Supplementary File of Symbolic Representation and Toolkit Development of Iterated Error-state Extended Kalman Filters on Manifolds

Dongjiao HE[1], *Student Member, IEEE,* Wei Xu[1], *Student Member, IEEE,* and Fu Zhang[1*], *Member, IEEE*

Please note that equation numbers and section numbers from the manuscript main texts are colored in this letter in red.

I. Important manifolds in practice and their derivatives

1. Euclidean space $\mathcal{M} = \mathbb{R}^n$:

$$\begin{aligned} \mathbf{x} \boxplus \mathbf{u} &= \mathbf{x} + \mathbf{u} \\ \mathbf{y} \boxminus \mathbf{x} &= \mathbf{y} - \mathbf{x} \\ \mathbf{x} \oplus \mathbf{v} &= \mathbf{x} + \mathbf{v} \\ \frac{\partial(((\mathbf{x}\boxplus\mathbf{u})\oplus\mathbf{v})\boxminus\mathbf{y})}{\partial\mathbf{u}} &= \mathbf{I}_{n\times n} \\ \frac{\partial(((\mathbf{x}\boxplus\mathbf{u})\oplus\mathbf{v})\boxminus\mathbf{y})}{\partial\mathbf{v}} &= \mathbf{I}_{n\times n} \end{aligned} \tag{1}$$

The homeomorphic space of $\mathbb{R}^n$ is itself at any point, based on which, the operations $\boxplus \backslash \boxminus$ conventional vector addition and substraction, and the local neighbor is $\mathbf{B}^n_\epsilon(\mathbf{x}) = \mathbb{R}^n, \forall \mathbf{x} \in \mathcal{M}$.

2. Special orthogonal group $\mathcal{M} = SO(3)$:

$$\begin{aligned} \mathbf{x} \boxplus \mathbf{u} &= \mathbf{x} \cdot \mathrm{Exp}_{SO(3)}(\mathbf{u}) \\ \mathbf{y} \boxminus \mathbf{x} &= \mathrm{Log}_{SO(3)}(\mathbf{x}^{-1} \cdot \mathbf{y}) \\ \mathbf{x} \oplus \mathbf{v} &= \mathbf{x} \cdot \mathrm{Exp}_{SO(3)}(\mathbf{v}) \\ \frac{\partial(((\mathbf{x}\boxplus\mathbf{u})\oplus\mathbf{v})\boxminus\mathbf{y})}{\partial\mathbf{u}} &= (\mathbf{J}^r_{SO(3)}(\mathbf{w}))^{-1}\mathrm{Exp}_{SO(3)}(-\mathbf{v})\mathbf{J}^r_{SO(3)}(\mathbf{u}) \\ \frac{\partial(((\mathbf{x}\boxplus\mathbf{u})\oplus\mathbf{v})\boxminus\mathbf{y})}{\partial\mathbf{v}} &= (\mathbf{J}^r_{SO(3)}(\mathbf{w}))^{-1}\mathbf{J}^r_{SO(3)}(\mathbf{v}) \end{aligned} \tag{2}$$

where $\mathbf{w} = ((\mathbf{x}\boxplus\mathbf{u})\oplus\mathbf{v}) \boxminus \mathbf{y}$. As known that $SO(3)$ is a matrix Lie group, $\mathrm{Exp}_{SO(3)}$ is the Exp map of $SO(3)$, and $\mathbf{J}^r_{SO(3)}$ is the according right Jacobian matrix. And the partial differentiations are derived by substituting the Jacobian matrix in (27):

$$\begin{aligned} \mathrm{Exp}_{SO(3)}(\mathbf{u}) &= \exp(\lfloor \mathbf{u} \rfloor) \\ &= \mathbf{I} + \sin(\|\mathbf{u}\|)\frac{\lfloor \mathbf{u} \rfloor}{\|\mathbf{u}\|} + (1-\cos(\|\mathbf{u}\|))\frac{\lfloor \mathbf{u} \rfloor^2}{\|\mathbf{u}\|^2} \\ \mathbf{J}^r_{SO(3)}(\mathbf{u}) &= \mathbf{I} - \left(\frac{1-\cos(\|\mathbf{u}\|)}{\|\mathbf{u}\|}\right)\frac{\lfloor \mathbf{u} \rfloor}{\|\mathbf{u}\|} + \left(1-\frac{\sin(\|\mathbf{u}\|)}{\|\mathbf{u}\|}\right)\frac{\lfloor \mathbf{u} \rfloor^2}{\|\mathbf{u}\|^2} \\ \mathbf{J}^r_{SO(3)}(\mathbf{u})^{-1} &= \mathbf{I} + \frac{1}{2}\lfloor \mathbf{u} \rfloor + (1-\alpha(\|\mathbf{u}\|))\frac{\lfloor \mathbf{u} \rfloor^2}{\|\mathbf{u}\|^2} \end{aligned} \tag{3}$$

where $\mathbf{u} \in \mathbb{R}^3$, $\lfloor \mathbf{u} \rfloor$ denotes the skew-symmetric matrix that maps the cross product of $\mathbf{u} \in \mathbb{R}^3$, and $\alpha(\|\mathbf{u}\|) = (\|\mathbf{u}\|/2) \cdot \cot(\|\mathbf{u}\|/2)$.

Finally, the local neighbor such that the $\boxplus/\boxminus$ in (2) exists is $\mathbf{B}^n_\epsilon(\mathbf{x}) = \{\mathbf{u} \in \mathbb{R}^3 | \|\mathbf{u}\| < \pi\}, \forall \mathbf{x} \in \mathcal{M}$.

3. Multiple direct spatial isometry $\mathcal{M} = SE_N(3)$:

$SE_N(3)$ is a matrix Lie group of the form:

$$\begin{bmatrix} \mathbf{R} & \mathbf{p}_1 & \cdots & \mathbf{p}_N \\ \mathbf{0}_{1\times 3} & 1 & \cdots & 0 \\ \mathbf{0}_{1\times 3} & 0 & \ddots & 0 \\ \mathbf{0}_{1\times 3} & 0 & \cdots & 1 \end{bmatrix} \tag{4}$$

where $\mathbf{R} \in SO(3)$, and $\mathbf{p}_i \in \mathbb{R}^3, i = 1, \cdots, N$.

$$\begin{aligned} \mathbf{x} \boxplus \mathbf{u} &= \mathbf{x} \cdot \mathrm{Exp}_{SE_N(3)}(\mathbf{u}) \\ \mathbf{y} \boxminus \mathbf{x} &= \mathrm{Log}_{SE_N(3)}(\mathbf{x}^{-1} \cdot \mathbf{y}) \\ \mathbf{x} \oplus \mathbf{v} &= \mathbf{x} \cdot \mathrm{Exp}_{SE_N(3)}(\mathbf{u}) \\ \frac{\partial(((\mathbf{x}\boxplus\mathbf{u})\oplus\mathbf{v})\boxminus\mathbf{y})}{\partial\mathbf{u}} &= (\mathbf{J}^r_{SE_N(3)}(\mathbf{w}))^{-1} \\ &\quad \cdot \mathbf{J}^r_{SE(3)}(\mathbf{v} + (\mathbf{J}^l_{SE_N(3)}(\mathbf{v}))^{-1}\mathbf{u}) \\ &\quad \cdot (\mathbf{J}^l_{SE_N(3)}(\mathbf{v}))^{-1} \\ \frac{\partial(((\mathbf{x}\boxplus\mathbf{u})\oplus\mathbf{v})\boxminus\mathbf{y})}{\partial\mathbf{v}} &= (\mathbf{J}^r_{SE_N(3)}(\mathbf{w}))^{-1} \\ &\quad \cdot \mathbf{J}^r_{SE_N(3)}(\mathbf{u} + (\mathbf{J}^r_{SE_N(3)}(\mathbf{u}))^{-1}\mathbf{v}) \\ &\quad \cdot (\mathbf{J}^r_{SE_N(3)}(\mathbf{u}))^{-1} \end{aligned} \tag{5}$$

