SMA 214
Mathematics (III)

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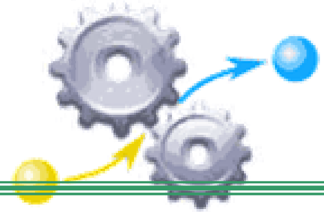
PART I

A review of what you studied previously

1- Functions

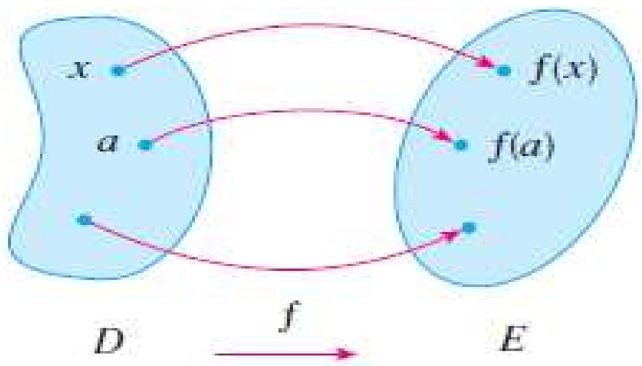
A function relates an input to an output.



It is like a machine that has an input and an output. And the out
put is related somehow to the input.

<u>Definition (**Function**)</u>: A rule for a relationship between an input, or independent, quantity and an output, or dependent, quantity in which each input value uniquely determines one output value.

We say "the output is a function of the input."



And write this symbolically as $f : D \to E$

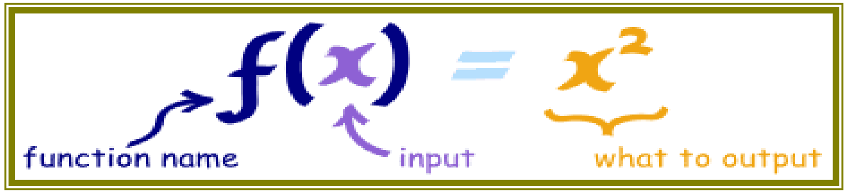
One-to-One Function

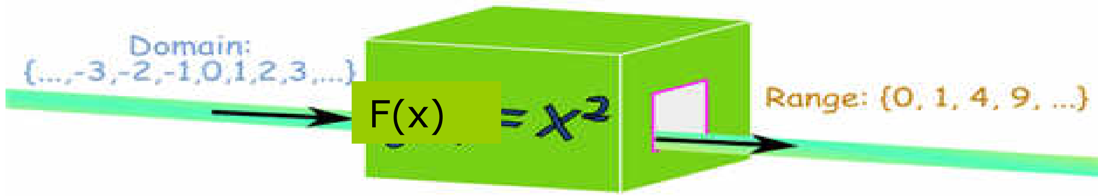
Sometimes in a relationship each input corresponds to exactly one output, and every output corresponds to exactly one input.

We call this kind of relationship a **one-to-one function (injective function)**.

Example: "**squaring**" **is a very simple function.** Here there are the three parts:

Input	*Relationship*	Output
0	0 × 0	0
1	1 × 1	1
-1	-1 × -1	1
…….	…….	…….
10	10 × 10	100





In fact we can write **f(0.3) = 0.09**

Is not one to one, why?

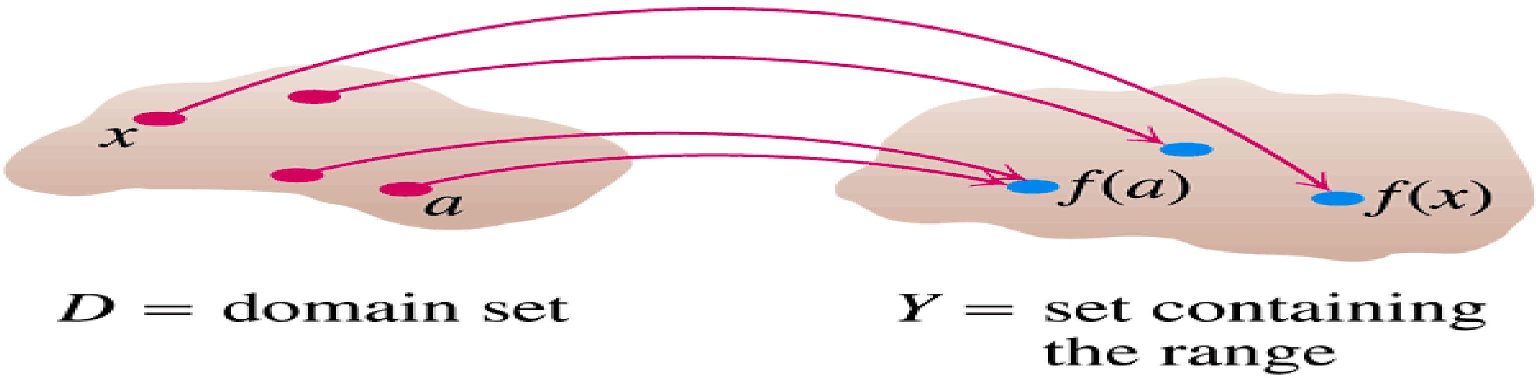
2) Domain, Codomain and Range

Definition Let $f : D \to Y$ be a function.

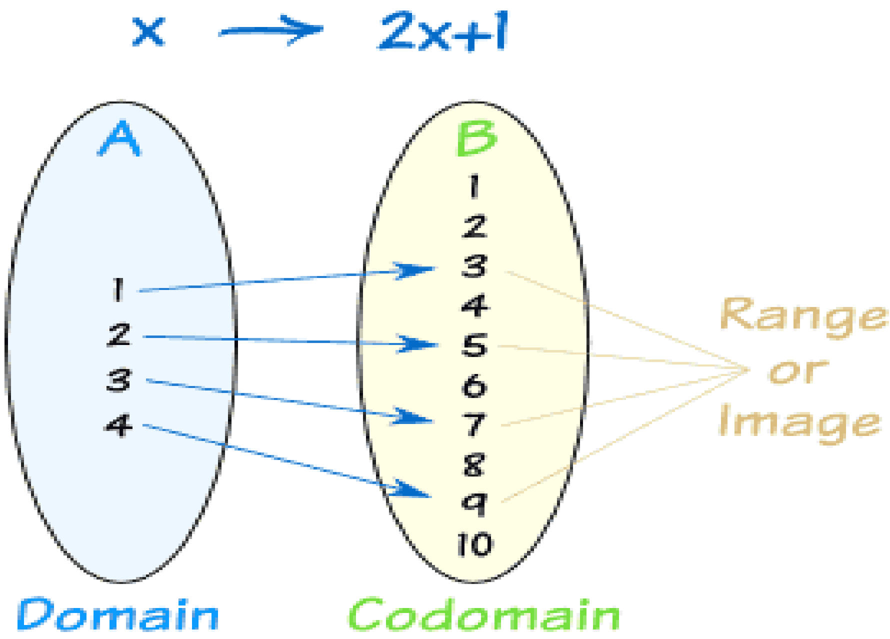
The set D is the domain of definition of the function f.

The set Y is the co-domain of the function f.

The set $f(D) = \{ f(a) \mid a \in D \} \subset Y$ is the range of the function f.



A function from a set D to a set Y assigns a unique element of Y to each element in D.



Definition

A *sequence* is a function whose domain is $\mathbb{N}$ (the set of all positive integers).

A *sequence of real numbers* is a sequence whose codomain is $\mathbb{R}$.

Let $f : \mathbb{N} \to \mathbb{R}$ be a sequence.

For each positive integer n, the value $f(n)$ is called the n^{th} *term* of the sequence and is usually denoted by a small letter together with n in the subscript,

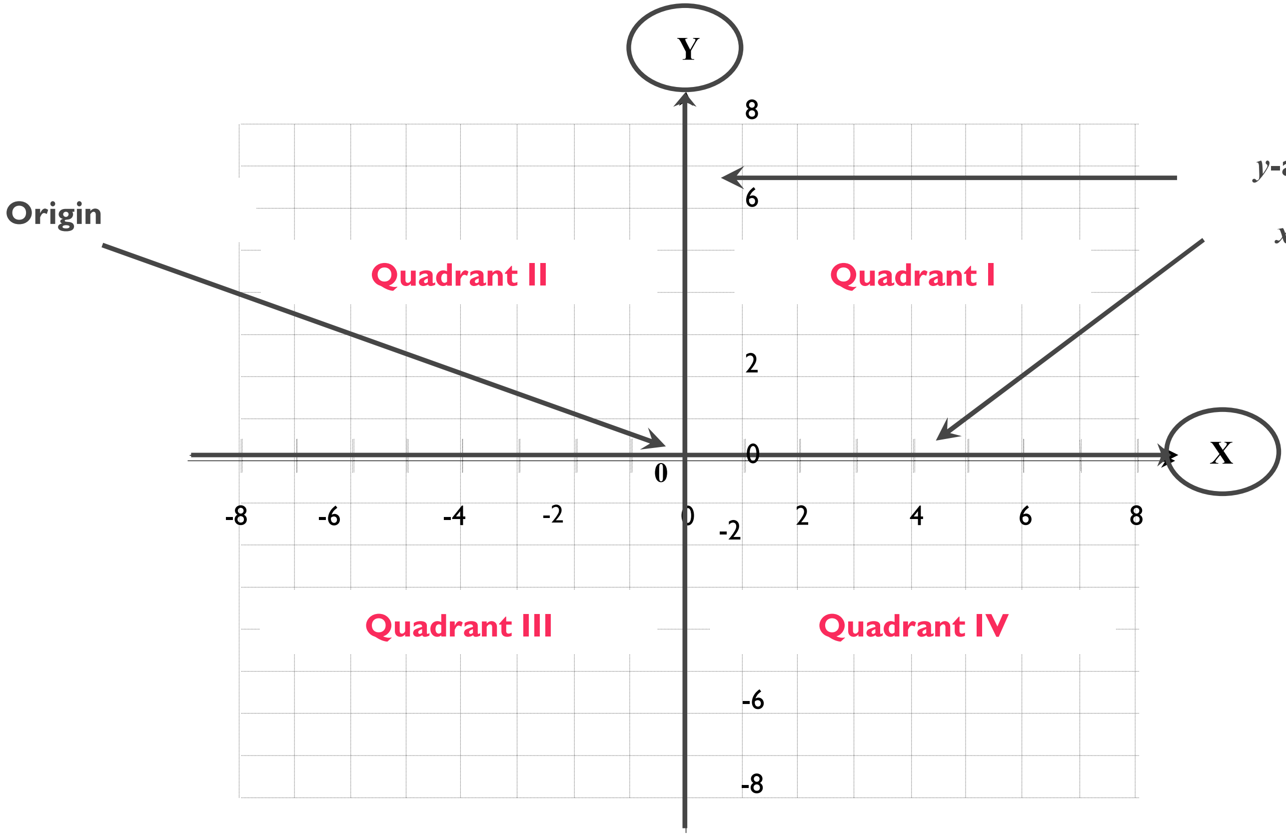
for example a_n. The sequence is also denoted by $\{a_n\}_{n=1,2,\ldots}$ because if we know all the a_n's, then we know the sequence.

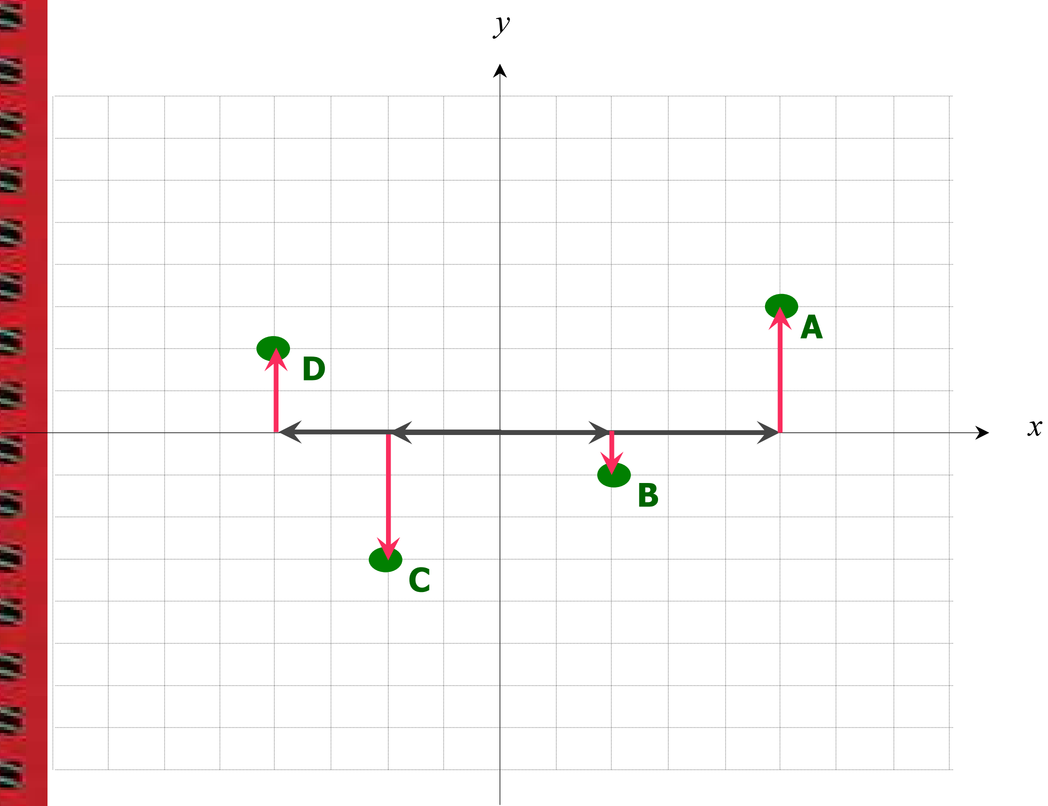
Example

Let $a_n = \dfrac{1}{n}$ The sequence $\{a_n\}_{n=1,2,\ldots}$ (or $\{a_n\}_{n=1}^{\infty}$)

can be represented by $\boxed{1, \dfrac{1}{2}, \dfrac{1}{3}, \dfrac{1}{4}, \ldots, \dfrac{1}{n}, \ldots}$

Rectangular (or Cartesian) Coordinate System





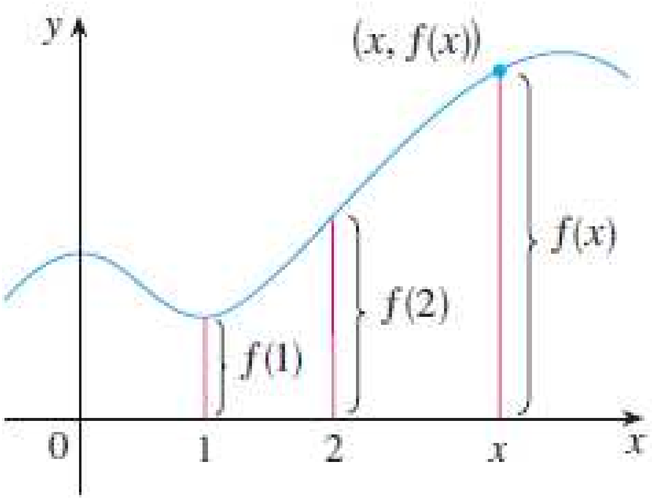
Ordered Pair (x, y)	Quadrant
A (5, 3)	Quadrant I
B (2, –1)	Quadrant IV
C (–2, –3)	Quadrant III
D (–4, 2)	Quadrant II

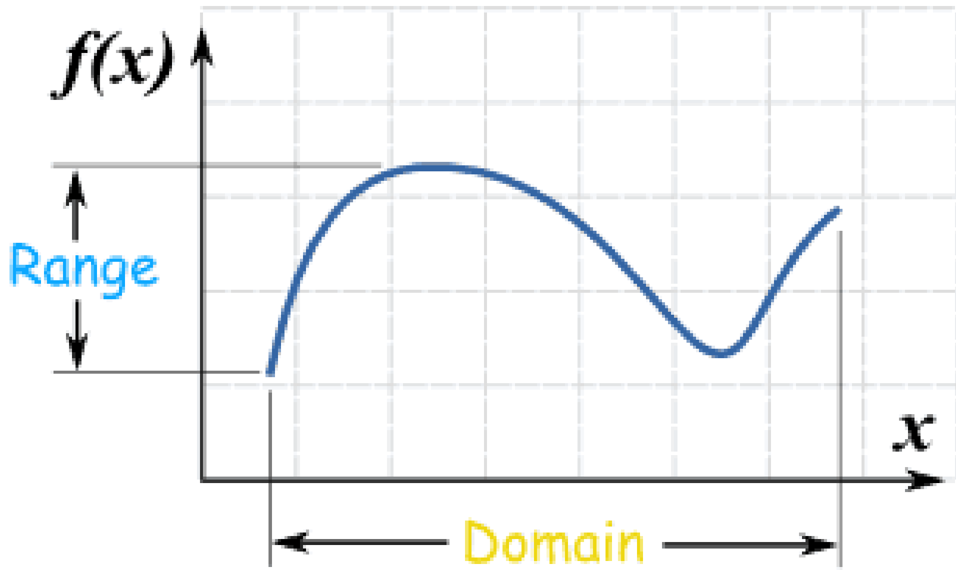
3) Graph of a Function

Graphs of Functions

If f is a function with domain D, its **graph** consists of the points in the Cartesian plane whose coordinates are the input-output pairs for f. In set notation, the graph is

$$\{(x, f(x)) \mid x \in D\}.$$

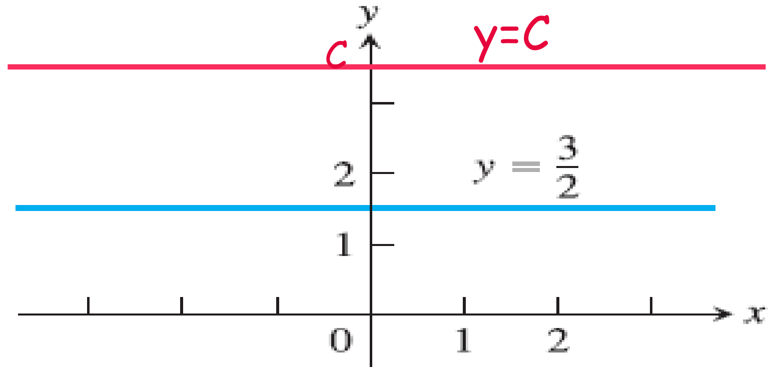




Linear functions

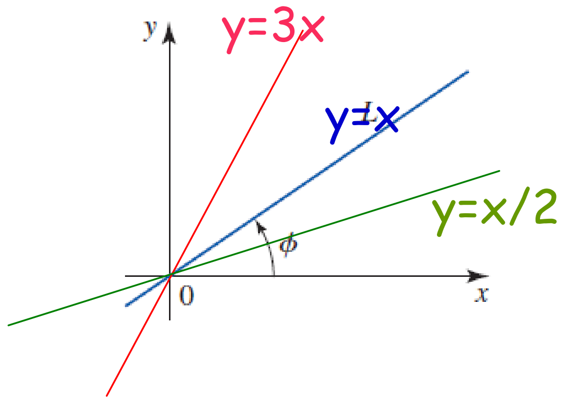
1- A constant function

$$\boxed{f(x)=C \text{ (constant)}}$$



2- The identity function

$$\boxed{f(x)=x}$$



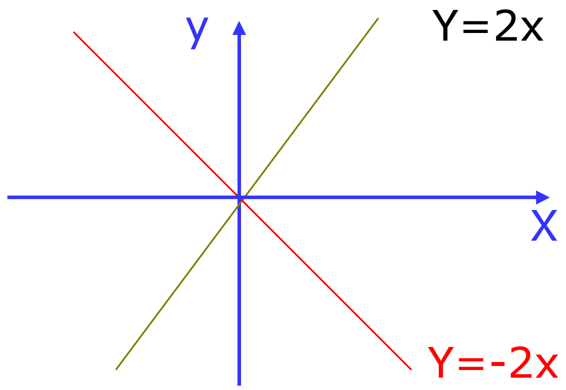
3- $y=mx$

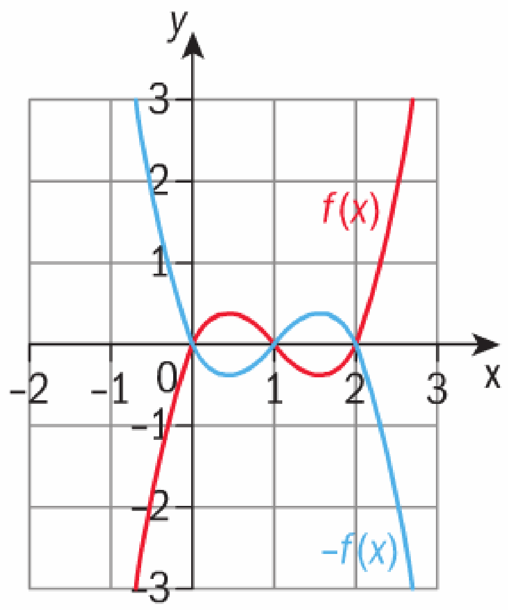
If $f(x) = 3x$

If $f(x) = x/2$

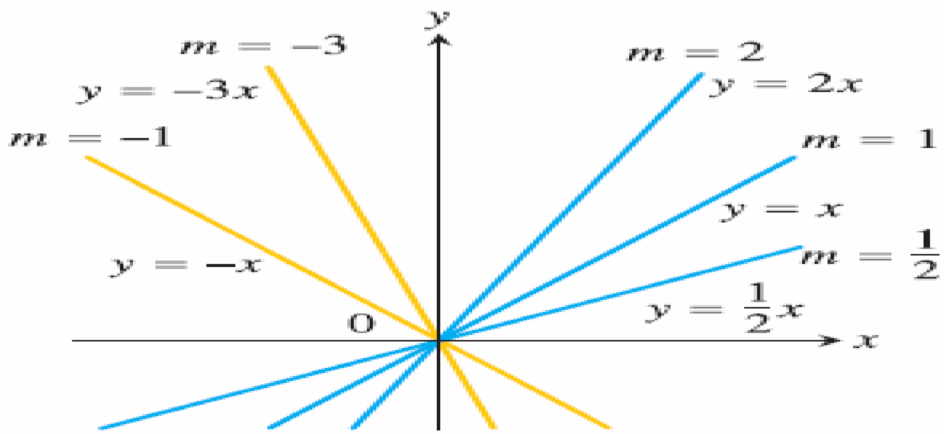
Reflections

$$-f(x) \text{ reflects } f(x) \text{ in the } x-\text{axis.}$$





$$\boxed{f(x) = mx}$$

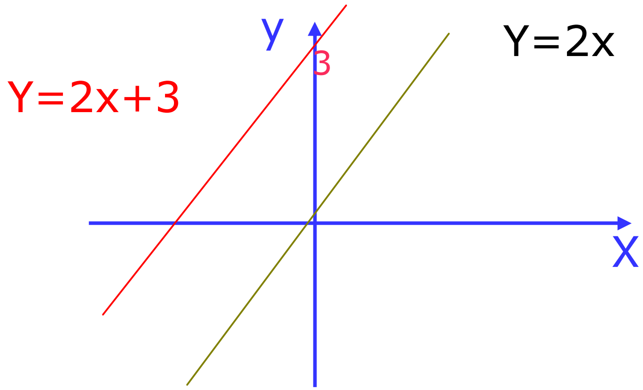


For $c > 1$, the graph is scaled:

$y = cf(x)$	Stretches the graph of f vertically by a factor of c.
$y = \frac{1}{c} f(x)$	Compresses the graph of f vertically by a factor of c.

