Multi-objective Optimization Approaches for Addressing Assignment Problems

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Abstract

Multi-objective optimization models stand as potent tools for balancing conflicting objectives to produce favorable outcomes in various domains. In academia, . We present multi-objective optimization approaches to address the optimal assignment of students under conflicting objectives. They represent

Key Words: Multi-objective optimization, linearization, assignment problem.

1 Introduction

2 Previous Works

3 Variables and Constraints

The multi-objective optimization models we are to present in the later section will utilize a similar set of variables and constraints as those employed in [4]. Consider a group of n first-year students, each of whom selects six seminars from m available options and ranks them by preference. Let $StudentChoice_{ij} = k$ denote Seminar k is the jth choice of Student i where $(i, j) \in I = \{1, 2, \cdots, n\} \times \{1, 2, 3, 4, 5, 6\}$. A binary variable X_{ij} is also used for each $(i, j) \in I$ to ensure the assignment process as follows:

$$X_{ij} = \begin{cases} 1 & \text{if Student } i \text{ is assigned to their } j\text{th ranked seminar,} \\ 0 & \text{otherwise.} \end{cases}$$

Now, we introduce the basic constraints used in models. First, we want to ensure that each student is assigned to one of his or her selections.

$$\sum_{j=1}^{6} X_{ij} = 1 \; \forall i = 1, 2, \ldots, n \tag{1}$$

Subsequently, we introduce "bookkeeping" variables to keep track of the distributions of student attributes within each seminar. For example, we can denote $MSEM_k = 10$ to implicate 10 male students in Seminar k. Similarly, we define the following variables

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that reflect the attributes of interest to the college.

$$MSEM_k : \text{number of male students assigned to Seminar } k \quad (\alpha)$$
$$FSEM_k : \text{number of female students assigned to Seminar } k \quad (\beta)$$
$$USSEM_k : \text{number of U.S. students assigned to Seminar } k \quad (\gamma)$$
$$NonUSSEM_k : \text{number of international students assigned to Seminar } k \quad (\delta)$$

With regard to these variables, we add constraints to ensure that they accurately corresponds to the distribution of student attributes in each seminar for given values of X_{ij}. Before addressing these characteristics, we denote simplified description of these constraints:

$$\begin{aligned} MaleI_k &= \{(i,j) \in I \mid \text{StudentChoice}_{ij} = k \text{ and Student } i \text{ is male}\}, \\ FemaleI_k &= \{(i,j) \in I \mid \text{StudentChoice}_{ij} = k \text{ and Student } i \text{ is female}\}, \\ USI_k &= \{(i,j) \in I \mid \text{StudentChoice}_{ij} = k \text{ and Student } i \text{ is U.S. citizen}\}, \text{and} \\ NonUSI_k &= \{(i,j) \in I \mid \text{StudentChoice}_{ij} = k \text{ and Student } i \text{ is an international student}\}. \end{aligned}$$

We can now use constraints (2) - (5) to ensure the definitions of variables α, β, γ, and δ.

$$MSEM_k = \sum_{(i,j) \in MaleI_k}^{n} X_{ij} \; \forall k = 1, 2, \ldots, m \tag{2}$$

$$FSEM_k = \sum_{(i,j) \in FemaleI_k}^{n} X_{ij} \; \forall k = 1, 2, \ldots, m \tag{3}$$

$$USSEM_k = \sum_{(i,j) \in USI_k}^{n} X_{ij} \; \forall k = 1, 2, \ldots, m \tag{4}$$

$$NonUSSEM_k = \sum_{(i,j) \in NonUSI_k}^{n} X_{ij} \; \forall k = 1, 2, \ldots, m \tag{5}$$

Finally, we add a constraint to ensure adherence to course capacities. In our selection process, we will obtain user-input values to determine the lower and upper bounds of these capacities. Define the user-input lower bound to be L and upper bound to be μ. We accomplish this constraint by using $MSEM_k$ and $FSEM_k$ as follows:

$$L \leq MSEM_k + FSEM_k \leq \mu \; \forall k = 1, 2, \ldots, m \tag{6}$$

Based on these six foundational constraints in conjunction with other constraints associated with the refinements made to the model from [4], we describe the feasible region of our assignment problem. Let us define the complete feasible region Ω as follows:

$$\Omega = \{\mathbf{x} \in \{0,1\}^n \mid \text{Constraints (1) - (12)}\}.$$

4 Methodology

The previous model adopted the deviation-based objective approach as denoted in [4] to formulate nonlinear balancing objectives. In contrast to this approach, we use the linerization of these nonlinear objectives along with the incorporation of additional constraints. Through this approach, we aim to obtain the highest seminar preferences, balancing gender and the number of international students using three objective functions as our previous work also aimed at.

$$f_1(\mathbf{x}) = \sum_{i=1}^{n} (R_1 X_{i1} + R_2 X_{i2} + R_3 X_{i3} + R_4 X_{i4} + R_5 X_{i5} + R_6 X_{i6}) = \text{Rank Value},$$

$$f_2(\mathbf{x}) = \sum_{k=1}^{m} |\,(MSEM_k - FSEM_k)\,| \ \forall k = 1, 2, \ldots, m = \text{Gender Penalty},$$

$$f_3(\mathbf{x}) = \sum_{k=1}^{m} |\,(USSEM_k - NonUSSEM_k)\,| \ \forall k = 1, 2, \ldots, m = \text{Citizenship Penalty}.$$

