Multibody dynamics (2)

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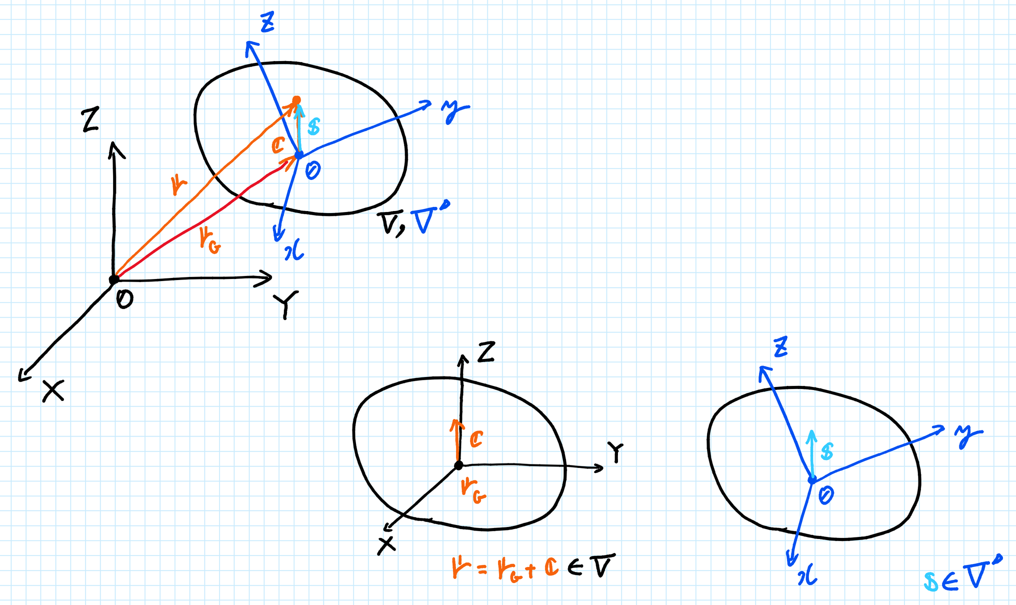
- Inertia

$\bullet\ m = \int_V dm,$

$\bullet\ \int_V c\, dm = 0 \Rightarrow \int_V d_T^{(n)} c\, dm = 0,$

$$\Rightarrow \int_V r\, dm = \int_V (r_G + c)\, dm = \int_V dm\, r_G + \int_V c\, dm = m r_G,$$

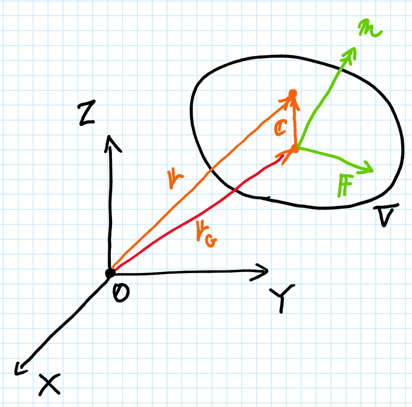
$$\Leftrightarrow r_G = \frac{1}{m}\int_V r\, dm = \frac{1}{m}\int_V \rho_m r\, dV.$$



- $m\, d_T^2 r = F,$

- $\int_V r \times d_T^2 r\, \rho_m\, dV = \int_V r \times \underbrace{f}_{[\text{体積力}]}\, dV$

$$= \int_V (r_G + c) \times f\, dV = r_G \times \int_V f\, dV + \int_V c \times f\, dV = r_G \times F + n.$$



$$\Leftrightarrow \int_V \tilde{r}\, d_T^2 r\, \rho_m\, dV = \tilde{r}_G F + n$$

$$\Leftrightarrow \int_V (\tilde{r}_G + \tilde{c})\, d_T^2 (r_G + c)\, \rho_m\, dV = \int_V (\tilde{r}_G d_T^2 r_G + \tilde{r}_G d_T^2 c + \tilde{c} d_T^2 r_G + \tilde{c}\, d_T^2 c)\, \rho_m\, dV = \tilde{r}_G F + n$$

$$\Leftrightarrow \int_V \rho_m dV \cdot \tilde{r}_G d_T^2 r_G + \tilde{r}_G \underbrace{\int_V d_T^2 c\, \rho_m dV}_{[=0]} + \underbrace{\left(\int_V c\, \rho_m dV\right)^{\sim}}_{[=0^{\sim}]} d_T^2 r_G + \int_V \tilde{c}\, d_T^2 c\, \rho_m dV = \tilde{r}_G F + n$$

