COMP 545: Advanced topics in optimization
From simple to complex ML systems

Lecture 1

Overview

$$
\begin{aligned}
\min_{x} \quad & f(x) \\
\text{s.t.} \quad & x \in \mathcal{C}
\end{aligned}
$$

Overview

- Different objective classes
- Different strategies within each problem
- Different approaches based on computational capabilities
- Different approaches based on constraints

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$$\min_{x} \quad f(x)$$
$$\text{s.t.} \quad x \in \mathcal{C}$$

And, always having in mind applications in machine learning, AI and signal processing

Motivation (no fancy images included)

Provable efficiency

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Provable efficiency

Lots of data

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Harder problems

Lots of data

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More complicated models

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Provable efficiency

"What shall we do?"

Motivation (no fancy images included)

Provable efficiency

"What shall we do?"

Set up algo nicely

Use prior knowledge

Converge faster

Exploit resources

Topics

- Continuous **optimization** (in general)
  - See syllabus
  - Both theory and practice

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- Recent applications that drive research

(We can discuss about it)

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- When no theory applies, some intuition

Topics NOT covered in this course

- Extensive coverage of convex optimization

  (Duality, KKT conditions, self-concordance, interior point methods, Lagrange multipliers)

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- Randomized algorithms

  (See Anshu’s course)

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- Online algorithms, like bandits
- Bayesian algorithms

Who is this class for?

- Definitely, for PhD students

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- Master or even undergraduates that want to start research

(but get in touch with me soon to assess your background)

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- Just auditing is fine by me

What is the vision for this course?

- For starters, have in mind that this is a first−time taught course

(Any feedback is more than welcome)

- My purpose and vision is to introduce a series of optimization courses in the CS (and Duncan Hall's in general) curriculum

(Feedback on GoogleDoc)

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- My purpose and vision is to introduce a series of optimization courses in the CS (and Duncan Hall's in general) curriculum

- The vision is for this course to be part of a sequence of courses that will focus on the theory+practice of methods

(Feedback on GoogleDoc)

Course format

- Lectures (slides) + whiteboard + in-class code running

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- "Interlude" lectures to provide some background (if needed)
- Your workload: scribing, presentations,
  reviews, final project

Goals + outcomes

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- Consider possible extensions of these works (project)
- Comprehend how optimization is key in ML/AI/SP
- Read and review recent papers

Prerequisites

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- A quiz will be given today for self-assessment

Grading policy

- 5% participation
- 10% scribing notes
- 15% reviews
- 20% presentation
- 50% project

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Usually there is scaling in final grades. For me, a good grade is given based on the overall performance of the students: I value self-motivation, being proactive and enthusiasm.

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- Deliverable in LaTEX

  (template available online)

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(Almost every week where you choose a paper)

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- You select on Tuesday – deliverables next Tuesday

  (not always true if the subject is too generic for paper reading)

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(btw, this is not your project presentation)

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- Grading: slides quality, clarity of main ideas

Final Project

(see syllabus)

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Please come find me the earliest to discuss projects

Final Project

(see syllabus)

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You should start reading papers soon, so that around mid-way you have a good project proposal

Any questions?

Quiz
(15 min. or NBO)

Setting up the background

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$$0 + x = x, \quad x \in \mathbb{R}^p$$

$$\alpha(x + y) = \alpha x + \alpha y, \quad x, y \in \mathbb{R}^p$$ (Distributive)

Vectors

– Span of a set of vectors:

$$\operatorname{span}\{x_1, x_2, \ldots, x_k\} = \{\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_k x_k \mid \alpha_i \in \mathbb{R}, i = [1, k]\}$$

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- Inner product:

$$x^\top y = \langle x, y \rangle = \sum_{i=1}^{p} x_i \cdot y_i$$

Vectors

- Inner production interpretation:

