SMA 214
Mathematics (III)

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Enroll_Access_Code	Course_ID	Course_Name

PART II

THE LAPLACE TRANSFORM & inverse

Laplace transform

DEFINITION. Laplace Transform

Let f be a function defined for $t > 0$. Then the integral

$$\mathcal{L}\{f(t)\} = \int_0^{\infty} f(t)e^{-st}\,dt = F(s)$$

is said to be the Laplace transform of *f*, *provided that the integral converges.*

$$f(t) \;\underset{\mathcal{L}^{-1}}{\overset{\mathcal{L}}{\rightleftarrows}}\; F(s)$$

$$\mathcal{L}\{f(t)\}=F(s)$$

$$\mathcal{L}^{-1}\{F(s)\}=f(t)$$

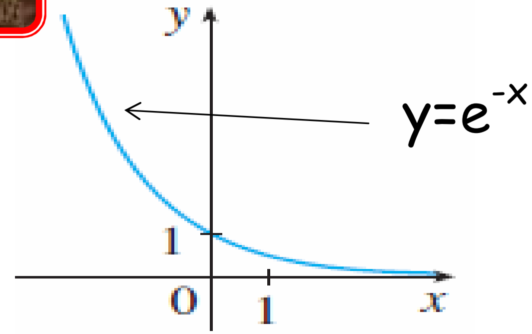
When the defining integral converges, the result is a function of s.

In general discussion we shall use a lowercase letter to denote the function being transformed and the corresponding capital letter to denote its Laplace transform—for example,

$$\mathscr{L}\{f(t)\} = F(s), \qquad \mathscr{L}\{g(t)\} = G(s), \qquad \mathscr{L}\{y(t)\} = Y(s).$$

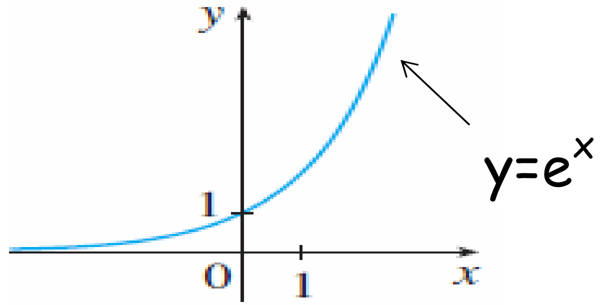
1- Laplace Transform of Standard functions

EXAMPLE Evaluate $\mathcal{L}\{1\}$



Solution

$$\mathcal{L}\{1\} = \int_0^\infty e^{-st}(1)\,dt$$



$$= \lim_{b\to\infty} \int_0^b e^{-st}\,dt$$

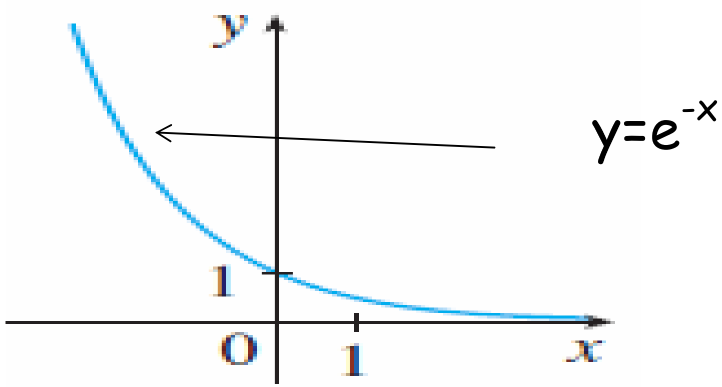
$$= \lim_{b\to\infty} \frac{-e^{-st}}{s}\Bigg|_0^b = \lim_{b\to\infty} \frac{-e^{-sb}+1}{s} = \frac{1}{s}$$

provided that $s > 0$. *In other words, when $s > 0$, the exponent $-sb$ is negative, and $e^{-sb} \to 0$* as $b\to\infty$. The integral diverges for $s < 0$.

The use of the limit sign becomes somewhat tedious, so we shall adopt the notation $|_0^\infty$ as a shorthand for writing $\lim_{b\to\infty}(\ )|_0^b$. For example,

$$\mathscr{L}\{1\} = \int_0^\infty e^{-st}(1)\,dt = \left.\frac{-e^{-st}}{s}\right|_0^\infty = \frac{1}{s}, \qquad s > 0.$$

At the upper limit, it is understood that we mean $e^{-st} \to 0$ as $t \to \infty$ for $s > 0$.



Thus $\quad 1 \xrightarrow{\ \mathcal{L}\ } \dfrac{1}{s}$

$\mathcal{L}\{1\}=1/s$

$\mathcal{L}^{-1}\{1/s\}=1$

EXAMPLE

Evaluate $\mathcal{L}\{ t \}$

Solution

$$\mathcal{L}\{ t \} = \int_0^\infty t\, e^{-st}\, dt$$

Integrating by parts

u=t dv=e^{-st}dt

du=dt v=(-1/s)e^{-st}

$$\therefore \mathcal{L}\{ t \} = \int_0^\infty t\, e^{-st}\, dt = \frac{-te^{-st}}{s}\Big|_{t=0}^{\infty} + \frac{1}{s}\int_0^\infty e^{-st}\, dt$$

$$= \frac{1}{s} F\{1\} = \frac{1}{s}\frac{1}{s} = \frac{1}{s^2}$$

Note that:

$$\boxed{\lim_{t \to \infty} te^{-st} = 0,\ s > 0}$$

$\mathcal{L}\{t\}=1/(s^2)$ $\mathcal{L}^{-1}\{1/(s^2)\}=t$

By Mathematical Induction, we can prove that

$$\mathcal{L}\{t^n\}=\frac{n!}{s^{n+1}},\ n=1,2\ldots \qquad \mathcal{L}^{-1}\left\{\frac{n!}{s^{n+1}}\right\}=t^n,\ n=1,2,\cdots$$

EXAMPLE **Evaluate** $\mathcal{L}\{e^{-3t}\}$

Solution

$$\mathcal{L}\{e^{-3t}\}=\int_0^\infty e^{-3t}e^{-st}dt=\int_0^\infty e^{-(s+3)t}\,dt$$

$$=\left.\frac{-e^{-(s+3)t}}{s+3}\right|_0^\infty$$

$$=\frac{1}{s+3},\quad s>-3.$$

∧

EXAMPLE

Evaluate $\mathcal{L}\left\{e^{at}\right\}$

Solution

$$\mathcal{L}\left\{e^{at}\right\} = \int_0^\infty e^{at}\, e^{-st} dt = \int_0^\infty e^{-(s-a)t}\, dt$$

$$= \left.\frac{-e^{-(s-a)t}}{s-a}\right|_{t=0}^{\infty} = \frac{1}{s-a} \quad , \quad s > a$$

$\mathcal{L}\{e^{at}\}=1/(s-a)$ $\qquad$ $\mathcal{L}^{-1}\{1/(s-a)\}=e^{at}$

EXAMPLE Evaluate $\mathcal{L}\{ \text{sinkt} \}$

Solution

$$\mathcal{L}\{\sin kt\} = \int_0^\infty \sin kt \, e^{-st} dt$$

u=sin kt dv=e^{-st}dt

du=k coskt dt v=(-1/s)e^{-st}

$$\mathcal{L}\{\sin kt\} = \left.\frac{-\sin kt \, e^{-st}}{s}\right|_{t=0}^{\infty} + \frac{k}{s}\int_0^\infty \cos kt \, e^{-st} dt$$

$$= \frac{k}{s}\int_0^\infty \cos kt \, e^{-st} dt$$

u=cos kt

du=-k sin kt dt

dv=e^{-st}dt

v=(-1/s)e^{-st}

$$\mathcal{L}\{\sin kt\} = \frac{k}{s}\left[\left.\frac{-\cos kt\, e^{-st}}{s}\right|_{t=0}^{\infty} - \frac{k}{s}\int_0^{\infty} \sin kt\, e^{-st}\,dt\right]$$

$$= \frac{k}{s^2} - \frac{k^2}{s^s}\mathcal{L}\{\sin kt\}$$

$$\Rightarrow \left(\frac{k^2}{s^s}+1\right)\mathcal{L}\{\sin kt\} = \frac{k}{s^2}$$

$$\Rightarrow (k^2+s^2)\mathcal{L}\{\sin kt\} = k$$

$$\Rightarrow \mathcal{L}\{\sin kt\} = \frac{k}{s^2+k^2}$$

$\mathcal{L}$\{sin kt\}=k/(s^2+k^2)

$\mathcal{L}^{-1}$\{k/(s^2+k^2\}=sinkt

Problem

Show that:

$$\mathcal{L}\{\cos kt\} = \frac{s}{s^2 + k^2}$$

Transform	Inverse Transform
$\mathcal{L}\{1\} = \frac{1}{s}$	$\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 1$
$\mathcal{L}\{t\} = \frac{1}{s^2}$	$\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = t$

Transform	Inverse Transform
$\mathcal{L}\{t^n\} = \dfrac{n!}{s^{n+1}}, n = 1,2.\cdots$	$\mathcal{L}^{-1}\left\{\dfrac{n!}{s^{n+1}}\right\} = t^n, n = 1,2,\cdots$
$\mathcal{L}\{e^{at}\} = \dfrac{1}{s-a}$	$\mathcal{L}^{-1}\left\{\dfrac{1}{s-a}\right\} = e^{at}$
$\mathcal{L}\{\sin kt\} = \dfrac{k}{s^2+k^2}$	$\mathcal{L}^{-1}\left\{\dfrac{k}{s^2+k^2}\right\} = \sin kt$
$\mathcal{L}\{\cos kt\} = \dfrac{s}{s^2+k^2}$	$\mathcal{L}^{-1}\left\{\dfrac{s}{s^2+k^2}\right\} = \cos kt$

$\mathcal{L}$, $\mathcal{L}^{-1}$ ARE LINEAR TRANSFORMS

For a linear combination of functions we can write:

$$\mathscr{L}\{\alpha f(t) + \beta g(t)\} = \alpha\mathscr{L}\{f(t)\} + \beta\mathscr{L}\{g(t)\} = \alpha F(s) + \beta G(s).$$

And,

$$\mathscr{L}^{-1}\{\alpha F(s) + \beta G(s)\} = \alpha\mathscr{L}^{-1}\{F(s)\} + \beta\mathscr{L}^{-1}\{G(s)\},$$

Examples,

1- $\mathscr{L}\{1 + 5t\} = \mathscr{L}\{1\} + 5\mathscr{L}\{t\} = \dfrac{1}{s} + \dfrac{5}{s^2}.$

2- $$\mathscr{L}\{4e^{-3t} - 10\sin 2t\} = 4\mathscr{L}\{e^{-3t}\} - 10\mathscr{L}\{\sin 2t\}$$
$$= \frac{4}{s+3} - \frac{20}{s^2+4}.$$

Evaluate (a) $\mathscr{L}^{-1}\left\{\frac{1}{s^5}\right\}$ (b) $\mathscr{L}^{-1}\left\{\frac{1}{s^2+7}\right\}$.

