

# MATHEMATICS 1S

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## CONTENTS

1	Vectors	2
1.1	Generalities . . . . .	2
1.2	Addition and Scalar Multiplication . . . . .	2
1.3	Parallel and Position vectors . . . . .	4
1.4	Collinearity and the section formula . . . . .	4
2	Logical Matters & Proof	5

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## 1 VECTORS

## 1.1 Generalities

Lecture 1  
January 7<sup>th</sup>, 2019

**1.1 definition.** A **scalar** is a one-component quantity that is invariant under rotations of the coordinate system, which describes the magnitude of something

**1.2 definition.** A **vector** is a two-component quantity, with magnitude (a positive real number) and direction

**1.3 remark.** If a vector has both a magnitude and direction of 0, then that vector is the zero vector. The zero vector can be thought as having no direction, or all directions

**1.4 definition. Equality:** Two vectors are equal if they have the same magnitude and the same direction

**1.5 remark.** Every vector is unique

*Proof.* Let  $\mathbf{u}, \mathbf{v}$  be two vectors with magnitude  $\lambda$  and the same direction. Hence by 1.1,  $\mathbf{u} = \mathbf{v}$   $\square$

**1.6 notation.** Generally, in printed text  $\vec{v}$  or  $\mathbf{v}$ . Handwritten  $\underline{v}$ . Magnitude  $|\mathbf{v}|$

**1.7 definition. Unit vector** is a vector of magnitude 1. There is exactly one for any given direction

**1.8 notation.** Generally, for a given vector  $\mathbf{v}$ ,  $\hat{\mathbf{v}}$

**1.9 proposition. Parallelogram**  $\vec{AB} = \vec{DC}$ , i.e Traversing left-up is the same as up-right (see proof below)

*Proof.* It follows from the fact that the opposite sides of a parallelogram are parallel and of equal length. Hence they are equal, by 1.1  $\square$

**1.10 proposition. Negative vectors**  $\vec{AB} = \mathbf{u} \iff \vec{BA} = -\mathbf{u}$  (see proof below)

*Proof.* It follows from the fact that they have the same magnitude but opposite directions  $\square$

**1.11 proposition. Zero vector**  $\vec{AA} = 0$  (see proof below)

*Proof.* For any point  $A$ ,  $|AA| = 0$  and so  $\vec{AA} = 0$   $\square$

## 1.2 Addition and Scalar Multiplication

**1.12 definition. Addition** Let  $\mathbf{u} = \vec{PQ}$ ,  $\mathbf{v} = \vec{QR}$ .  $\mathbf{u} + \mathbf{v} = \vec{PR}$ . nose-to-tail

**1.13 remark.** As per usual subtraction is simply,  $\mathbf{u} + (-\mathbf{v})$

**1.14 definition. Scalar Multiplication** For a vector  $\mathbf{u}$ , and a scalar  $\lambda$ .  $\lambda\mathbf{u}$  scales the vector's magnitude by  $\lambda$ , and if  $\lambda < 0$  inverts its direction

### Properties of Addition and Multiplication

**1.15 proposition. Commutative**  $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$  (see proof below)

*Proof.*

$$\begin{aligned}\mathbf{u} + \mathbf{v} &= [u_1, u_2, \dots, u_n] + [v_1, v_2, \dots, v_n] \\ &= [u_1 + v_1, u_2 + v_2, \dots, u_n + v_n] \text{ vector addition} \\ &= [v_1 + u_1, v_2 + u_2, \dots, v_n + u_n] \text{ commutative addition of real numbers} \\ &= [v_1, v_2, \dots, v_n] + [u_1, u_2, \dots, u_n] \\ &= \mathbf{v} + \mathbf{u}\end{aligned}$$

□

**1.16 proposition. Associative**  $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$  (see proof below)

*Proof.*

$$\begin{aligned}(\mathbf{u} + \mathbf{v}) + \mathbf{w} &= ([u_1, u_2, \dots, u_n] + [v_1, v_2, \dots, v_n]) + [w_1, w_2, \dots, w_n] \\ &= [u_1 + v_1, u_2 + v_2, \dots, u_n + v_n] + [w_1, w_2, \dots, w_n] \text{ (1.10)} \\ &= [(u_1 + v_1) + w_1, (u_2 + v_2) + w_2, \dots, (u_n + v_n) + w_n] \text{ commutative property of scalars} \\ &= [u_1 + (v_1 + w_1), u_2 + (v_2 + w_2), \dots, u_n + (v_n + w_n)] \text{ associative addition of real numbers} \\ &= [u_1, u_2, \dots, u_n] + [v_1 + w_1, v_2 + w_2, \dots, v_n + w_n] \\ &= \mathbf{u} + \mathbf{v} + \mathbf{w}\end{aligned}$$

□

**1.17 proposition. Distributive**  $\lambda(\mathbf{u} + \mathbf{v}) = \lambda\mathbf{u} + \lambda\mathbf{v}$  (see proof below)

*Proof.*

$$\begin{aligned}\lambda(\mathbf{u} + \mathbf{v}) &= \lambda([u_1 + v_1, u_2 + v_2, \dots, u_n + v_n] \text{ vector addition}) \\ &= [\lambda(u_1 + v_1), \lambda(u_2 + v_2), \dots, \lambda(u_n + v_n)] \text{ scalar multiplication} \\ &= [\lambda u_1 + \lambda v_1, \lambda u_2 + \lambda v_2, \dots, \lambda u_n + \lambda v_n] \text{ distributive for real numbers} \\ &= \lambda\mathbf{u} + \lambda\mathbf{v}\end{aligned}$$