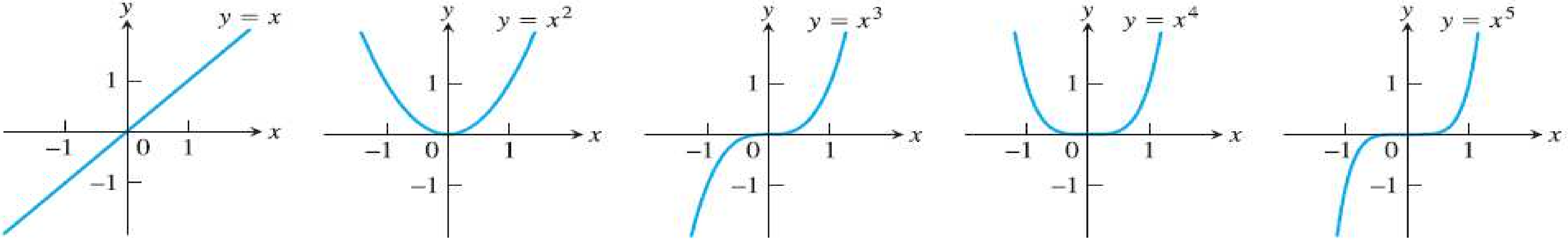
Vertical Shifts

$y = f(x) + k$	Shifts the graph of f *up* k units if $k > 0$
	Shifts it *down* $\lvert k \rvert$ units if $k < 0$

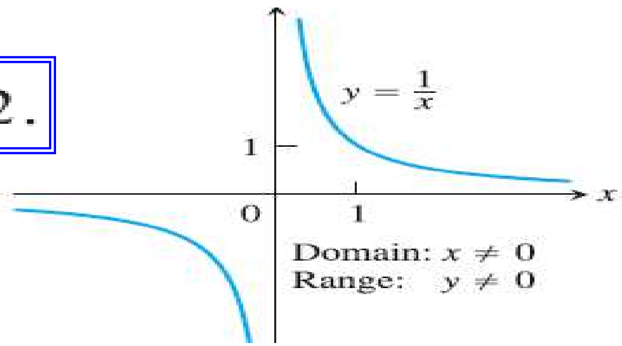


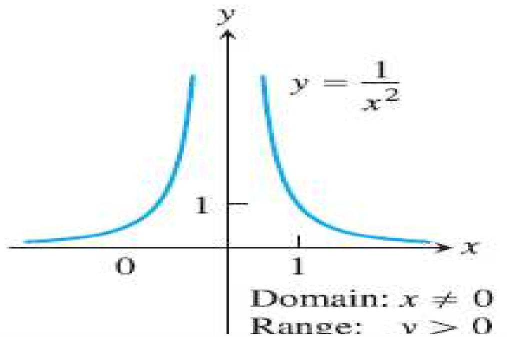
Power Functions A function $f(x) = x^a$, where a is a constant, is called a **power function**. There are several important cases to consider.

(a) $a = n$, a positive integer.

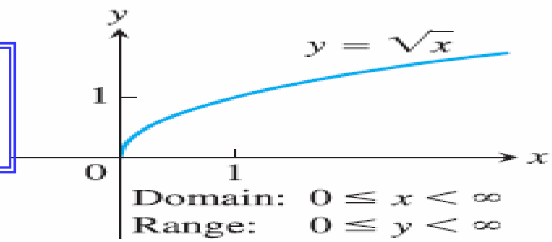


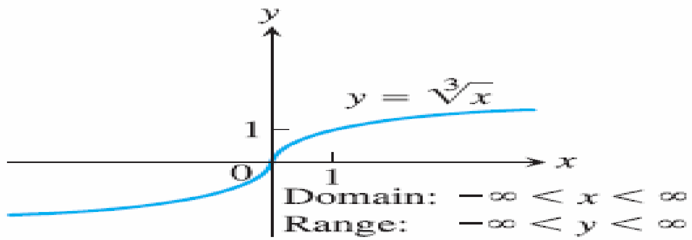
(b) $a = -1$ or $a = -2$.



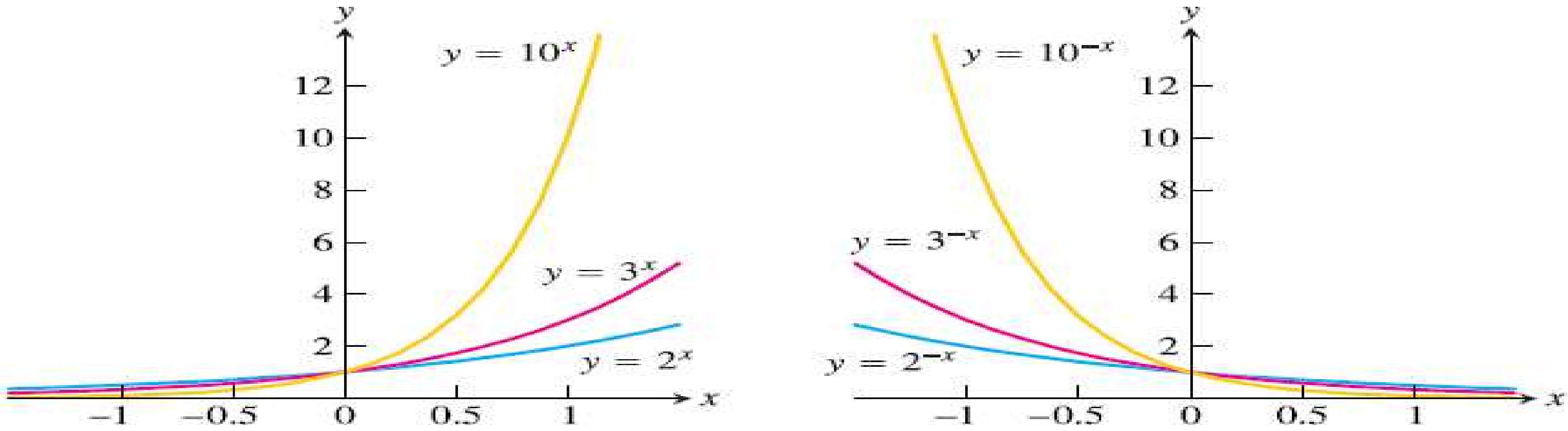


(c) $a = \frac{1}{2}, \frac{1}{3}, \frac{3}{2}$, and $\frac{2}{3}$.

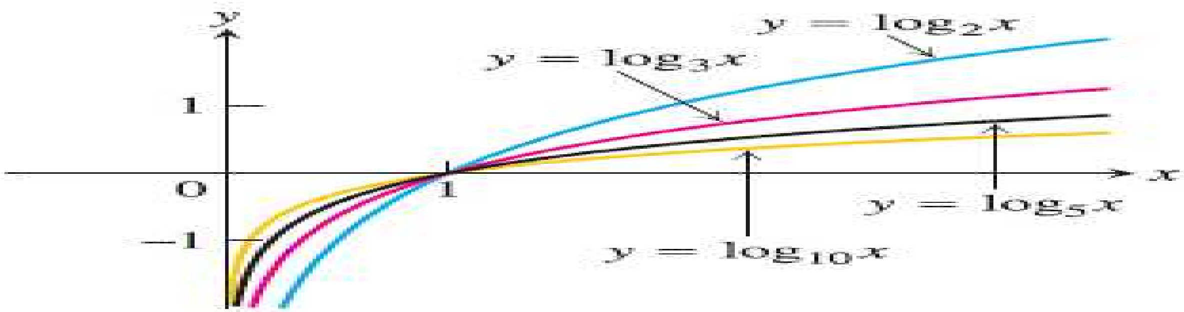




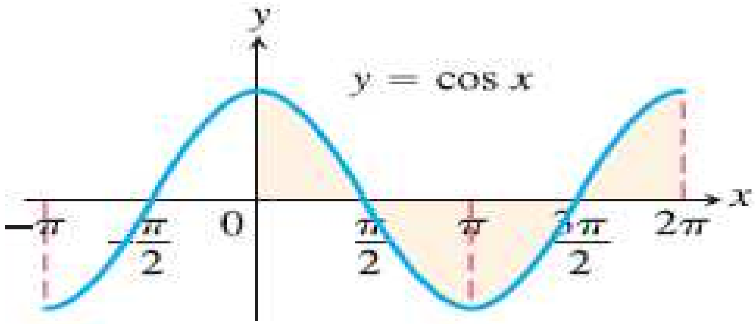
Exponential Functions Functions of the form $f(x) = a^x$, where the base $a > 0$ is a positive constant and $a \neq 1$, are called **exponential functions.**



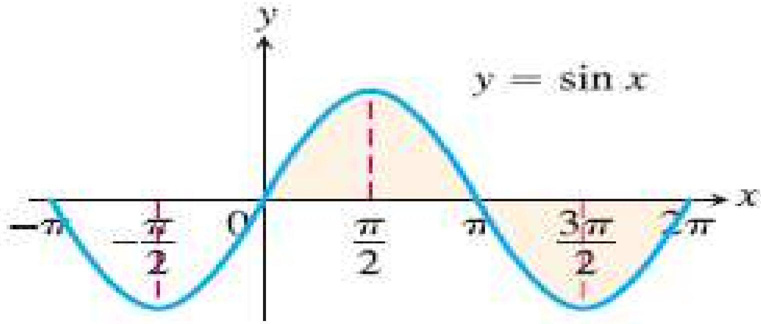
Logarithmic Functions These are the functions $f(x) = \log_a x$, where the base $a \neq 1$ is a positive constant. They are the *inverse functions* of the exponential functions,



Trigonometric Functions   The six basic trigonometric functions



Domain: $-\infty < x < \infty$
Range: $-1 \le y \le 1$
Period: 2π



Domain: $-\infty < x < \infty$
Range: $-1 \le y \le 1$
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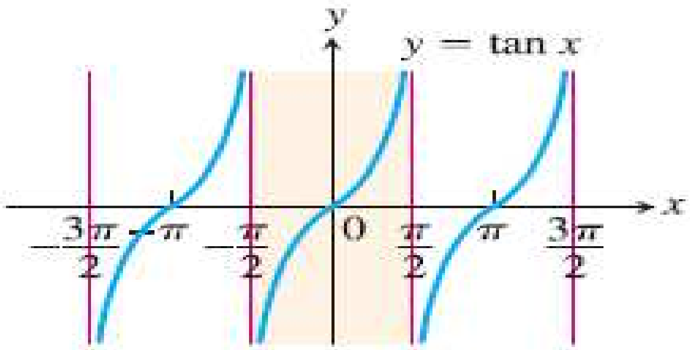
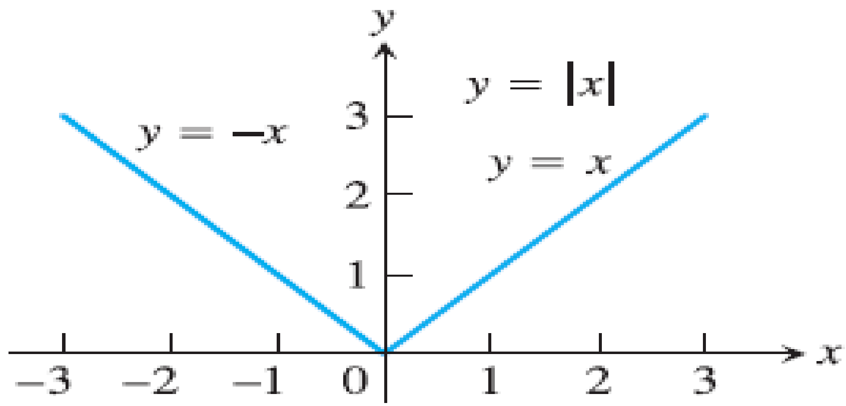


Domain: $x \neq \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$
Range: $-\infty < y < \infty$
Period: π

(c)

absolute value function

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0, \end{cases}$$



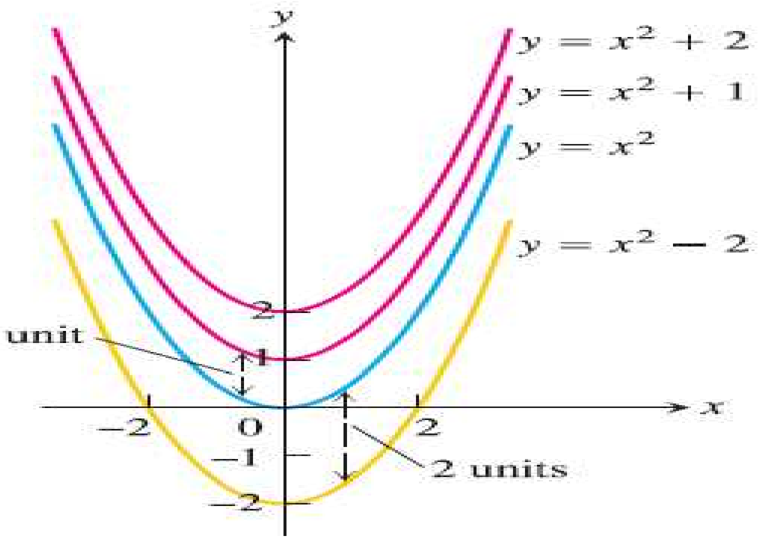
$| \quad | \leq b \iff -b \leq \quad \leq b$

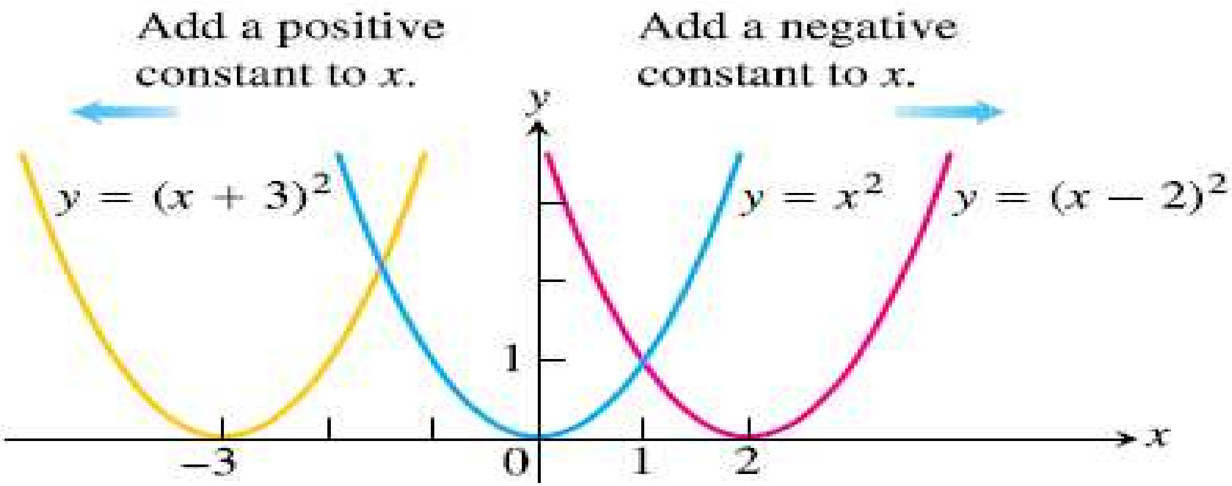
$| \quad | \geq b \iff \quad \leq -b$ **or** $\quad \geq b$

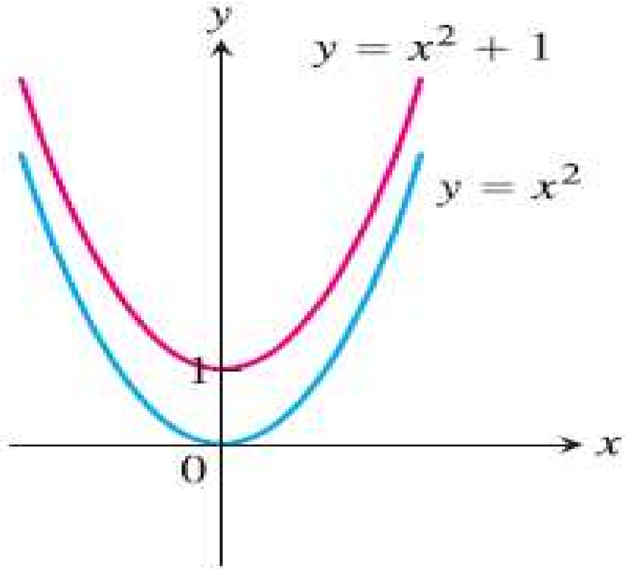
Shift Formulas

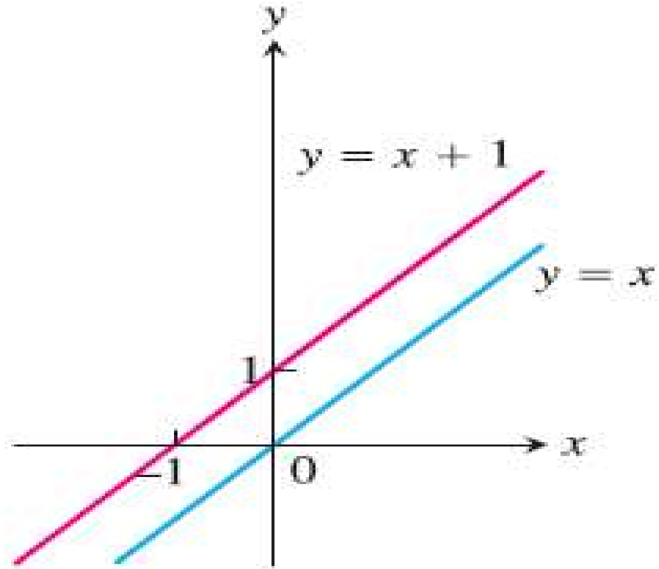
Vertical Shifts

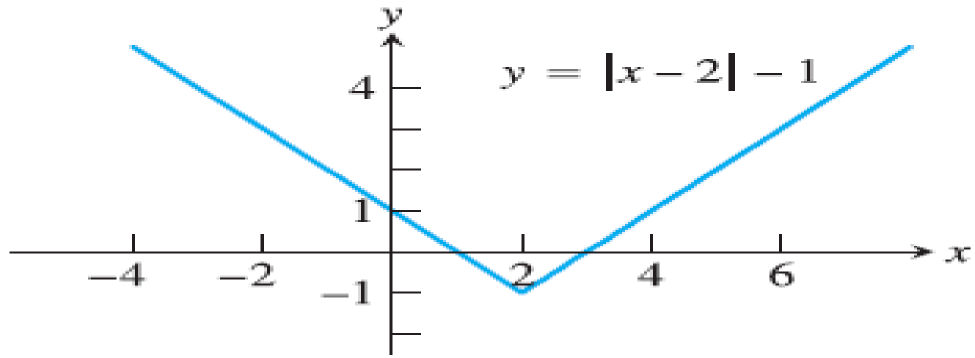
$y = f(x) + k$ Shifts the graph of f *up* k units if $k > 0$
Shifts it *down* $|k|$ units if $k < 0$






y
$y = x^2 + 1$
$y = x^2$
1
0
x


y
$y = x + 1$
$y = x$
1
−1
0
x


y
$y = |x - 2| - 1$
4
1
−1
−4
−2
2
4
6
x

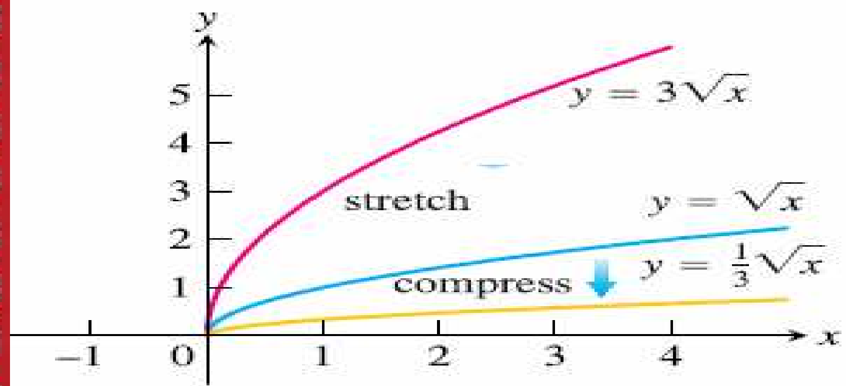
Vertical and Horizontal Scaling and Reflecting Formulas

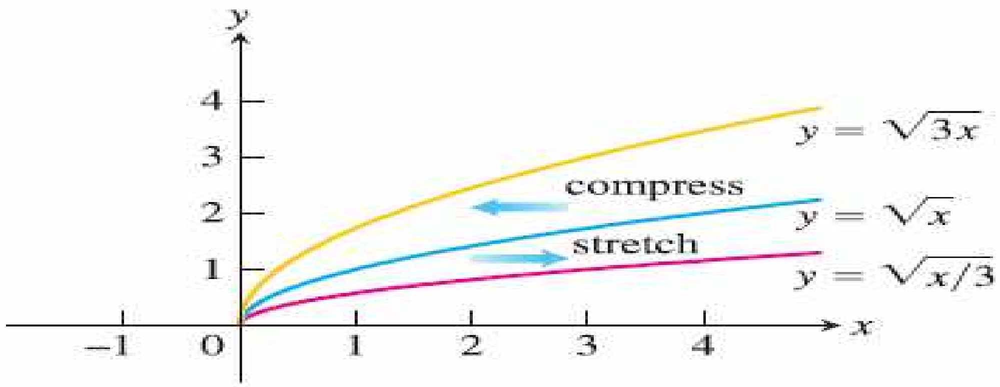
For $c > 1$, the graph is scaled:

$y = cf(x)$	Stretches the graph of f vertically by a factor of c.
$y = \frac{1}{c} f(x)$	Compresses the graph of f vertically by a factor of c.
$y = f(cx)$	Compresses the graph of f horizontally by a factor of c.
$y = f(x/c)$	Stretches the graph of f horizontally by a factor of c.

For $c = -1$, the graph is reflected:

$y = -f(x)$	Reflects the graph of f across the x-axis.
$y = f(-x)$	Reflects the graph of f across the y-axis.





