

# Abelian $\ell$ -adic Representations and Elliptic Curves

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June 16, 2024



# CONTENTS



## EDITORS' NOTES

We have tried to keep the book as similar to the original with minor changes. Here are some changes in notation:

Original	New	Meaning
$\Sigma_K$	$M_K^0$	Set of finite places of a number field $K$ .
$\ell$	$\lambda$	The residue field of a field $L$ relative to a finite place.
$R^*$	$R^\times$	The group of units of a ring $R$ .
$U^\circ$	$\mathring{U}$	The interior of a subset $U$ of a topological space.
$A_K$	$\mathcal{O}_K$	The ring of algebraic integers of a number field $K$ .
$\mathbf{N}v$	$\mathbf{N}v$	$= [\mathcal{O}_v : \mathfrak{m}_v]$ .
$\mathbb{G}_{m/K}$	$\mathbb{G}_{m,K}$	The multiplicative group of $K$ .
$\mathbb{P}_{n/K}$	$\mathbb{P}_K^n$	The $n$ -dimensional projective space over a field $K$ .
$X \times_K L$	$X \otimes_K L$	The base change of a $K$ -scheme $X$ by a field extension $L/K$ .

We also did some minor corrections and errata we found:

- Page ?? (I-3): it originally said “ $T'/T$ ”, and it should be “ $T/T'$ ”.
- Page ?? (IV-8): it originally said “ $\Delta_v = u^{12}\Delta'$ ”, and it should be “ $\Delta_v = u_v^{12}\Delta'$ ”.



# CHAPTER I

## $\ell$ -ADIC REPRESENTATIONS

### §1. The notion of an $\ell$ -adic representation

#### 1.1 Definition

Let  $K$  be a field, and let  $K_s$  be a separable algebraic closure of  $K$ . Let  $G = \text{Gal}(K_s/K)$  be the Galois group of the extension  $K_s/K$ . The group  $G$ , with the Krull topology, is compact and totally disconnected. Let  $\ell$  be a prime number, and let  $V$  be a finite-dimensional vector space over the field  $\mathbb{Q}_\ell$  of  $\ell$ -adic numbers. The full linear group  $\text{Aut}(V)$  is an  $\ell$ -adic Lie group, its topology being induced by the natural topology of  $\text{End}(V)$ ; if  $n = \dim(V)$ , we have  $\text{Aut}(V) \cong \text{GL}(n, \mathbb{Q}_\ell)$ . I-1

**Definition 1.1.** An  $\ell$ -adic representation of  $\mathfrak{G}$  (or, by abuse of language, of  $K$ ) is a continuous homomorphism  $\rho: G \rightarrow \text{Aut}(V)$ .

**Remark.** 1) A *lattice* of  $V$  is a sub- $\mathbb{Z}_\ell$ -module  $T$  which is free of finite rank, and generate  $V$  over  $\mathbb{Q}_\ell$ , so that  $V$  can be identified with  $T \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ . Notice that there exists a lattice of  $V$  which is stable under  $\mathfrak{G}$ . This follows from the fact that  $\mathfrak{G}$  is compact.

Indeed, let  $L$  be any lattice of  $V$ , and let  $H$  be the set of elements  $g \in \mathfrak{G}$  such that  $\rho(g)L = L$ . This is an open subgroup of  $G$ , and  $G/H$  is finite. The lattice  $T$  generated by the lattices  $\rho(g)L$ ,  $g \in G/H$ , is stable under  $G$ . I-2

Notice that  $L$  may be identified with the projective limit of the free  $(\mathbb{Z}/\ell^m\mathbb{Z})$ -modules  $T/\ell^m T$ , on which  $\mathfrak{G}$  acts; the vector space  $V$  may be reconstructed from  $T$  by  $V = T \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ .

- 2) If  $\rho$  is an  $\ell$ -adic representation of  $\mathfrak{G}$ , the group  $\mathfrak{G} = \text{Im}(\rho)$  is a closed subgroup of  $\text{Aut}(V)$ , and hence, by the  $\ell$ -adic analogue of Cartan's theorem (cf. [28])  $\mathfrak{G}$  is itself an  $\ell$ -adic Lie group. Its Lie algebra  $\mathfrak{g} = \text{Lie}(\mathfrak{G})$  is a subalgebra of  $\text{End}(V) = \text{Lie}(\text{Aut}(V))$ . The Lie algebra  $\mathfrak{g}$  is easily seen to be invariant under extensions of finite type of the ground field  $K$  (cf. [24], 1.2).

### Exercises.

- 1) Let  $V$  be a vector space of dimension 2 over a field  $k$  and let  $H$  be a subgroup of  $\text{Aut}(V)$ . Assume that  $\det(1 - h) = 0$  for all  $h \in H$ . Show the existence of a basis of  $V$  with respect to which  $H$  is contained either in the subgroup  $\begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix}$  or in the subgroup  $\begin{pmatrix} 1 & 0 \\ * & * \end{pmatrix}$  of  $\text{Aut}(V)$ .
- 2) Let  $\rho: G \rightarrow \text{Aut}(V_\ell)$  be an  $\ell$ -adic representation of  $\mathfrak{G}$ , where  $V_\ell$  is a  $\mathbb{Q}_\ell$ -vector space of dimension 2. Assume  $\det(1 - \rho(s)) = 0 \pmod{\ell}$  for all  $s \in G$ . Let  $T$  be a lattice of  $V_\ell$  stable by  $G$ . Show the existence of a lattice  $T'$  of  $V_\ell$  with the following two properties:
- (a)  $T'$  is stable by  $G$ .
- I-3 (b) Either  $T'$  is a sublattice of index  $\ell$  of  $T$  and  $G$  acts trivially on  $T/T'$  or  $T$  is a sublattice of index  $\ell$  of  $T'$  and  $G$  acts trivially on  $T/T'$ .
- (Apply exercise ?? above to  $k = F_\ell$  and  $V = T/\ell T$ .)
- 3) Let  $\rho$  be a semi-simple  $\ell$ -adic representation of  $G$  and let  $U$  be an invariant subgroup of  $G$ . Assume that, for all  $x \in U$ ,  $\rho(x)$  is unipotent (all its eigenvalues are equal to 1). Show that  $\rho(x) = 1$  for all  $x \in U$ . (Show that the restriction of  $\rho$  to  $U$  is semi-simple and use Kolchin's theorem to bring it to triangular form.)
- 4) Let  $\rho: G \rightarrow \text{Aut}(V_\ell)$  be an  $\ell$ -adic representation of  $G$ , and  $T$  a lattice of  $V_\ell$  stable under  $G$ . Show the equivalence of the following properties:
- (a) The representation of  $G$  in the  $F_\ell$ -vector space  $T/\ell T$  is irreducible.
- (b) The only lattices of  $V_\ell$  stable under  $G$  are the  $\ell^n T$ , with  $n \in \mathbb{Z}$ .



## 1.2 Examples

**1. Roots of unity.** Let  $\ell \neq \text{char}(K)$ . The group  $G = \text{Gal}(K_s/K)$  acts on the group  $\mu_m$  of  $\ell^m$ -th roots of unity, and hence also on  $T_\ell(\mu) = \varprojlim_{m \in \mathbb{N}} \mu_m$ . The  $\mathbb{Q}_\ell$ -vector space  $V_\ell(\mu) = T_\ell(\mu) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$  is of dimension 1, and the homomorphism  $\chi_\ell: G \rightarrow \text{Aut}(V_\ell) = \mathbb{Q}_\ell^\times$  defined by the action of  $G$  on  $V_\ell$  is a 1-dimensional  $\ell$ -adic representation of  $G$ . The character  $\chi_\ell$  takes its values in the group of units  $U$  of  $\mathbb{Z}_\ell$ ; by definition

$$g(z) = z^{\chi_\ell(g)} \quad \text{if } g \in G, \quad z^{\ell^m} = 1.$$

**2. Elliptic curves.** Let  $\ell \neq \text{char}(K)$ . Let  $E$  be an elliptic curve defined over  $K$  with a given rational point  $o$ . One knows that there is a unique I-4 structure of group variety on  $E$  with  $o$  as neutral element. Let  $E_m$  be the kernel of multiplication by  $\ell^m$  in  $E(K_s)$ , and let

$$T_\ell(E) = \varprojlim_m E_m, \quad V_\ell(E) = T_\ell(E) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell.$$

The Tate module  $T_\ell(E)$  is a free  $\mathbb{Z}_\ell$ -module on which  $G = \text{Gal}(K_s/K)$  acts (cf. [12], chap. VII). The corresponding homomorphism  $\pi_\ell: G \rightarrow \text{Aut}(V_\ell(E))$  is an  $\ell$ -adic representation of  $G$ . The group  $G_\ell = \text{Im}(\pi_\ell)$  is a closed subgroup of  $\text{Aut}(T_\ell(E))$ , a 4-dimensional Lie group isomorphic to  $\text{GL}(2, \mathbb{Z}_\ell)$ . (In chapter IV, we will determine the Lie algebra of  $G_\ell$ , under the assumption that  $K$  is a number field.)

Since we can identify  $E$  with its dual (in the sense of the duality of abelian varieties) the symbol  $(x, y)$  (cf. [12], *loc. cit.*) defines canonical isomorphisms

$$\bigwedge^2 T_\ell(E) = T_\ell(\mu), \quad \bigwedge^2 V_\ell(E) = V_\ell(\mu).$$

Hence  $\det(\pi_\ell)$  is the character  $\chi_\ell$  defined in example 1.

**3. Abelian varieties.** Let  $A$  be an abelian variety over  $K$  of dimension  $d$ . If  $\ell \neq \text{char}(K)$ , we define  $T_\ell(A)$ ,  $V_\ell(A)$  in the same way as in example 2. The group  $T_\ell(A)$  is a free  $\mathbb{Z}_\ell$ -module of rank  $2d$  (cf. [12], *loc. cit.*) on which  $G = \text{Gal}(K_s/K)$  acts.

**4. Cohomology representations.** Let  $X$  be an algebraic variety defined over the field  $K$ , and let  $X_s = X \times_K K_s$  be the corresponding variety over

$K_s$ . Let  $\ell \neq \text{char}(K)$ , and let  $i$  be an integer. Using the étale cohomology of **3** [3] we let

$$H^i(X_s, \mathbb{Z}_\ell) = \varprojlim_n H^i((X_s)_{\text{ét}}, \mathbb{Z}/\ell^n \mathbb{Z}), \quad H_\ell^i(X_s) = H^i(X_s, \mathbb{Z}_\ell) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell.$$

I-5 The group  $H_\ell^i(X_s)$  is a vector space over  $\mathbb{Q}_\ell$  on which  $G = \text{Gal}(K_s/K)$  acts (via the action of  $G$  on  $X_s$ ). It is finite dimensional, at least if  $\text{char}(K) = 0$  or if  $X$  is proper. We thus get an  $\ell$ -adic representation of  $G$  associated to  $H_\ell^i(X_s)$ ; by taking duals we also get homology  $\ell$ -adic representations. Examples 1, 2, 3 are particular cases of homology  $\ell$ -adic representations where  $i = 1$  and  $X$  is respectively the multiplicative group  $\mathbb{G}_m$ , the elliptic curve  $E$ , and the abelian variety  $A$ .

### Exercise.

- (a) Show that there is an elliptic curve  $E$ , defined over  $K_0 = \mathbb{Q}(T)$ , with  $j$ -invariant equal to  $T$ .
- (b) Show that for such a curve, over  $K = \mathbb{C}(T)$ , one has  $G_\ell = \text{SL}(T_\ell(E))$  (cf. **10** [10] for an algebraic proof).
- (c) Using ??, show that, over  $K_0$ , we have  $G_\ell = \text{GL}(T_\ell(E))$ .
- (d) Show that for any closed subgroup  $H$  of  $\text{GL}(2, \mathbb{Z}_\ell)$  there is an elliptic curve (defined over some field) for which  $G_\ell = H$ .

## §2. $\ell$ -adic representations of number fields

### 2.1 Preliminaries

(For the basic notions concerning number fields, see for instance **6** [6], **13** [13] or **44** [44].) Let  $K$  be a number field (i.e. a finite extension of  $\mathbb{Q}$ ). Denote by  $M_K^0$  the set of all finite places of  $K$ , i.e., the set of all normalized discrete valuations of  $K$  (or, alternatively, the set of prime ideals in the ring  $\mathcal{O}_K$  of integers of  $K$ ). The **residue field**  $k_v$  of a place  $v \in M_K^0$  is a finite field with  $\mathbf{N}(v) = p_v^{\deg(v)}$  elements, where  $p_v = \text{char}(k_v)$  and  $\deg(v)$  is the degree of  $k_v$  over  $F_{p_v}$ . The ramification index  $e_v$  of  $v$  is  $v(p_v)$ .

Let  $L/K$  be a finite Galois extension with Galois group  $G$ , and let  $w \in M_L^0$ . The subgroup  $D_w$  of  $G$  consisting of those  $g \in G$  for which  $gw = w$  is the **decomposition group** of  $w$ . The restriction of  $w$  to  $K$  is an integral multiple of an element  $v \in M_K^0$ ; by abuse of language, we also say that  $v$  is the restriction of  $w$  to  $K$ , and we write  $w \mid v$  (“ $w$  divides  $v$ ”). Let  $L$  (resp.  $K$ ) be the completion of  $L$  (resp.  $K$ ) with respect to  $w$  (resp.  $v$ ). We have  $D_w = \text{Gal}(L_w/K_v)$ . The group  $D_w$  is mapped homomorphically onto the Galois group  $\text{Gal}(\lambda_w/k_v)$  of the corresponding residue extension  $\lambda_w/k_v$ . The kernel of  $G \rightarrow \text{Gal}(\lambda_w/k_v)$  is the inertia group  $I_w$  of  $w$ . The quotient group  $D_w/I_w$  is a finite cyclic group generated by the **Frobenius element**  $F_w$ ; we have  $F(\lambda) = \lambda^{\mathbf{N}(v)}$  for all  $\lambda \in \lambda_w$ . The valuation  $w$  (resp.  $v$ ) is called **unramified** if  $I_w = \{1\}$ . Almost all places of  $K$  are unramified.

If  $L$  is an arbitrary algebraic extension of  $\mathbb{Q}$ , one defines  $M_K^0$  to be the projective limit of the sets  $M_{L_\alpha}^0$ , where  $L_\alpha$  ranges over the finite sub-extensions of  $L/\mathbb{Q}$ . Then, if  $L/K$  is an arbitrary Galois extension of the number field  $K$ , and  $w \in M_L^0$ , one defines  $D_w, I_w, F_w$  as before. If  $v$  is an unramified place of  $K$ , and  $w$  is a place of  $L$  extending  $v$ , we denote by  $F_v$  the conjugacy class of  $F_w$  in  $G = \text{Gal}(L/K)$ .

