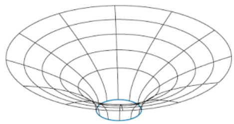


# Phys415

## General Relativity

*Lecture Notes*

# Contents

# 0 Physical Overview of General Relativity

In General Relativity, gravity is no longer a “force”...

> *Space tells matter how to move;*
> *Matter tells space how to curve.*
> — Misner, Thorne and Wheeler

Newtonian Gravity $G$ | Special Relativity $c$ | Quantum Mechanics $\hbar$

**General Relativity** $G, c$ | Quantum Field Theory $c, \hbar$

Quantum Gravity? $G, c, \hbar$

In Newtonian gravity, escape velocity is given by $\frac{1}{2}mv^2 = \frac{GMm}{R}$. Gravity is significant if $\frac{v^2}{2} \sim \frac{GM}{R}$. Special relativity is significant when $v^2 \sim c^2$. Hence, general relativity is significant when $\frac{c^2}{2} \sim \frac{GM}{R} \iff R \sim \frac{2GM}{c^2}$; this is the *Schwarzschild radius.*

When escape velocity $\sim$ light velocity, the existence of a black hole if implied.

# I Review of Special Relativity

*Background*

See (Schutz, 2009, ch 1) and (Doughty, 2018, ch 5, 12, 13).

Assumptions of Special Relativity:

1. The world is described by a 4-dimensional continuum, **spacetime**, or **Minkowski space** $\mathcal{M}^4$, which is the set of all **events** $x^\mu$,

$$x^\mu \equiv \underset{\sim}{\boldsymbol{x}} = \left(x^0, x^i\right) \equiv \left(x^0, \underset{\rightarrow}{x}\right) = (ct, \underset{\rightarrow}{x}).$$

*Notation*

Greek indices, $\mu, \nu$ run over *spacetime* index values; $\{0, 1, 2, 3\}$.
Latin indices (mid-alphabet), $i, j, k$ run over *spatial* index values; $\{1, 2, 3\}$.
$\underset{\sim}{\boldsymbol{x}}$ is a 4-vector (twiddle); $\underset{\rightarrow}{x}$ is a 3-vector (under bar).

2. There exist **inertial frames**; namely, frames in which the measured values of time $t$ and position $x^i$ of events result in *linear* equations of motion for *free* particles.

Postulates of Special Relativity:

1. Principle of Relativity: The laws of physics are invariant under transformations $x^\nu \to x^{\bar{\mu}}(x^\nu)$ from one inertial frame to another (and such transformations form a group).

2. Constancy of the Speed of Light: There exists an invariant upper bound on all velocities

$$\left|\frac{\mathrm{d}\underset{\rightarrow}{x}}{\mathrm{d}t}\right| \leq \left|\frac{\mathrm{d}\underset{\rightarrow}{x}}{\mathrm{d}t}\right|_{\max} = c \quad \text{(speed of light)}$$

and this value is the same in all inertial frames.

For photons,

$$c = \underset{\text{frame } K}{\left|\frac{\mathrm{d}\underset{\rightarrow}{x}}{\mathrm{d}t}\right|} = \underset{\text{frame } K'}{\left|\frac{\mathrm{d}\underset{\rightarrow}{x}'}{\mathrm{d}t}\right|}. \tag{I.1}$$

Rewriting (I.1) we see that, for photons,

$$\underbrace{(\mathrm{d}x)^2 + (\mathrm{d}y)^2 + (\mathrm{d}z)^2 - c^2(\mathrm{d}t)^2}_{\text{frame } K} = \underbrace{(\mathrm{d}x')^2 + (\mathrm{d}y')^2 + (\mathrm{d}z')^2 - c^2(\mathrm{d}t')^2}_{\text{frame } K'} = 0.$$

