

# MTH 9875 The Volatility Surface: Fall 2017

## Lecture 9: The SVI volatility surface parameterization

Jim Gatheral  
Department of Mathematics



### Outline of lecture 9

- No-arbitrage constraints on the tail behavior of implied volatility.
- The SVI parameterization of the volatility smile and its variants.
  - Sufficient conditions for no calendar-spread arbitrage.
  - Necessary and sufficient conditions for no calendar-spread arbitrage.
  - Arbitrage on a slice.
- Generating an implied volatility surface from raw index option data.
- Visualization:
  - Total variance plot.
  - 3D plot.
- SSVI and eSSVI
- Fitting SVI subject to no-arbitrage constraints.

### Roger Lee's Moment Formula

- [Roger Lee]<sup>[1]</sup> shows that implied variance is bounded above by a function linear in the log-strike  $k = \log(K/F)$  as  $|k| \rightarrow \infty$ .
  - The maximum slope of total implied variance  $w(k, T) = \sigma_{BS}^2(k, T) T$  is 2.
- He shows how to relate the gradients of the wings of the upper bound of the implied variance skew to the maximal finite moments of the underlying process.
- Lee's derivation assumes only the existence of a martingale measure: it makes no assumptions on the distribution of underlying returns. His result is completely model-independent.

## The Roger Lee Moment Formula

### Roger Lee's Lemma 3.32

There exists  $k^* > 0$  such that for all  $k > k^*$ ,

$$\sigma_{BS}^2(k) < 2 \frac{|k|}{T}.$$

### Proof.

We only need to show that  $C_{BS}(k, \sigma_{BS}(k) \sqrt{T}) < C_{BS}(k, \sqrt{2|k|})$  whenever  $k > k^*$ .

On the LHS, we have

$$\lim_{k \rightarrow \infty} C_{BS}(k, \sigma_{BS}(k) \sqrt{T}) = 0$$

and on the RHS, we have

$$\begin{aligned} \lim_{k \rightarrow \infty} C_{BS}(k, \sqrt{2|k|}) &= \lim_{k \rightarrow \infty} F\left\{N(d_1) - e^k N(d_2)\right\} \\ &= \lim_{k \rightarrow \infty} F\left\{N(0) - e^k N(-\sqrt{2|k|})\right\} = \frac{F}{2} \end{aligned}$$

## The left wing

Specifically, let  $q^* := \sup \{q : \mathbb{E} S_T^{-q} < \infty\}$  and

$$\beta^* := \limsup_{k \rightarrow -\infty} \frac{\sigma_{BS}^2(k, T) T}{|k|}$$

Then  $\beta^* \in [0, 2]$ ,

$$q^* = \frac{1}{2} \left( \frac{1}{\sqrt{\beta^*}} - \frac{\sqrt{\beta^*}}{2} \right)^2$$

and inverting this, we obtain  $\beta^* = g(q^*)$  with

$$g(x) = 2 - 4 \left[ \sqrt{x^2 + x} - x \right]$$

## The right wing

Similarly, let  $p^* := \sup \{p : \mathbb{E} S_T^{1+p} < \infty\}$  and

$$\alpha^* := \limsup_{k \rightarrow +\infty} \frac{\sigma_{BS}^2(k, T) T}{|k|}$$

Then  $\alpha^* \in [0, 2]$ ,

$$p^* = \frac{1}{2} \left( \frac{1}{\sqrt{\alpha^*}} - \frac{\sqrt{\alpha^*}}{2} \right)^2$$

and as for the left wing, it follows that  $\alpha^* = g(p^*)$ .

## Benaim and Friz

- With  $F(x)$  the CDF, [Benaim and Friz]<sup>[1]</sup> go on to show that Roger Lee's upper bound ( $\limsup$ ) may be replaced by a limit in most practical cases.
- Then we may write for the right tail

(1)

$$\frac{\sigma_{BS}(k, T)^2 T}{k} \sim g \left( -1 - \frac{\log [1 - F(k)]}{k} \right) \text{ as } k \rightarrow \infty$$

- and for the left tail

(2)

$$\frac{\sigma_{BS}(-k, T)^2 T}{k} \sim g \left( \frac{-\log F(-k)}{k} \right) \text{ as } k \rightarrow \infty$$

- By substituting the tail-behavior of  $F$  into equations (1) and (2), we can deduce the full tail behavior of the smile, not just Roger Lee's upper bound.

## Example: Black-Scholes

$$\begin{aligned} 1 - F(k) &= \int_k^\infty \frac{1}{\sqrt{2\pi\sigma^2 T}} e^{-y^2/(2\sigma^2 T)} dy \\ &\sim \frac{1}{\sqrt{2\pi}} \frac{e^{-k^2/(2\sigma^2 T)}}{k} \text{ as } k \rightarrow \infty \end{aligned}$$

Then

$$\log [1 - F(k)] \sim -\frac{k^2}{2\sigma^2 T} \text{ as } k \rightarrow \infty$$

and

$$\frac{\sigma_{BS}(k, T)^2 T}{k} \sim g \left( -1 + \frac{k}{2\sigma^2 T} \right) \sim \frac{2\sigma^2 T}{2k} \text{ as } k \rightarrow \infty$$

It follows that

$$\sigma_{BS}(k, T)^2 \sim \sigma^2 \text{ as } k \rightarrow \infty$$

## Implications of the moment formula

- Implied variance is linear in  $k$  as  $k \rightarrow \infty$  for stochastic volatility models.

- So, if we want a parametrization of the implied variance surface consistent with stochastic volatility, it needs to be linear in the wings!
  - and it needs to be curved in the middle - many conventional parameterizations of the volatility surface are quadratic for example.

## History of SVI

- SVI was originally devised at Merrill Lynch in 1999 and subsequently written up in detail in [Gatheral and Jacquier]<sup>[5]</sup>.
- SVI has two key properties that have led to its subsequent popularity with practitioners:
  - For a fixed time to expiry  $t$ , the implied Black-Scholes variance  $\sigma_{BS}^2(k, t)$  is linear in the log-strike  $k$  as  $|k| \rightarrow \infty$  consistent with Roger Lee's moment formula [Roger Lee]<sup>[7]</sup>.
  - It is relatively easy to fit listed option prices whilst ensuring no calendar spread arbitrage.
- As shown in [Gatheral and Jacquier]<sup>[4]</sup>, the SVI parameterization is not arbitrary in the sense that the large-maturity limit of the Heston implied volatility smile is exactly SVI.

## Characterisation of static arbitrage

### Definition 2.1

A volatility surface is free of static arbitrage if and only if the following conditions are satisfied: (i) it is free of calendar spread arbitrage; (ii) each time slice is free of butterfly arbitrage.

## Calendar spread arbitrage

### Lemma 2.2

If dividends are proportional to the stock price, the volatility surface  $w$  is free of calendar spread arbitrage if and only if  $\partial_t w(k, t) \geq 0$ , for all  $k \in \mathbb{R}$  and  $t > 0$ .

- Thus there is no calendar spread arbitrage if there are no crossed lines on a total variance plot.

### Proof of Lemma 2.2

- For any martingale  $X_t$  and  $t_2 \geq t_1$ ,

$$\mathbb{E} [(X_{t_2} - L)^+] \geq \mathbb{E} [(X_{t_1} - L)^+]$$

- Define  $F_{0,t} := \mathbb{E}[S_t | \mathcal{F}_0]$ .
- Let  $C_i$  be options with strikes  $K_i$  and expirations  $t_i$  with  $t_2 > t_1$ . Suppose the two options have the same moneyness so that

$$\frac{K_1}{F_{0,t_1}} = \frac{K_2}{F_{0,t_2}} =: e^k$$

- Then, if dividends are proportional,  $X_t := S_t/F_{0,t}$  is a martingale and

$$\frac{C_2}{K_2} = e^{-k} \mathbb{E} [(X_{t_2} - e^k)^+] \geq e^{-k} \mathbb{E} [(X_{t_1} - e^k)^+] = \frac{C_1}{K_1}.$$

- So, if dividends are proportional, keeping moneyness constant, option prices are non-decreasing in time to expiration.

- Let  $w_t := \sigma_{BS}^2(k, T) T$ .
- The Black-Scholes formula for the non-discounted value of an option may be expressed in the form  $C_{BS}(k, w_t)$  with  $C_{BS}(\cdot)$  strictly increasing in the total implied variance  $w_t$ .
- It follows that for fixed  $k$ , we must have the total implied variance  $w_t$  non-decreasing with respect to time to expiration:

## No calendar spread arbitrage condition

$$\partial_t w(k; \chi_t) \geq 0$$

where  $\chi_t$  denotes the  $t$ -dependent parameters of our fit.

## Butterfly arbitrage

### Definition 2.3

A slice is said to be free of butterfly arbitrage if the corresponding density is non-negative.

Now introduce the function  $g : \mathbb{R} \rightarrow \mathbb{R}$  defined by

$$g(k) := \left(1 - \frac{kw'(k)}{2w(k)}\right)^2 - \frac{w'(k)^2}{4} \left(\frac{1}{w(k)} + \frac{1}{4}\right) + \frac{w''(k)}{2}.$$

**Lemma 2.4**

A slice is free of butterfly arbitrage if and only if  $g(k) \geq 0$  for all  $k \in \mathbb{R}$  and  
 $\lim_{k \rightarrow +\infty} d_+(k) = -\infty$ .

**The raw SVI parameterization**

For a given parameter set  $\chi_R = \{a, b, \rho, m, \sigma\}$ , the *raw SVI parameterization* of total implied variance reads:

**Raw SVI parameterization**

$$w(k; \chi_R) = a + b \left\{ \rho(k - m) + \sqrt{(k - m)^2 + \sigma^2} \right\}$$

where  $a \in \mathbb{R}$ ,  $b \geq 0$ ,  $|\rho| < 1$ ,  $m \in \mathbb{R}$ ,  $\sigma > 0$ , and the obvious condition  $a + b \sigma \sqrt{1 - \rho^2} \geq 0$ , which ensures that  $w(k, \chi_R) \geq 0$  for all  $k \in \mathbb{R}$ . This condition ensures that the minimum of the function  $w(\cdot, \chi_R)$  is non-negative.

**Download some code**

```
In [1]: download.file(url="http://mfe.baruch.cuny.edu/wp-content/uploads/2016/10/987!
unzip(zipfile="9875-9.zip")

library(stinepack)
library(repr)
```

```
In [2]: source("BlackScholes.R")
source("plotSVI.R")
source("sviJW.R")
source("plotIvols.R")
source("sviarbitrage.R")
source("optionMetricsToIvols.R")
source("sviRoots.R")
source("sviFit0.R")
source("sviSqrtFit.R")
source("computeImpliedVols.R")
source("sviFitQuarticRoots.R")
source("sviVolSurface.R")
source("svi.R")
```

**An SVI example**

Here's a specific choice of parameters:

```
In [3]: sviparams <- list(a = 0.04, b = 0.4, sig = 0.1, rho = - 0.4, m = 0.1);
```

And here's how the SVI smile looks with these parameters:

```
In [4]: options(repr.plot.height=5)
curve(svi(sviparams,x),from=-1,to=1,col="red",xlab="Log-strike k",ylab=expres
```

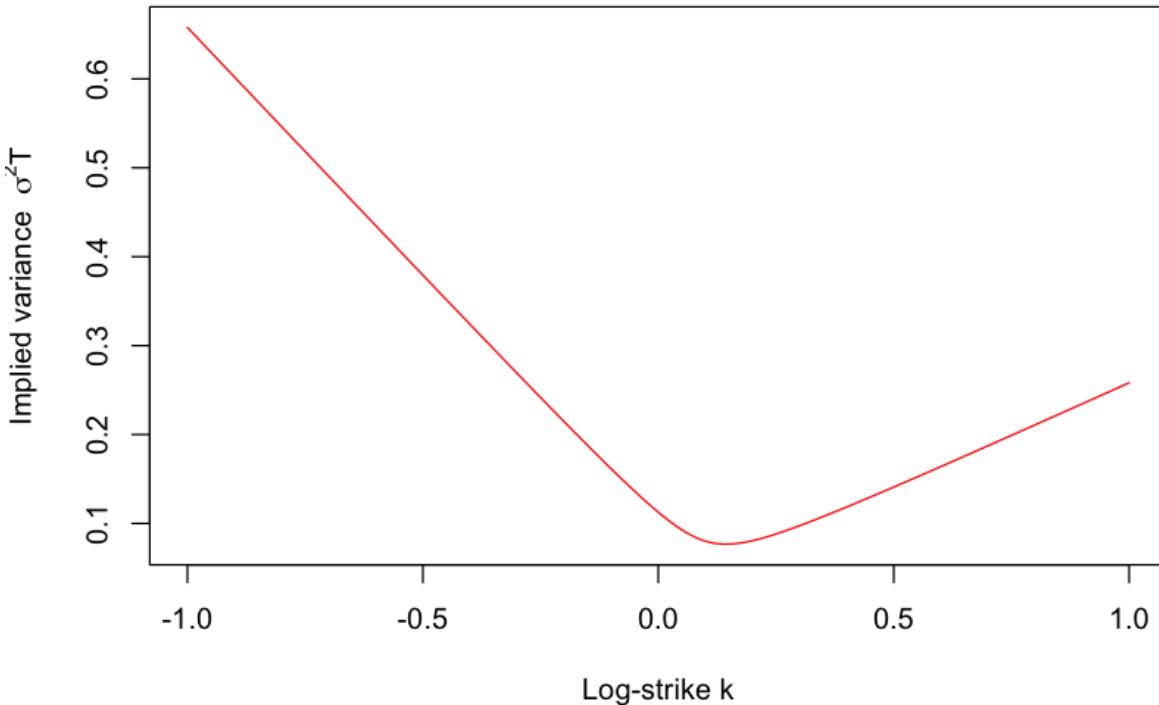


Figure 1: The SVI smile with  $a = 0.04$ ,  $b = 0.4$ ,  $\sigma = 0.1$ ,  $\rho = -0.4$ ,  $m = 0$

## Meaning of raw SVI parameters

Changes in the parameters have the following effects:

- Increasing  $a$  increases the general level of variance, a vertical translation of the smile;
- Increasing  $b$  increases the slopes of both the put and call wings, tightening the smile;
- Increasing  $\rho$  decreases (increases) the slope of the left(right) wing, a counter-clockwise rotation of the smile;
- Increasing  $m$  translates the smile to the right;
- Increasing  $\sigma$  reduces the at-the-money (ATM) curvature of the smile.

We can convince ourselves that if the wings are linear in  $k$ , we need 5 and only 5 parameters to

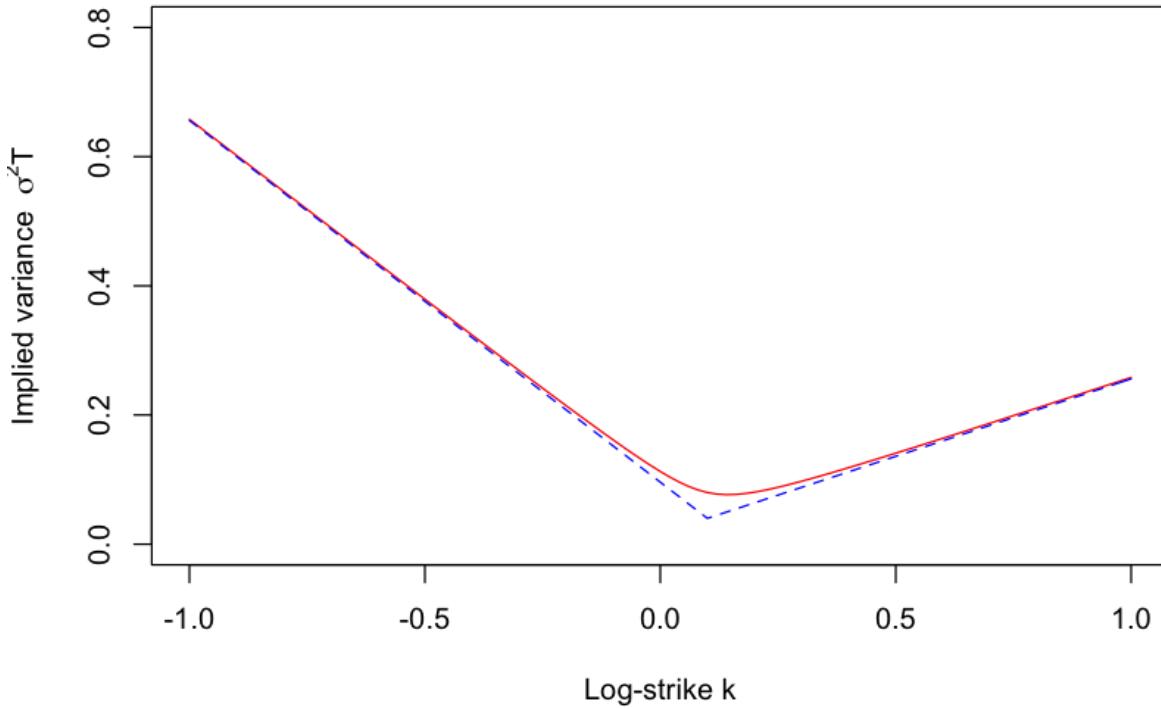
cover all reasonable transformations of the volatility smile.

### Demonstrate the effect of changing raw SVI parameters:

```
In [5]: sviparams <- list(a = 0.04, b = 0.4, sig = 0.1, rho = - 0.4, m = 0.1)
sviparams2 <- sviparams
```

```
In [6]: sviparams2$sig <- 0

curve(svi(sviparams,x),from=-1,to=1,col="red",
      xlab="Log-strike k",ylab=expression(paste("Implied variance ",sigma^2))
curve(svi(sviparams2,x),from=-1,to=1,col="blue",add=T, lty=2)
```



### The natural SVI parameterization

For a given parameter set  $\chi_N = \{\Delta, \mu, \rho, \omega, \zeta\}$ , the *natural SVI parameterization* of total implied variance reads:

#### Natural SVI parameterization

$$w(k; \chi_N) = \Delta + \frac{\omega}{2} \left\{ 1 + \zeta \rho (k - \mu) + \sqrt{(\zeta(k - \mu) + \rho)^2 + (1 - \rho^2)} \right\}$$

where  $\omega \geq 0$ ,  $\Delta \in \mathbb{R}$ ,  $\mu \in \mathbb{R}$ ,  $|\rho| < 1$  and  $\zeta > 0$ .

- This parameterization is a natural generalization of the time  $\infty$  Heston smile explored in [Gatheral and Jacquier]<sup>[4]</sup>.

## The SVI Jump-Wings (SVI-JW) parameterization

- Neither the raw SVI nor the natural SVI parameterizations are intuitive to traders.
- There is no reason to expect these parameters to be particularly stable.
- The *SVI-Jump-Wings (SVI-JW) parameterization* of the implied variance  $v$  (rather than the implied total variance  $w$ ) was inspired by a similar parameterization attributed to Tim Klassen, then at Goldman Sachs.

## An aside on Tim Klassen and Vola Dynamics

- Tim Klassen has continued to work on arbitrage-free parameterization of the volatility surface.
- Though the SVI parameterization used to fit various observed implied volatility surfaces remarkably well, it does not fit so well any more.
  - I am told that fits in other markets such as Crude Oil continue to be very good.
- His new startup Vola Dynamics (<https://www.voladynamics.com/>) offers clients the ability to fit much more highly parameterized arbitrage-free volatility surfaces to both index and single-stock implied volatilities

## SVI-JW

For a given time to expiry  $t > 0$ , and an SVI total variance smile  $w(k)$ , the SVI-JW parameters are defined as follows:

### SVI-JW parameterization

$$\begin{aligned}
 v_t &= w(0)/t, \\
 \psi_t &= \left. \frac{\partial \sqrt{w(k)}}{\partial k} \right|_{k=0} = \sqrt{t} \left. \frac{\partial \sigma_{BS}(k, t)}{\partial k} \right|_{k=0}, \\
 p_t &= \lim_{k \rightarrow -\infty} -\frac{1}{\sqrt{w(0)}} \frac{\partial w(k)}{\partial k}, \\
 c_t &= \lim_{k \rightarrow +\infty} \frac{1}{\sqrt{w(0)}} \frac{\partial w(k)}{\partial k}, \\
 \tilde{v}_t &= \min_{k \in \mathbb{R}} w(k)/t.
 \end{aligned}$$

## Interpretation of SVI-JW parameters

The SVI-JW parameters have the following interpretations:

- $\nu_t$  gives the ATM variance;
- $\psi_t$  gives the ATM skew;
- $p_t$  gives the slope of the left (put) wing;
- $c_t$  gives the slope of the right (call) wing;
- $\tilde{\nu}_t$  is the minimum implied variance.