**Definition 2.1.** Let  $\rho: \text{Gal}(\overline{K}/K) \rightarrow \text{Aut}(V)$  be an  $\ell$ -adic representation of  $K$ , and let  $v \in M_K^0$ . We say that  $\rho$  is unramified at  $v$  if  $\rho(I_w) = \{1\}$  for any valuation  $w$  of  $\overline{K}$  extending  $v$ .

If the representation  $\rho$  is unramified at  $v$ , then the restriction of  $\rho$  to  $D_w$  factors through  $D_w/I_w$  for any  $w \mid v$ ; hence  $\rho(F_w) \in \text{Aut}(V)$  is defined; we call  $\rho(F_w)$  the **Frobenius** of  $w$  in the representation  $\rho$ , and we denote it by  $F_{w,\rho}$ . The conjugacy class of  $F_{w,\rho}$  in  $\text{Aut}(V)$  depends only on  $v$ ; it is denoted by  $F_{v,\rho}$ . If  $L/K$  is the extension of  $K$  corresponding to  $H = \text{Ker}(\rho)$ , then  $\rho$  is unramified at  $v$  if and only if  $v$  is unramified in  $L/K$ . I-7

## 2.2 Čebotarev’s density theorem

Let  $P$  be a subset of  $M_K^0$ . For each integer  $n$ , let  $a_n(P)$  be the number of  $v \in P$  such that  $\mathbf{N}v \leq n$ . If  $a$  is a real number, one says that  $P$  **has density**  $a$  if

$$\lim_{n \rightarrow \infty} \frac{a_n(P)}{a_n(M_K^0)} = a \quad \text{when } n \rightarrow \infty.$$

Note that  $a_n(M_K^0) \sim n/\log(n)$ , by the prime number theorem (cf. Appendix, or [13], chap. VIII), so that the above relation may be rewritten:

$$a_n(P) = a \cdot \frac{n}{\log(n)} + o\left(\frac{n}{\log(n)}\right).$$

**Examples.** A finite set has density 0. The set of  $v \in M_K^0$  of degree 1 (i.e. such that  $\mathbf{N}v$  is prime) has density 1. The set of ordinary prime numbers whose first digit (in the decimal system, say) is 1 has no density.

**Theorem 1.** *Let  $L$  be a finite Galois extension of the number field  $K$ , with Galois group  $G$ . Let  $X$  be a subset of  $G$ , stable by conjugation. Let  $P_X$  have density equal to  $\text{Card}(X)/\text{Card}(G)$ .*

For the proof, see [7], [1], or the Appendix.

**Corollary 1.1.** *For every  $g \in G$ , there exist infinitely many unramified places  $w \in M_K^0$  such that  $F_w = g$ .*

For infinite extensions, we have:

**Corollary 1.2.** *Let  $L$  be a Galois extension of  $K$ , which is unramified outside a finite set  $S$ .*

- a) *The Frobenius elements of the unramified places of  $L$  are dense in  $\text{Gal}(L/K)$ .*
- b) *Let  $X$  be a subset of  $\text{Gal}(L/K)$ , stable by conjugation. Assume that the boundary of  $X$  has measure zero with respect to the Haar measure  $\mu$  of  $X$ , and normalize  $\mu$  such that its total mass is 1. Then the set of places  $v \notin S$  such that  $F_v \subset X$  has a density equal to  $\mu(X)$ .*

Assertion (b) follows from the theorem, by writing  $L$  as an increasing union of finite Galois extensions and passing to the limit (one may also use Prop. 1 of the Appendix). Assertion (a) follows from (b) applied to a suitable neighborhood of a given class of  $\text{Gal}(L/K)$ .

**Exercise.** Let  $G$  be an  $\ell$ -adic Lie group and let  $X$  be an analytic subset of  $G$  (i.e. a set defined by the vanishing of a family of analytic functions on  $G$ ). Show that the boundary of  $X$  has measure zero with respect to the Haar measure of  $G$ .

### 2.3 Rational $\ell$ -adic representations

Let  $\rho$  be an  $\ell$ -adic representation of the number field  $K$ . If  $v \in M_K^0$ , and if  $v$  is unramified with respect to  $\rho$ , we let  $P_{v,\rho}(T)$  denote the polynomial  $\det(1 - F_{v,\rho}T)$ .

**Definition 2.2.** The  $\ell$ -adic representation  $\rho$  is said to be **rational** (resp. integral) if there exists a finite subset  $S$  of  $M_K^0$  such that

- (a) Any element of  $M_K^0 \setminus S$ .
- (b) If  $v \notin S$ , the coefficients of  $P_{v,\rho}(T)$  belong to  $\mathbb{Q}$  (resp. to  $\mathbb{Z}$ ).

Remark (no sé cómo ponerlo, como sssec?)

Let  $K'/K$  be a finite extension. An  $\ell$ -adic representation  $\rho$  of  $K$  defines (by restriction) an  $\ell$ -adic representation  $\rho_{/K'}$  de  $K'$ . If  $\rho$  is rational (resp. integral), then the same is true for  $\rho_{/K'}$ ; this follows from the fact that the Frobenius elements relative to  $K'$  are powers of those relative to  $K$ .

### 2.4 Example

The  $\ell$ -adic representations of  $K$  given in examples ??, ??, ?? of section ?? are rational (even integral) representation. In example ??, one can take for  $S$  the set  $S_\ell$  of elements  $v$  of  $M_K^0$  with  $\rho_v = \ell$ ; In examples ??, ??, one can take for  $S$  the union of  $S_\ell$  and the set  $S_A$  where  $A$  has "bad reduction"; the fact that the corresponding Frobenius has an integral characteristic polynomial (which is independent of  $\ell$ ) is a consequence of Weil's results on endomorphisms of abelian varieties (cf. [40] and [12], chap. VII). The rationality of the cohomology representation is a well-known open question.

Ver si sigue siendo una pregunta abierta.

**Definition 2.3.** Let  $\ell'$  be a prime,  $\rho'$  an  $\ell'$ -adic representation of  $K$ , and assume that  $\rho, \rho'$  are rational. Then  $\rho, \rho'$  are said to be **compatible** if there exists a finite subset  $S$  of  $M_K^0$  such that  $\rho$  and  $\rho'$  are unramified outside of  $S$  and  $P_{v,\rho}(T) = P_{v,\rho'}(T)$  for  $v \in M_K^0 \setminus S$ .

(In other words, the characteristic polynomials of the Frobenius elements are the same for  $\rho$  and  $\rho'$ , at least for almost all  $v$ 's.)

If  $\rho: \text{Gal}(\overline{K}/K) \rightarrow \text{Aut}(V)$  is rational  $\ell$ -adic representation of  $K$ , then  $V$  has a composition series

$$V = V_0 \supset V_1 \supset \dots \supset V_q = 0$$

of  $\rho$ -invariants subspaces with  $V_i/V_{i+1}$  ( $0 \leq i \leq q-1$ ) simple (i.e. irreducible). The  $\ell$ -adic representation  $\rho'$  of  $K$  defined by  $V' = \sum_{i=0}^{q-1} V_i/V_{i+1}$  is semi-simple, rational, and compatible with  $\rho$ ; it is the "semi-simplification" of  $V$ .

**Theorem 2.** *Let  $\rho$  be a rational  $\ell$ -adic representation of  $K$ , let  $\ell'$  be a prime. Then there exists at most one (up to isomorphism)  $\ell'$ -adic rational representation  $\rho'$  of  $K$  which is semi-simple and compatible with  $\rho$ .*

(Hence there exists a unique (up to isomorphism) rational semi-simple  $\ell$ -adic representation compatible with  $\rho$ .)

I-10 *Proof.* Let  $\rho'_1, \rho'_2$  be semi-simple  $\ell$ -adic representations of  $K$  which are rational and compatible with  $\rho$ .

We first prove that  $\text{Tr}(\rho'_1(g)) = \text{Tr}(\rho'_2(g))$  for all  $g \in G$ . Let  $H = G/(\text{Ker}(\rho'_1) \cap \text{Ker}(\rho'_2))$ ; the representations  $\rho'_1, \rho'_2$  may be regarded as representations of  $H$ , and it suffices to show that  $\text{Tr}(\rho'_1(h)) = \text{Tr}(\rho'_2(h))$  for all  $h \in H$ . Let  $K' \subset \overline{K}$  be the fixed field of  $H$ . Then by the compatibility of  $\rho'_1, \rho'_2$  there is a finite subset  $S$  of  $M_K^0$  such that for all  $v \in M_K^0 \setminus S$ ,  $w \in M_K^0$ ,  $w \mid v$ , we have  $\text{Tr}(\rho'_1(F_w)) = \text{Tr}(\rho'_2(F_w))$ . But, by cor. ?? to Čebotarev's theorem (cf. ??) the  $F_w$  are dense in  $H$ . Hence  $\text{Tr}(\rho'_1(h)) = \text{Tr}(\rho'_2(h))$  for all  $h \in H$  since  $\text{Tr} \circ \rho'_1, \text{Tr} \circ \rho'_2$  are continuous.

The theorem now follows from the following result applied to the group ring  $\Lambda = \mathbb{Q}_\ell[H]$ . □

**Lemma 1.** *Let  $k$  be a field of characteristic zero, let  $\Lambda$  be a  $k$ -algebra, and let  $\rho_1, \rho_2$  be two finite-dimensional linear representations of  $\Lambda$ . If  $\rho_1, \rho_2$  are semi-simple and have the same trace ( $\text{Tr} \circ \rho_1 = \text{Tr} \circ \rho_2$ ), then they are isomorphic.*

For the proof see Bourbaki, Alg., ch. 8, §12, n°1, prop. 3.

Cómo citar esto

**Definition 2.4.** For each prime  $\ell$  let  $\rho_\ell$  be a rational  $\ell$ -adic representation of  $K$ . The system  $(\rho_\ell)$  is said **to be compatible** if  $\rho_\ell, \rho_{\ell'}$  are compatible for any two primes  $\ell, \ell'$ . The system  $(\rho_\ell)$  is said **to be strictly compatible** if there exists a finite subset  $S$  of  $M_K^0$  such that:

- (a) Let  $S_\ell = \{v | \rho_v = \ell\}$ . Then, for every  $v \notin S \cup S_\ell$ ,  $\rho_\ell$  is unramified at  $v$  and  $P_{v,\rho_\ell}(T)$  has rational coefficients.
- (b)  $P_{v,\rho_\ell}(T) = P_{v,\rho_{\ell'}}(T)$  if  $v \notin S \cup S_\ell \cup S_{\ell'}$ .

I-11

When a system  $(\rho_\ell)$  is strictly compatible, there is a smallest finite set  $S$  having properties (a) and (b) above. We call it the **exceptional set** of the system.

¿El strong e índice está bien aquí?

## 2.5 Examples

The systems of  $\ell$ -adic representations given in examples ??, ??, ?? of section ?? are strictly compatible. The exceptional set of the first one is empty. The exceptional set of example ?? (resp. ??) is the set of places where the elliptic curve (resp. the abelian variety) has "bad reduction", cf. [32].

## 2.6 Questions

- 1) Let  $\rho$  be a rational  $\ell$ -adic representation. Is true that  $P_{v,\rho}$  has coefficients for all  $v$  such that  $\rho$  is unramified at  $v$ ?

A somewhat similar question is:

Is any compatible system strictly compatible?

- 2) Can any rational  $\ell$ -adic representation be obtained (by tensor products, direct sums, etc.) from ones coming from  $\ell$ -adic cohomology?

¿es ones o one's?

- 3) Given a rational  $\ell$ -adic representation  $\rho$  of  $K$ , and a prime  $\ell'$ , does there exist a rational  $\ell'$ -adic representation  $\rho'$  of  $K$  compatible with  $\rho$ ?  
→ [no : easy counter-examples].

flecha loca apareció

- 4) Let  $\rho, \rho'$  be rational  $\ell, \ell'$ -adic representations of  $K$  which are compatible and semi-simple.

- (a) If  $\rho$  is abelian (i.e., if  $\text{Im}(\rho)$  is abelian), is it true that  $\rho'$  is abelian? (We shall see in chapter III that this is true at least if  $\rho$  is “locally algebraic”.)
- I-12 (b) Is it true that  $\text{Im}(\rho)$  and  $\text{Im}(\rho')$  are Lie groups of the same dimension? More optimistically, is it true that there exists a Lie algebra  $g$  over  $\mathbb{Q}$  such that  $\text{Lie}(\mathfrak{S}(\rho)) = g \otimes_{\mathbb{Q}} \mathbb{Q}_{\ell}$   $\text{Lie}(\mathfrak{S}(\rho')) = g \otimes_{\mathbb{Q}} \mathbb{Q}_{\ell'}$

No entiendo bien la puntuación y el Lie

- 5) Let  $X$  be a non-singular projective variety defined over  $K$ , and let  $i$  be an integer. Is the  $i$ -th cohomology representation  $H_{\ell}^i(X_S)$  semi-simple? Does its Lie algebra contain the homotheties if  $i \geq 1$ ? (When  $i = 1$ , an affirmative answer to either one of these questions would imply a positive solution for the “congruence subgroup problem” on abelian varieties, cf. [24], §3.)  $\rightarrow$  [yes for  $i = 1$ : see [48] and also [75].]

Remark (no sé cómo ponerlo, como sssec?)

The concept of an  $\ell$ -adic representation can be generalized by replacing the prime  $\ell$  by a place  $\lambda$  of a number field  $E$ . A  $\lambda$ -adic representation is then a continuous homomorphism  $\text{Gal}(K_S/K) \rightarrow \text{Aut}(V)$ , where  $V$  is a finite-dimensional vector space over the local field  $E_{\lambda}$ . The concepts of rational  $\lambda$ -adic representation, compatible representations, etc., can be defined in a way similar to the  $\ell$ -adic case.

### Exercise.

- 1) Let  $\rho$  and  $\rho'$  be two rational, semi-simple, compatible representations. Show that, if  $\mathfrak{S}(\rho)$  is finite, the same is true for  $\mathfrak{S}(\rho')$  and that  $\text{Ker}(\rho) = \text{Ker}(\rho')$ . (Apply exer. 3 of 1.1 to  $\rho'$  and to  $U = \text{Ker}(\rho)$ .) Generalize this to  $\lambda$ -adic representations (with respect to a number field  $E$ ).
- I-13 2) Let  $\rho$  (resp.  $\rho'$ ) be a rational  $\ell$ -adic (resp.  $\ell'$ -adic) representation of  $K$ , of degree  $n$ . Assume  $\rho$  and  $\rho'$  are compatible. If  $s \in G = \text{Gal}(\overline{K}/K)$ , let  $\sigma_i(s)$  (resp.  $\sigma'_i(s)$ ) be the  $i$ -th coefficient of the characteristic polynomial of  $\rho(s)$  (resp.  $\rho'(s)$ ). Let  $P(X_0, \dots, X_n)$  be a polynomial with rational coefficients, and let  $X_P$  (resp.  $X'_P$ ) be the set of  $s \in G$  such that  $P(\sigma_0(s), \dots, \sigma_n(s)) = 0$  (resp.  $P(\sigma'_0(s), \dots, \sigma'_n(s)) = 0$ ).



