

Supplementary File of Symbolic Representation and Toolkit Development of Iterated Error-state Extended Kalman Filters on Manifolds

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I. EXAMPLES FOR CASTING DIFFERENT ROBOTIC SYSTEMS INTO THE CANONICAL FORM

The canonical form shown in the paper is constructed through compactly representing the system model using the \oplus operation of the respective manifolds. In the following, we show how to incorporate the \oplus operation and cast a canonical form for different state components. Then with the composition property as stated in the paper, the complete state equation can be obtained by concatenating all components.

Example 1: Vectors in Euclidean space (e.g., position and velocity). Assume $\mathbf{x} \in \mathbb{R}^n$ subject to $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}, \mathbf{w})$. Using zero-order hold discretization, $\mathbf{f}(\mathbf{x}, \mathbf{u}, \mathbf{w})$ is assumed constant during the sampling period Δt , hence

$$\begin{aligned}\mathbf{x}_{k+1} &= \mathbf{x}_k + (\Delta t \mathbf{f}(\mathbf{x}_k, \mathbf{u}_k, \mathbf{w}_k)) \\ &= \mathbf{x}_k \oplus_{\mathbb{R}^n} (\Delta t \mathbf{f}(\mathbf{x}_k, \mathbf{u}_k, \mathbf{w}_k)).\end{aligned}\quad (1)$$

Example 2: Attitude kinematics in a global reference frame (e.g., the earth-frame). Let $\mathbf{x} \in SO(3)$ be the body attitude relative to the global frame and ${}^G\boldsymbol{\omega}$ be the global angular velocity which holds constant for one sampling period Δt , then

$$\begin{aligned}\dot{\mathbf{x}} &= [{}^G\boldsymbol{\omega}] \cdot \mathbf{x} \implies \mathbf{x}_{k+1} = \exp([\Delta t {}^G\boldsymbol{\omega}_k]) \cdot \mathbf{x}_k = \mathbf{x}_k \\ &\quad \cdot \exp([\Delta t (\mathbf{x}_k^T \cdot {}^G\boldsymbol{\omega}_k)]) = \mathbf{x}_k \oplus_{SO(3)} (\Delta t \mathbf{f}(\mathbf{x}_k, {}^G\boldsymbol{\omega}_k)), \\ &\quad \mathbf{f}(\mathbf{x}_k, {}^G\boldsymbol{\omega}_k) = \mathbf{x}_k^T \cdot {}^G\boldsymbol{\omega}_k.\end{aligned}\quad (2)$$

where $[w]$ is a skew-symmetric matrix mapping the cross product of $w \in \mathbb{R}^3$, and $\exp([w])$, the exponential operation of matrix $[w]$, denotes a rotation about the vector w .

Example 3: Attitude kinematics in body frame. Let $\mathbf{x} \in SO(3)$ be the body attitude relative to the global frame and ${}^B\boldsymbol{\omega}$ be the body angular velocity which holds constant for one sampling period Δt , then

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{x} \cdot [{}^B\boldsymbol{\omega}] \implies \mathbf{x}_{k+1} = \mathbf{x}_k \cdot \exp([\Delta t {}^B\boldsymbol{\omega}_k]) \\ &= \mathbf{x}_k \oplus_{SO(3)} (\Delta t \mathbf{f}({}^B\boldsymbol{\omega}_k)), \quad \mathbf{f}({}^B\boldsymbol{\omega}_k) = {}^B\boldsymbol{\omega}_k.\end{aligned}\quad (3)$$

Example 4: Vectors of known magnitude (e.g., gravity) in the global frame. Let $\mathbf{x} \in \mathbb{S}^2(g)$ be the gravity vector in the global frame with known magnitude g . Then,

$$\dot{\mathbf{x}} = \mathbf{0} \implies \mathbf{x}_{k+1} = \mathbf{x}_k = \mathbf{x}_k \oplus_{\mathbb{S}^2(g)} (\Delta t \mathbf{f}(\mathbf{x}_k)), \quad \mathbf{f}(\mathbf{x}_k) = \mathbf{0}. \quad (4)$$

Example 5: Vectors of known magnitude (e.g., gravity) in body frame. Let $\mathbf{x} \in \mathbb{S}^2(g)$ be the gravity vector in the

body frame and ${}^B\boldsymbol{\omega}$ be the body angular velocity which holds constant for one sampling period Δt . Then,

$$\begin{aligned}\dot{\mathbf{x}} &= -[{}^B\boldsymbol{\omega}] \mathbf{x} \implies \mathbf{x}_{k+1} = \exp(-[\Delta t {}^B\boldsymbol{\omega}_k]) \mathbf{x}_k \\ &= \mathbf{x}_k \oplus_{\mathbb{S}^2(g)} (\Delta t \mathbf{f}({}^B\boldsymbol{\omega}_k)), \quad \mathbf{f}({}^B\boldsymbol{\omega}_k) = -{}^B\boldsymbol{\omega}_k.\end{aligned}\quad (5)$$

Example 6: Bearing-distance parameterization of visual landmarks [1]. Let $\mathbf{x} \in \mathbb{S}^2(1)$ and $d(\rho) \in \mathbb{R}$ be the bearing vector and depth (with parameter ρ), respectively, of a visual landmark, and ${}^G\mathbf{R}_C, {}^G\mathbf{p}_C$ be the attitude and position of the camera. Then the visual landmark in the global frame is ${}^G\mathbf{R}_C(\mathbf{x}d(\rho)) + {}^G\mathbf{p}_C$, which is constant over time:

$$\begin{aligned}\frac{d({}^G\mathbf{R}_C(\mathbf{x}d(\rho)) + {}^G\mathbf{p}_C)}{dt} &= \mathbf{0} \implies \\ [{}^C\boldsymbol{\omega}] (\mathbf{x}d(\rho)) + \dot{\mathbf{x}}d(\rho) + \mathbf{x}d'(\rho)\dot{\rho} + {}^C\mathbf{v} &= \mathbf{0}.\end{aligned}\quad (6)$$

Left multiplying (6) by \mathbf{x}^T and using $\mathbf{x}^T \dot{\mathbf{x}} = 0$ yield $\dot{\rho} = -\mathbf{x}^T \cdot {}^C\mathbf{v} / d'(\rho)$. Substituting this to (6) leads to

$$\begin{aligned}\dot{\mathbf{x}} &= -[{}^C\boldsymbol{\omega} + \frac{1}{d(\rho)} [\mathbf{x}] \cdot {}^C\mathbf{v}] \cdot \mathbf{x} \implies \\ \mathbf{x}_{k+1} &= \exp\left([- \Delta t \left({}^C\boldsymbol{\omega}_k + \frac{1}{d(\rho)} [\mathbf{x}_k] \cdot {}^C\mathbf{v}_k\right)]\right) \mathbf{x}_k \\ &= \mathbf{x}_k \oplus_{\mathbb{S}^2(1)} (\Delta t \mathbf{f}(\mathbf{x}_k, {}^C\boldsymbol{\omega}_k, {}^C\mathbf{v}_k)), \\ \mathbf{f}(\mathbf{x}_k, {}^C\boldsymbol{\omega}_k, {}^C\mathbf{v}_k) &= -{}^C\boldsymbol{\omega}_k - \frac{1}{d(\rho)} [\mathbf{x}_k] \cdot {}^C\mathbf{v}_k.\end{aligned}\quad (7)$$

where ${}^C\boldsymbol{\omega} + \frac{1}{d(\rho)} [\mathbf{x}] \cdot {}^C\mathbf{v}$ is assumed constant for one sampling period Δt due to the zero-order hold assumption.