SOLUTION (a) Identify *$n + 1 = 5$ or $n = 4$ and then multiply and divide by $4!$:*

$$\mathscr{L}^{-1}\left\{\frac{1}{s^5}\right\} = \frac{1}{4!}\mathscr{L}^{-1}\left\{\frac{4!}{s^5}\right\} = \frac{1}{24}t^4.$$

(b) Identify $k^2 = 7$, so $k = \sqrt{7}$. We fix up the expression by multiplying and dividing by $\sqrt{7}$:

$$\mathscr{L}^{-1}\left\{\frac{1}{s^2+7}\right\} = \frac{1}{\sqrt{7}}\mathscr{L}^{-1}\left\{\frac{\sqrt{7}}{s^2+7}\right\} = \frac{1}{\sqrt{7}}\sin\sqrt{7}t.$$

Evaluate

$$\mathscr{L}^{-1}\left\{\frac{-2s+6}{s^2+4}\right\}.$$

SOLUTION ***We first rewrite the given function of s as two expressions by means of***
termwise division:

$$\mathscr{L}^{-1}\left\{\frac{-2s+6}{s^2+4}\right\} \overset{\text{termwise division}}{=} \mathscr{L}^{-1}\left\{\frac{-2s}{s^2+4}+\frac{6}{s^2+4}\right\}$$

$$\overset{\text{linearity and fixing up constants}}{=} -2\mathscr{L}^{-1}\left\{\frac{s}{s^2+4}\right\}+\frac{6}{2}\mathscr{L}^{-1}\left\{\frac{2}{s^2+4}\right\}$$

$$= -2\cos 2t + 3\sin 2t.$$

Evaluate $$\mathscr{L}^{-1}\left\{\frac{s^2+6s+9}{(s-1)(s-2)(s+4)}\right\}.$$

SOLUTION **There exist unique real constants** ***A*****,** ***B*****,** ***and C so that***

$$\frac{s^2+6s+9}{(s-1)(s-2)(s+4)} = \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{s+4}$$

$$= -\frac{16/5}{s-1} + \frac{25/6}{s-2} + \frac{1/30}{s+4},$$

$$\mathscr{L}^{-1}\left\{\frac{s^2+6s+9}{(s-1)(s-2)(s+4)}\right\} = -\frac{16}{5}\mathscr{L}^{-1}\left\{\frac{1}{s-1}\right\} + \frac{25}{6}\mathscr{L}^{-1}\left\{\frac{1}{s-2}\right\} + \frac{1}{30}\mathscr{L}^{-1}\left\{\frac{1}{s+4}\right\}$$

$$= -\frac{16}{5}e^{t} + \frac{25}{6}e^{2t} + \frac{1}{30}e^{-4t}.$$

2- TRANSFORMS OF DERIVATIVES

our immediate goal is to use the Laplace transform to solve differential equations.

To that end we need to evaluate quantities such as:

$\mathcal{L}(dy/dt)$ and $\mathcal{L}(d^2y/dt^2)$.

For example, if *f is continuous for $t > 0$,*

then integration by parts gives:

$e^{-st}f(t) \longrightarrow 0$ as $t \longrightarrow \infty$

$$\mathcal{L}\{f'(t)\} = \int_0^\infty e^{-st}f'(t)\,dt = e^{-st}f(t)\Big|_0^\infty + s\int_0^\infty e^{-st}f(t)\,dt$$

$$= -f(0) + s\mathcal{L}\{f(t)\}$$

or $$\boxed{\mathscr{L}\{f'(t)\} = sF(s) - f(0).}$$

Similarly, we can show that:

$$\boxed{\mathscr{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0).}$$

$$\boxed{\mathscr{L}\{f'''(t)\} = s^3F(s) - s^2f(0) - sf'(0) - f''(0).}$$

……

$$\boxed{\mathscr{L}\{f^{(n)}(t)\} = s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0),}$$

where $F(s) = \mathscr{L}\{f(t)\}$.

3- SOLVING LINEAR ODEs

It is apparent from the last general result that:

$$\mathscr{L}\{d^n y/dt^n\}$$

depends on $Y(s) = \mathscr{L}\{y(t)\}$ and the *(n - 1) derivatives* of *y(t)* *evaluated at* *t=0* .

This property makes the Laplace transform ideally suited for solving linear initial-value problems in which the differential equation has *constant coefficients*.

Such a differential equation is simply a linear combination of terms : $y, y', y'', \ldots, y^{(n)}$:

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_0 y = g(t),$$

where, $y(0) = y_0,\ y'(0) = y_1, \ldots, y^{(n-1)}(0) = y_{n-1},$

By the linearity property the Laplace transform of this linear combination is a linear combination of Laplace transforms:

$$a_n \mathscr{L}\left\{\frac{d^n y}{dt^n}\right\} + a_{n-1} \mathscr{L}\left\{\frac{d^{n-1} y}{dt^{n-1}}\right\} + \cdots + a_0 \mathscr{L}\{y\} = \mathscr{L}\{g(t)\}.$$

$$a_n[s^n Y(s) - s^{n-1} y(0) - \cdots - y^{(n-1)}(0)]$$

$$+ a_{n-1}[s^{n-1} Y(s) - s^{n-2} y(0) - \cdots - y^{(n-2)}(0)]$$

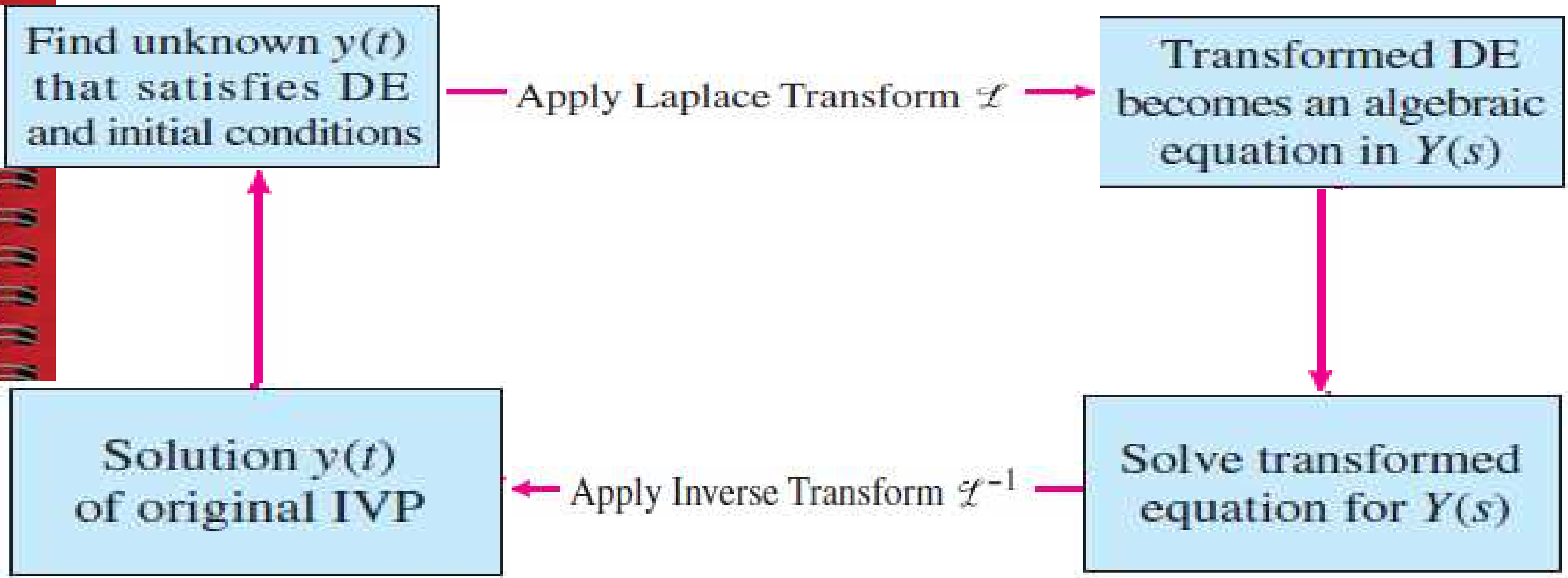
$$+ \cdots + a_0 Y(s) = G(s),$$ where $\mathscr{L}\{y(t)\} = Y(s)$ and $\mathscr{L}\{g(t)\} = G(s)$.

In other words, *the Laplace transform of a linear differential equation with constant coefficients* becomes an *algebraic equation in Y(s)*.

Finally, the solution $y(t)$ *of the original initial-value problem is:*

$$y(t) = \mathscr{L}^{-1}\{Y(s)\}$$

The procedure is summarized in the following diagram.