Vertical stretch or compression; reflection about x-axis if negative

Vertical shift

$$y = af(b(x + c)) + d$$

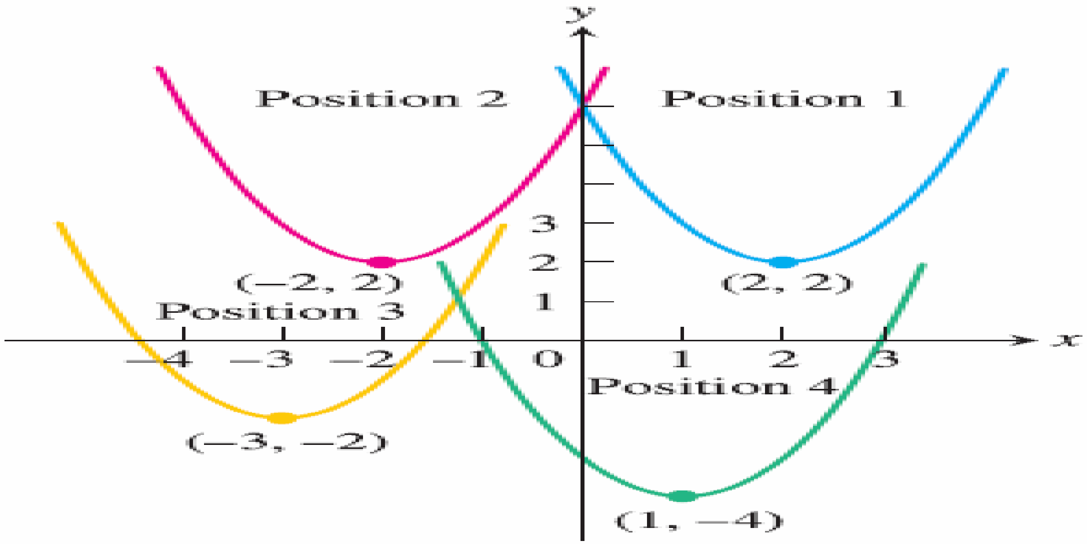
Horizontal stretch or compression; reflection about y-axis if negative

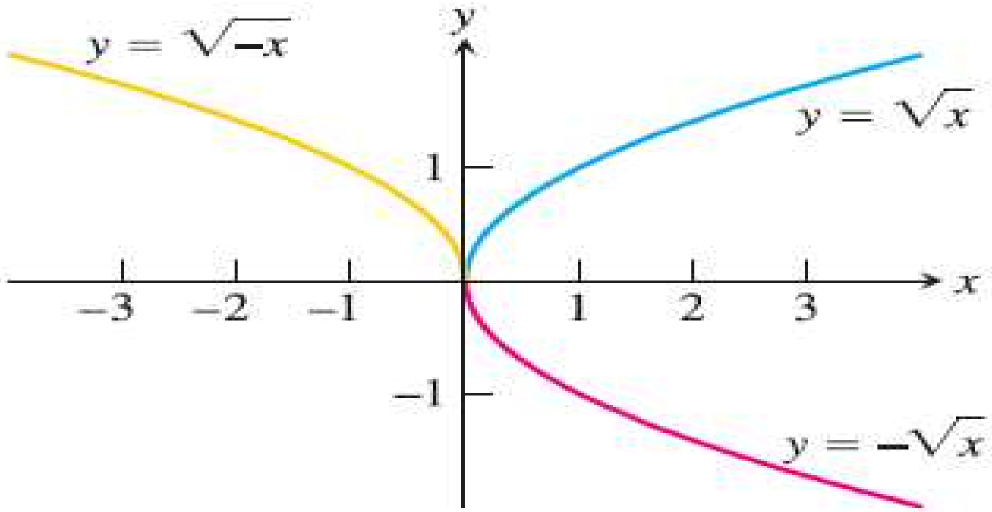
Horizontal shift

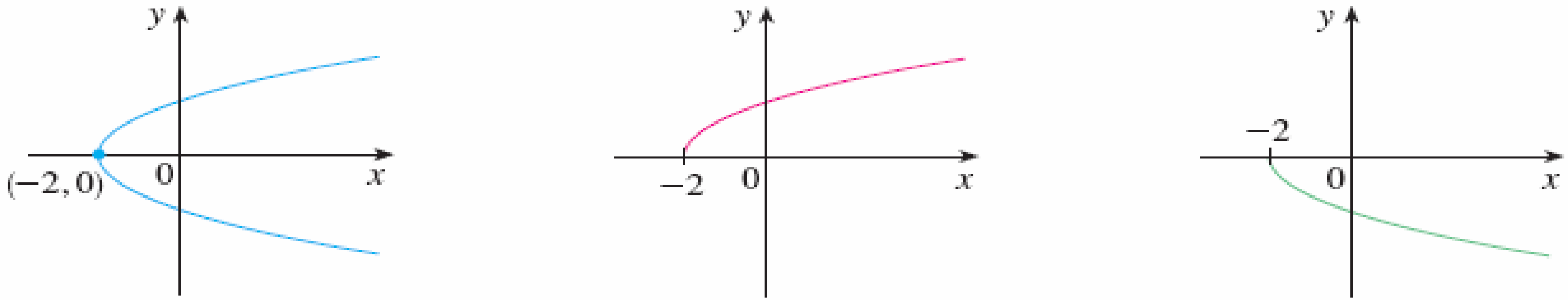
Match the equations listed in parts (a)–(d) to the graphs in the accompanying figure.

a. $y = (x - 1)^2 - 4$ **b.** $y = (x - 2)^2 + 2$

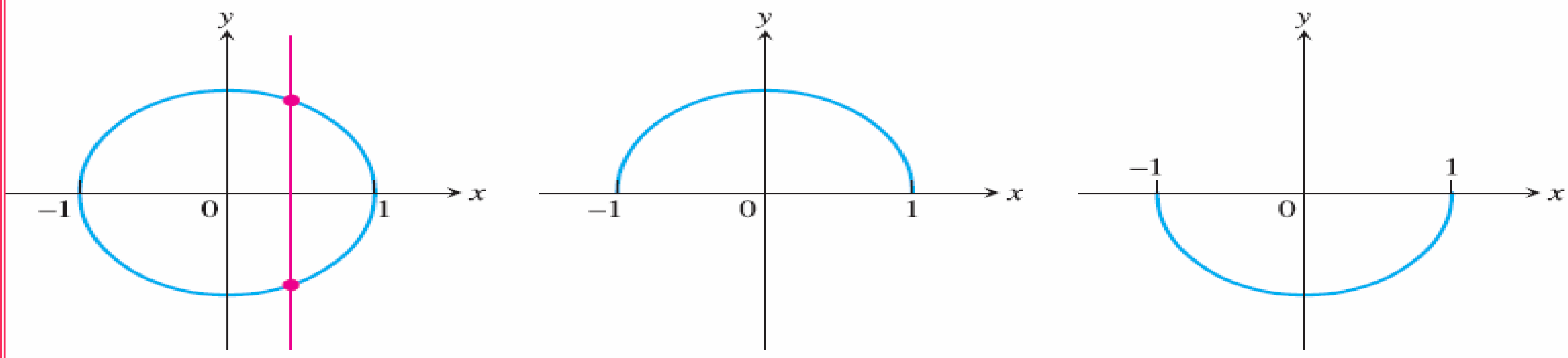
c. $y = (x + 2)^2 + 2$ **d.** $y = (x + 3)^2 - 2$







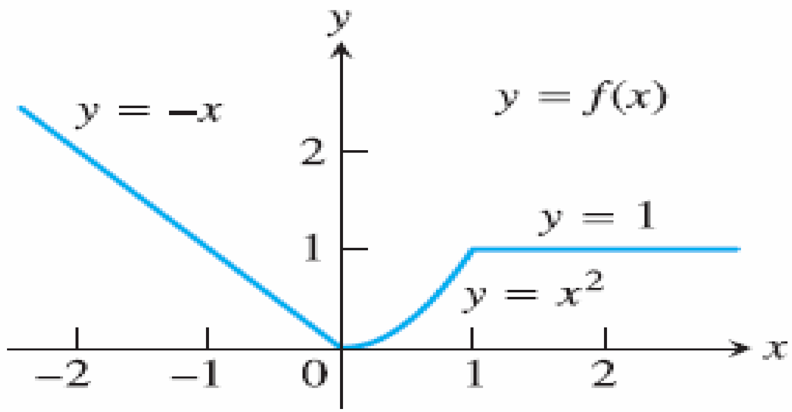
(a) $x = y^2 - 2$ (b) $y = \sqrt{x+2}$ (c) $y = -\sqrt{x+2}$



(a) $x^2 + y^2 = 1$ (b) $y = \sqrt{1 - x^2}$ (c) $y = -\sqrt{1 - x^2}$

Piecewise-Defined Functions

$$f(x) = \begin{cases} -x, & x < 0 \\ x^2, & 0 \le x \le 1 \\ 1, & x > 1 \end{cases}$$



Example For each of the following functions, find its (natural) domain.

(1) $f(x) = x^2 + 3$;

(2) $g(x) = \dfrac{1}{x-2}$;

(3) $h(x) = \sqrt{1+5x}$

(1) Since $f(x) = x^2 + 3$ is defined for all real numbers x, the domain of f is $\mathbb{R}$.

(2) Note that $g(x)$ is defined for all real numbers x except 2.

The domain of g is $\{x \in \mathbb{R} : x \neq 2\} = \mathbb{R} \setminus \{2\}$.

Remark The domain can also be written as $\{x \in \mathbb{R} : x < 2 \text{ or } x > 2\} = (-\infty, 2) \cup (2, \infty)$.

(3) Note that $\sqrt{1+5x}$ is defined if and only if $1 + 5x \geq 0$.

The domain of h is
$$\begin{aligned} \{x \in \mathbb{R} : 1 + 5x \geq 0\} &= \{x \in \mathbb{R} : x \geq -\tfrac{1}{5}\} \\ &= [-\tfrac{1}{5}, \infty). \end{aligned}$$

Let $f : A \longrightarrow B$ be a function. The range of f, denoted by ran (f) or R_f, is the image of A under f, that is, ran (f) = f [A].

Remark By definition, ran (f)= $\{y \in B : y = f(x) \text{ for some } x \in A\}$

The condition

Steps to find range of function

To find the range of a function f described by formula, where the domain is taken to be the natural domain:

(1) Put $y = f(x)$.

(2) Solve x in terms of y.

(3) The range of f is the set of all real numbers y such that x can be solved.

Example For each of the following functions, find its range.

(1) $f(x) = x^2 + 2$

(2) $g(x) = \dfrac{1}{x-2}$

(3) $h(x) = \sqrt{1 + 5x}$

Solution

(1) Put $y = f(x) = x^2 + 2$.

Solve for x. $x^2 = y - 2 \Rightarrow x = \pm\sqrt{y-2}$.

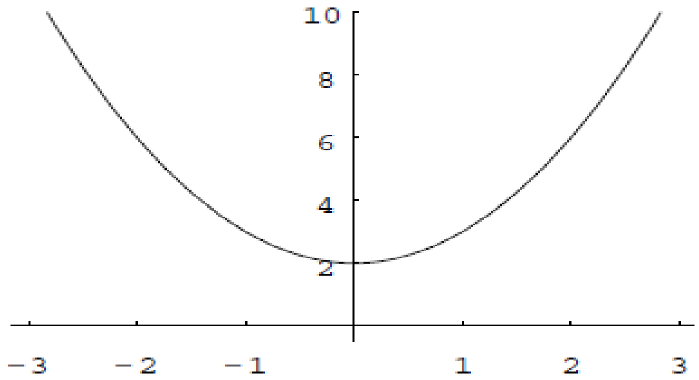
Note that x can be solved if and only if $y - 2 \geq 0$.

The range of f is $\{y \in \mathbb{R} : y - 2 \geq 0\} = \{y \in \mathbb{R} : y \geq 2\}$

$= [2, \infty)$.

Alternatively, to see that the range is $[2;\infty)$, we may use the graph of $y = x^2 + 2$ which is a parabola.

The lowest point (vertex) is (0,2). For any $y \geq 2$, we can always find $x \in R$ such that $f(x) = y$.



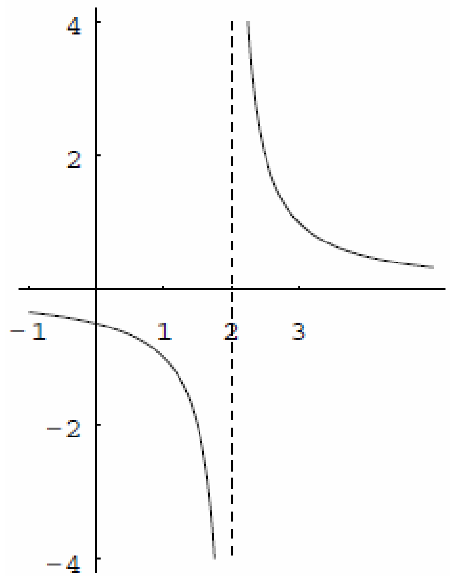
(2) Put $y = g(x) = \dfrac{1}{x-2}$.

Solve for x.

$$\begin{aligned} y &= \frac{1}{x-2} \\ x - 2 &= \frac{1}{y} \\ x &= \frac{1}{y} + 2. \end{aligned}$$

Note that x can be solved if and only if $y \neq 0$.

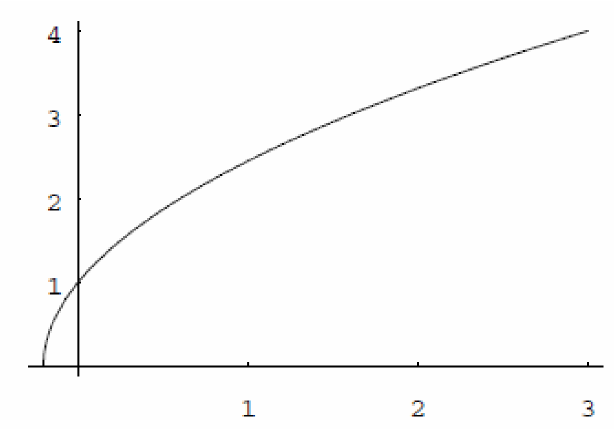
The range of g is $\{y \in \mathbb{R} : y \neq 0\} = \mathbb{R} \setminus \{0\}$.



(3) Put $y = h(x) = \sqrt{1+5x}$. Note that y cannot be negative.

Solve for x.

$$\begin{aligned} y &= \sqrt{1+5x}, & y \geq 0 \\ y^2 &= 1+5x, & y \geq 0 \\ x &= \frac{y^2-1}{5}, & y \geq 0. \end{aligned}$$



Note that x can always be solved for every $y \geq 0$.

The range of h is $\{y \in \mathbb{R} : y \geq 0\} = [0, \infty)$.

Remark $y = \sqrt{1+5x} \Longrightarrow y^2 = 1+5x$

but the converse is true only if $y \geq 0$.

Function	**Domain (x)**	**Range (y)**
$y = x^2$	$(-\infty, \infty)$	$[0, \infty)$
$y = 1/x$	$(-\infty, 0) \cup (0, \infty)$	$(-\infty, 0) \cup (0, \infty)$
$y = \sqrt{x}$	$[0, \infty)$	$[0, \infty)$
$y = \sqrt{4-x}$	$(-\infty, 4]$	$[0, \infty)$
$y = \sqrt{1-x^2}$	$[-1, 1]$	$[0, 1]$

4) Injective Functions, Invers of a function

Definition Let f be a function. We say that f is *injective* (*or one to one*) if the following condition is satisfied:

$$\boxed{x_1, x_2 \in \operatorname{dom}(f) \text{ and } x_1 \neq x_2 \Longrightarrow f(x_1) \neq f(x_2).}$$

This Condition means that different elements of the domain are mapped to different elements of the codomain.

It is equivalent to the following condition:

$$\boxed{x_1, x_2 \in \operatorname{dom}(f) \text{ and } f(x_1) = f(x_2) \Longrightarrow x_1 = x_2.}$$

Example Let $f(x) = 2^x$. The domain of f is $\mathbb{R}$. The function f is injective. This is because if $x_1, x_2 \in \mathbb{R}$ and $x_1 \neq x_2$, then $2^{x_1} \neq 2^{x_2}$.

Example Let $g(x) = x^2$. The domain of g is $\mathbb{R}$. The function g is not injective. This is because $-1 \neq 1$ (both are elements of $\mathbb{R}$), but $g(-1) = g(1)$.

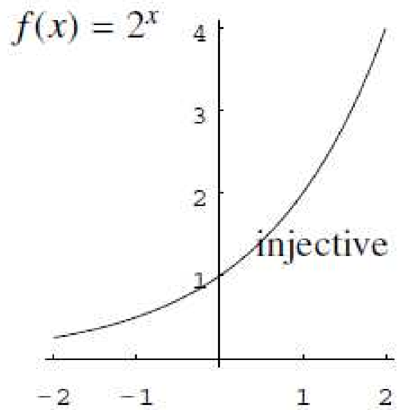
To show that a function f is injective, we have to consider all $x_1; x_2$ belonging to the domain with $x_1 \neq x_2$ and check that $f(x_1) \neq f(x_2)$.

However, to show that a function g is not injective, it suffices to find two different elements $x_1; x_2$ of the domain such that $g(x_1) = g(x_2)$.

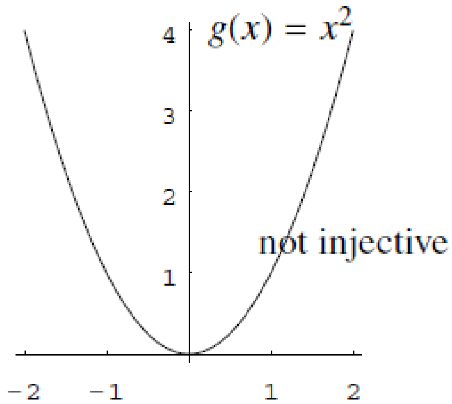
Below we give a geometric method to determine whether a function is injective or not.

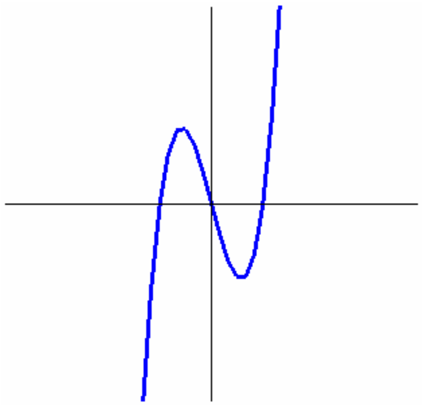
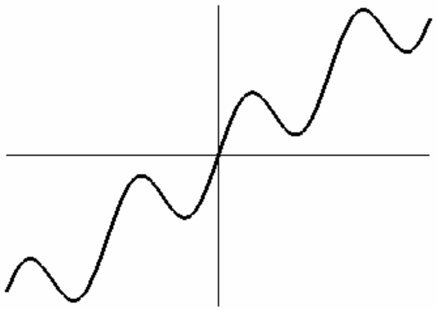
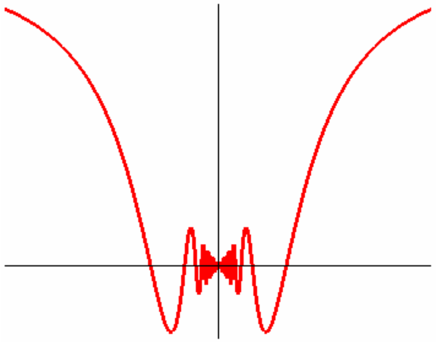
Horizontal Line Test

Let $f : X \longrightarrow \mathbb{R}$ be a function where $X \subseteq \mathbb{R}$. Then f is injective if and only if every horizontal line intersects the graph of f in at most one point.



Example The following two figures show the graphs of f and g in the last two examples. It is easy to see from the Horizontal Line Test that f is injective whereas g is not injective.



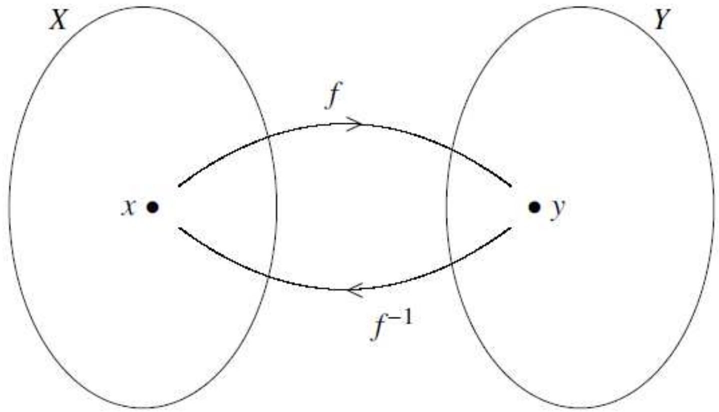
None of the above graphs are graphs of one-to-one functions since they correspond to rules which associate several *x*-values to some *y*-values. This follows since there are horizontal lines intersecting the graphs at more than 1 point.

Let f be an injective function. Then given any element y of ran (f), there is exactly one element x of dom (f) such that

$f(x) = y$.

This means that if we use an element y of ran (f) as input, we get one and only one output x.

The function obtained in this way is called the inverse of f .



Definition Let $f : X \longrightarrow Y$ be an injective function and let Y_1 be the range of f. The *inverse (function)* of f, denoted by f^{-1}, is the function from Y_1 to X such that for every $y \in Y_1$, $f^{-1}(y)$ is the unique element of X satisfying $f(f^{-1}(y)) = y$.

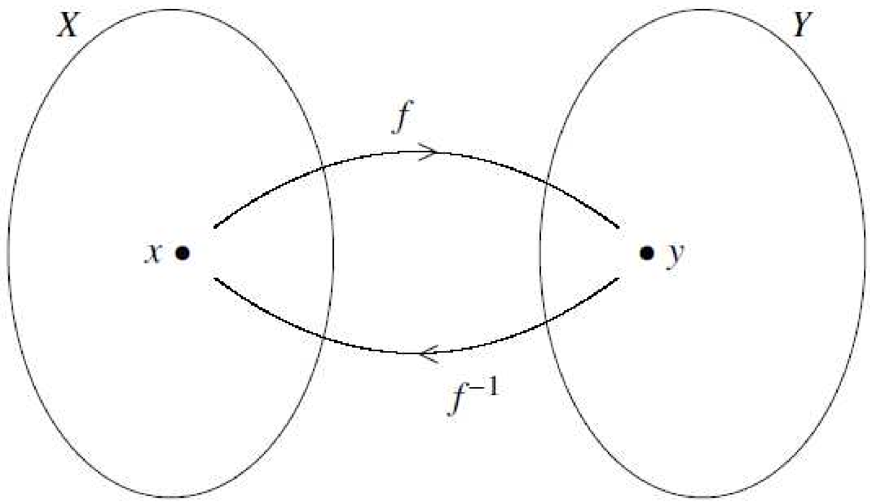
The following figure indicates a function f from a set X to a set Y.

Assuming that f is injective, for each y belonging to the range of f , there is one and only one element x of X such that

$f(x) = y$.

This element x is defined to be $f^{-1}(y)$. That is,

$f^{-1}(y) = x$ if and only if $f(x) = y$



Steps to find inverse functions

Let f be an injective function

Step 1 Put $y = f(x)$.

Step 2 Solve x in terms of y. The result will be in the form x = an expression in y.

Step 3 From the expression in y obtained in Step 2, the range of f can be determined. This is the domain of f^{-1}.

The required formula is $f^{-1}(y)$ = the expression in y obtained in Step 2.

Example Let $f(x) = 2x^3 + 1$. Find the inverse of f.

Solution The domain of f is R. It is not difficult to show that f is injective and that the range of f is R.

These two facts can also be seen from the following steps:

Put $y = f(x)$. That is, $y = 2x^3 + 1$.