$$\Leftrightarrow m \tilde{r}_G d_T^2 r_G + \int_V \tilde{c}\, d_T^2 c\, \rho_m dV = r_G \times F + n$$

$$\begin{aligned}
\Leftrightarrow m \tilde{r}_G d_T^2 r_G + \int_V \tilde{c}\, d_T (\omega \times c)\, \rho_m dV &= m \tilde{r}_G d_T^2 r_G - \int_V \tilde{c}\, d_T (c \times \omega)\, \rho_m dV \\
&= m \tilde{r}_G d_T^2 r_G - \int_V \tilde{c} \{ d_T c \times \omega + c \times d_T \omega \}\, \rho_m dV \\
&= m \tilde{r}_G d_T^2 r_G + \int_V \tilde{c}\, \omega \times \underbrace{(\omega \times c)}_{[=d_T c]}\, \rho_m dV - \int_V \tilde{c}\tilde{c}\, d_T\omega\, \rho_m dV \\
&= m \tilde{r}_G d_T^2 r_G + \int_V \underbrace{\tilde{c}\tilde{\omega}}\, \underbrace{\tilde{\omega} c}\, \rho_m dV - \int_V \tilde{c}\tilde{c}\, d_T\omega\, \rho_m dV \\
&= m \tilde{r}_G d_T^2 r_G + \int_V \underbrace{\{(\tilde{c}\omega)^{\sim} + \tilde{\omega}\tilde{c}\}}\, \underbrace{(\tilde{\omega} c)}\, \rho_m dV - \int_V \tilde{c}\tilde{c}\, d_T\omega\, \rho_m dV \\
&= m \tilde{r}_G d_T^2 r_G - \int_V \{(\tilde{c}\omega)^{\sim} + \tilde{\omega}\tilde{c}\}\, \underbrace{\tilde{c}\omega}_{[\tilde{\omega}c = \omega \times c = -c \times \omega = -\tilde{c}\omega]}\, \rho_m dV - \int_V \tilde{c}\tilde{c}\, d_T\omega\, \rho_m dV \\
&= m \tilde{r}_G d_T^2 r_G - \int_V \underbrace{(\tilde{c}\omega) \times (\tilde{c}\omega)}_{[=0]}\, \rho_m dV - \int_V \tilde{\omega}\tilde{c}\tilde{c}\omega\, \rho_m dV - \int_V \tilde{c}\tilde{c}\, d_T\omega\, \rho_m dV \\
&= m \tilde{r}_G d_T^2 r_G - \tilde{\omega} \left(\int_V \rho_m \tilde{c}\tilde{c}\, dV\right)\omega - \left(\int_V \rho_m \tilde{c}\tilde{c}\, dV\right) d_T\omega \\
&= r_G \times F + n
\end{aligned}$$

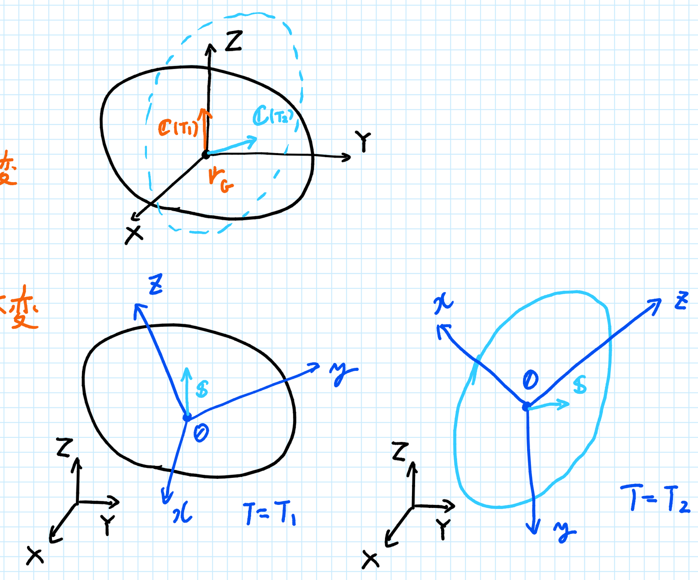
$$\Leftrightarrow \left.\begin{aligned}
&\bullet\ m \tilde{r}_G d_T^2 r_G + J d_T\omega + \tilde{\omega} J \omega = r_G \times F + n, \\
&\bullet\ J := -\int_V \rho_m \tilde{c}\tilde{c}\, dV = \int_V \rho_m \tilde{c}^T \tilde{c}\, dV.
\end{aligned}\right\}$$

