HOMEWORK 1:
BACKGROUND TEST

>>NAME HERE<<
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Minimum Background Test [80 pts]

1 Vectors and Matrices [20 pts]

Consider the matrix X and the vectors $\mathbf{y}$ and $\mathbf{z}$ below:

$$X = \begin{pmatrix} 9 & 8 \\ 7 & 6 \end{pmatrix} \qquad \mathbf{y} = \begin{pmatrix} 9 \\ 8 \end{pmatrix} \qquad \mathbf{z} = \begin{pmatrix} 7 \\ 6 \end{pmatrix}$$

1. What is the inner product of the vectors $\mathbf{y}$ and $\mathbf{z}$? (this is also sometimes called the *dot product*, and is sometimes written as $\mathbf{y}^T\mathbf{z}$)
   Solution goes here.

2. What is the product $X\mathbf{y}$?
   Solution goes here.

2 Calculus [20 pts]

1. If $y = 4x^3 - x^2 + 7$ then what is the derivative of y with respect to x?
   Solution goes here.

2. If $y = \tan(z)x^{6z} - \ln(\frac{7x+z}{x^4})$, what is the partial derivative of y with respect to x?
   Solution goes here.

3 Probability and Statistics [20 pts]

Consider a sample of data $S = \{0, 1, 1, 0, 0, 1, 1\}$ created by flipping a coin x seven times, where 0 denotes that the coin turned up heads and 1 denotes that it turned up tails.

1. What is the sample mean for this data?
   Solution goes here.

2. What is the sample variance for this data?
   Solution goes here.

3. What is the probability of observing this data, assuming it was generated by flipping a biased coin with $p(x = 1) = 0.7, p(x = 0) = 0.3$.
   Solution goes here.

4. Note that the probability of this data sample would be greater if the value of $p(x = 1)$ was not 0.7, but instead some other value. What is the value that maximizes the probability of the sample S? Please justify your answer.
   Solution goes here.

5. Consider the following joint probability table where both A and B are binary random variables:

A	B	$P(A, B)$
0	0	0.1
0	1	0.4
1	0	0.2
1	1	0.3

(a) What is $P(A = 0, B = 0)$?
Solution goes here.

(b) What is $P(A = 1)$?
Solution goes here.

(c) What is $P(A = 0|B = 1)$?
Solution goes here.

(d) What is $P(A = 0 \vee B = 0)$?
Solution goes here.

4 Big-O Notation [20 pts]

For each pair (f, g) of functions below, list which of the following are true: $f(n) = O(g(n))$, $g(n) = O(f(n))$, both, or neither. Briefly justify your answers.

1. $f(n) = \frac{n}{2}$, $g(n) = \log_2(n)$.
   Solution goes here.

2. $f(n) = \ln(n)$, $g(n) = \log_2(n)$.
   Solution goes here.

3. $f(n) = n^{100}$, $g(n) = 100^n$.
   Solution goes here.

Medium Background Test [20 pts]

5 Algorithm [5 pts]

Divide and Conquer: Assume that you are given a sorted array with n integers in the range $[-10, +10]$. Note that some integer values may appear multiple times in the array. Additionally, you are told that somewhere in the array the integer 0 appears exactly once. Provide an algorithm to locate the 0 which runs in $O(\log(n))$. Explain your algorithm in words, describe why the algorithm is correct, and justify its running time.

Solution goes here.

6 Probability and Random Variables [5 pts]

6.1 Probability

State true or false. Here Ω denotes the sample space and A^c denotes the complement of the event A.

1. For any $A, B \subseteq \Omega$, $P(A|B)P(B) = P(B|A)P(A)$.
   Solution goes here.
2. For any $A, B \subseteq \Omega$, $P(A \cup B) = P(A) + P(B) - P(A|B)$.
   Solution goes here.
3. For any $A, B, C \subseteq \Omega$ such that $P(B \cup C) > 0$, $\frac{P(A\cup B\cup C)}{P(B\cup C)} \geq P(A|B \cup C)P(B \cup C)$.
   Solution goes here.
4. For any $A, B \subseteq \Omega$ such that $P(B) > 0, P(A^c) > 0$, $P(B|A^C) + P(B|A) = 1$.
   Solution goes here.
5. For any n events $\{A_i\}_{i=1}^n$, if $P(\bigcap_{i=1}^n A_i) = \sum_{i=1}^n P(A_i)$, then $\{A_i\}_{i=1}^n$ are mutually independent.
   Solution goes here.