Solve for x:

$$y - 1 = 2x^3$$

$$\frac{y-1}{2} = x^3$$

$$\Rightarrow \quad \sqrt[3]{\frac{y-1}{2}} = x \quad (x \text{ can be solved for all real numbers } y)$$

Thus we have dom (f^{-1}) = R and $\boxed{f^{-1}(y) = \sqrt[3]{\frac{y-1}{2}}.}$

Example Let $g:[0,\infty) \longrightarrow R$ be the function given by

$g(x) = x^2$. Find the inverse of g.

Solution Because the domain of g is $[0,\infty)$, the function g is injective . Moreover, the range of g is $[0,\infty)$.

These two facts can also be seen from the following steps:

Put $y = g(x)$. That is, $y = x^2$. Note that $y \geq 0$ and that $x \geq 0$ since $x \varepsilon$ dom (f).

Solve for x: $y = x^2, \quad y \geq 0,\ x \geq 0$

$$\sqrt{y} = x \quad (x \text{ can be solved if and only if } y \geq 0,\ x = -\sqrt{y} \text{ is rejected})$$

Thus we have dom $(g^{-1}) = [0,\infty)$ and $g^{-1}(x) = \sqrt{x}.$

$$\boxed{\textbf{Caution} \quad f^{-1}(x) \neq \frac{1}{f(x)}}$$

Remark We use $\sin^{-1}$ or arcsin to denote the inverse of sin etc....

Although the sine function is not injective, we can make it injective by restricting the domain to $[-\pi/2 , \pi/2]$.

$$x = \sin^{-1} y \text{ means } \sin x = y \text{ and } -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}.$$

The domain of $\sin^{-1}$ is $[-1 , 1]$ because $-1 \leq \sin x \leq 1$.

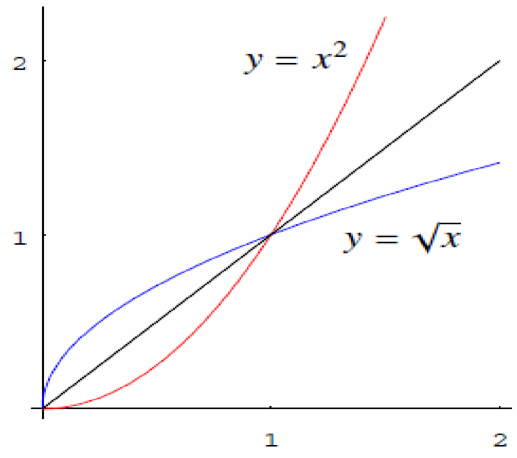
Graph of the inverse function

Let $f : X \longrightarrow R$ be a function, where $X \subseteq R$. Then its graph is a subset of the plane. If, in addition, f is injective, then f has an inverse and dom (f^{-1}) $\subseteq$ R.

Hence the graph of f^{-1} is also a subset of the plane.

There is a nice relationship between the graph of f and that of f^{-1}:

The graph of f and the graph of f^{-1} are symmetric about the line $y = x$.



1. For each of the following functions f, determine whether it is injective or not.

   (a) $f(x) = x^3 + 2x$ (b) $f(x) = x^2 - 5$

2. For each of the following functions f, find its inverse.

   (a) $f(x) = 3x - 2$ (b) $f(x) = x^5 + 3$

   (c) $f(x) = 1 + 2x^{\frac{1}{7}}$ (d) $f(x) = \sqrt[3]{2x^3 - 1}$

5) Even Functions and Odd Functions: Symmetry

Definition a subset $A \subseteq R$ is said to be *symmetric* about the origin if :

$$\forall x: \; x \in A \;\Rightarrow\; -x \in A$$

Examples

A = [-5,5] and B = {-2,-1,1,2} are *symmetric* about the origin, while, C = [-5,5) and D = [-2,5] are not *symmetric* about the origin (why ?)

Definition Let y=f(x) be a function with symmetric domain:

If f(-x) = f(x) , then f is called an **even function** **(of *x*)**

If f(-x) = - f(x) , then f is called an **odd function** **(of *x*)**

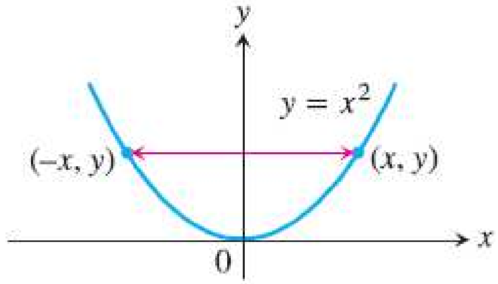
for every *x* in the function's domain.

The names even and odd come from powers of x.

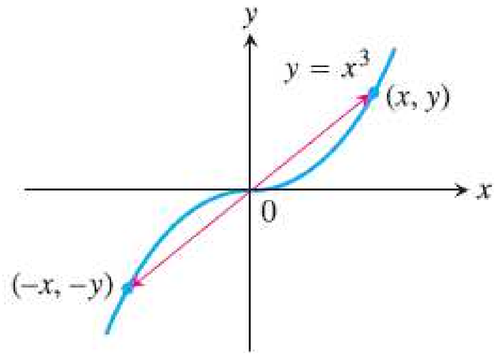
If y is an even power of x, as in $y = x^2$ or $y = x^4$ it is an even function of x because $(-x)^2 = x^2$ and $(-x)^4 = x^4$.

If y is an odd power of x, as in $y = x$ or $y = x^3$ it is an odd function of x because $(-x)^1 = -x$ and $(-x)^3 = -x^3$.

The graph of an even function is symmetric about they y-axis.



The graph of an odd function is symmetric about the origin



Example Determine whether each of the following functions is even, odd, or neither even nor odd.

(a) $f(x) = x^5 + x$ (b) $g(x) = 1 - x^4$ (c) $h(x) = 2x - x^2$

Solution

(a) Domain f = R is symmetric set (about the origin), and:

$$\begin{aligned} f(-x) &= (-x)^5 + (-x) = (-1)^5 x^5 + (-x) \\ &= -x^5 - x = -(x^5 + x) \\ &= -f(x) \end{aligned}$$

Therefore f is an odd function.

(b) Domain g = R is symmetric set (about the origin), and:

$$g(-x) = 1 - (-x)^4 = 1 - x^4 = g(x)$$

Therefore *g* is an even function.

(c) Domain h = R is symmetric set (about the origin), and:

$$h(-x) = 2(-x) - (-x)^2 = -2x - x^2$$

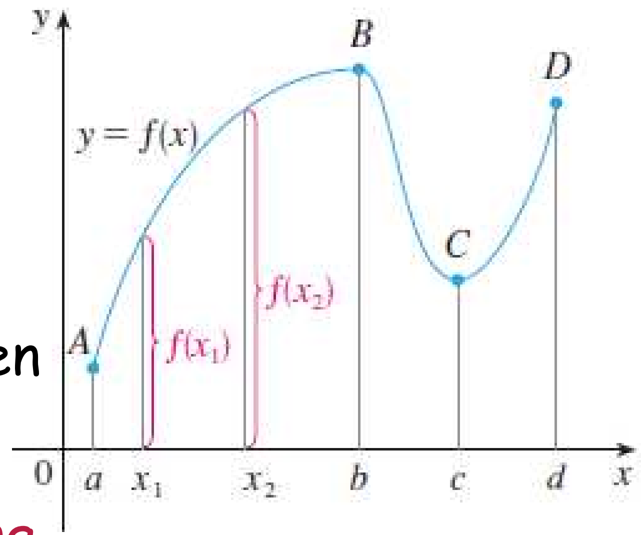
Since $h(-x) \neq h(x)$ and $h(-x) \neq -h(x)$, we conclude that *h* is neither even nor odd.

6) Monotonicity of Functions
(Increasing and Decreasing Functions)

If the graph of a function *climbs* or *rises* as you move from left to right, we say that the function is *increasing*.

If the graph *descends* or *falls* as you move from left to right, the function is *decreasing*.

The graph shown in Figure rises from *A* to *B*, falls from *B* to *C*, and rises again from *C* to *D*, so: The function f is increasing on the interval [a,b], decreasing on [b,c], and increasing again on [c,d].

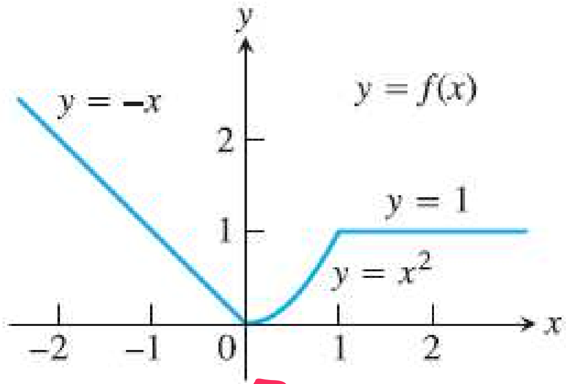


Notice that if x_1 and x_2 are any two numbers between a and b with $x_1 < x_2$, then $f(x_1) < f(x_2)$.

We use this as the defining property of an increasing function.

The function

$$f(x) = \begin{cases} -x, & x < 0 \\ x^2, & 0 \le x \le 1 \\ 1, & x > 1 \end{cases}$$



Monotonic decreasing on $(-\infty,0]$

Monotonic increasing on $[0,1]$

increasing on $[0,\infty)$

5- Algebraic Identities and Algebraic Expressions

Identities Let a and b be real numbers. Then we have:

Identity I

$(a + b)^2 = a^2 + 2ab + b^2$

Identity II

$(a - b)^2 = a^2 - 2ab + b^2$

Identity III

$a^2 - b^2 = (a + b)(a - b)$

Identity IV

$(x + a)(x + b) = x^2 + (a + b)x + ab$

Examples

$$\left(\sqrt{x}+2\right)^2 = \left(\sqrt{x}\right)^2 + 2\left(\sqrt{x}\right)(2) + 2^2 = x + 4\sqrt{x} + 4$$

$$\left(x - \frac{5}{x}\right)^2 = x^2 - 2(x)\left(\frac{5}{x}\right) + \left(\frac{5}{x}\right)^2 = x^2 - 10 + \frac{25}{x^2}$$

$$\left(\sqrt{x^2+1}+7\right)\left(\sqrt{x^2+1}-7\right) = \left(\sqrt{x^2+1}\right)^2 - 7^2 = \left(x^2+1\right) - 49$$
$$= x^2 - 48$$

$$(x+3y)(x+4y) = x^2 + (3y+4y)x + (3y)(4y)$$
$$= x^2 + 7yx + 12y^2$$

Caution <u>In general</u>

$$(a+b)^2 \neq a^2 + b^2$$

<u>Example</u> Simplify the following:

(1) $\dfrac{x^2 - x - 6}{x^2 - 6x + 9}$

(2) $\dfrac{x^2}{x^2 - 1} - 1$

(3) $\dfrac{2}{x^2 + 2x + 1} - \dfrac{1}{x^2 - x - 2}$

(4) $\left(x - y^{-1}\right)^{-1}$

(5) $\dfrac{3 + \dfrac{6}{x}}{x + \dfrac{x}{x+1}}$

<u>Solution</u>

(2) $$\frac{x^2}{x^2-1} - 1 = \frac{x^2 - (x^2 - 1)}{x^2 - 1} = \frac{1}{x^2 - 1}$$

(4) $$\left(x - y^{-1}\right)^{-1} = \left(x - \frac{1}{y}\right)^{-1} = \left(\frac{xy - 1}{y}\right)^{-1} = \frac{y}{xy - 1}$$

$$(5) \qquad \frac{3+\frac{6}{x}}{x+\frac{x}{x+1}} = \frac{\frac{3x+6}{x}}{\frac{x(x+1)+x}{x+1}}$$

$$= \frac{\frac{3x+6}{x}}{\frac{x^2+2x}{x+1}}$$

$$= \frac{3(x+2)}{x} \cdot \frac{x+1}{x(x+2)}$$

$$= \frac{3(x+1)}{x^2}$$

FAQ What is expected if we are asked to simplify an expression? For example, in (5), can we give $(3x + 3)/x^2$ as the answer?

Answer_ There is no definite rule to tell which expression is simpler. For (5), both $3(x + 1)/x^2$ and $(3x + 3)/x^2$ are acceptable. Use your own judgment.

Exercises

1. Expand the following:

(a) $(2x+3)^2$ (b) $(3x-y)^2$

(c) $(x+3y)(x-3y)$ (d) $(x+3y)(x+4y)$

(e) $\left(2\sqrt{x}-3\right)^2$ (f) $\left(\sqrt{x}+5\right)\left(\sqrt{x}-5\right)$

2. Factorize the following:

(a) $x^2-7x+12$ (b) x^2+x-6

(c) $x^2+8x+16$ (d) $9x^2+9x+2$

(e) $9x^2-6x+1$ (f) $5x^2-5$

(g) $3x^2-18x+27$ (h) $2x^2-12x+16$

3. Simplify the following:

(a) $\dfrac{x^2-x-6}{x^2-7x+12}$ (b) $\dfrac{x^2+3x-4}{2-x-x^2}$

(c) $\dfrac{2x}{x^2-1} \div \dfrac{4x^2+4x}{x-1}$ (d) $\dfrac{\dfrac{1}{x+h}-\dfrac{1}{x}}{h}$

6- Solving Linear Equations

A *linear equation* in one (real) unknown x is an equation that can be written in the form

$$\boxed{ax + b = 0}$$

where a and b are constants with $a \neq 0$

(in this course, we consider real numbers only; thus a "*constant*" means a real number that is fixed or given).

Definition A *solution* to the Equation $ax+b=0$ is a real number x_0 such that $a\,x_0 + b = 0$

Example The equation $2x + 3 = 0$ has exactly one solution, namely $x_0 = -3/2$

Properties of real numbers

Let *a*, *b* and *c* be real numbers. Then we have:

$$a=b \quad \Longrightarrow \quad a+c=b+c$$

means that if *a* = *b*, then *a+c* = *b+c*.

Also

$$a=b \quad \Longleftarrow \quad a+c=b+c$$

means that if *a+c* = *b+c* , then *a* = *b*.

i.e.

$$a=b \quad \Leftrightarrow \quad a+c=b+c$$ ✓

$$a=b \ \text{ iff } \ a+c=b+c$$ ✓

$$a=b \quad \text{if and only if} \quad a+c=b+c$$ ✓

Remark $\Longrightarrow$ is the symbol for "*implies*".

$\Longleftrightarrow$ is the symbol for $\Longrightarrow$ and $\Longleftarrow$

$\Longleftrightarrow$ **If and only if**

Or iff

Also

$$\boxed{a = b \iff ac = bc \quad , \quad c \neq 0}$$

Example Solve the following equations for x.

(1) $3x - 5 = 2(7 - x)$

(2) $a(b + x) = c - dx$, where a, b, c and d are real numbers with $a + d \neq 0$.

Solution

$$\begin{aligned} 3x - 5 &= 2(7 - x) \\ 3x - 5 &= 14 - 2x \\ 3x + 2x &= 14 + 5 \\ 5x &= 19 \\ x &= \frac{19}{5}. \end{aligned}$$

$$\boxed{\text{The solution is } \frac{19}{5}.}$$

Exercise

Solve the following equations for x.

(a) $2(x + 4) = 7x + 2$

(b) $\dfrac{5x + 3}{2} - 5 = \dfrac{5x - 4}{4}$

(c) $(a + b)x + x^2 = (x + b)^2$

(d) $\dfrac{x}{a} - \dfrac{x}{b} = c$

where a, b and c are constants with $a \neq b$.

7- Solving Quadratic Equations

A *quadratic equation* (in one unknown) is an equation that can be written in the form

$$ax^2 + bx + c = 0 \quad , \quad a \neq 0$$

where a , B , and c are constants

I- Factorization Method

Solve $x^2 + 2x - 15 = 0$.

$(x + 5)(x - 3) = 0$.

Hence $x = -5$ or $x = 3$.

II- Quadratic Formula

$$\alpha, \beta = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$ax^2 + bx + c = (x - \alpha)(x - \beta) = 0$$

III- Completing the Square

$$x^2 \pm Bx + c = \left[x \pm \frac{B}{2}\right]^2 - \left(\frac{B}{2}\right)^2 + c$$

I- Factorization Method

Solve $x^2 + 2x - 15 = 0$.

$(x+5)(x-3) = 0$.

Hence $x = -5$ or $x = 3$.

II- Quadratic Formula

$$\alpha, \beta = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$ax^2 + bx + c = (x-\alpha)(x-\beta) = 0$$

III- Completing the Square

$$x^2 \pm Bx + c = \left[x \pm \frac{B}{2}\right]^2 - \left(\frac{B}{2}\right)^2 + c$$

$$b^2 - 4ac$$

> 0

The equation has distinct real roots α and β,

$= 0$

The equation has a double root α, given by $\boxed{\alpha = \frac{-b}{2a}}$

< 0

The equation has no root.

Example Solve the following quadratic equations

$$(1) \quad 2x^2 - 9x + 10 = 0$$

$$(2) \quad x^2 + 2x + 3 = 0$$

Solution

(1) Using the quadratic formula, we see that the equation has two solutions given by

$$x = \frac{9 \pm \sqrt{(-9)^2 - 4(2)(10)}}{2(2)} = \frac{9 \pm 1}{4}.$$

Thus the solutions are $\frac{5}{2}$ and 2.

(2) Since $2^2 - 4(1)(3) = -8 < 0$, the equation has no solutions.

Exercise

1. Solve the following equations.

   (a) $4x - 4x^2 = 0$
   (b) $2 + x - 3x^2 = 0$
   (c) $4x(x - 4) = x - 15$
   (d) $x^2 + 2\sqrt{2}x + 2 = 0$
   (e) $x^2 + 2\sqrt{2}x + 3 = 0$
   (f) $x^3 - 7x^2 + 3x = 0$

2. Find the value(s) of k such that the equation $x^2 + kx + (k + 3) = 0$ has only one solution.

3. Find the positive number such that sum of the number and its square is 210.

Math 3 – SMA 214

CHAPTER 1

Some Topics of Linear Algebra

Idea

Many difficult problems can be handled easily once relevant information is organized in a certain way.

This text aims to teach you how to organize information in cases where certain mathematical structures are present.

Linear algebra is, in general, the study of those structures. Namely

Linear algebra is the study of vectors and linear functions.

1- Matrices and Determinants

A matrix is a rectangular array of numbers enclosed by a pair of bracket.

$$A = \begin{bmatrix} 2 & 3 & 7 \\ 1 & -1 & 5 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 1 & 4 \\ 4 & 7 & 6 \end{bmatrix}$$

Both A and B are examples of matrix.

Why matrix?

Consider the following set of equations:

$$\begin{cases} x + y = 7, \\ 3x - y = 5. \end{cases}$$

It is easy to show that $x = 3$ and $y = 4$.

How about solving

$$\begin{cases} x + y - 2z = 7, \\ 2x - y - 4z = 2, \\ -5x + 4y + 10z = 1, \\ 3x - y - 6z = 5. \end{cases}$$

Matrices can help…

In the matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

- numbers a_{ij} are called *elements (or entries)*. First subscript indicates the row;

Second subscript indicates the column. The matrix consists of mn elements

- It is called "the $m \times n$ matrix $A = [aij]$" or simply "the matrix A " if number of rows and columns are understood.

2. Square matrices

- When $m = n$, i.e.,

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & & \ddots & \\ a_{n1} & a_{n2} & & a_{nn} \end{bmatrix}$$

- A is called a "*square matrix of order* n" or "n-square matrix"
- elements $a_{11}, a_{22}, a_{33}, \ldots, a_{nn}$ called *diagonal elements*.
- $\sum_{i=1}^{n} a_{ii} = a_{11} + a_{22} + \ldots + a_{nn}$ is called the *trace* of A.

3. Equal matrices

- Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are said to be equal $(A = B)$ iff each element of A is equal to the corresponding element of B,

i.e., $a_{ij} = b_{ij}$ for $1 \le i \le m,\ 1 \le j \le n$.

- iff pronouns "if and only if"

if $A = B$, it implies $a_{ij} = b_{ij}$ for $1 \le i \le m,\ 1 \le j \le n$;

if $a_{ij} = b_{ij}$ for $1 \le i \le m,\ 1 \le j \le n$, it implies $A = B$.

Example: $A = \begin{bmatrix} 1 & 0 \\ -4 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Given that $A = B$, find a, b, c and d.

if $A = B$, then $a = 1$, $b = 0$, $c = -4$ and $d = 2$.

Zero matrices

- Every element of a matrix is zero, it is called a zero matrix, i.e.,

$$A = \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \vdots & 0 \\ \vdots & \vdots & O & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

4. Operations of matrices

Sums of matrices

- If $A = [a_{ij}]$ and $B = [b_{ij}]$ are $m \times n$ matrices, then $A + B$ is defined as a matrix $C = A + B$, where

$$C = [c_{ij}],\ c_{ij} = a_{ij} + b_{ij} \text{ for } 1 \le i \le m,\ 1 \le j \le n.$$

Example: if $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 & 0 \\ -1 & 2 & 5 \end{bmatrix}$

Evaluate $A + B$ and $A - B$.

$$A + B = \begin{bmatrix} 1+2 & 2+3 & 3+0 \\ 0+(-1) & 1+2 & 4+5 \end{bmatrix} = \begin{bmatrix} 3 & 5 & 3 \\ -1 & 3 & 9 \end{bmatrix}$$

$$A - B = \begin{bmatrix} 1-2 & 2-3 & 3-0 \\ 0-(-1) & 1-2 & 4-5 \end{bmatrix} = \begin{bmatrix} -1 & -1 & 3 \\ 1 & -1 & -1 \end{bmatrix}$$

- Two matrices of the same order are said to be conformable for addition or subtraction.
- Two matrices of different orders cannot be added or subtracted, e.g.,

$$\begin{bmatrix} 2 & 3 & 7 \\ 1 & -1 & 5 \end{bmatrix} \qquad \begin{bmatrix} 1 & 3 & 1 \\ 2 & 1 & 4 \\ 4 & 7 & 6 \end{bmatrix}$$

are NOT conformable for addition or subtraction.

5. Scalar multiplication

- Let λ be any scalar and $A = [a_{ij}]$ is an $m \times n$ matrix. Then $\lambda A = [\lambda a_{ij}]$ for $1 \le i \le m$, $1 \le j \le n$, i.e., each element in A is multiplied by λ.

Example: Evaluate $3A$, where $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \end{bmatrix}$

$$3A = \begin{bmatrix} 3\times 1 & 3\times 2 & 3\times 3 \\ 3\times 0 & 3\times 1 & 3\times 4 \end{bmatrix} = \begin{bmatrix} 3 & 6 & 9 \\ 0 & 3 & 12 \end{bmatrix}$$

- In particular, $\lambda = -1$, i.e., $-A = [-a_{ij}]$. It's called the *negative* of A.

Note: $A - A = 0$ is a zero matrix

Properties

Matrices A, B and C are conformable,

- $A + B = B + A$ (commutative law)
- $A + (B + C) = (A + B) + C$ (associative law)
- $\lambda(A + B) = \lambda A + \lambda B$, where λ is a scalar (distributive law)

Can you prove them?

Properties

Example: Prove $\lambda(A+B) = \lambda A + \lambda B$.

Proof

Let $C = A + B$, so $c_{ij} = a_{ij} + b_{ij}$.

Consider $\lambda c_{ij} = \lambda\,(a_{ij} + b_{ij}) = \lambda a_{ij} + \lambda b_{ij}$, we have, $\lambda C = \lambda A + \lambda B$.

