MATHEMATICS 1S
Andrew Wilson

Joao Almeida-Domingues[*]

University of Glasgow

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Contents

[*] 2334590D@student.gla.ac.uk

1 VECTORS

1.1 *Generalities*

Lecture 1
January 7th, 2019

1.1 definition. A scalar is a one-component quantity that is invariant under rotations of the coordinate system, which describes the magnitude of something

1.2 definition. A vector is a two-component quantity, with magnitude (a positive real number) and direction

1.3 remark. If a vector has both a magnitude and direction of 0, then that vector is the zero vector. The zero vector can be thought as having no direction, or all directions

1.4 definition. Equality: Two vectors are equal if they have the same magnitude and the same direction

1.5 remark. Every vector is unique

Proof. Let $\mathbf{u}, \mathbf{v}$ be two vectors with magnitude λ and the same direction. Hence by **??** , $\mathbf{u} = \mathbf{v}$ □

1.6 notation. Generally, in printed text $\overrightarrow{v}$ or $\mathbf{v}$. Handwritten $\underline{v}$. Magnitude $|\mathbf{v}|$

1.7 definition. Unit vector is a vector of magnitude 1. There is exactly one for any given direction

1.8 notation. Generally, for a given vector $\mathbf{v}$, $\hat{\mathbf{v}}$

1.9 proposition. ***Parallelogram*** $\overrightarrow{AB} = \overrightarrow{DC}$ *, i.e Traversing left-up is the same as up-right (see proof below)*

Proof. It follows from the fact that the opposite sides of a parallelogram are parallel and of equal length. Hence they are equal, by **??** □

1.10 proposition. ***Negative vectors*** $\overrightarrow{AB} = u \iff \overrightarrow{BA} = -u$ *(see proof below)*

Proof. It follows from the fact that they have the same magnitude but opposite directions □

1.11 proposition. ***Zero vector*** $\overrightarrow{AA} = 0$ *(see proof below)*

Proof. For any point A, $|AA| = 0$ and so $\overrightarrow{AA} = 0$ □

1.2 *Addition and Scalar Multiplication*

1.12 definition. Addition Let $\mathbf{u} = \overrightarrow{PQ}$, $\mathbf{v} = \overrightarrow{QR}$.$\mathbf{u} + \mathbf{v} = \overrightarrow{PR}$. *nose-to-tail*

1.13 remark. As per usual subtraction is simply, $\mathbf{u} + (-\mathbf{v})$

1.14 definition. Scalar Multiplication For a vector $\mathbf{u}$, and a scalar λ. $\lambda\mathbf{u}$ scales the vector's magnitude by λ, and if $\lambda < 0$ inverts its direction

Lecture 2
January 8^{th}, 2019

Properties of Addition and Multiplication

1.15 proposition. ***Commutative*** $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ *(see proof below)*

Proof.

$$\begin{aligned}
\mathbf{u} + \mathbf{v} &= [u_1, u_2, \ldots, u_n] + [v_1, v_2, \ldots, u_n] \\
&= [u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n] \text{ vector addition} \\
&= [v_1 + u_1, v_2 + u_2, \ldots, v_n + u_n] \text{ commutative adition of real numbers} \\
&= [v_1, v_2, \ldots, v_n] + [u_1, u_2, \ldots, u_n] \\
&= \mathbf{v} + \mathbf{u}
\end{aligned}$$

□

1.16 proposition. ***Associative*** $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$ *(see proof below)*

Proof.

$$\begin{aligned}
(\mathbf{u} + \mathbf{v}) + \mathbf{w} &= ([u_1, u_2, \ldots, u_n] + [v_1, v_2, \ldots, v_n]) + [w_1, w_2, \ldots, w_n] \\
&= [u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n] + [w_1, w_2, \ldots, w_n](1.10) \\
&= [(u_1 + v_1) + w_1, (u_2 + v_2) + w_2, \ldots, (u_n + v_n) + w_n] \text{ commutative property of scalars} \\
&= [u_1 + (v_1 + w_1), u_2 + (v_2 + w_2), \ldots, u_n + (v_n + w_n)] \text{ associative addition of real numbers} \\
&= [u_1, u_2, \ldots, u_n] + [v_1 + w_1, v_2 + w_2, \ldots, v_n + w_n] \\
&= \mathbf{u} + \mathbf{v} + \mathbf{w}
\end{aligned}$$

□

1.17 proposition. ***Distributive*** $\lambda(\mathbf{u} + \mathbf{v}) = \lambda\mathbf{u} + \lambda\mathbf{v}$ *(see proof below)*

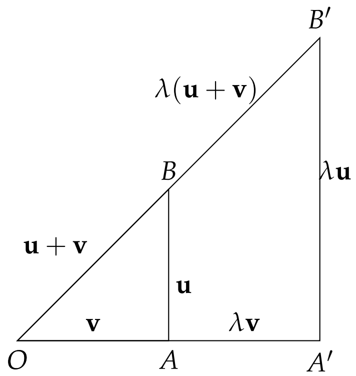
Proof.

$$\begin{aligned}
\lambda(\mathbf{u} + \mathbf{v}) &= \lambda([u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n] \text{ vector addition} \\
&= [\lambda(u_1 + v_1), \lambda(u_2 + v_2), \ldots, \lambda(u_n + v_n)] \text{ scalar multiplication} \\
&= [\lambda u_1 + \lambda v_1, \lambda u_2 + \lambda v_2, \ldots, \lambda u_n + \lambda v_n] \text{ distributive for real numbers} \\
&= \lambda\mathbf{u} + \lambda\mathbf{v}
\end{aligned}$$

