creates the whole curve
  x <- vsc$texp
  y <- vsc$vsQuote
  res <- stinterp(x,y,tau)$y
  return(res)
}

# Test the function
tau <- c(.25,.5,1,2)
date <- "2008-09-08"
varSwapForecast(date,tau,nu=nu,h=h,tscale=252,onFactor=1)
```

0.218807732159586   0.214898945069301   0.214162599659132
0.216497104926357

Constructing a time series of variance swap curves

For each of 2,658 days from Jan 27, 2003 to August 31, 2013:

- We compute proxy variance swaps from closing prices of SPX options sourced from OptionMetrics (www.optionmetrics.com) via WRDS.
- We form the forecasts $\mathbb{E}^{\mathbb{P}}\left[v_u \,|\, \mathcal{F}_t\right]$ using (3) with 500 lags of SPX RV data sourced from The Oxford-Man Institute of Quantitative Finance (http://realized.oxford-man.ox.ac.uk (http://realized.oxford-man.ox.ac.uk)).

- We note that the actual variance swap curve is a factor (of roughly 1.4) higher than the forecast, which we may attribute to a combination of overnight movements of the index and the price of volatility risk.
- Forecasts must therefore be rescaled to obtain close-to-close realized variance forecasts.

3-month forecast vs actual variance swaps

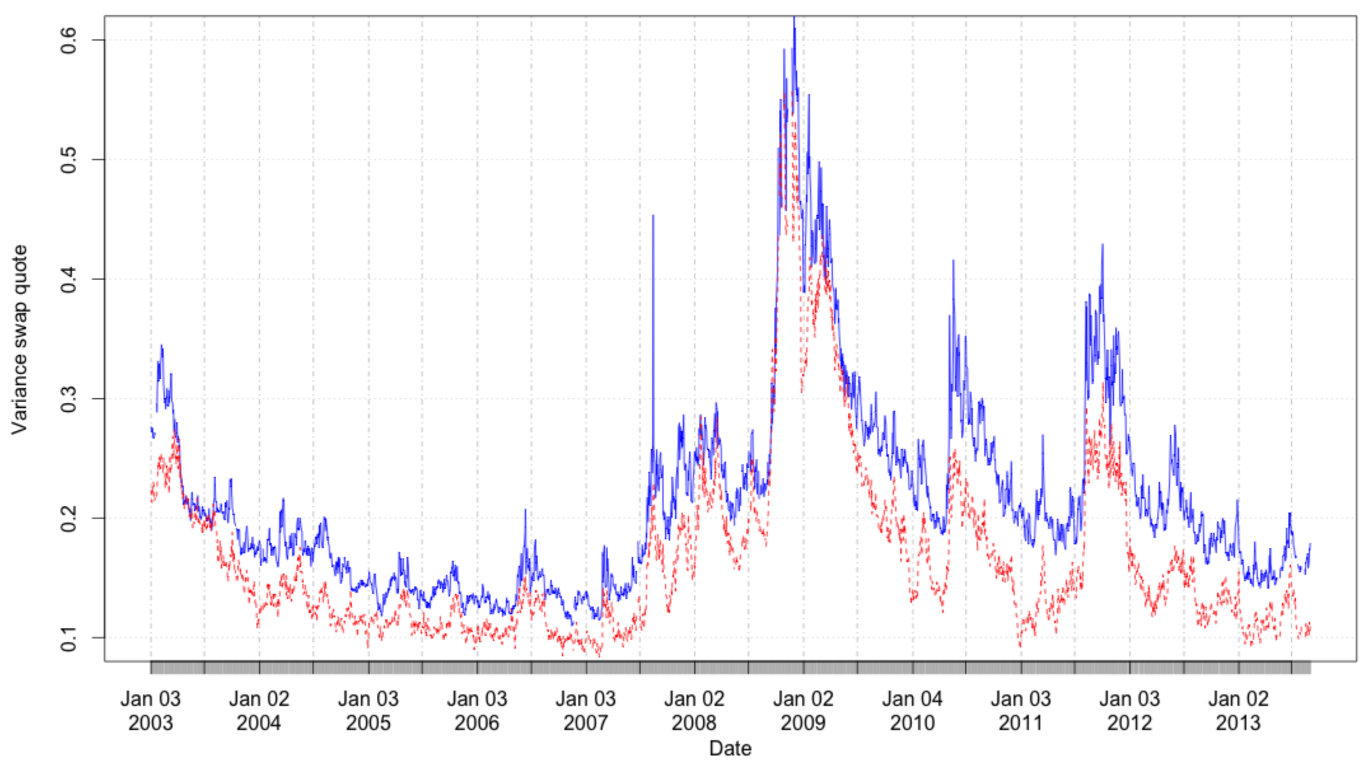