Since $\lambda C = \lambda(A+B)$, so

$$\lambda(A+B) = \lambda A + \lambda B$$

6. Matrix multiplication

▪If $A=[a_{ij}]$ is a $m\times p$ matrix and $B=[b_{ij}]$ is a $p\times n$ matrix, then AB is defined as a $m\times n$ matrix $C=AB$, where $C=[c_{ij}]$ with

$$c_{ij}=\sum_{k=1}^{p} a_{ik}b_{kj}=a_{i1}b_{1j}+a_{i2}b_{2j}+\ldots+a_{ip}b_{pj} \quad \text{for } 1\le i\le m,\ 1\le j\le n.$$

Example: $A=\begin{bmatrix}1 & 2 & 3\\ 0 & 1 & 4\end{bmatrix}$, $B=\begin{bmatrix}-1 & 2\\ 2 & 3\\ 5 & 0\end{bmatrix}$ and $C=AB$. Evaluate c_{21}.

$$\begin{bmatrix}1 & 2 & 3\\ 0 & 1 & 4\end{bmatrix}\begin{bmatrix}-1 & 2\\ 2 & 3\\ 5 & 0\end{bmatrix}$$

$$c_{21}=0\times(-1)+1\times 2+4\times 5=22$$

Example: $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 2 \\ 2 & 3 \\ 5 & 0 \end{bmatrix}$, Evaluate $C = AB$.

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 2 & 3 \\ 5 & 0 \end{bmatrix} \Rightarrow \begin{cases} c_{11} = 1 \times (-1) + 2 \times 2 + 3 \times 5 = 18 \\ c_{12} = 1 \times 2 + 2 \times 3 + 3 \times 0 = 8 \\ c_{21} = 0 \times (-1) + 1 \times 2 + 4 \times 5 = 22 \\ c_{22} = 0 \times 2 + 1 \times 3 + 4 \times 0 = 3 \end{cases}$$

$$C = AB = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 2 & 3 \\ 5 & 0 \end{bmatrix} = \begin{bmatrix} 18 & 8 \\ 22 & 3 \end{bmatrix}$$

- In particular, A is a $1 \times m$ matrix and B is a $m \times 1$ matrix, i.e.,

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1m} \end{bmatrix} \qquad B = \begin{bmatrix} b_{11} \\ b_{21} \\ \vdots \\ b_{m1} \end{bmatrix}$$

then $C = AB$ is a scalar.

$$\boxed{C = \sum_{k=1}^{m} a_{1k} b_{k1} = a_{11} b_{11} + a_{12} b_{21} + \dots + a_{1m} b_{m1}}$$

- BUT BA is a $m \times m$ matrix!

$$BA = \begin{bmatrix} b_{11} \\ b_{21} \\ \vdots \\ b_{m1} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1m} \end{bmatrix} = \begin{bmatrix} b_{11}a_{11} & b_{11}a_{12} & \dots & b_{11}a_{1m} \\ b_{21}a_{11} & b_{21}a_{12} & & b_{21}a_{1m} \\ \vdots & & \ddots & \\ b_{m1}a_{11} & b_{m1}a_{12} & & b_{m1}a_{1m} \end{bmatrix}$$

- So $AB \neq BA$ in general !

Properties

Matrices A, B and C are conformable,

- $A(B + C) = AB + AC$
- $(A + B)C = AC + BC$
- $A(BC) = (AB)\,C$

However

- $AB \neq BA$ in general
- $AB = 0$ NOT necessarily imply $A = 0$ or $B = 0$
- $AB = AC$ NOT necessarily imply $B = C$

Example: Prove $A(B + C) = AB + AC$

where A, B and C are n-square matrices

Proof

Let $X = B + C$, so $x_{ij} = b_{ij} + c_{ij}$. Let $Y = AX$, then

$$y_{ij} = \sum_{k=1}^{n} a_{ik} x_{kj} = \sum_{k=1}^{n} a_{ik}(b_{kj} + c_{kj})$$

$$= \sum_{k=1}^{n} (a_{ik} b_{kj} + a_{ik} c_{kj}) = \sum_{k=1}^{n} a_{ik} b_{kj} + \sum_{k=1}^{n} a_{ik} c_{kj}$$

So $Y = AB + AC$; therefore, $A(B + C) = AB + AC$

7. Types of matrices

- Identity matrix
- The inverse of a matrix
- The transpose of a matrix
- Symmetric matrix
- Orthogonal matrix

Identity matrix

- A square matrix whose elements $a_{ij} = 0$, for $i > j$ is called upper triangular, i.e.,

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & & a_{2n} \\ \vdots & & \ddots & \\ 0 & 0 & & a_{nn} \end{bmatrix}$$

- A square matrix whose elements $a_{ij} = 0$, for $i < j$ is called lower triangular, i.e.,

$$\begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & & 0 \\ \vdots & & \ddots & \\ a_{n1} & a_{n2} & & a_{nn} \end{bmatrix}$$

- Both upper and lower triangular,

i.e., $a_{ij} = 0$, for $i \neq j$, i.e.,

$$D = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & a_{nn} \end{bmatrix}$$

is called a diagonal matrix, simply

$$D = \mathrm{diag}[a_{11}, a_{22}, \ldots, a_{nn}]$$

In particular, $a_{11} = a_{22} = \ldots = a_{nn} = 1$,

The matrix I is called identity matrix.

Properties: AI = IA = A

Examples of identity matrices: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Special square matrix

- $AB \neq BA$ in general. However, if two square matrices A and B such that $AB = BA$, then

A and B are said to be commute.

Can you suggest two matrices that must commute with a square matrix A?

Ans: A itself, the identity matrix, ..

- If A and B such that $AB = -BA$, then

A and B are said to be anti-commute.

The inverse of a matrix

- If matrices A and B such that $AB = BA = I$, then B is called the inverse of A (symbol: A^{-1}); and A is called the inverse of B (symbol: B^{-1}).

Example:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 3 \\ 1 & 2 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 6 & -2 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

Show B is the the inverse of matrix A.

Ans: Note that

$$AB = BA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Can you show the details?

The transpose of a matrix

- The matrix obtained by interchanging the rows and columns of a matrix A is called the transpose of A (write A^T).

Example: $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$

The transpose of A is $A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$

- For a matrix $A = [a_{ij}]$, its transpose

$A^T = [b_{ij}]$, where $b_{ij} = a_{ji}$.

Symmetric matrix

- A matrix A such that $A^T = A$ is called symmetric,

  i.e., $a_{ji} = a_{ij}$ for all i and j.

- $A + A^T$ must be symmetric. Why?

Example: $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & -5 \\ 3 & -5 & 6 \end{bmatrix}$ is symmetric.

- A matrix A such that $A^T = -A$ is called skew-symmetric,

  i.e., $a_{ji} = -a_{ij}$ for all i and j.

- $A - A^T$ must be skew-symmetric. Why?

Orthogonal matrix

- A matrix A is called orthogonal if $AA^T = A^TA = I$, i.e., $A^T = A^{-1}$

Example: prove that

$$A = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{6} & -1/\sqrt{2} \\ 1/\sqrt{3} & -2/\sqrt{6} & 0 \\ 1/\sqrt{3} & 1/\sqrt{6} & 1/\sqrt{2} \end{bmatrix}$$

is orthogonal.

Since, $A^T = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{6} & -2/\sqrt{6} & 1/\sqrt{6} \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$. Hence, $AA^T = A^TA = I.$

Can you show the details?

Properties of matrix

- $(AB)^{-1} = B^{-1}A^{-1}$
- $(A^T)^T = A$ and $(\lambda A)^T = \lambda A^T$
- $(A + B)^T = A^T + B^T$
- $(AB)^T = B^T A^T$

Example: Prove $(AB)^{-1} = B^{-1}A^{-1}$.

Since $(AB)\,(B^{-1}A^{-1}) = A(B\,B^{-1})A^{-1} = I$ and

$(B^{-1}A^{-1})\,(AB) = B^{-1}(A^{-1}\,A)B = I$.

Therefore, $B^{-1}A^{-1}$ is the inverse of matrix AB.

TYPES OF MATRICES

NAME	DESCRIPTION	EXAMPLE
Rectangular matrix	No. of rows is not equal to no. of columns	$\begin{bmatrix} 6 & 2 & -1 \\ -2 & 0 & 5 \end{bmatrix}$
Square matrix	No. of rows is equal to no. of columns	$\begin{bmatrix} 2 & -1 & 3 \\ -2 & 0 & 1 \\ 1 & 2 & 4 \end{bmatrix}$
Diagonal matrix	Non-zero element in principal diagonal and zero in all other positions	$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 7 \end{bmatrix}$
Scalar matrix	Diagonal matrix in which all the elements on principal diagonal and same	$\begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

NAME	DESCRIPTION	EXAMPLE
Row matrix	A matrix with only 1 row	$\begin{bmatrix} 3 & 2 & 1 - 4 \end{bmatrix}$
Column matrix	A matrix with only I column	$\begin{bmatrix} 2 \\ 3 \end{bmatrix}$
Identity matrix	Diagonal matrix having each diagonal element equal to one (I)	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
Zero matrix	A matrix with all zero entries	$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

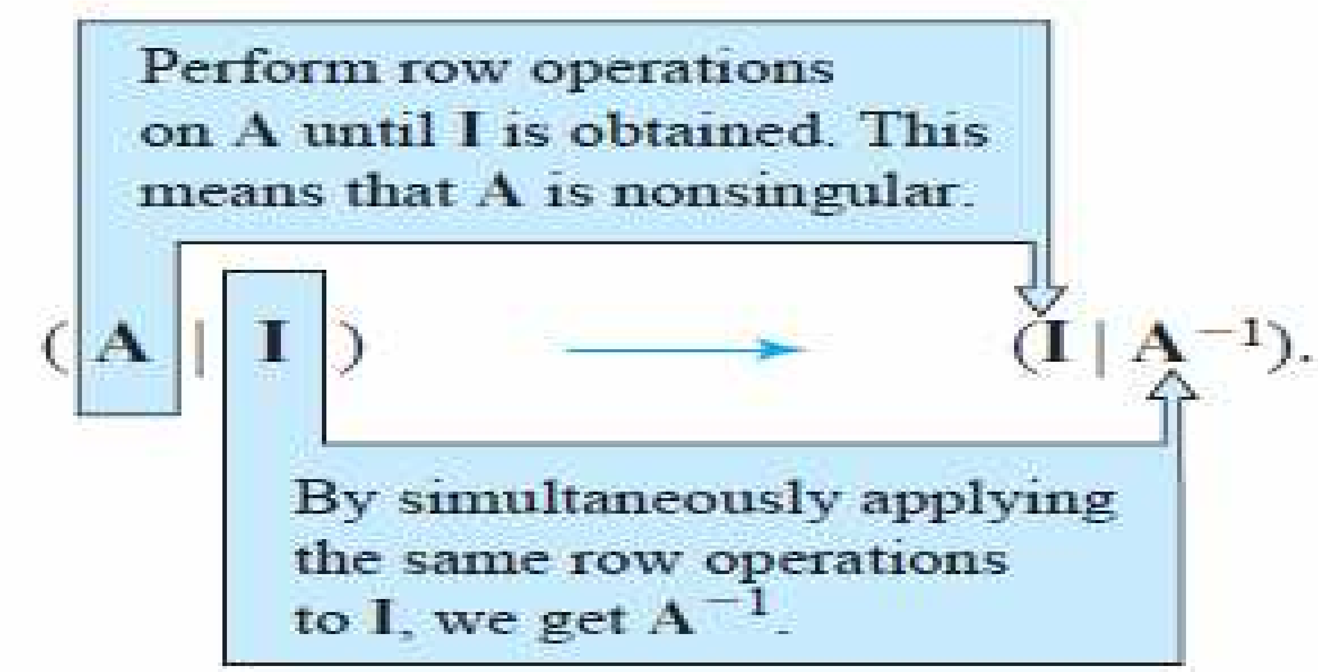
NAME	DESCRIPTION	EXAMPLE
Upper Triangular matrix	Square matrix having all the entries zero below the principal diagonal	$\begin{bmatrix} 2 & 5 & 3 \\ 0 & 4 & 6 \\ 0 & 0 & 7 \end{bmatrix}$
Lower Triangular matrix	Square matrix having all the entries zero above the principal diagonal	$\begin{bmatrix} 2 & 0 & 0 \\ 5 & 4 & 0 \\ 6 & 3 & 7 \end{bmatrix}$

8. Row Operations

Symbol	Meaning
R_{ij}	*Interchange rows i and j.*
cR_i	*Multiply the ith row by the nonzero constant c.*
cR_i+R_j	*Multiply the ith row by c and add to the jth row.*

Finding A^{-1} Using Elementary Row Operations

The procedure for finding A^{-1} is outlined in the following diagram:



EXAMPLE (Inverse by Elementary Row Operations)

Find the **multiplicative inverse** for

$$\mathbf{A} = \begin{pmatrix} 2 & 0 & 1 \\ -2 & 3 & 4 \\ -5 & 5 & 6 \end{pmatrix}.$$

SOLUTION

$$\left(\begin{array}{ccc|ccc} 2 & 0 & 1 & 1 & 0 & 0 \\ -2 & 3 & 4 & 0 & 1 & 0 \\ -5 & 5 & 6 & 0 & 0 & 1 \end{array}\right) \xrightarrow{\frac{1}{2}R_1} \left(\begin{array}{ccc|ccc} 1 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ -2 & 3 & 4 & 0 & 1 & 0 \\ -5 & 5 & 6 & 0 & 0 & 1 \end{array}\right) \xrightarrow[5R_1 + R_3]{2R_1 + R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 3 & 5 & 1 & 1 & 0 \\ 0 & 5 & \frac{17}{2} & \frac{5}{2} & 0 & 1 \end{array}\right)$$

$$\xrightarrow[\frac{1}{5}R_3]{\frac{1}{3}R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & \frac{5}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 1 & \frac{17}{10} & \frac{1}{2} & 0 & \frac{1}{5} \end{array}\right) \xrightarrow{-R_2 + R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & \frac{5}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{30} & \frac{1}{6} & -\frac{1}{3} & \frac{1}{5} \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & \frac{5}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{30} & \frac{1}{6} & -\frac{1}{3} & \frac{1}{5} \end{array}\right)$$

$$\xrightarrow{30R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & \frac{5}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 1 & 5 & -10 & 6 \end{array}\right) \xrightarrow[-\frac{5}{3}R_3 + R_2]{-\frac{1}{2}R_3 + R_1} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 5 & -3 \\ 0 & 1 & 0 & -8 & 17 & -10 \\ 0 & 0 & 1 & 5 & -10 & 6 \end{array}\right).$$

Because I appears to the left of the vertical line, we conclude that the matrix to the right of the line is A^{-1}

$$\boxed{\mathbf{A}^{-1} = \begin{pmatrix} -2 & 5 & -3 \\ -8 & 17 & -10 \\ 5 & -10 & 6 \end{pmatrix}}$$

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$$\left[\, A \mid I \,\right] \xrightarrow{\text{"Elementary Row Operations"}} \left[\, I \mid A^{-1} \,\right]$$

$$\left[\, 8 \mid 1 \,\right] \xrightarrow{\text{"Divide by 8"}} \left[\, 1 \mid \tfrac{1}{8} \,\right]$$

$$\left[\, A \mid I \,\right] \rightarrow \left[\, A^{-1}A \mid A^{-1}I \,\right] \rightarrow \left[\, I \mid A^{-1} \,\right]$$

9. Row echelon form, and reduced row echelon form

A matrix is in row echelon form if its entries satisfy the following conditions

1. The first nonzero entry in each row is a 1 (called a leading 1).
2. Each leading 1 comes in a column to the right of the leading 1s in rows above it.
3. All rows of all 0s come at the bottom of the matrix.

A matrix in row echelon form is in reduced row echelon form when every column that has a leading 1 has zeros in every position above and below its leading 1.

Example *Row-Echelon Form*

Determine whether each matrix is in row-echelon form. If it is, determine whether the matrix is in reduced row-echelon form.

A- $\begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -2 \end{bmatrix}$

B- $\begin{bmatrix} 1 & 2 & -1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & -4 \end{bmatrix}$

C- $\begin{bmatrix} 1 & -5 & 2 & -1 & 3 \\ 0 & 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

D- $\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

E- $\begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & 2 & 1 & -1 \\ 0 & 0 & 1 & -3 \end{bmatrix}$

F- $\begin{bmatrix} 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Solution:

- The matrices in (A), (C), (D), and (F) are in row-echelon form.
- The matrices in (D) and (F) are in *reduced* row-echelon form because every column that has a leading 1 has zeros in every position above and below its leading 1.
- The matrix in (B) is not in row-echelon form because the row of all zeros does not occur at the bottom of the matrix.
- The matrix in (E) is not in row-echelon form because the first nonzero entry in Row 2 is not a leading 1.

Example **Find the inverse (if exists)**

$$\begin{bmatrix} -7 & 2 & 9 \\ 2 & -4 & -6 \\ 3 & 5 & 2 \end{bmatrix}$$

Matrix Multiplication with Excel

Use the EXCEL MMULT function to multiply the matrices:

$$A = \begin{pmatrix} 1 & -1 & 2 \\ 2 & 1 & 3 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 1 \\ 2 & 3 \\ 1 & 2 \end{pmatrix}$$

Step 1 – Enter matrix A into cells B1:D2 and matrix B into cells B4:C6.

Step 2 – Select the output range (B8:C9) into which the product will be computed.

Step 3 – In the upper left-hand corner (B8) of this selected output range type the formula:

= MMULT(B1:D2,B4:C6).

Excel

Step 4 - Press **CONTROL SHIFT ENTER** (not just enter)

	A	B	C	D
1	**Matrix A**	1	-1	2
2		2	1	3
3				
4	**Matrix B**	1	1	
5		2	3	
6		1	2	
7				
8	**A B =**	1	2	
9		7	11	

10. Determinants

Determinant of order 2

Consider a 2×2 matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

- Determinant of A, denoted $|A|$, is a number and can be evaluated by

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

- easy to remember (for order 2 only)..

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = +a_{11}a_{22} - a_{12}a_{21}$$

Example: Evaluate the determinant: $\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$

$$\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 1 \times 4 - 2 \times 3 = -2$$

The following properties are true for determinants of *any* order.

1. If every element of a row (column) is zero, e.g., $\begin{vmatrix} 1 & 2 \\ 0 & 0 \end{vmatrix} = 1 \times 0 - 2 \times 0 = 0$, then $|A| = 0$.

2. $|A^T| = |A|$

   determinant of a matrix = that of its transpose

3. $|AB| = |A||B|$

Example: Show that the determinant of any orthogonal matrix is either $+1$ or -1.

Solution

For any orthogonal matrix, $A\,A^T = I$.

Since $|AA^T| = |A||A^T| = 1$ and $|A^T| = |A|$, so $|A|^2 = 1$ or $|A| = \pm 1$.

For any 2x2 matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

Its inverse can be written as $\boxed{A^{-1} = \dfrac{1}{|A|}\begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}}$

Example: Find the inverse of $A = \begin{bmatrix} -1 & 0 \\ 1 & 2 \end{bmatrix}$

The determinant of A is -2

Hence, the inverse of A is $A^{-1} = \dfrac{-1}{2}\begin{bmatrix} 2 & 0 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 1/2 & 1/2 \end{bmatrix}$

How to find an inverse for a 3x3 matrix?

Determinants of order 3

Consider an example:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Its determinant can be obtained by:

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 3\begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} - 6\begin{vmatrix} 1 & 2 \\ 7 & 8 \end{vmatrix} + 9\begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix}$$

$$= 3(-3) - 6(-6) + 9(-3) = 0$$

You are encouraged to find the determinant by using other rows or columns

Inverse of a 3×3 matrix

Each element a_{ij} of a square matrix has a *minor* which is the value of the determinant obtained from the matrix after eliminating the *i*th row and *j*th column to which the element is common.

The cofactor of element a_{ij} is then given as the minor of a_{ij} multiplied by

$$(-1)^{i+j}$$

Example

If $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 1 & 0 & 6 \end{bmatrix}$

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

The cofactor for each element of matrix A:

$$A_{11} = \begin{vmatrix} 4 & 5 \\ 0 & 6 \end{vmatrix} = 24 \qquad A_{12} = -\begin{vmatrix} 0 & 5 \\ 1 & 6 \end{vmatrix} = 5 \qquad A_{13} = \begin{vmatrix} 0 & 4 \\ 1 & 0 \end{vmatrix} = -4$$

$$A_{21} = \begin{vmatrix} 2 & 3 \\ 0 & 6 \end{vmatrix} = -12 \qquad A_{22} = \begin{vmatrix} 1 & 3 \\ 1 & 6 \end{vmatrix} = 3 \qquad A_{23} = -\begin{vmatrix} 1 & 2 \\ 1 & 0 \end{vmatrix} = 2$$

$$A_{31} = \begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = -2 \qquad A_{32} = -\begin{vmatrix} 1 & 3 \\ 0 & 5 \end{vmatrix} = -5 \qquad A_{33} = \begin{vmatrix} 1 & 2 \\ 0 & 4 \end{vmatrix} = 4$$

Adjoint of a square matrix

Let square matrix C be constructed from the square matrix A where the elements of C are the respective cofactors of the elements of A so that if:

$$\mathbf{A} = \left(a_{ij}\right) \text{ and } A_{ij} \text{ is the cofactor of } a_{ij} \text{ then } \mathbf{C} = \left(A_{ij}\right)$$

Then the transpose of C is called the adjoint of A, denoted by adjA (or Adj (A)).

So, the matrix C of $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 1 & 0 & 6 \end{bmatrix}$ is then given by:

$$C = \begin{bmatrix} 24 & 5 & -4 \\ -12 & 3 & 2 \\ -2 & -5 & 4 \end{bmatrix} \Rightarrow \operatorname{adj} A = \begin{bmatrix} 24 & 5 & -4 \\ -12 & 3 & 2 \\ -2 & -5 & 4 \end{bmatrix}^T = \begin{bmatrix} 24 & -12 & -2 \\ 5 & 3 & -5 \\ -4 & 2 & 4 \end{bmatrix}$$

Inverse matrix of $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 1 & 0 & 6 \end{bmatrix}$ is given by:

$$A^{-1} = \frac{1}{|A|} \operatorname{adj}(A) = \frac{1}{|A|} \begin{bmatrix} 24 & 5 & -4 \\ -12 & 3 & 2 \\ -2 & -5 & 4 \end{bmatrix}^T = \frac{1}{22} \begin{bmatrix} 24 & -12 & -2 \\ 5 & 3 & -5 \\ -4 & 2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 12/11 & -6/11 & -1/11 \\ 5/22 & 3/22 & -5/22 \\ -2/11 & 1/11 & 2/11 \end{bmatrix}$$

Example: An example in which cancellation is not valid

Show that $AC = BC$ but $A \neq B$

Solution: Let

$$A = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 4 \\ 2 & 3 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & -2 \\ -1 & 2 \end{bmatrix}$$

C is **noninvertible**, (*i.e.*, i.e., C^{-1} dos not exists)

$$AC = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix}$$

$$BC = \begin{bmatrix} 2 & 4 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix}$$

So $AC = BC$ But $A \neq B$

11. vectors

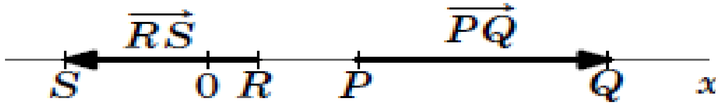
Definition. A (nonzero) **vector** is a directed line segment drawn from a point P (called its **initial point**) to a point Q (called its **terminal point**), with P and Q being distinct points.