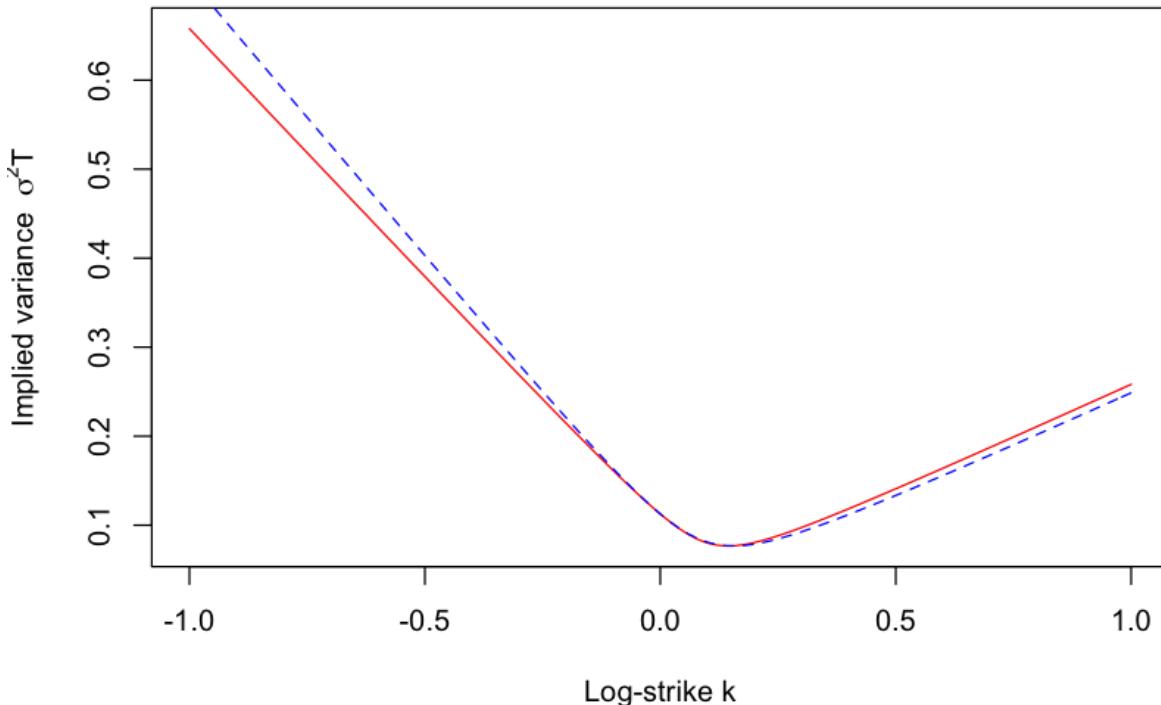
## Demonstrate the effect of changing SVI-JW parameters:

```
In [7]: (sviJWparams <- sviToJw(sviparams,1))
```

vt	psit	pt	ct	varmint	texp
0.1125685	-0.6599499	1.669089	0.7153239	0.07666061	1

```
In [8]: sviJWparams2 <- sviJWparams
sviJWparams2$pt <- sviJWparams$pt + 0.2
sviparams2 <- jwToSvi(sviJWparams2)
```

```
curve(svi(sviparams,x),from=-1,to=1,col="red",xlab="Log-strike k",ylab=expression("Implied variance " ~ \sigma^2 ~ T))
curve(svi(sviparams2,x),from=-1,to=1,col="blue",add=T, lty=2)
```



## Scaling of SVI Jump-Wings parameters with volatility

Note that, as defined here,

$$\psi_t = \left. \frac{\partial \sigma_{BS}(k)}{\partial k} \right|_{k=0}$$

The choice of volatility skew as the skew measure rather than variance skew for example, reflects the empirical observation that volatility is roughly lognormally distributed.

Specifically, we showed in our Asymptotics lecture and in Chapter 7 of *The Volatility Surface* that if the SDE for variance is of the form:

$$dv = -\lambda(v - \bar{v})dt + \eta\sqrt{v}\beta(v)dZ$$

we should have

$$\frac{\partial}{\partial k} \sigma_{BS}(k, T)^2 \approx \frac{\rho\eta\beta(v)}{\lambda' T} \left\{ 1 - \frac{(1 - e^{-\lambda' T})}{\lambda' T} \right\} \propto \beta(v)$$

with  $\lambda' = \lambda - \frac{1}{2}\rho\eta\beta(v)$ .

Thus

$$\left. \frac{\partial \sigma_{BS}(k)}{\partial k} \right|_{k=0} \approx \text{const.}$$

independent of volatility implies that

$$\beta(v) \sim \sqrt{v}$$

and therefore that the variance (volatility) process is lognormal.

This consistency of the SVI-JW parameterization with empirical volatility dynamics leads to greater parameter stability over time.

## Scaling of SVI Jump-Wings parameters with time to expiration

- If smiles scaled perfectly as  $1/\sqrt{w_t}$  (effectively  $1/\sqrt{t}$  in practice), SVI-JW parameters would be constant, independent of the slice  $t$ .
  - This makes it easy to extrapolate the SVI surface to expirations beyond the longest expiration in the data set.
- Since both scaling features are roughly consistent with empirical observation, we expect (and see) greater parameter stability over time.
  - Traders can keep parameters in their heads.

## SVI slices may cross at no more than four points

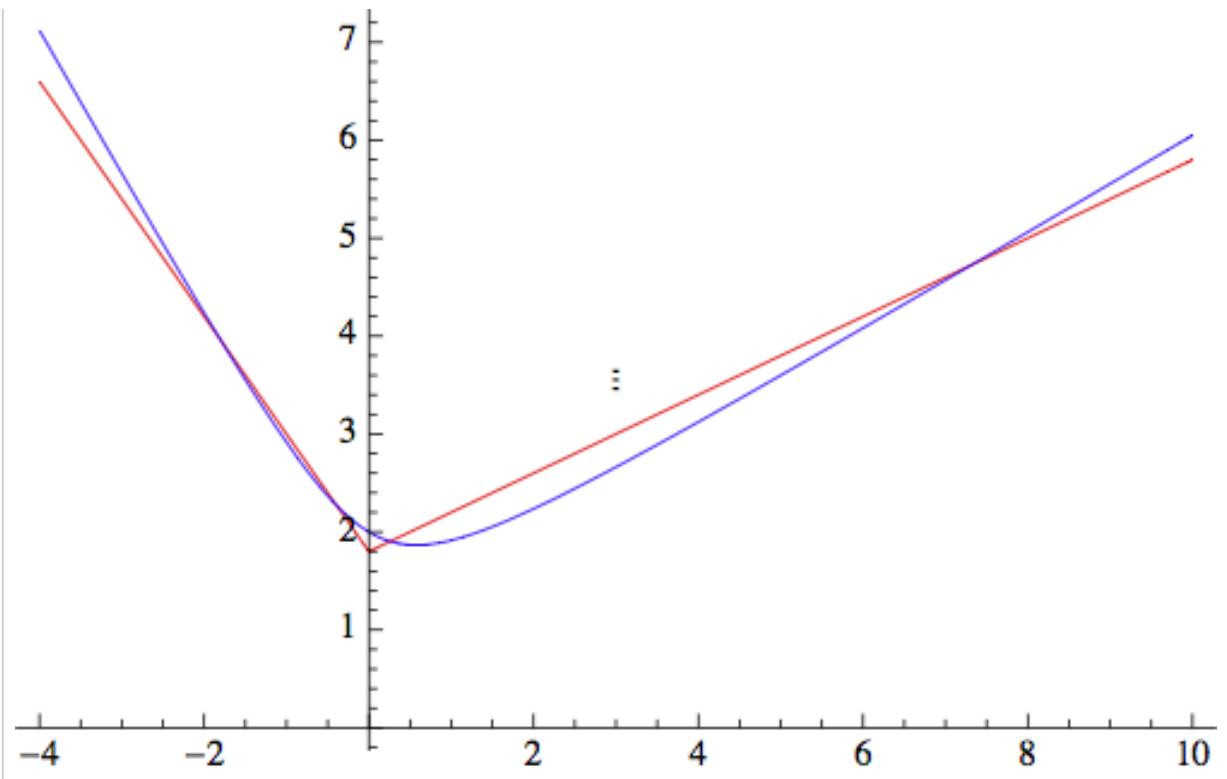


Figure 2: Lines cross at four points here.

### Condition for no calendar spread arbitrage

**Lemma 3.1**

Two raw SVI slices admit no calendar spread arbitrage if a certain quartic polynomial has no real root.

### Ferrari Cardano

The idea is as follows:

- Two total variance slices cross if

$$\begin{aligned} & a_1 + b_1 \left\{ \rho_1 (k - m_1) + \sqrt{(k - m_1)^2 + \sigma_1^2} \right\} \\ &= a_2 + b_2 \left\{ \rho_2 (k - m_2) + \sqrt{(k - m_2)^2 + \sigma_2^2} \right\} \end{aligned}$$

- Rearranging and squaring gives a quartic polynomial equation of the form

$$\alpha_4 k^4 + \alpha_3 k^3 + \alpha_2 k^2 + \alpha_1 k + \alpha_0 = 0,$$

where each of the coefficients are lengthy yet explicit expressions in terms of the raw SVI parameters.

- If this quartic polynomial has no real root, then the slices do not intersect.
- Roots of a quartic polynomial are known in closed-form thanks to [Cardano]<sup>[2]</sup>.

## SVI butterfly arbitrage

Recall the definition:

$$g(k) := \left(1 - \frac{kw'(k)}{2w(k)}\right)^2 - \frac{w'(k)^2}{4} \left(\frac{1}{w(k)} + \frac{1}{4}\right) + \frac{w''(k)}{2}.$$

- The highly nonlinear behavior of  $g$  makes it seemingly impossible to find general conditions on the parameters that would eliminate butterfly arbitrage.
- We now provide an example where the no-butterfly-arbitrage condition is violated.

```
In [9]: # Butterfly g(.) function
g <- function(sviparams,k){

  a <- sviparams$a;
  b <- sviparams$b;
  sig <- sviparams$sig;
  rho <- sviparams$rho;
  m <- sviparams$m;

  discr <- sqrt((k-m)*(k-m) + sig*sig);
  w <- a + b *(rho*(k-m)+ discr);
  dw <- b*rho + b *(k-m)/discr;
  d2w <- b*sig^2/(discr*discr*discr);

  return(1 - k*dw/w + dw*dw/4*(-1/w+k*k/(w*w)-4) +d2w/2);
}
```

## Axel Vogt post on Wilmott.com



**AVt**  
Senior Member

Posts: 971  
Joined: Dec 2001

Thu Apr 06, 06 08:37 PM

It works for observables and far beyond for extrapolation.

But for a (theoretical) experiment try the following data

a = -.40998372001772e-1,  
 b = .13308181151379,  
 m = .35858898335748,  
 rho = .30602086142471,  
 sigma = .41531878803777

## The Vogt smile

```
In [10]: # Axel Vogt butterfly arbitrage example
```

```
a <- -0.0410
b <- 0.1331
m <- 0.3586
rho <- 0.3060
sig <- 0.4153

vogtParams <- as.data.frame(list(a=a,b=b,m=m,rho=rho,sig=sig))
print(vogtJW <- sviToJw(vogtParams,1)) # SVI-JW parameters
```

	vt	psit	pt	ct	varmint	texp
1	0.01742625	-0.1752111	0.6997381	1.316798	0.0116249	1

Now let's take a look at the smile and graph the function  $g(k)$ :

```
In [11]: # First the Vogt smile
```

```
r <- 1.5
curve(svi(vogtParams,x),from=-r,to=r,col="blue",lwd=2,
      ylab=expression(w=sigma^2*T),xlab="Log-strike k")
```

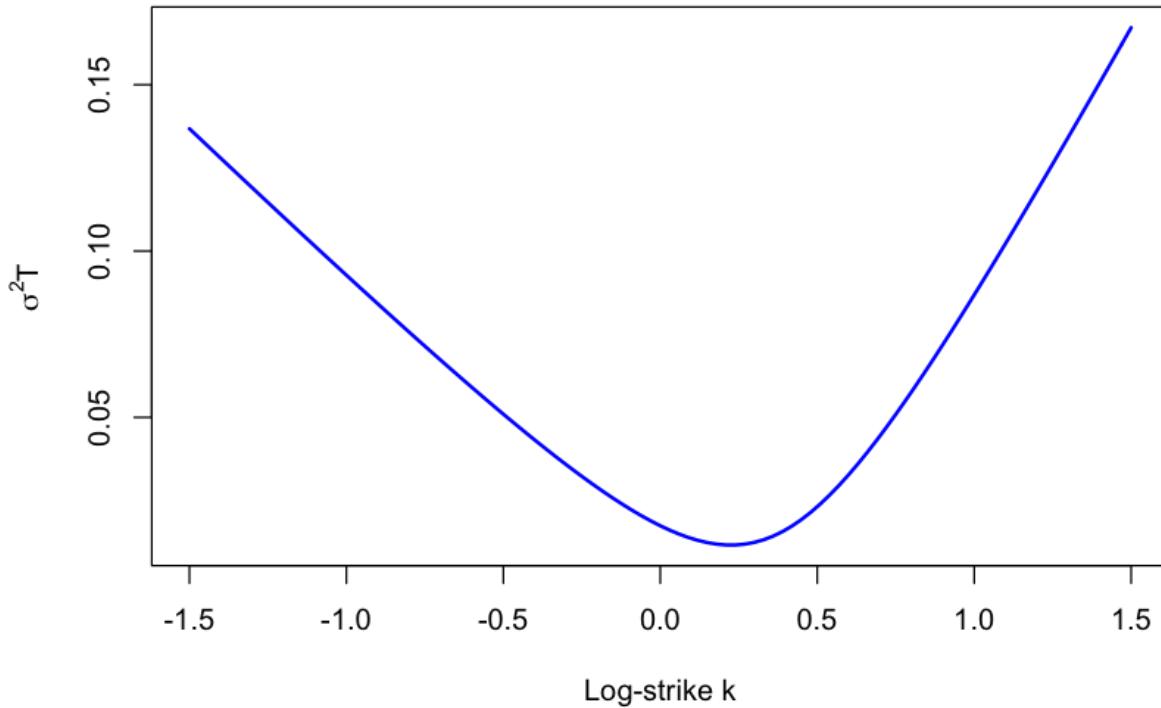


Figure 3: Plot of the total variance smile  $w$  using Axel Vogt's parameters.

```
In [12]: # Then the graph of g(.)
curve(g(vogtParams,x),from=-r,to=r,col="blue",n=500,lwd=2,ylab=expression(g[])
lines(c(-r,r),c(0,0),lty=2)
```

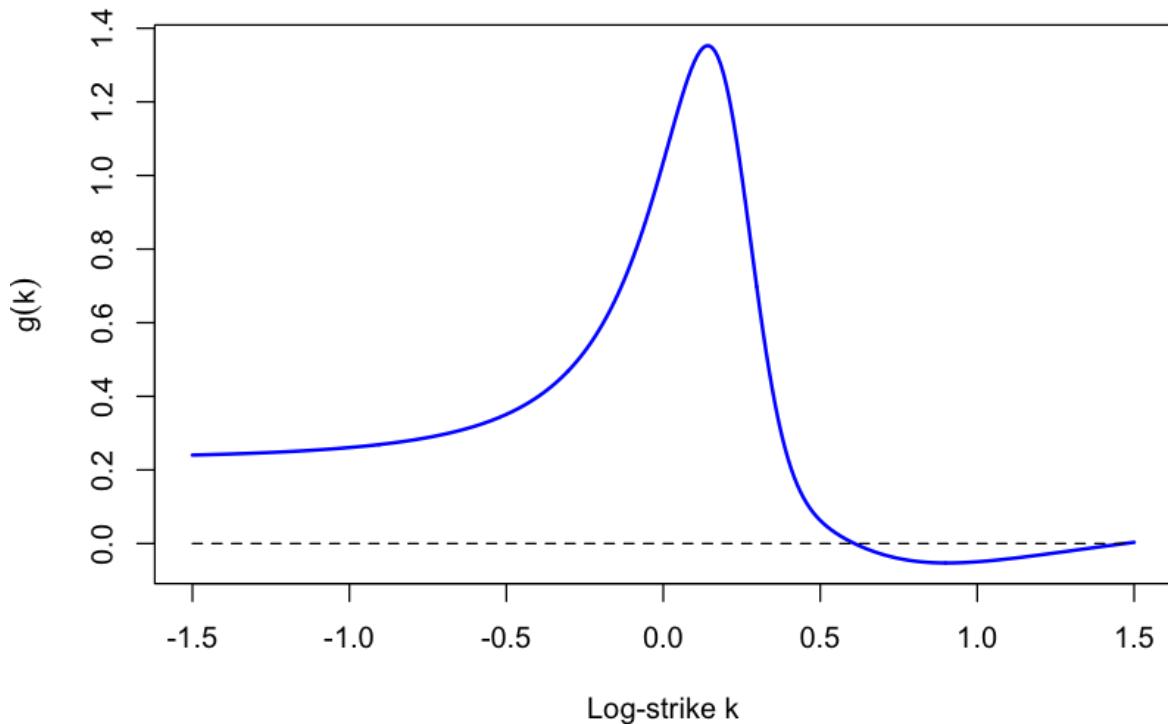


Figure 4: Plots of the function  $g$ , using Axel Vogt's parameters

- We see that the function  $g(\cdot)$  and so the density goes negative with the Vogt parameters.