Figure 17: Actual (proxy) 3-month variance swap quotes in blue vs forecast in red (with no scaling factor).

Ratio of actual to forecast

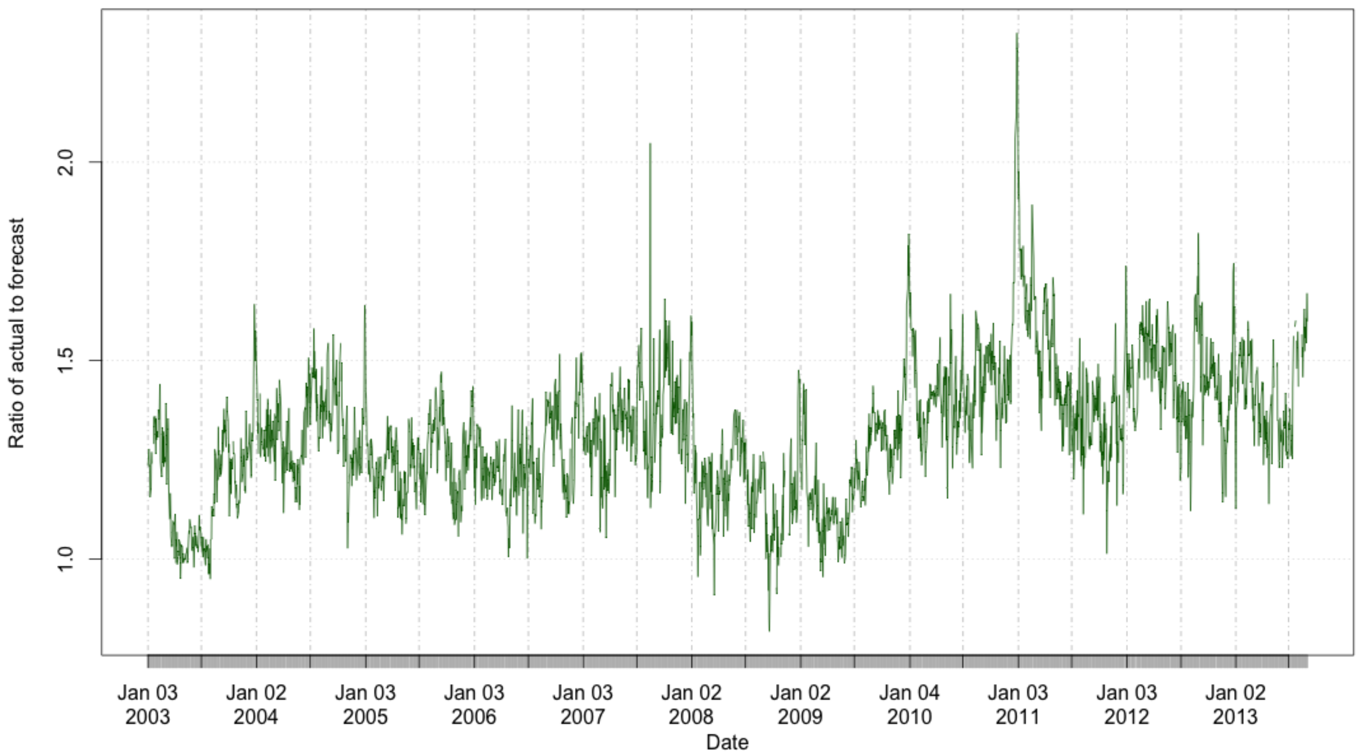


Figure 18: The ratio between 3-month actual variance swap quotes and 3-month forecasts.

The Lehman weekend

- Empirically, it seems that the variance curve is a simple scaling factor times the forecast, but that this scaling factor is time-varying.
  - We can think of this factor as having two multiplicative components: the overnight factor, and the price of volatility risk.
- Recall that as of the close on Friday September 12, 2008, it was widely believed that Lehman Brothers would be rescued over the weekend. By Monday morning, we knew that Lehman had failed.
- In Figure 19, we see that variance swap curves just before and just after the collapse of Lehman are just rescaled versions of the RFSV forecast curves.

We need variance swap estimates for 12-Sep-2008 and 15-Sep-2008

We proxy these by taking SVI fits for the two dates and computing the log-strips.

In [30]:
```
varSwaps12 <- c(
    0.2872021, 0.2754535, 0.2601864, 0.2544684, 0.2513854, 0.2515314,
    0.2508418, 0.2520099, 0.2502763, 0.2503309, 0.2580933, 0.2588361,
    0.2565093)