$$\langle x, y \rangle = \|x\| \cdot \|y\| \cdot \cos\theta$$

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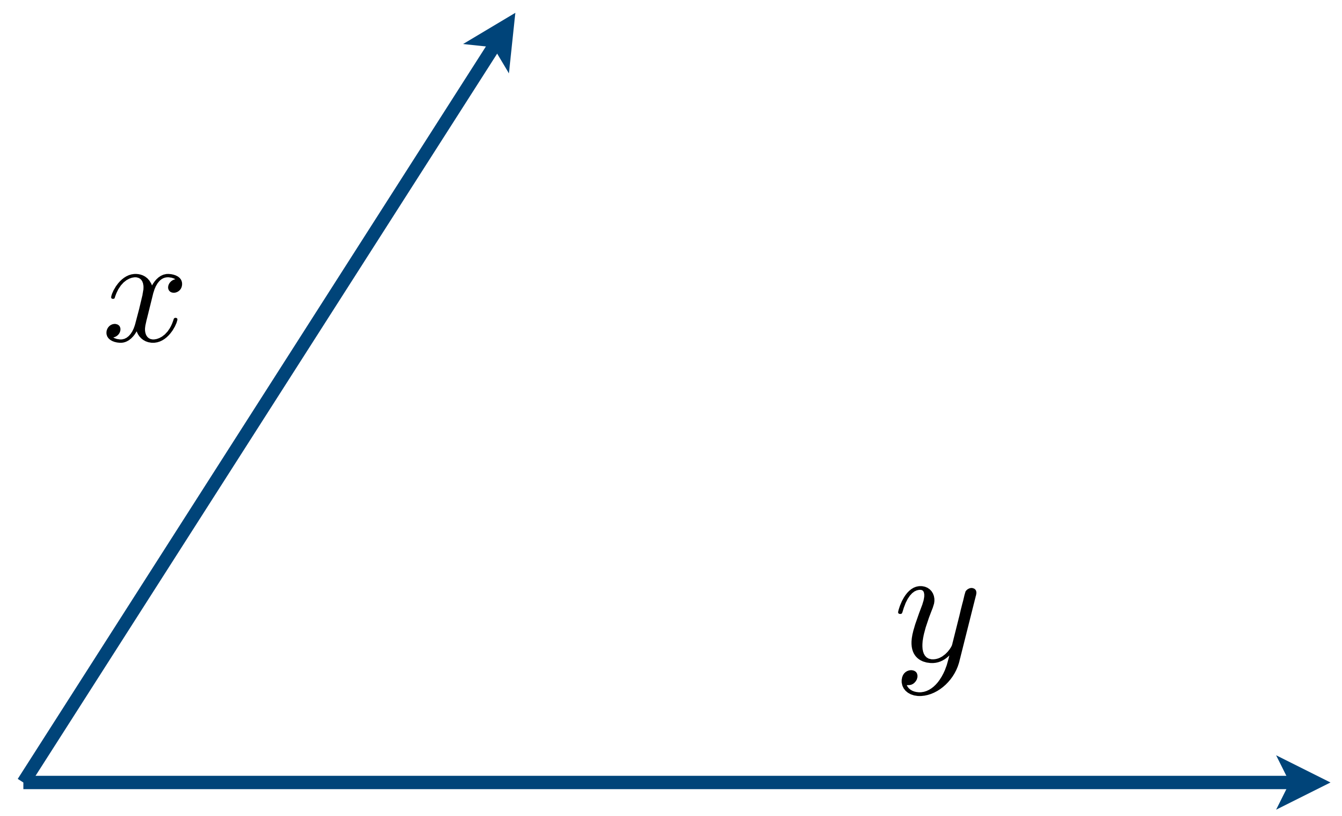

