CORRECTIONS FOR EXERCISES IN §1

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Exercise 1.43. Consider $\psi \in \mathcal{D}(\mathbb{R})$ such that
$$\psi(x) = 1 \quad \text{for all } x \in [-1, 1].$$
Such a test function exists by virtue of Lemma 1.19 since $[-1, 1]$ is compact, and thus
$$\left\langle \sum_{i=1}^{n} \delta_{1/n}, \psi \right\rangle = \sum_{i=1}^{n} \psi(1/n) = n \to \infty \quad \text{as } n \to \infty.$$
It follows that $\sum_{n=1}^{\infty} \delta_{1/n}$ does not converge in $\mathcal{D}'(\mathbb{R})$.

Now consider $\psi \in \mathcal{D}(0, +\infty)$. Since ψ has compact support in $(0, +\infty)$, it follows that there exists $\delta > 0$ for which
$$\delta = \min \big\{ x \in \mathbb{R} : x \in \operatorname{supp}(\psi) \big\}.$$
Let $N \in \mathbb{N}$ such that $N > 1/\delta$; then for all $m \geq N$, $1/m \leq 1/N < \delta$, hence $1/m \notin \operatorname{supp}(\psi)$ and $\psi(1/m) = 0$. It follows that, for any $m \geq N$,
$$\left\langle \sum_{n=1}^{m} \delta_{1/n}, \psi \right\rangle = \sum_{n=1}^{m} \psi(1/m) = \sum_{n=1}^{N} \psi(1/m),$$
and hence $\sum_{n=1}^{m} \delta_{1/n}$ converges in $\mathcal{D}'(0, +\infty)$.

Exercise 1.69. The map $T : \mathcal{D}(\mathbb{R}) \to \mathbb{R}$ defined to be
$$\langle T, \phi \rangle := \int_{\mathbb{R}} |\phi(t)| \mathrm{d}t$$
is not a distribution, since it is not linear:
$$\langle T, \phi \rangle + \langle T, -\phi \rangle = 2 \int_{\mathbb{R}} |\phi(t)| \mathrm{d}t \neq 0 = \langle T, \phi - \phi \rangle.$$
We claim that the map $T : \mathcal{D}(\mathbb{R}) \to \mathbb{R}$ defined to be
$$\langle T, \phi \rangle := \sum_{n=0}^{\infty} \phi^{(n)}(n)$$
is a distribution. First, we define T^m for any $m \in \mathbb{N}$ to be
$$\langle T^m, \phi \rangle = \sum_{n=0}^{m} \phi^{(n)}(n).$$
T^m is clearly well–defined since $\psi \in \mathrm{C}^{\infty}$, linear, and bounded since
$$\begin{aligned} \big|\langle T^m, \phi \rangle\big| &\leq \sum_{n=0}^{m} \sup \big\{ \big|\phi^{(n)}(x)\big| : x \in \operatorname{supp}(\phi) \big\} \\ &\leq (m+1) \sup \Big\{ \big|\phi^{(k)}(x)\big| : x \in \operatorname{supp}(\phi), k \leq m \Big\}. \end{aligned}$$
Here, implicitly, we have used the fact that $\phi(x) = 0$ for all $x \notin \operatorname{supp}(\phi)$, so by induction, it follows that $\phi^{(n)}(x) = 0$ for all $x \notin \operatorname{supp}(\phi)$. This entails that there exists $N \in \mathbb{N}$ such that
$$\operatorname{supp}(\phi) \subset B_N(0),$$
and therefore for all $m > N$, we have
$$\langle T^m, \phi \rangle = \sum_{n=0}^{m} \phi^{(n)}(n) = \sum_{n=0}^{N} \phi^{(n)}(n),$$
so T^m converges, and hence T is a distribution by Theorem 1.39.

Exercise 1.71. Let $\phi \in \mathcal{D}(\mathbb{R}^2)$; then there exists $K > 0$ such that $\phi(z, 2z) = 0$ for all $z > K$. Defining $f : [0, +\infty) \to \mathbb{R}$ as $f(z) := \phi(z, 2z)$, we note that f is C^{∞}, and
$$\int_0^{\infty} |f(z)| \mathrm{d}z = \int_0^{K} |f(z)| \mathrm{d}z \leq \sup \big\{ |\phi(x)| : x \in \operatorname{supp}(\phi) \big\} K,$$

hence f is integrable, and so

$$\langle T,\phi\rangle := \int_0^\infty \phi(z,2z)\mathrm{d}z$$

is well–defined as a distribution, since the integral is linear. Next, we compute