where $\mathbf{w} = ((\mathbf{x}\boxplus\mathbf{u})\oplus\mathbf{v}) \boxminus \mathbf{y}$. $\mathrm{Exp}_{SE_N(3)}$ is the Exp map of $SE_N(3)$, and $\mathbf{J}^r_{SE_N(3)}$ is the according right Jacobian matrix. And the partial differentiations are derived by substituting the Jacobian matrix in (27):

$$\begin{aligned} &\mathrm{Exp}_{SE_N(3)}(\mathbf{u}) = \exp(\lfloor \mathbf{u} \rfloor) = \exp\left(\begin{bmatrix} \lfloor \boldsymbol{\phi} \rfloor & \boldsymbol{\rho}_1 & \cdots & \boldsymbol{\rho}_N \\ \mathbf{0}_{N\times N} & \mathbf{0}_{N\times 1} & \cdots & \mathbf{0}_{N\times 1} \end{bmatrix}\right) \\ &= \begin{bmatrix} \exp(\lfloor \boldsymbol{\phi} \rfloor) & \mathbf{J}^l_{SO(3)}(\boldsymbol{\phi})\boldsymbol{\rho}_1 & \cdots & \mathbf{J}^l_{SO(3)}(\boldsymbol{\phi})\boldsymbol{\rho}_N \\ \mathbf{0}_{1\times 3} & 1 & \cdots & 0 \\ \mathbf{0}_{1\times 3} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0}_{1\times 3} & 0 & \cdots & 1 \end{bmatrix} \\ &\mathbf{J}^r_{SE_N(3)}(\mathbf{u}) \\ &= \begin{bmatrix} \mathbf{J}^r_{SO(3)}(\boldsymbol{\phi}) & \mathbf{0} & \cdots & \mathbf{0} \\ -\mathbf{J}^r_{SO(3)}(\boldsymbol{\phi})\mathbf{M}_1\mathbf{J}^r_{SO(3)}(\boldsymbol{\phi}) & \mathbf{J}^r_{SO(3)}(\boldsymbol{\phi}) & \cdots & \mathbf{0} \\ \vdots & \mathbf{0} & \ddots & \mathbf{0} \\ -\mathbf{J}^r_{SO(3)}(\boldsymbol{\phi})\mathbf{M}_N\mathbf{J}^r_{SO(3)}(\boldsymbol{\phi}) & \mathbf{0} & \cdots & \mathbf{J}^r_{SO(3)}(\boldsymbol{\phi}) \end{bmatrix} \\ &\mathbf{M}_i = \frac{1}{2}\lfloor \boldsymbol{\rho}_i \rfloor + \frac{1-\alpha(y)}{y^2}(\lfloor \boldsymbol{\rho}_i \rfloor \lfloor \boldsymbol{\phi} \rfloor + \lfloor \boldsymbol{\phi} \rfloor \lfloor \boldsymbol{\rho}_i \rfloor) \\ &\quad + \frac{2-\alpha(y)-(y/2)^2\sin(y/2)^{-2}}{2y^4}\lfloor \boldsymbol{\phi} \rfloor (\lfloor \boldsymbol{\rho}_i \rfloor \lfloor \boldsymbol{\phi} \rfloor + \lfloor \boldsymbol{\phi} \rfloor \lfloor \boldsymbol{\rho}_i \rfloor) \lfloor \boldsymbol{\phi} \rfloor \end{aligned} \tag{6}$$

where $\mathbf{u} = \begin{bmatrix} \boldsymbol{\phi} & \boldsymbol{\rho}_1 \cdots \boldsymbol{\rho}_N \end{bmatrix} \in \mathbb{R}^{3+3N}$ and $y = \|\boldsymbol{\phi}\|$.

The corresponding local neighbor is $\mathbf{B}^n_\epsilon(\mathbf{x}) = \{\begin{bmatrix} \boldsymbol{\phi} & \boldsymbol{\rho} \end{bmatrix} \in \mathbb{R}^{3+3N} | \|\boldsymbol{\phi}\| < \pi, \boldsymbol{\rho} \in \mathbb{R}^{3N}\}, \forall \mathbf{x} \in \mathcal{M}$.

4. Matrix Lie group:

*Corresponding author.
[1]All authors are with Department of Mechanical Engineering, University of Hong Kong. {hdj65822, xuweii, fuzhang}@hku.hk

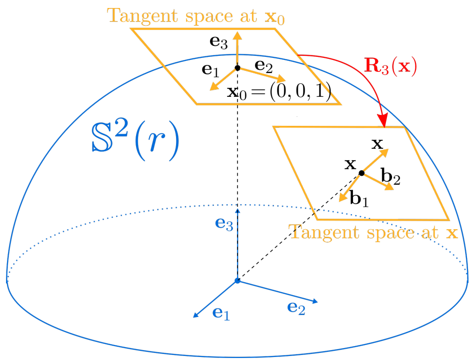

Fig. 1. The orthonormal basis ($\mathbf{b}_1$, $\mathbf{b}_2$) in the tangent plane of $\mathbf{x} \in \mathbb{S}^2(r)$. $\mathbf{R}_3(\mathbf{x})$ is defined by rotating $\mathbf{e}_3$ to $\mathbf{x}$, $\mathbf{b}_1$ and $\mathbf{b}_2$ are obtained through rotating $\mathbf{e}_1$ and $\mathbf{e}_2$ by $\mathbf{R}_3(\mathbf{x})$ respectively.

For a general matrix Lie group including the above $SO(3)$, $SE_N(3)$, its tangent space could always be chosen as the local homeomorphic space to minimally parameterize the state perturbation. Specifically, the tangent space permits a Lie algebraic structure denoted as $\mathfrak{m}$ and an exponential map $\exp : \mathfrak{m} \mapsto \mathcal{M}$ [1]. Let $\mathfrak{f} : \mathbb{R}^n \mapsto \mathfrak{m}$ be the map from the minimal parameterization space to the Lie algebra and $\mathrm{Exp} = \exp \circ \mathfrak{f}$ with inverse Log. Then the $\boxplus \backslash \boxminus$ and $\oplus$ operations with the according differentiations could be:

$$\begin{aligned}
\mathbf{x} \boxplus \mathbf{u} &= \mathbf{x} \cdot \mathrm{Exp}\left(\mathbf{u}\right) \ \text{ or } \mathrm{Exp}(\mathbf{u}) \cdot \mathbf{x} \\
\mathbf{y} \boxminus \mathbf{x} &= \mathrm{Log}\left(\mathbf{x}^{-1} \cdot \mathbf{y}\right) \ \text{ or } \mathrm{Log}(\mathbf{y} \cdot \mathbf{x}^{-1}) \\
\mathbf{x} \oplus \mathbf{v} &= \mathbf{x} \cdot \mathrm{Exp}\left(\mathbf{v}\right) \ \text{ or } \mathrm{Exp}(\mathbf{v}) \cdot \mathbf{x} \\
\tfrac{\partial(((\mathbf{x} \boxplus \mathbf{u}) \oplus \mathbf{v}) \boxminus \mathbf{y})}{\partial \mathbf{u}} &= \mathbf{J}_r^{-1}(\mathbf{w})\mathbf{J}_r(\mathbf{v} + \mathbf{J}_l^{-1}(\mathbf{v})\mathbf{u})\mathbf{J}_l^{-1}(\mathbf{v}) \\
&\quad \text{ or } \mathbf{J}_l^{-1}(\mathbf{w})\mathbf{J}_l(\mathbf{v} + \mathbf{J}_r^{-1}(\mathbf{v})\mathbf{u})\mathbf{J}_r^{-1}(\mathbf{v}) \\
\tfrac{\partial(((\mathbf{x} \boxplus \mathbf{u}) \oplus \mathbf{v}) \boxminus \mathbf{y})}{\partial \mathbf{v}} &= \mathbf{J}_r^{-1}(\mathbf{w})\mathbf{J}_r(\mathbf{v}) \ \text{ or } \mathbf{J}_l^{-1}(\mathbf{w})\mathbf{J}_l(\mathbf{v})
\end{aligned} \tag{7}$$