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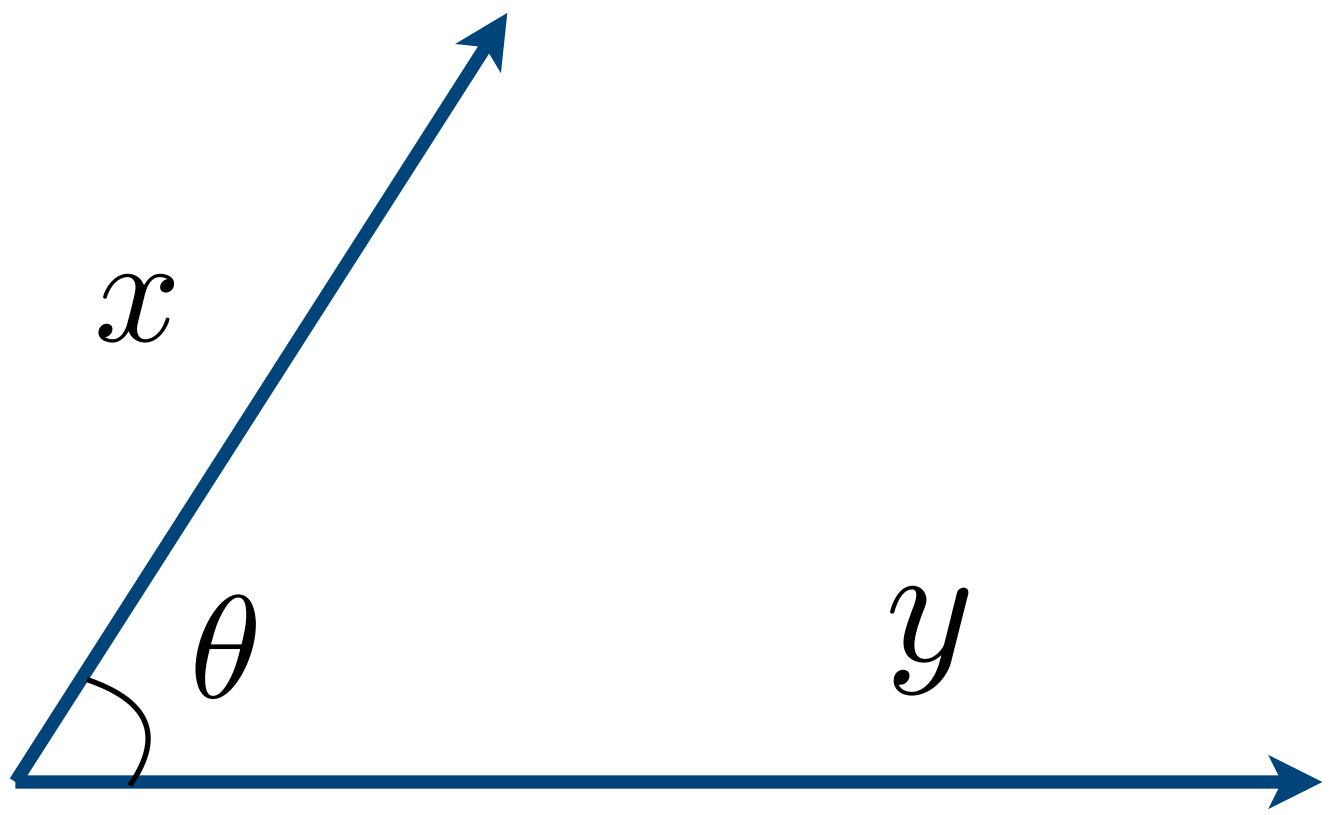