6.2 Discrete and Continuous Distributions

Match the distribution name to its probability density / mass function. Below, $|\boldsymbol{x}| = k$.

(a) Laplace Solution goes here.

(b) Multinomial Solution goes here.

(c) Poisson Solution goes here.

(d) Dirichlet Solution goes here.

(e) Gamma Solution goes here.

(f) $f(\boldsymbol{x}; \boldsymbol{\Sigma}, \boldsymbol{\mu}) = \frac{1}{\sqrt{(2\pi)^k \det(\boldsymbol{\Sigma})}} \exp\left(-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{\mu})\right)$

(g) $f(x; n, \alpha) = \binom{n}{x}\alpha^x(1-\alpha)^{n-x}$ for $x \in \{0, \ldots, n\}$; 0 otherwise

(h) $f(x; b, \mu) = \frac{1}{2b}\exp\left(-\frac{|x-\mu|}{b}\right)$

(i) $f(\boldsymbol{x}; n, \boldsymbol{\alpha}) = \frac{n!}{\Pi_{i=1}^k x_i!}\Pi_{i=1}^k \alpha_i^{x_i}$ for $x_i \in \{0, \ldots, n\}$ and $\sum_{i=1}^k x_i = n$; 0 otherwise

(j) $f(x; \alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1}e^{-\beta x}$ for $x \in (0, +\infty)$; 0 otherwise

(k) $f(\boldsymbol{x}; \boldsymbol{\alpha}) = \frac{\Gamma(\sum_{i=1}^k \alpha_i)}{\prod_{i=1}^k \Gamma(\alpha_i)} \prod_{i=1}^k x_i^{\alpha_i - 1}$ for $x_i \in (0, 1)$ and $\sum_{i=1}^k x_i = 1$; 0 otherwise

(l) $f(x; \lambda) = \lambda^x \frac{e^{-\lambda}}{x!}$ for all $x \in Z^+$; 0 otherwise

6.3 Mean and Variance

Consider a random variable which follows a Binomial distribution: $X \sim \text{Binomial}(n, p)$.

1. What is the mean of the random variable?
   Solution goes here.
2. What is the variance of the random variable?
   Solution goes here.

7 Linear algebra [5 pts]

7.1 Norm-enclature

Draw the regions corresponding to vectors $\mathbf{x} \in \mathbb{R}^2$ with the following norms:

1. $||\mathbf{x}||_1 \leq 1$ (Recall that $||\mathbf{x}||_1 = \sum_i |x_i|$)
2. $||\mathbf{x}||_2 \leq 1$ (Recall that $||\mathbf{x}||_2 = \sqrt{\sum_i x_i^2}$)
3. $||\mathbf{x}||_\infty \leq 1$ (Recall that $||\mathbf{x}||_\infty = \max_i |x_i|$)

   Solution figure goes here.

7.2 Geometry

Prove that these are true or false. Provide all steps.

1. The smallest Euclidean distance from the origin to some point $\mathbf{x}$ in the hyperplane $\mathbf{w}^T\mathbf{x} + b = 0$ is $\frac{|b|}{||\mathbf{w}||_2}$.
   Solution goes here.
2. The Euclidean distance between two parallel hyperplane $\mathbf{w}^T\mathbf{x} + b_1 = 0$ and $\mathbf{w}^T\mathbf{x} + b_2 = 0$ is $\frac{|b_1 - b_2|}{||\mathbf{w}||_2}$ (Hint: you can use the result from the last question to help you prove this one).

   Solution goes here.

8 Programming Skills - Test Yourself

You do not need to turn this in; it is on the honor system. Please attempt the following two problems on the HankerRank website.

Grading Students (Easy): https://www.hackerrank.com/challenges/grading/problem

Verify BST (Medium): https://www.hackerrank.com/challenges/ctci-is-binary-search-tree/problem