where two choices of operation definition are shown. $\mathbf{x}^{-1}$ is the inverse of $\mathbf{x}$ which always exists for an element on Lie groups. $\mathbf{J}_r$ and $\mathbf{J}_l$ are denoted as the right and left Jacobian matrices of the matrix lie group respectively, which satisfy [2]:

$$\begin{aligned}
\mathrm{Exp}(\boldsymbol{\tau} + \delta\boldsymbol{\tau}) &= \mathrm{Exp}(\boldsymbol{\tau})\mathrm{Exp}(\mathbf{J}_r(\boldsymbol{\tau})\delta\boldsymbol{\tau}) \\
\mathrm{Exp}(\boldsymbol{\tau} + \delta\boldsymbol{\tau}) &= \mathrm{Exp}(\mathbf{J}_l(\boldsymbol{\tau})\delta\boldsymbol{\tau})\mathrm{Exp}(\boldsymbol{\tau})
\end{aligned} \tag{8}$$

where $\mathbf{J}_l(\boldsymbol{\tau}) = \mathbf{J}_r(-\boldsymbol{\tau})$ as proved in [2]. The proof of partial differentiations of $(((\mathbf{x} \boxplus \mathbf{u}) \oplus \mathbf{v}) \boxminus \mathbf{y})$ with respect to $\mathbf{u}$ and $\mathbf{v}$ is shown in the following section III.

5. 2-sphere, $\mathcal{M} = \mathbb{S}^2(r) \triangleq \{\mathbf{x} \in \mathbb{R}^3 | \|\mathbf{x}\| = r, r > 0\}$:

$$\begin{aligned}
&\mathbf{x} \boxplus \mathbf{u} = \mathbf{R}(\mathbf{B}(\mathbf{x})\mathbf{u}) \cdot \mathbf{x} \\
&\mathbf{y} \boxminus \mathbf{x} = \begin{cases} \mathbf{B}\left(\mathbf{x}\right)^T \cdot \frac{\lfloor \mathbf{x} \rfloor \mathbf{y}}{\|\lfloor \mathbf{x} \rfloor \mathbf{y}\|} \cdot \mathrm{atan2}\left(\|\lfloor \mathbf{x} \rfloor \mathbf{y}\|, \mathbf{x}^{\mathrm{T}}\mathbf{y}\right), & \mathbf{y} \neq -\mathbf{x}; \\ \begin{bmatrix} \pi & 0 \end{bmatrix}^T, & \mathbf{y} = -\mathbf{x}; \end{cases} \\
&\mathbf{x} \oplus \mathbf{v} = \mathbf{R}(\mathbf{v}) \cdot \mathbf{x} \\
&\tfrac{\partial(((\mathbf{x} \boxplus \mathbf{u}) \oplus \mathbf{v}) \boxminus \mathbf{y})}{\partial \mathbf{u}} = \mathbf{N}((\mathbf{x} \boxplus \mathbf{u}) \oplus \mathbf{v}, \mathbf{y})\mathbf{R}(\mathbf{v})\mathbf{M}\left(\mathbf{x}, \mathbf{u}\right) \\
&\tfrac{\partial(((\mathbf{x} \boxplus \mathbf{u}) \oplus \mathbf{v}) \boxminus \mathbf{y})}{\partial \mathbf{v}} = -\mathbf{N}((\mathbf{x} \boxplus \mathbf{u}) \oplus \mathbf{v}, \mathbf{y})\mathbf{R}(\mathbf{v})\lfloor \mathbf{x} \boxplus \mathbf{u} \rfloor \mathbf{J}^r_{SO(3)}(\mathbf{v})
\end{aligned} \tag{9}$$

where $\mathbf{R}(\cdot) = \mathrm{Exp}_{SO(3)}(\cdot)$ denotes a rotation around the vector $\cdot$ and $\mathbf{J}^r_{SO(3)}$ is the right Jacobian matrix of $SO(3)$, and $\mathbf{B}(\mathbf{x}) = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 \end{bmatrix}$ denotes the basis vector on the tangent space of $\mathbb{S}^2$ at $\mathbf{x}$. The basis $\mathbf{B}(\mathbf{x})$ is not specified, which can be made arbitrary as long as it forms an orthonormal basis in the tangent plane of $\mathbf{x}$. As an example, we could adopt the method in [3] (see Fig. 1): rotate one of the three canonical basis $\mathbf{e}_i, i = 1, 2, 3$ of $\mathbb{R}^3$ to $\mathbf{x}$ (along the *geodesics*) and the rest two base vectors after the same rotation would form $\mathbf{B}(\mathbf{x})$. i.e.,

$$\begin{aligned}
\mathbf{R}_i(\mathbf{x}) &= \mathbf{R}\left(\tfrac{\lfloor \mathbf{e}_i \rfloor \mathbf{x}}{\|\lfloor \mathbf{e}_i \rfloor \mathbf{x}\|} \mathrm{atan2}\left(\|\lfloor \mathbf{e}_i \rfloor \mathbf{x}\|, \mathbf{e}_i^T \mathbf{x}\right)\right), \\
\mathbf{B}(\mathbf{x}) &= \mathbf{R}_i(\mathbf{x}) \begin{bmatrix} \mathbf{e}_j & \mathbf{e}_k \end{bmatrix}.
\end{aligned} \tag{10}$$

where $j = i + 1, k = i + 2$ but wrapped below 3. The $\mathbf{e}_i$ is chosen as:

$$\underset{\mathbf{e}_i}{\mathrm{argmin}} \|\mathbf{x} - \mathbf{e}_i\| \tag{11}$$

The local neighbor defining the $\boxplus / \boxminus$ in (9) is $\mathbf{B}^n_\epsilon(\mathbf{x}) = \{\mathbf{u} \in \mathbb{R}^2 | \|\mathbf{u}\| < \pi\}, \forall \mathbf{x} \in \mathcal{M}$ is the entire circle.