- (a) Show that the boundaries of  $X_P$  and  $X'_P$  have measure zero for the Haar measure  $\mu$  of  $G$  (use Exer. of 2.2).
- (b) Assume that  $\mu$  is normalized, i.e.  $\mu(G) = 1$ . Let  $T_P$  be the set of  $v \in M_K^0$  at which  $\rho$  is unramified, and for which the coefficients  $\sigma_0, \dots, \sigma_n$  of characteristic polynomial of  $F_{v,\rho}$  satisfy the equation  $P(\sigma_0, \dots, \sigma_n) = 0$ . Show that  $T_P$  has density equal to  $\mu(X_P)$ .
- (c) Show that  $\mu(X_P) = \mu(X'_P)$ .

## 2.7 Representations with values in a linear algebraic group

Let  $H$  be a linear algebraic group defined over a field  $K$ . If  $k'$  is a commutative  $k$ -algebra, let  $H(k')$  denote the group of points of  $H$  with values in  $k'$ . Let  $A$  denote the coordinate ring (or “affine ring”) of  $H$ . An element  $f \in A$  is said to be **central** if  $f(xy) = f(yx)$  for any  $x, y \in H(k')$  and any commutative  $k$ -algebra  $k'$ . If  $x \in H(k')$  we say that the conjugacy class of  $x$  in  $H$  is **rational over  $k$**  if  $f(x) \in k$  for any central element  $f$  of  $A$ .

**Definition 2.5.** Let  $H$  be a linear algebraic group over  $\mathbb{Q}$ , and let  $K$  be a field. A continuous homomorphism  $\rho: \text{Gal}(K_s/K) \rightarrow H(\mathbb{Q}_\ell)$  is called an  $\ell$ -adic representation of  $K$  with values in  $H$ .

(Note that  $H(\mathbb{Q}_\ell)$  is, in a natural way, a topological group and even an  $\ell$ -adic Lie group.)

If  $K$  is a number field, one defines in an obvious way what it means for  $\rho$  to be unramified at a place  $v \in M_K^0$ ; if  $w \mid v$ , one defines the Frobenius element  $F_{w,\rho} \in H(\mathbb{Q}_\ell)$  and its conjugacy class  $F_{v,\rho}$ . We say, as before, that  $\rho$  is **rational** if

- (a) there is a finite set  $S$  of  $M_K^0$  such that  $\rho$  is unramified outside  $S$ ,
- (b) if  $v \notin S$ , the conjugacy class  $F_{v,\rho}$  is rational over  $\mathbb{Q}$ .

Two rational representations  $\rho, \rho'$  (for primes  $\ell, \ell'$ ) are said to be **compatible** if there exists a finite subset  $S$  of  $M_K^0$  such that  $\rho$  and  $\rho'$  are unramified outside  $S$  and such that for any central element  $f \in A$  and any  $v \in M_K^0 \setminus S$  we have  $f(F_{v,\rho}) = f(F_{v,\rho'})$ . One defines in the same way the notions of **compatible** and **strictly compatible systems** of rational representations.

**Remark.** 1) If the algebraic group  $H$  is abelian, then condition ?? above means that  $F_{v,\rho}$  (which is now an element of  $H(\mathbb{Q}_\ell)$ ) is rational over  $\mathbb{Q}$ , i.e. belongs to  $H(\mathbb{Q})$ .

I-15 2) Let  $V_0$  be a finite-dimensional vector space over  $\mathbb{Q}$ , and let  $\mathrm{GL}_{V_0}$  be the linear algebraic group over  $\mathbb{Q}$  whose group of points in any commutative  $\mathbb{Q}$ -algebra  $k$  is  $\mathrm{Aut}(V_0 \otimes_{\mathbb{Q}} k)$ ; in particular, if  $V_\ell = V_0 \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$ , then  $\mathrm{GL}_{V_0}(\mathbb{Q}_\ell) = \mathrm{Aut}(V_\ell)$ . If  $\varphi: H \rightarrow \mathrm{GL}_{V_0}$  is a homomorphism of linear algebraic groups over  $\mathbb{Q}$ , call  $\varphi_\ell$  the induced homomorphism of  $H(\mathbb{Q}_\ell)$  into  $\mathrm{GL}_{V_0}(\mathbb{Q}_\ell) = \mathrm{Aut}(V_\ell)$ . If  $\rho$  is an  $\ell$ -adic representation of  $\mathrm{Gal}(\overline{K}/K)$  into  $H(\mathbb{Q}_\ell)$ , one gets by composition a linear  $\ell$ -adic representation  $\varphi_\ell \circ \rho: \mathrm{Gal}(\overline{K}/K) \rightarrow \mathrm{Aut}(V_\ell)$ . Using the fact that the coefficients of the characteristic polynomial are central functions, one sees that  $\varphi_\ell \circ \rho$  is *rational* if  $\rho$  is rational ( $K$  a number field). Of course, compatible representations in  $H$  give compatible linear representations. We will use this method of constructing compatible representations in the case where  $H$  is abelian (see ch. ??, ??).

## §A. Equipartition and $L$ -functions

### A.1 Equipartition

Let  $X$  be a compact topological space and  $C(X)$  the Banach space of continuous, complex-valued, functions on  $X$ , with its usual norm  $\|f\| = \sup_{x \in X} |f(x)|$ . For each  $x \in X$  let  $\delta_x$  be the Dirac measure associated to  $x$ ; if  $f \in C(X)$ , we have  $\delta_x(f) = f(x)$ .

Let  $(x_n)_{n \geq 1}$  be a sequence of points of  $X$ . For  $n \geq 1$ , let

$$\mu_n = \frac{\delta_{x_1} + \cdots + \delta_{x_n}}{n}$$

and let  $\mu$  be a Radon measure on  $X$  (i.e. a continuous linear form on  $C(X)$ , cf. Bourbaki, Int., chap. III, §1). The sequence  $(x_n)$  is said to be  **$\mu$ -equidistributed**, or  *$\mu$ -uniformly distributed*, if  $\mu_n \rightarrow \mu$  weakly as  $n \rightarrow \infty$ , i.e. if  $\mu_n(f) \rightarrow \mu(f)$  as  $n \rightarrow \infty$  for any  $f \in C(X)$ . Note that this implies that  $\mu$  is positive and of total mass 1. Note also that  $\mu_n(f) \rightarrow \mu(f)$  means that

$$\mu(f) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n f(x_i).$$

**Lemma 1.** *Let  $(\varphi_\alpha)$  be a family of continuous functions on  $X$  with the property that their linear combinations are dense in  $C(X)$ . Suppose that, for all  $\alpha$ , the sequence  $(\mu_n(\varphi_\alpha))_{n \geq 1}$  has a limit. Then the sequence  $(x_n)$  is equidistributed with respect to some measure  $\mu$  it is the unique measure such that  $\mu(\varphi_\alpha) = \lim_{n \rightarrow \infty} \mu_n(\varphi_\alpha)$  for all  $\alpha$ .*

If  $f \in C(X)$ , an argument using equicontinuity shows that the sequence  $(\mu_n(f))$  has a limit  $\mu(f)$ , which is continuous and linear in  $f$ ; hence the lemma.

**Proposition 1.** *Suppose that  $(x_n)$  is  $\mu$ -equidistributed. Let  $U$  be a subset of  $X$  whose boundary has  $\mu$ -measure zero, and, for all  $n$ , let  $n_U$  be the number of  $m \leq n$  such that  $x_m \in U$ . Then  $\lim_{n \rightarrow \infty} (n_U/n) = \mu(U)$ .*

Let  $\overset{\circ}{U}$  be the interior of  $U$ . We have  $\mu(\overset{\circ}{U}) = \mu(U)$ . Let  $\varepsilon > 0$ . By the definition of  $\mu(\overset{\circ}{U})$  there is a continuous function  $\varphi \in C(X)$ ,  $0 \leq \varphi \leq 1$ , with  $\varphi = 0$  on  $X \setminus \overset{\circ}{U}$  and  $\mu(\varphi) \geq \mu(U) - \varepsilon$ . Since  $\mu_n(\varphi) \leq n_U/n$  we have

$$\liminf_{n \rightarrow \infty} \frac{n_U}{n} \geq \lim_{n \rightarrow \infty} \mu_n(\varphi) = \mu(\varphi) \geq \mu(U) - \varepsilon,$$

from which we obtain  $\liminf n_U/n \geq \mu(U)$ . The same argument applied to  $X \setminus U$  shows that

$$\liminf_{n \rightarrow \infty} \frac{n - n_U}{n} \geq \mu(X \setminus U).$$

Hence  $\limsup_n n_U/n \leq \mu(U) \leq \liminf n_U/n$ , which implies the proposition.

**Examples.** 1. Let  $X = [0, 1]$ , and let  $\mu$  be the Lebesgue measure. A sequence  $(x_n)$  of points of  $X$  is  $\mu$ -equidistributed if and only if for each interval  $[a, b]$ , of length  $d > 0$  in  $[0, 1]$  the number of  $m \leq n$  such that  $x_m \in [a, b]$  is equivalent to  $dn$  as  $n \rightarrow \infty$ .

2. Let  $G$  be a compact group and let  $X$  be the space of conjugacy classes of  $G$  (i.e. the quotient space of  $G$  by the equivalence relation induced by inner automorphisms of  $G$ ). Let  $\mu$  be a measure on  $G$ ; its image of  $G \rightarrow X$  is a measure on  $X$ , which we also denote by  $\mu$ . We then have:

**Proposition 2.** *The sequence  $(x_n)$  of elements of  $X$  is  $\mu$ -equidistributed if and only if for any irreducible character  $\chi$  of  $G$  we have*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \chi(x_i) = \mu(\chi).$$

The map  $C(X) \rightarrow C(G)$  is an isomorphism of  $C(X)$  onto the space of central functions on  $G$ ; by the Peter-Weyl theorem, the irreducible characters  $\chi$  of  $G$  generate a dense subspace of  $C(X)$ . Hence the proposition follows from lemma ??.

**Corollary 2.1.** *Let  $\mu$  be the Haar measure of  $G$  with  $\mu(G) = 1$ . Then a sequence  $(x_n)$  of elements of  $X$  is  $\mu$ -equidistributed if and only if for any irreducible character  $\chi$  of  $G$ ,  $\chi \neq 1$  we have*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \chi(x_i) = 0.$$

This follows from Prop. ?? and the following facts:

$$\begin{aligned} \mu(\chi) &= 0 & \text{if } \chi \text{ is irreducible } \neq 1 \\ \mu(1) &= 1. \end{aligned}$$

**Corollary 2.2** (46 [46]). *Let  $G = \mathbb{R}/\mathbb{Z}$ , and let  $\mu$  be the normalized Haar measure on  $G$ . Then  $(x_n)$  is  $\mu$ -equidistributed if and only if for any integer  $m \neq 0$  we have*

$$\sum_{n \leq N} e^{2\pi m i x_n} = o(N) \quad (N \rightarrow \infty).$$

For the proof, it suffices to remark that the irreducible characters of  $\mathbb{R}/\mathbb{Z}$  are the mappings  $x \mapsto e^{2\pi m i x}$  ( $m \in \mathbb{Z}$ ).

## A.2 The connection with $L$ -functions

Let  $G$  and  $X$  be as in Example ?? above:  $G$  a compact group and  $X$  the space of its conjugacy classes. Let  $x_v, v \in M$ , be a family of elements of  $X$ , indexed by a denumerable set  $M$ , and let  $v \mapsto \mathbf{N}v$  be a function on  $M$  with values in the set of integers  $\geq 2$ . We make the following hypotheses:

- (1) The infinite product  $\prod_{v \in M} \frac{1}{1 - (\mathbf{N}v)^{-s}}$  converges for every  $s \in \mathbb{C}$  with  $\Re(s) > 1$ , and extends to a meromorphic function on  $\Re(s) > 1$  having neither zero nor pole except for a simple pole at  $s = 1$ .
- (2) Let  $\rho$  be an irreducible representation of  $G$ , with character  $\chi$ , and put

$$L(s, \rho) = \prod_{v \in M} \frac{1}{\det(1 - \rho(x_v)(\mathbf{N}v)^{-s})}.$$

Then this product converges for  $\Re(s) > 1$ , and extends to a meromorphic function on  $\Re(s) > 1$  having neither zero nor pole except possibly for  $s = 1$ .

The order of  $L(s, \rho)$  at  $s = 1$  will be denoted by  $-c_\chi$ . Hence, if  $L(s, \rho)$  has a pole (resp. a zero) of order  $m$  at  $s = 1$ , one has  $c_\chi = m$  (resp.  $c_\chi = -m$ ).

Under these assumptions, we have:

**Theorem 1.** (a) *The number of  $v \in M$  with  $\mathbf{N}v \leq n$  is equivalent to  $n/\log n$  (as  $n \rightarrow \infty$ ).*

(b) *For any irreducible character  $\chi$  of  $G$ , we have*

$$\sum_{\mathbf{N}v \leq n} \chi(x_v) = c_\chi \frac{n}{\log n} + o(n/\log n), \quad (n \rightarrow \infty).$$

The theorem results, by a standard argument, from the theorem of Wiener-Ikehara, cf. ?? below. Suppose now that the function  $v \mapsto \mathbf{N}v$  has the following property:

I-19

(3) *There exists a constant  $C$  such that, for every  $n \in \mathbb{Z}$ , the number of  $v \in M$  with  $\mathbf{N}v = n$  is  $\leq C$ .*

One may then arrange the elements of  $M$  as a sequence  $(v_i)_{i \geq 1}$ , so that  $i \leq j$  implies  $\mathbf{N}v_i \leq \mathbf{N}v_j$  (in general, this is possible in many ways). It then makes sense to speak about the equidistribution of the sequence of  $x_v$ 's; using (3), one shows easily that this does not depend on the chosen ordering of  $M$ . Applying theorem 1 and proposition 2, we obtain:

**Theorem 2.** *The elements  $x_v$  ( $v \in M$ ) are equidistributed in  $X$  with respect to a measure  $\mu$  such that for any irreducible character  $\chi$  of  $G$  we have*

$$\mu(\chi) = c_\chi.$$

**Corollary 2.1.** *The elements  $x_v$  ( $v \in M$ ) are equidistributed in  $X$  normalized Haar measure of  $G$  if and only if  $c_\chi = 0$  for every irreducible character  $\chi \neq 1$  of  $G$ , i.e., if and only if the  $L$ -functions relative to the non trivial irreducible characters of  $G$  are holomorphic and non zero at  $s = 1$ .*

**Examples.** 1) Let  $G$  be the Galois group of a *finite* Galois extension  $L/K$  of the number field  $K$ , let  $M$  be the set of unramified places of  $K$ , let  $x_v$  be the Frobenius conjugacy class defined by  $v \in M$ , and let  $\mathbf{N} v$  be the norm of  $v$ , cf. §??.