EXAMPLE

Use the Laplace transform to solve the initial-value problem

$$\frac{dy}{dt} + 3y = 13 \sin 2t, \quad y(0) = 6.$$

SOLUTION. We first take the transform of each member of the differential equation:

$$\mathscr{L}\left\{\frac{dy}{dt}\right\} + 3\mathscr{L}\{y\} = 13\mathscr{L}\{\sin 2t\}.$$

$$sY(s) - y(0) \quad + 3Y(s) \quad = (13)\ 2/(s^2 + 4)$$

$$sY(s) - 6 + 3Y(s) = \frac{26}{s^2 + 4} \qquad (s + 3)Y(s) = 6 + \frac{26}{s^2 + 4}.$$

$$Y(s) = \frac{6}{s+3} + \frac{26}{(s+3)(s^2+4)}$$

$$= \frac{6s^2+50}{(s+3)(s^2+4)}.$$

$$= \frac{A}{s+3} + \frac{Bs+C}{s^2+4}.$$

So, $6s^2 + 50 = A(s^2+4) + (Bs+C)(s+3)$

Put $s=-3$ $A=8$

Now, we equate the coefficients of s^2 and s:

$6 = 8 + B$ $B=-2$ $0 = (3)(-2)+C$ $C=6$

Thus,

$$Y(s) = \frac{6s^2 + 50}{(s+3)(s^2+4)} = \frac{8}{s+3} + \frac{-2s+6}{s^2+4}.$$

$$= \frac{8}{s+3} - \frac{2s}{s^2+4} + \frac{6}{s^2+4}$$

$$y(t) = 8\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\} - 2\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} + 3\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\}.$$

$$\boxed{= 8e^{-3t} - 2\cos 2t + 3\sin 2t.}$$

EXAMPLE. Solve

$$y'' - 3y' + 2y = e^{-4t}, \quad y(0) = 1, \quad y'(0) = 5.$$

SOLUTION.

$$\mathscr{L}\left\{\frac{d^2y}{dt^2}\right\} - 3\mathscr{L}\left\{\frac{dy}{dt}\right\} + 2\mathscr{L}\{y\} = \mathscr{L}\{e^{-4t}\}$$

$$s^2Y(s) - sy(0) - y'(0) - 3[sY(s) - y(0)] + 2Y(s) = \frac{1}{s + 4}$$

$$(s^2 - 3s + 2)Y(s) = s + 2 + \frac{1}{s + 4}$$

$$Y(s) = \frac{s + 2}{s^2 - 3s + 2} + \frac{1}{(s^2 - 3s + 2)(s + 4)}$$

$$= \frac{s^2 + 6s + 9}{(s - 1)(s - 2)(s + 4)}.$$

Thus,

$$Y(s) = -\frac{16/5}{s-1} + \frac{25/6}{s-2} + \frac{1/30}{s+4}$$

❾

$$\boxed{y(t) = \mathscr{L}^{-1}\{Y(s)\} = -\frac{16}{5}e^{t} + \frac{25}{6}e^{2t} + \frac{1}{30}e^{-4t}.}$$

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$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty f(t)e^{-st}dt$$

$$\boxed{\mathcal{L}\{\sinh(kt)\} = \frac{k}{s^2 - k^2}}$$

$$\mathcal{L}\{\sinh(kt)\} = \mathcal{L}\left\{\frac{1}{2}(e^{kt} - e^{-kt})\right\}$$

$$= \frac{1}{2}\left(\frac{1}{s-k} - \frac{1}{s+k}\right) = \frac{k}{s^2 - k^2}$$

$$\boxed{\mathcal{L}\{\cosh(kt)\} = \frac{s}{s^2 - k^2}}$$

$$\mathcal{L}\{\cosh(kt)\} = \mathcal{L}\left\{\frac{1}{2}(e^{kt} + e^{-kt})\right\}$$

$$= \frac{1}{2}\left(\frac{1}{s-k} + \frac{1}{s+k}\right) = \frac{s}{s^2 - k^2}$$

Exercise Evaluate $L\{\cos^2 t\}$.

$$\boxed{\cos^2 t = \frac{1}{2}(1 + \cos 2t),}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 1, \qquad \mathcal{L}^{-1}\left\{\frac{n!}{s^{n+1}}\right\} = t^n,$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}, \qquad \mathcal{L}^{-1}\left\{\frac{a}{s^2+a^2}\right\} = \sin at,$$

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos at, \qquad \mathcal{L}^{-1}\left\{\frac{a}{s^2-a^2}\right\} = \sinh at,$$

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2-a^2}\right\} = \cosh at.$$

Evaluate $\mathcal{L}\{t^2e^{at}\}$

$$\begin{aligned} \mathcal{L}\{t^2e^{at}\} &= \int_0^\infty e^{-st}t^2e^{at}dt \\ &= \int_0^\infty t^2e^{-(s-a)t}dt \end{aligned} = \ldots = \frac{2}{(s-a)^3}$$

4- The First Shifting Theorem

If $\mathcal{L}\{f(t)\} = F(s)$, then $\mathcal{L}\{e^{at} f(t)\} = F(s - a)$.

Example 1. Evaluate $\mathcal{L}\{e^{at}t^n\}$.

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}} \quad \text{so} \quad \boxed{\mathcal{L}\{e^{at}t^n\} = \frac{n!}{(s-a)^{n+1}}}\,.$$

Example 2. Evaluate $\mathcal{L}\{e^{at}\sin bt\}$.

$$\mathcal{L}\{\sin bt\} = \frac{b}{s^2+b^2} \quad \text{so} \quad \boxed{\mathcal{L}\{e^{at}\sin bt\} = \frac{b}{(s-a)^2+b^2}}\,.$$

Example 3. Evaluate $\mathcal{L}\{e^{at}\cos bt\}$.

$$\mathcal{L}\{\cos bt\} = \frac{s}{s^2+b^2} \qquad \text{so} \qquad \mathcal{L}\{e^{at}\cos bt\} = \frac{s-a}{(s-a)^2+b^2}\,.$$

Example 4. Evaluate $\mathcal{L}^{-1}\left\{\frac{2}{s^2+2s+2}\right\}$.

Now $s^2+2s+2 = (s+1)^2+1$, so we require

$$\mathcal{L}^{-1}\left\{\frac{2}{(s+1)^2+1}\right\} = 2\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+1}\right\} = 2e^{-t}\sin t.$$

The quantity inside { } is identical with the answer to Example 2 above with a = −1, b = 1.

$$\therefore \mathcal{L}^{-1}\left\{\frac{2}{s^2+2s+2}\right\} = 2e^{-t}\sin t.$$

Example 5. Evaluate $\mathcal{L}^{-1}\left\{\dfrac{3s+9}{s^2+2s+10}\right\}$.

Now

$$\frac{3s+9}{s^2+2s+10} = \frac{3(s+1)+6}{(s+1)^2+9} = \frac{3(s+1)}{(s+1)^2+9} + \frac{6}{(s+1)^2+9}.$$

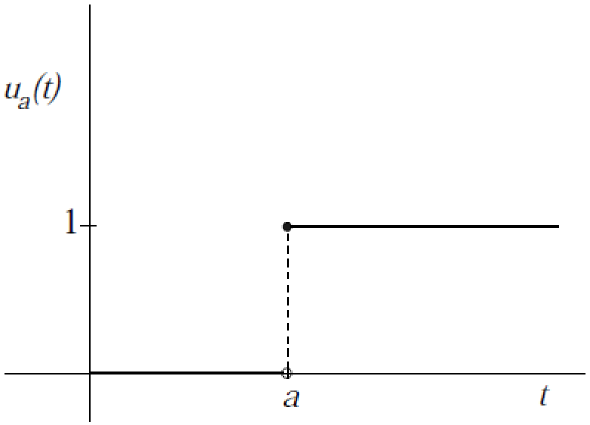
As in Examples 2 and 3 with a = −1, b = 3, hence

$$\mathcal{L}^{-1}\left\{\frac{3s+9}{s^2+2s+10}\right\} = 3e^{-t}\cos 3t + 2e^{-t}\sin 3t.$$

5- Step functions

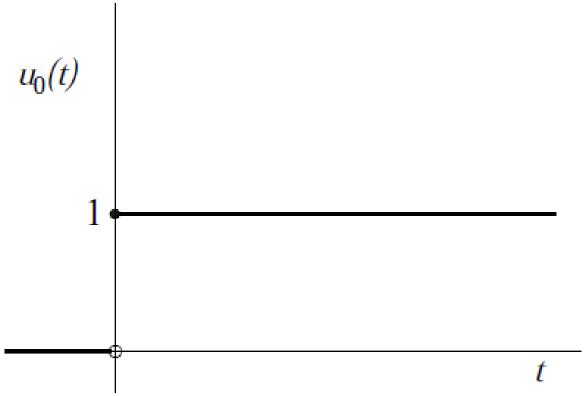
The unit step function $u_a(t)$ is defined as follows:

$$u_a(t) = \begin{cases} 0 & \text{when } t < a \\ 1 & \text{when } t \geq a \end{cases} \quad (a \geq 0).$$



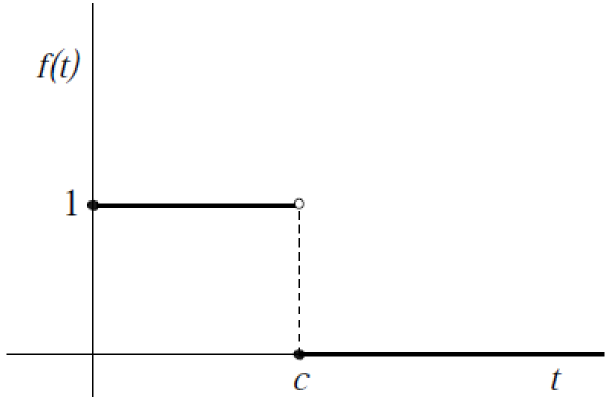
When a = 0 we have

$$u_0(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}.$$



Note that the function $f(t) = 1 - u_c(t)$ is given by

$$f(t) = \begin{cases} 1 & 0 \le t < c \\ 0 & t \ge c \end{cases}.$$

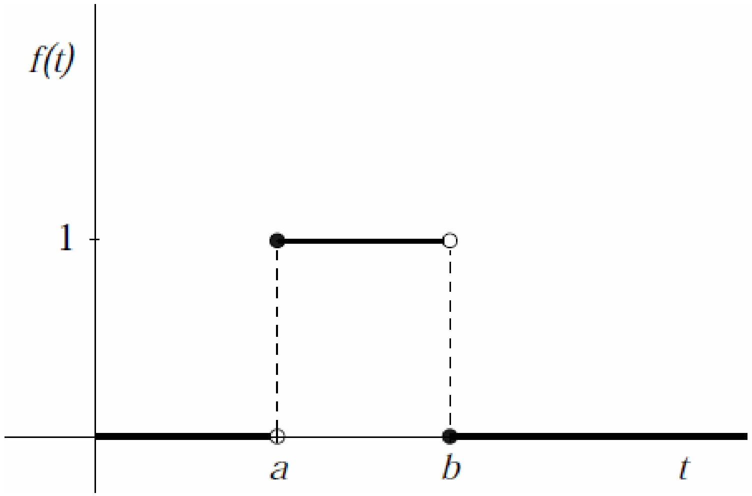


The difference between two step functions, i.e.,

$$f(t) = u_a(t) - u_b(t) \qquad (b > a)$$

$$= \begin{cases} 1 & ,0 \le t < a \\ 0 & , \quad t \ge a \end{cases} - \begin{cases} 1 & ,0 \le t < b \\ 0 & , \quad t \ge b \end{cases} ,b > a$$