texp12 <- c(
    0.01916496, 0.04654346, 0.09582478, 0.19164956, 0.26830938, 0.298425
74,
    0.51745380, 0.54483231, 0.76659822, 0.79397673, 1.26488706, 1.763175
91,
    2.26146475)

varSwaps15 <-  c(
    0.4410505, 0.3485560, 0.3083603, 0.2944378, 0.2756881, 0.2747838,
    0.2682212, 0.2679770, 0.2668113, 0.2706713, 0.2729533, 0.2689598,
    0.2733176)

texp15 <- c(
    0.01095140, 0.03832991, 0.08761123, 0.18343600, 0.26009582, 0.290212
18,
    0.50924025, 0.53661875, 0.75838467, 0.78576318, 1.25667351, 1.754962
35,
    2.25325120)
```

Actual vs predicted over the Lehman weekend

In [31]:
```
nu <- OxfordH$nu.est[1]
h <- OxfordH$h.est[1]

# Variance curve fV model forecasts
tau1000 <- seq(0,2.5,length.out=1001)[-1]
vs1 <- varSwapForecast("2008-09-12",tau1000,nu=nu,h=h,tscale=252,onFacto
r=1.29)
vs2 <- varSwapForecast("2008-09-15",tau1000,nu=nu,h=h,tscale=252,onFacto
r=1.29)
```

In [32]:
```
# Plot results
plot(texp12,varSwaps12,type="b",col="red",ylim=c(0.2,0.45),xlab="Maturit
y",ylab="Variance swap quote",lwd=2)
lines(texp15,varSwaps15,type="b",col="blue",lwd=2)
lines(tau1000,vs1,col="red",type="l",lty=2,lwd=2)
lines(tau1000,vs2,col="blue",type="l",lty=2,lwd=2)
```

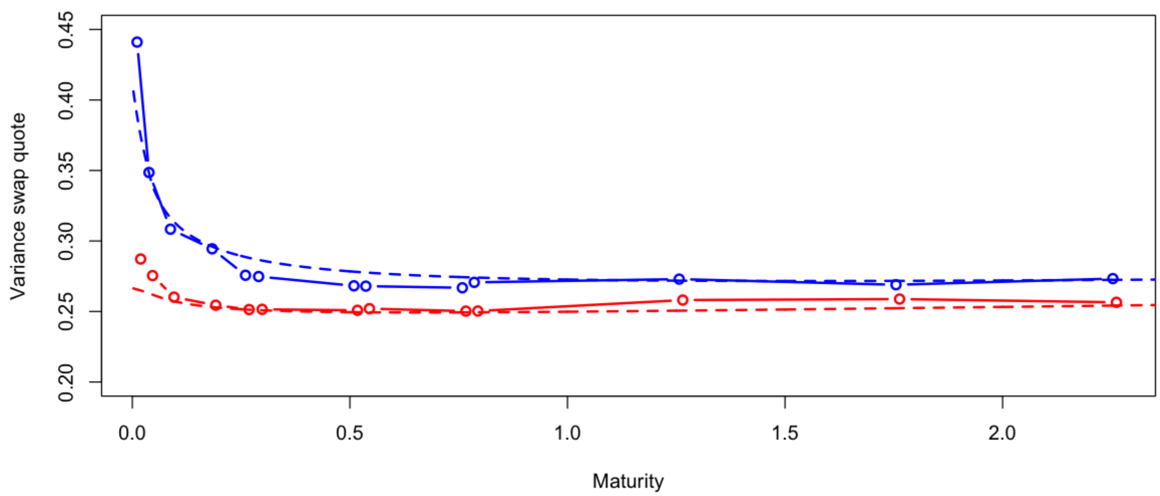


Figure 19: SPX variance swap curves as of September 12, 2008 (red) and September 15, 2008 (blue). The dashed curves are RFSV model forecasts rescaled by the 3-month ratio (1.29) as of the Friday close.

Remarks

We note that

- The actual variance swaps curves are very close to the forecast curves, up to a scaling factor.
- We are able to explain the change in the variance swap curve with only one extra observation: daily variance over the trading day on Monday 15-Sep-2008.
- The SPX options market appears to be backward-looking in a very sophisticated way.

The Flash Crash

- The so-called Flash Crash of Thursday May 6, 2010 caused intraday realized variance to be much higher than normal.
- In Figure 19, we plot the actual variance swap curves as of the Wednesday and Friday market closes together with forecast curves rescaled by the 3-month ratio as of the close on Wednesday May 5 (which was 2.52).
- We see that the actual variance curve as of the close on Friday is consistent with a forecast from the time series of realized variance that *includes* the anomalous price action of Thursday May 6.

Variance swap estimates

We again proxy variance swaps for 05-May-2010, 07-May-2010 and 10-May-2010 by taking SVI fits for the three dates and computing the log-strips.

In [27]:
```
varSwaps5 <- c(
    0.4250369, 0.2552473, 0.2492892, 0.2564899, 0.2612677, 0.2659618, 0.
2705928, 0.2761203,
    0.2828139, 0.2841165, 0.2884955, 0.2895839, 0.2927817, 0.2992602, 0.
3116500)