I-20 Properties (1), (2), (3) are satisfied with  $c_\chi = 0$  for all irreducible  $\chi \neq 1$ . This is trivial for (3). For (1), one remarks that  $L(s, l)$  is the zeta function of  $K$  (up to a finite number of terms), hence has a simple pole at  $s = 1$  and is holomorphic on the rest of the line  $\Re(s) = 1$ , cf. for instance **13** [13], chap. VII; for a proof of (2), cf. **1** [1]. Hence theorem 2 gives the equidistribution of the Frobenius elements, i.e. the Čebotarev density theorem, cf. ??.

2) Let  $C$  be the idèle class group of a number field  $K$ , and let  $\rho$  be a continuous homomorphism of  $C$  into a compact abelian Lie group  $G$ . An easy argument (cf. ch. III, 2.2) shows that  $\rho$  is almost everywhere unramified (i.e., if  $U_v$  denotes the group of units at  $v$ , then  $\rho(U_v) = 1$  for almost all  $v$ ). Choose  $\pi_v \in K$  with  $v(\pi_v) = 1$ . If  $\rho$  is unramified at  $v$ , then  $\rho(\pi_v)$  depends only on  $v$ , and we set  $x_v = \rho(\pi_v)$ . We make the following *assumption*:

(\*) *The homomorphism  $\rho$  maps the group  $C$  of idèles of volume 1 onto  $G$ .*

(Recall that the **volume** of an idèle  $\mathbf{a} = (a_v)$  is defined as the product of the normalized absolute values of its components  $a_v$ , cf. **13** [13] or **44** [44].)

Then, the elements  $x_v$  are *uniformly distributed* in  $G$  with respect to the normalized Haar measure. This follows from theorem 1 and the fact that the  $L$ -functions relative to the irreducible characters  $\chi$  of  $G$  are Hecke  $L$ -functions with Grössencharakteren; these  $L$ -functions are holomorphic and non-zero for  $\Re(s) \geq 1$  if  $\chi \neq 1$ , see [13], chap. VII.

**Remark.** This example (essentially due to Hecke) is given in Lang (*loc. cit.*, ch. VIII, §5) except that Lang has replaced the condition (\*) by the condition “ $\rho$  is surjective”, which is insufficient. This led him to affirm that, for example, the sequence  $(\log p)_p$  (and also the sequence  $(\log n)_n$ ) is uniformly distributed modulo 1; however, one knows that this sequence is not uniformly distributed for any measure on  $\mathbb{R}/\mathbb{Z}$  (cf. **22** [22]).

I-21

- 3) (Conjectural example). Let  $E$  be an elliptic curve defined over a number field  $K$  and let  $M$  be the set of finite places  $v$  of  $K$  such that  $E$  has good reduction at  $v$ , cf. 1.2 and chap. ?? . Let  $v \in M$ , let  $\ell \neq p_v$  and let  $F_v$  be the Frobenius conjugacy class of  $v$  in  $\text{Aut}(T_\ell(E))$ . The eigenvalues of  $F_v$  are algebraic numbers; when embedded into  $\mathbb{C}$  they give conjugate complex numbers  $\pi_v, \bar{\pi}_v$  with  $|\pi_v| = (\mathbf{N}v)^{1/2}$ . We may write then

$$\pi_v = (\mathbf{N}v)^{1/2} e^{i\phi_v}; \quad \bar{\pi}_v = (\mathbf{N}v)^{1/2} e^{-i\phi_v} \quad \text{with } 0 \leq \phi_v \leq \pi.$$

On the other hand, let  $G = \text{SU}(2)$  be the Lie group of  $2 \times 2$  unitary matrices with determinant 1. Any element of the space  $X$  of conjugacy classes of  $G$  contains a unique matrix of the form

$$\begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{pmatrix}, \quad 0 \leq \phi \leq \pi.$$

The image in  $X$  of the Haar measure of  $G$  is known to be  $\frac{2}{\pi} \sin^2 \phi d\phi$ . The irreducible representations of  $G$  are the  $m$ -th symmetric powers  $\rho_m$  of the natural representation  $\rho_1$  of degree 2.

Take now for  $x_v$  the element of  $X$  corresponding to the angle  $\phi = \phi_v$  defined above. The corresponding  $L$  function, relative to  $\rho_m$ , is:

$$L_{\rho_m}(s) = \prod_v \prod_{a=0}^{a=m} \frac{1}{1 - e^{i(m-2a)\phi_v} (\mathbf{N}v)^{-s}}.$$

If we put:

$$L_m^1(s) = \prod_v \prod_{a=0}^{a=m} \frac{1}{1 - \pi_v^{m-a} \bar{\pi}_v^a (\mathbf{N}v)^{-s}}$$

we have

$$L_{\rho_m}(s) = L_m^1(s - m/2).$$

I-22

The function  $L$  has been considered by **36** [36]. He conjectures that  $L_m^1$ , for  $m \geq 1$ , is holomorphic and non zero for  $\Re(s) \geq 1 + m/2$ , provided that  $E$  has no complex multiplication. Granting this conjecture, the corollary to theorem 2 would yield the uniform distribution of the  $x_v$ 's, or, equivalently, that the angles  $\phi_v$  of the Frobenius elements are uniformly distributed in  $[0, \pi]$  with respect to the measure  $\frac{2}{\pi} \sin^2 \phi d\phi$  ("conjecture of Sato-Tate").

One can expect analogous results to be true for other  $\ell$ -adic representations.

### A.3 Proof of theorem 1

The logarithmic derivative of  $L$  is

$$\frac{L'(s)}{L(s)} = - \sum_{\substack{v \geq 1 \\ m \geq 1}} \frac{\chi(x_v^m) \log(\mathbf{N} v)}{(\mathbf{N} v)^{ms}},$$

where  $x_v^m$  is the conjugacy class consisting of the  $m$ -th powers of elements in the class  $x_v$ . One sees this by writing  $L$  as the product

$$\prod_{j,v} \frac{1}{1 - \lambda_v^{(j)} (\mathbf{N} v)^{-s}}$$

I-23 where the  $\lambda_v^{(j)}$  are the eigenvalues of  $x_v$  in the given representation. Now the series

$$\sum_{\substack{v \geq 1 \\ m \geq 1}} \frac{\log(\mathbf{N} v)}{|(\mathbf{N} v)^{ms}|},$$

converges for  $\Re(s) > 1/2$ . Indeed it suffices to show that

$$\sum_v \frac{\log(\mathbf{N} v)}{(\mathbf{N} v)^\sigma} < \infty$$

if  $\sigma > 1$ ; but this series is majorized by

$$(\text{Constant}) \times \sum_v \frac{1}{(\mathbf{N} v)^{\sigma+\varepsilon}}, \quad (\varepsilon > 0).$$

On the other hand, the convergence for  $\sigma > 1$  of the product

$$\prod_v \frac{1}{1 - (\mathbf{N} v)^{-\sigma}}$$

shows that

$$\sum_v \frac{1}{(\mathbf{N} v)^\sigma} < \infty$$

for  $\sigma > 1$ ; hence our assertion. One can therefore write

$$\frac{L'(s)}{L(s)} = - \sum_v \frac{\chi(x_v) \log(\mathbf{N} v)}{(\mathbf{N} v)^s} + \phi(s)$$



I-24 where  $\phi(s)$  is holomorphic for  $\Re(s) > \frac{1}{2}$ . Moreover, by hypothesis,  $L'/L$  can be extended to a meromorphic function on  $\Re(s) \geq 1$  which is holomorphic except possibly for a simple pole at  $s = 1$  with residue  $-c_\chi$ . One may then apply the Wiener-Ikehara theorem (cf. [13]):

**Theorem 3.** *Let  $F(s) = \sum_{n=1}^{\infty} a_n/n^s$  be a Dirichlet series with complex coefficients. Suppose there exists a Dirichlet series  $F(s) = \sum_n a_n^+/n^s$  with positive real coefficients such that*

- (a)  $|a_n| \leq a_n^+$  for all  $n$ ;
- (b) The series  $F^+$  converges for  $\Re(s) > 1$ ;
- (c) The function  $F$  (resp.  $F^+$ ) can be extended to a meromorphic function on  $\Re(s) \geq 1$  having no poles except (resp. except possibly) for a simple pole at  $s = 1$  with residue  $c_+ > 0$  (resp.  $c$ ).

Then

$$\sum_{m \leq n} a_m = cn + o(n) \quad (n \rightarrow \infty),$$

(where  $c = 0$  if  $F$  is holomorphic at  $s = 1$ ).

One applies this theorem to

$$F(s) = - \sum_v \frac{\chi(x_v) \log(\mathbf{N} v)}{(\mathbf{N} v)^s},$$

and we take for  $F^+$  the series

$$d \sum_v \frac{\log(\mathbf{N} v)}{(\mathbf{N} v)^s},$$

where  $d$  is the degree of the given representation  $\rho$ ; this is possible since I-25  $\chi(x_v)$  is a sum of  $d$  complex numbers of absolute value 1, hence  $|\chi(x_v)| \leq d$ ; moreover, the series

$$\sum_v \frac{\log(\mathbf{N} v)}{(\mathbf{N} v)^s}$$

differs from the logarithmic derivative of

$$\prod_v \frac{1}{1 - (\mathbf{N} v)^{-s}}$$

by a function which is holomorphic for  $\Re(s) > 1/2$  as we saw above. Hence by the Wiener-Ikehara theorem we have

$$\sum_{\mathbf{N}v \leq n} \chi(x_v) \log(\mathbf{N}v) = c_\chi n + o(n) \quad (n \rightarrow \infty).$$

Consequently, by the Abel summation trick (cf. [13], Prop. 1),

$$\sum_{\mathbf{N}v \leq n} \chi(x_v) = c_\chi \frac{n}{\log n} + o(n/\log n) \quad (n \rightarrow \infty).$$

and in particular,

$$\sum_{\mathbf{N}v \leq n} 1 = \frac{n}{\log n} + o(n/\log n) \quad (n \rightarrow \infty).$$

Hence,

$$\frac{\sum_{\mathbf{N}v \leq n} \chi(x_v)}{\sum_{\mathbf{N}v \leq n} 1} \longrightarrow c_\chi \quad \text{as } n \rightarrow \infty,$$

and we may apply proposition 2 to conclude the proof.

q.e.d.

# CHAPTER II

## THE GROUPS $S_m$

Throughout this chapter,  $K$  denotes an algebraic number field. We as- II-1  
sociate to  $K$  a projective family  $(S_m)$  of commutative algebraic groups over  $\mathbb{Q}$ , and we show that each  $S_m$  gives rise to a strictly compatible system of rational  $\ell$ -adic representations of  $K$ .

In the next chapter, we shall see that all “locally algebraic” abelian rational representations are of the form described here.

### §1. Preliminaries

#### 1.1 The torus $\mathbb{T}$

Let  $\mathbb{T} = \mathfrak{R}_{K/\mathbb{Q}}(\mathbb{G}_{m,K})$  be the algebraic group over  $\mathbb{Q}$ , obtained from the multiplicative group  $\mathbb{G}_m$  by restriction of scalars from  $K$  to  $\mathbb{Q}$ , cf. **43** [43], §1.3. If  $A$  is a commutative  $\mathbb{Q}$ -algebra, the points of  $\mathbb{T}$  with values in  $A$  form by definition the multiplicative group  $(K \otimes_{\mathbb{Q}} A)^{\times}$  of invertible elements of  $K \otimes_{\mathbb{Q}} A$ . In particular,  $\mathbb{T}(\mathbb{Q}) = K^{\times}$ . If  $d = [K : \mathbb{Q}]$ , the group  $\mathbb{T}$  is a **torus** of dimension  $d$ ; this means that the group  $\mathbb{T}_{/\overline{\mathbb{Q}}} = \mathbb{T} \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$  obtained from  $\mathbb{T}$  by extending the scalars from  $\mathbb{Q}$  to  $\overline{\mathbb{Q}}$ , is isomorphic to...

II-2

Belen

#### 1.2 Cutting down $\mathbb{T}$

Let  $E$  be a subgroup of  $K = \mathbb{T}(\mathbb{Q})$  and let  $\overline{E}$  be the Zariski closure of  $E$  in  $\mathbb{T}$ . Using the formula  $\overline{E} \times \overline{E} = \overline{E \times E}$ , one sees that  $E$  is an algebraic subgroup of  $\mathbb{T}$ . Let  $\mathbb{T}_E$  be the quotient group  $\mathbb{T}/E$ ; then  $\mathbb{T}_E$  is also a torus

over  $\mathbb{Q}$ . Its character group  $X_E = X(\mathbb{T}_E)$  is the subgroup of  $X = X(T)$  consisting of those characters which take the value 1 on  $E$ . If  $\lambda = \prod_{\sigma \in \Gamma} [\sigma]^{n_\sigma}$  denotes a character of  $\mathbb{T}$ , then  $X_E$  is the subgroup of those  $\lambda \in X$  for which  $\prod_{\sigma \in \Gamma} [\sigma]^{n_\sigma} = 1$ , for all  $x \in E$ .

**Exercise.**

- a. Let  $K$  be quadratic over  $\mathbb{Q}$ , so that  $\dim T = 2$ . Let  $E$  be the group of units of  $K$ . Show that  $T$  is of dimension 2 (resp. 1) if  $K$  is imaginary (resp. real).
- b. Take for  $K$  a cubic field with one real place and one complex one, and let again  $E$  be its group of units (of rank 1). Show that  $\dim T = 3$  and  $\dim T_E = 1$ .

II-3 (For more examples, see 3.3.)

## 1.3 Enlarging groups

Belen

## §2. Construction of $T_m$ and $S_m$

### 2.1 Idèles and idèles-classes

We defined in Chapter ??, ?? the set  $M_K^0$  of finite places of the number field  $K$ . Let now  $M_K^\infty$  be the set of equivalence classes of archimedean absolute values of  $K$ , and let  $M_K$  be the union of  $M_K^0$  and  $M_K^\infty$ . If  $v \in M_K$  then  $K_v$  denotes the *completion* of  $K$  with respect to  $v$ . For  $v \in M_K^\infty$  we have  $K_v = \mathbb{R}$  or  $K_v = \mathbb{C}$ , and  $K$  is ultrametric if  $v \in M_K^0$ . For  $v \in M_K^0$ , the group of units of  $K_v$  is denoted by  $U_v$ . The **idèle group**  $I$  of  $K$  is the subgroup of

$$\prod_{v \in M_K} K_v^\times,$$

consisting of the families  $(a_v)$  with  $a_v \in U_v$ , for almost all  $v \in M_K^0$ ; it is given a topology by decreeing that the subgroup (with the product topology)

$$\prod_{v \in M_K^\infty} K_v^\times \times \prod_{v \in M_K^0} U_v$$

be open. We embed  $K^\times$  into  $I$  by sending  $a \in K^\times$  onto the idèle  $(a_v)$ , where  $a_v = a$  for all  $v$ . The topology induced on  $K$  is the discrete topology. The quotient group  $C_K = I/K^\times$  is called the **idèle class group** of  $K$ . (For all this, see **6** [6], **13** [13] or **44** [44].)