□

Proof. 2 , *by a diagram*

Let $\mathbf{u}, \mathbf{v}$ be two non-zero vectors, and A, B, C be 3 distinct points. Then:



Let the "prime" triangle represent a λ fold enlargement of the original triangle representing the original vectors and their addition. Hence we have that,

$$\begin{aligned} OB' = \lambda(\mathbf{u}+\mathbf{v}) &= OA' + A'B' \\ &= \lambda\mathbf{v} + \lambda\mathbf{u} \end{aligned}$$

□

1.3 *Parallel and Position vectors*

1.18 definition. Parallel: Let $\mathbf{u}$, $\mathbf{v}$ be two non-zero vectors. Then, $\mathbf{v}$ is parallel to $\mathbf{u}$ *iff* $\mathbf{v} = \lambda\mathbf{u}$ (i.e. if they share the same or opposite directions) and $\hat{\mathbf{u}} = \frac{1}{|\mathbf{u}|}\mathbf{u}$

1.19 remark. The non-zero vector is parallel to all vectors

Proof. The first part of the definition is self-evident as any scalar multiple of a vector will only alter its magnitude and/or reverse its direction. For the second part, we have that:

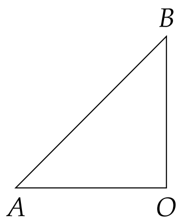
$$\left|\frac{1}{|\mathbf{u}|}\mathbf{u}\right| = \frac{1}{|\mathbf{u}|}|\mathbf{u}| = 1$$

Hence, we've shown that $\frac{1}{|\mathbf{u}|}|\mathbf{u}|$ is a unit vector of $\mathbf{u}$, which means that it only varies in magnitude, and is therefore parallel. □

1.20 definition. Position: Let O denote the origin, the vector from O to any point P ($\overrightarrow{OP}$) is called the position vector. For any points A and B, $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$

1.21 notation. $\mathbf{r}_p$

Proof.



$$\begin{aligned}\overrightarrow{AB} &= \overrightarrow{AO} + \overrightarrow{OB} \\ &= (-\mathbf{a}) + \mathbf{b} \\ &= \mathbf{b} - \mathbf{a}\end{aligned}$$ □

1.4 *Collinearity and the section formula*

1.22 definition. Collinear points, lie on a straight line

1.23 remark. One can test wether points are collinear by finding if their directed line segments, i.e. the vector formed starting at a point and ending at the other, are parallel.

1.24 example. Let $\overrightarrow{AB} = \mathbf{u}, \overrightarrow{BC} = 2\mathbf{u}, \overrightarrow{AC} = \mathbf{u} + \mathbf{2u} = \mathbf{3u}$. Hence they are all parallel to $\mathbf{u}$, it follows then that they are collinear.

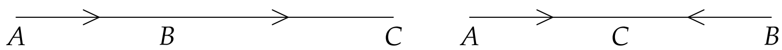
1.25 remark.

$$AB : BC = \beta : \alpha \implies \alpha\overrightarrow{AB} = \overrightarrow{BC}$$

Setting $\lambda = \frac{\beta}{\alpha}$,

$$\overrightarrow{AB} = \lambda\overrightarrow{BC} \quad AB : BC = \lambda : 1$$

Since A, B and C are collinear, we can deduce the distance between the points using their ratio $\left(\lambda = \frac{|AB|}{|BC|}\right)$. Furthermore, for $\lambda > 0$ we have that the vectors have the same direction, hence B lies between A and C. Note however that this is not true for $\lambda < 0$



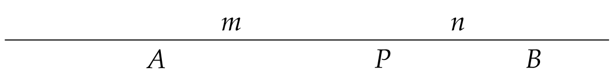
*******CLARIFY POST LECTURE********

1.26 proposition. ***Section Formula*** *Let A, B and P be collinear points, s.t:*

$$AP : PB = m : n$$

Then, P has position vector

$$\mathbf{p} = \frac{m\mathbf{b} + n\mathbf{a}}{m + n}$$



(see proof below)

Proof.

$$\begin{aligned} n\overrightarrow{AP} &= m\overrightarrow{PB} \\ n(\mathbf{p} - \mathbf{a}) &= m(\mathbf{b} - \mathbf{p}) \\ (m+n)\mathbf{p} &= m\mathbf{b} + n\mathbf{a} \\ \mathbf{p} &= \frac{m\mathbf{b} + n\mathbf{a}}{m+n} \end{aligned}$$

□

special case of 1.26, where $m = n = 1$

1.27 corollary. *The midpoint of AB has position vector* $\frac{1}{2}(\mathbf{a} + \mathbf{b})$

2 Logical Matters & Proof

2.1 definition. Direct Proof It consists of an argument that starts from the hypothesis and by a sequence of logical steps ends at the conclusion

2.2 remark. A common misconception is starting from the conclusion and finding something "true", by using both sides of the equation simultaneously

2.3 example. Prove that the product of two odd integers is also odd

Proof. Let a, b be arbitrary odd integers. Then a $a = 2k + 1$ and $b = 2l + 1$, for some $k, l \in \mathbb{Z}$. Hence,

$$\begin{aligned} ab &= (2k+1)(2l+1) \\ &= 4kl + 2k + 2l + 1 \\ &= 2(2kl + k + l) + 1 \end{aligned}$$

Therefore, since $2kl + k + l \in \mathbb{Z}$, ab is odd. □