A weak Lang-Weil bound and some cases of the Tate conjecture

José Cuevas Barrientos

Abstract. In this brief note, I expose some of the original material in the "little thesis" of my bachelor's degree. These include: an alternative proof of some weakening of Lang-Weil's bound for rational points and an application of Campana-Peternell's classification of threefolds to Tate's conjectures.

1. Weak Lang-Weil bounds

The following result is a corollary of some much more stronger bounds in Lang & Weil **[6]**:

Theorem 1.1: Let X be a quasi-projective, geometrically irreducible variety of dimension r over a finite field k. Then, for all extensions $\mathbb{F}_q/k$ we have the following bound

$$|X(\mathbb{F}_q)| = q^r + O(q^{r-1/2}).$$

Moreover, given an embedding $\iota\colon X \hookrightarrow \mathbb{P}^n_k$ then we actually have $|X(\mathbb{F}_q)| \leq q^r + Cq^{r-1/2}$, where C only depends on the Hilbert polynomial of the (schematic closure of) X and dimension n of the projective space where X is embedded.

Proof: Notice that if X isn't smooth, then as the smooth locus is open, their singularities are contained in a closed subset of X of (geometrical) irreducible components $Z_1, \dots, Z_n$. Therefore, by inductive hypothesis,

$$|X_{\mathrm{sm}}(\mathbb{F}_q)| = |X(\mathbb{F}_q)| - \sum_{i=1}^{n} |Z_i(\mathbb{F}_q)| = |X(\mathbb{F}_q)| - \underbrace{nq^{r-1} + O(q^{r-1-1/2})}_{\ll q^{r-1/2}},$$

where "$\ll$" designates Vinogradov notation and where X_{sm} is the smooth locus of X. Thus, we may assume X is smooth.

Let us proceed by induction on the dimension d. If $d = 1$, then we may apply Riemann's hypothesis for curves as proven by Weil. If otherwise $d \geq 2$, we may pass onto a dense open subscheme U since $X \smallsetminus U$ is of finite type, and thus has finitely many irreducible components of dimensions $< r$.

So, let U be an affine open. By Noether's normalization theorem we get a finite surjective morphism $U \twoheadrightarrow \mathbb{A}^r_k$, which we may compose by the canonical

Date: June 11, 2024.

projection $\mathbb{A}^r_k \twoheadrightarrow \mathbb{A}^1_k$ (since $r \geq 2$), and thus obtain a surjective morphism $U \twoheadrightarrow \mathbb{A}^1_k$.

Notice that the function field $L := K(U)$ is an extension of k, and we may take a *regular projective model* P of L, which exists and is birational to X. Then, apply Stein factorization (cf. HARTSHORNE [**4,** p. 280], Cor. III.11.5):

$$\begin{tikzcd} P \arrow[r, "f"] & C \arrow[d, "\text{finite}"] \\ X \arrow[u, dashed, "\text{birational}"] \arrow[r, two heads] & \mathbb{A}^1_k \end{tikzcd}$$

As asymptotic point-counting is birationally invariant, we'll substitute X by P, and suppose we have a dominant morphism $f\colon X \to C$ with geometrically integral fibres for some complete, geometrically integral curve C. As closed points go to closed points, we may count:

$$\begin{aligned} |X(\mathbb{F}_q)| &= \sum_{y \in C(\mathbb{F}_q)} |X_y(\mathbb{F}_q)| = \sum_{y \in C(\mathbb{F}_q)} \left(q^{r-1} + a_y q^{r-1-1/2}\right) \\ &= q^r + \left(\sum_{y \in C(\mathbb{F}_q)} a_y\right) q^{r-1/2}, \end{aligned}$$

where the a_y's depend on the geometrical properties of the fibre X_y. In particular, on the Hilbert polynomial by inductive hypothesis (relative to the fixed embedding $X_y \hookrightarrow X \hookrightarrow \mathbb{P}^n_k!$). Notice that since all non-constant morphisms from an integral scheme onto a normal irreducible curve is flat (cf. GÖRTZ & WEDHORN [**3,** p. 492], Prop. 15.4) and that since fibres of flat morphisms share Hilbert polynomial (cf. HARTSHORNE [**4,** p. 261], Thm. III.9.9); we may erase the singularities of C and conclude that the a_y's are uniformly bounded by B. Therefore, $|X(\mathbb{F}_q)| \leq q^r + B' q^{r-1-1/2}$ (where B' is a slight modification of B to take account for the points outside X).