Let  $S$  be a finite subset of  $M_K^0$ . Then by a **modulus of support**  $S$  we mean a family  $\mathfrak{m} = (m_v)_{v \in S}$  where the  $m_v$  are integers  $\geq 1$ . If  $v \in M_K$  and  $\mathfrak{m}$  is a modulus of support  $S$ , we let  $U_{v,\mathfrak{m}}$  denote the connected component of  $K_v^\times$  if  $v \in M_K^\infty$ , the subgroup of  $U_v$  consisting of those  $u \in U_v$  for which  $v(1-u) \geq m_v$  if  $v \in S$ , and  $U_v$  if  $v \in M_K^0 \setminus S$ . The group  $U_{\mathfrak{m}} = \prod_v U_{v,\mathfrak{m}}$  is an open subgroup of  $I$ . If  $E$  is the group of units of  $K$ , let  $E_{\mathfrak{m}} = E \cap U_{\mathfrak{m}}$ . The subgroup  $E_{\mathfrak{m}}$  is of finite index in  $E$ . (Conversely, by a theorem of Chevalley ([8], see also [24], n° 3.5) every subgroup of finite index in  $E$  contains an  $E_{\mathfrak{m}}$  for a suitable modulus  $\mathfrak{m}$ .)

Let  $I_{\mathfrak{m}}$  be the quotient  $I/U_{\mathfrak{m}}$  and  $C_{\mathfrak{m}}$  the quotient  $I/K^\times U_{\mathfrak{m}} = C/(\text{Image of } U_{\mathfrak{m}} \text{ in } C)$ . One then has the exact sequence:

$$1 \longrightarrow K^\times/E_{\mathfrak{m}} \longrightarrow I_{\mathfrak{m}} \longrightarrow C_{\mathfrak{m}} \longrightarrow 1$$

The group  $C_{\mathfrak{m}}$  is finite; in fact, the image of  $U_{\mathfrak{m}}$  in  $C$  is open, hence contains the connected component  $D$  of  $C$ , and the group  $C/D$  is known to be compact (see [13], [44]). Moreover, any open subgroup of  $I$  contains one of the  $U_{\mathfrak{m}}$ 's, hence  $C/D$  is the projective limit of the  $C_{\mathfrak{m}}$ 's. Class field theory (cf. for instance **6** [6]), gives an isomorphism of  $C/D = \varprojlim C_{\mathfrak{m}}$  onto the Galois group  $G^{\text{ab}}$  of the maximal abelian extension of  $K$ .

**Remark.** A more classical definition of  $C_{\mathfrak{m}}$  is as follows. Let  $\text{Id}_S$  be the group of fractional ideals of  $K$  prime to  $S$ , and  $P$  the subgroup of principal ideals  $(\gamma)$ , where  $\gamma$  is totally positive and  $\gamma \equiv 1 \pmod{\mathfrak{m}}$  (i.e.  $\gamma$  belongs to  $U_{v,\mathfrak{m}}$  for all  $v \in S$  and  $v \in M_K^\infty$ ). Let  $\text{Cl}_{\mathfrak{m}} = \text{Id}_S/P_{S,\mathfrak{m}}$ . We have the exact sequence:

$$1 \longrightarrow P_{S,\mathfrak{m}} \longrightarrow \text{Id}_S \longrightarrow \text{Cl}_{\mathfrak{m}} \longrightarrow 1.$$

For each  $a = \prod_{v \notin S} v^{a_v} \in \text{Id}_S$ , choose an idèle  $\alpha = (\alpha_v)$ , with  $\alpha_v \in U_{v,\mathfrak{m}}$  if  $v \in S$  or  $v \in M_K^\infty$ , and  $v(\alpha_v) = a_v$  if  $v \in M_K^0 \setminus S$ . The image of  $\alpha$  in  $I_{\mathfrak{m}} = I/U_{\mathfrak{m}}$  depends only on  $\mathbf{a}$ . We then get a homomorphism  $g: \text{Id}_S \rightarrow I_{\mathfrak{m}}$ .

One checks readily that  $g$  extends to a commutative diagram

$$\begin{array}{ccccccc}
 1 & \longrightarrow & P_{S,\mathfrak{m}} & \longrightarrow & \text{Id}_S & \longrightarrow & \text{Cl}_{\mathfrak{m}} \longrightarrow 1 \\
 & & \downarrow & & \downarrow g & & \downarrow f \\
 1 & \longrightarrow & K^\times/E_{\mathfrak{m}} & \longrightarrow & I_{\mathfrak{m}} & \longrightarrow & C_{\mathfrak{m}} \longrightarrow 1
 \end{array}$$

and that  $f: \text{Cl}_{\mathfrak{m}} \rightarrow C_{\mathfrak{m}}$  is an isomorphism: hence  $C$  can be identified with the ideal class group mod  $\mathfrak{m}$  (and this shows again that it is finite).

## 2.2 The groups $T_{\mathfrak{m}}$ and $S_{\mathfrak{m}}$

Belen.

## 2.3 The canonical $\ell$ -adic representation with values in $S_{\mathfrak{m}}$

Let  $\mathfrak{m}$  be a modulus, and let  $\ell$  be a prime number. Let  $\varepsilon: I \rightarrow I_{\mathfrak{m}} \rightarrow S_{\mathfrak{m}}(\mathbb{Q})$  be the homomorphism defined in ???. Let  $\pi: T \rightarrow S_{\mathfrak{m}}$  be the algebraic morphism  $T \rightarrow T_{\mathfrak{m}} \rightarrow S_{\mathfrak{m}}$ ; by taking points with values in  $\mathbb{Q}_{\ell}$ ,  $\pi$  defines a homomorphism

$$\pi_{\ell}: T(\mathbb{Q}_{\ell}) \longrightarrow S_{\mathfrak{m}}(\mathbb{Q}_{\ell})$$

Since  $K \otimes \mathbb{Q}_{\ell} = \prod_{v|\ell} K_v$ , the group  $T(\mathbb{Q}_{\ell})$  can be identified with  $K_{\ell}^{\times} = \prod_{v|\ell} K_v^{\times}$ , and is therefore a direct factor of the idele group  $I$ . Let  $\text{pr}_{\ell}$  denote the projection of  $I$  onto this factor. The map

$$\alpha_{\ell} = \pi_{\ell} \circ \text{pr}_{\ell}: I \longrightarrow T(\mathbb{Q}_{\ell}) \longrightarrow S_{\mathfrak{m}}(\mathbb{Q}_{\ell})$$

is a continuous homomorphism.

**Lemma 1.**  $\alpha_{\ell}$  and  $\varepsilon$  coincide on  $K^{\times}$ .

This is trivial from the commutativity of the diagram (\*\*) of ??.

II-6 Now, let  $\varepsilon_{\ell}: I \rightarrow S_{\mathfrak{m}}(\mathbb{Q}_{\ell})$  be defined by

$$\begin{aligned}
 \varepsilon_{\ell}(\mathbf{a}) &= \varepsilon(\mathbf{a})\alpha_{\ell}(\mathbf{a}^{-1}) & (***) \\
 \text{i.e. } \varepsilon_{\ell} &= \varepsilon \cdot \alpha_{\ell}^{-1}.
 \end{aligned}$$

(If  $\mathbf{a} \in I$ , write  $a_\ell$  the  $\ell$ -component of  $\mathbf{a}$ . Then

$$\varepsilon_\ell(\mathbf{a}) = \varepsilon(\mathbf{a})\pi_\ell(a_\ell^{-1}).)$$

By the lemma,  $\varepsilon_\ell$  is trivial on  $K$  and, hence, defines a map  $C \rightarrow S_{\mathfrak{m}}(\mathbb{Q}_\ell)$ ; since  $S_{\mathfrak{m}}(\mathbb{Q}_\ell)$  is totally disconnected (it is an  $\ell$ -adic Lie group), the latter homomorphism is trivial on the connected component  $D$  of  $C$ . We have already recalled that  $C/D$  may be identified with the Galois group  $G^{\text{ab}}$  of the maximal abelian extension of  $K$ . So we end up with a homomorphism  $\varepsilon_\ell: G^{\text{ab}} \rightarrow S_{\mathfrak{m}}(\mathbb{Q}_\ell)$ , i.e. with an  $\ell$ -adic representation of  $K$  with values in  $S_{\mathfrak{m}}$  (cf. Chap. ??, ??).

This representation is rational in the sense of Chapter ??, ??. More precisely, let  $v \notin \text{Supp}(\mathfrak{m})$ , and let  $f_v \in I$  be an idèle which is a uniformizing parameter at  $v$ , and which is equal to 1 everywhere else; let  $F_v = \varepsilon(f_v)$  be the image of  $f_v$  in  $S_{\mathfrak{m}}(\mathbb{Q}_\ell)$ . With these notations we have:

**Proposition 1.** *a) The representation  $\varepsilon_\ell: G^{\text{ab}} \rightarrow S_{\mathfrak{m}}(\mathbb{Q}_\ell)$  is a rational representation with values in  $S_{\mathfrak{m}}$ .*

*b)  $\varepsilon_\ell$  is unramified outside  $\text{Supp}(\mathfrak{m}) \cup S_\ell$ , where  $S_\ell = \{v : p_v = \ell\}$ .*

*c) If  $v \notin \text{Supp}(\mathfrak{m}) \cup S_\ell$ , then the Frobenius element  $F_{v, \varepsilon_\ell}$  (cf. Chap. ??, II-7 ??) is equal to  $F_v \in S_{\mathfrak{m}}(\mathbb{Q}_\ell)$ .*

*Proof.* It is known that the class field isomorphism  $C/D \xrightarrow{\sim} G^{\text{ab}}$  maps  $K_v^\times$  (resp.  $U_v$ ) onto a dense subgroup of the decomposition group of  $v$  in  $G^{\text{ab}}$  (resp. onto the inertia group of  $v$  in  $G^{\text{ab}}$ ), and that a uniformizing element  $f_v$  of  $K_v^\times$  is mapped onto the Frobenius class of  $v$ .

If  $v \notin \text{Supp}(\mathfrak{m})$  and  $a \in U_v$ , then  $\varepsilon(a) = 1$ ; if moreover  $p_v \neq \ell$ ,  $\alpha_\ell(a) = 1$ , hence  $\varepsilon_\ell(a) = 1$  and  $\varepsilon_\ell$  is unramified at  $v$ ; this proves b). For such a  $v$ , we have  $\varepsilon_\ell(f_v) = \varepsilon(f_v) = F_v$ ; hence c), and a) follows from c).  $\square$

**Corollary 1.1.** *The representations  $\varepsilon$  form a system of strictly compatible  $\ell$ -adic representations with values in  $S_{\mathfrak{m}}$ .*

We also see that the exceptional set of this system is contained in  $\text{Supp}(\mathfrak{m})$ ; for an example where it is different from  $\text{Supp}(\mathfrak{m})$ , see Exercise ??.

**Remark.** By construction,  $\varepsilon_\ell: I \rightarrow S_{\mathfrak{m}}(\mathbb{Q}_\ell)$  is given by  $x \mapsto \pi_\ell(x^{-1})$  on the open subgroup  $U_{\ell, \mathfrak{m}} = \prod_{v \nmid \ell} U_{v, \mathfrak{m}}$  of  $K_\ell^\times$ . Hence,  $\text{Im}(\varepsilon_\ell)$  contains  $\pi_\ell(U_{\ell, \mathfrak{m}}) \subset T_{\mathfrak{m}}(\mathbb{Q}_\ell) \subset S_{\mathfrak{m}}(\mathbb{Q}_\ell)$ , and is an open subgroup of  $S_{\mathfrak{m}}(\mathbb{Q}_\ell)$ . This open subgroup maps onto  $C_{\mathfrak{m}}$ , as remarked above. These properties imply, in particular, that  $\text{Im}(\varepsilon_\ell)$  is Zariski-dense in  $S_{\mathfrak{m}}$ .

II-8 **Exercises.**

(1) Let  $K = \mathbb{Q}$ ,  $\text{Supp}(\mathfrak{m}) = \emptyset$ .

a) Show that  $E_{\mathfrak{m}} = \{1\}$ ,  $C_{\mathfrak{m}} = \{1\}$ , hence  $T_{\mathfrak{m}} = S_{\mathfrak{m}} = \mathbb{G}_m$  and  $S_{\mathfrak{m}}(\mathbb{Q}) = \mathbb{Q}^{\times}$ ,  $S_{\mathfrak{m}}(\mathbb{Q}_{\ell}) = \mathbb{Q}_{\ell}^{\times}$ .

b) Show that  $I$  is the direct product of its subgroups  $I_{\mathfrak{m}}$  and  $\mathbb{Q}^{\times}$ ; hence any  $\mathbf{a} \in I$  may be written as

$$\mathbf{a} = u \cdot \gamma, \quad u \in U_{\mathfrak{m}}, \quad \gamma \in \mathbb{Q}^{\times}.$$

Show that, if  $\mathbf{a} = (a_p)$ , one has

$$\varepsilon(\mathbf{a}) = \gamma = \text{sgn}(a_{\infty}) \prod_p p^{v_p(a_p)}.$$

c) Show that

$$\rho_{\ell}(\mathbf{a}) = \gamma \cdot a_{\ell}^{-1},$$

and

$$F_p = p.$$

d) Show that  $\rho_{\ell}$  coincides with the character  $\chi_{\ell}$  of Chap. ??, ??.

(2) Let  $K = \mathbb{Q}$ ,  $\text{Supp}(\mathfrak{m}) = \{2\}$  and  $m_2 = 1$ . Show that the groups  $E_{\mathfrak{m}}$ ,  $C_{\mathfrak{m}}$ ,  $T_{\mathfrak{m}}$ ,  $S_{\mathfrak{m}}$  coincide with those of Exercise ??, hence that the exceptional set of the corresponding system is empty.

## 2.4 Linear representations of $S_{\mathfrak{m}}$

We recall first some well known facts on representations.

II-9 a) Let  $k$  be a field of characteristic 0; let  $H$  be an affine commutative algebraic group over  $k$ . Let  $X(H) = \text{Hom}_{\bar{k}}(H/\bar{k}, \mathbb{G}_{m, \bar{k}})$  be the group of characters of  $H$  (of degree 1). Here we write the characters of  $X(H)$  multiplicatively. The group  $G = \text{Gal}(\bar{k}/k)$  acts on  $X(H)$ .