The vector is denoted by $\overrightarrow{PQ}$.

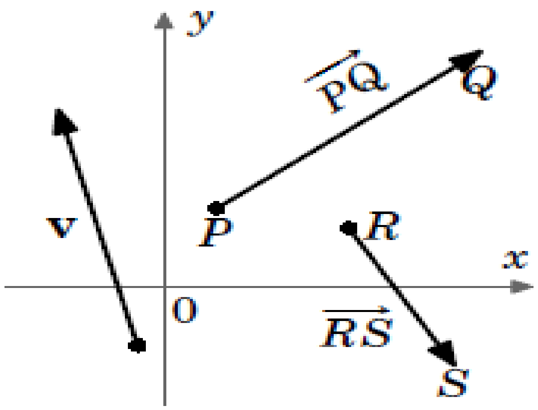
Its **magnitude** is the length of the line segment, denoted by $\|\overrightarrow{PQ}\|$

and its **direction** is the same as that of the directed line segment.

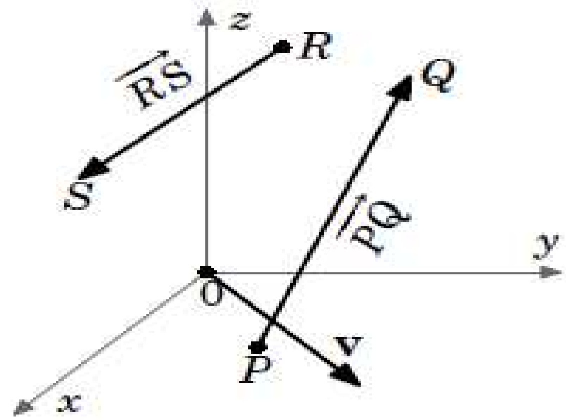
The **zero vector** is just a point, and it is denoted by $\vec{0}$.



(a) One dimension



(b) Two dimensions



(c) Three dimensions

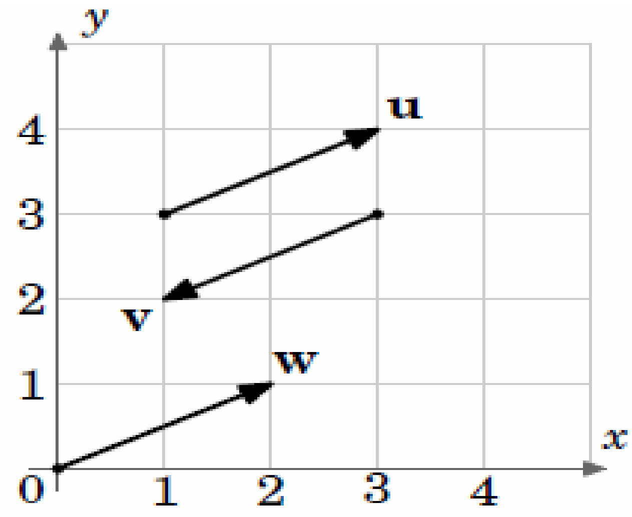
Definition Two nonzero vectors are **equal** if they have the same magnitude and the same direction.

Any vector with zero magnitude is equal to the zero vector.

By this definition, vectors with the same magnitude and direction but with different initial points would be equal.

For example, the vectors **u**, **v** and **w** all have the same magnitude $\sqrt{5}$ (by the Pythagorean Theorem).

And we see that **u** and **w** are parallel, since they lie on lines having the same slope 1/2 , and they point in the same direction. So **u** = **w**, even though they have different initial points.



We also see that **v** is parallel to **u** but points in the opposite direction. So **u** ≠ **v**.

So we can see that there are an infinite number of vectors for a given magnitude and direction, those vectors all being equal and differing only by their initial and terminal points.

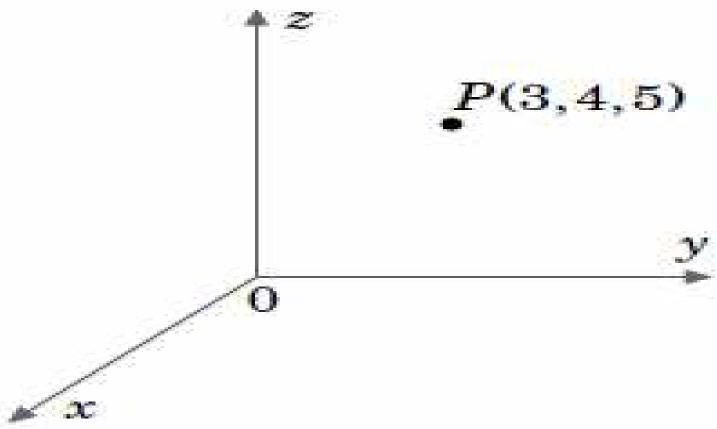
Is there a single vector which we can choose to represent all those equal vectors?

The answer is yes, and is suggested by the vector **w** in the previous figure.

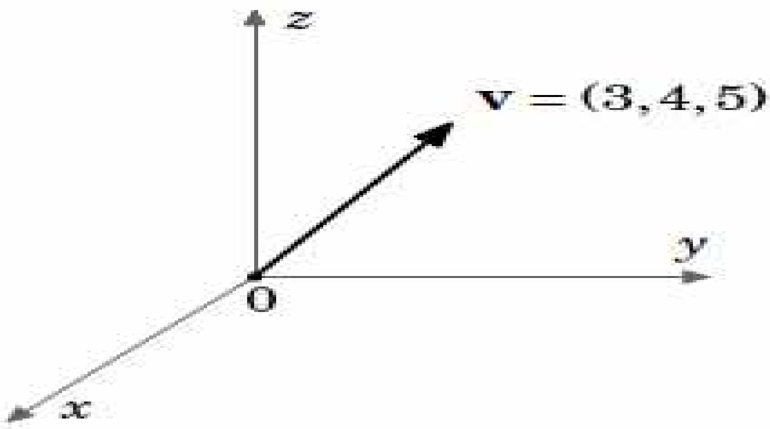
Unless otherwise indicated, when speaking of “the vector” with a given magnitude and direction, we will mean the one whose initial point is at the origin of the coordinate system.

Example . Let **v** be the vector in R^3 whose initial point is at the origin and whose terminal point is (3,4,5).

Though the *point* (3,4,5) and the vector **v** are different objects, it is convenient to write **v** = (3,4,5). When doing this, it is understood that the initial point of **v** is at the origin (0,0,0) and the terminal point is (3,4,5).



(a) The point (3,4,5)



(b) The vector (3,4,5)

The point-vector correspondence provides an easy way to check if two vectors are equal, without having to determine their magnitude and direction.

Similar to seeing if two points are the same, you are now seeing if the terminal points of vectors starting at the origin are the same.

For each vector, find the (unique!) vector it equals whose initial point is the origin. Then compare the coordinates of the terminal points of these "new" vectors:

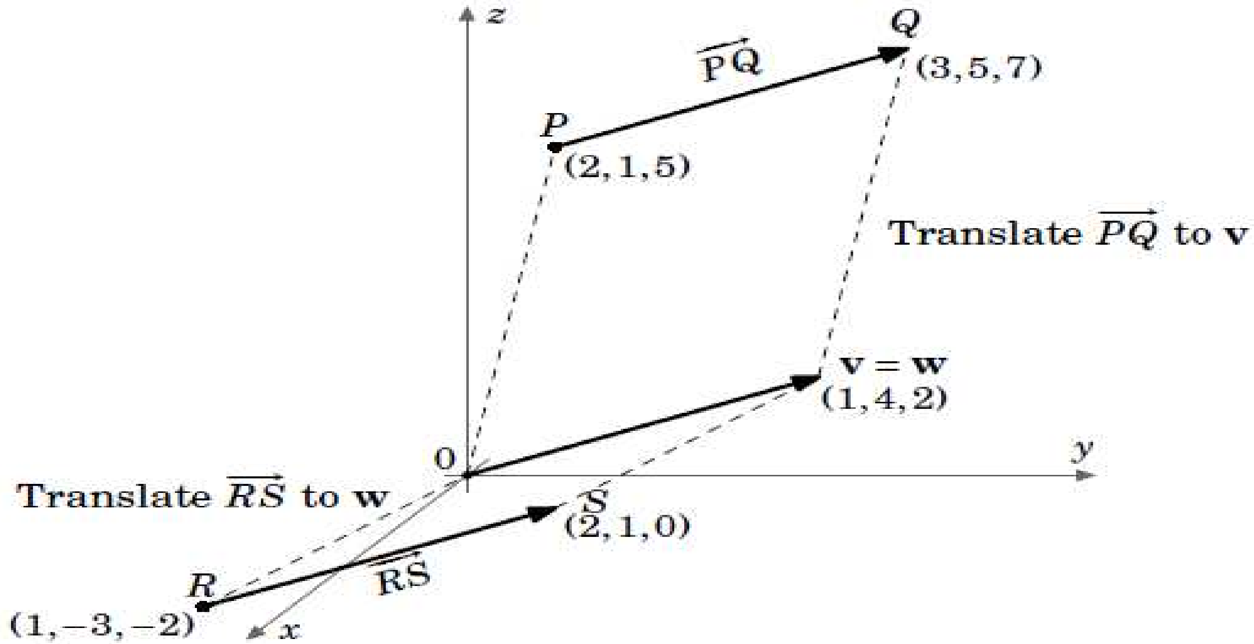
if those coordinates are the same, then the original vectors are equal.

To get the "new" vectors starting at the origin,
you *translate* each vector to start at the origin by subtracting the coordinates of the original initial point from the original terminal point.
The resulting point will be the terminal point of the "new" vector whose initial point is the origin. Do this for each original vector then compare.

Example. Consider the vectors $\overrightarrow{PQ}$ and $\overrightarrow{RS}$ in R^3, where

$$P = (2,1,5), Q = (3,5,7), R = (1,-3,-2) \text{ and } S = (2,1,0).$$

Does $\overrightarrow{PQ} = \overrightarrow{RS}$?



Solution: The vector $\overrightarrow{PQ}$ is equal to the vector **v** with initial point (0,0,0) and terminal point:

$$Q-P = (3,5,7)-(2,1,5) = (3-2,5-1,7-5) = (1,4,2).$$

Similarly, $\overrightarrow{RS}$ is equal to the vector **w** with initial point (0,0,0) and terminal point :

$S - R = (2,1,0) - (1,-3,-2) = (2-1, 1-(-3), 0-(-2)) = (1,4,2)$.

So $\overrightarrow{PQ}$ = **v** = (1,4,2) and $\overrightarrow{RS}$ = **w** = (1,4,2).

$\therefore\ \overrightarrow{PQ} = \overrightarrow{RS}$

Recall the distance formula for points in the Euclidean plane: For points $P = (x_1, y_1)$, $Q = (x_2, y_2)$ in R^2,

the distance d between P and Q is:

$$\boxed{d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}}$$

By this formula, we have the following result:

For a vector $\overrightarrow{PQ}$ in $\mathbb{R}^2$ with initial point $P=(x_1,y_1)$ and terminal point $Q=(x_2,y_2)$, the magnitude of $\overrightarrow{PQ}$ is:

$$\|\overrightarrow{PQ}\| = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

So:

For a vector $\mathbf{v}=(a,b)$ in $\mathbb{R}^2$, the magnitude of $\mathbf{v}$ is:

$$\|\mathbf{v}\| = \sqrt{a^2+b^2}$$

Also:

The distance d between points $P=(x_1,y_1,z_1)$ and $Q=(x_2,y_2,z_2)$ in $\mathbb{R}^3$ is:

$$d = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$$

And:

For a vector $\mathbf{v} = (a, b, c)$ in $\mathbb{R}^3$, the magnitude of $\mathbf{v}$ is:

$$\|\mathbf{v}\| = \sqrt{a^2 + b^2 + c^2}$$

Example 1.3. Calculate the following:

(a) The magnitude of the vector $\overrightarrow{PQ}$ in $\mathbb{R}^2$ with $P = (-1, 2)$ and $Q = (5, 5)$.
Solution: By formula (1.2), $\|\overrightarrow{PQ}\| = \sqrt{(5-(-1))^2 + (5-2)^2} = \sqrt{36+9} = \sqrt{45} = 3\sqrt{5}$.

(b) The magnitude of the vector $\mathbf{v} = (8, 3)$ in $\mathbb{R}^2$.
Solution: By formula (1.3), $\|\mathbf{v}\| = \sqrt{8^2 + 3^2} = \sqrt{73}$.

(c) The distance between the points $P = (2, -1, 4)$ and $Q = (4, 2, -3)$ in $\mathbb{R}^2$.
Solution: By formula (1.4), the distance $d = \sqrt{(4-2)^2 + (2-(-1))^2 + (-3-4)^2} = \sqrt{4+9+49} = \sqrt{62}$.

(d) The magnitude of the vector $\mathbf{v} = (5, 8, -2)$ in $\mathbb{R}^3$.
Solution: By formula (1.5), $\|\mathbf{v}\| = \sqrt{5^2 + 8^2 + (-2)^2} = \sqrt{25+64+4} = \sqrt{93}$.

12. Matrices and Vectors

Vectors – any matrix with only one column is a **column vector**. The number of rows in a column vector is the **dimension** of the column vector. An example of a 2 X 1 matrix or a two-dimensional column vector is shown to the right.

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

R^m will denote the set all m-dimensional column vectors

Any matrix with only one row (a 1 X n matrix) is a **row vector**. The dimension of a row vector is the number of columns.

$$\begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$$

Vectors appear in boldface type: for instance vector **v**.

Any m-dimensional vector (either row or column) in which all the elements equal zero is called a zero vector (written 0).

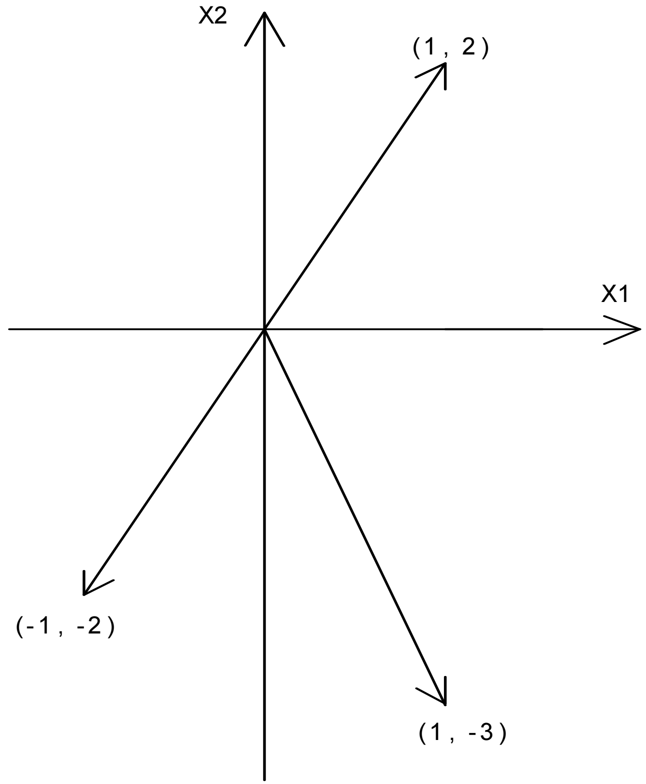
Examples are shown to the right.

$$\begin{pmatrix} 0 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Any m-dimensional vector corresponds to a directed line segment in the m-dimensional plane. For example, the two-dimensional vector **u** corresponds to the line segment joining the point (0,0) to the point (1,2)

The directed line segments (vectors **u**, **v**, **w**) are shown on the figure to the right.

$$\mathbf{u} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \qquad \mathbf{v} = \begin{pmatrix} 1 \\ -3 \end{pmatrix} \qquad \mathbf{w} = \begin{pmatrix} -1 \\ -2 \end{pmatrix}$$



The **scalar product** is the result of multiplying two vectors where one vector is a column vector and the other is a row vector. For the scalar product to be defined, the dimensions of both vectors must be the same.

The scalar product of u and v is written:

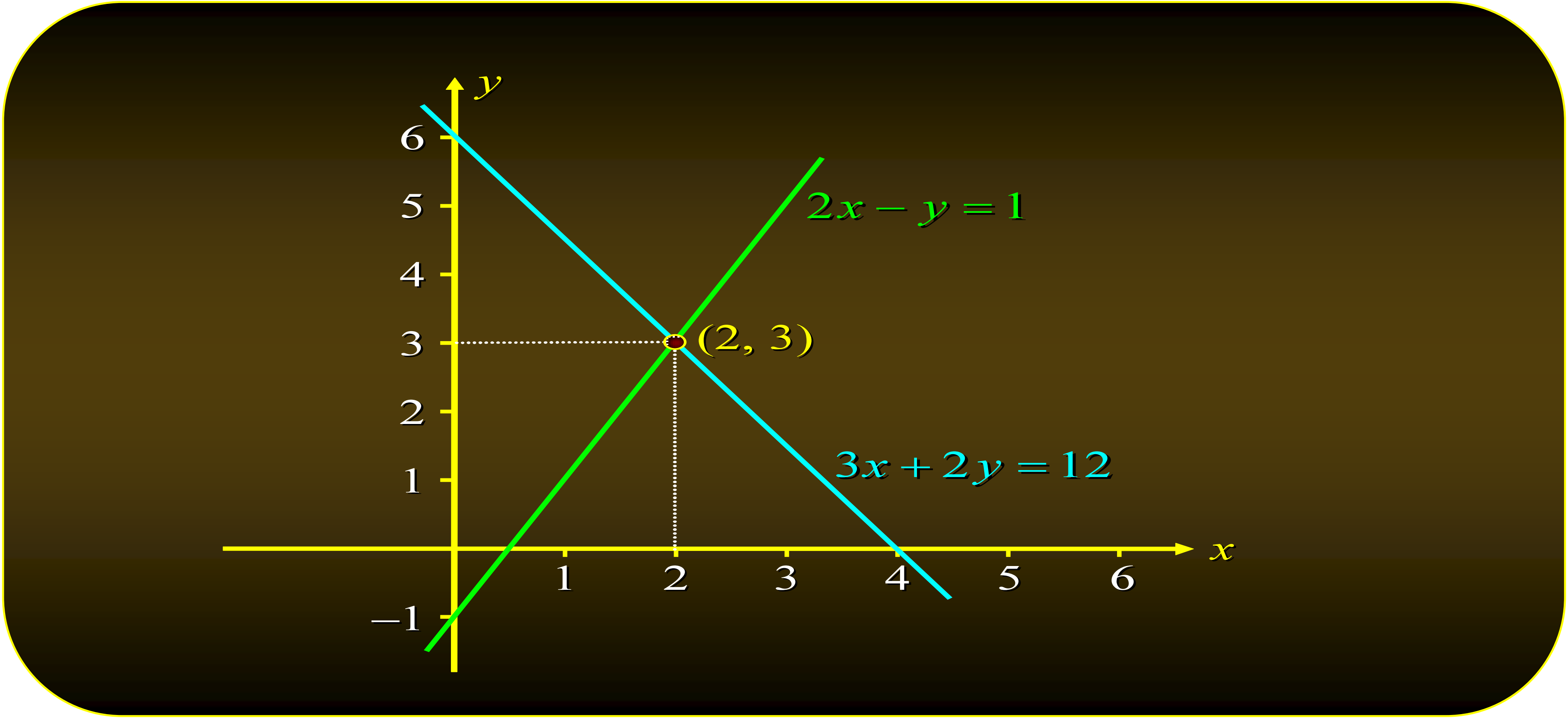
$$\mathbf{u} \cdot \mathbf{v} = (\mathbf{u}_1 \cdot \mathbf{v}_1) + (\mathbf{u}_2 \cdot \mathbf{v}_2) + \ldots + (\mathbf{u}_n \cdot \mathbf{v}_n)$$

Example: $\mathbf{u} = (1 \quad 2 \quad 3)$ $\quad \mathbf{v} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$

$$\mathbf{u} \cdot \mathbf{v} = (1 \cdot 2) + (2 \cdot 1) + (3 \cdot 2) = 10$$

26-7-2022

13. System of Linear Equations



Consider a system of *m* linear equations with *n* unknowns:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

m linear equations

<u>Matrix form of a system of linear equations</u>

$$\Downarrow$$

$$\underset{\substack{\| \\ A}}{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}} \underset{\substack{\| \\ \mathbf{x}}}{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}} = \underset{\substack{\| \\ \mathbf{b}}}{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}}$$

Single matrix equation

$$\underset{m \times n}{A}\ \underset{n \times 1}{\mathbf{x}} = \underset{m \times 1}{\mathbf{b}}$$

If there exists a solution of simultaneous equations then the system is called **consistent**; otherwise, the system is **inconsistent**.

$$\boxed{A\,x = b}$$

1- Gaussian-Jordan Elimination Method

2- matrix inverses Method

3- Cramer's Rule

1- Gauss-Jordan Elimination

- Solve $Ax = b$
- Consists of two phases:
  - **Forward elimination**
  - **Back substitution**
- *Forward Elimination*

  reduces $Ax = b$ to an upper triangular system $Tx = b'$
- *Back substitution* can then solve $Tx = b'$ for x

Augmented Matrix

$$\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{array}\right]$$

$$\Downarrow$$

$$\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ 0 & a'_{22} & a'_{23} & b'_2 \\ 0 & 0 & a''_{33} & b''_3 \end{array}\right]$$

Forward Elimination

$$\Downarrow$$

$$\left.\begin{array}{l} x_3 = \dfrac{b''_3}{a''_{33}} \qquad x_2 = \dfrac{b'_2 - a'_{23}x_3}{a'_{22}} \\ x_1 = \dfrac{b_1 - a_{13}x_3 - a_{12}x_2}{a_{11}} \end{array}\right\} \text{Back Substitution}$$

EXAMPLE Solve the system below by Gaussian elimination:

$$
\begin{aligned}
2x_1 + 6x_2 + x_3 &= 7 \\
x_1 + 2x_2 - x_3 &= -1 \\
5x_1 + 7x_2 - 4x_3 &= 9
\end{aligned}
$$

SOLUTION Using row operations on the augmented matrix of the system, we obtain:

$$
\left(\begin{array}{ccc|c} 2 & 6 & 1 & 7 \\ 1 & 2 & -1 & -1 \\ 5 & 7 & -4 & 9 \end{array}\right) \xrightarrow{R_{12}} \left(\begin{array}{ccc|c} 1 & 2 & -1 & -1 \\ 2 & 6 & 1 & 7 \\ 5 & 7 & -4 & 9 \end{array}\right)
$$

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & -1 \\ 2 & 6 & 1 & 7 \\ 5 & 7 & -4 & 9 \end{array}\right) \xrightarrow[-5R_1+R_3]{-2R_1+R_2} \left(\begin{array}{ccc|c} 1 & 2 & -1 & -1 \\ 0 & 2 & 3 & 9 \\ 0 & -3 & 1 & 14 \end{array}\right)$$

$$\xrightarrow{\frac{1}{2}R_2} \left(\begin{array}{ccc|c} 1 & 2 & -1 & -1 \\ 0 & 1 & \frac{3}{2} & \frac{9}{2} \\ 0 & -3 & 1 & 14 \end{array}\right) \xrightarrow{3R_2+R_3} \left(\begin{array}{ccc|c} 1 & 2 & -1 & -1 \\ 0 & 1 & \frac{3}{2} & \frac{9}{2} \\ 0 & 0 & \frac{11}{2} & \frac{55}{2} \end{array}\right)$$

$$\xrightarrow{\frac{2}{11}R_3} \left(\begin{array}{ccc|c} 1 & 2 & -1 & -1 \\ 0 & 1 & \frac{3}{2} & \frac{9}{2} \\ 0 & 0 & 1 & 5 \end{array}\right).$$

Back substitution

$$\begin{aligned} x_1 + 2x_2 - \quad x_3 &= -1 \\ x_2 + \frac{3}{2}x_3 &= \frac{9}{2} \\ x_3 &= 5. \end{aligned}$$

Substituting $x_3 = 5$ *into the second equation then gives:* $x_2 = 3$.