□

### 1.3 Parallel and Position vectors

**1.18 definition. Parallel:** Let  $\mathbf{u}$ ,  $\mathbf{v}$  be two non-zero vectors. Then,  $\mathbf{v}$  is parallel to  $\mathbf{u}$  iff  $\mathbf{v} = \lambda \mathbf{u}$  and  $\hat{\mathbf{u}} = \frac{1}{|\mathbf{u}|} \mathbf{u}$

*Proof.* The first part of the definition is self-evident as any scalar multiple of a vector will only alter its magnitude and/or reverse its direction. For the second part, we have that:

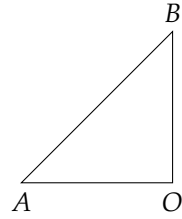
$$\left| \frac{1}{|\mathbf{u}|} \mathbf{u} \right| = \frac{1}{|\mathbf{u}|} |\mathbf{u}| = 1$$

Hence, we've shown that  $\frac{1}{|\mathbf{u}|} \mathbf{u}$  is a unit vector of  $\mathbf{u}$ , which means that it only varies in magnitude, and is therefore parallel.  $\square$

**1.19 definition. Position:** Let  $O$  denote the origin, the vector from  $O$  to any point  $P$  ( $\overrightarrow{OP}$ ) is called the position vector. For any points  $A$  and  $B$ ,  $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$

**1.20 notation.**  $\mathbf{r}_p$

*Proof.*



$$\begin{aligned} \overrightarrow{AB} &= \overrightarrow{AO} + \overrightarrow{OB} \\ &= (-\mathbf{a}) + \mathbf{b} \\ &= \mathbf{b} - \mathbf{a} \end{aligned}$$

$\square$

### 1.4 Collinearity and the section formula

**1.21 definition. Collinear** points, lie on a straight line

**1.22 remark.** One can test whether points are collinear by finding if their directed line segments, i.e. the vector formed starting at a point and ending at the other, are parallel.

**1.23 example.** Let  $\overrightarrow{AB} = \mathbf{u}$ ,  $\overrightarrow{BC} = 2\mathbf{u}$ ,  $\overrightarrow{AC} = \mathbf{u} + 2\mathbf{u} = 3\mathbf{u}$ . Hence they are all parallel to  $\mathbf{u}$ , it follows then that they are collinear.

**1.24 remark.**

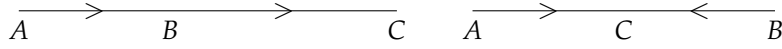
$$AB : BC = \beta : \alpha \implies \alpha \overrightarrow{AB} = \beta \overrightarrow{BC}$$

Setting  $\lambda = \frac{\beta}{\alpha}$ ,

$$\overrightarrow{AB} = \lambda \overrightarrow{BC} \quad AB : BC = \lambda : 1$$

Since  $A, B$  and  $C$  are collinear, we can deduce the distance between the

points using their ratio  $\left(\lambda = \frac{|AB|}{|BC|}\right)$ . Furthermore, for  $\lambda > 0$  we have that the vectors have the same direction, hence  $B$  lies between  $A$  and  $C$ . Note however that this is not true for  $\lambda < 0$



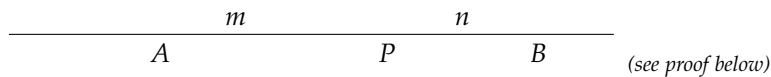
\*\*\*\*\*CLARIFY POST LECTURE\*\*\*\*\*

**1.25 proposition.** *Section Formula* Let  $A, B$  and  $P$  be collinear points, s.t:

$$AP : PB = m : n$$

Then,  $P$  has position vector

$$\mathbf{p} = \frac{m\mathbf{b} + n\mathbf{a}}{m + n}$$



*Proof.*

$$\begin{aligned} n\overrightarrow{AP} &= m\overrightarrow{PB} \\ n(\mathbf{p} - \mathbf{a}) &= m(\mathbf{b} - \mathbf{p}) \\ (m + n)\mathbf{p} &= m\mathbf{b} + n\mathbf{a} \\ \mathbf{p} &= \frac{m\mathbf{b} + n\mathbf{a}}{m + n} \end{aligned}$$

□

**1.26 corollary.** *The midpoint of  $AB$  has position vector  $\frac{1}{2}(\mathbf{a} + \mathbf{b})$*

special case of 1.4, where  
 $m = n = 1$

## 2 LOGICAL MATTERS & PROOF

**2.1 definition. Direct Proof** It consists of an argument that starts from the hypothesis and by a sequence of logical steps ends at the conclusion

**2.2 remark.** A common misconception is starting from the conclusion and finding something "true", by using both sides of the equation simultaneously

**2.3 example.** Prove that the product of two odd integers is also odd

*Proof.* Let  $a, b$  be arbitrary odd integers. Then  $a = 2k + 1$  and  $b = 2l + 1$ , for some  $k, l \in \mathbb{Z}$ . Hence,

$$\begin{aligned}ab &= (2k + 1)(2l + 1) \\&= 4kl + 2k + 2l + 1 \\&= 2(2kl + k + l) + 1\end{aligned}$$

Therefore, since  $2kl + k + l \in \mathbb{Z}$ ,  $ab$  is odd.

□