The first objective f_1 remains consistent with the previous model, quantifying the success in obtaining student seminar preferences. This objective aims to minimize the rank values, implying that a lower value of $f_1(\mathbf{x})$ represents higher student satisfaction in seminar assignments. The rank coefficients $R_1, R2, R3, R4, R5, \text{and} R6$ are structured to reflect seminar preferences as outlined in the previous work [4]. In our program, the arrangement of rank coefficients, starting with the lowest value for R_1 and escalating in value as j approaches 6 ($R_2 \leq R_3 \leq R_4 \leq R_5 \leq R_6$), encapsulates our intention to prioritize the assignment of higher-ranked seminars to students.

Note that f_2 and f_3 are nonlinear functions. As shown in [3], the previous model utilized convex quadratic objectives for constructing f_2 and f_3. Though these objectives were instrumental in minimizing gender imbalances in each seminar, the program necessitated approximately 10 minutes of runtime. Hence, we attempted to minimize the sum of the quantities of $|MSEM_k - FSEM_k|$ and $|USSEM_k - NonUSSEM_k|$ which represent the imbalance in gender and the number of international students in Seminar k. However, these operations do not represent the linearization of quadratic objectives. Therefore, we added two of the constraints to ensure the linerization process. Let $|MSEM_k - FSEM_k| = g$ and $|USSEM_k - NonUSSEM_k| = c$. Then, the constraints are defined as follows:

$$g \geq MSEM_k - FSEM_k \tag{7}$$

$$g \geq FSEM_k - MSEM_k \tag{8}$$

$$c \geq USSEM_k - NonUSSEM_k \tag{9}$$

$$c \geq NonUSSEM_k - USSEM_k \tag{10}$$

Furthermore, we imposed additional constraints on the variables g and c to address the minimized gender imbalances and ensure a relevant distribution of international students. We intended to keep the values of g and c as small as possible to ensure a balanced distribution while still achieving a feasible solution. Thus, we introduce two new constants g' and c' obtained through iterative executions of the program with varying constants to identify the minimum values that still lead to a feasible solution. The additional constraints are then formulated as follows:

$$g \leq g' \tag{11}$$
$$c \leq c' \tag{12}$$

5 Multi-Objective Models

Building upon the methodology presented in [4], we employ normalized objectives combined with their associated coefficients to scale each objective function for ensuring its comparative representation within the multi-objective context.

Let $f_i(x)$ symbolize the original objective function pertaining to the i-th goal. Further, let z_i^U denote the smallest value of $f_i(x)$ across domain Ω, and z_i^N represent the greatest value amongst these minima for all objectives.

For each i encompassed in the set $\{1, 2, 3\}$:

Optimal solution corresponding to the i-th objective:
$x_i^* = \text{argmin }_x \{f_i(x) \mid x \in \Omega\}$

Utopian points:
$z_i^U = f_i(x_i^*)$

Objective function normalized to the i-th goal:
$f_i^{\text{norm}}(x) = \dfrac{f_i(x) - z_i^U}{z_i^N - z_i^U}$

Nadir points:
$z_i^N = \max\left\{f_i(x_j^*) \mid 1 \leq j \leq 3\right\}$

Based on the formulations above, we introduce three multi-objective models that underpin our seminar student assignment process.

5.1 Blended Optimization Model

The Blended optimization model combines multiple objectives and integrate them into a single objective with each one assigned a specific coefficient weight. Our previous work used the blended optimization model using quadratic objectives. These objectives were constructed using the sum of squared differences. This approach was adopted as those quadratic objectives allowed us to leverage commercial convex quadratic solvers.

In the context of our seminar assignment, our focus has shifted towards the linearized forms of f_2 and f_3. Since linear objectives are convex, we follow the approach demonstrated in [4] to construct the blended optimization model.

$$\text{minimize} \left\{ \sum_{i=1}^{3} w_i \cdot f_i^{norm}(x) \mid x \in \Omega \right\}$$

In our seminar assignment, the blended optimization approach aims to balance conflicting objectives that follow from student and faculty preferences. We can assign weights that reflect the significance of each objective to ensure the comprehensive representation of conflicting objectives for equitable and qualitative assignment.

5.2 Hierarchical Optimization Model

While the blended optimization model combines multiple objectives into a single one that consists of linear combination of those objectives with given weights, the hierarchical optimization method takes a sequential approach in our case. This approach ranks the objectives based on their priorities. Then, the optimization procedure addresses the objectives in the order of their ranking for prioritizing higher-ranked goals before considering the next level of objectives.

Given a list of our objectives $f_1(x), f_2(x),$ and $f_3(x)$, suppose that they are prioritized based on their importance in our seminar assignment where the priorities of each objective are denoted as $p_1 < p_2 < p_3$ implicating that p_1 has the highest priority. First, solve the highest priority objective and obtain z_1^U.