We knew already that there could be arbitrage on a slice with SVI and Axel Vogt has given us an embarrassing example. In fact, we can generate arbitrage even with  $\rho = 0$ . Let's redo the plots:

```
In [13]: vogtParams0 <- as.data.frame(list(a=a,b=b,m=m,rho=0,sig=sig))

# Plot Vogt smile (smileSVI)
r <- 1.5
curve(svi(vogtParams0,x),from=-r,to=r,col="blue",lwd=2,
      ylab=expression(w=sigma^2*T),xlab="Log-strike k")
```

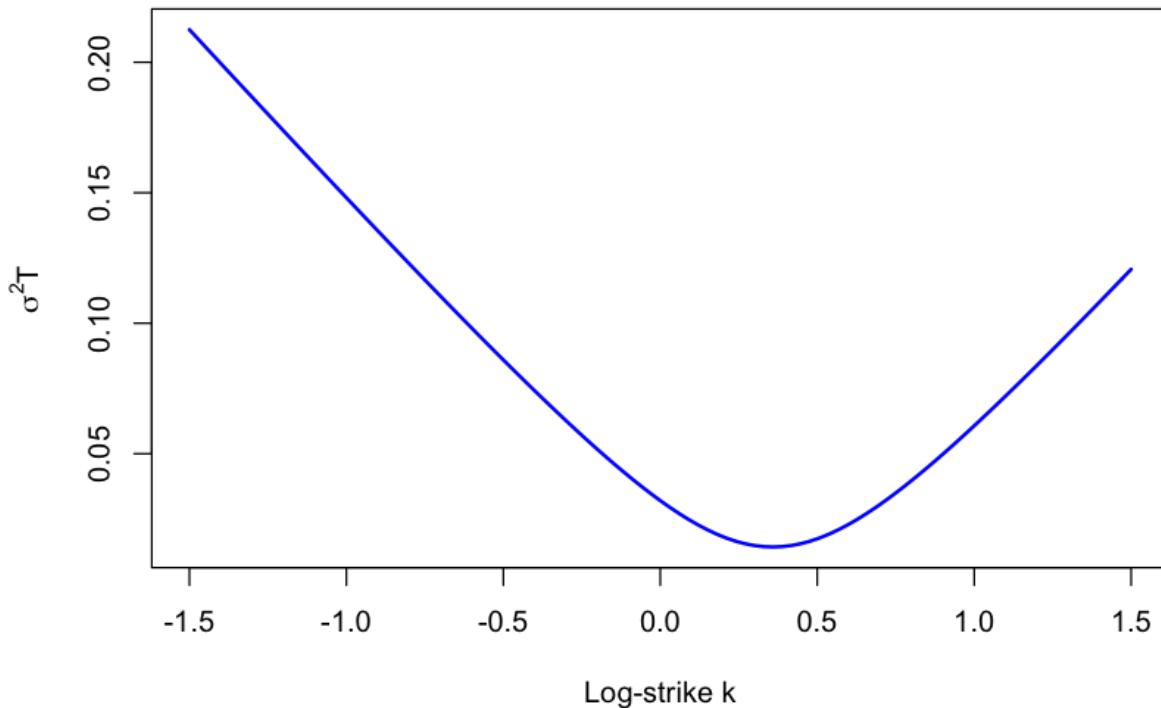


Figure 5: Plots of the total variance smile  $w(k, T)$  using Axel Vogt's parameters, except with  $\rho = 0$ .

```
In [14]: # Plot graph of g (ggraphSVI)
curve(g(vogtParams0,x),from=-r,to=r,col="blue",n=500,lwd=2,ylab=expression(g
lines(c(-r,r),c(0,0),lty=2)
```

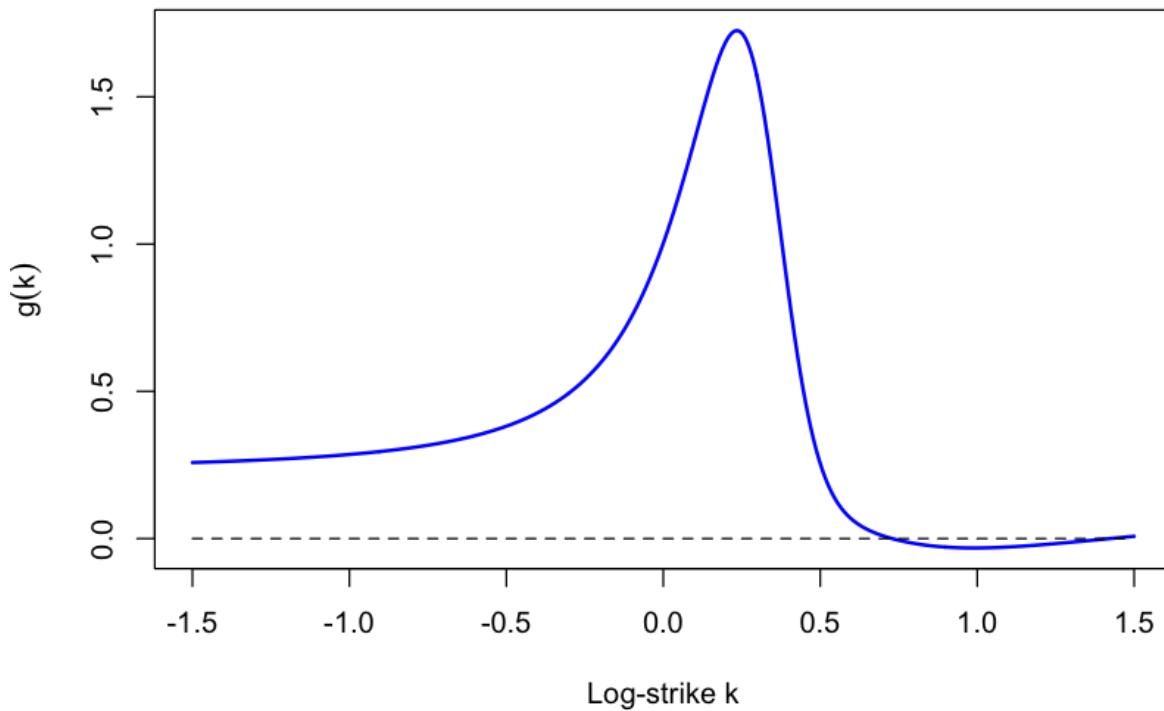


Figure 6: Plots of the function  $g(k)$  (bottom), using Axel Vogt's parameters, but with  $\rho = 0$ .

Until 2012 or so, I thought there was no simple way to guarantee the absence of butterfly arbitrage.

## SSVI (or Surface SVI)

Consider now the following extension of the natural SVI parameterization:

### Surface SVI (SSVI) parameterization

(3)

$$w(k, \theta_t) = \frac{\theta_t}{2} \left\{ 1 + \rho \varphi(\theta_t) k + \sqrt{(\varphi(\theta_t) k + \rho)^2 + (1 - \rho^2)} \right\}$$

with  $\theta_t > 0$  for  $t > 0$ , and where  $\varphi$  is a smooth function from  $(0, \infty)$  to  $(0, \infty)$  such that the limit  $\lim_{t \rightarrow 0} \theta_t \varphi(\theta_t)$  exists in  $\mathbb{R}$ .

## Interpretation of SSVI

- This representation amounts to considering the volatility surface in terms of ATM variance time, instead of standard calendar time.
- The ATM total variance is  $\theta_t = \sigma_{BS}^2(0, t) t$  and the ATM volatility skew is given by

$$\partial_k \sigma_{BS}(k, t)|_{k=0} = \frac{1}{2\sqrt{\theta_t t}} \partial_k w(k, \theta_t) \Big|_{k=0} = \frac{\rho\sqrt{\theta_t}}{2\sqrt{t}} \varphi(\theta_t).$$

- The smile is symmetric around at-the-money if and only if  $\rho = 0$ , a well-known property of stochastic volatility models.

## Conditions on SSVI for no calendar spread arbitrage

### Theorem 4.1

The SSVI surface (3) is free of calendar spread arbitrage if and only if

1.  $\partial_t \theta_t \geq 0$ , for all  $t \geq 0$ ;
2.  $0 \leq \partial_\theta(\theta \varphi(\theta)) \leq \frac{1}{\rho^2} (1 + \sqrt{1 - \rho^2}) \varphi(\theta)$ , for all  $\theta > 0$ , where the upper bound is infinite when  $\rho = 0$ .

- In particular, SSVI is free of calendar spread arbitrage if:
  - the skew in total variance terms is monotonically increasing in trading time and
  - the skew in implied variance terms is monotonically decreasing in trading time.
- In practice, any reasonable skew term structure that a trader defines will have these properties.

## Conditions on SSVI for no butterfly arbitrage

### Theorem 4.2

The volatility surface (3) is free of butterfly arbitrage if the following conditions are satisfied for all  $\theta > 0$ :

1.  $\theta \varphi(\theta) (1 + |\rho|) < 4$ ;
2.  $\theta \varphi(\theta)^2 (1 + |\rho|) \leq 4$ .

### Remark

Condition 1 needs to be a strict inequality so that  $\lim_{k \rightarrow +\infty} d_+(k) = -\infty$  and the SVI density integrates to one.

## Are these conditions necessary?

### Lemma 4.2

The volatility surface (3) is free of butterfly arbitrage only if

$$\theta\varphi(\theta)(1+|\rho|) \leq 4, \quad \text{for all } \theta > 0.$$

Moreover, if  $\theta\varphi(\theta)(1+|\rho|) = 4$ , the surface (3) is free of butterfly arbitrage only if

$$\theta\varphi(\theta)^2(1+|\rho|) \leq 4.$$

So the theorem is almost if-and-only-if.

## The Roger Lee arbitrage bounds

- The asymptotic behavior of the surface (3) as  $|k|$  tends to infinity is

$$w(k, \theta_t) = \frac{(1 \pm \rho)\theta_t}{2} \varphi(\theta_t)|k| + \mathcal{O}(1), \quad \text{for any } t > 0.$$

- Thus the condition  $\theta\varphi(\theta)(1+|\rho|) \leq 4$  of Theorem 4.2 corresponds to the upper bound of 2 on the asymptotic slope established by Roger Lee.
  - Again, Condition 1 of the theorem is necessary.

## No static arbitrage with SSVI

### Corollary 4.1

The SSVI surface (3) is free of static arbitrage if the following conditions are satisfied:

1.  $\partial_t \theta_t \geq 0$ , for all  $t > 0$
2.  $0 \leq \partial_\theta(\theta\varphi(\theta)) \leq \frac{1}{\rho^2} (1 + \sqrt{1 - \rho^2}) \varphi(\theta)$ , for all  $\theta > 0$ ;
3.  $\theta\varphi(\theta)(1+|\rho|) < 4$ , for all  $\theta > 0$ ;
4.  $\theta\varphi(\theta)^2(1+|\rho|) \leq 4$ , for all  $\theta > 0$ .

- A large class of simple closed-form arbitrage-free volatility surfaces!

## A Heston-like surface

### Example 4.1

The function  $\varphi$  defined as

$$\varphi(\theta) = \frac{1}{\lambda\theta} \left\{ 1 - \frac{1 - e^{-\lambda\theta}}{\lambda\theta} \right\},$$

with  $\lambda \geq (1 + |\rho|)/4$  satisfies the conditions of [Corollary 4.1](#).

- This function is consistent with the implied variance skew in the Heston model as shown in [The Volatility Surface]<sup>[3]</sup> (equation 3.19).

## A power-law surface

### Example 4.2

The choice

$$\varphi(\theta) = \frac{\eta}{\theta^\gamma (1 + \theta)^{1-\gamma}}$$

gives a surface that is completely free of static arbitrage provided that  $\gamma \in (0, 1/2]$  and  $\eta(1 + |\rho|) \leq 2$ .

- This function is more consistent with the empirically-observed term structure of the volatility skew.

## An even simpler power-law surface

- The choice

$$\varphi(\theta) = \frac{\eta}{\theta^\gamma}$$

gives a surface that is completely free of static arbitrage for  $\theta < \theta_{\max}$  provided that  $\gamma \in (0, 1/2]$  where  $\theta_{\max}$  is greater than the longest-dated expiration in practice.

- In particular,  $\gamma = 1/2$  is a possible choice. This is the so-called *square-root* parameterization.
- Also  $\gamma = \frac{1}{2} - H$  would correspond (roughly) to rough volatility.

## Even more flexibility...

### Theorem 4.3

Let  $(k, t) \mapsto w(k, t)$  be a SSVI volatility surface (3) satisfying the conditions of [Corollary 4.1](#) (in particular free of static arbitrage), and  $\alpha : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  a non-negative and increasing function of time. Then the volatility surface  $(k, t) \mapsto w_\alpha(k, \theta_t) := w(k, \theta_t) + \alpha_t$  is also free of static arbitrage.

- [Corollary 4.1](#) gives us the freedom to match three features of one smile (level, skew, and curvature say) but only two features of all the other smiles (level and skew say), subject of course to the given smiles being themselves arbitrage-free.
- [Theorem 4.3](#) may allow us to match an additional feature of each smile through  $\alpha_t$ .

### eSSVI (or Extended SSVI)

- More recently, [Hendriks and Martini]<sup>[6]</sup> showed how to extend SSVI by allowing  $\rho$  to depend on  $\theta$ .
- The eSSVI parameterization then looks like

#### Surface SVI (SSVI) parameterization

$$w(k, \theta_t) = \frac{\theta_t}{2} \left\{ 1 + \rho(\theta_t) \varphi(\theta_t) k + \sqrt{(\varphi(\theta_t) k + \rho(\theta_t))^2 + (1 - \rho(\theta_t)^2)} \right\}.$$

- Obviously, fixing  $\theta_t$ , the no butterfly arbitrage conditions are just the same as before.
- However, the no calendar spread arbitrage conditions have to change.
- Theorem 4.1 of [Hendriks and Martini]<sup>[6]</sup> reads as follows.
  - Define  $\gamma := \frac{1}{\varphi(\theta)}$  and  $\delta := \theta \partial_\theta(\rho(\theta))$ . The eSSVI surface is free of calendar-spread arbitrage if and only if  $\partial_t \theta_t > 0$  and

$$(\delta + \rho \gamma)^2 \leq \begin{cases} \gamma^2, & \text{if } 0 \leq \gamma \leq 1, \\ 2\gamma - 1, & \text{if } \gamma > 1. \end{cases}$$

### Admissible parametric families for $\rho(\theta)$

- [Hendriks and Martini]<sup>[6]</sup> exhibit various admissible parametric families for  $\rho(\theta)$ .
- Recall that the choice

$$\varphi(\theta) = \frac{\eta}{\theta^\lambda}$$

is arbitrage-free for  $\theta < \theta_{\max}$  where  $\theta_{\max}$  is rarely relevant in practice.

- In particular, they then show that the choice

$$\rho(\theta) = \rho_0 + (\rho_m - \rho_0) \left( \frac{\theta}{\theta_{\max}} \right)^a$$

is admissible with the extra condition that

$$a \leq \begin{cases} \frac{(1-\lambda)(1-\rho_m)}{\rho_0 - \rho_m}, & \text{if } \rho_0 < \rho_m, \\ \frac{(1-\lambda)(1+\rho_m)}{\rho_m - \rho_0}, & \text{if } \rho_0 > \rho_m. \end{cases}$$

## How to eliminate butterfly arbitrage

- We have shown how to define a volatility smile that is free of butterfly arbitrage.
- This smile is completely defined given three observables.
  - The ATM volatility and ATM skew are obvious choices for two of them.
  - The most obvious choice for the third observable in equity markets would be the asymptotic slope for  $k$  negative and in FX markets and interest rate markets, perhaps the ATM curvature of the smile might be more appropriate.

## How to fix butterfly arbitrage

- Supposing we choose to fix the SVI-JW parameters  $v_t$ ,  $\psi_t$  and  $p_t$  of a given SVI smile, we may guarantee a smile with no butterfly arbitrage by choosing the remaining parameters  $c'_t$  and  $\tilde{v}'_t$  according to SSVI as

$$c'_t = p_t + 2\psi_t, \quad \text{and} \quad \tilde{v}'_t = v_t \frac{4p_t c'_t}{(p_t + c'_t)^2}.$$

- That is, given a smile defined in terms of its SVI-JW parameters, we are guaranteed to be able to eliminate butterfly arbitrage by changing the call wing  $c_t$  and the minimum variance  $\tilde{v}_t$ , both parameters that are hard to calibrate with available quotes in equity options markets.

## Example: Fixing the Vogt smile

- The SVI-JW parameters corresponding to the Vogt smile are:

$$\begin{aligned} & (v_t, \psi_t, p_t, c_t, \tilde{v}_t) \\ &= (0.01742625, -0.1752111, 0.6997381, 1.316798, 0.0116249). \end{aligned}$$

- We know then that choosing  $(c_t, \tilde{v}_t) = (0.3493158, 0.01548182)$  must give a smile free of butterfly arbitrage.

- There must exist some pair of parameters  $\{c_t, \tilde{v}_t\}$  with  $c_t \in (0.349, 1.317)$  and  $\tilde{v}_t \in (0.0116, 0.0155)$  such that the new smile is free of butterfly arbitrage and is as close as possible to the original one in some sense.

The R-code follows:

```
In [15]: t <- 1
w <- a+b*(-rho*m + sqrt(m^2+sig^2))
v <- w/t
psi <- 1/sqrt(w)*b/2*(-m/sqrt(m^2+sig^2)+rho)
p <- 1/sqrt(w)*b*(1-rho)
c <- 1/sqrt(w)*b*(1+rho)
wtilde <- (a+b*sig*sqrt(1-rho^2))
vtilde <- wtilde/t

(cprime <- vogtJW$pt + 2*vogtJW$psit)
(vtildeprime <- vogtJW$vt*4*vogtJW$pt*cprime/(vogtJW$pt+cprime)^2)

0.349315822498558
0.0154818248407312
```

```
In [16]: vogtFixedJW <- vogtJW
vogtFixedJW$ct <- cprime
vogtFixedJW$varmint <- vtildeprime

print(as.data.frame(rbind(vogtJW,vogtFixedJW)))

# Convert SVI-JW parameters back to raw SVI parameters
vogtFixedParams <- jwToSvi(vogtFixedJW)
print(vogtFixedParams)
```

	vt	psit	pt	ct	varmint	texp
1	0.01742625	-0.1752111	0.6997381	1.3167982	0.01162490	1
2	0.01742625	-0.1752111	0.6997381	0.3493158	0.01548182	1
	a	b	sig	rho	m	
1	0.007740912	0.06924203	0.1186078	-0.3340365	0.04203375	

## Numerical optimization

- In this particular case, choosing the objective function as the sum of squared option price differences plus a large penalty for butterfly arbitrage, we arrive at the following “optimal” choices of the call wing and minimum variance parameters that still ensure no butterfly arbitrage:

$$(c_t, \tilde{v}_t) = (0.8564763, 0.0116249).$$

- Note that the optimizer has left  $\tilde{v}_t$  unchanged but has decreased the call wing.

- The resulting smiles and plots of the function  $g$  are shown in [Figure 8](#).

```
In [17]: # Now find values of ct and varmint that minimize distance from the original
strikes <- (-10:10)/10;

callVals <- function(sviparams,k){
  BSFormula(1, exp(k), 1, 0, sigma=sqrt(svi(sviparams,k)))
}

callValsVogt <- callVals(vogtParams,strikes);

obj <- function(par){
  c <- par[1]
  varmin <- par[2]
  vogtNewJW <- vogtJW
  vogtNewJW$ct <- c
  vogtNewJW$varmint <- varmin
  vogtNew <- jwToSvi(vogtNewJW)
  callValsNew <- callVals(vogtNew,strikes)
  # We need to also heavily penalize negative g
  arbVal<- sviSliceArbitrageCompute(vogtNew)$gArb

  return(sum((callValsNew-callValsVogt)^2)+100*arbVal^2)
}
```

```
In [18]: obj(par=c(vogtJW$ct,vogtJW$varmint))
```

0.278426534133911

```
In [19]: upperC=max(vogtJW$ct,vogtFixedJW$ct)
lowerC=min(vogtJW$ct,vogtFixedJW$ct)
upperV=max(vogtJW$varmint,vogtFixedJW$varmint)
lowerV=min(vogtJW$varmint,vogtFixedJW$varmint)

res <- optim(par=c(vogtJW$ct,vogtJW$varmint),
             fn=obj,
             method="L-BFGS-B",
             lower=c(lowerC,lowerV),
             upper=c(upperC,upperV))

vogtOptimalJW <- vogtJW
(vogtOptimalJW$ct <- res$par[1])
(vogtOptimalJW$varmint <- res$par[2])
```

0.856476171350731

0.0116249032354779

```
In [20]: (vogtOptimalParams <- jwToSvi(vogtOptimalJW))
```

a	b	sig	rho	m
-0.03051987	0.1027168	0.4123978	0.1007175	0.272344

## The Vogt smile fixed

Now we have found the "optimal" choices  $(c_t, \tilde{v}_t) = (0.8564763, 0.0116249)$ , let's plot the resulting smile:

```
In [21]: r <- 1.5
curve(svi(vogtParams,x),from=-r,to=r,col="blue",lwd=2,ylab=NA,xlab=NA)
ylab=expression(w=sigma^2*T)
curve(svi(vogtFixedParams,x),from=-r,to=r,col="red",lwd=2,lty=2,add=T)
curve(svi(vogtOptimalParams,x),from=-r,to=r,col="green4",lwd=2,lty=3,add=T)
```

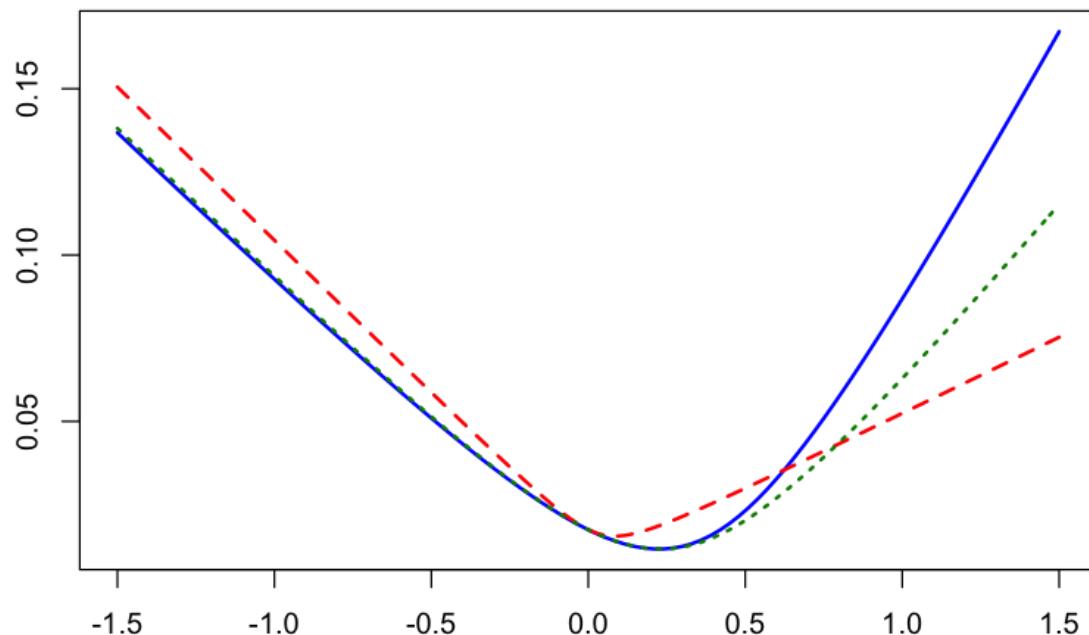


Figure 7: Plots of the total variance smile. The graph corresponding to the original Axel Vogt parameters is solid, to the guaranteed butterfly-arbitrage-free parameters dashed, and to the “optimal” choice of parameters dotted.

```
In [22]: # Plot graphs of g (ggraphSVI)
curve(g(vogtParams,x),from=-r,to=r,col="blue",n=500,lwd=2,ylab=NA,xlab=NA)
lines(c(-r,r),c(0,0),lty=2);
curve(g(vogtFixedParams,x),from=-r,to=r,col="red",n=500,add=T,lwd=2,lty=2)
curve(g(vogtOptimalParams,x),from=-r,to=r,col="green4",n=500,add=T,lwd=2,lty=2)
```