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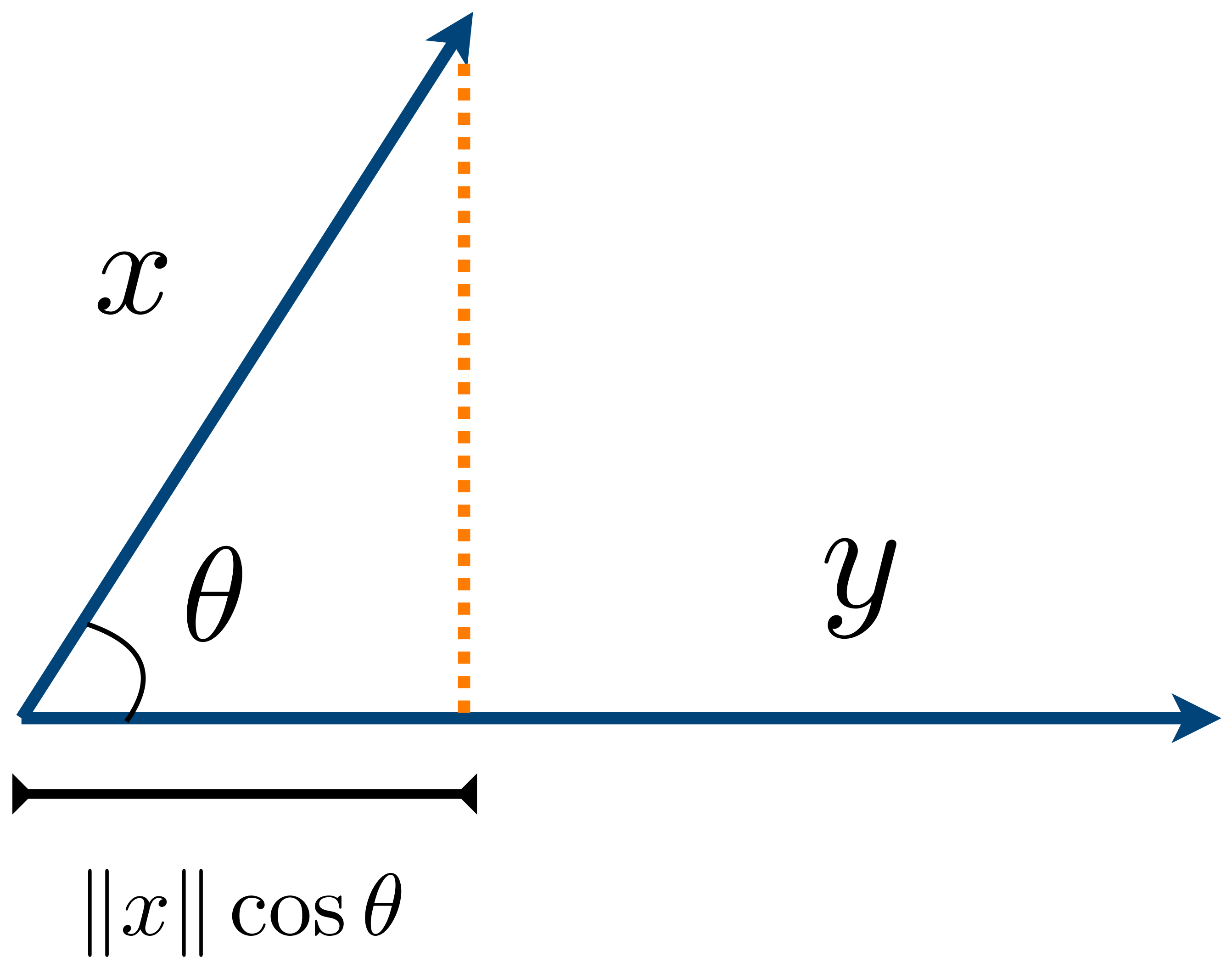

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Vectors

– Norms = notion of distance in multiple dimensions

Properties:

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\begin{aligned}
\|x\| &\geq 0, \forall x \in \mathbb{R}^p \\
\|x\| &= 0, \text{ iff } x = 0 \\
\|\alpha x\| &= |\alpha| \|x\|, \forall \alpha \in \mathbb{R} \\
\|x + y\| &\leq \|x\| + \|y\| \qquad \text{(Triangle inequality)} \\
|x^\top y| &\leq \|x\| \|y\| \qquad \text{(Cauchy–Schwarz)}
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$$\|x + y\| \leq \|x\| + \|y\|$$ (Triangle inequality)

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- Standard vector norms:

$$\|x\|_2 = \sqrt{\sum_i x_i^2} \qquad \|x\|_1 = \sum_i |x_i| \qquad \|x\|_\infty = \max_i |x_i|$$

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- Famous wanna-be norms: $\|x\|_0 = \operatorname{card}(x)$

Example – Sparse projection

– Find:

$$\widehat{x} \in \underset{x \in \mathbb{R}^p}{\operatorname{argmin}} \quad \|x - y\|_2^2 \qquad \text{for given } \; y \in \mathbb{R}^p$$

$$\text{s.t.} \quad \|x\|_0 \leq k < p$$

Example – Sparse projection

– Find: $\widehat{x} \in \underset{x \in \mathbb{R}^p}{\operatorname{argmin}} \quad \|x - y\|_2^2$ for given $y \in \mathbb{R}^p$

$\quad$ s.t. $\quad \|x\|_0 \leq k < p$

1. What is this problem?
2. How to solve this problem?

– Step 1: Assume we know $\mathcal{S} := \operatorname{supp}(\widehat{x}) \subset \{1, \ldots, p\}$

$$\widehat{x}_{\mathcal{S}} = y_{\mathcal{S}}$$

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– Step 3: Find efficient ways to solve the last part!