$$\Leftrightarrow J d_T\omega + \tilde{\omega} J \omega = -r_G \times \underbrace{(m d_T^2 r_G - F)}_{[=0]} + n = n.$$

$$\boxed{J d_T\omega + \tilde{\omega} J \omega = n.}$$

- $J = \int_V \rho_m \tilde{c}^T \tilde{c}\, dV = -\int_V \rho_m \tilde{c}\tilde{c}\, dV = -\int_{V^\circ} \rho_m \begin{bmatrix} 0 & -c_Z & c_Y \\ c_Z & 0 & -c_X \\ -c_Y & c_X & 0 \end{bmatrix}\begin{bmatrix} 0 & -c_Z & c_Y \\ c_Z & 0 & -c_X \\ -c_Y & c_X & 0 \end{bmatrix} dV^\circ = \int_{V^\circ} \rho_m \begin{bmatrix} c_Y^2 + c_Z^2 & -c_X c_Y & -c_X c_Z \\ -c_Y c_X & c_X^2 + c_Z^2 & -c_Y c_Z \\ -c_Z c_X & -c_Z c_Y & c_X^2 + c_Y^2 \end{bmatrix} dV^\circ$: 時変

- $J^{\triangle} = \int_{V^\circ} \rho_m \tilde{s}^T \tilde{s}\, dV^{\circ} = -\int_{V^\circ} \rho_m \tilde{s}\tilde{s}\, dV^{\circ} = -\int_{V^\circ} \rho_m \begin{bmatrix} 0 & -s_z & s_y \\ s_z & 0 & -s_x \\ -s_y & s_x & 0 \end{bmatrix}\begin{bmatrix} 0 & -s_z & s_y \\ s_z & 0 & -s_x \\ -s_y & s_x & 0 \end{bmatrix} dV^\circ = \int_{V^\circ} \rho_m \begin{bmatrix} s_y^2 + s_z^2 & -s_x s_y & -s_x s_z \\ -s_y s_x & s_x^2 + s_z^2 & -s_y s_z \\ -s_z s_x & -s_z s_y & s_x^2 + s_y^2 \end{bmatrix} dV^\circ$: 時不変



$$T = \frac{1}{2}\int_V (d_T r)^T d_T r\, dm = \frac{1}{2}\int_V \rho_m (d_T r)^T d_T r\, dV = \frac{1}{2}\int_V \rho_m (d_T r_G + d_T c)^T (d_T r_G + d_T c)\, dV$$

$$= \frac{1}{2}\int_V \rho_m (d_T r_G^T d_T r_G + 2 d_T r_G^T \cdot d_T c + d_T c^T d_T c)\, dV = \frac{1}{2}\int_V \rho_m dV \cdot d_T r_G^T d_T r_G + \underbrace{\frac{1}{2}\int_V \rho_m d_T c^T dV\, d_T r_G}_{[=0]} + \frac{1}{2}\int_V \rho_m d_T c^T d_T c\, dV$$

$$= \frac{1}{2} m \|d_T r_G\|^2 + \frac{1}{2}\int_V \rho_m (\omega \times c)^T (\omega \times c)\, dV = \frac{1}{2} m \|d_T r_G\|^2 + \frac{1}{2}\int_V \rho_m (-c \times \omega)^T (-c \times \omega)\, dV = \frac{1}{2} m \|d_T r_G\|^2 + \omega^T \left(\frac{1}{2}\int_V \rho_m \tilde{c}^T \tilde{c}\, dV\right)\omega.$$

$$\Rightarrow T = \frac{1}{2} m \|d_T r_G\|^2 + \frac{1}{2}\omega^T J \omega = \frac{1}{2} m \|d_T r_G\|^2 + \frac{1}{2}\omega^{\triangle T} J^{\triangle} \omega^{\triangle} \Leftrightarrow \omega^T J \omega = \omega^{\triangle T} \underbrace{A^T J A}\, \omega \equiv \omega^{\triangle T} \underbrace{J^{\triangle}}\, \omega^{\triangle}$$

$$\Leftrightarrow A^T J A = J^{\triangle} \Leftrightarrow A (A^T J A) A^T = A J^{\triangle} A^T.$$