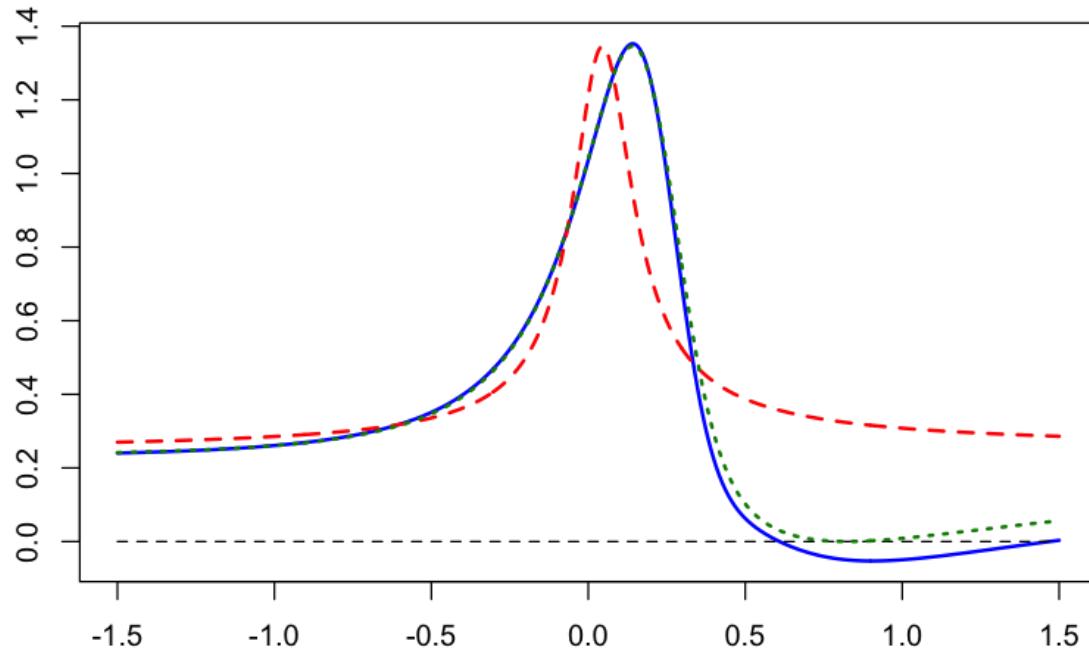


Figure 8: Plots of the function  $g$ . The graph corresponding to the original Axel Vogt parameters is solid, to the guaranteed butterfly-arbitrage-free parameters dashed, and to the “optimal” choice of parameters dotted.

### Why extra flexibility may not help

- The additional flexibility potentially afforded to us through the parameter  $\alpha_t$  of [Theorem 4.3](#) sadly does not help us with the Vogt smile.
- For  $\alpha_t$  to help, we must have  $\alpha_t > 0$ ; it is straightforward to verify that this translates to the condition  $v_t (1 - \rho^2) < \tilde{v}_t$  which is violated in the Vogt case.

```
In [23]: print(c(vogtJW$vt*(1-vogtParams$rho^2), vogtJW$varmint))
```

### Generating implied vols

- We source raw index option price data from OptionMetrics In this case, the data file is spxOptionMetrics.rData.

- We use the function *generateOptionMetricsIvols* from optionMetricsToIvols.R to generate *spxIvols050915*.
- We use the function *plotIvols* from plotIvols.R to plot the results.

```
spxIvols050915 <- generateOptionMetricsIvols(spxData050915)
plotIvols(spxIvols050915)
```

## Raw data

Raw OptionMetrics price data looks like this:

```
In [24]: load("spxOptionMetrics.rData")
head(spxData050915)
```

	secid	date	symbol	symbol_flag	exdate	last_date	cp_flag	strike_price
545518	108105	20050915	SPB.FT	0	20060617	20050908	C	1600000
545519	108105	20050915	SPB.LT	0	20051217	20050809	C	1600000
545520	108105	20050915	SPB.RT	0	20060617	20050830	P	1600000
545521	108105	20050915	SPB.XT	0	20051217	20050902	P	1600000
545522	108105	20050915	SPL.FA	0	20070616	NA	C	700000
545523	108105	20050915	SPL.FJ	0	20070616	NA	C	850000

Note that the strike prices are out by a factor of 1,000! So let's fix them:

```
In [25]: spxData050915$strike_price <- spxData050915$strike_price/1000
```

## Implied volatility computation for index options

- We compute all implied volatilities from option price data
  - We don't need external estimates of interest rates and dividends
- We use put-call parity to get implied forward prices and discount factors.
  - Find the unique forward price and discount factor that minimize implied forward pricing errors.
- In this way, we can avoid errors due to non-synchronous parameter estimates and typically generate very smooth implied volatility curves.

## Detail of forward and PV estimation

We choose the forward and the PV factor for each expiration so as to minimize the squared error in the put-call parity relationship:

```

pvGuess <- 1;
fGuess <- mean(imid+strikes);
nearTheMoneyStrikes <- strikes[order(abs(imid))][1:6];

include <- (strikes%in%nearTheMoneyStrikes);
obj <- function(params){
  f <- params[1]; pv <- params[2];
  ifit <- pv*(f-strikes); # This is a vector length nk
  errmid <- (ifit-imid)*include;
  return(sum(errmid^2));
}
fit <- optim(c(fGuess,pvGuess),obj,method="L-BFGS-B",
             lower=c(min(strike
s),0.5),upper=c(max(strikes),2));
ffit <- fit$par[1];
pvfit <- fit$par[2];

```

## Implied volatility output

The resulting implied volatility output looks like this:

```
In [26]: spxIvols050915 <- generateOptionMetricsIvols(spxData050915)
spxIvols050915[30:50, ]
```

	<b>Expiry</b>	<b>Texp</b>	<b>Strike</b>	<b>Bid</b>	<b>Ask</b>	<b>Fwd</b>	<b>CallMid</b>
<b>30</b>	20050917	0.005475702	1150	NA	0.37535440	1227.798	NA
<b>31</b>	20050917	0.005475702	1155	NA	0.32547806	1227.798	NA
<b>32</b>	20050917	0.005475702	1160	0.30490494	0.37678130	1227.798	67.95133827
<b>33</b>	20050917	0.005475702	1165	NA	0.32676025	1227.798	NA
<b>34</b>	20050917	0.005475702	1170	0.26360852	0.34657446	1227.798	58.00243316
<b>35</b>	20050917	0.005475702	1175	0.24285638	0.26499148	1227.798	52.87469594
<b>36</b>	20050917	0.005475702	1180	0.24263809	0.28715960	1227.798	48.00243316
<b>37</b>	20050917	0.005475702	1185	NA	0.23352259	1227.798	NA
<b>38</b>	20050917	0.005475702	1190	0.19745139	0.20977679	1227.798	37.92579083
<b>39</b>	20050917	0.005475702	1195	0.15864560	0.19484019	1227.798	32.92579083
<b>40</b>	20050917	0.005475702	1200	0.16142793	0.17666020	1227.798	28.00243316
<b>41</b>	20050917	0.005475702	1205	0.15018761	0.15579053	1227.798	23.07907550
<b>42</b>	20050917	0.005475702	1210	0.12791572	0.13656493	1227.798	18.15571784
<b>43</b>	20050917	0.005475702	1215	0.09898340	0.10967283	1227.798	13.18126528
<b>44</b>	20050917	0.005475702	1220	0.08676468	0.09128081	1227.798	8.51338207
<b>45</b>	20050917	0.005475702	1225	0.06220211	0.08685587	1227.798	4.33090031
<b>46</b>	20050917	0.005475702	1230	0.06835615	0.08318531	1227.798	1.78832116
<b>47</b>	20050917	0.005475702	1235	0.06703419	0.07488402	1227.798	0.43430657
<b>48</b>	20050917	0.005475702	1240	0.08106528	0.09051137	1227.798	0.20437956
<b>49</b>	20050917	0.005475702	1245	0.10671125	0.15155820	1227.798	0.43430657
<b>50</b>	20050917	0.005475702	1250	0.11062406	0.12265544	1227.798	0.07664234

## Implied volatility smiles: September 15, 2005

Finally plot the smiles:

```
In [27]: options(repr.plot.width=10,repr.plot.height=6)
```

```
In [28]: res05 <- plotIvols(spxIvols050915)
```

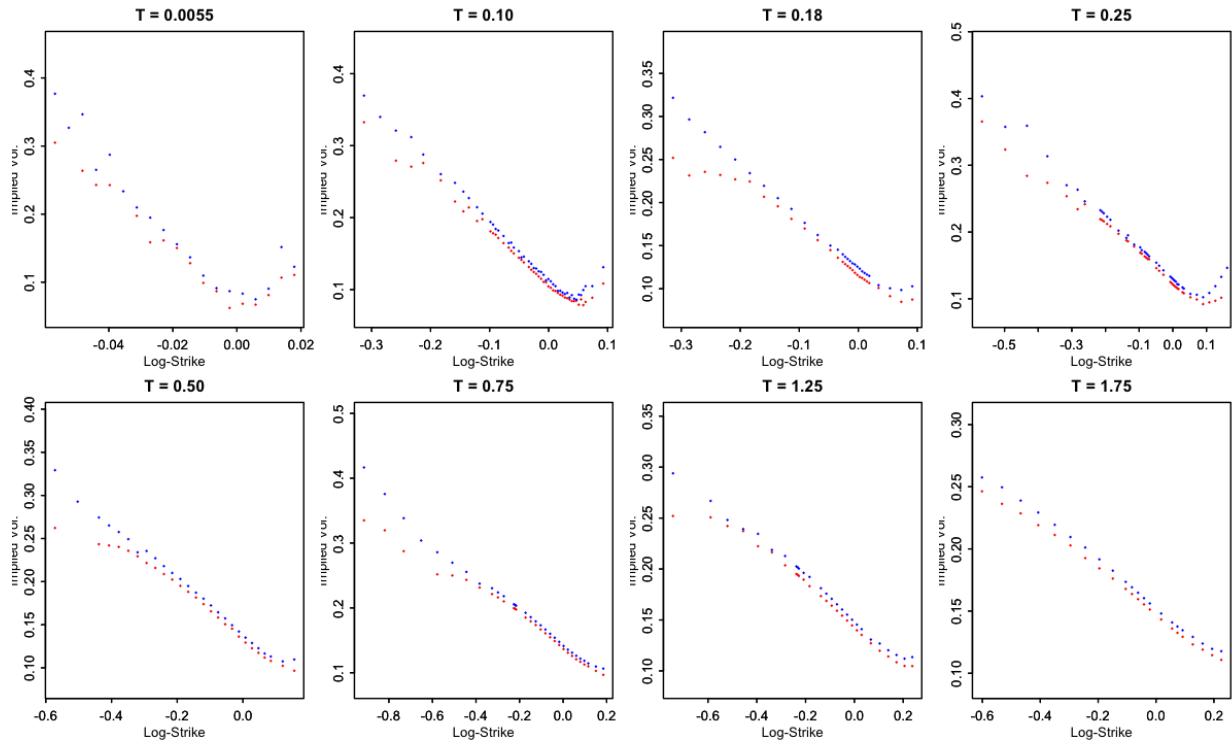


Figure 9: Here are the SPX implied volatility smiles corresponding to the output

## Single-stock options

- That's fine for index options which are European-style. What about single-stock options which are American-style?
- We can't use the same put-call parity trick we used before to impute forwards and PV factors.
- In principle, we have to take the whole volatility surface into account when pricing American options.

- Ignoring this however, given some American option model that takes discrete dividends as an input, we can impute interest rates and borrow costs for single-stocks.
  - We choose these to minimize the difference between call and put implied volatilities.

## SVI parameters used in *The Volatility Surface*

Here are the SVI parameters corresponding to the SVI fit shown in Figures 3.2 and 3.3 of *The Volatility Surface*:

```
In [29]: texp <- sort(unique(spxIvols050915$Texp))
```

```
svidata <- c(
  c(-0.0001449630, 0.0092965440, 0.0196713280, -0.2941176470, -0.0054273230),
  c(-0.000832134 , 0.024439766 , 0.069869455 , -0.299975308 , 0.02648364 ),
  c(-0.0008676750, 0.0282906450, 0.0873835580, -0.2892204290, 0.0592703000),
  c(-0.0000591593, 0.0331790820, 0.0812872370, -0.3014043240, 0.0652549210),
  c(0.0011431940 , 0.0462796440 , 0.1040682980 , -0.3530782140 , 0.0942000770),
  c(0.0022640980 , 0.0562604150 , 0.1305339330 , -0.4387409470 , 0.1111230690),
  c(0.0040335530 , 0.0733707550 , 0.1707947600 , -0.4968970370 , 0.1496609160),
  c(0.0034526910 , 0.0917230540 , 0.2236814130 , -0.4942213210 , 0.1854128490));

sviMatrix <- as.data.frame(t(array(svidata, dim=c(5,8))));
colnames(sviMatrix)<-c("a","b","sig","rho","m")
```

```
In [30]: # Inspect the matrix
sviMatrix
```

<b>a</b>	<b>b</b>	<b>sig</b>	<b>rho</b>	<b>m</b>
-0.0001449630	0.009296544	0.01967133	-0.2941176	-0.005427323
-0.0008321340	0.024439766	0.06986945	-0.2999753	0.026483640
-0.0008676750	0.028290645	0.08738356	-0.2892204	0.059270300
-0.0000591593	0.033179082	0.08128724	-0.3014043	0.065254921
0.0011431940	0.046279644	0.10406830	-0.3530782	0.094200077
0.0022640980	0.056260415	0.13053393	-0.4387409	0.111123069
0.0040335530	0.073370755	0.17079476	-0.4968970	0.149660916
0.0034526910	0.091723054	0.22368141	-0.4942213	0.185412849

### Equivalent SVI-JW parameters.

We note that for longer expirations, JW parameters are almost independent of expiration:

```
In [31]: sviToJw(sviMatrix, texp)
```

<b>vt</b>	<b>psit</b>	<b>pt</b>	<b>ct</b>	<b>varmint</b>	<b>texp</b>
0.005461444	-0.02393127	2.1999949	1.1999972	0.005446544	0.005475702
0.011729194	-0.23199502	0.9217058	0.4963300	0.007865899	0.101300479
0.014634977	-0.23575331	0.7146813	0.3940218	0.008422155	0.177960301
0.015913797	-0.24169930	0.6783336	0.3641304	0.009867464	0.254620123
0.018220317	-0.24736379	0.6536137	0.3125000	0.011214052	0.503764545
0.019460020	-0.25260651	0.6687175	0.2608696	0.011772234	0.752908966
0.020901180	-0.26222826	0.6791513	0.2282609	0.011915278	1.251197810
0.022010229	-0.26465446	0.6984348	0.2364130	0.012168504	1.749486653

### SVI fit example: September 15, 2005

Now check the quality of the SVI fit:

```
In [32]: res05 <- plotIvols(spxIvols050915, sviMatrix)
```

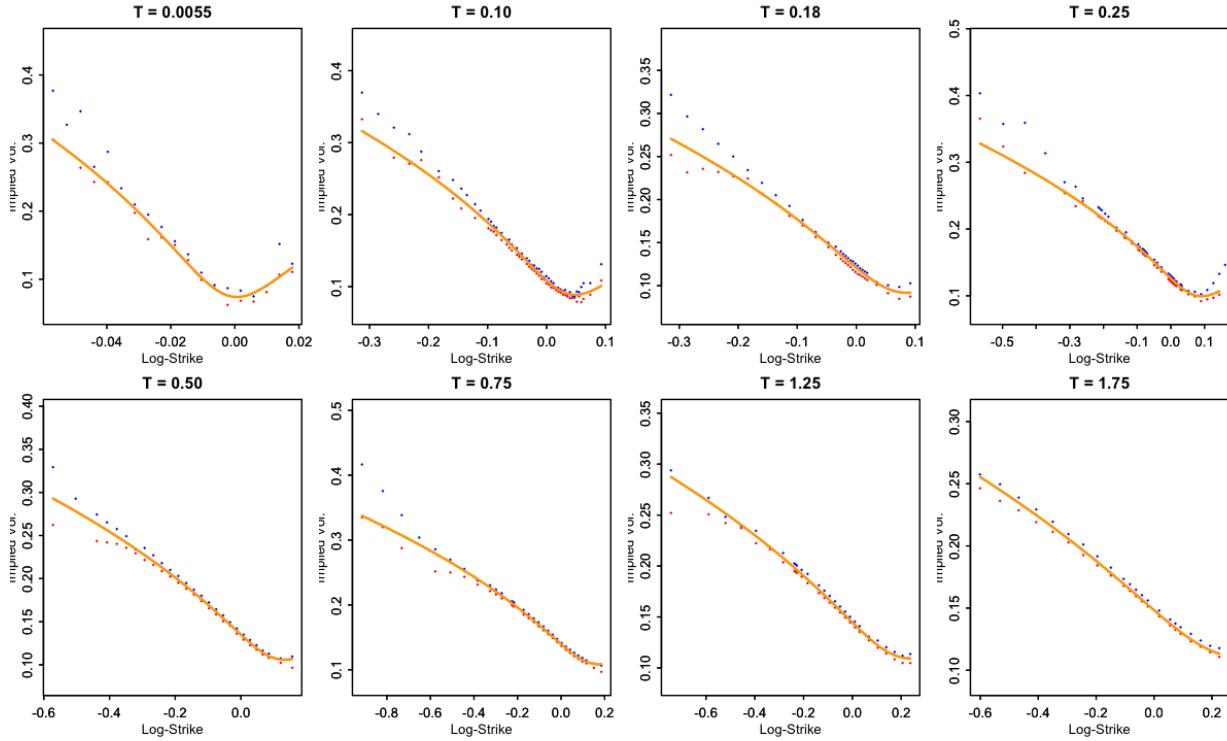


Figure 10: This fit is the one shown in Figure 3.3 of *The Volatility Surface*

### Comments on the SVI fit

- It's hard to deny that this SVI fit is strikingly good.

- At least in this particular case, the SVI parameterization gives an arbitrage-free surface with very good fits to the data.
- We would like to achieve a similar fit quality using only simple code written in R.

### SPX total variance plot as of 15-Sep-2005

```
In [33]: options(repr.plot.width=7,repr.plot.height=5)
```

```
In [34]: plotSVI(sviMatrix)
```

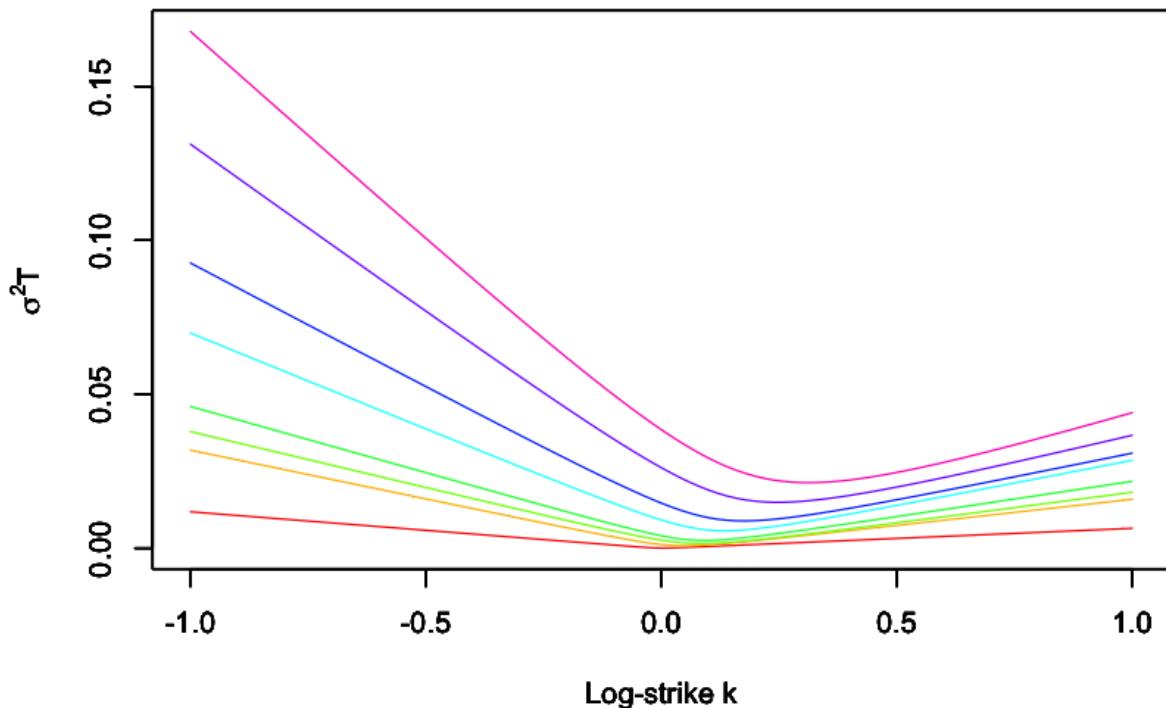


Figure 11: SVI fits to the SPX implied volatility surface as of the close on September 15, 2005, the day before triple witching.