Let  $\Lambda$  be the affine algebra of  $H$ , and let  $\bar{\Lambda} = \Lambda \otimes_k \bar{k}$  be the one of  $H/\bar{k}$ . Every element  $\chi \in X(H)$  can be identified with an invertible element of  $\bar{\Lambda}$ . Hence, by linearity, a homomorphism

$$\alpha: \bar{k}[X(H)] \longrightarrow \bar{\Lambda}$$

where  $\bar{k}[X(H)]$  is the group algebra of  $X(H)$  over  $\bar{k}$ . This is a  $G$ -homomorphism if the action of  $G$  is defined by

$$s \left( \sum_{\chi} a_{\chi} \chi \right) = \sum s(a_{\chi}) s(\chi)$$

for  $a_{\chi} \in \bar{k}$  and  $\chi \in X(H)$ . It is well-known (linear independence of characters) that  $\alpha$  is injective. It is bijective if and only if  $H$  is a group of multiplicative type (cf. ??, remark 2). Hence we may identify  $\bar{k}[X(H)]$  with a subalgebra of  $\bar{\Lambda}$ .

b) Let  $V$  be a finite-dimensional  $k$ -vector space and let

$$\phi: H \longrightarrow \mathrm{GL}_V$$

be a *linear representation* of  $H$  into  $V$ . Assume  $\phi$  is *semi-simple* (this is always the case if  $H$  is of multiplicative type). We associate to  $\phi$  its **trace**

$$\theta_{\phi} = \sum_{\chi} n_{\chi}(\phi) \chi$$

in  $\mathbb{Z}[X(H)]$ , where  $n_{\chi}(\phi)$  is the multiplicity of  $\chi$  in the decomposition of  $\chi$  over  $\bar{k}$ .

We have  $\theta_{\phi}(h) = \mathrm{Tr}(\phi(h))$  for any point  $h$  of  $H$  (with value in any commutative  $k$ -algebra). Let  $\mathrm{Rep}_k(H)$  be the set of isomorphism classes of linear semi-simple representations of  $H$ . If  $k_1$  is an extension of  $k$ , then scalar extension from  $k$  to  $k_1$  defines a map  $\mathrm{Rep}_k(H) \rightarrow \mathrm{Rep}_{k_1}(H/k_1)$  which is easily seen to be *injective*. We say that an element of  $\mathrm{Rep}_{k_1}(H/k_1)$  *can be defined over  $k$* , if it is in the image of this map. II-10

**Proposition 2.** *The map  $\phi \mapsto \theta_{\phi}$  defines a bijection between  $\mathrm{Rep}_k(H)$  and the set of elements  $\theta = \sum n_{\chi} \chi$  of  $\mathbb{Z}[X(H)]$  which satisfy:*

(a)  *$\theta$  is invariant by  $G$  (i.e.  $n_{\chi} = n_{s(\chi)}$  for all  $s \in G$ ,  $\chi \in X(H)$ ).*

(b)  $n_\chi \geq 0$  for every  $\chi \in X(H)$ .

*Proof.* The injectivity of the map  $\phi \mapsto \theta_\phi$  is well-known (and does not depend on the commutativity of  $H$ ). To prove surjectivity, consider first the case where  $\theta$  has the form  $\theta = \sum_i \chi^{(i)}$  where  $\chi^{(i)}$  is a full set of different conjugates of a character  $\chi \in X(H)$ . If  $G(\chi)$  is the subgroup of  $G$  fixing  $\chi$ , then

$$\theta = \sum_{s \in G/G(\chi)} s(\chi). \quad (*)$$

II-11 The fixed field  $k_\chi$  of  $G(\chi)$  in  $k$  is the smallest subfield of  $k$  such that  $\chi \in \Lambda \otimes k_\chi$ . Consider  $\chi$  as a representation of degree 1 of  $H/k_\chi$ . One gets, by restriction of scalars to  $k$ , a representation  $\phi$  of  $H$  of degree  $[k_\chi : k]$ . One sees easily that the trace  $\theta_\phi$  of  $\phi$  is equal to  $\theta$ . The surjectivity of  $\phi \mapsto \theta_\phi$  now follows from the fact that any  $\theta$  satisfying (a) and (b) is a sum of elements of the form (??) above.  $\square$

**Corollary 2.1.** *In order that  $\phi_1 \in \text{Rep}_{k_1}(H/k_1)$  can be defined over  $k$ , it is necessary and sufficient that  $\theta_{\phi_1} \in \Lambda \otimes_k k_1$  belongs to  $k_1$ .*

c) We return now to the groups  $S_{\mathfrak{m}}$ :

**Proposition 3.** *Let  $k_1$  be an extension of  $k$  and let  $\phi \in \text{Rep}_{k_1}(S_{\mathfrak{m}/k_1})$ . The following properties are equivalent:*

- (i)  $\phi$  can be defined over  $k$ ,
- (ii) for every  $v \notin \text{Supp}(\mathfrak{m})$ , the coefficients of the characteristic polynomial  $\phi(F_v)$  belong to  $k$ ,
- (iii) there exists a set  $M$  of places of  $k$  of density 1 (cf. Chapter ??, ??) such that  $\text{Tr}(\phi(F_v)) \in k$  for all  $v \in M$ .

*Proof.* The implications (i)  $\implies$  (ii)  $\implies$  (iii) are trivial. To prove (iii)  $\implies$  (i) we need the following lemma.  $\square$

**Lemma 2.** *The set of Frobeniuses  $F_v$ ,  $v \in M$ , is dense in  $S$  for the Zariski topology.*

*Proof.* Let  $X$  be the set of all  $F_v$ 's,  $v \in M$ , and let  $\ell$  be a prime number. Let  $\overline{X} \subseteq S_m$  (resp.  $\overline{X}_\ell \subseteq S_m(\mathbb{Q}_\ell)$ ) the closure of  $X$  in the Zariski topology (resp.  $\ell$ -adic topology). It is clear that  $\overline{X} \subseteq \overline{X}(\mathbb{Q}_\ell)$ . On the other hand, II-12 Cebotarev's theorem (cf. Chapter ??, ??) implies that  $\overline{X} = \text{Im}(\varepsilon_\ell)$  (cf. ??). The set  $\text{Im}(\varepsilon_\ell)$ , however, is Zariski dense in  $S_m$  (cf. Remark in ??). Hence  $\overline{X} = S_m$ , which proves the lemma.  $\square$

Let us now prove that (iii)  $\implies$  (i). Let  $\theta_\phi$  be the trace of  $\theta$  in  $\Lambda \otimes_k k_1$ , where  $\Lambda$  is the affine algebra of  $H = S_{m/k}$ . Let  $\{\ell_\alpha\}$  be a basis of the  $k$ -vector space  $k_1$ , with  $\ell_{\alpha_0} = 1$  for some index  $\alpha_0$ . We have  $\theta_\phi = \sum_\alpha \lambda_\alpha \otimes \ell_\alpha$  ( $\lambda_\alpha \in \Lambda$ ); hence  $\text{Tr}(\phi(h)) = \theta_\phi(h) = \sum_\alpha \lambda_\alpha(h) \ell_\alpha$  for all  $h \in H(k_1)$ . Take  $h = F_v$ , with  $v \in M$ . Since  $F_v$  belongs to  $H(k)$  we have  $\lambda_\alpha(F_v) \in k$  for all  $\alpha$ ; since  $\text{Tr}(\phi(F_v)) \in k$ , we get  $\lambda_\alpha(F_v) = 0$  for all  $\alpha \neq \alpha_0$ . By the lemma, the  $F_v$ 's,  $v \in M$ , are Zariski-dense in  $H$ ; hence  $\lambda_\alpha = 0$  for  $\alpha \neq \alpha_0$  and  $\theta_\phi = \lambda_{\alpha_0}$  belongs to  $\Lambda$  and (i) follows from the corollary to Proposition 1.  $\square$

**Exercise.** Show that the characters of  $S_m$  correspond in a one-one way to the homomorphisms  $\chi: I \rightarrow \overline{\mathbb{Q}}^\times$  having the following two properties:

- (a)  $\chi(x) = 1$  if  $x \in U_m$ .
- (b) For each embedding  $\sigma$  of  $K$  into  $\overline{\mathbb{Q}}$ , there exists an integral number  $n(\sigma)$  such that

$$\chi(x) = \prod_{\sigma \in \Gamma} \sigma(x)^{n(\sigma)}$$

for all  $x \in K^\times$ .

## 2.5 $\ell$ -adic representations associated to a linear representation of $S_m$

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## 2.6 Alternative construction

Let  $\phi_0: S_m \rightarrow \text{GL}_{V_0}$  be as in ??. If we compose  $\phi_0$  with the map  $\varepsilon: I \rightarrow S_m(\mathbb{Q})$  defined in ??, we obtain a homomorphism

$$\phi_0 \circ \varepsilon: I \longrightarrow \text{GL}_{V_0}(\mathbb{Q}) = \text{Aut}(V_0).$$

Conversely:

II-13

**Proposition 4.** *Let  $f: I \rightarrow \text{Aut}(V_0)$  be a homomorphism. There exists a  $\phi_0: S_{\mathfrak{m}} \rightarrow \text{GL}_{V_0}$  such that  $\phi_0 \circ \varepsilon = f$  if and only if the following conditions are satisfied:*

- 1) *The kernel of  $f$  contains  $U_{\mathfrak{m}}$ .*
- 2) *There exists an algebraic homomorphism  $\psi: T \rightarrow \text{GL}_{V_0}$  such that  $\psi(x) = f(x)$  for every  $x \in K^\times = T(\mathbb{Q})$ .*

Moreover, such a  $\phi_0$  is unique.

*Proof.* The necessity of the conditions (a) and (b) is trivial. Conversely, if  $f$  has properties (a), (b), it defines a homomorphism  $I/U_{\mathfrak{m}} \rightarrow \text{Aut}(V_0)$ . On the other hand, since  $f$  and  $\psi$  agree on  $K^\times$  the morphism  $\psi$  is equal to 1 on  $E_{\mathfrak{m}} = K^\times \cap U_{\mathfrak{m}}$ , hence on its Zariski-closure  $\overline{E}_{\mathfrak{m}}$ . This means that  $\psi$  factors through

$$T \longrightarrow T_{\mathfrak{m}} \longrightarrow \text{GL}_{V_0}.$$

By the universal property of  $S_{\mathfrak{m}}$  (cf. ?? and ??), the maps  $I/U_{\mathfrak{m}} \rightarrow \text{GL}_{V_0}(\mathbb{Q})$  and  $T_{\mathfrak{m}} \rightarrow \text{GL}_{V_0}$  define an algebraic morphism  $\phi_0: S_{\mathfrak{m}} \rightarrow \text{GL}_{V_0}$ , and one checks easily that  $\phi_0$  has the required properties, and is unique.  $\square$

**Remark.** Since  $U$  is open, property (a) implies that  $f$  is *continuous* with respect to the discrete topology of  $\text{Aut}(V_0)$ . Conversely, any continuous homomorphism  $f: I \rightarrow \text{Aut}(V_0)$  is trivial on some  $U_{\mathfrak{m}}$ ; moreover, there is a smallest such  $\mathfrak{m}$ ; it is called the **conductor** of  $f$ .

II-14

**Exercise.** Let  $\mathfrak{m}$  be a modulus and let  $V_0$  be a finite dimensional  $\mathbb{Q}$ -vector space. For each  $v \notin \text{Supp}(\mathfrak{m})$  let  $F_v$  be an element of  $\text{Aut}(V_0)$ . Assume:

- 1) The  $F_v$ 's commute pairwise.
- 2) There exists an algebraic morphism  $\psi: T \rightarrow \text{GL}_{V_0}$  such that  $\psi(\alpha) = \prod F_v^{v(\alpha)}$  for  $\alpha \in K^\times$ ,  $\alpha \equiv 1 \pmod{\mathfrak{m}}$ , and  $\alpha > 0$  at each real place.

Show that there exists an algebraic morphism  $\phi_0: S_{\mathfrak{m}} \rightarrow \text{GL}_{V_0}$  for which the Frobenius elements are equal to the  $F_v$ 's.

## 2.7 The real case

Belen.

## 2.8 An example: complex multiplication of abelian varieties

(We give here only a brief sketch of the theory, with a few indications on the proofs. For more details, see **34** [34], **35** [35], **41** [41], [42] and **32** [32].)

Let  $A$  be an abelian variety of dimension  $d$  defined over  $K$ . Let  $\text{End}_K(A)$  be its ring of endomorphisms and put  $\text{End}_K(A)_0 = \text{End}_K(A) \otimes \mathbb{Q}$ . Let  $E$  be a number field of degree  $2d$ , and

II-15

$$i: E \rightarrow \text{End}_K(A)_0$$

be an injection of  $E$  into  $\text{End}_K(A)_0$ . The variety  $A$  is then said to have “complex multiplication” by  $E$ ; in the terminology of Shimura-Taniyama, it is a variety of “type (CM)”.

Let  $\ell$  be a prime integer and define  $T_\ell(A)$  and  $V_\ell = T_\ell(A) \otimes \mathbb{Q}_\ell$  as in Chapter ??, ??. These are free modules over  $\mathbb{Z}_\ell$  and  $\mathbb{Q}_\ell$ , of rank  $2d$ . The  $\mathbb{Q}$ -algebra  $\text{End}_K(A)_0$  acts on  $V_\ell$ ; hence the same is true for  $E$ , and, by linearity, for  $E_\ell = E \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$ . One proves easily:

**Lemma 3.**  *$V_\ell$  is a free  $E_\ell$ -module of rank 1.*

Let  $\rho: \text{Gal}(\overline{K}/K) \rightarrow \text{Aut}(V_\ell)$  be the  $\ell$ -adic representation defined by  $A$ . If  $s \in \text{Gal}(\overline{K}/K)$ , it is clear that  $\rho(s)$  commutes with  $E$ , hence with  $E_\ell$ . But the lemma above implies that the commuting algebra of  $E_\ell$  in  $\text{End}_K(V_\ell)$  is  $E_\ell$  itself. Hence,  $\rho$  may be identified with a homomorphism

$$\rho_\ell: \text{Gal}(\overline{K}/K) \longrightarrow E_\ell^\times$$

Let now  $T_E$  be the  $2d$ -dimensional torus attached to  $E$  (as  $\mathbb{T}$  is attached to  $K$ ), so that  $T_E(\mathbb{Q}_\ell) = E_\ell^\times$ , and  $\rho$  takes values in  $T_E(\mathbb{Q}_\ell)$ .