II. PARTIAL DIFFERENTIATIONS OF MATRIX LIE GROUPS

As shown in the paper, \boxplus and \oplus of the matrix Lie group could be defined as:

$$\mathbf{x} \boxplus \mathbf{u} = \mathbf{x} \cdot \text{Exp}(\mathbf{u}) \text{ or } \text{Exp}(\mathbf{u}) \cdot \mathbf{x} \quad (8)$$

$$\mathbf{y} \boxminus \mathbf{x} = \text{Log}(\mathbf{x}^{-1} \cdot \mathbf{y}) \text{ or } \text{Log}(\mathbf{y} \cdot \mathbf{x}^{-1}) \quad (9)$$

$$\mathbf{x} \oplus \mathbf{v} = \mathbf{x} \cdot \text{Exp}(\mathbf{v}) \text{ or } \text{Exp}(\mathbf{v}) \cdot \mathbf{x} \quad (10)$$

where two choices of operation definition are shown and \oplus is defined the same as \boxplus . Exp is a map from \mathbb{R}^n to the matrix Lie group, with its inverse as Log . Denote \mathbf{J}_r and \mathbf{J}_l as the right and left Jacobian matrix of the matrix lie group respectively, which are defined as [2]:

$$\begin{aligned}\text{Exp}(\boldsymbol{\tau} + \delta\boldsymbol{\tau}) &= \text{Exp}(\boldsymbol{\tau}) \text{Exp}(\mathbf{J}_r(\boldsymbol{\tau})\delta\boldsymbol{\tau}) \\ \text{Exp}(\boldsymbol{\tau} + \delta\boldsymbol{\tau}) &= \text{Exp}(\mathbf{J}_l(\boldsymbol{\tau})\delta\boldsymbol{\tau}) \text{Exp}(\boldsymbol{\tau})\end{aligned}\quad (11)$$

where $\mathbf{J}_l(\boldsymbol{\tau}) = \mathbf{J}_r(-\boldsymbol{\tau})$ as proved in [2].

The partial differentiations of $((\mathbf{x} \boxplus \mathbf{u}) \oplus \mathbf{v}) \boxminus \mathbf{y}$ w.r.t. \mathbf{u} and \mathbf{v} are directly calculated without using the chain rule. Here we only derive differentiations of matrix Lie groups whose \boxplus and \oplus are both defined as right multiplication, i.e., $\mathbf{x} \boxplus \mathbf{u} =$

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$\mathbf{x} \oplus \mathbf{u} = \mathbf{x} \cdot \text{Exp}(\mathbf{u})$, and the counterparts for left multiplication could be derived using the same procedure.

Firstly, we prove the partial differentiations of $(\mathbf{x} \boxplus \mathbf{u}) \boxminus \mathbf{y}$ w.r.t \mathbf{u} . Denote $\mathbf{z} = (\mathbf{x} \boxplus \mathbf{u}) \boxminus \mathbf{y}$, we have:

$$\mathbf{y} \cdot \text{Exp}(\mathbf{z}) = \mathbf{x} \cdot \text{Exp}(\mathbf{u}) \quad (12)$$

Hence a small variation $\Delta \mathbf{u}$ in \mathbf{u} causes a small variation $\Delta \mathbf{z}$ in \mathbf{z} , which is subject to:

$$\mathbf{y} \cdot \text{Exp}(\mathbf{z} + \Delta \mathbf{z}) = \mathbf{x} \cdot \text{Exp}(\mathbf{u} + \Delta \mathbf{u}) \quad (13)$$

Using the fact $\text{Exp}(\mathbf{u} + \Delta \mathbf{u}) = \text{Exp}(\mathbf{u}) \cdot \text{Exp}(\mathbf{J}_r(\mathbf{u}) \Delta \mathbf{u})$ as shown in [2], it is derived that the left hand side of (13):

$$\mathbf{y} \cdot \text{Exp}(\mathbf{z} + \Delta \mathbf{z}) = \mathbf{y} \cdot \text{Exp}(\mathbf{z}) \cdot \text{Exp}(\mathbf{J}_r(\mathbf{z}) \Delta \mathbf{z}) \quad (14)$$

and the right hand side of (13)

$$\mathbf{x} \cdot \text{Exp}(\mathbf{u} + \Delta \mathbf{u}) = \mathbf{x} \cdot \text{Exp}(\mathbf{u}) \cdot \text{Exp}(\mathbf{J}_r(\mathbf{u}) \Delta \mathbf{u}) \quad (15)$$

Equating two sides of (13) leads to

$$\mathbf{J}_r(\mathbf{z}) \Delta \mathbf{z} = \mathbf{J}_r(\mathbf{u}) \Delta \mathbf{u} \quad (16)$$

and as a result

$$\frac{\partial((\mathbf{x} \boxplus \mathbf{u}) \boxminus \mathbf{y})}{\partial \mathbf{u}} = \frac{\Delta \mathbf{z}}{\Delta \mathbf{u}} = (\mathbf{J}_r(\mathbf{z}))^{-1} \cdot \mathbf{J}_r(\mathbf{u}) \quad (17)$$

Afterwards, using the fact $\text{Exp}(\mathbf{u} + \Delta \mathbf{u}) = \text{Exp}(\mathbf{u}) \cdot \text{Exp}(\mathbf{J}_r(\mathbf{u}) \Delta \mathbf{u}) = \text{Exp}(\mathbf{J}_l(\mathbf{u}) \Delta \mathbf{u}) \cdot \text{Exp}(\mathbf{u})$, denote $\mathbf{w} = (\mathbf{x} \boxplus \mathbf{u}) \oplus \mathbf{v} \boxminus \mathbf{y}$, we have:

$$\begin{aligned} \mathbf{y} \cdot \text{Exp}(\mathbf{w}) &= \mathbf{x} \cdot \text{Exp}(\mathbf{u}) \cdot \text{Exp}(\mathbf{v}) \\ &= \mathbf{x} \cdot \text{Exp}(\mathbf{u} + (\mathbf{J}_r(\mathbf{u}))^{-1} \mathbf{v}) \\ &= \mathbf{x} \cdot \text{Exp}(\mathbf{v} + (\mathbf{J}_l(\mathbf{v}))^{-1} \mathbf{u}) \end{aligned} \quad (18)$$