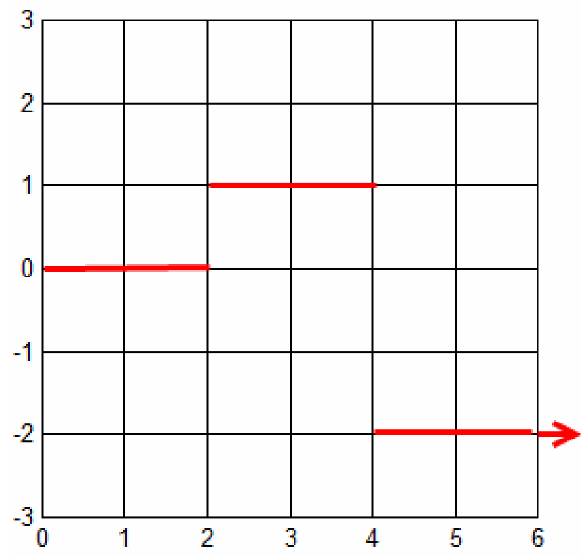
has a graph of the form:



Note (express f(t) in terms of unit step functions)

If there are 3 intervals to consider [0, 2) , [2, 4) , [4, ∞)

We need to figure out the height of the graph on each interval.



We know that $u_2(t) = 0$ on [0, 2).

Let a and b be the heights for the other two intervals. There will be a "jump" at $t = 2$ and $t = 4$.

At $t = 2$ the jump = a – 0 = coefficient of $u_2(t)$ in f(t); ⇨ a – 0 = 1 ⇨ a = 1.

At $t = 4$ the jump = b – a = coefficient of $u_4(t)$ in f(t); ⇨ b – a = -3 ⇨ b = -2.

$$f(t) = \begin{cases} 0 , & 0 \le t < 2 \\ a , & 2 \le t < 4 \\ b , & t \ge 4 \end{cases}$$

Thus $\boxed{f(t) = u_2(t) - 2u_4(t).}$

General case:

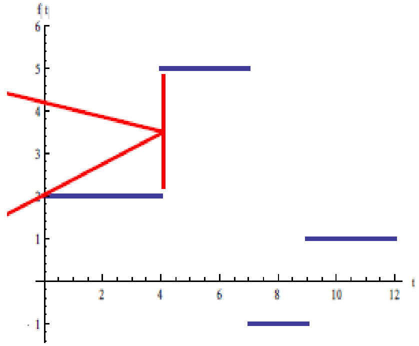
$$f(t)=\begin{cases} f_1(t), & t< a \\ f_2(t), & a< t <b \\ f_3(t), & t > b \end{cases}$$

is given by :

$$f(t) = f_1(t) + (f_2(t) - f_1(t))u_a(t) + (f_3(t) - f_2(t))\, u_b(t)$$

For the function

$$h(t)=\begin{cases} 2 & ,0\le t<4 \\ 5 & ,4\le t<7 \\ -1 & ,7\le t<9 \\ 1 & ,\quad t\ge 9 \end{cases}$$



To write h(t) in terms of $u_c(t)$, we will need $u_4(t)$, $u_7(t)$, **and** $u_9(t)$. (The definition changes at 4, 7, and 9.)

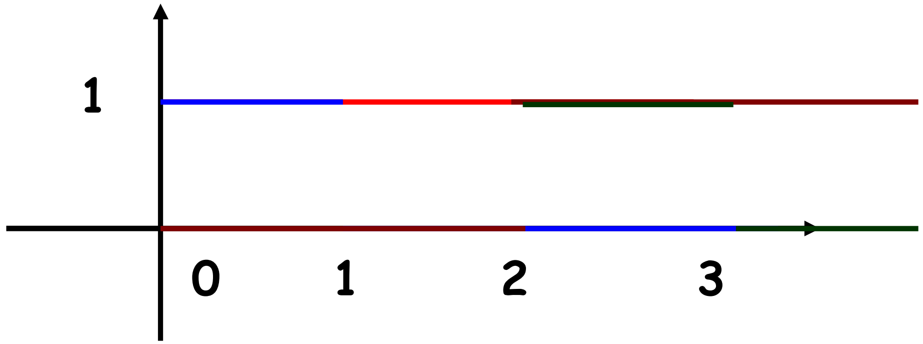
$$h(t) =2 + (5-2)u_4 + (-1-5)u_7 + (1-(-1))u_9$$

$$=2 + 3u_4 - 6u_7 + 2u_9$$

Write f(t) in terms of unit step functions

$$f(t) = \begin{cases} 1 & 0 \le t < 1 \\ 0 & 1 \le t < 2 \\ 1 & 2 \le t < 3 \\ 0 & t \ge 3 \end{cases}.$$

$f(t) = u_0(t) - u_1(t) + u_2(t) - u_3(t)$

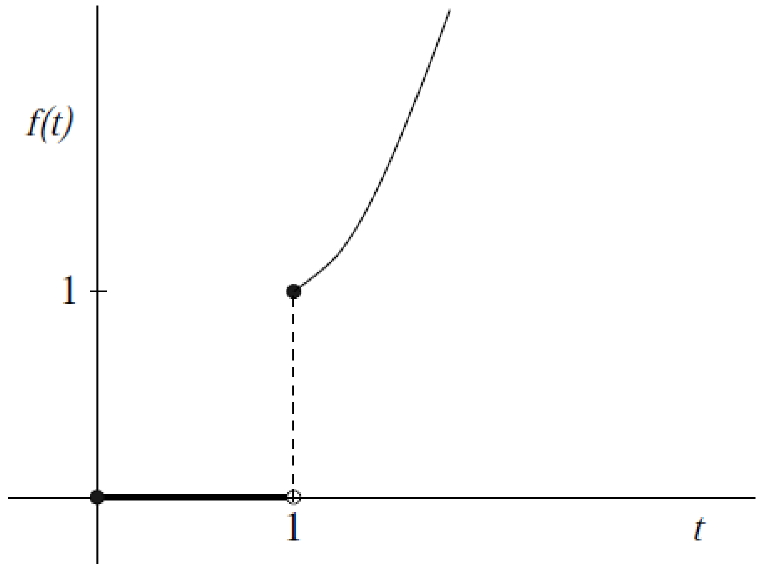


Step functions can be used to turn on or turn off portions of the graph of a function.

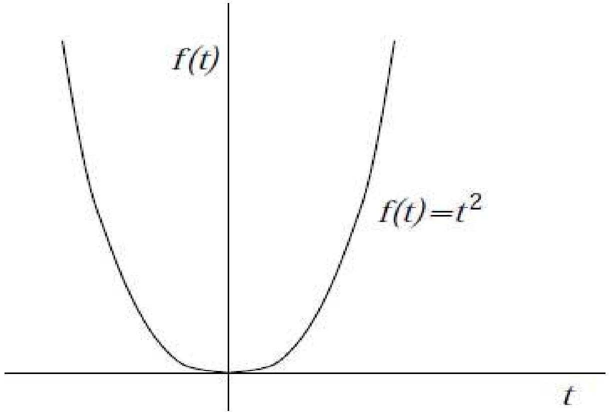
For example, the function $f(t) = t^2$ when multiplied by $u_1(t)$ becomes

$$f(t) = t^2 u_1(t) = \begin{cases} 0 & 0 \le t < 1 \\ t^2 & t \ge 1 \end{cases},$$

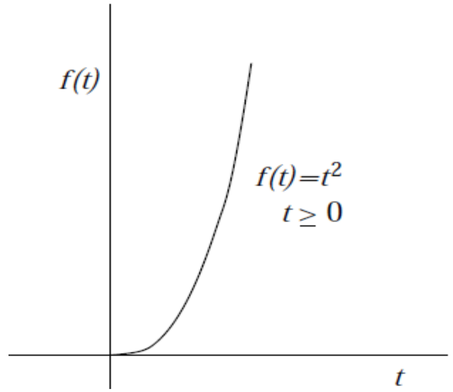
so that its graph is



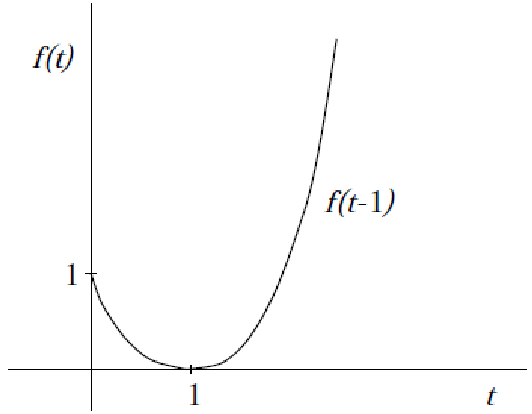
Given the function $f(t) = t^2$ note the graphs of the following functions:



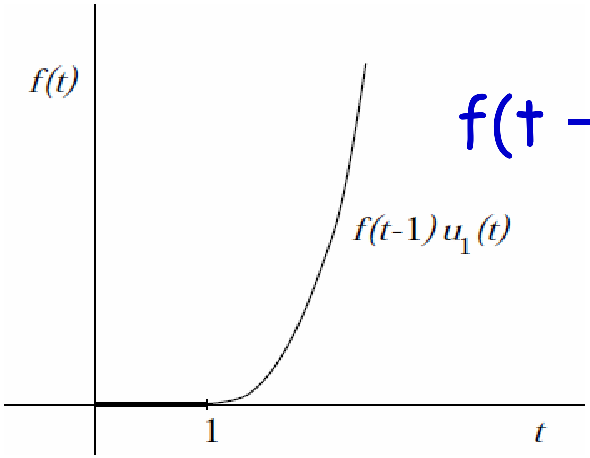
$f(t) = t^2$



$f(t) = t^2 \, , \, t \geq 0$



$f(t - 1) \, , \, t \geq 0$



$f(t - 1)u_1(t) \, ,$
$t \geq 0$

Hence, given a function f(t), defined for $t \geq 0$, the graph of the function

$f(t - a)u_a(t)$

consists of the graph of f(t) translated through a distance a to the right with the portion from 0 to a 'turned off', i.e., put equal to zero.