The differentiations of $\mathbb{S}^2$ is derived as follows using the chain rules, and in respective step of the chain rule, $\mathbf{N}\left(\mathbf{x}, \mathbf{y}\right)$ and $\mathbf{M}\left(\mathbf{x}, \mathbf{u}\right)$ are defined as:

$$\begin{aligned}
\mathbf{N}(\mathbf{x}, \mathbf{y}) &= \tfrac{\partial(\mathbf{x} \boxminus \mathbf{y})}{\partial \mathbf{x}} = \mathbf{B}(\mathbf{y})^T \Big(\tfrac{\theta}{\|\lfloor \mathbf{y} \rfloor \mathbf{x}\|} \lfloor \mathbf{y} \rfloor + \lfloor \mathbf{y} \rfloor \mathbf{x} \cdot \mathbf{P}(\mathbf{x}, \mathbf{y}) \Big) \\
\mathbf{M}(\mathbf{x}, \mathbf{u}) &= \tfrac{\partial(\mathbf{x} \boxplus \mathbf{u})}{\partial \mathbf{u}} = -\mathbf{R}(\mathbf{B}(\mathbf{x})\mathbf{u}) \lfloor \mathbf{x} \rfloor \mathbf{J}^r_{SO(3)}(\mathbf{B}(\mathbf{x})\mathbf{u})\, \mathbf{B}(\mathbf{x}) \\
\mathbf{P}(\mathbf{x}, \mathbf{y}) &= \tfrac{1}{r^4} \left(\tfrac{-\mathbf{y}^T \mathbf{x} \|\lfloor \mathbf{y} \rfloor \mathbf{x}\| + r^4 \theta}{\|\lfloor \mathbf{y} \rfloor \mathbf{x}\|^3} \mathbf{x}^T \lfloor \mathbf{y} \rfloor^2 - \mathbf{y}^T \right)
\end{aligned} \tag{12}$$

It is noticed that $\mathbf{N}(\mathbf{x}, \mathbf{y})$ has a singularity at point $\mathbf{x} = -\mathbf{y}$, however it will not be encountered in the calculation of this paper, since only the value of $\mathbf{N}(\mathbf{x}, \mathbf{x}) = \frac{\mathbf{B}(\mathbf{x})^T \lfloor \mathbf{x} \rfloor}{r^2}$ will be used.

II. Examples for casting different robotic systems into the canonical form

The canonical form shown in the paper is constructed through compactly representing the system model using the $\oplus$ operation of the respective manifolds. In the following, we show how to incorporate the $\oplus$ operation and cast a canonical form for different state components. Then with the composition property as stated in the paper, the complete state equation can be obtained by concatenating all components.

Example 1: Vectors in Euclidean space (e.g., position and velocity). Assume $\mathbf{x} \in \mathbb{R}^n$ subject to $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}, \mathbf{w})$. Using zero-order hold discretization, $\mathbf{f}(\mathbf{x}, \mathbf{u}, \mathbf{w})$ is assumed constant during the sampling period Δt, hence

$$\begin{aligned}
\mathbf{x}_{k+1} &= \mathbf{x}_k + \left(\Delta t \mathbf{f}\left(\mathbf{x}_k, \mathbf{u}_k, \mathbf{w}_k\right)\right) \\
&= \mathbf{x}_k \oplus_{\mathbb{R}^n} \left(\Delta t \mathbf{f}\left(\mathbf{x}_k, \mathbf{u}_k, \mathbf{w}_k\right)\right).
\end{aligned} \tag{13}$$

Example 2: Attitude kinematics in a global reference frame (e.g., the earth-frame). Let $\mathbf{x} \in SO(3)$ be the body attitude relative to the global frame and ${}^G\boldsymbol{\omega}$ be the global angular velocity which holds constant for one sampling period Δt, then

$$\begin{aligned}
&\dot{\mathbf{x}} = \lfloor {}^G\boldsymbol{\omega} \rfloor \cdot \mathbf{x} \Longrightarrow \mathbf{x}_{k+1} = \exp(\lfloor \Delta t\ {}^G\boldsymbol{\omega}_k \rfloor) \cdot \mathbf{x}_k = \mathbf{x}_k \\
&\quad \cdot \exp\left(\lfloor \Delta t (\mathbf{x}_k^T \cdot {}^G\boldsymbol{\omega}_k) \rfloor\right) = \mathbf{x}_k \oplus_{SO(3)} \left(\Delta t \mathbf{f}\left(\mathbf{x}_k, {}^G\boldsymbol{\omega}_k\right)\right), \\
&\qquad \mathbf{f}\left(\mathbf{x}_k, {}^G\boldsymbol{\omega}_k\right) = \mathbf{x}_k^T \cdot {}^G\boldsymbol{\omega}_k.
\end{aligned} \tag{14}$$

where $\lfloor \mathbf{w} \rfloor$ is a skew-symmetric matrix mapping the cross product of $\mathbf{w} \in \mathbb{R}^3$, and $\exp(\lfloor \mathbf{w} \rfloor)$, the exponential operation of matrix $\lfloor \mathbf{w} \rfloor$, denotes a rotation about the vector $\mathbf{w}$.

Example 3: Attitude kinematics in body frame. Let $\mathbf{x} \in SO(3)$ be the body attitude relative to the global frame and ${}^B\boldsymbol{\omega}$ be the body angular velocity which holds constant for one sampling period Δt, then

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{x} \cdot \lfloor {}^B\boldsymbol{\omega} \rfloor \implies \mathbf{x}_{k+1} = \mathbf{x}_k \cdot \exp(\lfloor \Delta t \ {}^B\boldsymbol{\omega}_k \rfloor) \\ &= \mathbf{x}_k \oplus_{SO(3)} \left(\Delta t \mathbf{f} \left({}^B\boldsymbol{\omega}_k \right) \right), \ \mathbf{f} \left({}^B\boldsymbol{\omega}_k \right) = \ {}^B\boldsymbol{\omega}_k. \end{aligned} \tag{15}$$

Example 4: Vectors of known magnitude (e.g., gravity) in the global frame. Let $\mathbf{x} \in \mathbb{S}^2(g)$ be the gravity vector in the global frame with known magnitude g. Then,

$$\dot{\mathbf{x}} = \mathbf{0} \Longrightarrow \mathbf{x}_{k+1} = \mathbf{x}_k = \mathbf{x}_k \oplus_{\mathbb{S}^2(g)} (\Delta t \mathbf{f}(\mathbf{x}_k)), \mathbf{f}(\mathbf{x}_k) = \mathbf{0}. \tag{16}$$

Example 5: Vectors of known magnitude (e.g., gravity) in body frame. Let $\mathbf{x} \in \mathbb{S}^2(g)$ be the gravity vector in the body frame and ${}^B\boldsymbol{\omega}$ be the body angular velocity which holds constant for one sampling period Δt. Then,

$$\begin{aligned} \dot{\mathbf{x}} &= -\lfloor {}^B\boldsymbol{\omega} \rfloor \mathbf{x} \Longrightarrow \mathbf{x}_{k+1} = \exp(\lfloor -\Delta t \ {}^B\boldsymbol{\omega}_k \rfloor) \mathbf{x}_k \\ &= \mathbf{x}_k \oplus_{\mathbb{S}^2(g)} \left(\Delta t \mathbf{f} \left({}^B\boldsymbol{\omega}_k \right) \right), \mathbf{f} \left({}^B\boldsymbol{\omega}_k \right) = -{}^B\boldsymbol{\omega}_k. \end{aligned} \tag{17}$$

Example 6: Bearing-distance parameterization of visual landmarks [4]. Let $\mathbf{x} \in \mathbb{S}^2(1)$ and $d(\rho) \in \mathbb{R}$ be the bearing vector and depth (with parameter ρ), respectively, of a visual landmark, and ${}^G\mathbf{R}_C, {}^G\mathbf{p}_C$ be the attitude and position of the camera. Then the visual landmark in the global frame is ${}^G\mathbf{R}_C(\mathbf{x}d(\rho)) + {}^G\mathbf{p}_C$, which is constant over time:

$$\begin{aligned} &\tfrac{d({}^G\mathbf{R}_C(\mathbf{x}d(\rho)) + {}^G\mathbf{p}_C)}{dt} = \mathbf{0} \implies \\ &\lfloor {}^C\boldsymbol{\omega} \rfloor (\mathbf{x}d(\rho)) + \dot{\mathbf{x}}d(\rho) + \mathbf{x}d'(\rho)\dot{\rho} + {}^C\mathbf{v} = \mathbf{0}. \end{aligned} \tag{18}$$