This suggests the definition of the **spacetime interval** between any neighbouring events $x^\mu$ and $x^\mu + \mathrm{d}x^\mu$ as

$$\begin{aligned}\mathrm{d}s^2 &\equiv -c^2\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 \\ &= \eta_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu \qquad \textit{...pseudo-Riemannian structure}\end{aligned}$$

where $x^\mu = (x^0, x^i) = (ct, x, y, z)$ and where

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & +1 & & \\ & & +1 & \\ & & & +1 \end{pmatrix} \equiv \mathrm{diag}\,(-1, +1, +1, +1) \equiv -1 \oplus \mathbb{1}_3.$$

For photons, $\mathrm{d}s^2 = 0$. Using the two postulates, one may show that the interval $\mathrm{d}s^2$ is invariant with respect to coordinates based in any inertial frame (Schutz, 2009, §1.6).

$$\mathrm{d}s^2 = \eta_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu = \eta_{\bar{\mu}\bar{\nu}}\mathrm{d}x^{\bar{\mu}}\mathrm{d}x^{\bar{\nu}}. \tag{I.2}$$

*Note*

- The symbol $\mathrm{d}s^2$ for the interval is purely notational convention, since we may have $\mathrm{d}s^2 < 0$ is some cases.
- The first postulate alone implies either S.R. or its $c \to \infty$ limit (a.k.a. Galilean relativity).
- Formally, $\mathrm{d}x^\mu\mathrm{d}x^\nu$ is shorthand for the *tensor product* $\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu$, and you will see that $\mathrm{d}s^2 = \underset{\sim}{\boldsymbol{\eta}}$ is really the *metric tensor*...

# I.1 Lorentz Transformations

In an inertial frame $K$, the equations of motion of a free particle are linear;

$$\frac{\mathrm{d}^2x^\mu}{\mathrm{d}\lambda^2} = 0 \qquad \Longleftrightarrow \qquad x^\mu = x_0^\mu + u^\mu\lambda,$$

where $x_0^\mu, u^\mu$ are constant and $\lambda$ is a parameter. Similarly, in any other inertial frame $\bar{K}$,

$$\frac{\mathrm{d}^2 x^{\bar\mu}}{\mathrm{d}\lambda^2} = 0 \qquad \Longleftrightarrow \qquad x^{\bar\mu} = x_0^{\bar\mu} + u^{\bar\mu}\lambda.$$

Now,

$$\begin{aligned}
\frac{\mathrm{d}x^{\bar\mu}}{\mathrm{d}\lambda} &= \frac{\partial x^{\bar\mu}}{\partial x^\nu}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda} = \frac{\partial x^{\bar\mu}}{\partial x^\nu}u^\nu && \textit{...chain rule}\\
\implies 0 &= \frac{\mathrm{d}^2 x^{\bar\mu}}{\mathrm{d}\lambda^2} = \frac{\partial}{\partial x^\alpha}\left(\frac{\partial x^{\bar\mu}}{\partial x^\nu}u^\nu\right)u^\alpha = \frac{\partial^2 x^{\bar\mu}}{\partial x^\alpha \partial x^\nu}u^\nu u^\alpha && \because\ u^\nu \textit{ constant}\\
\implies 0 &= \frac{\partial^2 x^{\bar\mu}}{\partial x^\alpha \partial x^\nu}. && \textit{...since true } \forall u^\alpha
\end{aligned}$$

So the required transformation between inertial frames is *linear*;

$$\boxed{x^{\bar\mu} = L^{\bar\mu}{}_\nu x^\nu + a^{\bar\mu},} \tag{I.3}$$

where $a^{\bar\mu}$ and $L^{\bar\mu}{}_\nu \equiv \frac{\partial x^{\bar\mu}}{\partial x^n u}$ are constants. Differentiate (I.3) and substitute into (I.2) to give $\eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = \eta_{\bar\mu\bar\nu}L^{\bar\mu}{}_\mu L^{\bar\nu}{}_\nu \mathrm{d}x^\mu \mathrm{d}x^\nu$. Since this is true $\forall\, \mathrm{d}x^\mu$,