Finally, we must prove that B' only depends on the Hilbert polynomial and the dimension of the embedding. To do so, notice that $\mathbb{P}^n_k$ has finitely many sub-varieties over k of given dimension r and given Hilbert polynomial $Q := Q_X(t)$. This is due to the fact that they're parametrized by k-rational points of the Hilbert scheme $\mathrm{Hilb}_Q(\mathbb{P}^n/k)$, which is a projective scheme (cf. KOLLÁR [**5,** p. 10], Thm. I.1.4). So we define A to be the maximum of the $B'(X)$. □

In general, if X is a complete, geometrically equidimensional variety over a finite field k, then its base change $X_{k^{\mathrm{alg}}}$ is a quasi-compact algebraic scheme over k^{alg}, and as such has $c < \infty$ irreducible components. Then we have the bound

$$|X(\mathbb{F}_q)| = cq^r + O(q^{r-1/2}), \qquad r = \dim X,$$

where $\mathbb{F}_q/k$ runs over the finite extensions of k and the error term is regarding $q \to \infty$. The original paper of LANG & WEIL gives the sharper bound (when

X is geometrically irreducible of degree d, cf. [**6**], Thm. 1):

$$|X(\mathbb{F}_q)| \leq q^r + (d-1)(d-2)q^{r-1-1/2} + O(q^{r-1}).$$

Our demonstration, through the Hasse-Weil bounds, gives the same result on surfaces, but to use the induction method on the sharp bound it would be necessary to control the degree on the fibres as a function of the degree of the domain.

2. A remark on Tate's Conjectures

First, let's settle *the* Tate conjecture which we'll treat in this section, it is the following, also known as *Tate's conjecture on algebraic cycles*:

> **Conjeture $T^i_k(X)$ 2.1:** The Gysin map $\gamma_X\colon \mathrm{CH}^i(X) \to V^i(X)^{\mathfrak{G}}$ is such that $\mathrm{Img}(\gamma_X) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell = V^i(X)$.

We'll say "$T^i(X)$ holds" if the previous statement holds for the indicated index i and scheme X. In this section we'll analyse closely the $i = 1$ case (also known as *Tate's conjecture on divisors*), which translates into the statement that every element $y \in H^2_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell)$ in ℓ-adic cohomology comes as a $\mathbb{Q}_\ell$-linear combination of Weil divisors.

For convenience of the reader, we repeat the facts that we'll use.

Proposition 2.2: Fix a geometrically integral, projective, smooth variety X over a field k which is of finite type over its prime sub-field. The following statements are equivalent:

(1) $T^1_{k^{\text{sep}}}(X_{k^{\text{sep}}})$ holds.
(2) $T^1_K(X_K)$ holds for all sufficiently large extensions K/k.

Proof: Cf. Tate [**8**, p. 98]. □

Theorem 2.3: Let X and Y be projective equidimensional schemes over a field k. Then:

(1) $T^1(X) + T^1(Y) \iff T^1(X \times_k Y)$.
(2) $T^1(X)$ is birationally invariant.
(3) Let $X \to Y$ be a dominant k-rational map, therefore $T^1(Y) \implies T^1(X)$.

Proof: Cf. Tate [**9**], (5.2). □

Lemma 2.4: Let X be a projective variety over a field k, and let $\mathscr{E}$ be a locally free $\mathscr{O}_X$-module of (constant) rank r. Then the projective bundle $\mathbb{P}_X(\mathscr{E})$ is birational to $\mathbb{P}^{r-1}_X$.

PROOF: By definition, there is a dense open subset $U \subseteq X$ such that $\mathscr{E}|_U \simeq \mathscr{O}_U^r$. Therefore

$$\mathbb{P}_X(\mathscr{E}) \times_X U = \mathbb{P}_U(\mathscr{E}|_U) \cong \mathbb{P}_U(\mathscr{O}_U^r|_U) \cong \mathbb{P}_U^{r-1}.$$

Since they are isomorphic in a dense open, they are birational. □

Corollary 2.4.1: Let X be a projective variety over a field k and let $\mathscr{E}$ be a locally free $\mathscr{O}_X$-module Then $T^1(X)$ holds iff $T^1\big(\mathbb{P}_X(\mathscr{E})\big)$ holds.

Let's see in which cases the conjecture holds:

Example.
(1) It holds for projective curves: indeed, since $\dim_{\mathbb{Q}_\ell} H^2(X) = 1$ by Poincaré's duality, the Gysin map is non-trivial.
(2) For projective spaces and, thus, for unirational varieties:[1] this is due to the fact that $\mathrm{CH}^1(\mathbb{P}^n) = \mathrm{Pic}(\mathbb{P}^n) \cong \mathbb{Z}$. Then, by the weak Lefschetz theorem on ℓ-adic cohomology and an inductive argument it follows that $V^1(\mathbb{P}^n) = H^2(\mathbb{P}^n, \mathbb{Q}_\ell) \cong \mathbb{Q}_\ell$.
(3) For abelian varieties over number fields: this is a corollary of a stronger Tate conjecture, proven by FALTINGS [**2**].