$$\left\{\begin{aligned}
&\bullet\ J = A J^{\triangle} A^T, \\
&\bullet\ d_T\omega = d_T (A \omega^{\triangle}) = (d_T A)\omega^{\triangle} + A\, d_T\omega^{\triangle} \\
&\qquad = A \tilde{\omega}^{\triangle} \omega^{\triangle} + A\, d_T\omega^{\triangle} = A (\omega^{\triangle} \times \omega^{\triangle}) + A\, d_T\omega^{\triangle} \\
&\qquad = A\, d_T\omega^{\triangle}
\end{aligned}\right.$$

$$\begin{aligned}
\Rightarrow J d_T\omega + \tilde{\omega} J \omega &= A J^{\triangle} A^T A\, d_T\omega^{\triangle} + \underbrace{(\dot{A} A^T)}_{[\tilde{\omega} = \dot{A} A^T = A \tilde{\omega}^{\triangle} A^T]} A J^{\triangle} A^T A \omega^{\triangle} \\
&= A J^{\triangle} d_T\omega^{\triangle} + A \tilde{\omega}^{\triangle} A^T A J^{\triangle} \omega^{\triangle} \\
&= A \{ J^{\triangle} d_T\omega^{\triangle} + \tilde{\omega}^{\triangle} J^{\triangle} \omega^{\triangle} \} \\
&= A n^{\triangle},
\end{aligned}$$

$$\Leftrightarrow \boxed{J^{\triangle} d_T\omega^{\triangle} + \tilde{\omega}^{\triangle} J^{\triangle} \omega^{\triangle} = n^{\triangle}.}$$

- Virtual work theorem

$$\left.\begin{aligned}
&m_i d_T^2 r_{Gi} - F_i = 0, \\
&J_i^{\triangle} d_T\omega_i^{\triangle} + \tilde{\omega}_i^{\triangle} J_i^{\triangle} \omega_i^{\triangle} - n_i^{\triangle} = 0,
\end{aligned}\right\} i \in \{1, \dots, N\}.$$

$$\Leftrightarrow \left.\begin{aligned}
&M_G d_T^2 \bar{r}_G - F_G = 0, \\
&J_G^{\triangle} d_T^2 \bar{\omega}^{\triangle} + \bar{\tilde{\omega}}^{\triangle} J_G^{\triangle} \bar{\omega}^{\triangle} - n_G^{\triangle} = 0.
\end{aligned}\right\}$$

$M_G := \mathrm{blkdiag}\{m_1 I, \dots, m_N I\} \in \mathbb{R}^{3N \times 3N}$, $r_G := [r_{G1}^T \cdots r_{GN}^T]^T \in \mathbb{R}^{3N}$, $F_G := [F_1^T \cdots F_N^T]^T \in \mathbb{R}^{3N}$;
$J_G := \mathrm{blkdiag}\{J_1^{\triangle}, \dots, J_N^{\triangle}\} \in \mathbb{R}^{3N \times 3N}$, $\bar{\omega}^{\triangle} := [\omega_1^{\triangle T} \cdots \omega_N^{\triangle T}]^T \in \mathbb{R}^{3N}$, $\bar{\tilde{\omega}}^{\triangle} := \mathrm{blkdiag}\{\tilde{\omega}_1^{\triangle}, \dots, \tilde{\omega}_N^{\triangle}\} \in \mathbb{R}^{3N \times 3N}$.

$$\Leftrightarrow \bullet\ \delta \bar{r}_G^T \cdot (M_G d_T^2 \bar{r}_G - F_G) + \delta \bar{\pi}^{\triangle T} (J_G^{\triangle} d_T \bar{\omega}^{\triangle} + \bar{\tilde{\omega}}^{\triangle} J_G^{\triangle} \bar{\omega}^{\triangle} - n_G^{\triangle}) = 0,$$

$$\Leftrightarrow \left.\begin{array}{l} \bullet\ \delta \bar{r}_G^T \cdot (M_G d_T^2 \bar{r}_G - \mathbb{F}_G) + \delta \bar{\pi}^{\triangle T} (\mathbb{J}_G^\triangle d_T \bar{\omega}^\triangle + \tilde{\bar{\omega}}^\triangle \mathbb{J}_G^\triangle \bar{\omega}^\triangle - n_G^\triangle) = 0, \\ \circ\ \delta \mathbb{C} = 0. \end{array}\right\}$$