Left multiplying (18) by $\mathbf{x}^T$ and using $\mathbf{x}^T\dot{\mathbf{x}} = 0$ yield $\dot{\rho} = -\mathbf{x}^T \cdot {}^C\mathbf{v}/d'(\rho)$. Substituting this to (18) leads to

$$\begin{aligned} \dot{\mathbf{x}} &= -\lfloor {}^C\boldsymbol{\omega} + \tfrac{1}{d(\rho)} \lfloor \mathbf{x} \rfloor \cdot {}^C\mathbf{v} \rfloor \cdot \mathbf{x} \implies \\ \mathbf{x}_{k+1} &= \exp\left(\lfloor -\Delta t \left({}^C\boldsymbol{\omega}_k + \tfrac{1}{d(\rho)} \lfloor \mathbf{x}_k \rfloor \cdot {}^C\mathbf{v}_k \right) \rfloor \right) \mathbf{x}_k \\ &= \mathbf{x}_k \oplus_{\mathbb{S}^2(1)} \left(\Delta t \mathbf{f} \left(\mathbf{x}_k, {}^C\boldsymbol{\omega}_k, {}^C\mathbf{v}_k \right) \right), \\ &\mathbf{f} \left(\mathbf{x}_k, {}^C\boldsymbol{\omega}_k, {}^C\mathbf{v}_k \right) = -{}^C\boldsymbol{\omega}_k - \tfrac{1}{d(\rho)} \lfloor \mathbf{x}_k \rfloor \cdot {}^C\mathbf{v}_k. \end{aligned} \tag{19}$$

where ${}^C\boldsymbol{\omega} + \frac{1}{d(\rho)} \lfloor \mathbf{x} \rfloor \cdot {}^C\mathbf{v}$ is assumed constant for one sampling period Δt due to the zero-order hold assumption.

III. Partial differentiations of matrix Lie groups

The partial differentiations of $(((\mathbf{x} \boxplus \mathbf{u}) \oplus \mathbf{v}) \boxminus \mathbf{y})$ w.r.t. $\mathbf{u}$ and $\mathbf{v}$ are directly calculated without using the chain rule. Here we only derive differentiations of matrix Lie groups whose $\boxplus$ and $\oplus$ are both defined as right multiplication, i.e., $\mathbf{x} \boxplus \mathbf{u} = \mathbf{x} \oplus \mathbf{u} = \mathbf{x} \cdot \mathrm{Exp}(\mathbf{u})$, and the counterparts for left multiplication could be derived using the same procedure.

Firstly, we prove the partial differentiations of $(\mathbf{x} \boxplus \mathbf{u}) \boxminus \mathbf{y}$ w.r.t $\mathbf{u}$. Denote $\mathbf{z} = (\mathbf{x} \boxplus \mathbf{u}) \boxminus \mathbf{y}$, we have:

$$\mathbf{y} \cdot \mathrm{Exp}(\mathbf{z}) = \mathbf{x} \cdot \mathrm{Exp}(\mathbf{u}) \tag{20}$$

Hence a small variation $\Delta \mathbf{u}$ in $\mathbf{u}$ causes a small variation $\Delta \mathbf{z}$ in $\mathbf{z}$, which is subject to:

$$\mathbf{y} \cdot \mathrm{Exp}(\mathbf{z} + \Delta \mathbf{z}) = \mathbf{x} \cdot \mathrm{Exp}(\mathbf{u} + \Delta \mathbf{u}) \tag{21}$$

Using the fact $\mathrm{Exp}(\mathbf{u} + \Delta \mathbf{u}) = \mathrm{Exp}(\mathbf{u}) \cdot \mathrm{Exp}\left(\mathbf{J}_r(\mathbf{u})\Delta \mathbf{u}\right)$ as shown in [2], it is derived that the left hand side of (21):

$$\mathbf{y} \cdot \mathrm{Exp}(\mathbf{z} + \Delta \mathbf{z}) = \mathbf{y} \cdot \mathrm{Exp}(\mathbf{z}) \cdot \mathrm{Exp}(\mathbf{J}_r(\mathbf{z})\Delta \mathbf{z}) \tag{22}$$

and the right hand side of (21)

$$\mathbf{x} \cdot \mathrm{Exp}(\mathbf{u} + \Delta \mathbf{u}) = \mathbf{x} \cdot \mathrm{Exp}(\mathbf{u}) \cdot \mathrm{Exp}(\mathbf{J}_r(\mathbf{u})\Delta \mathbf{u}) \tag{23}$$

Equating two sides of (21) leads to

$$\mathbf{J}_r(\mathbf{z})\Delta \mathbf{z} = \mathbf{J}_r(\mathbf{u})\Delta \mathbf{u} \tag{24}$$

and as a result

$$\tfrac{\partial((\mathbf{x} \boxplus \mathbf{u}) \boxminus \mathbf{y})}{\partial \mathbf{u}} = \tfrac{\Delta \mathbf{z}}{\Delta \mathbf{u}} = (\mathbf{J}_r(\mathbf{z}))^{-1} \cdot \mathbf{J}_r(\mathbf{u}) \tag{25}$$

Afterwards, using the fact $\mathrm{Exp}(\mathbf{u} + \Delta \mathbf{u}) = \mathrm{Exp}(\mathbf{u}) \cdot \mathrm{Exp}(\mathbf{J}_r(\mathbf{u})\Delta \mathbf{u}) = \mathrm{Exp}(\mathbf{J}_l(\mathbf{u})\Delta \mathbf{u}) \cdot \mathrm{Exp}(\mathbf{u})$, denote $\mathbf{w} = ((\mathbf{x} \boxplus \mathbf{u}) \oplus \mathbf{v}) \boxminus \mathbf{y}$, we have:

$$\begin{aligned} \mathbf{y} \cdot \mathrm{Exp}(\mathbf{w}) &= \mathbf{x} \cdot \mathrm{Exp}(\mathbf{u}) \cdot \mathrm{Exp}(\mathbf{v}) \\ &= \mathbf{x} \cdot \mathrm{Exp}(\mathbf{u} + (\mathbf{J}_r(\mathbf{u}))^{-1}\mathbf{v}) \\ &= \mathbf{x} \cdot \mathrm{Exp}(\mathbf{v} + (\mathbf{J}_l(\mathbf{v}))^{-1}\mathbf{u}) \end{aligned} \tag{26}$$

Based on the result of (25), using the chain rule,

$$\begin{aligned} \tfrac{\partial(((\mathbf{x} \boxplus \mathbf{u}) \oplus \mathbf{v}) \boxminus \mathbf{y})}{\partial \mathbf{u}} &= (\mathbf{J}_r(\mathbf{w}))^{-1} \mathbf{J}_r(\mathbf{v} + (\mathbf{J}_l(\mathbf{v}))^{-1}\mathbf{u})(\mathbf{J}_l(\mathbf{v}))^{-1} \\ \tfrac{\partial(((\mathbf{x} \boxplus \mathbf{u}) \oplus \mathbf{v}) \boxminus \mathbf{y})}{\partial \mathbf{v}} &= (\mathbf{J}_r(\mathbf{w}))^{-1} \mathbf{J}_r(\mathbf{u} + (\mathbf{J}_r(\mathbf{u}))^{-1}\mathbf{v})(\mathbf{J}_r(\mathbf{u}))^{-1} \end{aligned} \tag{27}$$