***Substituting both* these values back into the first equation finally yield $x_1 = 10$.**

2- Matrix inverses Method

Since: $$A\,x = b$$

then

$$A^{-1}\,A x = A^{-1}\,b$$

That is $$I\,x = x = A^{-1}\,b$$

The solution is then:

$$\boxed{x = A^{-1}\,b}$$

EXAMPLE

Solve

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 8 \\ 4x_2 + 5x_3 &= 13 \\ x_1 \qquad + 6x_3 &= 7 \end{aligned}$$

SOLUTION

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 1 & 0 & 6 \end{bmatrix} \qquad b = \begin{bmatrix} 8 \\ 13 \\ 7 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 12/11 & -6/11 & -1/11 \\ 5/22 & 3/22 & -5/22 \\ -2/11 & 1/11 & 2/11 \end{bmatrix}$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = A^{-1}b = \frac{1}{22}\begin{bmatrix} 24 & -12 & -2 \\ 5 & 3 & -5 \\ -4 & 2 & 4 \end{bmatrix}\begin{bmatrix} 8 \\ 13 \\ 7 \end{bmatrix} = \frac{1}{22}\begin{bmatrix} 22 \\ 44 \\ 22 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

3- Cramer's Rule

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

If $D = \det A \neq 0$, then the system has a unique solution as shown below (Cramer's Rule).

where

$$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{32} & a_{33} \end{vmatrix} \qquad D_{x_1} = \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}$$

$$D_{x_2} = \begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{12} & b_2 & a_{23} \\ a_{13} & b_3 & a_{33} \end{vmatrix} \qquad D_{x_3} = \begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{12} & a_{22} & b_2 \\ a_{13} & a_{32} & b_3 \end{vmatrix}$$

$$\boxed{x_1 = \frac{D_{x_1}}{D}, \quad x_2 = \frac{D_{x_2}}{D}, \quad x_3 = \frac{D_{x_3}}{D}}$$

EXAMPLE Consider the following equations:

$$2x_1 - 4x_2 + 5x_3 = 36$$
$$-3x_1 + 5x_2 + 7x_3 = 7$$
$$5x_1 + 3x_2 - 8x_3 = -31$$

where

$$A = \begin{bmatrix} 2 & -4 & 5 \\ -3 & 5 & 7 \\ 5 & 3 & -8 \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 36 \\ 7 \\ -31 \end{bmatrix}$$

$$D = \begin{vmatrix} 2 & -4 & 5 \\ -3 & 5 & 7 \\ 5 & 3 & -8 \end{vmatrix} = -336 \neq 0 \qquad D_{x_1} = \begin{vmatrix} 36 & -4 & 5 \\ 7 & 5 & 7 \\ -31 & 3 & -8 \end{vmatrix} = -672$$

$$D_{x_2} = \begin{vmatrix} 2 & 36 & 5 \\ -3 & 7 & 7 \\ 5 & -31 & -8 \end{vmatrix} = 1008$$

$$D_{x_3} = \begin{vmatrix} 2 & -4 & 36 \\ -3 & 5 & 7 \\ 5 & 3 & -31 \end{vmatrix} = -1344$$

$$\boxed{\begin{aligned} \therefore\ x_1 &= \frac{D_{x_1}}{D} = \frac{-672}{-336} = 2 \\ x_2 &= \frac{D_{x_2}}{D} = \frac{1008}{-336} = -3 \\ x_3 &= \frac{D_{x_3}}{D} = \frac{-1344}{-336} = 4 \end{aligned}}$$

The Gauss-Jordan Method (ERO)

Solve $Ax = b$ where:

$$A = \begin{pmatrix} 2 & 2 & 1 \\ 2 & -1 & 2 \\ 1 & -1 & 2 \end{pmatrix} \qquad b = \begin{pmatrix} 9 \\ 6 \\ 5 \end{pmatrix} \quad \xrightarrow{\text{ERO}} \quad A'|b' = \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

Special Cases: After application of the Gauss-Jordan method, linear systems having no solution or infinite number of solutions can be recognized.

No solution example:

$$\left(\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & 0 & -2 \end{array}\right)$$

Infinite solutions example:

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

1-8-2022

Basic variables and solutions to linear equation systems

For any linear system, a variable that appears with a coefficient of 1 in a single equation and a coefficient of 0 in all other equations is called a **basic variable.**

Any variable that is not a basic variable is called a **nonbasic** variable.

Let BV be the set of basic variables for $A'x=b'$ and NBV be the set of nonbasic variables for $A'x=b'$.

The character of the solutions to $A'x=b'$ (and $Ax=b$) depends upon which of following **cases occur**.

Case 1

$A'\mathbf{x}=\mathbf{b}'$ has at least one row of the form $[0, 0, \ldots, 0 \mid c]$ ($c \neq 0$).

Then $A'\mathbf{x}=\mathbf{b}'$ (and $A\mathbf{x} = \mathbf{b}$) has **no solution**.

In the matrix to the right, row 5 meets this Case 1 criteria.

Variables x_1, x_2, and x_3 are basic while x_4 is a nonbasic variable.

$$A'|\mathbf{b}' = \left(\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 2 & 3 \\ 0 & 0 & 1 & 3 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{array}\right)$$

Case 2

Suppose Case 1 does not apply and NBV, the set of nonbasic variables is empty.

Then **A'x=b'** will have a **unique solution.**

The matrix to the right has a unique solution.

The set of basic variables is x_1, x_2, and x_3 while the NBV set is empty.

$$A'|b' = \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

Case 3

Suppose Case 1 does not apply and NBV is not empty.

Then $A'x=b'$ (and $Ax = b$) will have an infinite number of solutions.

BV = {x_1, x_2, and x_3} while

NBV = {x_4 and x_5}.

$$A' \mid b' = \left(\begin{array}{ccccc|c} 1 & 0 & 0 & 1 & 1 & 3 \\ 0 & 1 & 0 & 2 & 0 & 2 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

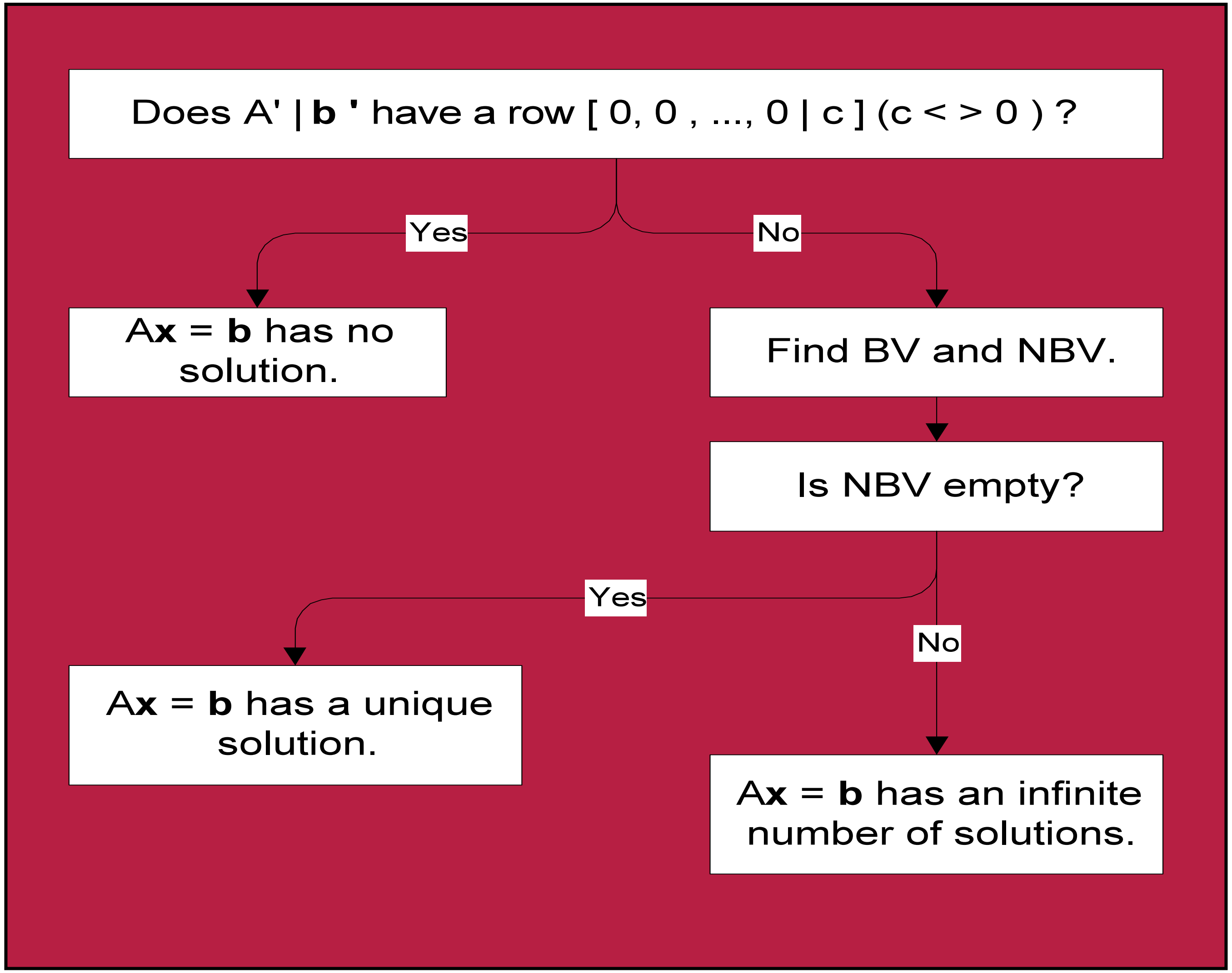
A summary of Gauss-Jordan method is shown to the right.

The end result of the Gauss-Jordan method will be one either

Case 1,

Case 2, or

Case 3.



EXAMPLE Solve:

$$\begin{aligned} x + 3y - 2z &= -7 \\ 4x + y + 3z &= 5 \\ 2x - 5y + 7z &= 19. \end{aligned}$$

SOLUTION

$$\left(\begin{array}{ccc|c} 1 & 3 & -2 & -7 \\ 4 & 1 & 3 & 5 \\ 2 & -5 & 7 & 19 \end{array}\right) \xrightarrow[-2R_1 + R_3]{-4R_1 + R_2} \left(\begin{array}{ccc|c} 1 & 3 & -2 & -7 \\ 0 & -11 & 11 & 33 \\ 0 & -11 & 11 & 33 \end{array}\right)$$

$$\xrightarrow[-\frac{1}{11}R_3]{-\frac{1}{11}R_2} \left(\begin{array}{ccc|c} 1 & 3 & -2 & -7 \\ 0 & 1 & -1 & -3 \\ 0 & 1 & -1 & -3 \end{array}\right) \xrightarrow[-R_2 + R_3]{-3R_2 + R_1} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & -3 \\ \boxed{0} & \boxed{0} & \boxed{0} & \boxed{0} \end{array}\right).$$

$\therefore$ BV = {x, y} while NBV = {z}. ⇒ The system has an infinite number of solutions

Now convert this reduced matrix back into equations. We have

$$x + z = 1$$

$$y - z = -3$$

or, equivalently,

$$x = 1 - z$$

$$y = -3 + z \quad \text{and } z \text{ is free}$$

These two equations tell us that the values of x and y depend on what z is. z is "free" to take on the value of any real number. Once z is chosen, we have a solution.

Since we have infinite choices for the value of z, we have infinite solutions.

As examples, $x = 2$, $y = -3$, $z = 0$ is one solution; $x = -1$, $y = -1$, $z = 2$ is another solution.

Try plugging these values back into the original equations to verify that these indeed are solutions.

EXAMPLE Solve:

$$\begin{aligned} x_1 + x_2 + x_3 &= 1 \\ x_1 + 2x_2 + x_3 &= 2 \\ 2x_1 + 3x_2 + 2x_3 &= 0 \end{aligned}$$

SOLUTION

$$[A\,|\,b] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 2 \\ 2 & 3 & 2 & 0 \end{array}\right] \xrightarrow[-2R_1+R_3]{-R_1+R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -2 \end{array}\right] \xrightarrow[-R_2+R_3]{-R_2+R_1} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & -3 \end{array}\right]$$

Since $-3 \neq 0$, so by case 1, the system has no solution

EXAMPLE Solve:

$$\begin{aligned} -3x_1 - 3x_2 + 9x_3 &= 12 \\ 2x_1 + 2x_2 - 4x_3 &= -2 \\ -2x_2 - 4x_3 &= -8 \end{aligned}$$

SOLUTION We start by converting the linear system into an augmented matrix.

$$[A \mid b] = \left[\begin{array}{ccc|c} -3 & -3 & 9 & 12 \\ 2 & 2 & -4 & -2 \\ 0 & -2 & -4 & -8 \end{array}\right] \xrightarrow[\frac{1}{2}R_2,\ \frac{-1}{2}R_3]{-\frac{1}{3}R_1} \left[\begin{array}{ccc|c} 1 & 1 & -3 & -4 \\ 1 & 1 & -2 & -1 \\ 0 & 1 & 2 & 4 \end{array}\right] \xrightarrow{-R_1 + R_2} \left[\begin{array}{ccc|c} 1 & 1 & -3 & -4 \\ 0 & 0 & 1 & 3 \\ 0 & 1 & 2 & 4 \end{array}\right]$$

$$\xrightarrow{R_{23}} \left[\begin{array}{ccc|c} 1 & 1 & -3 & -4 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right] \xrightarrow{-R_2 + R_1} \left[\begin{array}{ccc|c} 1 & 0 & -5 & -8 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right] \xrightarrow[5R_3 + R_1]{-2R_3 + R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Thus the system has a unique solution $x_1 = 7$, $x_2 = -2$ and $x_3 = 3$.

14. A Vector Space

Definition of a vector space

A vector space is a subset $V \subseteq \mathbb{R}^n$ s.t. the following conditions hold:

1. If v_1 , $v_2 \in V$, then v_1 , $v_2 \in V$.
2. If $c \in \mathbb{R}$, $v \in \mathbb{R}^2$ then $cv \in V$.

The former condition is called closed under vector addition while the latter is called closed under multiplication by a scalar.

Example

Let $n = 2$, so we are looking at $\mathbb{R}^2$. Let $V = \{(x; y) : y = 2x\}$.
In other words V consists of all the points on the line $y = 2x$.
Lets show that V is a vector space by checking that conditions (1) and (2) in the definition are true. This is done as follows:

1- Let (a_1,a_2); $(b_1; b_2) \in V$ be points on the line $y = 2x$.
By definition $a_2 = 2a_1$ and $b_2 = 2b_1$. It follows that

$$a_2 + b_2 = 2a_1 + 2b_1 = 2(a_1 + b_1).$$

In other words,

$$(a_1 , a_2) + (b_1 , b_2) = (a_1 + b_1 , a_2 + b_2) \in V:$$

2- Now let $(a_1 , a_2) \in V$ and $c \in \mathbb{R}$. Since $a_2 = 2a_1$, we also have that $ca_2 = 2ca_1$, so $c(a_1 , a_2) = (ca_1 , ca_2) \in V$.

Example Let $V = \{(x; y) : y = 2x + 1\}$ i.e. the points on the line

$$y = 2x+1.$$

Now $(1,3)$ is on the line, but $(-1)(1,3) = (-1,-3)$ is not. Thus V is not closed under multiplication by a scalar,

so it's **not a vector space**.

15. A Linear combination

Definition A vector v that can be written as $c_1v_1+ \dots +c_kv_k$ is called **a linear combination** of the vectors $v_1, \dots, v_k$.

Example Assume we are given one vector $v = (1,0) \in \mathbb{R}^2$. Then we may write $(2,0) = 2v$,

so $(2 , 0)$ is a linear combination of v.

However, say we look at the vector (1,1). Clearly,

$$(1,1) \neq cx = c(1;\ 0) = (c;\ 0)$$

for any choice of $c \in \mathbb{R}$.

In other words **(1,1) is not a linear combination of x**.

<u>**Example**</u> Assume we are given vectors

$$v_1 = (1,0,1)\ ,\ v_2 = (2,1,0)\ ,\ v_3 = (0,0,1).$$

Let $v \in \mathbb{R}^3$ be another vector defined by $v=(1,1,1)$

Is v a linear combination of v1 , v2 and v3 ? In order to answer this question, note that a linear combination of v_1 ,v_2 and v_3 with coefficients c_1 , c_2 and c_3 has the following form:

$$c_1v_1+c_2v_2+c_3v_3=c_1(1,0,1)+c_2(2,1,0)+c_3(0,0,1)=(c_1+2c_2,c_2,c_1+c_3)$$

Now, v is a linear combination of v_1 , v_2 and v_3 if and only if we can find c_1 , c_2 and c_3 such that

$$v = c_1 v_1 + c_2 v_2 + c_3 v_3$$

which is equivalent to

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} c_1 + 2c_2 \\ c_2 \\ c_1 + c_3 \end{bmatrix} \Rightarrow \begin{aligned} c_1 + 2c_2 &= 1 \\ c_2 &= 1 \\ c_1 + c_3 &= 1 \end{aligned} \Rightarrow \begin{aligned} c_1 &= -1 \\ c_2 &= 1 \\ c_3 &= 2 \end{aligned}$$

You can easily check that these values really constitute a solution to our problem:

$$(-1)v_1 + (1)v_2 + (2)v_3 = -(1,0,1) + (2,1,0) + 2(0,0,1)$$

$$= (1,1,1) = v$$

Thus v is a linear combination of v_1 , v_2 and v_3

Let $V = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ be a set of row vectors all having the same dimension.

I) A linear combination of the vectors in V is any vector of the form

$$c_1v_1 + c_2v_2 + \ldots + c_k v_k$$

where $c_1, c_2, \ldots, c_k$ are arbitrary scalars.

Example:

If $V = \{ [1 , 2] , \{2 , 1] \}$

$2v_1 - v_2 = 2([1 \;\; 2]) - [2 \;\; 1] = [0 \;\; 3]$

Thus: $[0 \;\; 3]$ is a linear combination of the vectors in $V = \{ [1 , 2] , \{2 , 1] \}$.

and

$v_1 + 3v_2 = [1 \;\; 2] + 3([2 \;\; 1]) = [6 \;\; 3]$

Thus: $[6 \;\; 3]$ is a linear combination of the vectors in $V = \{ [1 , 2] , \{2 , 1] \}$.

16. Linear Independence and Dependence of Vectors

Given any set of m vectors $\mathbf{v}_1, \ldots, \mathbf{v}_m$ (with the same number of components), a **linear combination** of these vectors is an expression of the form

$$c_1\mathbf{v}_1, + c_2\mathbf{v}_2, + \ldots + c_m\mathbf{v}_m$$

where $c_1, c_2, \ldots, c_m$ are any scalars. Now consider the equation

$$c_1\mathbf{v}_1, + c_2\mathbf{v}_2, + \ldots + c_m\mathbf{v}_m = \mathbf{0} \qquad (1)$$

Clearly, this vector equation (1) holds if we choose all c_j's zero, because then it becomes $\mathbf{0} = \mathbf{0}$. If this is the only m-tuple of scalars for which (1) holds, then our vectors $\mathbf{v}_1, \ldots, \mathbf{v}_m$ are said to form a *linearly independent set* or, more briefly, we call them **linearly independent**.

Otherwise, if (1) also holds with scalars not all zero, we call these vectors **linearly dependent**.

This means that we can express at least one of the vectors as a linear combination of the other vectors. For instance, if **(1)** holds with, say, $c_1 \neq 0$, we can solve **(1)** for $\mathbf{v}_1$:

$$\mathbf{v}_1 = k_2\mathbf{v}_2 + \dots + k_m\mathbf{v}_m \qquad \text{where } k_j = -c_j / c_1.$$

(Some k_j's may be zero. Or even all of them, namely, if $\mathbf{v}_1 = \mathbf{0}$).

A set of vectors is called **linearly independent** if the only linear combination of vectors in V that equals 0 is the trivial linear combination ($c_1 = c_2 = \dots = c_k = 0$).

A set of vectors is called **linearly dependent** if there is a nontrivial linear combination of vectors in V that adds up to 0.

Example

Show that V = {[1 , 2] , [2 , 4]} is a linearly dependent set of vectors.

Since 2([1 , 2]) – 1([2 , 4 }) = (0 0), there is a nontrivial linear combination with $c_1 = 2$ and $c_2 = -1$ that yields **0**.

Thus V is a **linear dependent** set of vectors.

Definition

A set $B=\{b_1,b_2,\dots,b_k\}$ of vectors is a **basis** for a vector space *V*

iff

(1) The vectors in the set B are linearly independent, and

(2) Any vector in *V* can be expressed as a linear combination of the vectors in the set B.

If (2) holds, we also say that the set of vectors **spans** the vector space *V*.

Example **Let $B=\{ (2,4),(1,1) \}$**

1- B is linearly independent

$$c_1\begin{pmatrix}2\\4\end{pmatrix}+c_2\begin{pmatrix}1\\1\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix} \Rightarrow \begin{aligned}2c_1+1c_2&=0\\4c_1+1c_2&=0\end{aligned} \Rightarrow c_1=c_2=0$$

2- B spans $\mathbb{R}^2$

If $(x,y) \in \mathbb{R}^2$,then we can find c_1 and c_2 s.t.

$$(x,y) = c_1(2,4) + c_2(1,1) \Rightarrow \begin{aligned} 2c_1 + 1c_2 &= x \\ 4c_1 + 1c_2 &= y \end{aligned}$$

$$\Rightarrow c_2 = 2x - y \text{ and } c_1 = (y - x)/2$$

Example The space $\mathbb{R}^2$ has many bases. Another one is this.

B={ (1,0),(0,1) }

The verification is easy (Exercise)

Definition For any $\mathbb{R}^n$

$$\mathbf{e}_1 = (1, 0, \cdots, 0),$$

$$\mathbf{e}_2 = (0, 1, \cdots, 0),$$

$$\mathbf{e}_3 = (0, 0, 1, \cdots, 0),$$

$$\cdots\cdots$$

$$\mathbf{e}_n = (0, 0, \cdots, 1),$$

is the standard (or natural) basis. We denote these vectors $e_1, e_2, e_3, \ldots\ldots, e_n$.