**Theorem 1.** (a) *The system  $(\rho_\ell)$  is a strictly compatible system of rational  $\ell$ -adic representations of  $K$  with values in  $T_E$  (in the sense of Chap. ??, II-16 ??).*

(b) *There is a modulus  $\mathfrak{m}$  and a morphism*

$$\varphi: S_{\mathfrak{m}} \longrightarrow T_E$$

*such that  $\rho$  is the image by  $\varphi$  of the canonical system  $(\varepsilon_\ell)$  attached to  $S_{\mathfrak{m}}$ , cf. ??.*

Moreover, the restriction of  $\varphi$  to  $T_{\mathfrak{m}}$  can be given explicitly:

Let  $t$  be the tangent space at the origin of  $A$ . It is a  $K$ -vector space on which  $E$  acts, i.e. an  $(E, K)$ -bimodule. If we view it as an  $E$ -vector space, the action of  $K$  is given by a homomorphism  $j: K \rightarrow \text{End}_E(t)$ . In particular, if  $x \in K^\times$ ,  $\det_E j(x)$  is an element of  $E^\times$ ; the map  $\det_E j: K^\times \rightarrow E^\times$  is clearly the restriction of an algebraic morphism  $\delta: \mathbb{T} \rightarrow T_E$ .

**Theorem 2.** *The map  $\delta: \mathbb{T} \rightarrow T_E$  coincides with the composition map  $\mathbb{T} \rightarrow T_{\mathfrak{m}} \rightarrow S_{\mathfrak{m}} \xrightarrow{\varphi} T_E$*

**Examples.** If  $A$  is an elliptic curve,  $E$  is an imaginary quadratic field, and the action of  $E$  on the one-dimensional  $K$ -vector space  $t$  defines an embedding  $E \rightarrow K$ . The map  $\det_E j: K^\times \rightarrow E^\times$  is just the *norm* relative to this embedding.

**Indications on the proofs of Theorems 1 and 2.** Part ?? of Theorem ?? is proved as follows: Let  $S$  denote the finite set of  $v \in M_K^0$  where  $A$  has “bad reduction”. If  $v \notin S$ , and  $\ell \neq p_v$ , one shows easily that  $p_\ell$  is unramified at  $v$  (the converse is also true, see [32]); moreover the corresponding Frobenius element  $F_{v, \rho_\ell}$  may be identified with the Frobenius endomorphism  $F_v$  of the reduced variety  $\tilde{A}_v$ . But  $F_v$  commutes with  $E$  in  $\text{End}(\tilde{A}_v)_0$  and the commuting algebra of  $E$  in  $\text{End}(\tilde{A}_v)_0$  is  $E$  itself (cf. [34]). Hence  $F_v$  belongs to  $E^\times = T_E(\mathbb{Q})$  and this implies ??.

Theorem ?? and part ?? of Theorem ?? are less easy; they are proved, in a somewhat different form in 34 [34] (see also [32]). Note that one could express them (as in ??) by saying that *there exists a homomorphism  $f: I \rightarrow E^\times$  (where  $I$  denotes, as usual, the group of idèles of  $K$ ) having the following properties:*

- 1)  $f$  is trivial on  $U_{\mathfrak{m}}$ , for some modulus  $\mathfrak{m}$  with support  $S$ .
- 2) If  $v \notin S$ , the image by  $f$  of a uniformizing parameter at  $v$  is the Frobenius element  $F_v \in E^\times$ .

3) If  $x \in K^\times$  is a principal idèle, one has  $f(x) = \det_E j(x)$ .

This is essentially what is proved in [34], formula (3), except that the result is expressed in terms of ideals instead of ideles, and  $\det_E j(x)$  is written in a different form, namely “ $\prod_\alpha N_{K/K^\times}(x)^{\psi_\alpha}$ ”.

**Remark.** Another possible way of proving Theorems ?? and ?? is the following:

Let  $\ell$  be a prime integer distinct from any of the  $p_v$ ,  $v \in S$ . One then sees that the Galois-module  $V_\ell$  is of Hodge-Tate type in the sense of Chapter III, 1.2 (indeed, the corresponding local modules are associated with  $\ell$ -divisible II-18 groups, and one may apply Tate’s theorem [39]). Hence  $\rho_\ell$  is “locally algebraic” (Chapter III, *loc. cit.*), and using the theorem of Chapter III, 2.3 one sees it defines a morphism  $\varphi: S_m \rightarrow T_E$ . One has  $\varphi \circ \varepsilon_\ell = \rho_\ell$  by construction; the same is true for any prime number  $\ell'$ , since  $\varphi \circ \varepsilon_{\ell'}$  and  $\rho_{\ell'}$  have the same Frobenius elements for almost all  $v$ . This proves part ?? of Theorem ??. As for Theorem ??, one uses the explicit form of the Hodge-Tate decomposition of  $V_\ell$ , as given by 39 [39], combined with the results of the Appendix to Chapter III.

### §3. Structure of $T_m$ and applications

#### 3.1 Structure of $X(T_m)$

Belen.

#### 3.2 The morphism $j^*: \mathbb{G}_m \rightarrow T_m$

Belen.

#### 3.3 Structure of $T_m$

We need first some notations:

Let  $H_c$  be the closed subgroup of  $G = \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$  generated by  $C_\infty$  (cf. ??). There is a unique continuous homomorphism  $\varepsilon: H_c \rightarrow \{\pm 1\}$  such that II-19  $\varepsilon(c) = -1$  for all  $c \in C_\infty$ . Indeed the unicity of  $\varepsilon$  is clear, and one proves its existence by taking the restriction to  $H_c$  of the homomorphism  $G \rightarrow \{\pm 1\}$

associated with an imaginary quadratic extension of  $\mathbb{Q}$ . We let  $H = \text{Ker}(\varepsilon)$ . The groups  $H$  and  $H_c$  are closed invariant subgroups of  $G$ , and  $(H : H_c) = 2$ .

Let now  $K$  be, as before, a finite extension of  $\mathbb{Q}$ ; we identify it with a subfield of  $\mathbb{Q}$ ; let  $G_K = \text{Gal}(\overline{\mathbb{Q}}/K)$  be the corresponding subgroup of  $G$ . The field  $K$  is *totally real* if and only if all the elements  $c$  of  $C_\infty$  act trivially on  $K$ , i.e. if and only if  $G_K$  contains  $G_c$ . Hence, there exists a *maximal totally real subfield*  $K_0$  of  $K$ , whose Galois group is  $G_{K_0} = G_K \cdot H_c$ . We let  $K_1$ , be the field corresponding to  $G_K \cdot H$ . We have

$$K_0 \subset K_1 \subset K \quad \text{and} \quad [K_1 : K_0] = 1 \text{ or } 2.$$

As shown by Weil (cf. [47]) the fields  $K_0$  and  $K_1$  are closely connected to the groups  $T_m$  relative to  $K$ . Indeed, if  $\chi = \sum_\sigma b_\sigma[\sigma]$  is an element of the group denoted by  $Y^-$  in ??, we have  $b_{c\sigma} = -b_\sigma$  for all  $c \in C_\infty$ . If  $h = c_1 \cdots c_n$ , this gives

$$b_{h\sigma} = (-1)^n b_\sigma = \varepsilon(h) b_\sigma$$

and by continuity the same holds for all  $h \in H_c$ . One deduces from this:

**Proposition 1.** *The norm map defines an isomorphism of the space  $Y_{K_1}^0$  relative to  $K$  onto the space  $Y_K^-$  relative to  $K$ .*

II-20 More precisely, if  $\chi_1 = \sum b_{\sigma_1}[\sigma_1]$  belongs to  $Y_{K_1}^-$ , where  $\sigma_1 \in \Gamma_{K_1}$ , the image of  $\chi_1$ , by the norm map is

$$N_{K_1/K_0}^*(\chi_1) = \sum_\sigma b_{\sigma/K_1}[\sigma], \quad \sigma \in \Gamma_K,$$

where  $\sigma/K_1$  is the restriction of  $\sigma$  to  $K$ . It is clear that this map is injective. Conversely, if  $\chi = \sum_\sigma b_\sigma[\sigma]$  belongs to  $Y_K^-$ , we saw above that  $b_{h\sigma} = \varepsilon(h)b_\sigma$  for all  $h \in H_c$ , hence  $b_{h\sigma} = b_\sigma$  for  $h \in H$  and of course also for  $h \in H \cdot G_K$ . This shows that  $b_\sigma$  depends only on the restriction of  $\sigma$  to  $K_1$ , and hence that  $\chi$  belongs to the image of the norm map.

**Corollary 1.1.** *The tori  $T_m$  attached to  $K$  and  $K_1$  are isogenous to each other.*

There remains to describe the tori  $T_m$  attached to  $K_1$ . There are two cases:



- (1)  $K_1 = K_0$ . In this case, we have  $Y^- = 0$  and  $T_m$  is one-dimensional, and isomorphic to  $\mathbb{G}_m$ .

Indeed, if  $\chi = \sum_{\sigma} b_{\sigma}[\sigma]$  belongs to  $Y^-$ , and  $c \in C_{\infty}$ , we have  $b_{c\sigma} = -b_{\sigma}$  (cf. ??) but also  $b_{c\sigma} = b_{\sigma}$  since  $c \in G_K \cdot H_c = G_K \cdot H$ . This shows that  $b_{\sigma} = 0$  for all  $\sigma$ , hence  $Y^- = 0$ .

- (2)  $[K_1 : K_0] = 2$ . The field  $K_1$  is then a *totally imaginary quadratic extension* of  $K_0$  (and it is the only one contained in  $K$ , as one checks readily). In this case  $Y^-$  is of dimension  $d = [K_0 : \mathbb{Q}]$  and  $T_m$  is  $(d+1)$ -dimensional.

More precisely, the space  $Y$  attached to  $K_1$  is  $2d$ -dimensional and the involution  $\sigma$  of  $K_1$  corresponding to  $K_0$  decomposes  $Y$  in two eigenspaces of dimension  $d$  each; the space  $Y^-$  is the one corresponding to the eigenvalue  $-1$  of  $\sigma$ . This is proved by the same argument as above, once one remarks that all  $c \in C_{\infty}$  induce  $\sigma$  on  $K_1$ . II-21

**Remark.** In this last case (which is the most interesting one), the torus  $T_m$  is isogenous to the product of  $\mathbb{G}_m$  by the  $d$ -dimensional torus kernel of the norm map from  $K_1$  to  $K_0$ .



## CHAPTER III

### $\ell$ -ADIC REPRESENTATIONS ATTACHED TO ELLIPTIC CURVES

Let  $K$  be a number field and let  $E$  be an elliptic curve over  $K$ . If  $\ell$  is a prime number, let

$$\rho_\ell: \text{Gal}(\overline{K}/K) \longrightarrow \text{Aut}(V_\ell(E))$$

be the corresponding  $\ell$ -adic representation of  $K$ , cf. chap. ??, ??. The main result of this Chapter is the determination of the Lie algebra of the  $\ell$ -adic Lie group  $G_\ell = \text{Im}(\rho_\ell)$ . This is based on a finiteness theorem of Šafarevič (1.4) combined with the properties of locally algebraic abelian representations (chap. III) and Tate's local theory of elliptic curves with non-integral modular invariant (Appendix, A1). The variation of  $G_\ell$  with  $\ell$  is studied in §??.

The Appendix gives analogous results in the local case (i.e. when  $K$  is a local field).

### §1. Preliminaries

III-2

#### 1.1 Elliptic curves (cf. 5 [5], 9 [9], 10 [10])

By an elliptic curve, we mean an abelian variety of dimension 1, i.e. a complete, non singular, connected curve of genus 1 with a given rational point  $P_0$ , taken as an origin for the composition law (and often written  $o$ ).

Let  $E$  be such a curve. It is well known that  $E$  may be embedded, as a non-singular cubic, in the projective plane  $\mathbb{P}_K^2$ , in such a way that  $P_0$  becomes a "flex" (one takes the projective embedding defined by the complete linear series containing the divisor  $3 \cdot P_0$ ). In this embedding, three points  $P_1, P_2$ ,

$P_3$  have sum 0 if and only if the divisor  $P_1 + P_2 + P_3$  is the intersection of  $E$  with a line. By choosing a suitable coordinate system, the equation of  $E$  can be written in Weierstrass form

$$y^2 = 4x^3 - g_2x - g_3$$

where  $x, y$  are non-homogeneous coordinates and the origin  $P_0$  is the point at infinity on the  $y$ -axis. The discriminant

$$\Delta = g_2^3 - 27g_3^2$$

is non-zero.

The coefficients  $g_2, g_3$  are determined up to the transformations  $g_2 \mapsto u^4g_2, g_3 \mapsto u^6g_3, u \in K^\times$ . The modular invariant  $j$  of  $E$  is

$$j = 2^6 3^3 \frac{g_2^3}{g_2^3 - 27g_3^2} = 2^6 3^3 \frac{g_2^3}{\Delta}.$$

III-3 Two elliptic curves have the same  $j$  invariant if and only if they become isomorphic over the algebraic closure of  $K$ .

(All this remains valid over an arbitrary field, except that, when the characteristic is 2 or 3, the equation of  $E$  has to be written in the more general form

$$y^2 + a_1xy + a_3y + x^3 + a_2x^2 + a_4x + a_6 = 0.$$

Here again, 0 is the point at infinity on the  $y$ -axis and the corresponding tangent is the line at infinity. There are corresponding definitions for  $\Delta$  and  $j$ , for which we refer to **9** [9] or **20** [20]; note, however, that there is a misprint in Ogg's formula for  $\Delta$ : the coefficient of  $\beta_4^3$  should be  $-8$  instead of  $-1$ .)

## 1.2 Good reduction

Let  $v \in M_K^0$  be a finite place of the number field  $K$ . We denote by  $\mathcal{O}_v$  (resp.  $\mathfrak{m}_v, k_v$ ) the corresponding local ring in  $K$  (resp. its maximal ideal, its residue field).

Let  $E$  be an elliptic curve over  $K$ . One says that  $E$  has **good reduction at**  $v$  if one can find a coordinate system in  $\mathbb{P}_K^2$  such that the corresponding equation  $f$  for  $E$  has coefficient in  $\mathcal{O}_v$  and its reduction  $\tilde{f} \bmod \mathfrak{m}_v$  defines a

non-singular cubic  $\tilde{E}_v$  (hence an elliptic curve) over the residue field  $k_v$  (in other words, the discriminant  $\Delta(f)$  of  $f$  must be an invertible element of  $\mathcal{O}_v$ ). The curve  $\tilde{E}_v$  is called the **reduction** of  $E$  at  $v$ ; it does not depend on the choice of  $f$ , provided, of course, that  $\Delta(f) \in \mathcal{O}_v^\times$ .

One can prove that the above definition is equivalent to the following one: there is an abelian scheme  $E_v$  over  $\text{Spec}(\mathcal{O}_v)$ , in the sense of **19** [19], chap. VI, whose generic fiber is  $E$ ; this scheme is then unique, and its special fiber is  $\tilde{E}_v$ . Note that  $\tilde{E}_v$  is defined over the finite field  $k_v$ ; we denote its **Frobenius endomorphism** by  $F_v$ .

On either definition, one sees that  $E$  has **good reduction for almost all places of  $K$** .