Objective:

$$\min \quad f_1(\mathbf{x})$$

Subject to:

$$\mathbf{x} \in \Omega$$

Then, the solve next highest priority objective while ensuring that the solution does not deviate from z_1^U by the specified relative tolerance, denoted ϵ. In the context of our seminar assignment, relative tolerance will ensure that while we try to accommodate student preferences, we will not significantly compromise gender balance and citizenship balance. For example, if the $\epsilon = 0.1$, this implies that we only allow 10 percent of the less prioritized objectives compared to high-priority objectives.

Objective:

$$\min \quad f_2(\mathbf{x})$$

Subject to:

$$f_1(\mathbf{x}) \leq z_1^U(1+\epsilon)$$
$$\mathbf{x} \in \Omega$$

Finally, we can optimize the third priority objective, while ensuring the solutions do not deviate from the previous optimal solutions for the first two objectives by the specified relative tolerance values.

Objective:

$$\min \quad f_3(\mathbf{x})$$

Subject to:

$$\begin{aligned} f_1(\mathbf{x}) &\leq z_1^U(1+\epsilon) \\ f_2(\mathbf{x}) &\leq z_2^U(1+\epsilon) \\ \mathbf{x} &\in \Omega \end{aligned}$$

This formulation ensures that solutions for the subsequent objectives are within the range defined by the relative tolerance from the optimal solutions of the higher priority objectives. By adopting the hierarchical optimization model, we can dynamically adjust the assignment outcome to better reflect the evolving nature of changing student and faculty preferences. For example, if a significant proportion of students express dissatisfaction with their seminars due to a mismatch between their assigned seminars and potential majors, we could prioritize the significance of the student preference objective by assigning $f_1(x)$ the highest priority for the subsequent year. This adaptability ensures that the assignment system remains responsive to student feedback and continuously refines itself for enhanced satisfaction.

5.3 Combined Blended and Hierarchical Approaches

In our continuous pursuit of optimizing the seminar assignment process aiming for equitable and satisfactory assignment, we propose a hybrid approach that combines features of both blended and hierarchical optimization models. This approach incorporates the weight attributes of the blended approach with the priority-ranking scheme of the hierarchical method. This approach can bring about a nuanced methodology to address the challenges of balancing conflicting objectives in our seminar assignment.

Given the three objectives $f_1(x), f_2(x),$ and $, f_3(x)$ within the context of our seminar assignment, we leverage both weights and hierarchical priorities of each objective to offer a structured and flexible way to produce the harmonized assignment outcome. The hybrid approach is operated as follows.

1. **Hierarchical Approach:**

   Suppose priorities $p_1 < p_2 = p_3$ correspond to the objectives $f_1(x)$, $f_2(x)$, and $f_3(x)$ respectively and let w_i be the weight of the i^{th} objective. We first start by minimizing the highest-priority objective.

   $$\begin{aligned} &\textbf{Objective:} \quad \min\ f_1(\mathbf{x}) \\ &\textbf{Subject to:} \quad \mathbf{x} \in \Omega \end{aligned}$$

2. **Blending Within the Same Priority Level:**

   Recall the hierarchical approach used to address this hybrid method. Note that $f_2(x)$ and $f_3(x)$ share an equivalent priority as denoted in the hypothesis above. Then, we can employ a blending technique. We can blend these objectives into a single one using a linear combination of weights and normalized objectives to comprehensively balance them, as described in section 5.1.

$$\begin{aligned} \textbf{Objective}: &\quad \min\ w_2 \cdot f_2^{norm}(\mathbf{x}) + w_3 \cdot f_3^{norm}(\mathbf{x}) \\ \textbf{Subjectto}: &\quad f_1(\mathbf{x}) \leq z_1^U(1+\epsilon) \\ &\quad \mathbf{x} \in \Omega \end{aligned}$$

Similarly, alternative sequencing of our objectives for the hybrid approach in our seminar assignment can be accomplished. Specifically, we can blend objectives with identical priorities first and then optimize the less-prioritized objective. The following describes the mathematical formulation and procedure that implements this approach.

$$\begin{aligned} \textbf{Objective}: &\quad \min\ w_2 \cdot f_2^{norm}(\mathbf{x}) + w_3 \cdot f_3^{norm}(\mathbf{x}) \\ \textbf{Subject to:} &\quad \mathbf{x} \in \Omega \end{aligned}$$

Once the blending is complete, we direct our focus to the optimization of $f_1(x)$, which has a lesser priority. Let the blended objective and the utopian point of it be denoted as $f(x)$ and z^U . Then, we can apply the hierarchical approach as follows:

$$\begin{aligned} \textbf{Objective}: &\quad \min\ f_1(\mathbf{x}) \\ \textbf{Subject to:} &\quad f(x) \leq z^U \\ &\quad \mathbf{x} \in \Omega \end{aligned}$$

By synthesizing the strengths of both optimization techniques, this proposed model allows a seminar assignment process both adaptable and precise in addressing the balance among conflicting objectives.

6 Discussions on Results

7 Conclusions

8 Future Works

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