texp5 <- c(
    0.002737851, 0.043805613, 0.120465435, 0.150581793, 0.197125257, 0.2
92950034,
    0.369609856, 0.402464066, 0.618754278, 0.654346338, 0.867898700, 0.9
00752909,
    1.117043121, 1.615331964, 2.631074606)

varSwaps7 <- c(
    0.5469727, 0.4641713, 0.3963352, 0.3888213, 0.3762354, 0.3666858, 0.
3615814, 0.3627013,
    0.3563324, 0.3573946, 0.3495730, 0.3533829, 0.3521515, 0.3506186, 0.
3594066)

texp7 <- c(
    0.01642710, 0.03832991, 0.11498973, 0.14510609, 0.19164956, 0.287474
33, 0.36413415,
    0.39698836, 0.61327858, 0.64887064, 0.86242300, 0.89527721, 1.111567
42, 1.60985626,
    2.62559890)

varSwaps10 <- c(
    0.3718439, 0.3023223, 0.2844810, 0.2869835, 0.2886912, 0.2905637, 0.
2957070, 0.2960737,
    0.3005086, 0.3031188, 0.3058492, 0.3065815, 0.3072041, 0.3122905, 0.
3299425)

texp10 <- c(
    0.008213552, 0.030116359, 0.106776181, 0.136892539, 0.183436003, 0.2
79260780,
    0.355920602, 0.388774812, 0.605065024, 0.640657084, 0.854209446, 0.8
87063655,
    1.103353867, 1.601642710, 2.617385352)
```

In [28]:
```
# Variance curve fV model forecasts
vsf5 <- varSwapCurve("2010-05-05",bigT=2.5,nSteps=100,nu=nu,h=h,tscale=2
52,onFactor=2.52)
vsf7 <- varSwapCurve("2010-05-07",bigT=2.5,nSteps=100,nu=nu,h=h,tscale=2
52,onFactor=2.52)

plot(texp5,varSwaps5,type="b",col="red",xlab=expression(paste("Time to m
aturity ",tau)),ylab="Variance swap quote",lwd=2,ylim=c(0.2,.55))
lines(texp7,varSwaps7,type="b",col="green4",lwd=2)
legend("topright",inset=.02,c("May 5","May 7"),lty=1,col=c("red","green
4"))