Based on the result of (17), using the chain rule,

$$\begin{aligned} \frac{\partial(((\mathbf{x} \boxplus \mathbf{u}) \oplus \mathbf{v}) \boxminus \mathbf{y})}{\partial \mathbf{u}} &= (\mathbf{J}_r(\mathbf{w}))^{-1} \mathbf{J}_r(\mathbf{v} + (\mathbf{J}_l(\mathbf{v}))^{-1} \mathbf{u}) (\mathbf{J}_l(\mathbf{v}))^{-1} \\ \frac{\partial(((\mathbf{x} \boxplus \mathbf{u}) \oplus \mathbf{v}) \boxminus \mathbf{y})}{\partial \mathbf{v}} &= (\mathbf{J}_r(\mathbf{w}))^{-1} \mathbf{J}_r(\mathbf{u} + (\mathbf{J}_r(\mathbf{u}))^{-1} \mathbf{v}) (\mathbf{J}_r(\mathbf{u}))^{-1} \end{aligned} \quad (19)$$

III. PARTIAL DIFFERENTIATION OF compound manifolds

Lemma 1. If $\mathbf{x}_1, \mathbf{y}_1 \in \mathcal{M}_1$; $\mathbf{x}_2, \mathbf{y}_2 \in \mathcal{M}_2$; $\mathbf{u}_1 \in \mathbb{R}^{n_1}$ and $\mathbf{u}_2 \in \mathbb{R}^{n_2}$; where n_1, n_2 are dimensions of $\mathcal{M}_1, \mathcal{M}_2$ respectively, define compound manifold $\mathcal{M} = \mathcal{M}_1 \times \mathcal{M}_2$, and its elements $\mathbf{x} = [\mathbf{x}_1 \ \mathbf{x}_2]^T \in \mathcal{M}$; $\mathbf{y} = [\mathbf{y}_1 \ \mathbf{y}_2]^T \in \mathcal{M}$; $\mathbf{u} = [\mathbf{u}_1 \ \mathbf{u}_2]^T \in \mathbb{R}^{n_1+n_2}$ and $\mathbf{v}_1 \in \mathbb{R}^{l_1}$, $\mathbf{v}_2 \in \mathbb{R}^{l_2}$, $\mathbf{v} = [\mathbf{v}_1 \ \mathbf{v}_2]^T \in \mathbb{R}^{l_1+l_2}$, then

$$\begin{aligned} \frac{\partial(((\mathbf{x} \boxplus \mathbf{u}) \oplus \mathbf{v}) \boxminus \mathbf{y})}{\partial \mathbf{u}} &= \begin{bmatrix} \frac{\partial(((\mathbf{x}_1 \boxplus \mathbf{u}_1) \oplus \mathbf{v}_1) \boxminus \mathbf{y}_1)}{\partial \mathbf{u}_1} & \mathbf{0} \\ \mathbf{0} & \frac{\partial(((\mathbf{x}_2 \boxplus \mathbf{u}_2) \oplus \mathbf{v}_2) \boxminus \mathbf{y}_2)}{\partial \mathbf{u}_2} \end{bmatrix} \end{aligned}$$

Proof. Define $\mathbf{w} = ((\mathbf{x} \boxplus \mathbf{u}) \oplus \mathbf{v}) \boxminus \mathbf{y}$, then according to the composition of operation \boxplus , \boxminus and \oplus in the paper, we have

$$\begin{aligned} \mathbf{w} &= ((\mathbf{x} \boxplus \mathbf{u}) \oplus \mathbf{v}) \boxminus \mathbf{y} \\ &= \left(\left(\begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} \boxplus \mathbf{u} \right) \oplus \mathbf{v} \right) \boxminus \mathbf{y} \\ &= \left(\begin{bmatrix} \mathbf{x}_1 \boxplus_{\mathcal{M}_1} \mathbf{u}_1 \\ \mathbf{x}_2 \boxplus_{\mathcal{M}_2} \mathbf{u}_2 \end{bmatrix} \oplus \mathbf{v} \right) \boxminus \mathbf{y} \\ &= \begin{bmatrix} (\mathbf{x}_1 \boxplus_{\mathcal{M}_1} \mathbf{u}_1) \oplus_{\mathcal{M}_1} \mathbf{v}_1 \\ (\mathbf{x}_2 \boxplus_{\mathcal{M}_2} \mathbf{u}_2) \oplus_{\mathcal{M}_2} \mathbf{v}_2 \end{bmatrix} \boxminus \mathbf{y} \\ &= \begin{bmatrix} ((\mathbf{x}_1 \boxplus_{\mathcal{M}_1} \mathbf{u}_1) \oplus_{\mathcal{M}_1} \mathbf{v}_1) \boxminus_{\mathcal{M}_1} \mathbf{y}_1 \\ ((\mathbf{x}_2 \boxplus_{\mathcal{M}_2} \mathbf{u}_2) \oplus_{\mathcal{M}_2} \mathbf{v}_2) \boxminus_{\mathcal{M}_2} \mathbf{y}_2 \end{bmatrix} \triangleq \begin{bmatrix} \mathbf{w}_1 \\ \mathbf{w}_2 \end{bmatrix} \end{aligned}$$

As a result, the differentiation is

$$\begin{aligned} \frac{\partial \mathbf{w}}{\partial \mathbf{u}} &= \begin{bmatrix} \frac{\partial \mathbf{w}_1}{\partial \mathbf{u}_1} & \frac{\partial \mathbf{w}_1}{\partial \mathbf{u}_2} \\ \frac{\partial \mathbf{w}_2}{\partial \mathbf{u}_1} & \frac{\partial \mathbf{w}_2}{\partial \mathbf{u}_2} \end{bmatrix} = \begin{bmatrix} \frac{\partial \mathbf{w}_1}{\partial \mathbf{u}_1} & \mathbf{0} \\ \mathbf{0} & \frac{\partial \mathbf{w}_2}{\partial \mathbf{u}_2} \end{bmatrix} \\ &= \begin{bmatrix} \frac{\partial(((\mathbf{x}_1 \boxplus_{\mathcal{M}_1} \mathbf{u}_1) \oplus_{\mathcal{M}_1} \mathbf{v}_1) \boxminus_{\mathcal{M}_1} \mathbf{y}_1)}{\partial \mathbf{u}_1} & \mathbf{0} \\ \mathbf{0} & \frac{\partial(((\mathbf{x}_2 \boxplus_{\mathcal{M}_2} \mathbf{u}_2) \oplus_{\mathcal{M}_2} \mathbf{v}_2) \boxminus_{\mathcal{M}_2} \mathbf{y}_2)}{\partial \mathbf{u}_2} \end{bmatrix} \end{aligned}$$

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The partial differentiation of $((\mathbf{x} \boxplus \mathbf{u}) \oplus \mathbf{v}) \boxminus \mathbf{y}$ with respect to the \mathbf{v} on the *compound manifold* \mathcal{M} is able to be proved in the same way as above.