The Laplace transform of $u_a(t)$ is

$$\mathcal{L}\{u_a(t)\} = \int_0^{\infty} e^{-st} u_a(t)\, dt = \int_a^{\infty} e^{-st} dt$$

$$= \left[\frac{e^{-st}}{-s}\right]_a^{\infty} = \frac{e^{-as}}{s},$$

$$\boxed{\text{i.e., } \mathcal{L}\{u_a(t)\} = \frac{e^{-as}}{s}, \quad (s > 0).}$$

For example

$$\boxed{\mathcal{L}\{u_3(t)\} = \frac{e^{-3s}}{s}}$$

Example 1. Write f(t) in terms of unit step functions and find its Laplace transform . where

$$f(t) = \begin{cases} 1 & 0 \le t < 1 \\ 0 & 1 \le t < 2 \\ 1 & 2 \le t < 3 \\ 0 & t \ge 3 \end{cases}.$$

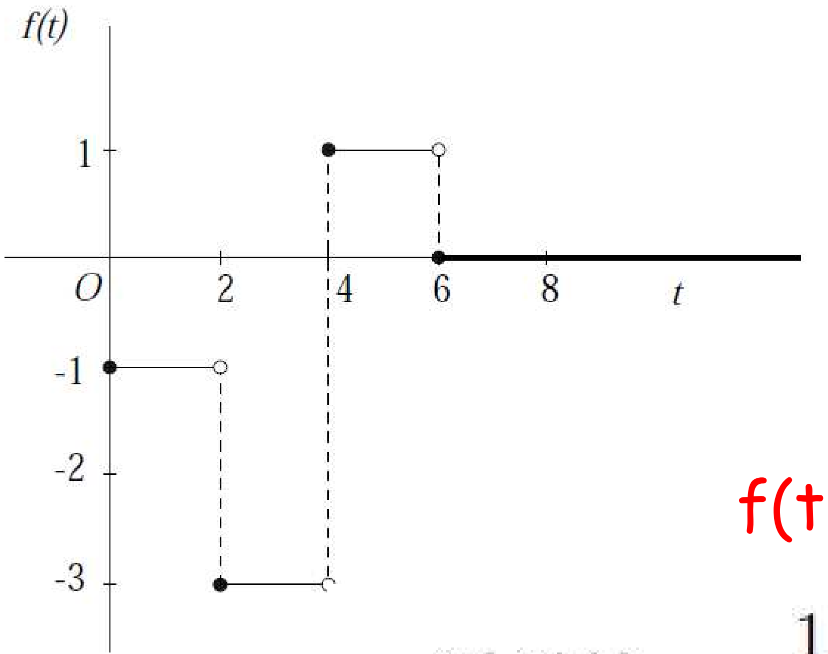
Since f(t) can be expressed as

$$f(t) = u_0(t) - u_1(t) + u_2(t) - u_3(t),$$

we then have

$$\mathcal{L}\{f(t)\} = \frac{1}{s} - \frac{e^{-s}}{s} + \frac{e^{-2s}}{s} - \frac{e^{-3s}}{s} = \frac{1}{s}(1 - e^{-s})(1 + e^{-2s}).$$

Example 2. Represent the function shown in the diagram below in terms of unit step functions and find its Laplace transform.



or

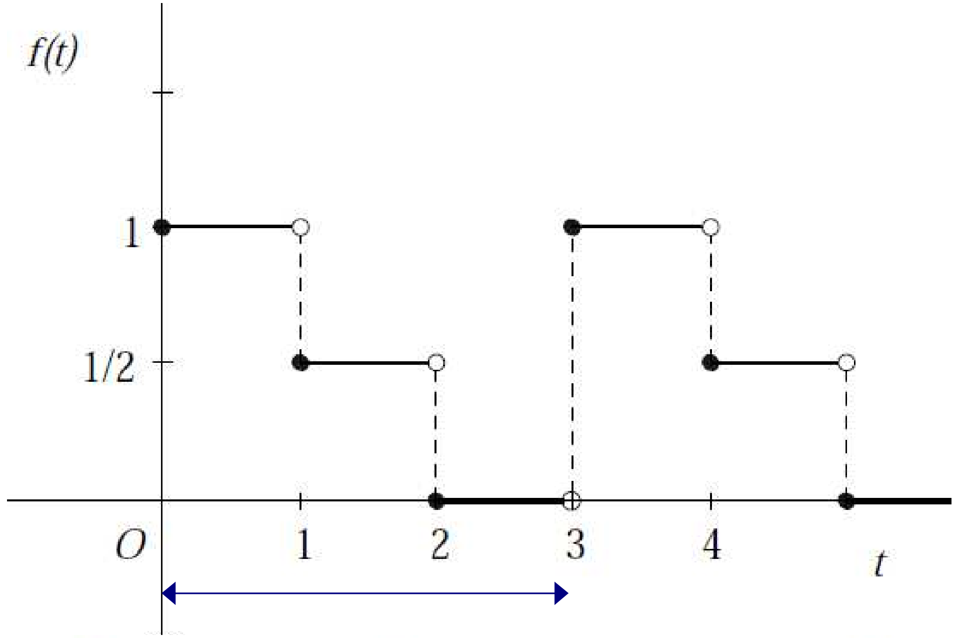
$$h(t)=\begin{cases}-1 & ,0\le t<2\\ -3 & ,2\le t<4\\ 1 & ,4\le t<6\\ 0 & ,\quad t\ge 6\end{cases}$$

$$f(t) = -u_0(t) - 2u_2(t) + 4u_4(t) - u_6(t).$$

$$\therefore \mathcal{L}\{f(t)\} = -\frac{1}{s} - \frac{2}{s}e^{-2s} + \frac{4}{s}e^{-4s} - \frac{1}{s}e^{-6s}$$

$$= -\frac{1}{s}(1-e^{-2s})(1+3e^{-2s}-e^{-4s}).$$

Example 3. The function shown in the diagram below is periodic with period 3. Write the function in terms of unit step functions and find its Laplace transform.



$$f(t) = u_0(t) - \frac{1}{2}u_1(t) - \frac{1}{2}u_2(t) + u_3(t) - \frac{1}{2}u_4(t) - \frac{1}{2}u_5(t)\ldots$$
$$= (u_0 + u_3 + u_6 + \ldots) - \frac{1}{2}(u_1 + u_4 + u_7 + \ldots) - \frac{1}{2}(u_2 + u_5 + u_8 + \ldots).$$

$$
\begin{aligned}
\mathcal{L}\{f(t)\} &= \frac{1}{s}(1+e^{-3s}+e^{-6s}+\dots) - \frac{1}{2s}(e^{-s}+e^{-4s}+e^{-7s}+\dots) \\
&\quad -\frac{1}{2s}(e^{-2s}+e^{-5s}+e^{-8s}+\dots) \\
&= \frac{1}{s}\,\frac{1}{1-e^{-3s}} - \frac{1}{2s}\frac{e^{-s}}{1-e^{-3s}} - \frac{1}{2s}\,\frac{e^{-2s}}{1-e^{-3s}} \quad \text{(Geometric Series)} \\
&= \frac{2-e^{-s}-e^{-2s}}{2s(1-e^{-3s})}\,.
\end{aligned}
$$

6- The Second Shifting Theorem

If $\mathcal{L}\{f(t)\} = F(s)$, **then**

$$\begin{cases} \mathcal{L}\{e^{at} f(t)\} = F(s - a). \\ \mathcal{L}\{f(t-a)\, u_a(t)\} = e^{-as} F(s), \quad \text{for } a > 0 \end{cases}$$

First Shifting

Second Shifting

$$\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t-a)u_a(t).$$

Example 4. Evaluate $\mathcal{L}\{(t-\pi)u_\pi(t)\}$.

$$f(t) = t, \qquad \text{so } F(s) = \frac{1}{s^2} \qquad \text{and} \qquad \mathcal{L}\{(t-\pi)u_\pi(t)\} = \frac{e^{-\pi s}}{s^2}.$$

Example 5. Evaluate $\mathcal{L}\{tu_2(t)\}$.

Because the step function is $u_2(t)$, i.e., has suffix 2, the function t must be written as a function of $(t-2)$, i.e., $t = (t-2) + 2$.

$$\begin{aligned} \therefore \mathcal{L}\{tu_2(t)\} &= \mathcal{L}\{(t-2)u_2(t) + 2u_2(t)\} \\ &= \frac{e^{-2s}}{s^2} + \frac{2e^{-2s}}{s} = \frac{(1+2s)}{s^2}e^{-2s}, \end{aligned}$$

because $f(t) = t$ in the first term.