lines(vsf5,col="red",type="l",lty=2,lwd=2)
lines(vsf7,col="green4",type="l",lty=2,lwd=2)
```

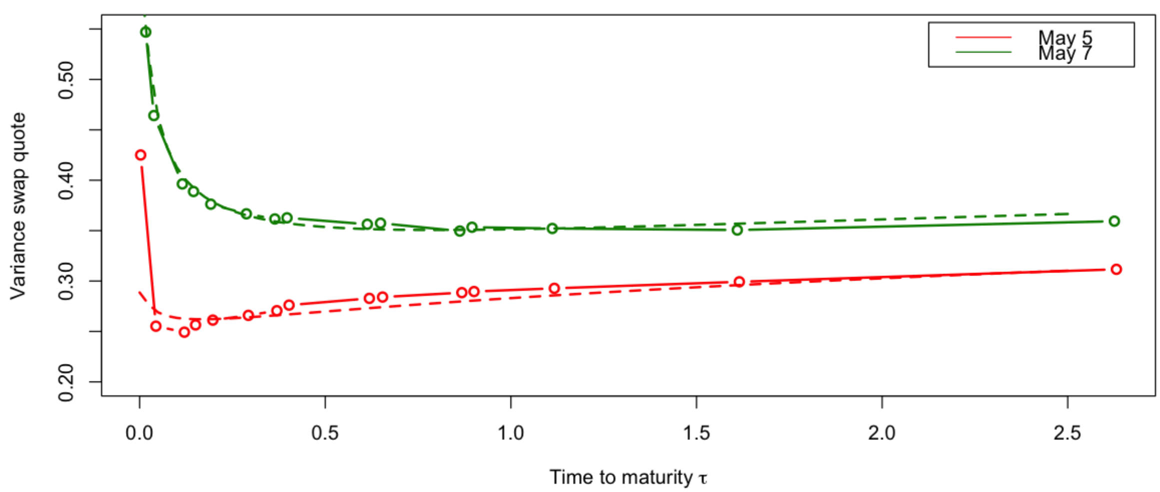


Figure 20: SPX variance swap curves as of May 5, 2010 (red) and May 7, 2010 (green). The dashed curves are RFSV model forecasts rescaled by the 3-month ratio (2.52) as of the close on Wednesday May 5. The curve as of the close on May 7 is consistent with the forecast **including** the crazy moves on May 6.

The weekend after the Flash Crash

Now we plot forecast and actual variance swap curves as of the close on Friday May 7 and Monday May 10.

In [29]:
```
# Variance curve fV model forecasts
vsf7 <- varSwapCurve("2010-05-07",bigT=2.5,nSteps=100,nu=nu,h=h,tscale=2
52,onFactor=2.52)
vsf10 <- varSwapCurve("2010-05-10",bigT=2.5,nSteps=100,nu=nu,h=h,tscale=
252,onFactor=2.52)
```

In [30]:
```
plot(texp7,varSwaps7,type="b",col="green4",xlab=expression(paste("Time t
o maturity ",tau)),ylab="Variance swap quote",lwd=2,ylim=c(0.2,.55))
lines(texp10,varSwaps10,type="b",col="orange",lwd=2)
legend("topright",inset=.02,c("May 7","May 10"),lty=1,col=c("green4","or
ange"))