$$\begin{aligned}\Big\langle \frac{\partial T}{\partial x}+2\frac{\partial T}{\partial y},\phi\Big\rangle &= -\Big\langle T,\frac{\partial \phi}{\partial x}+2\frac{\partial \phi}{\partial y}\Big\rangle\\
&= -\int_0^\infty \frac{\partial \phi}{\partial x}(z,2z)+2\frac{\partial \phi}{\partial y}(z,2z)\mathrm{d}z,\\
&= -\int_0^\infty f'(z)\mathrm{d}z,\\
&= -\int_0^K f'(z)\mathrm{d}z,\\
&= f(0)-f(K)=\phi(0,0)=\langle\delta_0,\phi\rangle,\end{aligned}$$

as required. To carry out the above computation, we have used the definition of differentiation for a distribution, the chain rule, the Fundamental Theorem of Calculus, and the fact that f is compactly supported.

Exercise 1.72. Let $v_n(x) := \frac{n}{1+n^2x^2}$, and $w_n(x) := \arctan(nx)$; we remark that since v_n and w_n are uniformly bounded by n and 1 respectively, they are in $\mathrm{L}^1_{\mathrm{loc}}$, and hence may be identified with distributions.

1. For $x\neq 0$, and $n\geq 1$, we estimate

$$0\leq \frac{n}{1+n^2x^2}=\frac{1}{1/n+nx^2}\leq\frac{1}{nx^2}\to 0 \quad \text{as } n\to\infty,$$

which demonstrates that $v_n(x)\to 0$ for all $x\neq 0$, and hence $v_n(x)$ converges to $v(x):=0$ almost everywhere.

2. By the above estimate, for any $\alpha>0$,

$$\int_\alpha^\infty |v_n(x)-v(x)|\mathrm{d}x=\int_\alpha^\infty |v_n(x)|\mathrm{d}x\leq\int_\alpha^\infty\frac{1}{nx^2}\mathrm{d}x\leq\frac{1}{n\alpha}\to 0,$$

and a similar argument holds for the integral over $(-\infty,-\alpha)$. Therefore $v_n\to v$ in $\mathrm{L}^1\big((-\infty,-\alpha)\cup(\alpha,+\infty)\big)$.

3. Let $\psi\in\mathcal{D}(\mathbb{R})$; then

$$\langle w_n,\psi\rangle=\int_{\mathbb{R}} w_n(x)\psi(x)\mathrm{d}x=\int_{\mathbb{R}}\arctan(nx)\psi(x)\mathrm{d}x.$$

We note that $|\arctan(nx)\psi(x)|\leq\frac{\pi}{2}|\psi(x)|$, which is integrable, and moreover

$$\arctan(nx)\psi(x)\to\frac{\pi}{2}\mathrm{sgn}(x)\psi(x)=\begin{cases}\frac{\pi}{2}\psi(x) & x>0,\\ 0 & x=0,\\ \frac{\pi}{2}\psi(x) & x<0,\end{cases}$$

so by applying the Dominated Convergence Theorem, we obtain that

$$\langle w_n,\psi\rangle\to\Big\langle\frac{\pi}{2}\mathrm{sgn},\psi\Big\rangle\quad\text{as } n\to\infty.$$

4. We note $w_n'(x)=v_n(x)$ so that

$$\lim_{n\to\infty}\langle v_n,\psi\rangle=\lim_{n\to\infty}\langle w_n',\psi\rangle=\Big\langle\frac{\pi}{2}\mathrm{sgn}',\psi\Big\rangle=\langle\pi\delta_0,\psi\rangle,$$

where we have used the jump formula along with the fact that $\frac{\pi}{2}\mathrm{sgn}$ is piecewise C^1.

Alternatively, consider the change of variable $y=nx$; then:

$$\int_{\mathbb{R}} v_n(x)\psi(x)\mathrm{d}x=\int_{\mathbb{R}}\frac{n\psi(x)}{1+n^2x^2}\mathrm{d}x=\int_{\mathbb{R}}\frac{\psi(y/n)}{1+y^2}\mathrm{d}y.$$

We note next that $\frac{\psi(y/n)}{1+y^2}\to\frac{\psi(0)}{1+y^2}$ as $n\to\infty$, and hence applying the Dominated Convergence Theorem once more, we find that

$$\int_{\mathbb{R}}\frac{\psi(y/n)}{1+y^2}\mathrm{d}y\to\int_{\mathbb{R}}\frac{\psi(0)}{1+y^2}=\pi\psi(0)=\langle\pi\delta_0,\psi\rangle.$$

Exercise 1.73. 1. Let $f\in\mathcal{D}(\mathbb{R})$, and set $f_n(x):=f(x-n)$. We claim that $f_n(x)\to 0$ almost everywhere, $f_n\to 0$ in $\mathcal{D}'(\mathbb{R})$, and f_n does not converge in $\mathrm{L}^1(\mathbb{R})$, unless $f=0$.