Example 6. Evaluate $\mathcal{L}\{\sin(t-3)u_3(t)\}$.

$$f(t) = \sin t, \quad \Rightarrow \quad \mathcal{L}\{\sin(t-3)u_3(t)\} = \frac{1}{s^2+1}e^{-3s}.$$

Example 7. Find the Laplace transform of the function

$$g(t) = \begin{cases} 0 & t < 1 \\ t^2 - 2t + 2 & t \geq 1 \end{cases}.$$

Solution

$$t^2 - 2t + 2 = (t-1)^2 + 1,$$

$$\text{i.e.,} \quad f(t-1) = (t-1)^2 + 1 \quad \text{so} \quad f(t) = t^2 + 1.$$

$$\therefore \mathcal{L}\{g(t)\} = \mathcal{L}\{f(t-1)u_1(t)\} = e^{-s}\mathcal{L}\{f(t)\} = e^{-s}\left(\frac{2}{s^3} + \frac{1}{s}\right).$$

Example 8. Evaluate $\mathcal{L}^{-1}\left\{\frac{se^{-\pi s}}{s^2+4}\right\}$.

Solution We have:

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} = \cos 2t = f(t).$$

$$\therefore \mathcal{L}^{-1}\left\{\frac{se^{-\pi s}}{s^2+4}\right\} = f(t-\pi)u_\pi(t) = \cos 2(t-\pi)u_\pi(t) = \cos 2tu_\pi(t).$$

Example 9. Evaluate $\mathcal{L}^{-1}\{\frac{e^{-s}}{s^4}\}$.

The e^{-s} indicates that $u_1(t)$ appears in the function and so does $f(t-1)$.

$$\mathcal{L}^{-1}\{\frac{1}{s^4}\} = \frac{1}{6}t^3 = f(t),$$ so $$f(t-1) = \frac{1}{6}(t-1)^3$$ and

$$\boxed{\mathcal{L}^{-1}\{\frac{e^{-s}}{s^4}\} = \frac{1}{6}(t-1)^3 u_1(t).}$$

Example 10. Evaluate $$\mathcal{L}^{-1}\left\{\frac{1+e^{-\pi s/2}}{s^2+1}\right\}.$$

Solution

This is the sum of the inverse transforms

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} \qquad \text{and} \qquad \mathcal{L}^{-1}\left\{\frac{e^{-\pi s/2}}{s^2+1}\right\}.$$

The first equals sin t;
the second changes
sin t to sin(t − π/2) = −cos t and multiplies by $u_{\pi/2}(t)$.

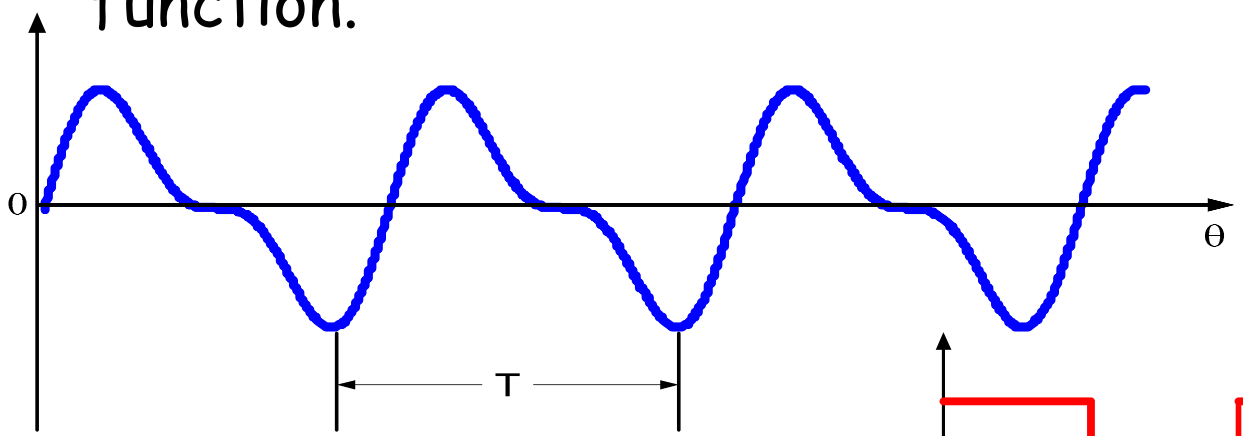
$$\therefore \mathcal{L}^{-1}\left\{\frac{1+e^{-\pi s/2}}{s^2+1}\right\} = \sin t - u_{\frac{\pi}{2}}(t)\cos t.$$

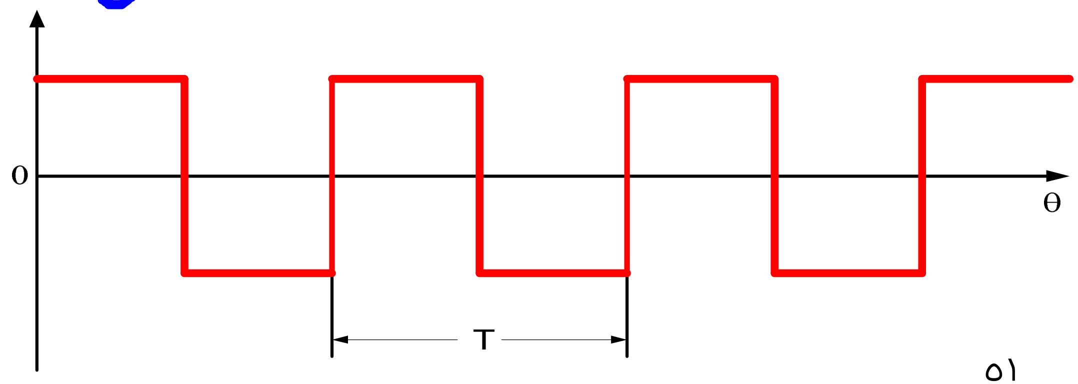
FOURIER SERIES

Periodic Functions:

- Any function that satisfies $f(t) = f(t+T)$

where T is a constant and is called the ***period*** of the function.





For example:

$$\cos x = \cos(x + 2\pi) = \ldots = \cos(x + 2n\pi),$$

thus the period of cos x is 2π.

Also the period of **sin x** is 2π,

the period of tan x is π

and the period of a constant function is any positive number.

Odd and Even Functions:

A function $f(x)$ is said to be odd if:

$f(-x) = -f(x)$ for all x.

It is easy to see that: x^5, $x^3 - 2x$, $\sin x$, $\tan x$, $(e^x - e^{-x})$ are odd functions

A function $f(x)$ is said to be even if :

$f(-x) = f(x)$ for all x .

Thus x^2 , $x^4 + 3x^2 + 7$, $\cos x$,

$(e^x + e^{-x})$ are even functions.

These definitions imply that the graph of an even function is symmetric with respect to the y-axis of cordinates and

the graph of an odd function is symmetric with respect to the origin.

Remarks

1) If f(x) is integrable on the interval [-a , a], then

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx$$

Replacing x by -x in the integrals, we get:

$$\int_{-a}^{a} f(x)dx = \int_{0}^{a}[f(x)+f(-x)]dx = \begin{cases} 2\int_{0}^{a} f(x)dx, & \text{if } f(x) \text{ is an even function,} \\ 0 & \text{if } f(x) \text{ is an odd function.} \end{cases}$$

2- In general, any function $f(x)$ can be represented as the sum of even and odd functions. It is obvious that:

$$f(x) = \tfrac{1}{2}[f(x) + f(-x)] + \tfrac{1}{2}[f(x) - f(-x)]$$
$$= F(x) + G(x),$$

where $F(x) = \tfrac{1}{2}[f(x) + f(-x)]$, -- even function

$G(x) = \tfrac{1}{2}[f(x) - f(-x)]$, -- odd function

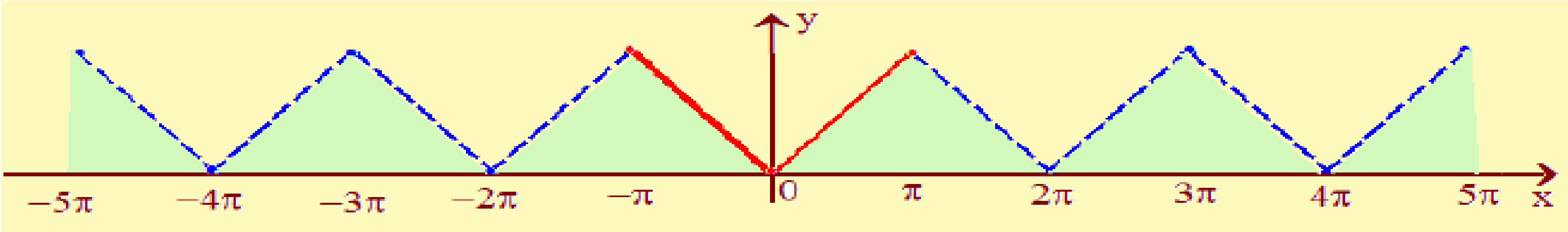
Every function $f(x)$ defined on the interval of the form $[-a, a]$ can be represented as the sum of the corresponding even and odd functions.

Example

Graph following functions:

$$f(x) = |x| \quad , \quad x \in [-\pi, \pi]$$

and has a period 2π .



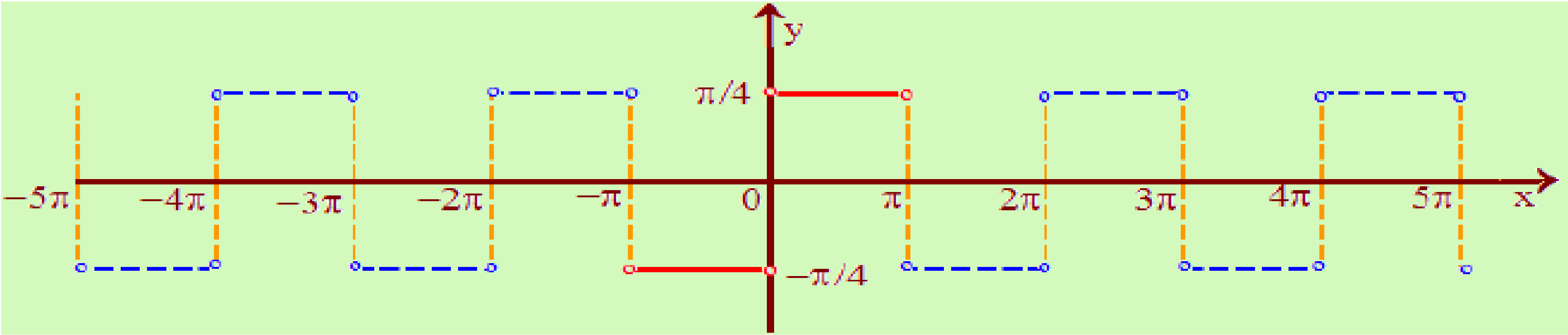
Since the period is 2π , that portion of the graph in $-\pi \leq x \leq \pi$ is extended periodically outside this range.