$$\Leftrightarrow \delta \bar{r}_G^T \cdot (M_G d_T^2 \bar{r}_G - \mathbb{F}_G) + \delta \bar{\pi}^{\triangle T} (\mathbb{J}_G^\triangle d_T \bar{\omega}^\triangle + \tilde{\bar{\omega}}^\triangle \mathbb{J}_G^\triangle \bar{\omega}^\triangle - n_G^\triangle) + \lambda^T \cdot \delta \mathbb{C} = 0$$

$$\Leftrightarrow \delta \bar{r}_G^T \cdot (M_G d_T^2 \bar{r}_G - \mathbb{F}_G) + \delta \bar{\pi}^{\triangle T} (\mathbb{J}_G^\triangle d_T \bar{\omega}^\triangle + \tilde{\bar{\omega}}^\triangle \mathbb{J}_G^\triangle \bar{\omega}^\triangle - n_G^\triangle) + \lambda^T (\partial_{\bar{r}_G} \mathbb{C} \cdot \delta \bar{r}_G + \partial_{\bar{\pi}^\triangle} \mathbb{C} \cdot \delta \bar{\pi}^\triangle) = 0$$

$$\Leftrightarrow \delta \bar{r}_G^T \cdot (M_G d_T^2 \bar{r}_G - \mathbb{F}_G + \partial_{\bar{r}_G} \mathbb{C}^T \cdot \lambda) + \delta \bar{\pi}^{\triangle T} (\mathbb{J}_G^\triangle d_T \bar{\omega}^\triangle + \tilde{\bar{\omega}}^\triangle \mathbb{J}_G^\triangle \bar{\omega}^\triangle - n_G^\triangle + \partial_{\bar{\pi}^\triangle} \mathbb{C}^T \cdot \lambda) = 0$$

$$\Leftrightarrow \left.\begin{array}{l} \bullet\ M_G d_T^2 \bar{r}_G + \partial_{\bar{r}_G} \mathbb{C}^T \cdot \lambda = \mathbb{F}_G, \\ \bullet\ \mathbb{J}_G^\triangle d_T \bar{\omega}^\triangle + \partial_{\bar{\pi}^\triangle} \mathbb{C}^T \cdot \lambda = n^\triangle - \tilde{\bar{\omega}}^\triangle \mathbb{J}_G^\triangle \bar{\omega}^\triangle. \end{array}\right\}$$

- $\mathbb{C}(T, \bar{r}_G, \bar{\omega}^\triangle) = 0 \quad \Leftrightarrow d_T \mathbb{C} = \partial_T \mathbb{C} + \partial_{\bar{r}_G} \mathbb{C} \cdot d_T \bar{r}_G + \partial_{\bar{\pi}^\triangle} \mathbb{C} \cdot d_T \bar{\pi} = 0$

$$\Leftrightarrow \partial_{\bar{r}_G} \mathbb{C} \cdot d_T \bar{r}_G + \partial_{\bar{\pi}^\triangle} \mathbb{C} \cdot \bar{\omega}^\triangle = -\partial_T \mathbb{C}$$

$$\begin{aligned} \Rightarrow d_T^2 \mathbb{C} &= d_T (\partial_T \mathbb{C} + \partial_{\bar{r}_G} \mathbb{C} \cdot d_T \bar{r}_G + \partial_{\bar{\pi}^\triangle} \mathbb{C} \cdot \bar{\omega}^\triangle) \\ &= d_T(\partial_T \mathbb{C}) + d_T (\partial_{\bar{r}_G} \mathbb{C} \cdot d_T \bar{r}_G + \partial_{\bar{\pi}^\triangle} \mathbb{C} \cdot \bar{\omega}^\triangle) \\ &= d_T(\partial_T \mathbb{C}) + d_T(\partial_{\bar{r}_G} \mathbb{C}) d_T \bar{r}_G + \partial_{\bar{r}_G} \mathbb{C} \cdot d_T^2 \bar{r}_G + d_T(\partial_{\bar{\pi}^\triangle} \mathbb{C}) \cdot \bar{\omega}^\triangle + \partial_{\bar{\pi}^\triangle} \mathbb{C} \cdot d_T \bar{\omega}^\triangle \\ &= 0 \end{aligned}$$

$$\Leftrightarrow \partial_{\bar{r}_G} \mathbb{C} \cdot d_T^2 \bar{r}_G + \partial_{\bar{\pi}^\triangle} \mathbb{C} \cdot d_T \bar{\omega}^\triangle = - d_T(\partial_{\bar{r}_G} \mathbb{C}) d_T \bar{r}_G - d_T(\partial_{\bar{\pi}^\triangle} \mathbb{C}) \bar{\omega}^\triangle - d_T(\partial_T \mathbb{C})$$