IV. Partial differentiation of *compound manifolds*

Lemma 1. *If $\mathbf{x}_1, \mathbf{y}_1 \in \mathcal{M}_1$; $\mathbf{x}_2, \mathbf{y}_2 \in \mathcal{M}_2$; $\mathbf{u}_1 \in \mathbb{R}^{n_1}$ and $\mathbf{u}_2 \in \mathbb{R}^{n_2}$; where n_1, n_2 are dimensions of $\mathcal{M}_1$, $\mathcal{M}_2$ respectively, define compound manifold $\mathcal{M} = \mathcal{M}_1 \times \mathcal{M}_2$, and its elements $\mathbf{x} = \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 \end{bmatrix}^T \in \mathcal{M}$; $\mathbf{y} = \begin{bmatrix} \mathbf{y}_1 & \mathbf{y}_2 \end{bmatrix}^T \in \mathcal{M}$; $\mathbf{u} = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{bmatrix}^T \in \mathbb{R}^{n_1+n_2}$ and $\mathbf{v}_1 \in \mathbb{R}^{l_1}$, $\mathbf{v}_2 \in \mathbb{R}^{l_2}$, $\mathbf{v} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix}^T \in \mathbb{R}^{l_1+l_2}$, then*

$$\begin{aligned} &\tfrac{\partial(((\mathbf{x} \boxplus_{\mathcal{M}} \mathbf{u}) \oplus_{\mathcal{M}} \mathbf{v}) \boxminus_{\mathcal{M}} \mathbf{y})}{\partial \mathbf{u}} \\ &= \begin{bmatrix} \frac{\partial(((\mathbf{x}_1 \boxplus_{\mathcal{M}_1} \mathbf{u}_1) \oplus_{\mathcal{M}_1} \mathbf{v}_1) \boxminus_{\mathcal{M}_1} \mathbf{y}_1)}{\partial \mathbf{u}_1} & \mathbf{0} \\ \mathbf{0} & \frac{\partial(((\mathbf{x}_2 \boxplus_{\mathcal{M}_2} \mathbf{u}_2) \oplus_{\mathcal{M}_2} \mathbf{v}_2) \boxminus_{\mathcal{M}_2} \mathbf{y}_2)}{\partial \mathbf{u}_2} \end{bmatrix} \end{aligned}$$

Proof. Define $\mathbf{w} = ((\mathbf{x} \boxplus_{\mathcal{M}} \mathbf{u}) \oplus_{\mathcal{M}} \mathbf{v}) \boxminus_{\mathcal{M}} \mathbf{y}$, then according to the composition of operation $\boxplus$, $\boxminus$ and $\oplus$ in the paper, we have

$$\begin{aligned} \mathbf{w} &= ((\mathbf{x} \boxplus_{\mathcal{M}} \mathbf{u}) \oplus_{\mathcal{M}} \mathbf{v}) \boxminus_{\mathcal{M}} \mathbf{y} \\ &= \left(\left(\begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} \boxplus_{\mathcal{M}} \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \end{bmatrix} \right) \oplus_{\mathcal{M}} \mathbf{v} \right) \boxminus_{\mathcal{M}} \mathbf{y} \\ &= \left(\begin{bmatrix} \mathbf{x}_1 \boxplus_{\mathcal{M}_1} \mathbf{u}_1 \\ \mathbf{x}_2 \boxplus_{\mathcal{M}_2} \mathbf{u}_2 \end{bmatrix} \oplus_{\mathcal{M}} \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \end{bmatrix} \right) \boxminus_{\mathcal{M}} \mathbf{y} \\ &= \begin{bmatrix} (\mathbf{x}_1 \boxplus_{\mathcal{M}_1} \mathbf{u}_1) \oplus_{\mathcal{M}_1} \mathbf{v}_1 \\ (\mathbf{x}_2 \boxplus_{\mathcal{M}_2} \mathbf{u}_2) \oplus_{\mathcal{M}_2} \mathbf{v}_2 \end{bmatrix} \boxminus_{\mathcal{M}} \begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \end{bmatrix} \\ &= \begin{bmatrix} ((\mathbf{x}_1 \boxplus_{\mathcal{M}_1} \mathbf{u}_1) \oplus_{\mathcal{M}_1} \mathbf{v}_1) \boxminus_{\mathcal{M}_1} \mathbf{y}_1 \\ ((\mathbf{x}_2 \boxplus_{\mathcal{M}_2} \mathbf{u}_2) \oplus_{\mathcal{M}_2} \mathbf{v}_2) \boxminus_{\mathcal{M}_2} \mathbf{y}_2 \end{bmatrix} \triangleq \begin{bmatrix} \mathbf{w}_1 \\ \mathbf{w}_2 \end{bmatrix} \end{aligned}$$

As a result, the differentiation is

$$\frac{\partial \mathbf{w}}{\partial \mathbf{u}} = \begin{bmatrix} \frac{\partial \mathbf{w}_1}{\partial \mathbf{u}_1} & \frac{\partial \mathbf{w}_1}{\partial \mathbf{u}_2} \\ \frac{\partial \mathbf{w}_2}{\partial \mathbf{u}_1} & \frac{\partial \mathbf{w}_2}{\partial \mathbf{u}_2} \end{bmatrix} = \begin{bmatrix} \frac{\partial \mathbf{w}_1}{\partial \mathbf{u}_1} & \mathbf{0} \\ \mathbf{0} & \frac{\partial \mathbf{w}_2}{\partial \mathbf{u}_2} \end{bmatrix}$$
$$= \begin{bmatrix} \frac{\partial(((\mathbf{x}_1 \boxplus_{\mathcal{M}_1} \mathbf{u}_1) \oplus_{\mathcal{M}_1} \mathbf{v}_1) \boxminus_{\mathcal{M}_1} \mathbf{y}_1)}{\partial \mathbf{u}_1} & \mathbf{0} \\ \mathbf{0} & \frac{\partial(((\mathbf{x}_2 \boxplus_{\mathcal{M}_2} \mathbf{u}_2) \oplus_{\mathcal{M}_2} \mathbf{v}_2) \boxminus_{\mathcal{M}_2} \mathbf{y}_2)}{\partial \mathbf{u}_2} \end{bmatrix}$$

■

The partial differentiation of $(((\mathbf{x} \boxplus_{\mathcal{M}}) \oplus_{\mathcal{M}}) \mathbf{v}) \boxminus_{\mathcal{M}} \mathbf{y})$ with respect to the $\mathbf{v}$ on the *compound manifold* $\mathcal{M}$ is able to be proved in the same way as above.

V. Proof of Theorem 1

Proof. To prove that the state parameterization in the iterated EKF is non-singular, we only need to show that each step of the EKF suffers from no singularity issue caused by the error-state $\delta\mathbf{x}$. A general requirement of EKF is that the manifold and system should be both smooth (meaning that $\mathbf{x} \boxplus \mathbf{u}$ is differentiable with respect to (w.r.t.) $\mathbf{u}$, $\mathbf{y} \boxminus \mathbf{x}$ is differentiable w.r.t. $\mathbf{y}$, $\mathbf{f}(\mathbf{x}, \mathbf{u}, \mathbf{w})$ in the state equation is differentiable w.r.t. $\mathbf{x}, \mathbf{u}$ and $\mathbf{w}$, and $\mathbf{x} \oplus \mathbf{v}$ is differentiable w.r.t. $\mathbf{x}$ and $\mathbf{v}$), otherwise, there will exist no Jacobian matrix required in the EKF (e.g., $\frac{\partial \mathbf{f}(\mathbf{x},\mathbf{u},\mathbf{w})}{\partial \mathbf{x}}$). This requirement is necessary even for systems operating in normal Euclidean space $\mathbb{R}^n$ and hence is not pertain to the state parameterization.