It is clear that $|-x| = |x|$, then f(x) is an even function.

Example

Graph following functions:

$$f(x) = \begin{cases} -\dfrac{\pi}{4}, & x \in (-\pi, 0) \\ \dfrac{\pi}{4}, & x \in (0, \pi) \end{cases} \quad \text{period } 2\pi.$$



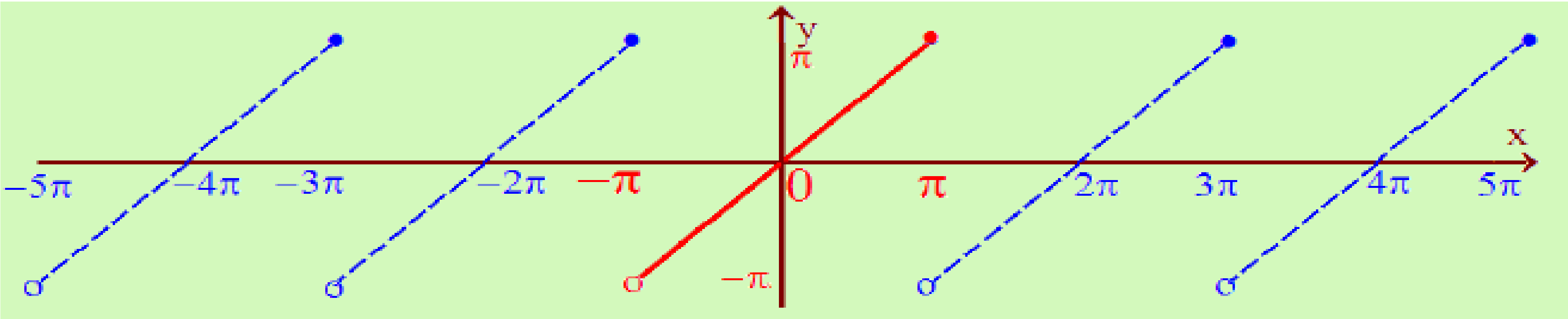
It is clear that, $f(x)$ is not defined at $x = 0, \pm\pi, \pm2\pi, \ldots$ etc. These values are the discontinuities of $f(x)$.

It is obvious that f(x) is an odd function.

Example

Graph following functions:

$$f(x) = x \,, \quad -\pi < x \leq \pi, \text{ period} = 2\pi.$$



1- FOURIER SERIES

Let $f(x)$ be defined in the interval $(-T, T)$ and outside of this interval by

$$f(x + 2T) = f(x),$$

i.e. assume that $f(x)$ has the period **2T**.

The Fourier expansion or Fourier series $S(x)$ corresponding to $f(x)$ is given by:

$$\boxed{f(x) \cong S(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{T}\right) + b_n \sin\left(\frac{n\pi x}{T}\right)\right]}$$

where a_0, a_n, b_n, $n = 1, 2, \ldots$ are called Fourier coefficients and given by :

$$a_n = \frac{1}{T} \int_{-T}^{T} f(x) \cos\left(\frac{n\pi x}{T}\right) dx$$

$$b_n = \frac{1}{T} \int_{-T}^{T} f(x) \sin\left(\frac{n\pi x}{T}\right) dx$$

To determine a_0 , put $n=0$, hence:

$$a_0 = \frac{1}{T} \int_{-T}^{T} f(x)\, dx$$

It is clear that $b_0 = 0$, since $\sin 0 = 0$.

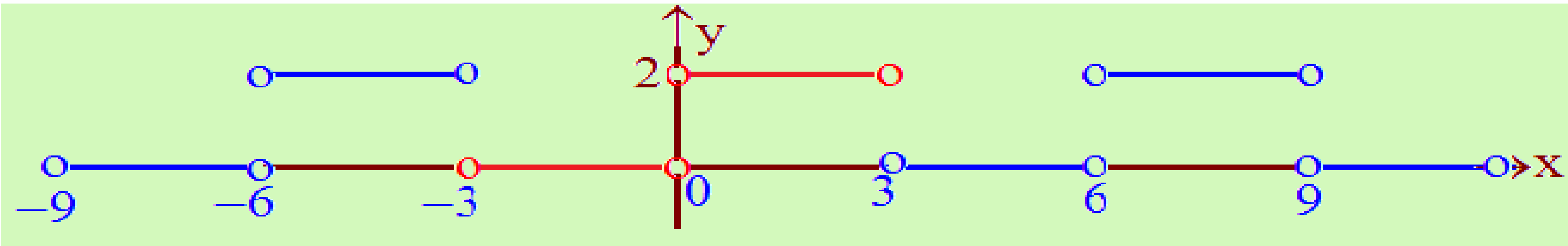
Example

Find the Fourier coefficients corresponding to the function:

$$f(x) = \begin{cases} 0 & -3 < x < 0 \\ 2 & 0 < x < 3 \end{cases}, \qquad \text{period} = 6$$

Solution:

The graph of f(x) is shown below:



Period = 2T = 6, and T = 3. Then

$$a_0 = \frac{1}{T}\int_{-T}^{T} f(x)\,dx = \frac{1}{3}\int_{-3}^{3} f(x)\,dx$$

$$= \frac{1}{3}\left[\int_{-3}^{0} 0\,dx + \int_{0}^{3} 2\,dx\right] = \frac{1}{3}(6) = 2$$

And,

$$a_n = \frac{1}{T}\int_{-T}^{T} f(x)\cos(\frac{n\pi x}{T})\,dx = \frac{1}{3}\int_{0}^{3} 2\cos(\frac{n\pi x}{3})\,dx$$

$$= \frac{2}{3}\left[\frac{3}{n\pi}\right]\left[\sin\frac{n\pi x}{3}\right]\Bigg|_{x=0}^{x=3} = \frac{2}{n\pi}[0-0] = 0 \quad ,n = 1,2,\cdots$$

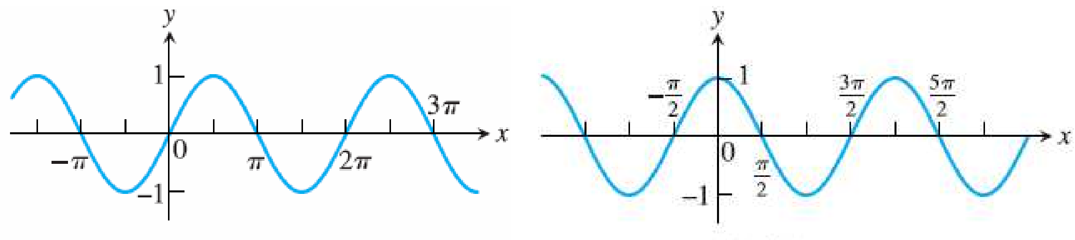
$$b_n = \frac{1}{T}\int_{-T}^{T} f(x)\ \sin(\frac{n\pi\, x}{T})\, dx = \frac{1}{3}\int_{-3}^{3} f(x)\ \sin(\frac{n\pi\, x}{3})\, dx$$

$$= \frac{1}{3}\int_{0}^{3} 2\sin(\frac{n\pi\, x}{3})\, dx$$

$$= \frac{2}{3}\frac{3}{n\pi}\left[-\cos\frac{n\pi\, x}{3}\right]\Bigg|_{x=0}^{x=3} = \frac{2}{n\pi}(-\cos(n\,\pi\,) + \cos\ 0)$$

$$= \frac{2}{n\pi}(1-(-1)^n)$$

$$= \begin{cases} 0 & \text{if n is even} \\ \dfrac{4}{n\pi} & \text{if n is odd} \end{cases}$$



(a) $f(x) = \sin x$ (b) $f(x) = \cos x$

i.e. $b_1 = 4/\pi$, $b_2 = 0$, $b_3 = 4/3\pi$, $b_4 = 0$,
$b_5 = 4/5\pi$, $b_6 = 0$, ... etc.

Therefore $b_2 = b_4 = b_6 = \ldots\ldots = b_{2m} = 0$,
$b_{2m+1} = 4/(2m+1)\pi$, for all $m=0,1,2,\ldots.$,

And $a_o = 2$, $a_1 = a_2 = a_3 = \ldots. = a_m = 0$.

$$f(x) \approx S(x) = \frac{a_0}{2} + \left[a_1 \cos\frac{\pi x}{3} + a_2 \cos\frac{2\pi x}{3} + \cdots \right] + \left[b_1 \sin\frac{\pi x}{3} + b_2 \sin\frac{2\pi x}{3} + \cdots \right]$$

$$= \frac{2}{2} + \left[\frac{4}{\pi}\sin\frac{\pi x}{3} + \frac{4}{3\pi}\sin\frac{3\pi x}{3} + \frac{4}{5\pi}\sin\frac{5\pi x}{3} + \cdots \right]$$

$$= 1 + \frac{4}{\pi}\left[\sin\frac{\pi x}{3} + \frac{1}{3}\sin\frac{3\pi x}{3} + \frac{1}{5}\sin\frac{5\pi x}{3} + \cdots \right]$$

Example Compute the Fourier series for:

$$f(x) = \begin{cases} 0, & -\pi < x < 0, \\ x, & 0 < x < \pi. \end{cases}$$

Solution Here $L = \pi$.