- Geometrical constraint condition : $\partial_T \mathbb{C} \equiv 0$

$$\Rightarrow \boxed{\begin{array}{l} \partial_{\bar{r}_G} \mathbb{C} \cdot d_T \bar{r}_G + \partial_{\bar{\pi}^\triangle} \mathbb{C} \cdot \bar{\omega}^\triangle = 0 \\ \partial_{\bar{r}_G} \mathbb{C} \cdot d_T^2 \bar{r}_G + \partial_{\bar{\pi}^\triangle} \mathbb{C} \cdot d_T \bar{\omega}^\triangle = - d_T(\partial_{\bar{r}_G} \mathbb{C}) d_T \bar{r}_G - d_T(\partial_{\bar{\pi}^\triangle} \mathbb{C}) \bar{\omega}^\triangle \end{array}}$$

- Constraint condition by Euler parameter

  - $\mathbb{C} = 0 \Leftrightarrow d_T \mathbb{C} = \partial_T \mathbb{C} + \partial_{\bar{r}_G} \mathbb{C} \cdot d_T \bar{r}_G + \partial_{\bar{\varepsilon}} \mathbb{C} \cdot d_T \bar{\varepsilon} = 0$

    $\Leftrightarrow \partial_{\bar{r}_G} \mathbb{C} \cdot d_T \bar{r}_G + \partial_{\bar{\varepsilon}} \mathbb{C} \cdot d_T \bar{\varepsilon} = -\partial_T \mathbb{C}, \quad [\, \bar{\varepsilon} := [\varepsilon_1^T \dots \varepsilon_N^T]^T \in \mathbb{R}^{4N} \,]$

  - $d_T^2 \mathbb{C} = 0 \Leftrightarrow d_T(\partial_{\bar{r}_G} \mathbb{C}) d_T \bar{r}_G + \partial_{\bar{r}_G} \mathbb{C} \cdot d_T^2 \bar{r}_G + d_T(\partial_{\bar{\varepsilon}} \mathbb{C}) \cdot d_T \bar{\varepsilon} + \partial_{\bar{\varepsilon}} \mathbb{C} \cdot d_T^2 \bar{\varepsilon} = -d_T(\partial_T \mathbb{C})$

    $\Leftrightarrow \partial_{\bar{r}_G} \mathbb{C} \cdot d_T^2 \bar{r}_G + \partial_{\bar{\varepsilon}} \mathbb{C} \cdot d_T^2 \bar{\varepsilon} = -d_T(\partial_T \mathbb{C}) - d_T(\partial_{\bar{r}_G} \mathbb{C}) d_T \bar{r}_G - d_T(\partial_{\bar{\varepsilon}} \mathbb{C}) \cdot d_T \bar{\varepsilon} =: \gamma.$

  - $\mathbb{C}_E := \begin{bmatrix} \varepsilon_1^T \varepsilon_1 - 1 \\ \vdots \\ \varepsilon_N^T \varepsilon_N - 1 \end{bmatrix} = 0 \Leftrightarrow d_T \mathbb{C}_E = \partial_{\bar{\varepsilon}} \mathbb{C}_E \cdot d_T \bar{\varepsilon} = 0$

$$\begin{aligned} &\Leftrightarrow d_T^2 \mathbb{C}_E = d_T(\partial_{\bar{\varepsilon}} \mathbb{C}_E) \cdot d_T \bar{\varepsilon} + \partial_{\bar{\varepsilon}} \mathbb{C}_E \cdot d_T^2 \bar{\varepsilon} = 0 \\ &\Leftrightarrow \partial_{\bar{\varepsilon}} \mathbb{C}_E \cdot d_T^2 \bar{\varepsilon} = - d_T(\partial_{\bar{\varepsilon}} \mathbb{C}_E) \cdot d_T \bar{\varepsilon} \\ &\Leftrightarrow 2 \begin{bmatrix} \varepsilon_1^T & & \\ & \ddots & \\ & & \varepsilon_N^T \end{bmatrix} \cdot d_T^2 \bar{\varepsilon} = - d_T \left(2 \begin{bmatrix} \varepsilon_1^T & & \\ & \ddots & \\ & & \varepsilon_N^T \end{bmatrix} \right) \cdot d_T \bar{\varepsilon} \\ &\qquad\qquad = -2 \begin{bmatrix} d_T \varepsilon_1^T d_T \varepsilon_1 \\ \vdots \\ d_T \varepsilon_N^T d_T \varepsilon_N \end{bmatrix} =: \gamma_E \\ &\Leftrightarrow \boxed{\partial_{\bar{\varepsilon}} \mathbb{C}_E \cdot d_T^2 \bar{\varepsilon} = \gamma_E} \end{aligned}$$