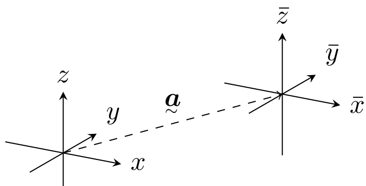
$$\boxed{\eta_{\bar\mu\bar\nu}L^{\bar\mu}{}_\mu L^{\bar\nu}{}_\nu = \eta_{\mu\nu}.} \tag{I.4}$$

In matrix form, (I.3) and (I.4) are

$$\boxed{\begin{aligned} \bar{\underset{\sim}{x}} &= L\underset{\sim}{x} + \underset{\sim}{a}, && \text{(I.3a)}\\ L^{\mathsf{T}}\eta L &= \eta. && \text{(I.4a)} \end{aligned}}$$

Transformations $\underset{\sim}{x} \mapsto \bar{\underset{\sim}{x}}$ defined by (I.3), (I.4) are the *inhomogeneous* Lorentz transformations, or **Poincaré transformations**, and form the Poincaré group IO$(1,3)$ (pronounced Inhomogeneous Orthogonal group).

"Inhomogeneous" refers to the inclusion of spacetime translations $x^\nu \mapsto x^{\bar\mu} = \delta^{\bar\mu}{}_\nu x^\nu + a^{\bar\mu}$, which form a subgroup $\mathrm{T}^4$ of the Poincaré group. If we set $a^{\bar\mu} = 0$ in (I.3), we are left with *homogeneous* transformations, called simply the **Lorentz transformations**.



*(homogeneity of spacetime)*

In G.R., our task is to generalise these ideas to general coordinate frames for which $L^{\bar\mu}{}_\nu$ are not necessarily constant.

### I.1.1 Examples of Lorentz Transformations

A transformation belonging to the (homogeneous) Lorentz group $O(1,3)$ can be represented as a matrix acting on coordinates $x^\mu$ when they are *viewed as vectors.*

$$x^\mu \cong \underset{\sim}{\boldsymbol{x}} = \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}$$

> *Note*
>
> Spacetime events (i.e., *points* in spacetime) are not themselves vectors—neither addition nor scalar multiplication of events makes physical sense (i.e., spacetime itself is not a *vector space*). However, in S.R. we may represent events by their associated displacement vector relative to a chosen orthogonal inertial frame.

The Lorentz group $O(1,3)$ consists of (combinations of) the following:

- **Rotations,** e.g., about the $z$-axis (in the $xy$-plane) by an angle $\theta$;

$$L_R(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

- **Boosts,** e.g., by a velocity $\underset{\rightarrow}{v}$ in the $x$-direction;

$$L_B(\alpha) = \begin{pmatrix} \cosh\alpha & -\sinh\alpha & 0 & 0 \\ -\sinh\alpha & \cosh\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos i\alpha & i\sin i\alpha & 0 & 0 \\ i\sin i\alpha & \cos i\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

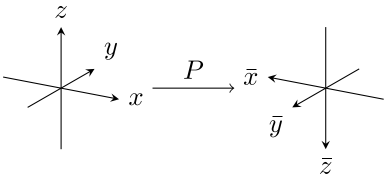
where $\alpha = \tanh^{-1}\frac{v}{c}$ is the *rapidity parameter.* In terms of the velocity $\beta \equiv \frac{v}{c}$, one has $\cosh\alpha = \frac{1}{\sqrt{1-\beta^2}} \equiv \gamma$ and $\sinh\alpha = \frac{\beta}{\sqrt{1-\beta^2}} \equiv \beta\gamma$.

> *Note*
>
> Boosts appear similar to rotations, but differ as a consequence of the indefiniteness of the metric. Formally, an $x$-boost of rapidity $\alpha$ is equivalent to a rotation by an 'imaginary angle' $\alpha' = i\alpha$ through the $\tau x$-plane, where $\tau = it$ is 'imaginary time'... though this is not a good picture physically!

An important difference between rotations and boosts is that, where $0 \leq \theta < 2\pi$ for rotations, we have $-\infty < \alpha < \infty$ for boosts, i.e., rotations form a *compact*[1] subgroup of the Lorentz (or Poincaré) group, whereas boosts are *non-compact*—and in fact do not form a subgroup (because, in general, the composition of two boosts forms a combination of a rotation and a boost).