### SPX total variance plot as of 15-Sep-2005: zoomed version

```
In [35]: plotSVI(sviMatrix,xrange=c(-.25,.25),yrange=c(0,0.01))

# Add a circle around the area of interest
y19 <- svi(sviMatrix[2,],.19)
points(x=.18,y=y19,pch=1,cex=10,pty=2,col="dark green")
```

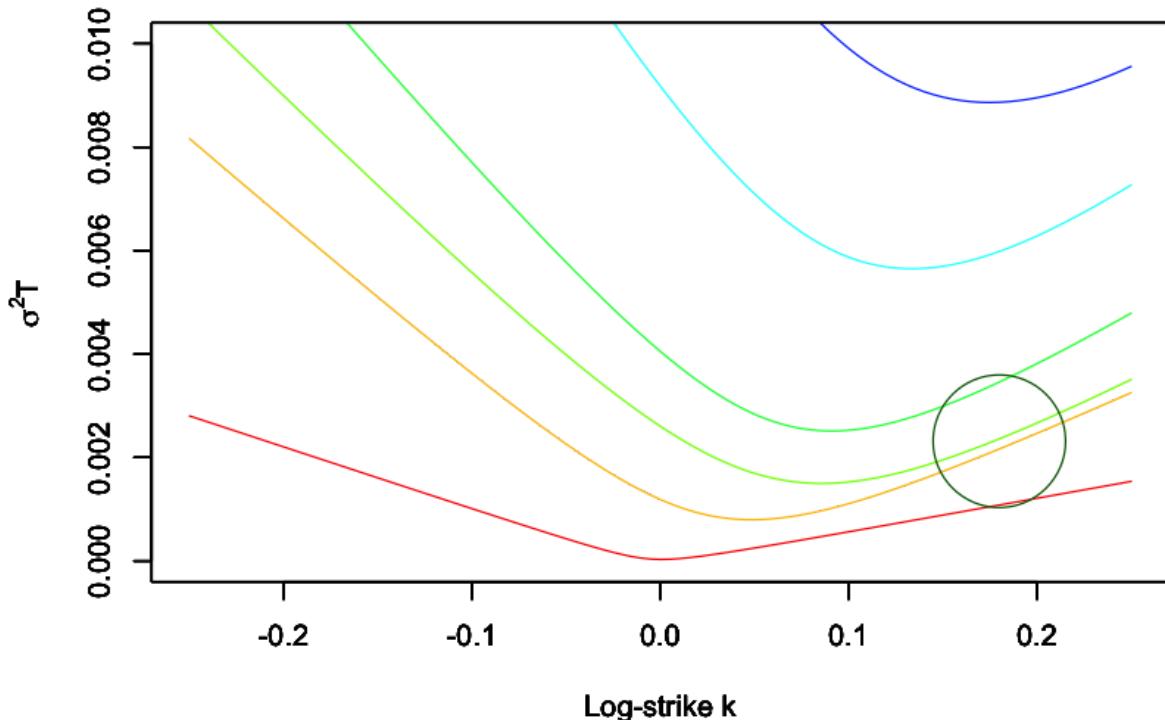


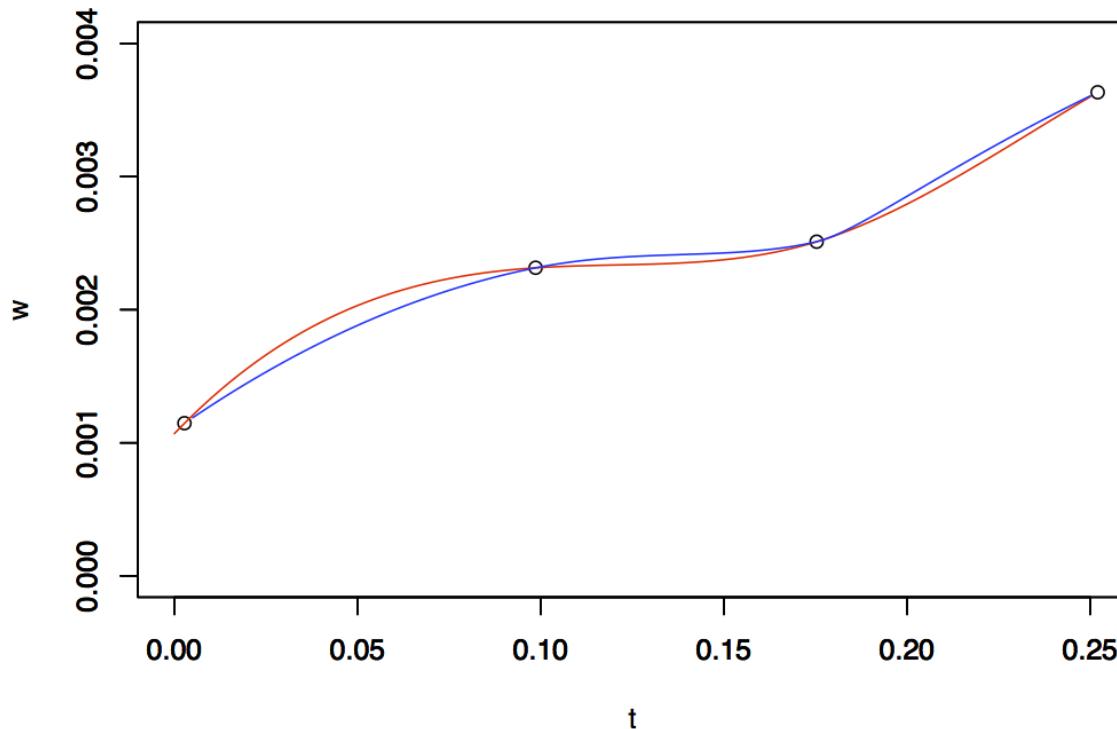
Figure 12: A zoomed view of the same plot showing lines getting very close to each other: small local variances!

### The total variance plot: observations

- Note that none of the lines cross:  $w$  is a non-decreasing function of  $t$  for every log-strike  $k$
- No calendar spread arbitrage is equivalent to no lines crossing.
  - Negative calendar spreads (lines crossing) are equivalent to negative local variances
  - Recall that (equation (1.6) on page 11):
 
$$v_{loc}(k, T) = \frac{\partial_T C}{\frac{1}{2} K^2 \partial_{K,K} C}$$
  - Lines moving together mean low local variance
  - Lines moving apart mean high local variance

### Stineman interpolation

- Monotonic spline interpolation is essential. Stineman spline is a particularly attractive choice.
- Here's the term structure of total variance at constant  $k = 0.19$  (red is cubic spline, blue is Stineman spline):



- We see that the cubic spline generates local variances that are too small – they can even go negative.

### Fitting SVI slice-by-slice

Fitting slice-by-slice is not a good idea. Let's see why:

```
In [36]: spxOptData <- spxIvols050915  
  
# Fit each slice individually  
fit0 <- svifit(spxOptData)  
plotSVI(fit0,yrange=c(0,.1))
```

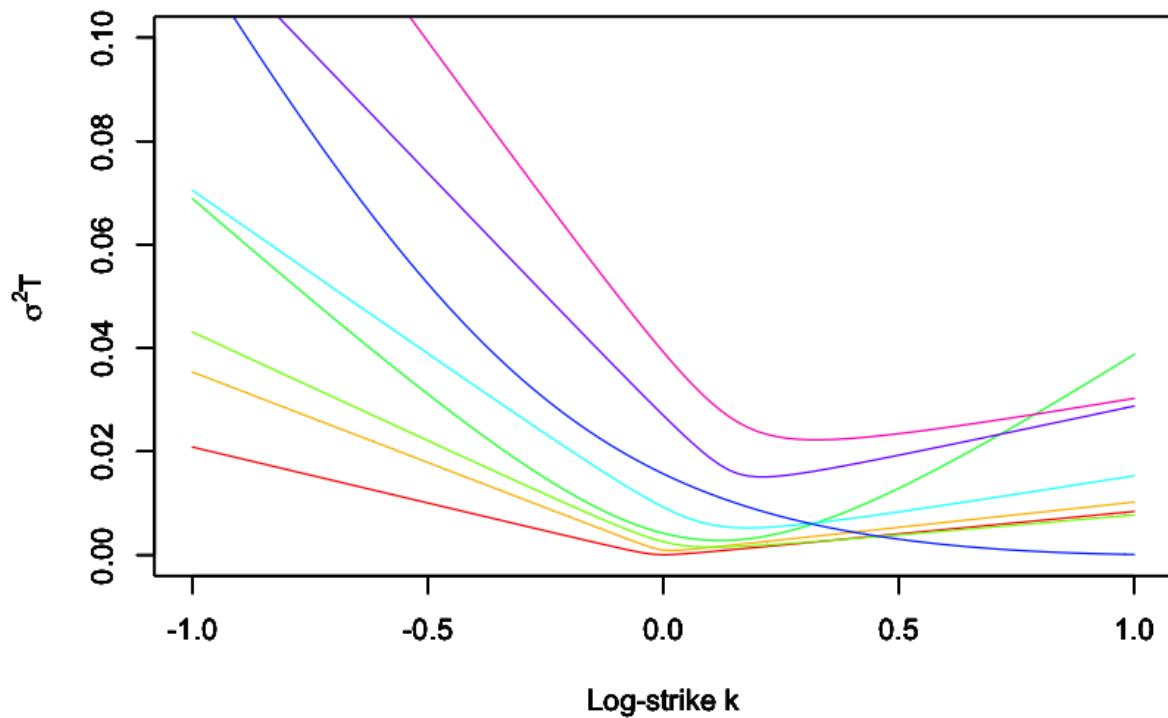


Figure 13: If we fit the slices independently, we see arbitrage everywhere!

```
In [37]: options(repr.plot.width=10,repr.plot.height=6)
```

```
In [38]: plotIvols(spxOptData, fit0)
```

### \$expiries

0.0054757015742642	0.101300479123888	0.177960301163587
0.254620123203285	0.503764544832307	0.752908966461328
1.74948665297741		1.25119780971937

### \$atmVol

0.0753317389474099	0.109284017684344	0.119883508623873
0.125678282197117	0.134805847236067	0.139563468055532
0.145130261684339	0.149268986510335	

### \$atmSkew

-0.798770818522045	-0.74115989314962	-0.553487395374889
-0.472758942307923	-0.342787380535833	-0.30404615721671
-0.243667774447752	-0.205278718439569	

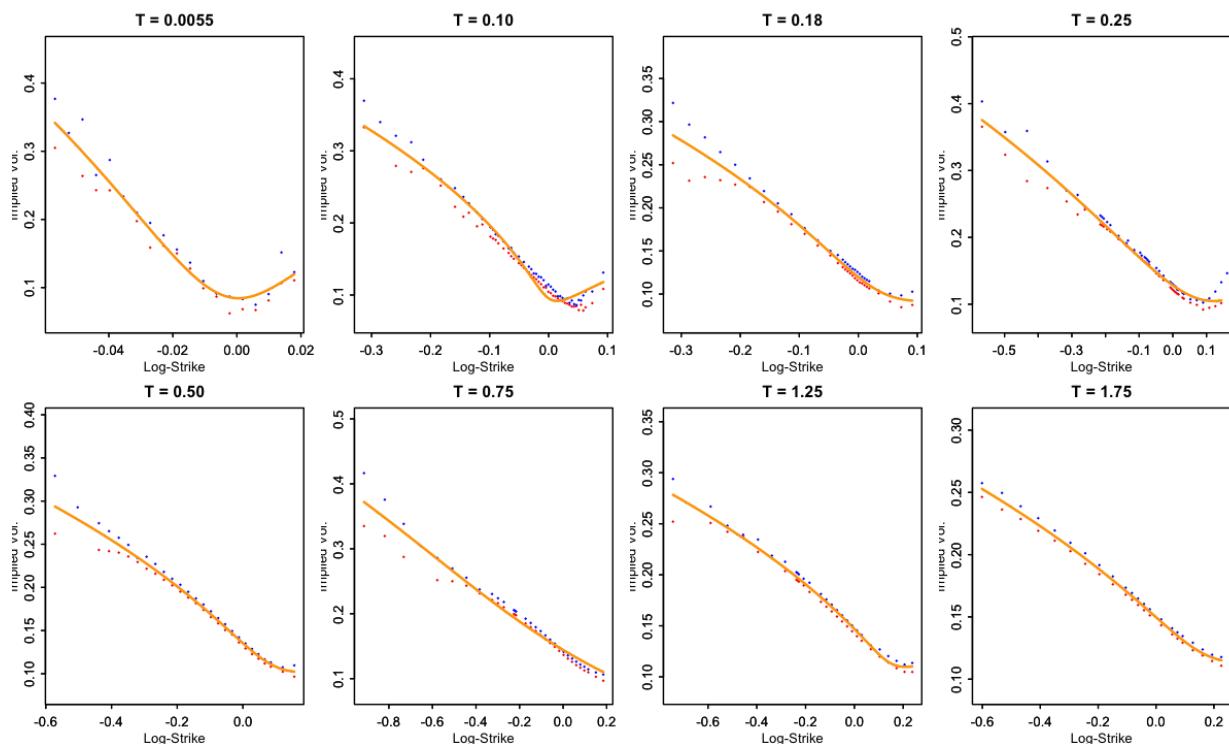


Figure 14: ...but fits to individual slices are good

### Fitting with no calendar spread arbitrage

- To achieve a fit with no calendar spread arbitrage:
  - We need to find explicit constraints on the parameters.
  - We need a good initial guess.

### The SVI square root parameterization

Consider the following special case of SSVI:

**SVI square-root parameterization**

$$w(k, \theta_t) = \frac{\theta_t}{2} \left\{ 1 + \rho \varphi(\theta_t) k + \sqrt{(\varphi(\theta_t)k + \rho)^2 + (1 - \rho^2)} \right\}$$

with

$$\varphi(\theta) = \frac{\eta}{\sqrt{\theta}}.$$

- The SVI square-root parameterization is free of static arbitrage for up to some (very long in practice) expiration.

**Square-root fit example: September 15, 2005**

```
In [39]: texp <- unique(spxOptData$Texp)
fitSqrt <- sviSqrtFit(spxOptData)
plotIvols(spxOptData, fitSqrt)
```

**\$expiries**

0.0054757015742642	0.101300479123888	0.177960301163587
0.254620123203285	0.503764544832307	0.752908966461328
1.74948665297741		

**\$atmVol**

0.0753317389474099	0.109284017684344	0.119883508623873
0.125678282197117	0.134805847236067	0.139563468055532
0.145130261684339	0.149268986510335	

**\$atmSkew**

-0.798770818522045	-0.74115989314962	-0.553487395374889
-0.472758942307923	-0.342787380535833	-0.30404615721671
-0.243667774447752	-0.205278718439569	

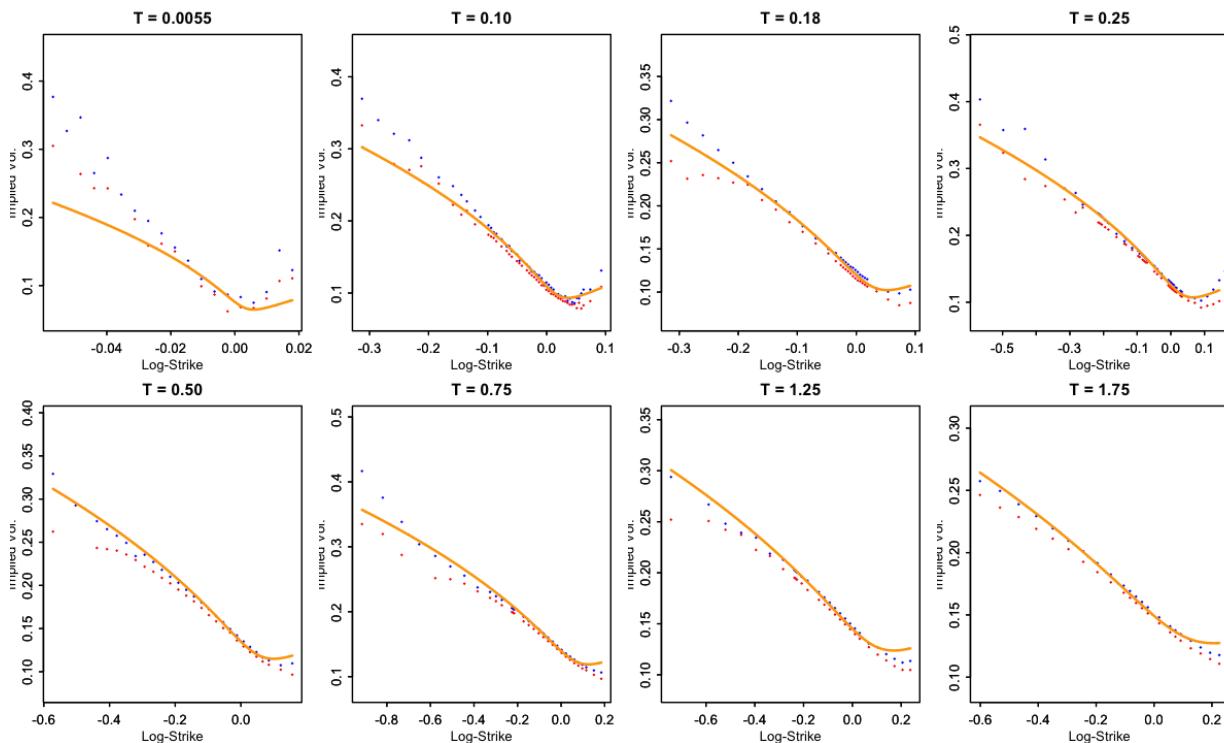


Figure 15: SVI square-root fits surprisingly well! ... except for the front month.