If  $E$  has good reduction at a given place  $v$ , its  $j$  invariant is **integral at  $v$**  (i.e. belongs to  $\mathcal{O}_v$ ) and its reduction  $\tilde{j} \bmod \mathfrak{m}_v$  is the  $j$  invariant of the reduced curve  $\tilde{E}_v$ .

The converse is almost true, but not quite: if  $j$  belongs to  $\mathcal{O}_v$ , there is a finite extension  $L$  of  $K$  such that  $E \otimes_K L$  has good reduction at all the places of  $L$  dividing  $v$  (this is the “potential good reduction” of **32** [32], §2). For the proof of this, see **29** [29], §4, n° 3.

**Remark.** The definitions and results of this section have nothing to do with number fields. They apply to every field with a discrete valuation.

### 1.3 Properties of $V_\ell$ related to good reduction

Let  $\ell$  be a prime number. We define, as in chap. ??, ??, the Galois modules  $T_\ell$  and  $V_\ell$  by:

$$V_\ell = T_\ell \otimes \mathbb{Q}_\ell, \quad T_\ell = \varprojlim_n E_{\ell^n}$$

where  $E_{\ell^n}$  is the kernel of  $\ell^n: E(\overline{K}) \rightarrow E(\overline{K})$ .

III-5

We denote by  $\rho_\ell$  the corresponding homomorphism of  $\text{Gal}(\overline{K}/K)$  into  $\text{Aut}(T_\ell)$ . Recall that  $E_{\ell^n}$ ,  $T_\ell$  and  $V_\ell$  are of rank 2 over  $\mathbb{Z}/\ell^n\mathbb{Z}$ ,  $\mathbb{Z}_\ell$  and  $\mathbb{Q}_\ell$ , respectively.

Let now  $v$  be a place of  $K$ , with  $p_v \neq \ell$  and let  $\tilde{v}$  be some extension of  $v$  to  $\overline{K}$ ; let  $D$  (resp.  $I$ ) be the corresponding decomposition group (resp. inertia group), cf. chap. ??, ??. If  $E$  has good reduction at  $v$ , one easily sees that reduction at  $v$  defines an *isomorphism* of  $E_{\ell^n}$  onto the corresponding module for the reduced curve  $\tilde{E}_v$ . In particular,  $E_{\ell^n}$ ,  $T_\ell$ ,  $V_\ell$  are *unramified*

at  $v$  (chap. ??, ??) and the Frobenius automorphism  $F_{v,\rho_\ell}$  of  $T_\ell$  corresponds to the Frobenius endomorphism  $F_v$  of  $\tilde{E}_v$ . Hence:

$$\det(F_{v,\rho_\ell}) = \det(F_v) = \mathbf{N} v$$

and

$$\det(1 - F_{v,\rho_\ell}) = \det(1 - F_v) = 1 - \text{tr}(F_v) + \mathbf{N} v$$

is equal to the number of  $k_v$ -points of  $\tilde{E}_v$ .

Conversely:

**Theorem 1** (Criterion of Néron-Ogg-Šafarevič). *If  $V$  is unramified at  $v$  for some  $\ell \neq p_v$ , then  $E$  has good reduction at  $v$ .*

For the proof, see **32** [32], §1.

**Corollary 1.1.** *Let  $E$  and  $E'$  be two elliptic curves which are isogenous (over  $K$ ). If one of them has good reduction at a place  $v$ , the same is true for the other one.*

III-6 (Recall that  $E$  and  $E'$  are said to be **isogenous** if there exists a non-trivial morphism  $E \rightarrow E'$ .)

This follows from the theorem, since the  $\ell$ -adic representations associated with  $E$  and  $E'$  are isomorphic.

**Remark.** For a direct proof of this corollary, see **11** [11].

**Exercise.** Let  $S$  be the finite set of places where  $E$  does not have good reduction. If  $v \in M_K^0 \setminus S$ , we denote by  $t_v$  the number of  $k_v$ -points of the reduced curve  $\tilde{E}_v$ .

(a) Let  $\ell$  be a prime number and let  $m$  be a positive integer. Show that the following properties are equivalent:

- (i)  $t_v \equiv 0 \pmod{\ell^m}$  for all  $v \in M_K^0 \setminus S$ ,  $p_v \neq \ell$ .
- (ii) The set of  $v \in M_K^0 \setminus S$  such that  $t_v \equiv 0 \pmod{\ell^m}$  has density one (cf. chap. ??, 2.2).
- (iii) For all  $s \in \text{Im}(\rho)$ , one has  $\det(1 - s) \equiv 0 \pmod{\ell^m}$ .

(The equivalence of ?? and ?? follows from Čebotarev's density theorem. The implications ??  $\implies$  ?? and ??  $\implies$  ?? are easy.)

(b) We take now  $m = 1$ . Show that the properties ??, ?? and ?? are equivalent to:

(iv) There exists an elliptic curve  $E'$  over  $K$  such that:

( $\alpha$ ) Either  $E'$  is isomorphic to  $E$ , or there exist an isogeny  $E' \rightarrow E$  of degree  $\ell$ .

( $\beta$ ) The group  $E'(K)$  contains an element of order  $\ell$ .

(The implication ??  $\implies$  ?? is easy. For the proof of the converse, use Exer. ?? of chap. ??, ??.) [For  $m > 2$ , see 64 [64].]

## 1.4 Šafarevič's theorem

III-7

It is the following (cf. [23]):

**Theorem 2.** *Let  $S$  be a finite set of places of  $K$ . The set of isomorphism classes of elliptic curves over  $K$ , with good reduction at all places not in  $S$ , is finite.*

Since isogenous curves have the same bad reduction set (cf. ??), this implies:

**Corollary 2.1.** *Let  $E$  be an elliptic curve over  $K$ . Then, up to isomorphism, there are only a finite number of elliptic curves which are  $K$ -isogenous to  $E$ .*

To prove the theorem, we use the following criterion for good reduction:

**Lemma 1.** *Let  $S$  be a finite set of places of  $K$  containing the divisors of 2 and 3, and such that the ring  $\mathcal{O}_S$  of  $S$ -integers is principal. Then, an elliptic curve  $E$  defined over  $K$  has good reduction outside  $S$  if and only if its equation can be put in the Weierstrass form  $y^2 = 4x^3 - g_2x - g_3$  with  $g_i \in \mathcal{O}_S$  and  $\Delta = g_2^3 - 27g_3^2 \in \mathcal{O}_S^\times$  (the group of units of  $\mathcal{O}_S$ ).*

*Proof.* The sufficiency is trivial. To prove necessity, we write the curve  $E$  in the form

$$y^2 = 4x^3 - g'_2x - g'_3 \quad (*)$$

with  $g'_i \in K$ . Let  $v$  be a place of  $K$  not in  $S$ . Then, since there is good reduction at  $v$ , and since the divisors of 2 and 3 do not belong to  $S$ , the III-8 curve  $E$  can be written in the form

$$y^2 = 4x^3 - g'_{2,v}x - g'_{3,v}$$

with  $g_{i,v}$  in the local ring at  $v$  and the discriminant  $\Delta_v$  a unit in this ring. Using the properties of the Weierstrass form, there is an element  $u_v \in K$  such that  $g_{2,v} = u_v^4 g'_2$ ,  $g_{3,v} = u_v^6 g'_3$ ,  $\Delta_v = u_v^{12} \Delta'$ ; moreover, as we can take  $g_{i,v} = g'_i$  for almost all  $v$ , we see that we can assume that  $u_v = 1$  for almost all  $v \notin S$ . Since the ring  $\mathcal{O}_S$  is principal, there is an element  $u \in K^\times$  with  $v(u) = v(u_v)$  for all  $v \notin S$ . Then, if we replace  $x$  by  $u^{-2}x$  and  $y$  by  $u^{-3}y$  in (??), the curve  $E$  takes the form

$$y^2 = 4x^3 - g'_2x - g'_3$$

with  $g_2 = u^4 g'_2$ ,  $g_3 = u^6 g'_3$  and  $\Delta = u^{12} \Delta'$ . Since, by construction,  $g_i \in \mathcal{O}_S$  and  $\Delta \in \mathcal{O}_S^\times$  the lemma is established.  $\square$

*Proof of the theorem.* After possibly adding a finite number of places of  $K$  to  $S$ , we may assume that  $S$  contains all the divisors of 2 and 3, and that the ring  $\mathcal{O}_S$  is principal. If  $E$  is an elliptic curve defined over  $K$  having good reduction outside  $S$ , the above lemma tells us that we can write  $E$  in the form

$$y^2 = 4x^3 - g'_2x - g'_3 \quad (*)$$

with  $g_i \in \mathcal{O}_S$  and  $\Delta = g_2^3 - 27g_3^2 \in \mathcal{O}_S$ . But, since we are free to multiply  $\Delta$  by any  $u \in (\mathcal{O}_S^\times)^{12}$ , and since  $\mathcal{O}_S^\times / (\mathcal{O}_S^\times)^{12}$  is a finite group, we see that there  
 III-9 is a finite set  $X \subset \mathcal{O}_S^\times$  such that any elliptic curve of the above type can be written in the form (??) with  $g_i \in \mathcal{O}_S$  and  $\Delta \in X$ . But, for a given  $\Delta$ , the equation

$$U^3 - 27V^2 = \Delta$$

represents an affine elliptic curve. Using a theorem of Siegel (generalized by Mahler and Lang, cf. **14** [14], chap. VII), one sees that this equation has only a *finite* number of solutions in  $\mathcal{O}_S$ . This finishes the proof of the theorem.  $\square$

**Remark.** There are many ways in which one can deduce Šafarevič's theorem from Siegel's. The one we followed has been shown to us by Tate.

## §2. The Galois module attached to $E$

In this section,  $E$  denotes an elliptic curve over  $K$ . We are interested in the structure of the Galois modules  $E_{\ell^n}$ ,  $T_\ell$ ,  $V_\ell$  defined in ??.



## 2.1 The irreducibility theorem

Recall first that the ring  $\text{End}_K(E)$  of  $K$ -endomorphisms of  $E$  is either  $\mathbb{Z}$  or of rank 2 over  $\mathbb{Z}$ . In the first case, we say that  $E$  has “no complex multiplication over  $K$ .” If the same is true for any finite extension of  $K$ , we say that  $E$  has “no complex multiplication.”

**Theorem 1.** *Assume that  $E$  has no complex multiplication over  $K$ . Then:* III-10

- (a)  $V_\ell$  is irreducible for all primes  $\ell$ ;
- (b)  $E_\ell$  is irreducible for almost all primes  $\ell$ .

We need the following elementary result:

**Lemma 1.** *Let  $E$  be an elliptic curve defined over  $K$  with  $\text{End}_K(E) = \mathbb{Z}$ . Then, if  $E' \rightarrow E$ ,  $E'' \rightarrow E$  are  $K$ -isogenies with non-isomorphic cyclic kernels, the curves  $E'$  and  $E''$  are non-isomorphic over  $K$ .*

*Proof.* Let  $n'$  and  $n''$  be respectively the orders of the kernels of  $E' \rightarrow E$  and  $E'' \rightarrow E$ . Suppose that  $E'$  and  $E''$  are isomorphic over  $K$ , and let  $E' \rightarrow E''$  be an isomorphism. If  $E \rightarrow E'$  is the transpose of the isogeny  $E' \rightarrow E$ , it has a cyclic kernel of order  $n'$ , and hence the isogeny  $E \rightarrow E''$ , obtained by composition of  $E \rightarrow E'$ ,  $E' \rightarrow E''$ ,  $E'' \rightarrow E$ , has for kernel an extension of  $\mathbb{Z}/n''\mathbb{Z}$  by  $\mathbb{Z}/n'\mathbb{Z}$ . But, since  $\text{End}_K(E) = \mathbb{Z}$ , this isogeny must be multiplication by an integer  $a$ , and its kernel must therefore be of the form  $\mathbb{Z}/a\mathbb{Z} \times \mathbb{Z}/a\mathbb{Z}$ . Hence  $n'$  and  $n''$  divide  $a$ . Since  $a^2 = n'n''$ , we obtain  $a = n' = n''$ , a contradiction.  $\square$

*Proof of the theorem.*

- (a) It suffices to show that, if  $\text{End}_K(E) = \mathbb{Z}$ , there is no one-dimensional  $\mathbb{Q}_\ell$ -subspace of  $V_\ell$  stable under  $\text{Gal}(\overline{K}/K)$ . Suppose there were one; its intersection  $X$  with  $T_\ell$  would be a submodule of  $T_\ell$  with  $X$  and  $T_\ell/X$  free  $\mathbb{Z}_\ell$ -modules of rank 1. For  $n \geq 0$ , consider the image  $X(n)$  of  $X$  in  $E_{\ell^n} = T/\ell^n T$ . This is a submodule of  $E_\ell$  which is cyclic of order  $\ell^n$  and stable by  $\text{Gal}(\overline{K}/K)$ . Hence it corresponds to a finite  $K$ -algebraic subgroup of  $E$  and one can define the quotient curve  $E(n) = E/X(n)$ . III-11  
The kernel of the isogeny  $E \rightarrow E(n)$  is cyclic of order  $\ell^n$ . The above lemma then shows that the curves  $E(n)$ ,  $n \geq 0$ , are pairwise non-isomorphic, contradicting the corollary to Šafarevič’s theorem (??).

- (b) If  $E$  is not irreducible, there exists a Galois submodule  $X$  of  $E$  which is one-dimensional over  $\mathbb{F}_\ell$ . In the same way as above, this defines an isogeny  $E \rightarrow E/X_\ell$  whose kernel is cyclic of order  $\ell$ . The above lemma shows that the curves which correspond to different values of  $\ell$  are non-isomorphic, and one again applies the corollary to Šafarevič's theorem.  $\square$

**Remark.** One can prove part ?? of the above theorem by a quite different method (cf. [25], §3.4); instead of the Šafarevič's theorem, one uses the properties of the decomposition and inertia subgroups of  $\text{Im}(\rho_\ell)$ , cf. Appendix.

# INDEX

- $C_K$ , 23
- $C_{\mathfrak{m}}$ , 23
- compatible (system  $(\rho_\ell)$ ), 8
- Conductor, 30
- $D$ , 23
- $\varepsilon_\ell$ , 24
- $E_{\mathfrak{m}}$ , 23
- Equidistribution, 12
- exceptional set (of a system), 9
- $\mathbb{G}_m$ , 21
- $I$ , 22
- Idèle, 22
- Idèle class, 23
- $I_{\mathfrak{m}}$ , 23
- Integral (representation), 7
- $M_K$ , 22
- $M_K^\infty$ , 22
- Rational (representation), 7
- strictly compatible (system  $(\rho_\ell)$ ),  
8
- Torus, 21
- Trace, 27
- $\mathbb{T} = \mathfrak{R}_{K/\mathbb{Q}}(\mathbb{G}_{m,K})$ , 21
- $U_v$ , 22
- $U_{v,\mathfrak{m}}$ , 23