Both rotations and boosts depend on continuous parameters ($\theta$ or $\alpha$). However, the Lorentz group $\mathrm{O}(1,3)$ also contains *discrete* transformations...

- **Parity inversion**; $P = \mathrm{diag}(1, -\mathbb{1}_3) \equiv \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$.




Notice that $P$ is *not* equivalent to a rotation; it transforms a right-handed frame into a left and vice versa, since $\det P = -1$.

> *Note*
>
> In even spatial dimensions, the transformation $R = \mathrm{diag}(1, -\mathbb{1}_{2n})$ is *not* a parity transformation; it is a rotation by $\pi$ and $\det R = 1$. In these cases, inversions $x_i \mapsto -x_i$ of a *single* spatial coordinate are parity transformations.

- **Time reversal**; $T = \mathrm{diag}(-1, \mathbb{1}_3)$.

The Lorentz matrix condition (I.4) implies that $(\det L)^2 = 1 \iff \det L = \pm 1$ for any Lorentz transformation $L \in \mathrm{O}(1,3)$. Those with $\det L = +1$ and those with $\det L = -1$ form two disconnected pieces of $\mathrm{O}(1,3)$, but only the first piece contains the identity transformation $\delta^{\bar{\mu}}{}_{\nu}$.

Inspecting the $\bar{\mu}\nu = \bar{0}0$ component of (I.4) gives

$$\left(L^{\bar{0}}{}_0\right)^2 - \sum_{\bar{k}=1}^{3} \left(L^{\bar{k}}{}_0\right)^2 = 1 \implies \left(L^{\bar{0}}{}_0\right)^2 \geq 1,$$

[1] A *compact* set is one for which any infinite sequence of elements contains a convergent subsequence. E.g., $[0, 2\pi)$ is compact, but $\mathbb{R}$ is not (consider the sequence $\{1, 2, 3, ...\} \subset \mathbb{R}$).

which shows that there exists two disconnected classes of Lorentz transformation with $L^{\bar{0}}{}_0 \geq 1$ and $L^{\bar{0}}{}_0 \leq -1$. Those with $L^{\bar{0}}{}_0 \geq 1$ are called **orthochronous**.

### I.1.2 The Restricted Lorentz Group

We define the subgroup of restricted Lorentz transformations by adding two conditions to the Lorentz matrix condition (I.4); that they be 1) orthochronous and 2) have determinant unity.

$$\mathrm{SO}^{+}(1,3) \equiv \left\{ \Lambda \,\middle|\, \Lambda^{\intercal}\eta\Lambda = \eta, \Lambda^{\bar{0}}{}_0 \geq 1, \det\Lambda = 1 \right\}$$

We have removed the discrete transformations involving $P$ and $T$, so that $\mathrm{SO}^{+}(1,3)$ is *continuous* and *connected*, unlike $\mathrm{O}(1,3)$.

*Notation*

The S in $\mathrm{SO}^{+}(1,3)$ refers to the condition $\det\Lambda = 1$, and the $^{+}$ refers to the orthochronous condition. Sometimes $\mathrm{SO}^{+}(1,3)$ is simply written as $\mathrm{SO}(1,3)$.

Groups whose elements may be continuously parametrised are **Lie groups**. The translation group $\mathrm{T}^4$ (parametrised continuously by $\Delta x^{\mu}$) and the rotation group SO(3) (parametrised continuously by three angles) are examples of *connected* Lie groups.