Proposition 2.5: Let X be a geometrically irreducible, complete, smooth and rationally connected (e.g. Fano) variety over a number field k. Then $T^1(X)$ holds.

PROOF: As $\operatorname{car} k = 0$, then rationally connected is the same as *separably* rationally connected (cf. KOLLÁR [**5,** p. 200], Prop. IV.3.3.1) and thus $H^0(X, \Omega^m_{X/k}) = 0$ for all $m > 0$ (cf. [**5,** p. 202], Cor. IV.3.8). Now, by ℓ-adic and Betti cohomology comparison, we obtain that

$$H^2_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell) \cong H^2_{\text{sing}}(X^{\text{an}}) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell \cong H^{1,1}(X) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell,$$

where the last isomorphism comes from Hodge decomposition (cf. VOISIN [**10,** p. 142]). Therefore, it suffices to prove that the Chern class homomorphism $\operatorname{Pic} X \to H^{1,1}(X^{\text{an}})$ is surjective, and this is precisely the content of Lefschetz theorem on $(1,1)$-classes (cf. [**10,** p. 280]).

Finally, that geometrically integral Fano varieties are rationally connected follows from the fact that, after base change, the variety $X_{k^{\text{alg}}}$ is rationally chain connected (cf. [**5,** p. 254], Thm. V.2.13) and, in particular, is rationally connected (cf. [**5,** p. 204], Thm. IV.3.10). □

Remark 2.5.1: The reader might notice that the proof only uses that irregularity $h^{0,j}(X) = 0$ vanishes for all j and, thus, the statement would be

[1] Following KOLLÁR [**5,** p. 199], we say a variety X is *unirational* if there exists a dominant k-rational map $\mathbb{P}^n_k \dashrightarrow X$. Kollár himself warns that certain authors use the word *unirational* for "geometrically unirational".

true for this class of varieties. However, a folklore Mumford conjecture (cf. [**5,** p. 202], Conj. IV.3.8.1) implies that this is *equivalent* to being rationally connected in characteristic 0; Miyaoka proved this on dimension ≤ 3.

At last, the following theorem follows from the classification on Campana & Peternell [**1**]:

Theorem 2.6: Tate's conjecture on divisors T^1 holds for geometrically irreducible, projective, smooth varieties of dimension ≤ 3 whose tangent bundle is nef over a number field k.

Proof: Let X be a variety as in the statement. To use the classification, we'd like to base change into the algebraic closure $\mathbb{Q}^{\mathrm{alg}}$, but it suffices to go into a sufficiently large finite extension K/k.

If X is a curve, then we're done by Example 1. If $X_{K^{\mathrm{alg}}}$ is a surface, then it's minimal (cf. [**1,** p. 176], Thm. 3.1) and, by the Kodaira-Enriques' classification we're in one of the following cases:

(a) X is an abelian surface: where it follows from Faltings' theorem.
(b) X_K is hyper-elliptic, i.e., the quotient of a product of elliptic curves. Since there is a surjective morphism $E_1 \times_K E_2 \to X$ it suffices to check T^1 on the domain, which follows from being a product of elliptic curves.
(c) $X_K \cong \mathbb{P}^2_K$: this is the Example 2.
(d) $X_K \cong \mathbb{P}^1_K \times_k \mathbb{P}^1_K$: this is the Example 2 with the fact that T^1 holds in a product iff it holds on each factor.
(e) $X_K \cong \mathbb{P}_C(\mathscr{E})$, where C is an elliptic curve and $\mathscr{E}$ is locally free of rank 2: this follows from the previous corollary since T^1 holds for curves.

If X_K is a threefold, then there is an étale covering $\widetilde{X}$ which belongs to the following list (cf. [**1,** p. 185], Thm. 10.1):

(a) and (b) X_K is Fano.
(c) $\widetilde{X} \cong \mathbb{P}(\mathscr{E})$, where $\mathscr{E}$ is locally free over an elliptic curve.
(d) $\widetilde{X} \cong \mathbb{P}(\mathscr{F}) \times_E \mathbb{P}(\mathscr{G})$, where $\mathscr{F}, \mathscr{G}$ are locally free over an elliptic curve E.
(e) $\widetilde{X} \cong \mathbb{P}(\mathscr{E})$, where $\mathscr{E}$ is locally free over an abelian surface.
(f) $\widetilde{X}$ is an abelian variety.

To prove that X_K satisfies T^1, it suffices to show that $T^1(\widetilde{X})$ holds. By the previous proposition, it holds on cases (a) and (b); by Faltings' theorem, it holds on the case (f) and, by the corollary 2.4.1 so do the cases (c) and (e).