IV. IMPORTANT MANIFOLDS IN PRACTICE AND THEIR DERIVATIVES

1. Euclidean space $\mathcal{M} = \mathbb{R}^n$:

$$\begin{aligned} \mathbf{x} \boxplus \mathbf{u} &= \mathbf{x} + \mathbf{u} \\ \mathbf{y} \boxminus \mathbf{x} &= \mathbf{y} - \mathbf{x} \\ \mathbf{x} \oplus \mathbf{v} &= \mathbf{x} + \mathbf{v} \\ \frac{\partial((\mathbf{x} \boxplus \mathbf{u}) \oplus \mathbf{v}) \boxminus \mathbf{y}}{\partial \mathbf{u}} &= \mathbf{I}_{n \times n} \\ \frac{\partial((\mathbf{x} \boxplus \mathbf{u}) \oplus \mathbf{v}) \boxminus \mathbf{y}}{\partial \mathbf{v}} &= \mathbf{I}_{n \times n} \end{aligned} \quad (20)$$

The homeomorphic space of \mathbb{R}^n is itself at any point, based on which, the operations \boxplus , \boxminus conventional vector addition and subtraction, and $U_{\mathbf{x}} = \mathbf{B}_{\epsilon}^n(\mathbf{x}) = \mathbb{R}^n$.

2. Special orthogonal group $\mathcal{M} = SO(3)$:

$$\mathbf{x} \boxplus \mathbf{u} = \mathbf{x} \cdot \text{Exp}_{SO(3)}(\mathbf{u}) \quad (21)$$

$$\mathbf{y} \boxminus \mathbf{x} = \text{Log}_{SO(3)}(\mathbf{x}^{-1} \cdot \mathbf{y}) \quad (22)$$

$$\mathbf{x} \oplus \mathbf{v} = \mathbf{x} \cdot \text{Exp}_{SO(3)}(\mathbf{v}) \quad (23)$$

$$\frac{\partial((\mathbf{x} \boxplus \mathbf{u}) \oplus \mathbf{v}) \boxminus \mathbf{y}}{\partial \mathbf{u}} = (\mathbf{J}_{SO(3)}^r(\mathbf{w}))^{-1} \text{Exp}_{SO(3)}(-\mathbf{v}) \mathbf{J}_{SO(3)}^r(\mathbf{u}) \quad (24)$$

$$\frac{\partial((\mathbf{x} \boxplus \mathbf{u}) \oplus \mathbf{v}) \boxminus \mathbf{y}}{\partial \mathbf{v}} = (\mathbf{J}_{SO(3)}^r(\mathbf{w}))^{-1} \mathbf{J}_{SO(3)}^r(\mathbf{v}) \quad (25)$$

where $\mathbf{w} = ((\mathbf{x} \boxplus \mathbf{u}) \oplus \mathbf{v}) \boxminus \mathbf{y}$. As known that $SO(3)$ is a matrix Lie group, $\text{Exp}_{SO(3)}$ is the Exp map of $SO(3)$, and $\mathbf{J}_{SO(3)}^r$ is the according right Jacobian matrix. And the partial differentiations are derived by substituting the Jacobian matrix in (19). Because the exponential map and its inverse of $SO(3)$ represent the geodesics (i.e., the shortest path) of the

rotation, the homeomorphic neighborhood is $\mathbf{B}_\epsilon^3(\mathbf{x}) = \{\mathbf{a} | \mathbf{a} \in \mathbb{R}^3, \|\mathbf{a}\| < \pi\}$. And $\text{Exp}_{SO(3)}$ and $\mathbf{J}_{SO(3)}^r$ are defined as:

$$\begin{aligned}\text{Exp}_{SO(3)}(\mathbf{u}) &= \exp([\mathbf{u}]) \\ &= \mathbf{I} + \sin(\|\mathbf{u}\|) \frac{[\mathbf{u}]}{\|\mathbf{u}\|} + (1 - \cos(\|\mathbf{u}\|)) \frac{[\mathbf{u}]^2}{\|\mathbf{u}\|^2} \\ \mathbf{J}_{SO(3)}^r(\mathbf{u}) &= \mathbf{I} - \left(\frac{1 - \cos(\|\mathbf{u}\|)}{\|\mathbf{u}\|} \right) \frac{[\mathbf{u}]}{\|\mathbf{u}\|} + \left(1 - \frac{\sin(\|\mathbf{u}\|)}{\|\mathbf{u}\|} \right) \frac{[\mathbf{u}]^2}{\|\mathbf{u}\|^2} \\ \mathbf{J}_{SO(3)}^r(\mathbf{u})^{-1} &= \mathbf{I} + \frac{1}{2} [\mathbf{u}] + (1 - \alpha(\|\mathbf{u}\|)) \frac{[\mathbf{u}]^2}{\|\mathbf{u}\|^2}\end{aligned}\quad (26)$$

where $\mathbf{u} \in \mathbb{R}^3$, $[\mathbf{u}]$ denotes the skew-symmetric matrix that maps the cross product of $\mathbf{u} \in \mathbb{R}^3$. And $\alpha(\|\mathbf{u}\|) = (\|\mathbf{u}\|/2) \cdot \cot(\|\mathbf{u}\|/2)$.