To show that each step of the EKF suffers from no singularity issue caused by the error-state $\delta\mathbf{x}$, we investigate the prediction and update step of the EKF separately as follows:

In the prediction step, the prediction of $\delta\mathbf{x}$ is directly given by (10), which is always valid. For the matrices $\mathbf{F_x}, \mathbf{F_w}$ required in the covariance propagation (15), their existence relies on the $\mathbf{G_x}$ and $\mathbf{G_f}$ defined in (14). To show their existence, we rewrite $\mathbf{G_x}$ and $\mathbf{G_f}$ into a simpler form as below:

$$\begin{aligned} \mathbf{G_x} &= \left. \frac{\partial\Big(\big((\mathbf{x} \boxplus \mathbf{u}) \oplus \mathbf{v}\big) \boxminus \big(\mathbf{x} \oplus \mathbf{v}\big)\Big)}{\partial \mathbf{u}} \right|_{\mathbf{u}=\mathbf{0}} \\ \mathbf{G_f} &= \left. \frac{\partial\Big(\big(\mathbf{x} \oplus \mathbf{v}\big) \boxminus \big(\mathbf{x} \oplus \bar{\mathbf{v}}\big)\Big)}{\partial \mathbf{v}} \right|_{\mathbf{v}=\bar{\mathbf{v}}} \end{aligned} \tag{28}$$

To show that $\mathbf{G_x}$ exists, we need to prove that $\big((\mathbf{x} \boxplus \mathbf{u}) \oplus \mathbf{v}\big) \boxminus \big(\mathbf{x} \oplus \mathbf{v}\big)$ is non-singular for a certain open set containing the point of evaluation $\mathbf{u} = \mathbf{0}$ (within this open set, we could freely vary $\mathbf{u}$ to obtain the differentiation $\mathbf{G_x}$). To prove this, we note that $\mathbf{x} \oplus \mathbf{v}$ is differentiable to $\mathbf{x}$ and $\mathbf{x} \boxplus \mathbf{u}$ is differentiable to $\mathbf{u}$, therefore, $(\mathbf{x} \boxplus \mathbf{u}) \oplus \mathbf{v}$ is differentiable to $\mathbf{u}$ according to the chain rule. Since $(\mathbf{x} \boxplus \mathbf{u}) \oplus \mathbf{v}$ is a differentiable (hence smooth) function of $\mathbf{u}$, according to the epsilon-delta definition, for any ϵ-neighborhood of $\big((\mathbf{x} \boxplus \mathbf{u}) \oplus \mathbf{v}\big)|_{\mathbf{u}=\mathbf{0}} = \mathbf{x} \oplus \mathbf{v}$, there always exists a small open neighborhood of the origin, denoted by $\mathcal{N}_\delta = \{\mathbf{u} \in \mathbb{R}^n \mid \|\mathbf{u}\| < \delta\}$, such that when $\mathbf{u} \in \mathcal{N}_\delta$, $(\mathbf{x} \boxplus \mathbf{u}) \oplus \mathbf{v}$ is within the ϵ-neighborhood. In particularly, we choose $U_{\mathbf{x} \oplus \mathbf{v}}$ as the ϵ-neighborhood, then the resultant $\mathcal{N}_\delta$ will ensure $(\mathbf{x} \boxplus \mathbf{u}) \oplus \mathbf{v}$ is always within the neighbor $U_{\mathbf{x} \oplus \mathbf{v}}$ of $\mathbf{x} \oplus \mathbf{v}$, which guarantees the non-singularity of the $\big((\mathbf{x} \boxplus \mathbf{u}) \oplus \mathbf{v}\big) \boxminus \big(\mathbf{x} \oplus \mathbf{v}\big)$ according to (1). Similarly, for $\mathbf{G_f}$, since $\mathbf{x} \oplus \mathbf{v}$ is differentiable (hence smooth) to $\mathbf{v}$, an open set $\mathcal{N}_\delta$ around $\bar{\mathbf{v}}$ always exists such that $\big(\mathbf{x} \oplus \mathbf{v}\big) \boxminus \big(\mathbf{x} \oplus \bar{\mathbf{v}}\big)$ is non-singular for the evaluation of $\mathbf{G_f}$.

In the update step, we only need to show the matrix $\mathbf{J}$ defined in (21) is non-singular and the state update in (26) is well defined. The latter condition always holds since each state update is restricted within the neighbor of the current state estimate as shown in (25). To show that $\mathbf{J}$ is non-singular, we rewrite it into a simpler form as below

$$\mathbf{J} = \left. \frac{\partial\big((\mathbf{x} \boxplus \mathbf{u}) \boxminus (\mathbf{x} \boxplus \mathbf{u}^\star)\big)}{\partial \mathbf{u}} \right|_{\mathbf{u}=\mathbf{u}^\star}, \quad \text{where } \mathbf{u}^\star \in \mathbf{B}_\epsilon^n(\mathbf{x}) \tag{29}$$

Since $\mathbf{x} \boxplus \mathbf{u}$ is differentiable w.r.t. $\mathbf{u} \in \mathbf{B}_\epsilon^n(\mathbf{x})$ (hence smooth around the point $\mathbf{u}^\star \in \mathbf{B}_\epsilon^n(\mathbf{x})$, following the derivation similar to above, there always exists an open set $\mathcal{N}_\delta$ around $\mathbf{u}^\star$ such that $\mathbf{x} \boxplus \mathbf{u}$ is within the neighbor $U_{\mathbf{x} \boxplus \mathbf{u}^\star}$ of $\mathbf{x} \boxplus \mathbf{u}^\star$, meaning that $(\mathbf{x} \boxplus \mathbf{u}) \boxminus (\mathbf{x} \boxplus \mathbf{u}^\star)$ is non-singular over this open set hence allows valid differentiation.

Since both the prediction and update step of the EKF suffer from no singularity issue caused by the error-state parameterization (encapsulated by $\boxplus$ and $\boxminus$), the state parameterization is singularity-free. Moreover, this holds in the whole state workspace since the above proof is applicable to all possible state $\mathbf{x}$. ■

VI. Further Experiment Results

In this section, we show the results of implementing our developed toolkit on the tightly-coupled lidar-inertial navigation system for two datasets collected in this work, i.e., V1-01, V3-01.

A. Indoor UAV flight

For the dataset V1-01, the experiment is conducted in an indoor environment (see Fig. 2 (A)) where the UAV took off from the ground and flied in a circle path. During the path following, the UAV is constantly facing at a cluttered office area behind a safety net (see Fig. 2 (B)). After the path following, a human pilot took over the UAV and landed it manually to ensure that the landing position coincides with the take-off point.

Fig. 2 (C) shows the real-time mapping results overlaid with the 3D trajectory estimated by our system. It can be seen that our system achieves consistent mapping even in the cluttered indoor environment. The position drift is less than 0.414% (i.e., $0.185m$ drift over the $44.66m$ path, see Fig. 2 (C1)). This drift is caused, in part, by the accumulation of odometry error, which is common in SLAM systems, and in part by inaccurate manual landing.

We show the estimated trajectory of position (${}^G\mathbf{p}_I$), rotation (${}^G\mathbf{R}_I$), and velocity (${}^G\mathbf{v}_I$) of *IKFoM* in Fig. 3, where the experiment starts from $80.7994s$ and ends at $174.6590s$. Our system achieves smooth state estimation that is suitable for onboard feedback control. All the estimated state variables agree well with the actual motions. For further experiment demonstration, we refer readers to the videos at https://youtu.be/MByP2twPFtA.