So, case (d) is the only treated in isolation, for which we apply Prop. 7.2 of [**1,** p. 181], which asserts that there is an unramified surjective morphism $f\colon \widetilde{E} \to E$ where E is an elliptic curve, such that $X \times_E \widetilde{E} \cong \widetilde{S}_1 \times_{\widetilde{E}} \widetilde{S}_2$ (as schemes over $\widetilde{E}$), and where each $\varrho_i\colon \widetilde{S}_i \to \widetilde{E}$ is a ruled elliptic surface.

Notice that by Hurwitz's formula (cf. Liu [**7,** p. 290], Thm. 7.4.16) we have that

$$2p_a(\widetilde{E}) - 2 = (\deg f)(2p_a(E) - 2) + \sum_{x\in X_K^0} (e_x - 1)[\Bbbk(x) : K],$$

where e_x is the ramification index of each $f_x^\sharp\colon \mathscr{O}_{Y,f(x)} \to \mathscr{O}_{X_K,x}$. As f is unramified, then $p_a(\widetilde{E}) = p_a(E) = 1$. Moreover f must be smooth, since it's flat (because the codomain is a normal integral curve, and the domain is integral, cf. Görtz & Wedhorn [**3,** p. 492], Prop. 15.4); f has fibres of pure dimension 0, is of finite type and $\Omega^1_{\tilde{E}/E} \simeq 0$ (cf. [**7,** p. 221], Cor. 6.2.3). So, necessarily, $\widetilde{E}$ is an integral, projective, smooth curve of genus 1; that is, an elliptic curve.

Finally, all ruled surfaces $\varrho_i\colon \widetilde{S}_i \to \widetilde{E}$ are of the form $\widetilde{S}_i \cong \mathbb{P}_{\tilde{E}}(\mathscr{F}_i)$ for some locally free $\mathscr{F}_i$ sheaf of rank 2 (cf. Hartshorne [**4,** p. 370], Prop. V.2.2) Therefore, there exist some open subsets $U_i \subseteq \widetilde{E}$ such that $\mathbb{P}_{\tilde{E}}(\mathscr{F}_i) \times_{\tilde{E}} U_i \cong \mathbb{P}^1_K \times_K U_i$. Taking $V := U_1 \cap U_2$, we verify that

$$\begin{aligned}\mathbb{P}_{\tilde{E}}(\mathscr{F}_1) \times_{\tilde{E}} \mathbb{P}_{\tilde{E}}(\mathscr{F}_2) \times_{\tilde{E}} V &\cong (\mathbb{P}^1_K \times_K U_1) \times_{\tilde{E}} (\mathbb{P}^1_K \times_K U_2) \\ &\cong \mathbb{P}^1_K \times_K \mathbb{P}^1_K \times_K V.\end{aligned}$$

In resume, we show that $\widetilde{X}$ (which is an étale covering of X_K) is birational to $\mathbb{P}^1_K \times_K \mathbb{P}^1_K \times_K \widetilde{E}$, where $\widetilde{E}$ is an elliptic curve, and conclude by birational invariance of T^1. $\square$

Acknowledgements

The thesis was written under the supervision of professor Héctor Pastén, to whom I am very thankful for his support and explanations.

References

1. Campana, F. & Peternell, T. Projective manifolds whose tangent bundles are numerically effective. *Math. Ann.* **289,** 169–187. doi:10.1007/BF01446566 (1991).
2. Faltings, G. Endlichkeitssätze für abelsche Varietäten über Zahlkörpern. *Invent. math.* **73,** 349–366. doi:10.1007/BF01388432 (1983).
3. Görtz, U. & Wedhorn, T. *Schemes. With Examples and Exercises* 2nd ed. (Springer Spektrum Wiesbaden, 2010).
4. Hartshorne, R. *Algebraic Geometry Graduate Texts in Mathematics* **52** (Springer-Verlag New York, 1977).
5. Kollár, J. *Rational curves on algebraic varieties* (Springer-Verlag, 2001).
6. Lang, S. & Weil, A. Number of points of varieties in finite fields. *Amer. J. Math.* **76,** 819–827. doi:10.2307/2372655 (1954).
7. Liu, Q. *Algebraic Geometry and Arithmetic Curves* (Oxford University Press, 2002).
8. Tate, J. *Algebraic cycles and poles of zeta functions* in *Arithmetical Algebraic Geometry* (ed Schilling, O. F. G.) (Harper & Row, 1965), 93–110.

9. Tate, J. *Conjectures on Algebraic Cycles in ℓ-adic Cohomology* in *Motives* (eds Jannsen, U., Kleiman, S. L. & Serre, J.-P.) (American Mathematical Society, 1994), 71–83.
10. Voisin, C. *Hodge theory and complex algebraic geometry I* (Cambridge University Press, 2003).

Email address: josecuevasbtos@uc.cl

Departamento de Matemáticas, Pontificia Universidad Católica de Chile. Facultad de Matemáticas, 4860 Av. Vicuña Mackenna, Macul, RM, Chile
URL: josecuevas.xyz