3. Multiple direct spatial isometry $\mathcal{M} = SE_N(3)$:

$SE_N(3)$ is a matrix Lie group of the form:

$$\begin{bmatrix} \mathbf{R} & \mathbf{p}_1 & \cdots & \mathbf{p}_N \\ \mathbf{0}_{1 \times 3} & 1 & \cdots & 0 \\ \mathbf{0}_{1 \times 3} & 0 & \ddots & 0 \\ \mathbf{0}_{1 \times 3} & 0 & \cdots & 1 \end{bmatrix} \quad (27)$$

where $\mathbf{R} \in SO(3)$, and $\mathbf{p}_i \in \mathbb{R}^3, i = 1, \dots, N$.

$$\begin{aligned}\mathbf{x} \boxplus \mathbf{u} &= \mathbf{x} \cdot \text{Exp}_{SE_N(3)}(\mathbf{u}) \\ \mathbf{y} \boxminus \mathbf{x} &= \text{Log}_{SE_N(3)}(\mathbf{x}^{-1} \cdot \mathbf{y}) \\ \mathbf{x} \oplus \mathbf{v} &= \mathbf{x} \cdot \text{Exp}_{SE_N(3)}(\mathbf{v}) \\ \frac{\partial((\mathbf{x} \boxplus \mathbf{u}) \oplus \mathbf{v}) \boxminus \mathbf{y}}{\partial \mathbf{u}} &= (\mathbf{J}_{SE_N(3)}^r(\mathbf{w}))^{-1} \\ &\quad \cdot \mathbf{J}_{SE_N(3)}^r(\mathbf{v} + (\mathbf{J}_{SE_N(3)}^l(\mathbf{v}))^{-1} \mathbf{u}) \\ &\quad \cdot (\mathbf{J}_{SE_N(3)}^l(\mathbf{v}))^{-1} \\ \frac{\partial((\mathbf{x} \boxplus \mathbf{u}) \oplus \mathbf{v}) \boxminus \mathbf{y}}{\partial \mathbf{v}} &= (\mathbf{J}_{SE_N(3)}^r(\mathbf{w}))^{-1} \\ &\quad \cdot \mathbf{J}_{SE_N(3)}^r(\mathbf{u} + (\mathbf{J}_{SE_N(3)}^l(\mathbf{u}))^{-1} \mathbf{v}) \\ &\quad \cdot (\mathbf{J}_{SE_N(3)}^r(\mathbf{u}))^{-1}\end{aligned}\quad (28)$$

where $\mathbf{w} = ((\mathbf{x} \boxplus \mathbf{u}) \oplus \mathbf{v}) \boxminus \mathbf{y}$. $\text{Exp}_{SE_N(3)}$ is the Exp map of $SE_N(3)$, and $\mathbf{J}_{SE_N(3)}^r$ is the according right Jacobian matrix. And the partial differentiations are derived by substituting the Jacobian matrix in (19). The homeomorphic neighborhood $\mathbf{B}_\epsilon^{3+3N}(\mathbf{x}) = \{[\mathbf{a} \ \mathbf{b}] | \mathbf{a} \in \mathbb{R}^3, \|\mathbf{a}\| < \pi, \mathbf{b} \in \mathbb{R}^{3 \times N}\}$. And the $\text{Exp}_{SE_N(3)}$ and $\mathbf{J}_{SE_N(3)}^r$ are defined as:

$$\begin{aligned}\text{Exp}_{SE_N(3)}(\mathbf{u}) &= \exp([\mathbf{u}]) = \exp \left(\begin{bmatrix} [\phi] & \rho_1 & \cdots & \rho_N \\ \mathbf{0}_{N \times N} & \mathbf{0}_{N \times 1} & \cdots & \mathbf{0}_{N \times 1} \end{bmatrix} \right) \\ &= \begin{bmatrix} \exp([\phi]) & \mathbf{J}_{SO(3)}^l(\phi) \rho_1 & \cdots & \mathbf{J}_{SO(3)}^l(\phi) \rho_N \\ \mathbf{0}_{1 \times 3} & 1 & \cdots & 0 \\ \mathbf{0}_{1 \times 3} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0}_{1 \times 3} & 0 & \cdots & 1 \end{bmatrix} \\ \mathbf{J}_{SE_N(3)}^r(\mathbf{u}) &= \begin{bmatrix} \mathbf{J}_{SO(3)}^r(\phi) & \mathbf{0} & \cdots & \mathbf{0} \\ -\mathbf{J}_{SO(3)}^r(\phi) \mathbf{M}_1 \mathbf{J}_{SO(3)}^r(\phi) & \mathbf{J}_{SO(3)}^r(\phi) & \cdots & \mathbf{0} \\ \vdots & \mathbf{0} & \ddots & \mathbf{0} \\ -\mathbf{J}_{SO(3)}^r(\phi) \mathbf{M}_N \mathbf{J}_{SO(3)}^r(\phi) & \mathbf{0} & \cdots & \mathbf{J}_{SO(3)}^r(\phi) \end{bmatrix} \\ \mathbf{M}_i &= \frac{1}{2} [\rho_i] + \frac{1 - \alpha(y)}{y^2} ([\rho_i] [\phi] + [\phi] [\rho_i]) \\ &\quad + \frac{2 - \alpha(y) - (y/2)^2 \sin(y/2)^{-2}}{2y^4} [\phi] ([\rho_i] [\phi] + [\phi] [\rho_i]) [\phi]\end{aligned}\quad (29)$$

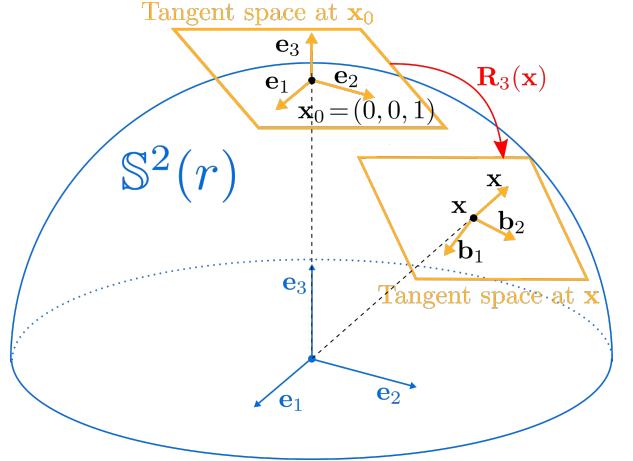


Fig. 1. The orthonormal basis $(\mathbf{b}_1, \mathbf{b}_2)$ in the tangent plane of $\mathbf{x} \in \mathbb{S}^2(r)$. $\mathbf{R}_3(\mathbf{x})$ is defined by rotating \mathbf{e}_3 to \mathbf{x} , \mathbf{b}_1 and \mathbf{b}_2 are obtained through rotating \mathbf{e}_1 and \mathbf{e}_2 by $\mathbf{R}_3(\mathbf{x})$ respectively.

where $\mathbf{u} = [\phi \ \rho_1 \cdots \rho_N] \in \mathbb{R}^{3+3N}$ and $y = \|\phi\|$.

