2B : Linear Algebra
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Contents

These lecture notes were collated by me from a mixture of sources , the two main sources being the lecture notes provided by the lecturer and the content presented in-lecture. All other referenced material (if used) can be found in the *Bibliography* and *References* sections.

The primary goal of these notes is to function as a succinct but comprehensive revision aid, hence if you came by them via a search engine , please note that they're not intended to be a reflection of the quality of the materials referenced or the content lectured.

Lastly, with regards to formatting, the pdf doc was typeset in LaTeX, using a modified version of Stefano Maggiolo's class

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1 VECTORS

1.1 *Basics*

Most of this has been covered already, if too succinct check 1R/1S notes

1.1 definition. Vector displacement from one point to another in space ; geometrically represented by a directed line segment

1.2 remark. In general we use the origin of the cartesian coordinate system as the displacement origin

1.3 definition. $\mathbb{R}^n$ the set of all n-tuples of real number , for $n \in \mathbb{R}$

$$\mathbb{R}^n = \{(x_1, \ldots, x_n) : x_1, \ldots, x_n \in R\}$$

1.4 remark. remember that () are used for ordered objects

1.5 notation. $\mathbf{v}$; $\underline{v}$; $\mathbf{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$; $\mathbf{v} = \begin{bmatrix} v_1 \ldots v_n \end{bmatrix}$

1.6 definition. Vector Addition $\mathbf{u} + \mathbf{v} = (u_1 + v_1, \ldots, u_n + v_n)$

1.7 definition. Scalar Multiplication $\lambda\mathbf{v} = (\lambda v_1, \ldots, \lambda v_n)$

Algebraic Properties of Vector Addition

$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ (commutativity of vector addition)
$(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$ (commutativity of vector addition)
$\mathbf{u} + \mathbf{0} = \mathbf{u}$
$\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$
$c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$ (distributivity of vector addition)
$(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$ (distributivity of scalar addition)
$c(d\mathbf{u}) = (cd)\mathbf{u}$
$\mathbf{1u} = \mathbf{u}$

1.2 *Linear Combination and Independence*

1.8 definition. Linear Combination Sum of the members of a set, where each member is multiplied by a constant

It follows from **??** that for a set composed of vectors , we can say that an arbitary vector $\mathbf{v}$ is a linear combination of $\mathbf{u}, \mathbf{w}$ *iff*

$$\mathbf{v} = j\mathbf{u} + k\mathbf{w} \quad , \quad \text{for } j, k \in \mathbb{R}$$

Systems of Linear Equations and Matrices

Note that for a vector in $\mathbb{R}^n$ we have a linear equation in n variables whose solution is a vector of size n. In other words, the linear equation $a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$ has solution $a_1s_1 + a_2s_2 + \cdots + a_ns_n = b$

Hence, we can find the values which satisfy **??** by finding the solution vector which satisfies all linear equations simultaneously, i.e by constructing and solving the appropriate system of equations. This becomes clearer if we rewrite **??** in vector form

$$\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = j \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} + k \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix}$$

Recall also that we can transform the above in an augment matrix, where each column represents a vector and its unknown coefficient, and by performing successive EROs we can simplify the system enough so as to hopefully be able to glean the solution vector from the matrix

1.9 definition. Pivot leading non-zero entry

1.10 definition. Row Echelon Form Matrix which satisfies the following:

1. All pivots in lower rows are strictly to the right of those of the rows above it
2. All non-zero rows are above zero rows

$$\begin{bmatrix} 1 & 2 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & 5 \end{bmatrix}$$

1.11 definition. Reduced Row Echelon Form Matrix which is in REF and

1. All pivots in lower rows are equal to 1
2. Every element above the pivots is equal to 0

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

2

2.1 definition. Span

$$\operatorname{span}(S) = \left\{ \sum_{i=1}^{k} \lambda_i v_i | k \in \mathbb{N}, v_i \in S, \lambda_i \in K \right\}$$

To do (??)