First, we note that since f is compactly supported, there exists K such that $\mathrm{supp}(f)\subset B_K(0)$. Then whenever $n>K+|x|$, $|x-n|\geq|n|-|x|\geq K$, and hence $f(x-n)=0$. It follows that $f_n(x)\to 0$ almost everywhere as $n\to\infty$.

Secondly, let $\psi \in \mathcal{D}(\mathbb{R})$. Since ψ is compactly supported,

$$\int_{\mathbb{R}} f_n(x)\psi(x)\mathrm{d}x = \int_{\operatorname{supp}(\psi)} f_n(x)\psi(x)\mathrm{d}x.$$

Now, since $|f_n(x)\psi(x)| \leq \sup\big\{|f(x)| : x \in \operatorname{supp}(f)\big\}|\psi(x)|$, which is integrable, and $f_n\psi \to 0$ almost everywhere, applying the Dominated Convergence Theorem implies that

$$\int_{\mathbb{R}} f_n(x)\psi(x)\mathrm{d}x \to 0,$$

which entails that $f_n \to 0$ in $\mathcal{D}'(\mathbb{R})$.

Finally, suppose that f_n converges in $\mathrm{L}^1(\mathbb{R})$. Then it follows that the limit of the sequence f_n in $\mathrm{L}^1(\mathbb{R})$ agrees with the limit in the sense of distributions. Hence, it would follow that

$$\int_{\mathbb{R}} |f_n|\mathrm{d}x \to 0,$$

but

$$\int_{\mathbb{R}} |f_n(x)|\mathrm{d}x = \int_{\mathbb{R}} |f(x-n)|\mathrm{d}x = \int_{\mathbb{R}} |f(y)|\mathrm{d}y,$$

and this integral is zero if and only if $f = 0$.

3. We begin by noting that (contrary to the definition of an approximation of the identity given in Exercise 1.32), here

$$\chi_{1/n^2}(x) := n^2\chi(n^2x) \quad \text{for some } \chi \in \mathcal{D}(\mathbb{R}) \text{ such that } \int_{\mathbb{R}} \chi = 1.$$

Next, we note that for any $x > 0$ and $k \leq n$,

$$n^2x - n^2/k \geq n^2x - n \to \infty \quad \text{as } n \to \infty.$$

Similarly, if $x < 0$ and $k \leq n$,

$$n^2x - n^2/k \leq n^2x \to -\infty.$$

It follows that for each $x \neq 0$, and any K, there exists $N(x, K)$ such that whenever $n > N(x, K)$,

$$\min\{|n^2x - n^2/k| : k = 1, \dots, n\} > K.$$

Moreover, choosing K such that $\chi(x) = 0$ for all $|x| \geq K$, we find that for $n \geq N(x, K)$,

$$\sum_{k=1}^{n} \chi_{1/n^2}(x - 1/k) = \sum_{k=1}^{n} n^2\chi(n^2x - n^2/k) = 0,$$

and hence the sequence converges to 0 almost everywhere.

On the other hand, suppose that $\operatorname{supp}(\chi) \subset B_R(0)$, let $\psi \in \mathcal{D}(\mathbb{R})$ such that $\psi(x) = 1$ for all $x \in [-R-1, R+1]$, and write

$$\left\langle \sum_{k=1}^{n} \chi_{1/n^2}(x - 1/k), \psi \right\rangle = \sum_{k=1}^{n} \int_{\mathbb{R}} n^2\chi(n^2x - n^2/k)\psi(x)\mathrm{d}x.$$

Consider a single term from this sum with the change of variable $y = n^2x - n^2/k$, we find that

$$\int_{\mathbb{R}} n^2\chi(n^2x - n^2/k)\psi(x)\mathrm{d}x = \int_{\mathbb{R}} \chi(y)\psi(y/n^2 + 1/k)\mathrm{d}y.$$

Next, we note that if $|y| < R$, we have that

$$\left|\frac{y}{n^2} + \frac{1}{k}\right| \leq \frac{R}{n^2} + 1 \leq R + 1.$$

By our choice of ψ, it follows that

$$\int_{\mathbb{R}} \chi(y)\psi(y/n^2 + 1/k)\mathrm{d}y = \int_{\mathbb{R}} \chi(y)\mathrm{d}y = 1,$$

but then

$$\sum_{k=1}^{n} \int_{\mathbb{R}} n^2\chi(n^2x - n^2/k)\psi(x)\mathrm{d}x = n,$$

and hence the sequence does not converge in $\mathcal{D}'(\mathbb{R})$.