4. 2-sphere, $\mathcal{M} = \mathbb{S}^2(r) \triangleq \{\mathbf{x} \in \mathbb{R}^3 | \|\mathbf{x}\| = r, r > 0\}$:

$$\begin{aligned}\mathbf{x} \boxplus \mathbf{u} &= \mathbf{R}(\mathbf{B}(\mathbf{x})\mathbf{u}) \cdot \mathbf{x} \\ \mathbf{y} \boxminus \mathbf{x} &= \begin{cases} \mathbf{B}(\mathbf{x})^T \cdot \frac{[\mathbf{x}]\mathbf{y}}{\|[\mathbf{x}]\mathbf{y}\|} \cdot \text{atan2}([\mathbf{x}]\mathbf{y}, \mathbf{x}^T \mathbf{y}), & \mathbf{y} \neq -\mathbf{x}; \\ [\pi \ 0]^T, & \mathbf{y} = -\mathbf{x}; \end{cases} \\ \mathbf{x} \oplus \mathbf{v} &= \mathbf{R}(\mathbf{v}) \cdot \mathbf{x} \\ \frac{\partial((\mathbf{x} \boxplus \mathbf{u}) \oplus \mathbf{v}) \boxminus \mathbf{y}}{\partial \mathbf{u}} &= \mathbf{N}((\mathbf{x} \boxplus \mathbf{u}) \oplus \mathbf{v}, \mathbf{y}) \mathbf{R}(\mathbf{v}) \mathbf{M}(\mathbf{x}, \mathbf{u}) \\ \frac{\partial((\mathbf{x} \boxplus \mathbf{u}) \oplus \mathbf{v}) \boxminus \mathbf{y}}{\partial \mathbf{v}} &= -\mathbf{N}((\mathbf{x} \boxplus \mathbf{u}) \oplus \mathbf{v}, \mathbf{y}) \mathbf{R}(\mathbf{v}) [\mathbf{x} \boxplus \mathbf{u}] \mathbf{J}_{SO(3)}^r(\mathbf{v})\end{aligned}\quad (30)$$

where $\mathbf{R}(\cdot) = \text{Exp}_{SO(3)}(\cdot)$ denotes a rotation around the vector \cdot and $\mathbf{J}_{SO(3)}^r$ is the right Jacobian matrix of $SO(3)$. And the homeomorphic neighborhood $\mathbf{B}_\epsilon^2(\mathbf{x}) = \{\mathbf{a} | \mathbf{a} \in \mathbb{R}^3, \|\mathbf{a}\| = r\}$ is the entire sphere surface. $\mathbf{B}(\mathbf{x}) = [\mathbf{b}_1 \ \mathbf{b}_2]$ denotes the basis vector on the tangent space of \mathbb{S}^2 at \mathbf{x} . The basis $\mathbf{B}(\mathbf{x})$ is not specified, which can be made arbitrary as long as it forms an orthonormal basis in the tangent plane of \mathbf{x} . As an example, we could adopt the method in [3] (see Fig. 1): rotate one of the three canonical basis $\mathbf{e}_i, i = 1, 2, 3$ of \mathbb{R}^3 to \mathbf{x} (along the geodesics) and the rest two base vectors after the same rotation would form $\mathbf{B}(\mathbf{x})$. i.e.,

$$\begin{aligned}\mathbf{R}_i(\mathbf{x}) &= \mathbf{R} \left(\frac{[\mathbf{e}_i]\mathbf{x}}{\|[\mathbf{e}_i]\mathbf{x}\|} \text{atan2}([\mathbf{e}_i]\mathbf{x}, \mathbf{e}_i^T \mathbf{x}) \right), \\ \mathbf{B}(\mathbf{x}) &= \mathbf{R}_i(\mathbf{x}) [\mathbf{e}_j \ \mathbf{e}_k].\end{aligned}\quad (31)$$

where $j = i + 1, k = i + 2$ but wrapped below 3. The \mathbf{e}_i is chosen as:

$$\underset{\mathbf{e}_i}{\operatorname{argmin}} \|\mathbf{x} - \mathbf{e}_i\| \quad (32)$$

The differentiations of \mathbb{S}^2 is derived as follows using the chain rules, and in respective step of the chain rule, $\mathbf{N}(\mathbf{x}, \mathbf{y})$ and $\mathbf{M}(\mathbf{x}, \mathbf{u})$ are defined as:

$$\begin{aligned}\mathbf{N}(\mathbf{x}, \mathbf{y}) &= \frac{\partial(\mathbf{x} \boxplus \mathbf{y})}{\partial \mathbf{x}} = \mathbf{B}(\mathbf{y})^T \left(\frac{\theta}{\|\mathbf{y}\| \mathbf{x}} [\mathbf{y}] + [\mathbf{y}] \mathbf{x} \cdot \mathbf{P}(\mathbf{x}, \mathbf{y}) \right) \\ \mathbf{M}(\mathbf{x}, \mathbf{u}) &= \frac{\partial(\mathbf{x} \boxplus \mathbf{u})}{\partial \mathbf{u}} = -\mathbf{R}(\mathbf{B}(\mathbf{x})\mathbf{u}) [\mathbf{x}] \mathbf{J}_{SO(3)}^r(\mathbf{B}(\mathbf{x})\mathbf{u}) \mathbf{B}(\mathbf{x}) \\ \mathbf{P}(\mathbf{x}, \mathbf{y}) &= \frac{1}{r^4} \left(\frac{-\mathbf{y}^T \mathbf{x} \|\mathbf{y}\| \mathbf{x}}{\|\mathbf{y}\| \mathbf{x}^3} \mathbf{x}^T [\mathbf{y}]^2 - \mathbf{y}^T \right)\end{aligned}\quad (33)$$

It is noticed that $N(\mathbf{x}, \mathbf{y})$ has a singularity at point $\mathbf{x} = -\mathbf{y}$, however it will not be encountered in the calculation of this paper, since only the value of $N(\mathbf{x}, \mathbf{x}) = \frac{\mathbf{B}(\mathbf{x})^T |\mathbf{x}|}{r^2}$ will be used.