2.2 definition. Linearly Independent

2.3 corollary. *if* $\mathbf{v}$ *is a linear combination of* $\mathbf{w}, \mathbf{z}$ *, then they're linearly* ***de****pendent*

2.4 corollary. *if* $\mathbf{v}$ *is a multiple of* $\mathbf{w}$ *then,* $\mathbf{v}, \mathbf{w}$ *are linearly* ***de****pendent*

2.5 remark. (2.4) is just a special case of (**??**) , where by definition of scalar multiple and linear combination we have that $\mathbf{v} = \mathbf{0} + k\mathbf{w}$, $k \in \mathbb{R}$

2.6 remark. In sum , if any vector within a vector set can be rewritten as a linear combination of any other vector within the same set. Then that vector is redundant, and the set is linearly dependent.

2.7 remark. Note that $S = \text{span}(\mathbf{w}, \mathbf{z}, \mathbf{v}) = \text{span}(\mathbf{w}, \mathbf{z})$, since as noted above $\mathbf{v}$ is redundant because it is accounted for by $\lambda\mathbf{w} + \theta\mathbf{z}$

$dim(P)$ is the dimension of the vector space e.g. $dim(R^3) = 3$

Cheatsheet: Does V span P

Where V is a vector set $\mathbf{v_1}, \ldots, \mathbf{v_n}$, and P is a vector space R^n

Check $V\mathbf{x} = \mathbf{p}$, for arbitrary $\mathbf{p} \in P$
Check that $rk(V) = dim(P)$ **To do (??)**
Check if V contains the standard basis. If it does, then it must contain all of P

3 Elementary Matrices & Subspaces

3.1 definition. Space

Informally, a space is a set of vectors composed of all possible linear combinations of all vector(s) in that space. In other words, I can choose one or more vectors in that space and I can add and multiply them and never leave that space. Crucially, this must include a way to get back to the origin, hence the 0 vector must be part of every space.

3.2 remark. All objects within a space must go through the origin (e.g. line in $\mathbb{R}^2$, plane in $\mathbb{R}^3$)

We call a space entirely contained within another space S, a *subspace* of S. So a subspace , S_b of S , is a space within which all the vector operations on S are defined on S_b but where usually some other conditions have to be satisfied, so that performing those operations on the members of S_b imply never leaving S_b and S

3.3 remark. A common example of subsets which are not subspaces are those which do not go through the origin of S. Given that they are defined for all operations on S_b , however scalar multiplication by 0 is not defined

Note that for a matrix $A_{m \times n}$, its columns linear combinations form a subspace of R^m. In order to picture why, think of R^3. Each vector represents *"one part of the 3D"* , but by themselves they can't represent a space. Geometrically they represent 3 finite individual line segments. However, by including all possible linear combinations we *fill in* the space between them, effectively representing all possible planes within R^3 , or all possible planes which satisfy some part of R^3 given some condition (e.g. positive numbers)

3.4 definition. Column Space

Note then, that the above means that we are essentially asking whether $A\mathbf{x} = \mathbf{b}$ is within R^m, and if so then for what $\mathbf{x}$. Now, given that the column space *by definition* contain all possible combinations, then it contains all possible $A\mathbf{x}$; Hence, $A\mathbf{x} = \mathbf{b} \iff \mathbf{b} \in C(A)$

3.5 example. mit:null
A specially useful corollary which follows from above, is that one can find all possible solutions for an homogenous system if we set $\mathbf{b} = \mathbf{0}$.

3.6 proposition. *The set of solutions of an homogenous system A is the null space of the augmented matrix $[A|\mathbf{0}]$ (see proof below)*

Proof. It follows directly from above, and from the definition of an homogenous system QED

4 Basis, Dimensions, Rank

To do…

- ☐ 1 (p. **??**): `https://en.wikipedia.org/wiki/Linear_span`
- ☐ 2 (p. **??**): ADD REF TO RANK SEC4
- ☐ 3 (p. **??**): Colum Space example MIT lecture
- ☐ 4 (p. **??**): Bibtex: `http://sites.millersville.edu/bikenaga/linear-algebra/inverse/inverse.html` ; `https://ocw.mit.edu/courses/mathematics/18-06-linear-algebra-spring-2010/video-lectures/lecture-6-column-space-and-nullspace/` ; `https://math.stackexchange.com/questions/56201/how-to-tell-if-a-set-of-vect`