(1) $$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{\pi} x\,dx = \frac{x^2}{2\pi}\bigg|_0^{\pi} = \frac{\pi}{2},$$

(2) $$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{\pi} x\cos nx\,dx$$

u=x
du=dx

dv=cos nx dx
v=(1/n) sin nx

$$a_n = \frac{1}{\pi}\left[\frac{1}{n}x\sin nx\bigg|_{x=0}^{x=\pi} - \frac{1}{n}\int_0^{\pi}\sin nx dx\right] = \frac{1}{n^2\pi}[\cos n\pi - 1]$$

$$= \frac{1}{\pi n^2}(\cos n\pi - 1) = \frac{1}{\pi n^2}[(-1)^n - 1], \quad n = 1, 2, 3, \ldots$$

(3) $$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{\pi} x\sin nx\,dx$$

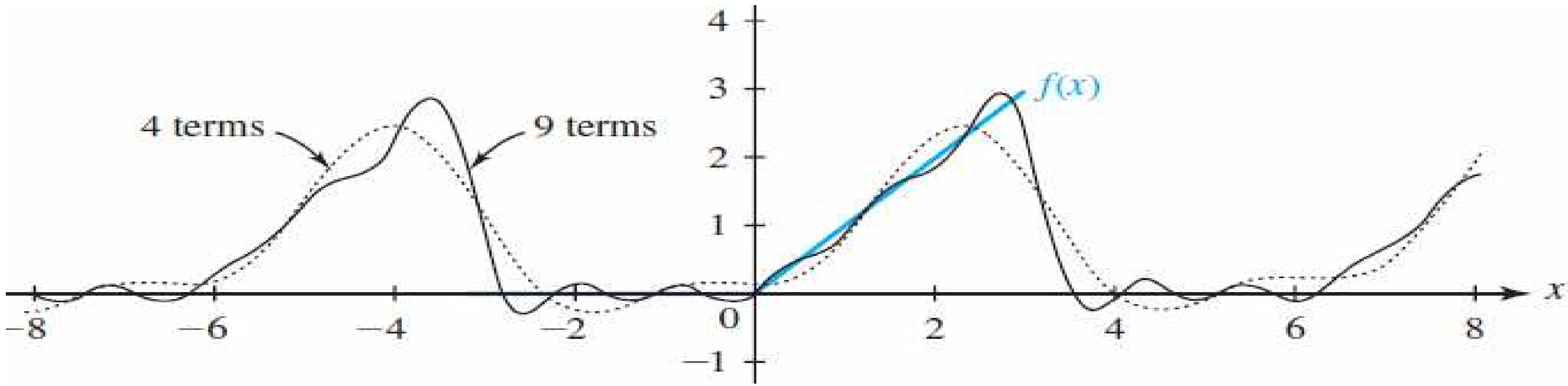
$$= \frac{-\cos n\pi}{n} = \frac{(-1)^{n+1}}{n}, \quad n = 1, 2, 3, \ldots.$$

(4) Therefore,

$$f(x) \sim \frac{\pi}{4} + \sum_{n=1}^{\infty}\left\{\frac{1}{\pi n^2}[(-1)^n - 1]\cos nx + \frac{(-1)^{n+1}}{n}\sin nx\right\}$$

$$= \frac{\pi}{4} - \frac{2}{\pi}\left\{\cos x + \frac{1}{9}\cos 3x + \frac{1}{25}\cos 5x + \cdots\right\}$$

$$+ \left\{\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \cdots\right\}.$$



Partial sums of Fourier series in Example

Example Compute the Fourier series for:

$$f(x) = \begin{cases} -1\,, & -\pi < x < 0\,, \\ 1\,, & 0 < x < \pi\,. \end{cases}$$

Solution *Again, $L = \pi$. Notice that f is an odd function. Since the product of an odd function and an* even function is odd, *f(x) cos(nx) is also an odd function. Thus:*

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = 0\,, \qquad n = 0, 1, 2, \ldots$$

Furthermore, *f(x) sin(nx) is the product of two odd functions and therefore is even, so:*

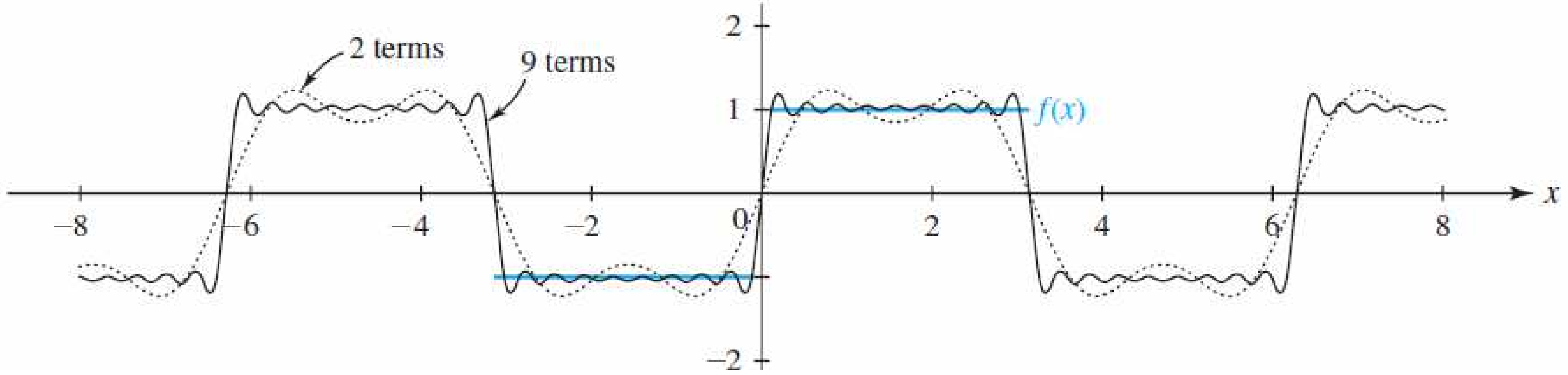
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{2}{\pi}\int_{0}^{\pi} \sin nx\,dx$$

$$= \frac{2}{\pi}\left[\frac{-\cos nx}{n}\right]\Bigg|_0^{\pi} = \frac{2}{\pi}\left[\frac{1}{n} - \frac{(-1)^n}{n}\right], \qquad n = 1, 2, 3, \ldots,$$

$$= \begin{cases} 0\,, & n \text{ even}\,, \\ \dfrac{4}{\pi n}\,, & n \text{ odd}\,. \end{cases}$$

Thus,

$$f(x) \sim \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{[1-(-1)^n]}{n}\sin nx = \frac{4}{\pi}\left[\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right].$$



Partial sums of Fourier series

Example Compute the Fourier series for:

$$f(x) = |x|, -1 < x < 1.$$

Solution Here $L = 1$

Since f is an even function, $f(x)\sin(n\pi x)$ is an odd function. Therefore,

$$b_n = \int_{-1}^{1} f(x)\sin(n\pi x)\,dx = 0, \qquad n = 1, 2, 3, \ldots$$

$$a_0 = \int_{-1}^{1} f(x)\,dx = 2\int_{0}^{1} x\,dx = x^2\Big|_0^1 = 1,$$

$$a_n = \int_{-1}^{1} f(x)\cos(n\pi x)\,dx = 2\int_{0}^{1} x\cos(n\pi x)\,dx$$

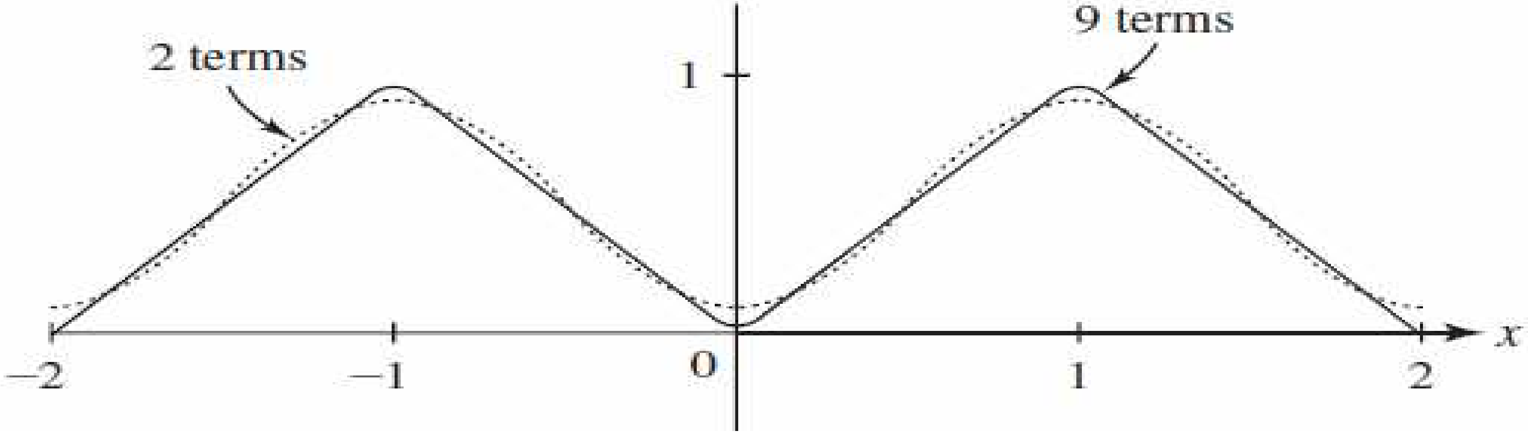
$$= \frac{2}{\pi^2 n^2}(\cos n\pi - 1)$$

$$= \frac{2}{\pi^2 n^2}[(-1)^n - 1], \qquad n = 1, 2, 3, \ldots .$$

Therefore,

$$f(x) \sim \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{\pi^2 n^2}[(-1)^n - 1]\cos(n\pi x)$$

$$= \frac{1}{2} - \frac{4}{\pi^2}\left\{\cos(\pi x) + \frac{1}{9}\cos(3\pi x) + \frac{1}{25}\cos(5\pi x) + \cdots\right\}$$



Partial sums of Fourier series