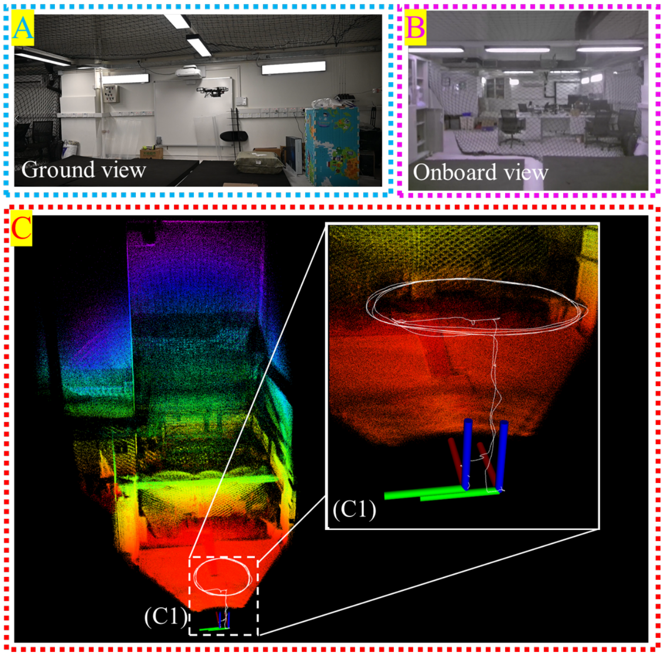

Fig. 2. Real-time mapping results of dataset V1-01, an indoor UAV flight experiment of lidar-inertial navigation system. A: Photo from ground view; B: Snapshot of onboard FPV video; C: Map result of *IKFoM*, (C1) Trajectory and poses of the UAV at beginning and end of the experiment.

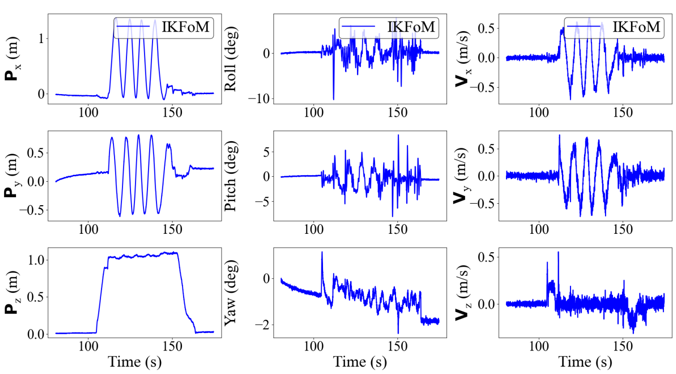

Fig. 3. Estimates of *IKFoM* for the position, rotation in Euler angles and velocity of the IMU on dataset V1-01, an indoor UAV flight experiment of lidar-inertial navigation system.

B. Indoor quick shake

Fig. 4 showing the data measured by gyroscope is supplemented to illustrate the large rotation up to $496.45deg/s$ during this experiment.

C. Outdoor random walk

The experiment of V3-01 is conducted in a structured outdoor environment which is a corridor between a slope and the Hawking Wong building of the University of Hong Kong. In the experiment, the UAV is handheld to move along the road and then return to the beginning position (see Fig. 5 (A)).

The real-time mapping results of dataset V3-01 estimated by our toolkit is shown in Fig. 5 (B), which clearly shows the building on one side and the cars and bricks on the slope. The position drift is less than 0.0627% (i.e., $0.131m$ drift over $209.5m$ path, see Fig. 5 (B3)). This small drift supports the efficacy of our system.

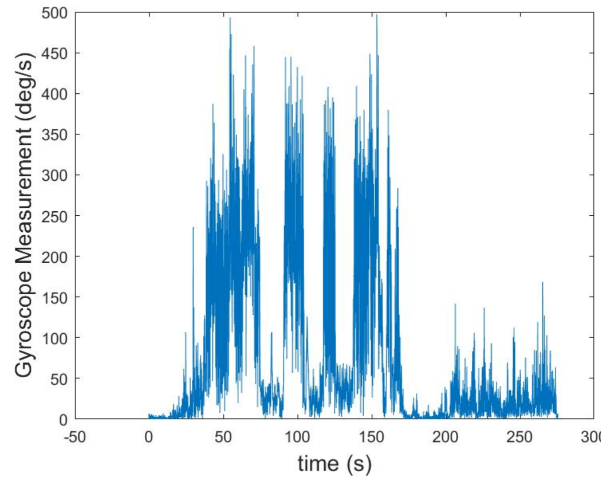

Fig. 4. Magnitude of gyroscope measurements for dataset V2-01, an indoor quick-shake experiment of lidar-inertial navigation system.

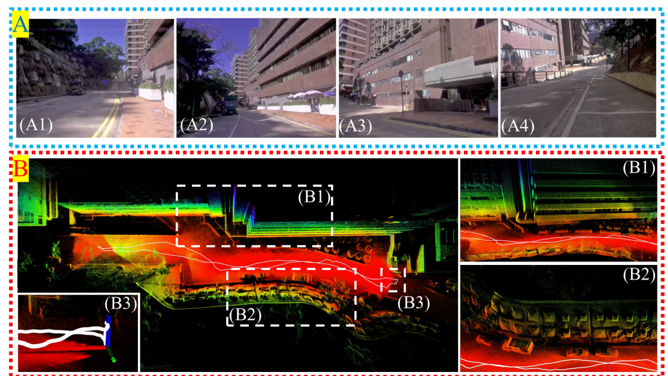

Fig. 5. Real-time mapping results of dataset V3-01, an outdoor random walk experiment of lidar-inertial navigation system. A: Photos of the environment of this experiment; B: Map result of *IKFoM*, (B1) Local zoom-in map result of one side of the road, (B2) Local zoom-in map result of the other side of the road, (B3) Poses of the UAV at the beginning and end of the experiment.

The estimations of the kinematics parameters are shown in Fig. 6, where the experiment starts from $353.000s$ and ends at $509.999s$. The trajectory estimated by *IKFoM* is approximately symmetric about the middle time in X and Z direction, which agrees with the actual motion profile where the sensor is moved back on the same road. For further experiment demonstration, we refer readers to the videos at https://youtu.be/MByP2twPFtA.

REFERENCES

[1] R. M. Murray, Z. Li, and S. S. Sastry, *A mathematical introduction to robotic manipulation*. CRC press, 2017.

[2] J. Sola, J. Deray, and D. Atchuthan, "A micro lie theory for state estimation in robotics," *arXiv preprint arXiv:1812.01537*, 2018.

[3] C. Hertzberg, R. Wagner, U. Frese, and L. Schröder, "Integrating generic sensor fusion algorithms with sound state representations through encapsulation of manifolds," *Information Fusion*, vol. 14, no. 1, pp. 57–77, 2013.

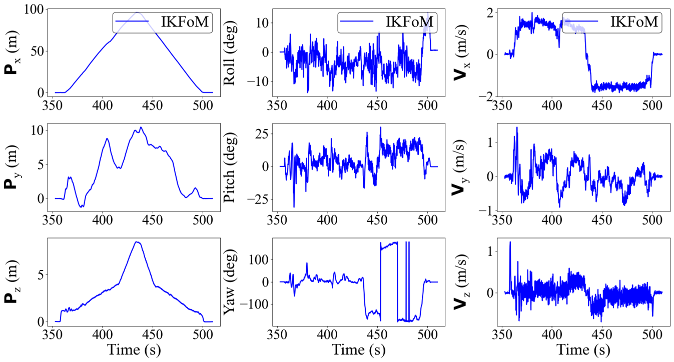

Fig. 6. Estimates of *IKFoM* for the position, rotation in Euler angles and velocity of the IMU on dataset V3-01, an outdoor random walk experiment of lidar-inertial navigation system.

[4] M. Bloesch, M. Burri, S. Omari, M. Hutter, and R. Siegwart, "Iterated extended kalman filter based visual-inertial odometry using direct photometric feedback," *The International Journal of Robotics Research*, vol. 36, no. 10, pp. 1053–1072, 2017.