Reintroducing translations to the restricted Lorentz group gives the restricted Poincaré group $\mathrm{ISO}^{+}(1,3)$—also a connected Lie group. Unrestricted groups may be reconstructed by reintroducing the discrete transformations;

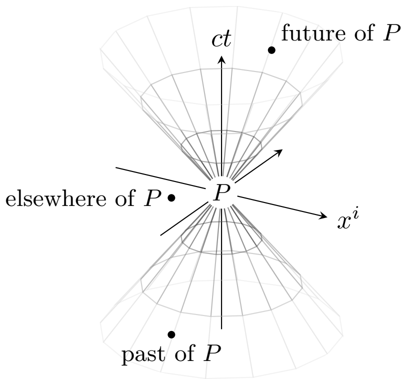
$$\mathrm{O}(1,3) = \left\{ \Lambda, \Lambda P, \Lambda T, \Lambda PT \mid \Lambda \in \mathrm{SO}^{+}(1,3) \right\}.$$

## I.2 The Scalar Product in Minkowski Spacetime

## I.3 The Causal Structure of Minkowski Spacetime

Because of the indefiniteness of the Minkowski metric (and hence of the Minkowski scalar product), 4-vectors can have positive, zero or negative norm.

A 4-vector $\underset{\sim}{\boldsymbol{V}}$ is called $\begin{cases}\text{timelike} \\ \text{null} \\ \text{spacelike}\end{cases}$ if $\underset{\sim}{\boldsymbol{V}} \cdot \underset{\sim}{\boldsymbol{V}} = V_\mu V^\mu = \eta_{\mu\nu} V^\mu V^\nu \begin{cases} < 0 \\ = 0 \\ > 0 \end{cases}.$




One spatial dimension suppressed; lightcone in $(3+1)$-d spacetime is a continuum of spheres.

By the second postulate, all *causally connected* events relative to an event $P$ lie in its future or past lightcone, because $P$ cannot causally influence events outside the lightcone without transmitting superluminal data.

A particle's **worldline** is a curve $x^\mu = x^\mu(\lambda)$ whose tangent $\underset{\sim}{\boldsymbol{V}}$ given by $V^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}$ is everywhere timelike; $V^\mu V_\mu < 0$. This implies that, in an inertial frame, the particles velocity $\underset{\rightarrow}{v}$ is subluminal $|\underset{\rightarrow}{v}| < c$ (Ex: $\square$).

Consider a freely moving particle whose (timelike) worldline unit tangent vector is $\underset{\sim}{\boldsymbol{V}}$. There always exists a Lorentz transformation which sends a timelike unit vector to $V^\mu \mapsto V^{\bar{\mu}} = (1, 0, 0, 0)$ (Ex: $\square$). The **proper time interval** $\mathrm{d}\tau$between two neighbouring events along the particle's worldline is defined as the interval of time measured in the particle's instantaneous rest frame.

$$c^2 \mathrm{d}\tau^2 = -\eta_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu \tag{I.5}$$

*Note*

Since $\mathrm{d}x$ is timelike, $\eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu < 0$, hence the minus sign. Proper time is not defined for spacelike intervals, since the above definition would yield an imaginary time. Instead, proper distance is defined by $\mathrm{d}\ell^2 = \eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu$.

Proper time is the time *experienced* by the particle; i.e., the time that would be measured by a clock moving on the same worldline.

Suppose we have a worldline $x^\mu(\lambda)$. Integrate $\mathrm{d}\tau$ via (I.5) along the path to get the total proper time elapsed between events $x^\mu(\lambda_i)$ and $x^\mu(\lambda_f)$.

$$\Delta\tau = \frac{1}{c}\int_{\lambda_i}^{\lambda_f} \mathrm{d}\lambda \sqrt{-\eta_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}} \tag{I.6}$$

*Note*

The proper time elapsed depends on the path taken between the two events.

Particle worldlines can be characterised by an action principle: variation of (I.6) with respect to the trajectories, $\delta x^\mu(\lambda)$, yields the equations of motion $\frac{\partial^2 x^\mu}{\partial \lambda^2} = 0$ (with $\delta x^\mu = 0$ fixed at both ends) (Ex: $\square$). In fact, the extremised trajectory minimises the proper time: freely falling particles take the path of maximum proper time.

# II The Equivalence Principle

# References

N. Doughty. *Lagrangian interaction: an introduction to relativistic symmetry in electrodynamics and gravitation.* CRC Press, 2018.

B. Schutz. *A first course in general relativity.* Cambridge university press, 2009.