V. SYSTEM-SPECIFIC PARTIAL DIFFERENTIATIONS OF THE TIGHTLY-COUPLED LIDAR NAVIGATION SYSTEM

To calculate the system-specific partial differentiations, the following equations are provided for user-friendly purpose:

For the state or measurement $\mathbf{x} \in SO(3)$ and a vector $\mathbf{a} \in \mathbb{R}^3$:

$$\begin{aligned} \frac{\partial((\mathbf{x} \boxplus \mathbf{u}) \cdot \mathbf{a})}{\partial \mathbf{u}} \Big|_{\mathbf{u}=0} &= \lim_{\mathbf{u} \rightarrow 0} \frac{(\mathbf{x} \boxplus \mathbf{u}) \cdot \mathbf{a} - \mathbf{x} \cdot \mathbf{a}}{\mathbf{u}} = \lim_{\mathbf{u} \rightarrow 0} \frac{\mathbf{x} \cdot \text{Exp}(\mathbf{u}) \cdot \mathbf{a} - \mathbf{x} \cdot \mathbf{a}}{\mathbf{u}} \\ &= \lim_{\mathbf{u} \rightarrow 0} \frac{\mathbf{x} \cdot (\mathbf{I} + [\mathbf{u}]) \cdot \mathbf{a} - \mathbf{x} \cdot \mathbf{a}}{\mathbf{u}} \\ &= \lim_{\mathbf{u} \rightarrow 0} \frac{\mathbf{x} \cdot \mathbf{a} + \mathbf{x} \cdot [\mathbf{u}] \cdot \mathbf{a} - \mathbf{x} \cdot \mathbf{a}}{\mathbf{u}} = \lim_{\mathbf{u} \rightarrow 0} \frac{\mathbf{x} \cdot [\mathbf{u}] \cdot \mathbf{a}}{\mathbf{u}} \\ &= \lim_{\mathbf{u} \rightarrow 0} \frac{-\mathbf{x} \cdot [\mathbf{a}] \cdot \mathbf{u}}{\mathbf{u}} \\ &= -\mathbf{x} \cdot [\mathbf{a}] \end{aligned} \quad (34)$$

For the state or measurement $\mathbf{x} \in \mathbb{S}^2(r)$,

$$\frac{\partial(\mathbf{x} \boxplus \mathbf{y})}{\partial \mathbf{x}} = \frac{\mathbf{B}^T(\mathbf{x}) |\mathbf{x}|}{r^2}, \mathbf{x} = \mathbf{y} \quad (35)$$

The system-specific partial differentiations for the lidar-inertial navigation system in the paper are therefore:

Partial differentiations for $\mathbf{f}(\mathbf{x}, \mathbf{u}, \mathbf{w})$:

$$\begin{aligned} \frac{\partial \mathbf{f}(\mathbf{x} \boxplus \delta \mathbf{x}, \mathbf{u}, \mathbf{0})}{\partial \delta \mathbf{x}} \Big|_{\delta \mathbf{x}=0} &= \begin{bmatrix} \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{U}_{23}^F & -{}^G\mathbf{R}_I & \mathbf{0} & \mathbf{U}_{26}^F & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \\ \frac{\partial \mathbf{f}(\mathbf{x}, \mathbf{u}, \mathbf{w})}{\partial \mathbf{w}} \Big|_{\mathbf{w}=0} &= \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -{}^G\mathbf{R}_I & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \end{aligned} \quad (36)$$

where $\mathbf{U}_{23}^F = -{}^G\mathbf{R}_I [\mathbf{a}_m - \mathbf{b}_a]$ and $\mathbf{U}_{26}^F = -[{}^G\mathbf{g}] \mathbf{B}({}^G\mathbf{g})$, $\mathbf{B}(\cdot)$ is defined in the equation (31).

And partial differentiations for $\mathbf{h}(\mathbf{x}, \mathbf{v})$:

$$\begin{aligned} \frac{\partial(\mathbf{h}(\mathbf{x} \boxplus \delta \mathbf{x}, \mathbf{0}) \boxplus \mathbf{h}(\mathbf{x}, \mathbf{0}))}{\partial \delta \mathbf{x}} \Big|_{\delta \mathbf{x}=0} &= \\ \begin{bmatrix} \vdots & \vdots \\ \mathbf{G}_i & \mathbf{0} & \mathbf{U}_{i3}^H & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{U}_{i7}^H & \mathbf{G}_i {}^G\mathbf{R}_I \\ \vdots & \vdots \end{bmatrix}, & (37) \end{aligned}$$

$$\frac{\partial(\mathbf{h}(\mathbf{x}, \mathbf{v}) \boxplus \mathbf{h}(\mathbf{x}, \mathbf{0}))}{\partial \mathbf{v}} \Big|_{\mathbf{v}=0} = \text{diag}(\dots, -\mathbf{G}_i {}^G\mathbf{R}_I {}^I\mathbf{T}_L {}^L\mathbf{p}_{f_i}, \dots)$$

where $\mathbf{U}_{i3}^H = -\mathbf{G}_i {}^G\mathbf{R}_I [{}^I\mathbf{T}_L {}^L\mathbf{p}_{f_i}]$, and $\mathbf{U}_{i7}^H = -\mathbf{G}_i {}^G\mathbf{R}_I {}^I\mathbf{T}_L [{}^L\mathbf{p}_{f_i}]$.

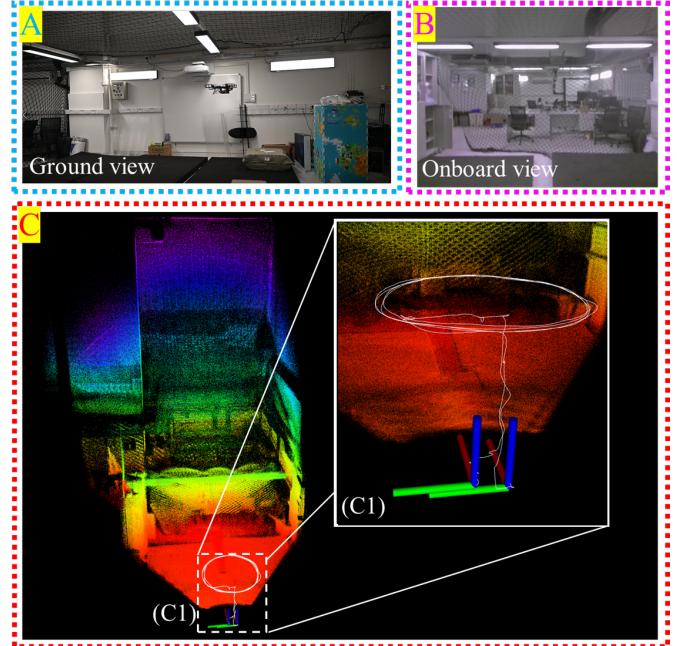


Fig. 2. Real-time mapping results of dataset V1-01, an indoor UAV flight experiment of lidar-inertial navigation system. A: Photo from ground view; B: Snapshot of onboard FPV video; C: Map result of IKFoM, (C1) Trajectory and poses of the UAV at beginning and end of the experiment.