lines(vsf7,col="green4",type="l",lty=2,lwd=2)
lines(vsf10,col="orange",type="l",lty=2,lwd=2)
```

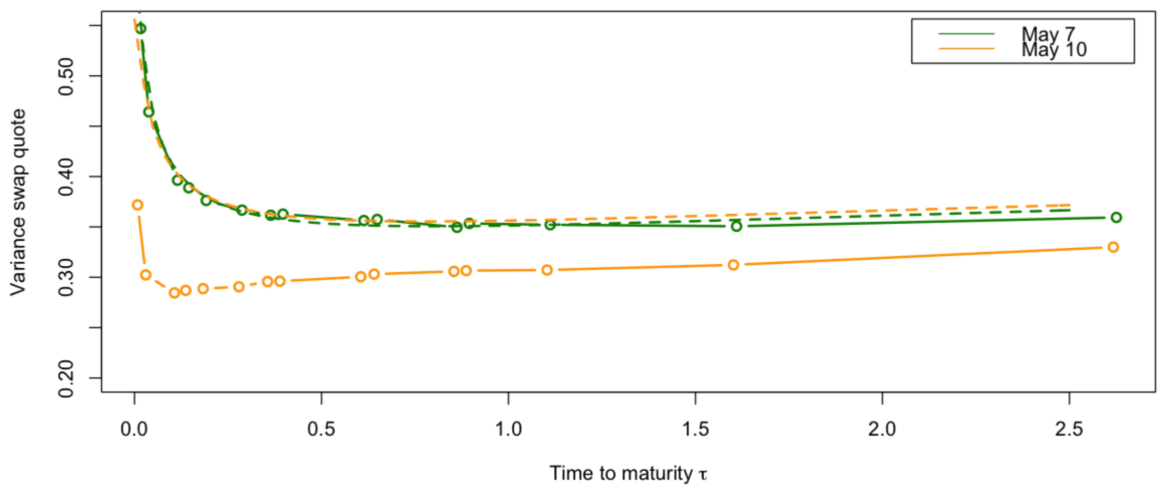


Figure 21: The May 10 actual curve is inconsistent with a forecast that includes the Flash Crash.

Now let's see what happens if we exclude the Flash Crash from the time series used to generate the variance curve forecast.

In [31]:
```
flash.day <- which(index(spx.rk)=="2010-05-06")
spx.rk.p <- spx.rk[-flash.day]
plot(spx.rk["2010-05-04::2010-05-10"],type="b")
lines(spx.rk.p["2010-05-04::2010-05-10"],col="red",type="b")
```

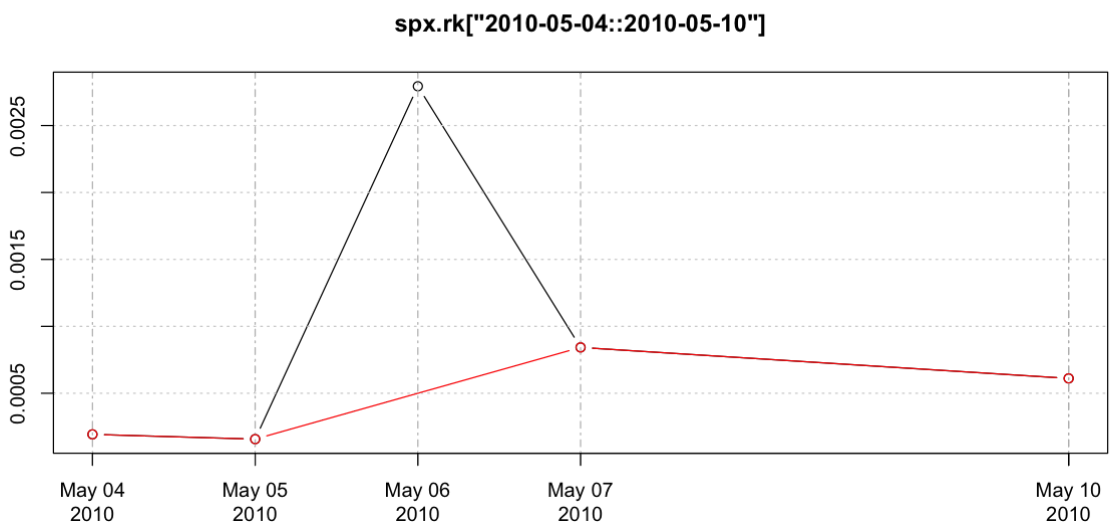


Figure 22: spx.rk.p has the May 6 realized variance datapoint eliminated. Notice the crazy realized variance estimate for May 6!

We need a new variance curve forecast function that uses the new time series.

In [32]:
```
xi1p <- function(date,nu,h,dt, tscale){ # dt=(u-t) is in units of years
  xi <- rv.forecast.XTS(spx.rk.p,h=h,date=date,nLags=500,delta=dt*tscale
,nu)
  return(xi)
}

# Forward variance curve (again the array tt should be in units of year
s)
xip <- function(date,tt,nu,h, tscale){sapply(tt,function(x){xi1p(date,nu
=nu,h=h,x,tscale)})}

varSwapCurve.p <- function(date,bigT,nSteps,nu,h,tscale,onFactor){
  # Make vector of fwd variances
  tt <- seq(0,bigT,length.out=(nSteps+1))
  dt <- tt[2]
  xicurve <- xip(date,tt,nu,h,tscale)
  xicurve.mid <- (xicurve[1:nSteps]+xicurve[2:(nSteps+1)])/2
  int.xicurve <- cumsum(xicurve.mid)*dt
  varcurve <- int.xicurve/tt[-1]
  varcurve <- c(xicurve[1], varcurve)*onFactor*tscale #onFactor is to co
mpensate for overnight moves
  res <- data.frame(tt,sqrt(varcurve))
  names(res) <- c("texp","vsQuote")
  return(res)
}

varSwapForecast.p <- function(date,tau,nu,h,tscale,onFactor){
  vsc <- varSwapCurve.p(date,bigT=2.5,nSteps=100,nu=nu,h=h,tscale,onFact
or) # Creates the whole curve
  x <- vsc$texp
  y <- vsc$vsQuote
  res <- stinterp(x,y,tau)$y
  return(res)
}