- Virtual work theorem by Euler parameter

$$\left.\begin{array}{l} \bullet\ \delta \bar{r}_G^T \cdot (M_G d_T^2 \bar{r}_G - \mathbb{F}_G) + \delta \bar{\pi}^{\triangle T} (\mathbb{J}_G^\triangle d_T \bar{\omega}^\triangle + \tilde{\bar{\omega}}^\triangle \mathbb{J}_G^\triangle \bar{\omega}^\triangle - n_G^\triangle) = 0, \\ \circ\ \delta \mathbb{C} = 0, \\ \circ\ \delta \mathbb{C}_E = 0. \end{array}\right\}$$

$$\Leftrightarrow \delta \bar{r}_G^T \cdot (M_G d_T^2 \bar{r}_G - \mathbb{F}_G) + \delta \bar{\pi}^{\triangle T} (\mathbb{J}_G^\triangle d_T \bar{\omega}^\triangle + \tilde{\bar{\omega}}^\triangle \mathbb{J}_G^\triangle \bar{\omega}^\triangle - n_G^\triangle) + \lambda^T \cdot \delta \mathbb{C} + \lambda_E^T \cdot \delta \mathbb{C}_E = 0$$

  - $\delta \bar{\pi}^\triangle = \begin{bmatrix} \delta \pi_1^\triangle \\ \vdots \\ \delta \pi_N^\triangle \end{bmatrix} = \begin{bmatrix} 2 \mathbb{L}_1 \delta \varepsilon_1 \\ \vdots \\ 2 \mathbb{L}_N \delta \varepsilon_N \end{bmatrix} = 2 \mathbb{L}_G \delta \bar{\varepsilon}, \quad \mathbb{L}_G := \mathrm{blkdiag}\{ \mathbb{L}_1, \dots, \mathbb{L}_N \},$

  - $d_T \bar{\omega}^\triangle = d_T(2 \mathbb{L}_G d_T \bar{\varepsilon}) = 2(d_T \mathbb{L}_G) \cdot d_T \bar{\varepsilon} + 2 \mathbb{L}_G d_T^2 \bar{\varepsilon}$

    $= 2 \begin{bmatrix} d_T \mathbb{L}_1 \cdot d_T \varepsilon_1 \\ \vdots \\ d_T \mathbb{L}_N \cdot d_T \varepsilon_N \end{bmatrix} + 2 \mathbb{L}_G d_T^2 \bar{\varepsilon} = 2 \mathbb{L}_G d_T^2 \bar{\varepsilon},$

  - $\tilde{\bar{\omega}}^\triangle = \mathrm{blkdiag}\{ \tilde{\omega}_1^\triangle, \dots, \tilde{\omega}_N^\triangle \} = \mathrm{blkdiag}\{ 2(\mathbb{L}_1 d_T \varepsilon_1)^\sim, \dots, 2(\mathbb{L}_N d_T \varepsilon_N)^\sim \}$

    $= \mathrm{blkdiag}\{ 2(\underbrace{\mathbb{L}_1 d_T \mathbb{L}_1^T}_{[(\mathbb{L} d_T \varepsilon)^\sim = \mathbb{L} d_T \mathbb{L}^T]}), \dots, 2(\mathbb{L}_N d_T \mathbb{L}_N^T) \} = 2 \mathbb{L}_G \cdot d_T \mathbb{L}_G^T$