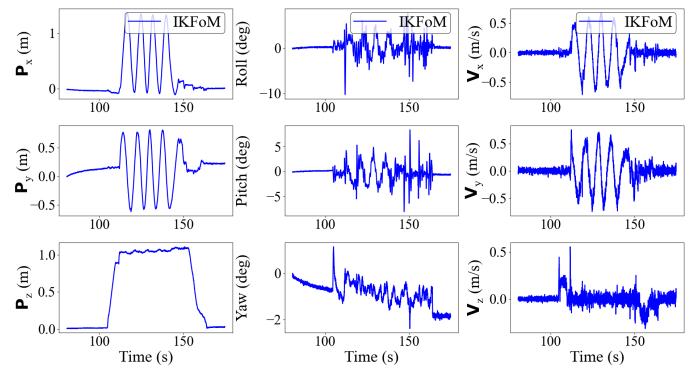


Fig. 3. Estimates of IKFoM for the position, rotation in Euler angles and velocity of the IMU on dataset V1-01, an indoor UAV flight experiment of lidar-inertial navigation system.

VI. FURTHER EXPERIMENT RESULTS

In this section, we show the results of implementing our developed toolkit on the tightly-coupled lidar-inertial navigation system for two datasets collected in this work, i.e., V1-01, V3-01.

A. Indoor UAV flight

For the dataset V1-01, the experiment is conducted in an indoor environment (see Fig. 2 (A)) where the UAV took off from the ground and flew in a circle path. During the path following, the UAV is constantly facing at a cluttered office area behind a safety net (see Fig. 2 (B)). After the path following, a human pilot took over the UAV and landed it manually to ensure that the landing position coincides with the take-off point.

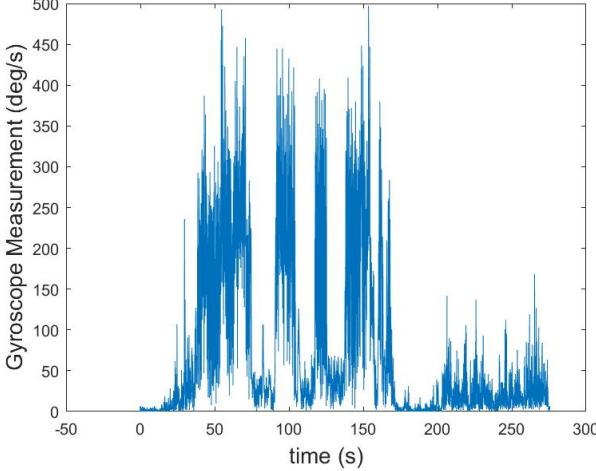


Fig. 4. Magnitude of gyroscope measurements for dataset V2-01, an indoor quick-shake experiment of lidar-inertial navigation system.

Fig. 2 (C) shows the real-time mapping results overlaid with the 3D trajectory estimated by our system. It can be seen that our system achieves consistent mapping even in the cluttered indoor environment. The position drift is less than 0.414% (i.e., 0.185m drift over the 44.66m path, see Fig. 2 (C1)). This drift is caused, in part, by the accumulation of odometry error, which is common in SLAM systems, and in part by inaccurate manual landing.

We show the estimated trajectory of position (${}^G\mathbf{p}_I$), rotation (${}^G\mathbf{R}_I$), and velocity (${}^G\mathbf{v}_I$) of IKFoM in Fig. 3, where the experiment starts from 80.7994s and ends at 174.6590s. Our system achieves smooth state estimation that is suitable for onboard feedback control. All the estimated state variables agree well with the actual motions. For further experiment demonstration, we refer readers to the videos at <https://youtu.be/MByP2twPFtA>.

B. Indoor quick shake

Fig. 4 showing the data measured by gyroscope is supplemented to illustrate the large rotation up to 496.45deg/s during this experiment.

C. Outdoor random walk

The experiment of V3-01 is conducted in a structured outdoor environment which is a corridor between a slope and the Hawking Wong building of the University of Hong Kong. In the experiment, the UAV is handheld to move along the road and then return to the beginning position (see Fig. 5 (A)).

The real-time mapping results of dataset V3-01 estimated by our toolkit is shown in Fig. 5 (B), which clearly shows the building on one side and the cars and bricks on the slope. The position drift is less than 0.0627% (i.e., 0.131m drift over 209.5m path, see Fig. 5 (B3)). This small drift supports the efficacy of our system.

The estimations of the kinematics parameters are shown in Fig. 6, where the experiment starts from 353.000s and ends at 509.999s. The trajectory estimated by IKFoM is

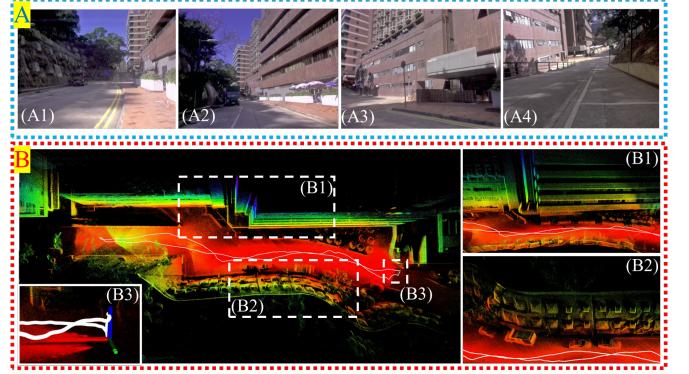


Fig. 5. Real-time mapping results of dataset V3-01, an outdoor random walk experiment of lidar-inertial navigation system. A: Photos of the environment of this experiment; B: Map result of IKFoM, (B1) Local zoom-in map result of one side of the road, (B2) Local zoom-in map result of the other side of the road, (B3) Poses of the UAV at the beginning and end of the experiment.

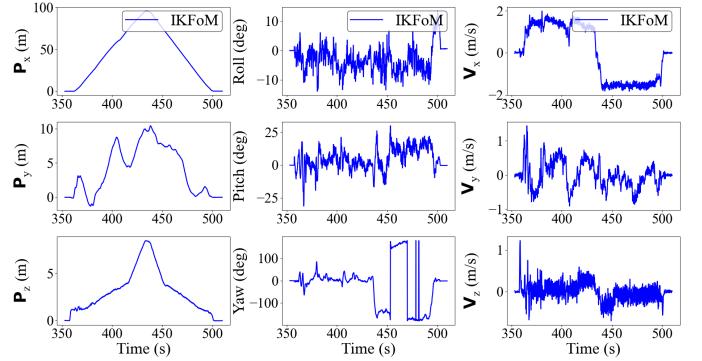


Fig. 6. Estimates of IKFoM for the position, rotation in Euler angles and velocity of the IMU on dataset V3-01, an outdoor random walk experiment of lidar-inertial navigation system.

approximately symmetric about the middle time in X and Z direction, which agrees with the actual motion profile where the sensor is moved back on the same road. For further experiment demonstration, we refer readers to the videos at <https://youtu.be/MByP2twPFtA>.

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