And here's the total variance plot:

```
In [40]: options(repr.plot.width=7,repr.plot.height=5)
```

```
In [41]: plotSVI(fitSqrt)
```

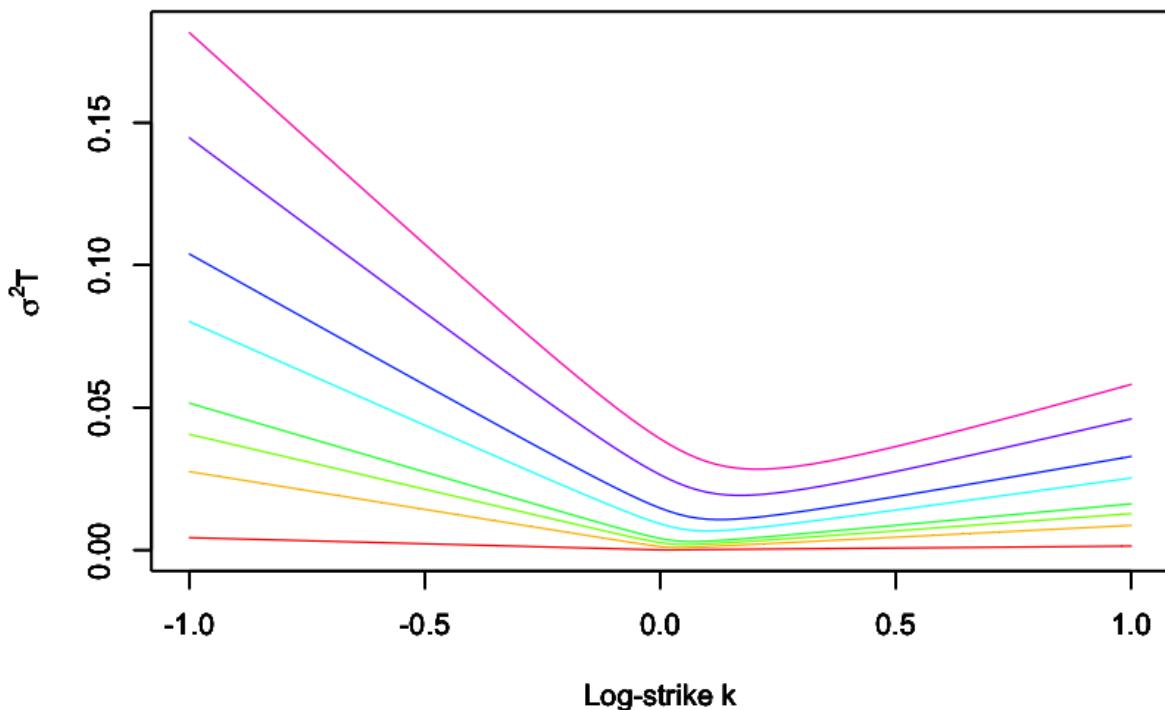


Figure 16: ... and there is no calendar spread arbitrage

### Quote-driven vs order-driven markets

- In quote-driven markets, we have tight bid-offer prices for all listed options. Our fit must necessarily go through all of the bids and offers.
- In order-driven markets, it's not so clear what the current market price of an option is:
  - We may have a bid with no ask
  - Or an ask with no bid.
  - Or neither a bid nor an ask.
  - Or the bid-ask spread may just be very wide.
- In order-driven options markets therefore, the square-root fit may be more than adequate – even preferable because it has so few parameters and is therefore very robust.
- In quote-driven markets, we need to do better than the square-root fit.

### Quantifying lines crossing

- Consider two SVI slices with parameters  $\chi_1$  and  $\chi_2$  where  $t_2 > t_1$ .

- We first compute the points  $k_i$  ( $i = 1, \dots, n$ ) with  $n \leq 4$  at which the slices cross, sorting them in increasing order. If  $n > 0$ , we define the points  $\tilde{k}_i$  as

$$\begin{aligned}\tilde{k}_1 &:= k_1 - 1, \\ \tilde{k}_i &:= \frac{1}{2}(k_{i-1} + k_i), \quad \text{if } 2 \leq i \leq n, \\ \tilde{k}_{n+1} &:= k_n + 1.\end{aligned}$$

- For each of the  $n + 1$  points  $\tilde{k}_i$ , we compute the amounts  $c_i$  by which the slices cross:

$$c_i = \max [0, w(\tilde{k}_i, \chi_1) - w(\tilde{k}_i, \chi_2)].$$

## Crossedness

### Definition 4.4

The crossedness of two SVI slices is defined as the maximum of the  $c_i$  ( $i = 1, \dots, n$ ). If  $n = 0$ , the crossedness is null.

## Computing crossedness

To see if lines cross and to compute crossedness, call `ComputeSviRoots` from `sviRoots.R`. For example if we have:

```
In [42]: # Make the lines cross
svi2 <- sviMatrix[1:2,]
svi2$a[2] <- svi2$a[2]-0.0005
plotsVI(svi2)
```

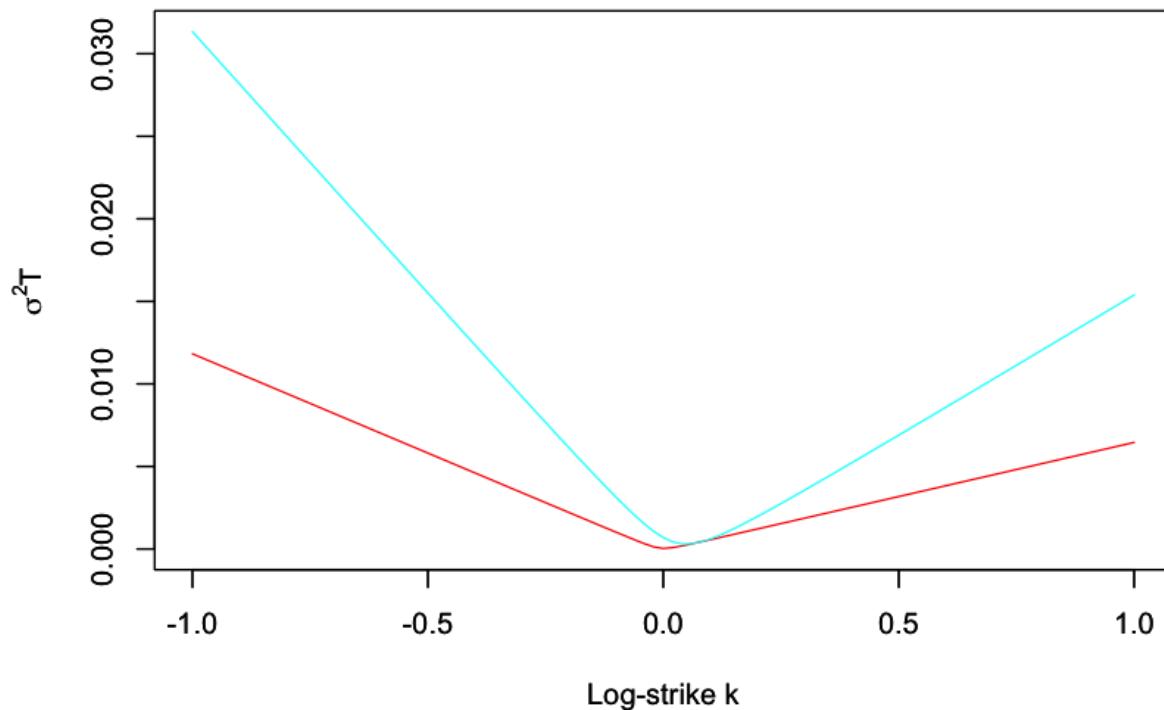
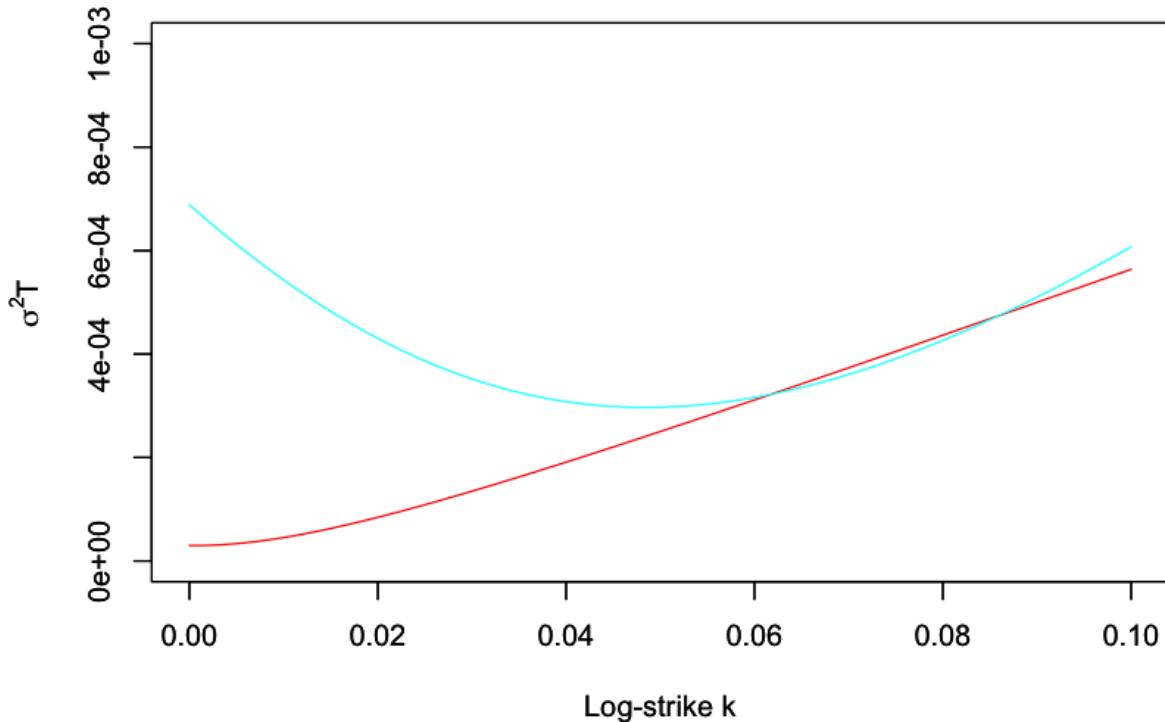


Figure 17: We lower slice 2 in order to introduce calendar spread arbitrage (crossed lines).

```
In [43]: # Zoom in to the problem area
plotSVI(svi2,xrange=c(0,0.1),yrange=c(0,0.001))
```



Calling sviRoots() on these two slices gives:

```
In [44]: sviRoots(svi2)
```

```
$roots
0.0617520329996525  0.085905064286618
$crossedness
1.40209913616877e-05
```

## QR fit

Finally, using *fitSqrt* as an initial guess, change SVI parameters slice-by slice, working backwards, so as to minimize the sum of squared distances between the fitted prices and the mid option prices with a big penalty for crossing either the previous slice or the next slice (as quantified by the crossedness).

```
In [45]: options(repr.plot.width=10,repr.plot.height=6)
```

```
In [46]: fitQR <- sviFitQR(spxOptData,fitSqrt)
```

```
In [47]: res <- plotIvols(spxOptData, fitQR)
```

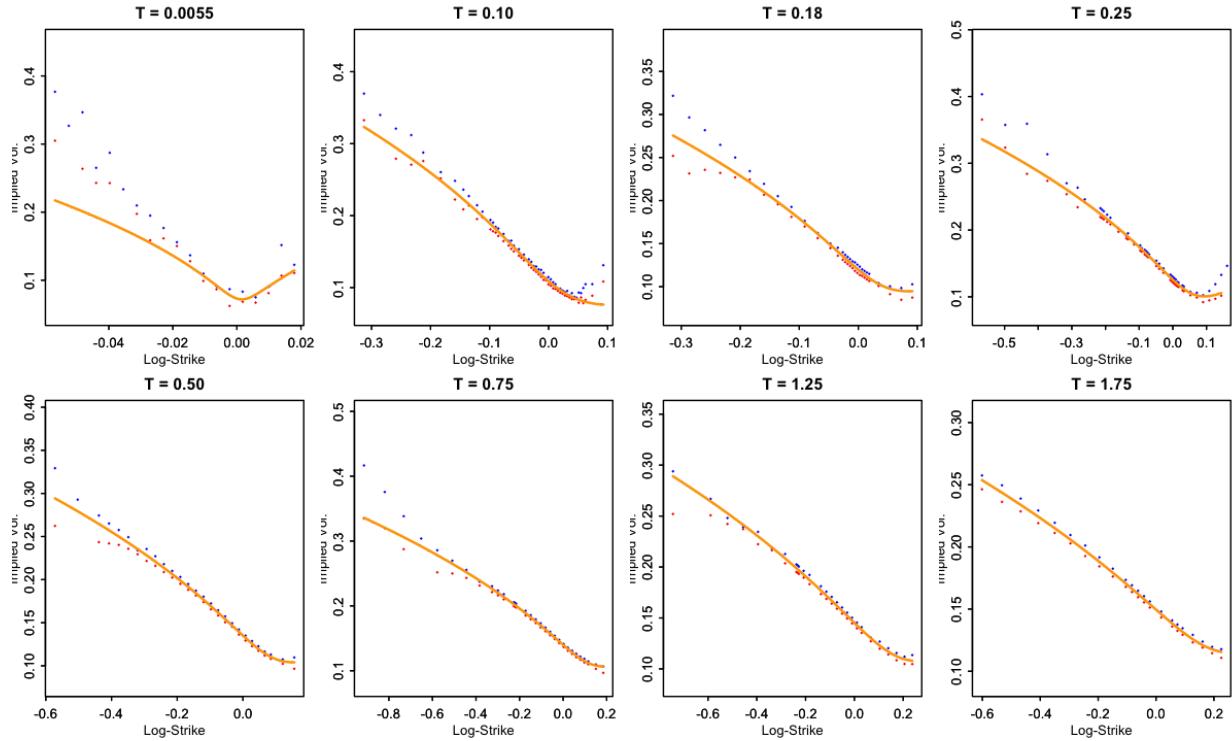


Figure 18: fitQR slice-by-slice fit quality

Let's check one of the slices:

```
In [48]: res <- plotIvols(spxOptData, fitQR, slices=4)
```

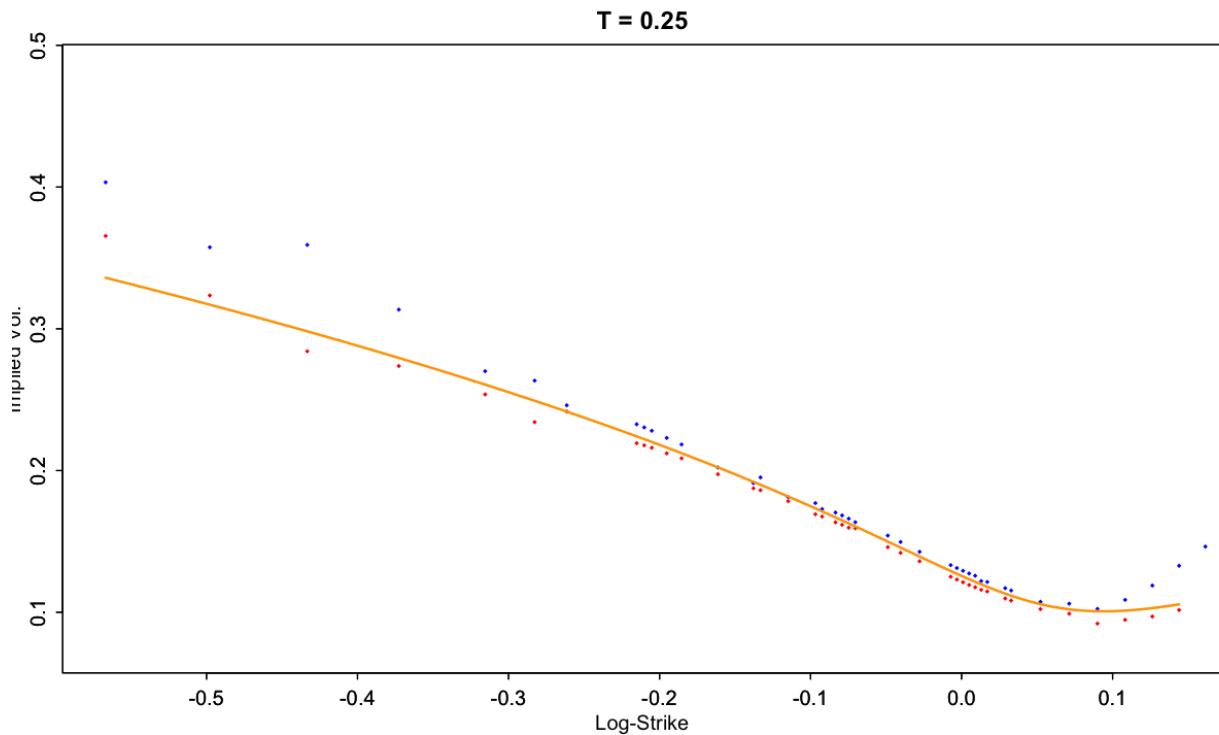


Figure 20: SPX Dec-2011 option quotes as of 3pm on 15-Sep-2011. Red triangles are bid implied

volatilities; blue triangles are offered implied volatilities; the orange solid line is the QR SVI fit (*fitQR*).

And now for the total variance plot:

```
In [49]: options(repr.plot.width=7,repr.plot.height=5)
```

```
In [50]: plotSVI(fitQR)
```

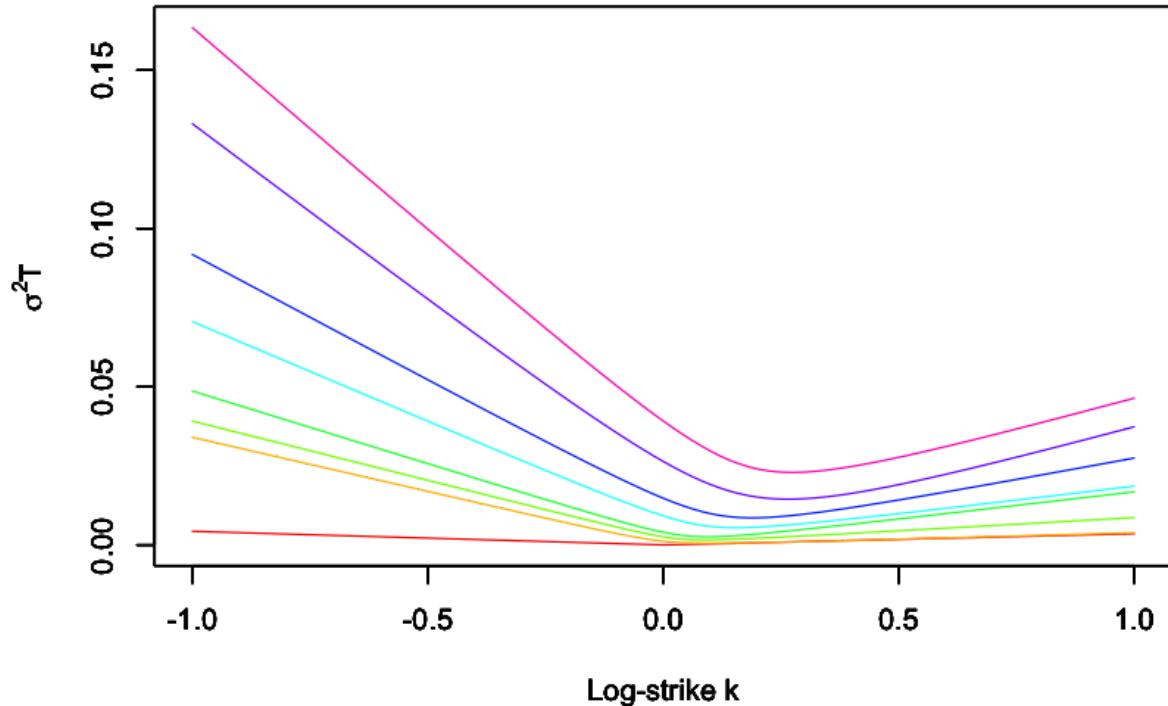


Figure 21: *fitQR* total variance plot: No calendar spread arbitrage!

However, it does look as if there are some very small local variances...

```
In [51]: plotSVI(fitQR,yrange=c(0,0.05))
```

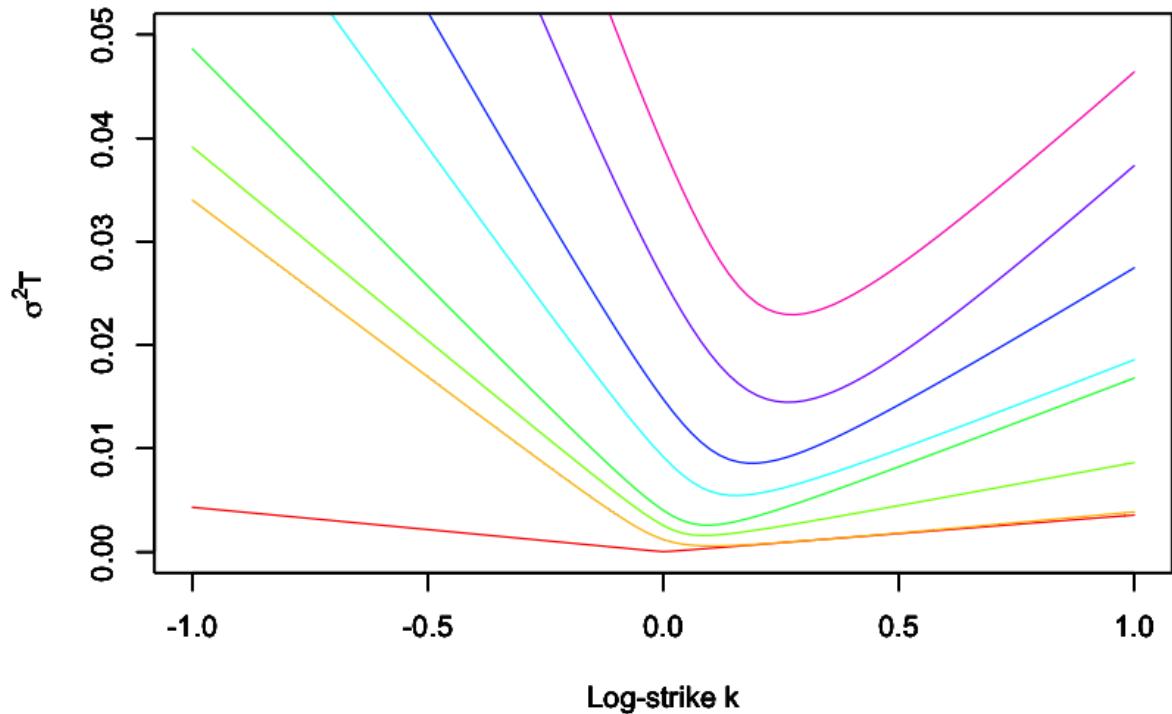


Figure 22: Closeup of the *fitQR* total variance plot: No calendar spread arbitrage, but very close...