# Test the function
tau <- c(.25,.5,1,2)
date <- "2010-05-10"
varSwapForecast.p(date,tau,nu=nu,h=h,tscale=252,onFactor=1/(1-.35))
```

0.240523008484307 0.240994982045739 0.247513149936132 0.260165004204

Finally, we compare our new forecast curves with the actuals.

In [33]:
```
# Variance curve fV model forecasts
vsf7 <- varSwapCurve("2010-05-07",bigT=2.5,nSteps=100,nu=nu,h=h,tscale=2
52,onFactor=2.52)
vsf10p <- varSwapCurve.p("2010-05-10",bigT=2.5,nSteps=100,nu=nu,h=h,tsca
le=252,onFactor=2.52)
```

In [34]:
```
plot(texp7,varSwaps7,type="b",col="green4",xlab=expression(paste("Time t
o maturity ",tau)),ylab="Variance swap quote",lwd=2,ylim=c(0.2,.55))
lines(texp10,varSwaps10,type="b",col="orange",lwd=2)
legend("topright",inset=.02,c("May 7","May 10"),lty=1,col=c("green4","or
ange"))

lines(vsf7,col="green4",type="l",lty=2,lwd=2)
lines(vsf10p,col="orange",type="l",lty=2,lwd=2)
```

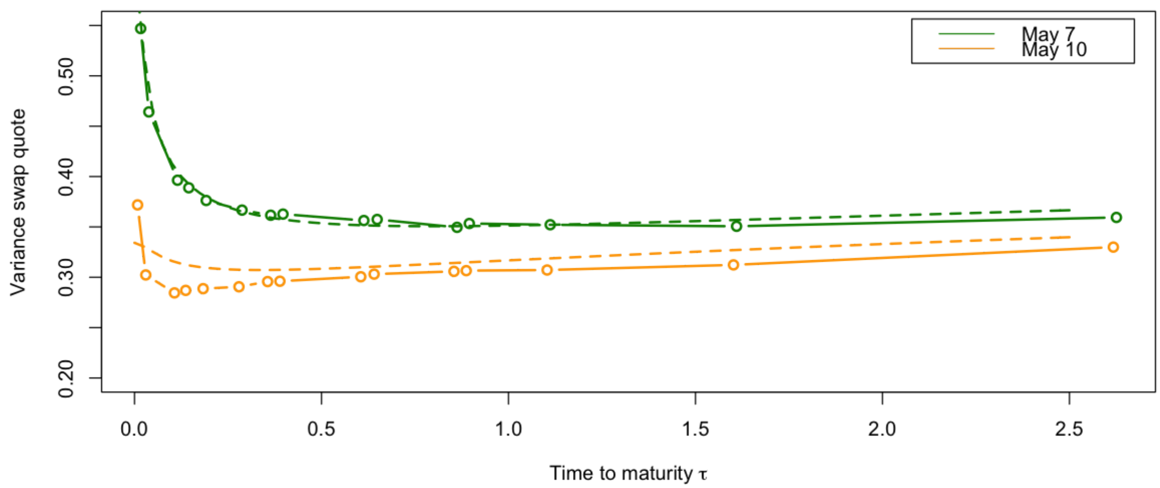


Figure 23: The May 10 actual curve is consistent with a forecast that excludes the Flash Crash.

Resetting of expectations over the weekend

- In Figures 21 and 23, we see that the actual variance swap curve on Monday, May 10 is consistent with a forecast that excludes the Flash Crash.
- Volatility traders realized that the Flash Crash should not influence future realized variance projections.

Summary

- We uncovered a remarkable monofractal scaling relationship in historical volatility.
  - A corollary is that volatility is not a long memory process, as widely believed.
- This leads to a natural non-Markovian stochastic volatility model under $\mathbb{P}$.
- The simplest specification of $\frac{d\mathbb{Q}}{d\mathbb{P}}$ gives a non-Markovian generalization of the Bergomi model.

- The history of the Brownian motion $\lbrace W_s, s
- This model fits the observed volatility surface surprisingly well with very few parameters.
- For perhaps the first time, we have a simple consistent model of historical and implied volatility.

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