Matrices

- Matrix in m, n dimensions: $A \in \mathbb{R}^{m \times n}$

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & & \ddots & \vdots \\ A_{m1} & A_{m2} & \cdots & A_{mn} \end{bmatrix}$$

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- Properties:

$$A + B = B + A,\ \forall A, B \in \mathbb{R}^{m\times n}$$
$$(A + B) + C = A + (B + C),\ \forall A, B, C \in \mathbb{R}^{m\times n}$$
$$A + 0 = 0 + A,\ \forall A \in \mathbb{R}^{m\times n}$$
$$(A + B)^\top = A^\top + B^\top,\ \forall A, B \in \mathbb{R}^{m\times n}$$

Matrices

– Matrix multiplication： $C = AB$ where $C \in \mathbb{R}^{m \times p}$, $A \in \mathbb{R}^{m \times n}$, and $B \in \mathbb{R}^{n \times p}$

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– Properties：

$$(AB)C = A(BC),\ \forall A, B, C$$

$$\alpha(AB) = (\alpha A)B,\ \forall A, B$$

$$A(B + C) = AB + AC,\ \forall A, B, C$$

$$(AB)^\top = B^\top A^\top,\ \forall, A, B$$

$$AB \neq BA$$

Matrices

- Inner product:

$$\langle A, B \rangle = \mathrm{Tr}(A^\top B) = \mathrm{Tr}(B^\top A), \forall A, B \in \mathbb{R}^{m \times n}$$

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- Rank of a matrix: maximum # of independent columns or rows

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$$\langle A, B \rangle = \mathrm{Tr}(A^\top B) = \mathrm{Tr}(B^\top A), \forall A, B \in \mathbb{R}^{m \times n}$$

- Rank of a matrix: maximum # of independent columns or rows

- Nullspace of a matrix: $\{x \mid Ax = 0\}$

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- Positive semi-definite matrices：　$A \succeq 0$

1. $A \in \mathbb{R}^{n\times n}$
2. A is symmetric
3. $x^\top Ax \geq 0,\ \forall x \in \mathbb{R}^n,\ x \neq 0$

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- Matrix singular value decomposition: $A \in \mathbb{R}^{m \times n}$

$$A = U\Sigma V^\top = \sum_{i=1}^{r} \sigma_i u_i v_i^\top,\ U \in \mathbb{R}^{m\times r}, \Sigma \in \mathbb{R}^{r\times r}, V \in \mathbb{R}^{n\times r} \qquad r \le \{m, n\}$$

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- Left and right singular vectors are orthogonal: $U^\top U = I$ and $V^\top V = I$

Matrices

- Norms:

$$\|A\|_F = \sqrt{\sum_{ij} A_{ij}^2}$$

(Frobenius norm)

$$\|A\|_* = \sum_{i}^{r} \sigma_i$$

(Nuclear norm)

$$\|A\|_2 = \max_i \sigma_i$$

(Spectral norm)

..there are more norms to worry about (e.g., operator norms)
but we will skip them here..

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  - The columns and rows of A are linearly independent
  - There exists a square matrix, A^{-1}, such that $A^{-1}A = AA^{-1} = I$

Example – Low-rank projection

– Find:

$$\begin{aligned} \widehat{X} \in \min_{X} \quad & \frac{1}{2}\|X - Y\|_F^2 \\ \text{s.t.} \quad & \operatorname{rank}(X) \le r \end{aligned}$$

for given $Y \in \mathbb{R}^{m \times n}$

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– Step 2: Transform original problem

$$\min_{X} \quad \frac{1}{2}\|C - \Sigma\|_F^2 \qquad \text{where} \qquad C = U^\top X V$$

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How does this relate to sparse projection?

Demo

Papers to review – due next Thursday

(Select one of the following papers)

- “Efficient projections onto the ℓ_1–norm ball for learning in high dimensions”, Duchi et al., 2008.

- “Stay on path: PCA along graph paths”, Asteris et al., 2015.

- “CUR matrix decompositions for improved data analysis”, Mahoney and Drineas, 2008.

- “Simple and Deterministic Matrix Sketching”, Liberty, 2013.

(Rule: as you read, think of extensions – feel free to find me for more discussions)

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- We have set up background and notation w.r.t. linear algebra
- We saw a toy example where non−convex operations happen

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Next lecture

- Brief introduction to convex optimization and related topics

Intuition for inner product