Now we plot the empirical term structure of ATM volatility skew  $\partial_k \sigma_{\text{BS}}(k, T)|_{k=0}$

```
In [52]: plot(texp,res05$atmSkew,col="red",pch=20)
```

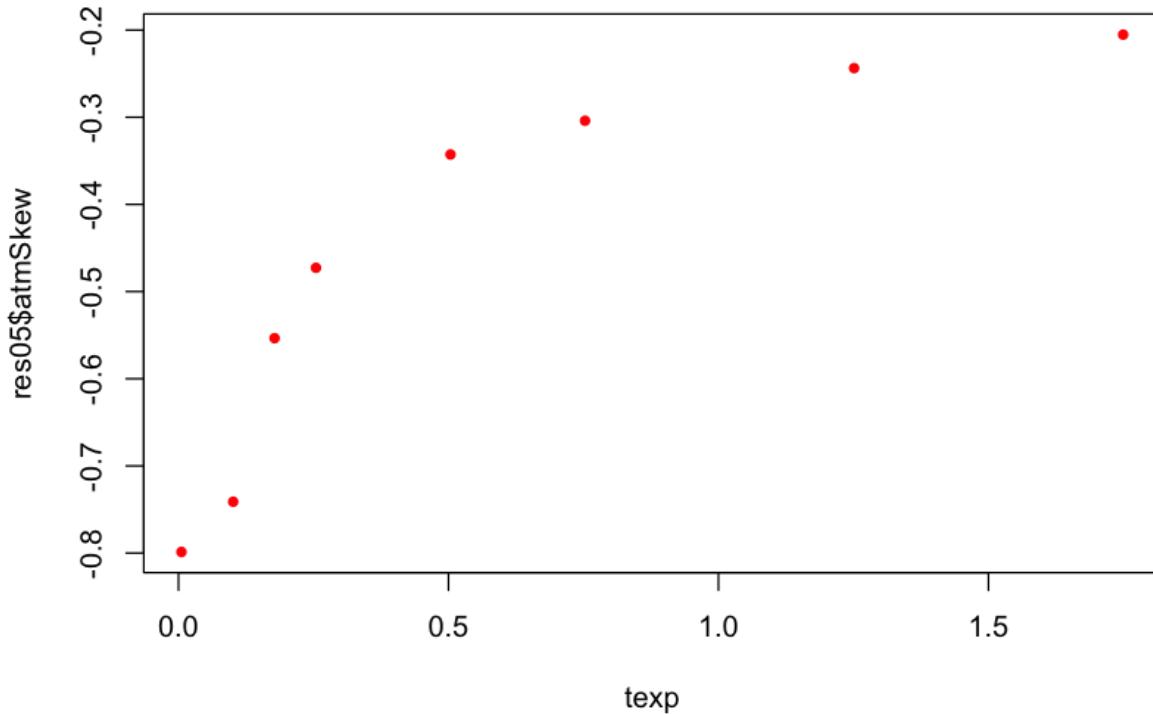


Figure 23: Empirical term structure of ATM volatility skew  $\partial_k \sigma_{\text{BS}}(k, T)|_{k=0}$

Now we check the various fitted surfaces for calendar spread arbitrage:

```
In [53]: print(sviCalendarArbitrageCheck(sviMatrix)) # The pre-cooked fit
print(sviCalendarArbitrageCheck(fit0)) # The fit slice-by-slice
print(sviCalendarArbitrageCheck(fitSqrt)) # The sqrt fit
print(sviCalendarArbitrageCheck(fitQR)) # The final optimized fit
```

```
[1] 0
[1] 0.1272942
[1] 0
[1] 0
```

and then for butterfly arbitrage:

```
In [54]: print(sviSliceArbitrageCheck(sviMatrix))
print(sviSliceArbitrageCheck(fit0))
print(sviSliceArbitrageCheck(fitSqrt))
print(sviSliceArbitrageCheck(fitQR))

numeric(0)
[1] 1
numeric(0)
numeric(0)
```

## Summary of the calibration recipe

- Given mid implied volatilities  $\sigma_{ij} = \sigma_{BS}(k_i, t_j)$ , compute mid option prices using the Black-Scholes formula.
- Fit the square-root SVI surface by minimizing sum of squared distances between the fitted prices and the mid option prices. This is now the initial guess.
- Starting with the square-root SVI initial guess, change SVI parameters slice-by slice so as to minimize the sum of squared distances between the fitted prices and the mid option prices with a big penalty for crossing either the previous slice or the next slice (as quantified by the crossedness from [Definition 4.4](#)).

Finally, let's compare the pre-cooked surface with *fitQR*:

```
In [55]: volTVS <- function(k,t){sqrt(sviW(sviMatrix,texp,k,t)/t)}

# 3D plot of TVS volatility surface
  k <- seq(-.5,.5,0.01) # Vector of log-strikes
  t <- seq(0.04,1.74,0.02) # Vector of times
  z <- t(volTVS(k,t)); # Array of volatilities

  # Add colors
  nbcoll <- 100;
  color <- rainbow(nbcoll,start=.3,end=.5);
  nrz <- nrow(z)
  nc <- ncol(z)
  # Compute the z-value at the facet centres
  zfacet <- z[-1, -1] + z[-1, -nc] + z[-nrz, -1] + z[-nrz, -nc]
  # Recode facet z-values into color indices
  facetcol <- cut(zfacet, nbcoll);
```

```
In [56]: options(repr.plot.width=14,repr.plot.height=10)
```

In [57]:

```
# Generate 3D plot
persp(k, t, z, col=color[facetcol], phi=30, theta=30,
      r=1/sqrt(3)*20,d=5,expand=.5,ltheta=-135,lphi=20,ticktype="simple",
      shade=.8,border=NA,xlab="Log-strike k",ylab="Expiration t",zlab="Imp."
```

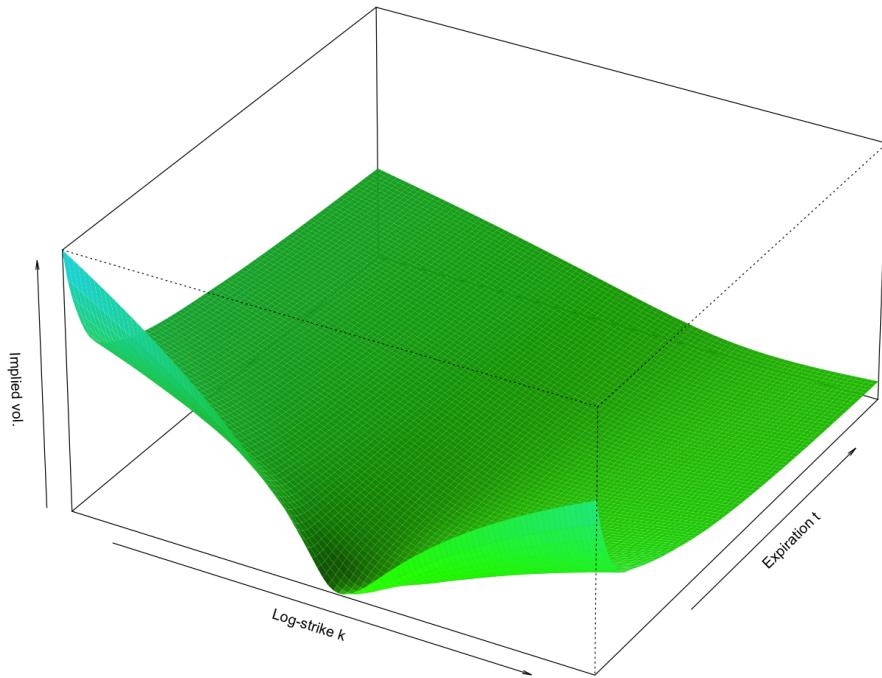


Figure 24: 3D plot of the pre-cooked SVI fit to the SPX implied volatilities as of September 15, 2005

In [58]:

```
volTVS <- function(k,t){sqrt(sviW(fitQR,texp,k,t)/t)}

# 3D plot of TVS volatility surface
k <- seq(-.5,.5,0.01) # Vector of log-strikes
t <- seq(0.04,1.74,0.02) # Vector of times
z <- t(volTVS(k,t)); # Array of volatilities

# Add colors
nbcoll <- 100;
color <- rainbow(nbcoll,start=.3,end=.5);
nrz <- nrow(z)
ncz <- ncol(z)
# Compute the z-value at the facet centres
zfacet <- z[-1, -1] + z[-1, -ncz] + z[-nrz, -1] + z[-nrz, -ncz]
# Recode facet z-values into color indices
facetcol <- cut(zfacet, nbcoll);
```

In [59]:

```
# Generate 3D plot
persp(k, t, z, col=color[facetcol], phi=30, theta=30,
      r=1/sqrt(3)*20,d=5,expand=.5,ltheta=-135,lphi=20,ticktype="simple",
      shade=.8,border=NA,xlab="Log-strike k",ylab="Expiration t",zlab="Imp."
```

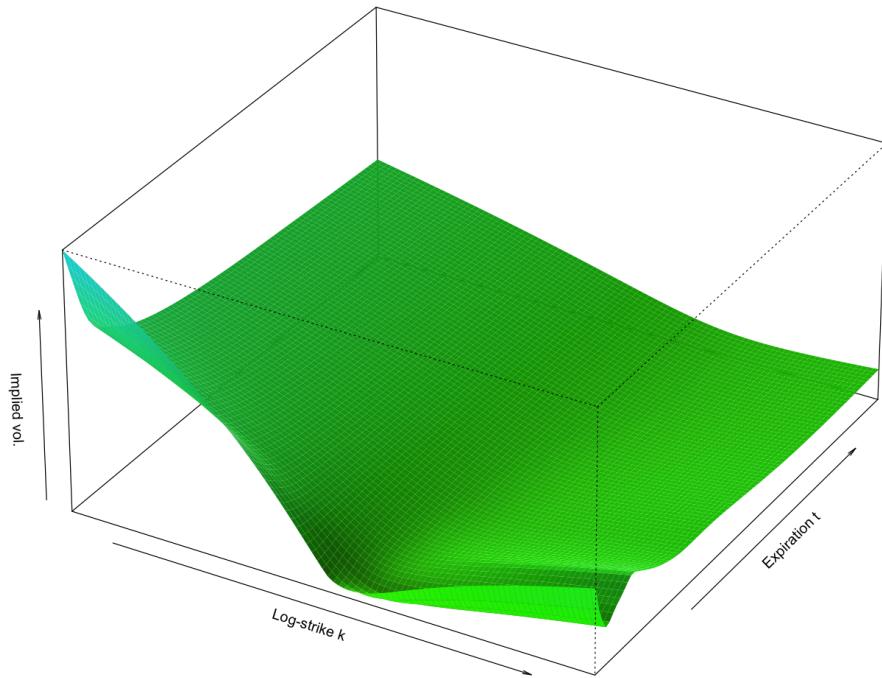


Figure 25: 3D plot of the QR fit to the SPX implied volatilities as of September 15, 2005

Note that the QR fit looks almost as good as the pre-cooked one.

### Choice of objective function

- A 0.1% volatility error on an ATM option is a much bigger pricing error than on a very far out-of-the-money option.
- Practitioners take this into consideration by vega-weighting their fits.
- From the Black-Scholes formula:

$$\text{vega} = \sqrt{T} K \frac{e^{-d_2^2/2}}{\sqrt{2\pi}} = \sqrt{T} F \frac{e^{-d_1^2/2}}{\sqrt{2\pi}}$$

- Alternatively, weight by the reciprocal of the bid-ask volatility spread.

**Now we repeat the whole calibration recipe for 15-Sep-2011**

In [60]: `head(spxData110915)`

	<b>secid</b>	<b>date</b>	<b>symbol</b>	<b>symbol_flag</b>	<b>exdate</b>	<b>last_date</b>	<b>cp_flag</b>	<b>st</b>
<b>2559920</b>	108105	20110915	SPX 110917C100000	1	20110917	20110815	C	10
<b>2559921</b>	108105	20110915	SPX 110917C1000000	1	20110917	20110915	C	10
<b>2559922</b>	108105	20110915	SPX 110917C1005000	1	20110917	20110829	C	10
<b>2559923</b>	108105	20110915	SPX 110917C1010000	1	20110917	20110829	C	10
<b>2559924</b>	108105	20110915	SPX 110917C1015000	1	20110917	20110912	C	10
<b>2559925</b>	108105	20110915	SPX 110917C1020000	1	20110917	20110909	C	10

In [61]: `spxData110915$strike_price <- spxData110915$strike_price/1000  
spxIvols110915 <- generateOptionMetricsIvols(spxData110915)  
fitSqrt110915 <- sviSqrtFit(spxIvols110915)  
fitQR110915 <- sviFitQR(spxIvols110915,fitSqrt110915)`

In [62]: `options(repr.plot.height=9)`

```
In [63]: res <- plotIvols(spxIvols110915, fitQR110915)
```

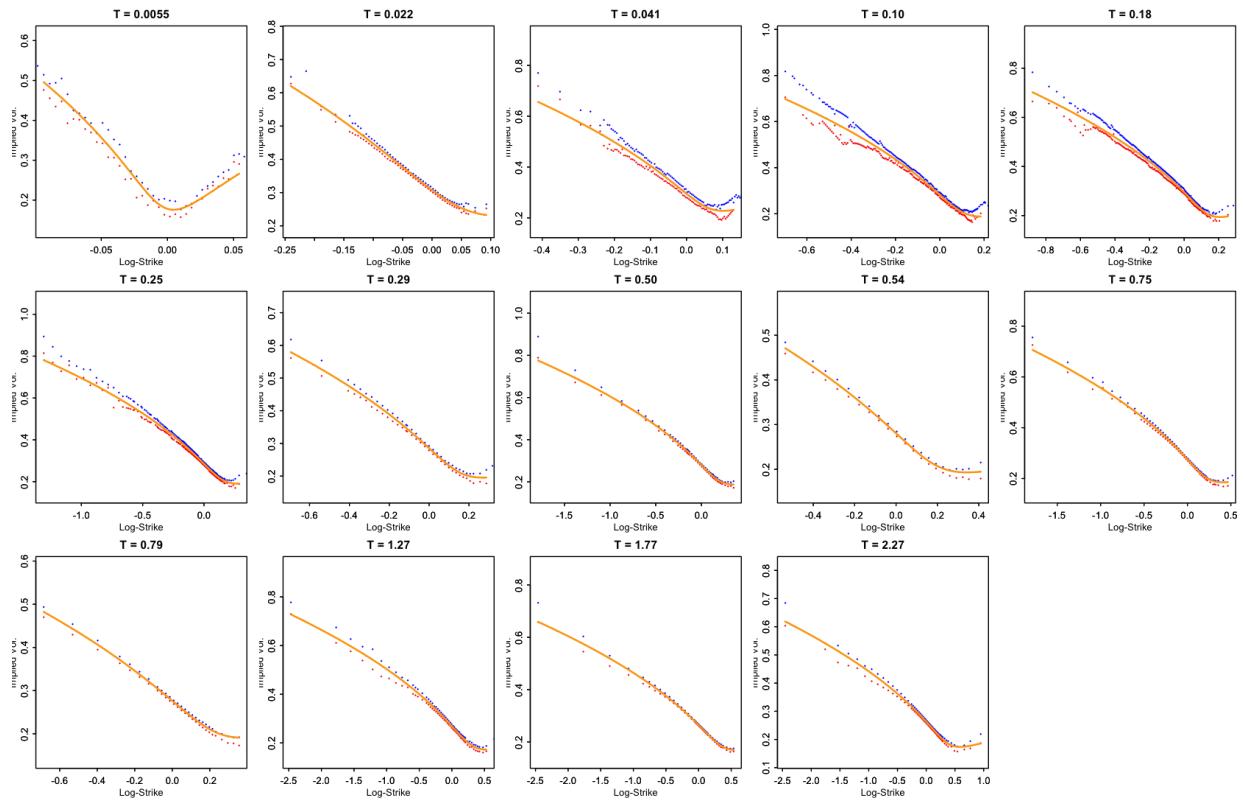


Figure 26: SPX option quotes as of 3pm on 15-Sep-2011. Red triangles are bid implied volatilities; blue triangles are offered implied volatilities; the orange solid line is the quartic roots (QR) SVI fit.

### SVI square-root calibration: December 2011 detail

```
In [64]: options(repr.plot.width=7,repr.plot.height=5)
```

```
In [65]: foo <- plotIvols(spxIvols110915,fitQR110915,slices=7)
```

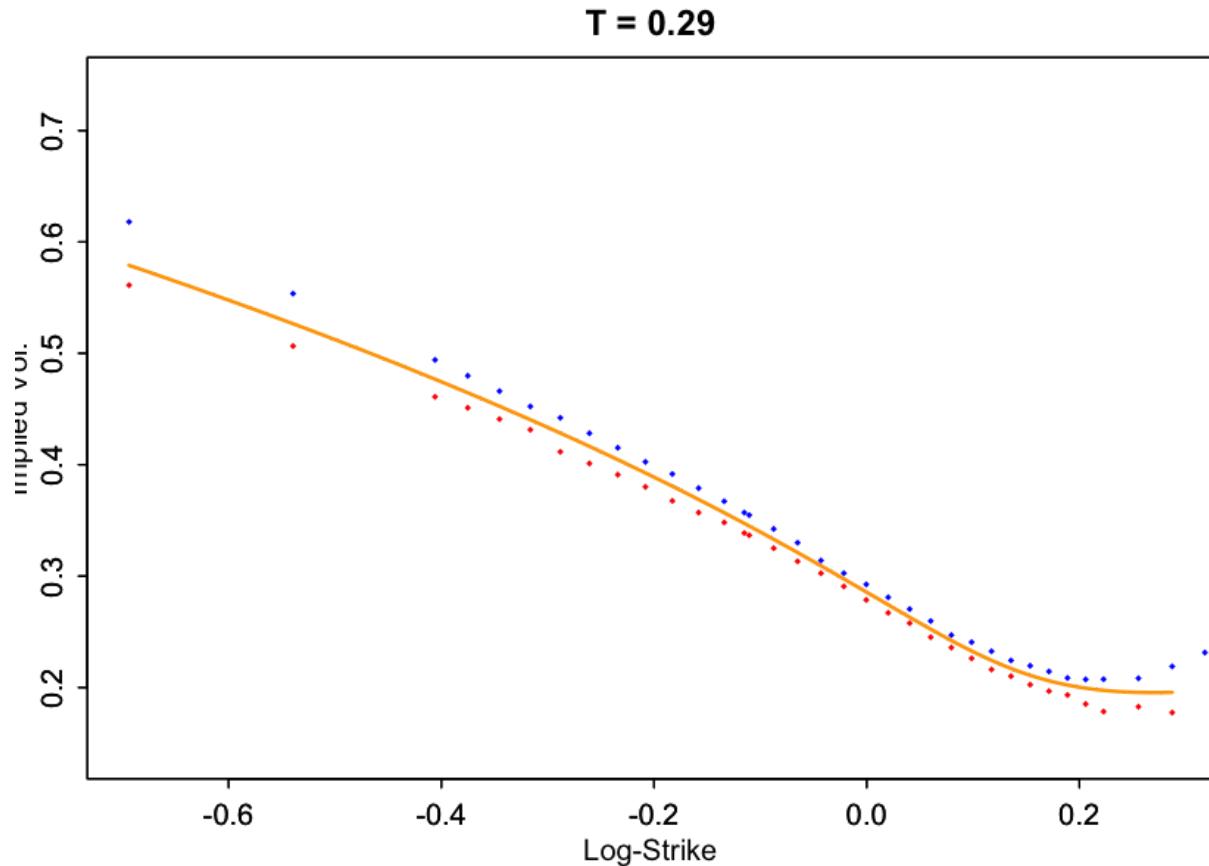


Figure 27: SPX Dec-2011 option quotes as of 3pm on 15-Sep-2011. Red triangles are bid implied volatilities; blue triangles are offered implied volatilities; the orange solid line is the quartic roots SVI fit

```
In [66]: # 15-Sep-2011 SPX smile to Mar-2011
foo <- plotIvols(spxIvols110915, fitQR110915, slices=8)
```