$$\Leftrightarrow \delta \bar{r}_G^T \cdot (M_G d_T^2 \bar{r}_G - \mathbb{F}_G) + 2 \delta \bar{\varepsilon}^T \cdot \mathbb{L}_G^T \{ \mathbb{J}_G^\triangle (2 \mathbb{L}_G \cdot d_T^2 \bar{\varepsilon}) + (2 \mathbb{L}_G \cdot d_T \mathbb{L}_G^T) \mathbb{J}_G^\triangle \cdot (\underbrace{-2 d_T \mathbb{L}_G \bar{\varepsilon}}_{[\mathbb{L} d_T \varepsilon = -(d_T \mathbb{L}) \varepsilon]}) - n_G^\triangle \} + \lambda^T (\partial_{\bar{r}_G} \mathbb{C} \cdot \delta \bar{r}_G + \partial_{\bar{\varepsilon}} \mathbb{C} \cdot \delta \bar{\varepsilon}) + \lambda_E^T \cdot \partial_{\bar{\varepsilon}} \mathbb{C}_E \cdot \delta \bar{\varepsilon} = 0,$$

$$\Leftrightarrow \delta \bar{r}_G^T (M_G d_T^2 \bar{r}_G - \mathbb{F}_G + \partial_{\bar{r}_G} \mathbb{C}^T \cdot \lambda) + \delta \bar{\varepsilon}^T (4 \mathbb{L}_G^T \mathbb{J}_G^\triangle \mathbb{L}_G \cdot d_T^2 \bar{\varepsilon} - 8 d_T \mathbb{L}_G^T \mathbb{J}_G^\triangle d_T \mathbb{L}_G \bar{\varepsilon} - 2 \mathbb{L}_G n_G^\triangle + \partial_{\bar{\varepsilon}} \mathbb{C}^T \cdot \lambda + \partial_{\bar{\varepsilon}} \mathbb{C}_E^T \cdot \lambda_E) = 0.$$

$$\Leftrightarrow \left.\begin{array}{l} \circ\ M_G d_T^2 \bar{r}_G + \partial_{\bar{r}_G} \mathbb{C}^T \cdot \lambda = \mathbb{F}_G, \\ \bullet\ 4 \mathbb{L}_G^T \mathbb{J}_G^\triangle \mathbb{L}_G \cdot d_T^2 \bar{\varepsilon} + \partial_{\bar{\varepsilon}} \mathbb{C}^T \cdot \lambda + \partial_{\bar{\varepsilon}} \mathbb{C}_E^T \cdot \lambda_E = 8 d_T \mathbb{L}_G^T \mathbb{J}_G^\triangle d_T \mathbb{L}_G \bar{\varepsilon} + 2 \mathbb{L}_G n_G^\triangle, \\ \bullet\ \partial_{\bar{r}_G} \mathbb{C} \cdot d_T^2 \bar{r}_G + \partial_{\bar{\varepsilon}} \mathbb{C} \cdot d_T^2 \bar{\varepsilon} = \gamma, \quad [\, \gamma := -d_T(\partial_T \mathbb{C}) - d_T(\partial_{\bar{r}_G} \mathbb{C}) d_T \bar{r}_G - d_T(\partial_{\bar{\varepsilon}} \mathbb{C}) \cdot d_T \bar{\varepsilon} \,] \\ \bullet\ \partial_{\bar{\varepsilon}} \mathbb{C}_E \cdot d_T^2 \bar{\varepsilon} = \gamma_E, \quad [\, \gamma_E := -2 [\dot{\varepsilon}_1^T \dot{\varepsilon}_1 \dots \dot{\varepsilon}_N^T \dot{\varepsilon}_N] \,] \end{array}\right\}$$

$$\Leftrightarrow \boxed{\begin{bmatrix} M_G & 0 & \partial_{\bar{r}_G} \mathbb{C}^T & 0 \\ 0 & 4 \mathbb{L}_G^T \mathbb{J}_G^\triangle \mathbb{L}_G & \partial_{\bar{\varepsilon}} \mathbb{C}^T & \partial_{\bar{\varepsilon}} \mathbb{C}_E^T \\ \partial_{\bar{r}_G} \mathbb{C} & \partial_{\bar{\varepsilon}} \mathbb{C} & 0 & 0 \\ 0 & \partial_{\bar{\varepsilon}} \mathbb{C}_E & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} \ddot{\bar{r}}_G \\ \ddot{\bar{\varepsilon}} \\ \lambda \\ \lambda_E \end{bmatrix} = \begin{bmatrix} \mathbb{F}_G \\ 8 d_T \mathbb{L}_G^T \mathbb{J}_G^\triangle d_T \mathbb{L}_G \bar{\varepsilon} + 2 \mathbb{L}_G n_G^\triangle \\ \gamma \\ \gamma_E \end{bmatrix}}$$