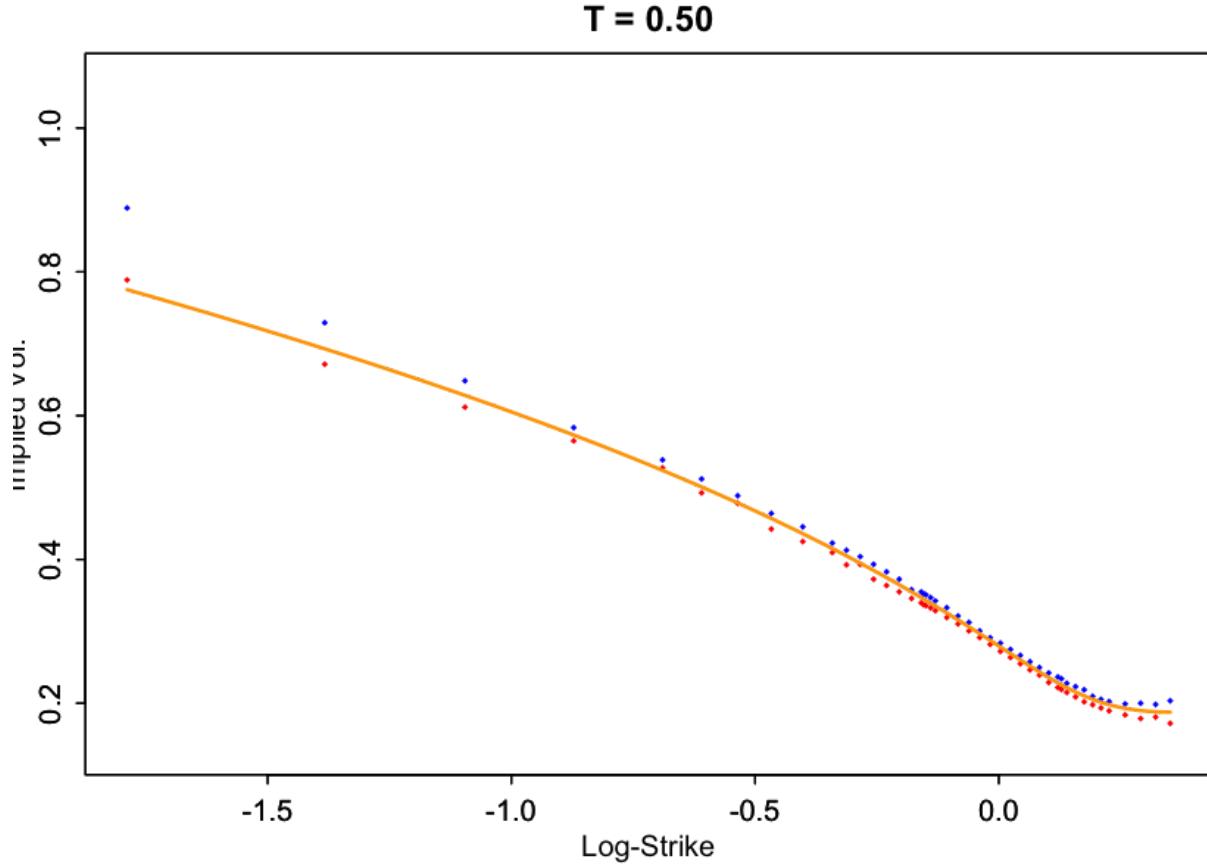


Figure 28: SPX Mar-2012 option quotes as of the close on 15-Sep-2011. Red triangles are bid implied volatilities; blue triangles are offered implied volatilities; the orange solid line is the SVI fit

## Now we repeat the whole calibration recipe for 19-Aug-2015

```
In [67]: spxData150819$strike_price <- spxData150819$strike_price/1000
spxIvols150819 <- generateOptionMetricsIvols(spxData150819)
```

```
In [68]: fitSqrt150819 <- sviSqrtFit(spxIvols150819)
fitQR150819 <- sviFitQR(spxIvols150819, fitSqrt150819)
```

```
In [69]: options(repr.plot.width=14, repr.plot.height=9)
```

```
In [70]: res <- plotIvols(spxIvols150819, fitQR150819)
```

Warning message:

In sqrt(svi(sviMatrix[slice, ], k)/t): NaNs produced

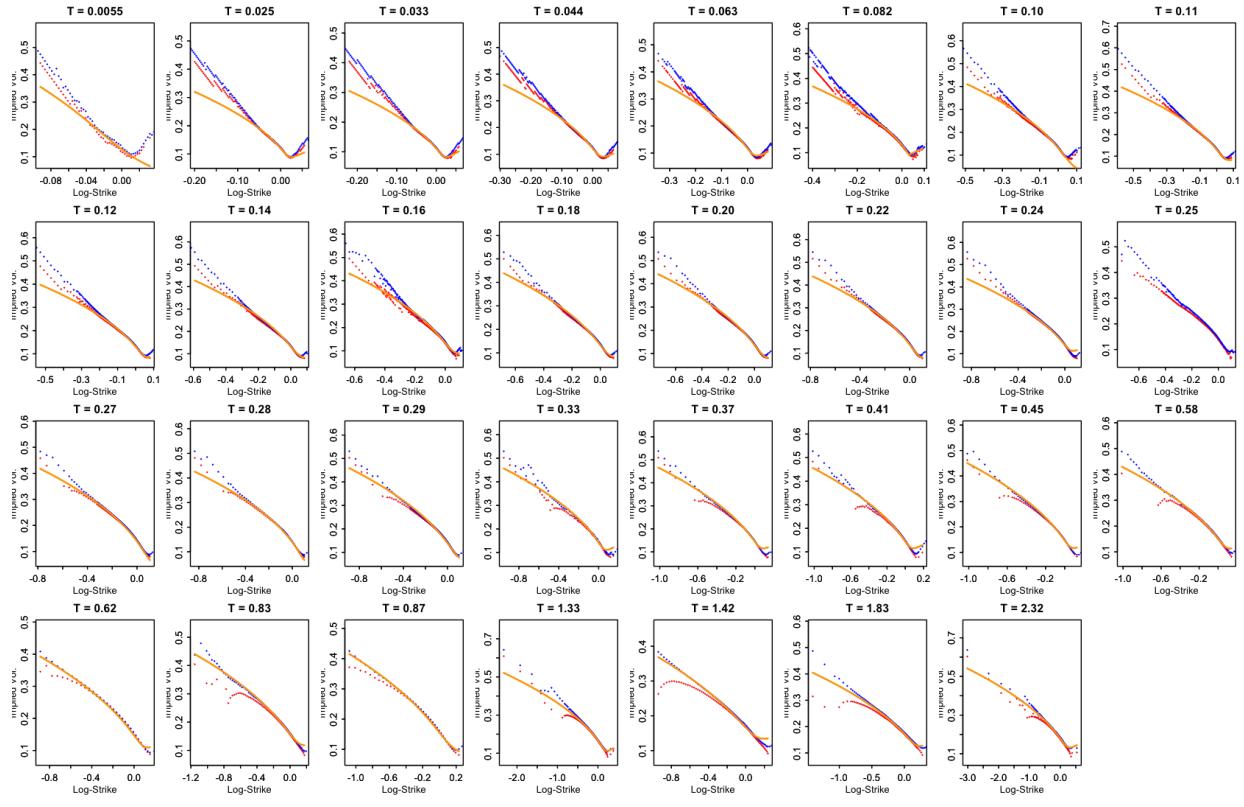


Figure 29: SPX option quotes as of the close on 19-Aug-2015. Red triangles are bid implied volatilities; blue triangles are offered implied volatilities; the orange solid line is the quartic roots (QR) SVI fit.

## SVI fit: August 2015 detail

```
In [71]: options(repr.plot.width=7, repr.plot.height=5)
```

```
In [72]: foo <- plotIvols(spxIvols150819,fitSqrt150819,slices=6)
```

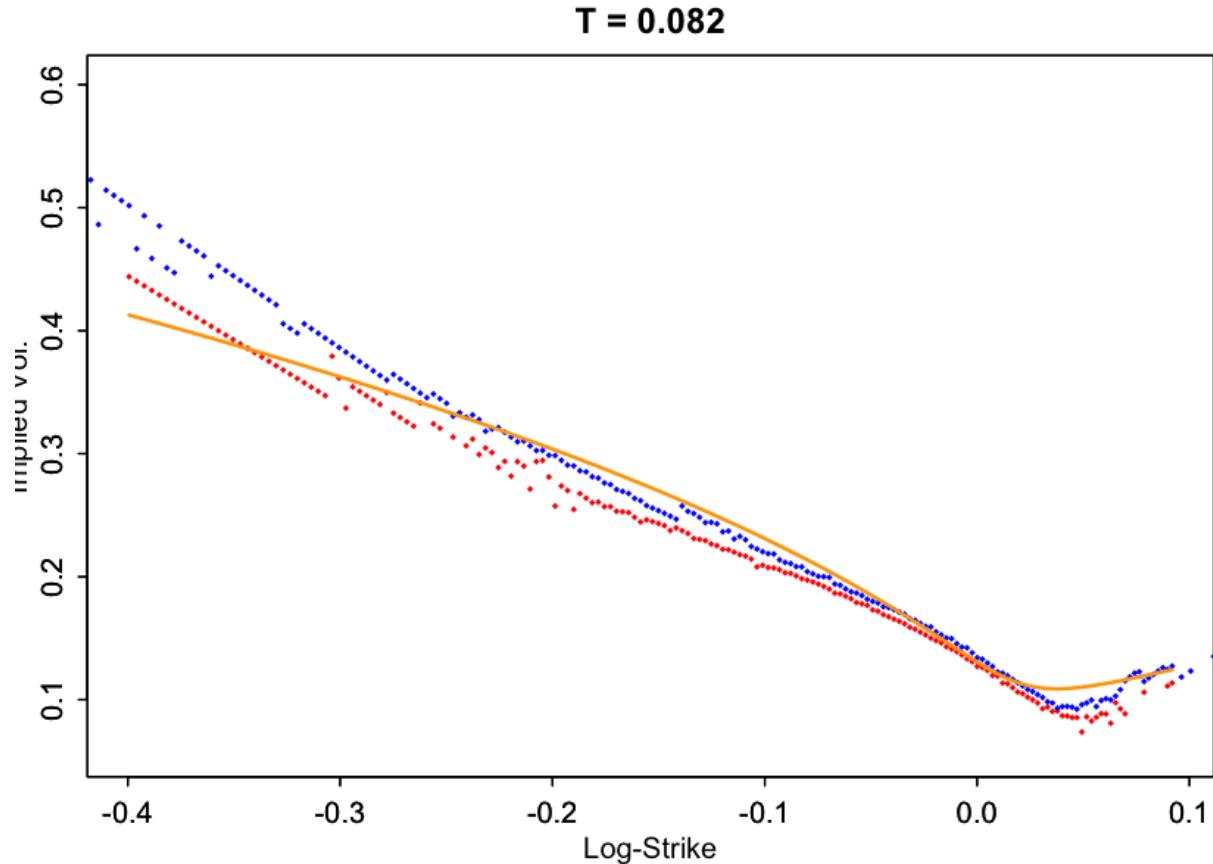


Figure 30: SPX Sep-2015 option quotes as of the close on 19-Aug-2015. Red triangles are bid implied volatilities; blue triangles are offered implied volatilities; the orange solid line is the quartic roots SVI fit

```
In [73]: foo <- plotIvols(spxIvols150819,fitSqrt150819,slices=20)
```

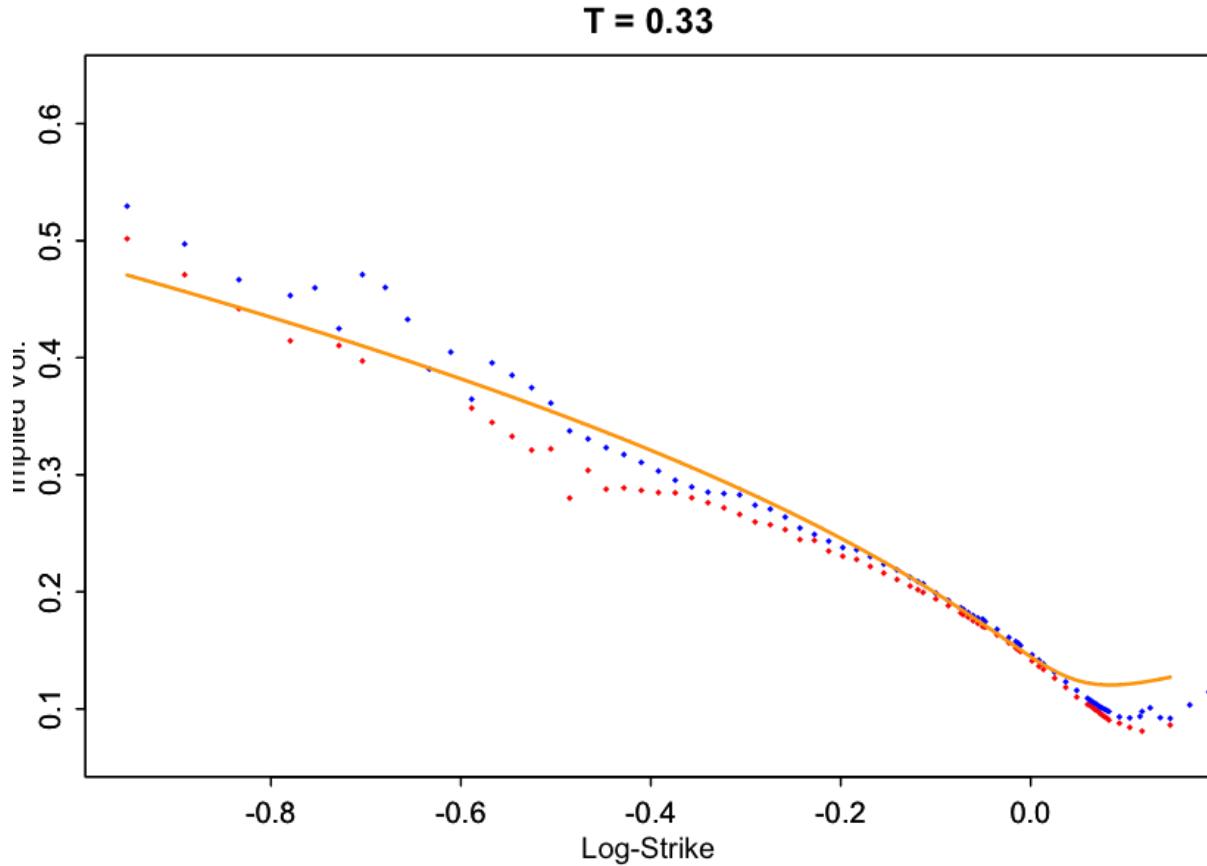


Figure 31: SPX Dec-2015 option quotes as of the close on 19-Aug-2015. Red triangles are bid implied volatilities; blue triangles are offered implied volatilities; the orange solid line is the quartic roots SVI fit

```
In [74]: vol150819 <- function(k,t){sqrt(sviW(fitSqrt150819,res$expiries,k,t)/t)}

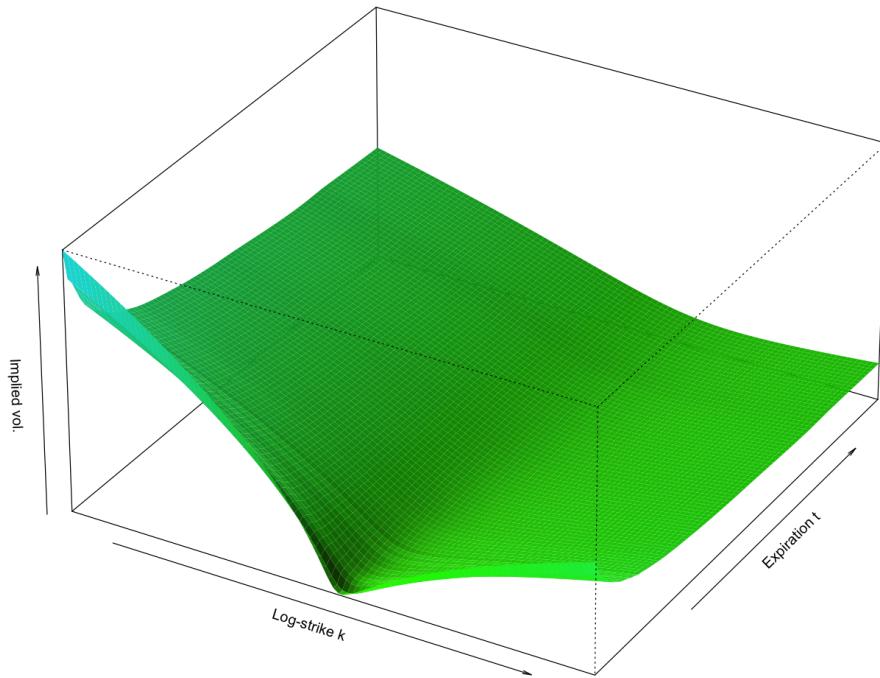
# 3D plot of TVS volatility surface
k <- seq(-.5,.5,0.01) # Vector of log-strikes
t <- seq(0.04,1.74,0.02) # Vector of times
z <- t(vol150819(k,t)); # Array of volatilities

# Add colors
nbcoll <- 100;
color <- rainbow(nbcoll,start=.3,end=.5);
nrz <- nrow(z)
ncz <- ncol(z)
# Compute the z-value at the facet centres
zfacet <- z[-1, -1] + z[-1, -ncz] + z[-nrz, -1] + z[-nrz, -ncz]
# Recode facet z-values into color indices
facetcol <- cut(zfacet, nbcoll);
```

```
In [75]: options(repr.plot.width=14,repr.plot.height=10)
```

In [76]:

```
# Generate 3D plot
persp(k, t, z, col=color[facetcol], phi=30, theta=30,
      r=1/sqrt(3)*20,d=5,expand=.5,ltheta=-135,lphi=20,ticktype="simple",
      shade=.8,border=NA,xlab="Log-strike k",ylab="Expiration t",zlab="Imp."
```



## Historical success of the SVI parameterization

- There is no reason in principle why a simple parameterization like SVI should fit market smiles.
- However, we see above that the fit quality in 2011 was amazing.
  - Could it be that market makers were using SVI to make markets?
- Now however, as option markets have got tighter, we see that SVI no longer fits as well.
  - It's still the only guaranteed arbitrage-free parameterization in the public domain.
  - And it's still good as a low-dimensional approximation to market smiles.
  - But it's no longer good enough for market making.

## Postscript

You now have code to generate an SVI fit to market European option prices in OptionMetrics format or alternatively, implied volatilities in standard format.

## References

1.  $\Delta$  Shalom Benaim and Peter Friz, Regular variation and smile asymptotics, *Mathematical Finance* **19**(1), 1–12, (2009).
2.  $\Delta$  Gerolamo Cardano, *Ars magna or The Rules of Algebra*, Dover (1545).
3.  $\Delta$  Jim Gatheral, *The Volatility Surface: A Practitioner's Guide*, John Wiley and Sons, Hoboken, NJ (2006).
4.  $\Delta$  Jim Gatheral and Antoine Jacquier, Convergence of Heston to SVI, *Quantitative Finance* **11**(8) 1129–1132 (2011).
5.  $\Delta$  Jim Gatheral and Antoine Jacquier, Arbitrage-free SVI volatility surfaces, *Quantitative Finance* **14**(1) 59-71 (2014).
6.  $\Delta$  Sebas Hendriks and Claude Martini, The extended SSVI volatility surface, available at [\(https://papers.ssrn.com/sol3/papers.cfm?abstract\\_id=2971502\)](https://papers.ssrn.com/sol3/papers.cfm?abstract_id=2971502) (2017).
7.  $\Delta$  Roger Lee, The moment formula for implied volatility at extreme strikes, *Mathematical Finance*, **14**(3), 469–480 (2004).

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