17. The Rank of a Matrix

Let A be any m x n matrix, and denote the rows of A by $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_m$. Define $R = \{\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_m\}$.

The **rank** of A is the number of vectors in the **largest linearly independent** subset of R.

To find the rank of matrix A,

apply the Gauss-Jordan method to matrix A.

Let A' be the final result. It can be shown that the rank of A' = rank of A.

The rank of A' = the number of **nonzero rows in A'**.

Therefore, the rank A = rank A' = number of nonzero rows in A'.

Given V = {[1 0 0], [0 1 0], [1 1 0]}

Form matrix A

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}$$

After the Gauss-Jordan method:

$$\boxed{\text{rank } A = \text{rank } A' = 2}$$

A method of determining whether a set of vectors

$V = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\}$ is linearly dependent is to from a matrix A whose i^{th} row is v_i.

If the rank of $A = m$, then V is a linearly independent set of vectors.

If the rank $A < m$, then V is a linearly dependent set of vectors.

See the example to the right.

Given V = {[1 0 0], [0 1 0], [1 1 0]}

Form matrix A

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}$$

After the Gauss-Jordan method:

$$A' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Since the rank of $A' = 2$ which is < 3, the set of vectors in V are linearly dependent.

18. Linear Transformations

Sometimes we want to consider functions between $\mathbb{R}^n$ and $\mathbb{R}^m$. Such a function associates with each vector in $\mathbb{R}^n$ a vector in $\mathbb{R}^m$, according to a rule defined by the function.

Definition

A function T that maps n-vectors to m-vectors is called *a linear transformation* if the following two conditions are satisfied:

$$1\text{-} \quad T(u+v) = T(u) + T(v)$$

$$2\text{-} \quad T(cu) = cT(u).$$

These two conditions can be rolled into the single requirement that

$$T(au + bv) = aT(u) + bT(v)$$

EXAMPLE

Determine whether or not the given function is a linear transformation.

$$T(x, y) = (x + y,\ x - y,\ 2x).$$

T maps vectors in $\mathbb{R}^2$ to vectors in $\mathbb{R}^3$. For example,

$$T(2,-3) = (-1,\ 5,\ 4) \quad \text{and} \quad T(1,\ 1) = (2,\ 0,\ 2).$$

Let $\mathbf{u}=(a, b)$ and $\mathbf{v}=(c, d)$. $\Rightarrow$ $\mathbf{u}+\mathbf{v}=(a+c,\ b+d)$

$$T(\mathbf{u}+\mathbf{v}) = T(a+c,\ b+d) = (a+c+b+d,\ a+c-b-d,\ 2a+2c) \qquad (\mathrm{I})$$

While, $T(\mathbf{u})+T(\mathbf{v}) = (a+b,\ a-b,\ 2a) + (c+d,\ c-d,\ 2c)$

$= (a+b+c+d,\ a-b+c-d,\ 2a+2c)$

$= (a+c+b+d,\ a+c-b-d,\ 2a+2c)$

$= T(a+c,\ b+d)$

$= (a+c+b+d, a+c-b-d, 2a+2c) = T(\mathbf{u}+\mathbf{v})$.

This verifies condition (1) of the definition.

For condition (2), let α be any number. Then

$$
\begin{aligned}
T(\alpha \mathbf{u}) &= T(\alpha a, \alpha b) \\
&= (\alpha a + \alpha b, \alpha a - \alpha b, 2\alpha a) \\
&= \alpha (a + b, a - b, 2a) = \alpha T(\mathbf{u}).
\end{aligned}
$$

It is easy to check that the function

$$P(a, b, c) = (a^2, 1, 1, \sin(a))$$

from $\mathbb{R}^2$ to $\mathbb{R}^4$ is not linear

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EXAMPLE Let $T(x, y) = (x - y, 0, 0)$.

Then T is a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^3$.

T is certainly **not one-to-one**, since, for example,

$$T(1, 1) = T(2, 2) = (0, 0, 0).$$

In fact, $T(x, x) = (0, 0, 0)$ for every number x.

Thus T maps many vectors to the origin in $\mathbb{R}^3$.

T is also not onto $\mathbb{R}^3$, since no vector in $\mathbb{R}^3$ with a nonzero second or third component is the image of any vector in $\mathbb{R}^2$ under T.

EXAMPLE 2 Let $S : R^3 \to R^2$ be defined by

$$S(x, y, z) = (x, y).$$

S is onto, since every vector in $\mathbb{R}^2$ is the image of a vector in $\mathbb{R}^3$ under S.
For example, $(-3,\sqrt{97}) = S(-3,\sqrt{97}, 0)$.

But S is not one-to-one. For example, $S(-3,\sqrt{97}, 22)$ also equals $(-3,\sqrt{97})$

The standard (associated) matrix A_T for the Linear Transformation T

Every linear transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ can be associated with a matrix A_T that carries all of the information about the transformation.

The standard basis for R^n consists of the n orthogonal unit vectors

$\mathbf{e}_1 = (1, 0, \cdots, 0)$, $\mathbf{e}_2 = (0, 1, \cdots, 0)$, $\mathbf{e}_3 = (0, 0, 1, \cdots, 0)$,

.... $\mathbf{e}_n = (0, 0, \cdots, 1)$,

with a similar basis (with m components) for $\mathbb{R}^m$.

Now let A_T be the matrix whose columns are of the images in R^m of $T(\mathbf{e}_1), T(\mathbf{e}_2), \cdots, T(\mathbf{e}_n)$ with coordinates written in terms of the standard basis in R^m.

The A_T is an $m \times n$ matrix that represents T in the sense that

i.e:

$$A_T = \begin{bmatrix} \begin{bmatrix} \vdots \\ T(e_1) \\ \vdots \end{bmatrix} & \begin{bmatrix} \vdots \\ T(e_2) \\ \vdots \end{bmatrix} & \cdots & \begin{bmatrix} \vdots \\ T(e_n) \\ \vdots \end{bmatrix} \end{bmatrix} \qquad (m \times n \text{ matrix})$$

and

$$T(x_1, x_2, \ldots, x_n) = A_T \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad (m \times 1 \text{ matrix}) \in \mathbb{R}^m$$

EXAMPLE

Find the standard matrix A for the linear transformation $T: R^2 \rightarrow R^2$ defined by the formula

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 + x_2 \\ x_1 - 3x_2 \\ -x_1 + x_2 \end{bmatrix}$$

Solution We leave it for you to verify that

$$T(\mathbf{e}_1) = T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \quad \text{and} \quad T(\mathbf{e}_2) = T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix}$$

Thus, the standard matrix is

$$A = [T(\mathbf{e}_1) \mid T(\mathbf{e}_2)] = \begin{bmatrix} 2 & 1 \\ 1 & -3 \\ -1 & 1 \end{bmatrix}$$

EXAMPLE

Rewrite the transformation $T(x_1, x_2) = (3x_1 + x_2, 2x_1 - 4x_2)$ in column-vector form and find its standard matrix.

Solution

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 3x_1 + x_2 \\ 2x_1 - 4x_2 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Thus, the standard matrix is

$$\begin{bmatrix} 3 & 1 \\ 2 & -2 \end{bmatrix}$$

$T : R^n \rightarrow R^m$

For a matrix transformation $T(X) = AX$.

Let B denote the reduced echelon form of A.

1- If $m > n$, then T cannot be onto.
2- If $m < n$, then T cannot be one-to-one.
3- T is onto if and only if B has a pivot in every row.
4- T is one-to-one if and only if B has a pivot in every column.

Example. Let $T(\mathbf{X}) = A\mathbf{X}$, where

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \\ 1 & 3 & 2 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

- T is **not** one-to-one, as every column does not have a pivot.
- T **is** onto, as every row has a pivot.

EXAMPLE

Determine whether or not the given function is a linear transformation. If it is, write the matrix representation of T (using the standard bases)

$$T(x, y) = (x - y, 0, 0)$$

$T(1, 0) = (1, 0, 0)$ and $T(0, 1) = (-1, 0, 0)$

so

$$A_T = \begin{bmatrix} 1 & -1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Now observe that

$$T(x,y) = A_T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x - y \\ 0 \\ 0 \end{bmatrix}$$

EXAMPLE

Write the matrix representation of S, where

$S(x, y, z, w) = (x - y + 2z + 8w,\ y - z,\ x - w,\ y + 4w,\ 5x + 5y - z,\ 0,\ 0)$

$S(1, 0, 0, 0) = (1, 0, 1, 0, 5, 0, 0)$, $S(0, 1, 0, 0) = (-1, 1, 0, 1, 5, 0, 0)$,

$S(0, 0, 1, 0) = (2, -1, 0, 0, -1, 0, 0)$, $S(0, 0, 0, 1) = (8, 0, -1, 4, 0, 0, 0)$,

so

$$\therefore \quad A_S = \begin{bmatrix} 1 & -1 & 2 & 8 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 4 \\ 5 & 5 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Find its RANK

EXAMPLE

Write the matrix representation of S, where

$S(x, y, z, w) = (x - y + 2z + 8w,\ y - z,\ x - w,\ y + 4w,\ 5x + 5y - z,\ 0,\ 0)$

$S(1, 0, 0, 0) = (1, 0, 1, 0, 5, 0, 0)$, $S(0, 1, 0, 0) = (-1, 1, 0, 1, 5, 0, 0)$,

$S(0, 0, 1, 0) = (2, -1, 0, 0, -1, 0, 0)$, $S(0, 0, 0, 1) = (8, 0, -1, 4, 0, 0, 0)$,

so

$$\therefore \quad A_S = \begin{bmatrix} 1 & -1 & 2 & 8 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 4 \\ 5 & 5 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Find its RANK

Basic Matrix Transformations in R^2 and R^3

we will continue our study of linear transformations by considering some basic types of matrix transformations in R^2 and R^3 that have simple geometric interpretations. The transformations we will study here are important in such fields as computer graphics, engineering, and physics.

Some of the most basic matrix operators on R^2 and R^3 are those that map each point into its symmetric image about a fixed line or a fixed plane that contains the origin; these are called ***reflection operators***.

Table 1 shows the standard matrices for the reflections about the coordinate axes in R^2, and

Table 2 shows the standard matrices for the reflections about the coordinate planes in R^3.

Table 1

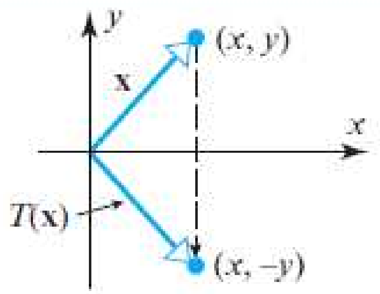
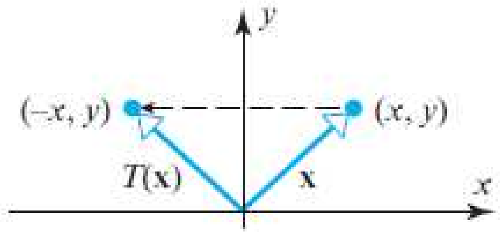
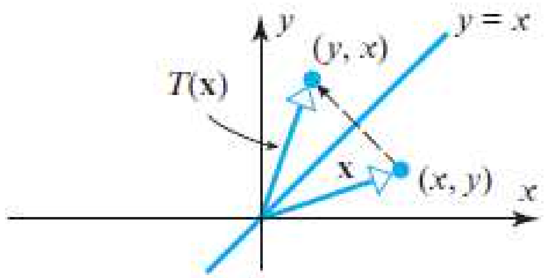
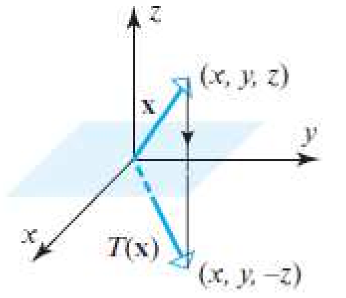
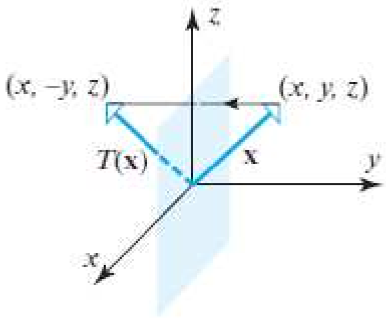
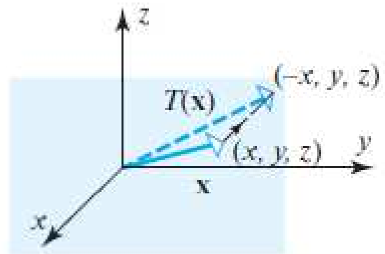
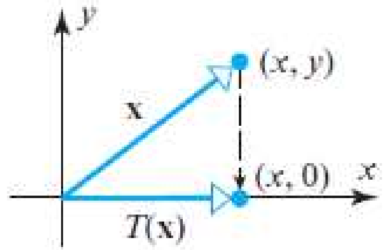
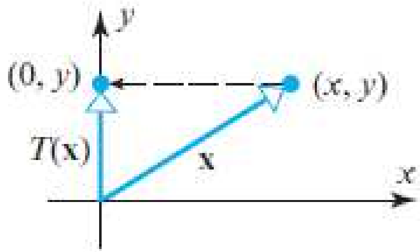
Operator	Illustration	Images of $\mathbf{e}_1$ and $\mathbf{e}_2$	Standard Matrix
Reflection about the x-axis $T(x, y) = (x, -y)$		$T(\mathbf{e}_1) = T(1, 0) = (1, 0)$ $T(\mathbf{e}_2) = T(0, 1) = (0, -1)$	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
Reflection about the y-axis $T(x, y) = (-x, y)$		$T(\mathbf{e}_1) = T(1, 0) = (-1, 0)$ $T(\mathbf{e}_2) = T(0, 1) = (0, 1)$	$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$
Reflection about the line $y = x$ $T(x, y) = (y, x)$		$T(\mathbf{e}_1) = T(1, 0) = (0, 1)$ $T(\mathbf{e}_2) = T(0, 1) = (1, 0)$	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Table 2

Operator	**Illustration**	**Images of $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$**	**Standard Matrix**
Reflection about the xy-plane $T(x, y, z) = (x, y, -z)$		$T(\mathbf{e}_1) = T(1, 0, 0) = (1, 0, 0)$ $T(\mathbf{e}_2) = T(0, 1, 0) = (0, 1, 0)$ $T(\mathbf{e}_3) = T(0, 0, 1) = (0, 0, -1)$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$
Reflection about the xz-plane $T(x, y, z) = (x, -y, z)$		$T(\mathbf{e}_1) = T(1, 0, 0) = (1, 0, 0)$ $T(\mathbf{e}_2) = T(0, 1, 0) = (0, -1, 0)$ $T(\mathbf{e}_3) = T(0, 0, 1) = (0, 0, 1)$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
Reflection about the yz-plane $T(x, y, z) = (-x, y, z)$		$T(\mathbf{e}_1) = T(1, 0, 0) = (-1, 0, 0)$ $T(\mathbf{e}_2) = T(0, 1, 0) = (0, 1, 0)$ $T(\mathbf{e}_3) = T(0, 0, 1) = (0, 0, 1)$	$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Matrix operators on R^2 that map each point into its orthogonal projection onto a fixed line through the origin are called ***projection operators*** (or more precisely, ***orthogonal projection*** operators).
Table 3 shows the standard matrices for the orthogonal projections onto the coordinate axes in R^2

Table 3

Operator	Illustration	Images of $\mathbf{e}_1$ and $\mathbf{e}_2$	Standard Matrix
Orthogonal projection onto the x-axis $T(x, y) = (x, 0)$		$T(\mathbf{e}_1) = T(1, 0) = (1, 0)$ $T(\mathbf{e}_2) = T(0, 1) = (0, 0)$	$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$
Orthogonal projection onto the y-axis $T(x, y) = (0, y)$		$T(\mathbf{e}_1) = T(1, 0) = (0, 0)$ $T(\mathbf{e}_2) = T(0, 1) = (0, 1)$	$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

19. Eigenvalues and Eigenvectors

Definition 1: A nonzero vector x is an *eigenvector* (or *characteristic vector*) of a square matrix A if there exists a scalar λ such that: $Ax = \lambda x$.

Then λ is an *eigenvalue* (or *characteristic value*) of A.

Note: The zero vector can not be an eigenvector.

But $\lambda = 0$ can be an eigenvalue.

Example:

$$Show\ x = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad is\ an\ eigenvector\ for\ A = \begin{bmatrix} 2 & -4 \\ 3 & -6 \end{bmatrix}$$

$$Solution\ :\ Ax = \begin{bmatrix} 2 & -4 \\ 3 & -6 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$But\ \ for\ \lambda = 0,\ \lambda x = 0\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$Thus\ ,x\ is\ an\ eigenvector\ of\ A,\ and\ \lambda = 0\ is\ an\ eigenvalue\ .$$

Let x be an eigenvector of the matrix A. Then there must exist an eigenvalue λ such that: $Ax = \lambda x$ or, equivalently,

$$Ax - \lambda x = \mathbf{0} \quad \text{or} \quad (A - \lambda I)x = \mathbf{0}$$

If we define a new matrix $B = A - \lambda I$, then $Bx = 0$

If B has an inverse then: $x = B^{-1}0 = 0$.

But an **eigenvector** cannot be **zero**.

Thus, it follows that x will be an eigenvector of A if and only if

B does not have an inverse, or equivalently

$$\det(B) = 0 \quad \text{or} \quad \det(A - \lambda I) = 0$$

This is called **the characteristic equation** of A.

Its roots determine the **eigenvalues** of A.

Example : Find the eigenvalues of $A = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}$

$$|A - \lambda I| = \left\| \begin{bmatrix} 2-\lambda & -12 \\ 1 & -5-\lambda \end{bmatrix} \right\| = \begin{vmatrix} 2-\lambda & -12 \\ 1 & -5-\lambda \end{vmatrix} = (2-\lambda)(-5-\lambda)+12$$

$$= (\lambda-2)(\lambda+5)+12 = \lambda^2 + 3\lambda + 2 = (\lambda+1)(\lambda+2) = 0$$

Thus there exist two **eigenvalues**: **−1** , **− 2**

To each distinct eigenvalue of a matrix A there will correspond at least one eigenvector which can be found by solving the appropriate set of homogenous equations $(A - \lambda I)x$.

If λ_i is an eigenvalue then the corresponding eigenvector x_i is the solution of $(A - \lambda_i I)x_i = 0$

Example (cont.):

$$A=\begin{bmatrix}2 & -12\\ 1 & -5\end{bmatrix}$$

$$\lambda=-1:\quad (A-\lambda I)x=\left(\begin{bmatrix}2 & -12\\ 1 & -5\end{bmatrix}-(-1)\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right)\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}3 & -12\\ 1 & -4\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}3x_1-12x_2\\ x_1-4x_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix} \qquad \Rightarrow \begin{matrix}3x_1-12x_2=0\,,\\ x_1-4x_2=0\end{matrix}$$

$$x_1-4x_2=0\Rightarrow x_1=4s,\ x_2=s$$

$$\therefore\quad \mathbf{x}_1=\begin{bmatrix}x_1\\ x_2\end{bmatrix}=s\begin{bmatrix}4\\ 1\end{bmatrix},\quad s\neq 0$$

$$\lambda=-2:(-2)I-A=\begin{bmatrix}-4 & 12\\ -1 & 3\end{bmatrix}\Rightarrow\begin{bmatrix}1 & -3\\ 0 & 0\end{bmatrix}$$

$$\therefore\quad \mathbf{x}_2=\begin{bmatrix}x_1\\ x_2\end{bmatrix}=s\begin{bmatrix}3\\ 1\end{bmatrix},\ s\neq 0$$

Note: The roots of the characteristic equation can be repeated.

That is, $\lambda_1 = \lambda_2 = \ldots = \lambda_k$. If that happens, the eigenvalue is said to be of multiplicity k.

Example : Find the eigenvalues of $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

$$|A - \lambda I| = \left| \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \right| = \begin{vmatrix} 2-\lambda & 1 & 0 \\ 0 & 2-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{vmatrix}$$

$$= (2-\lambda)^3 = 0$$

$\lambda = 2$ is an eigenvalue of multiplicity 3.

Example (cont.): Find the eigenvectors of

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Recall that $\lambda = 2$ is an *eigenvalue* of multiplicity 3.

Solve the homogeneous linear system represented by

$$(A - 2I)\mathbf{x} = \left(\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \right) \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore \quad x_2 = 0$, and the other two variables are any two parameters (i.e. $x_1 = s, x_3 = t$).

So, the eigenvectors of λ = 2 are of the form

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} s \\ 0 \\ t \end{bmatrix} = s\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

where, s and *t* not both zero.

Find the matrix representation of $T\colon \mathbb{C}^3 \to \mathbb{C}^4$, $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 3x+2y+z \\ x+y+z \\ x-3y \\ 2x+3y+z \end{bmatrix}$.

Answer: $A_T = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & -3 & 0 \\ 2 & 3 & 1 \end{bmatrix}$

Define the linear transformation

$$T\colon \mathbb{C}^3 \to \mathbb{C}^2, \quad T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 - x_2 + 5x_3 \\ -4x_1 + 2x_2 - 10x_3 \end{bmatrix}$$

Verify that T is a linear transformation.

C31 (Chris Black) Given that the linear transformation $T\colon \mathbb{C}^3 \to \mathbb{C}^3$, $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 2x+y \\ 2y+z \\ x+2z \end{bmatrix}$ is injective, show directly that $\{T(\mathbf{e}_1), T(\mathbf{e}_2), T(\mathbf{e}_3)\}$ is a linearly independent set.

Solution (Chris Black) We have

$$T(\mathbf{e}_1) = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \qquad T(\mathbf{e}_2) = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \qquad T(\mathbf{e}_3) = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

Let's put these vectors into a matrix and row reduce to test their linear independence.

$$\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & 2 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & 0 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & \boxed{1} \end{bmatrix}$$

so the set of vectors $\{T(\mathbf{e}_1), T(\mathbf{e}_1), T(\mathbf{e}_1)\}$ is linearly independent.

C32 (Chris Black) Given that the linear transformation $T\colon \mathbb{C}^2 \to \mathbb{C}^3$, $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+y \\ 2x+y \\ x+2y \end{bmatrix}$ is injective, show directly that $\{T(\mathbf{e}_1), T(\mathbf{e}_2)\}$ is a linearly independent set.

Solution (Chris Black) We have $T(\mathbf{e}_1) = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ and $T(\mathbf{e}_2) = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$. Putting these into a matrix as columns and row-reducing, we have

$$\begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 1 & 2 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 \\ 0 & \boxed{1} \\ 0 & 0 \end{bmatrix}$$

Thus, the set of vectors $\{T(\mathbf{e}_1), T(\mathbf{e}_2)\}$ is linearly independent.

C33 (Chris Black) Given that the linear transformation $T\colon \mathbb{C}^3 \to \mathbb{C}^5$, $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 0 & 1 \\ 3 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is injective, show directly that $\{T(\mathbf{e}_1), T(\mathbf{e}_2), T(\mathbf{e}_3)\}$ is a linearly independent set.

Solution (Chris Black) We have

$$T(\mathbf{e}_1) = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 3 \end{bmatrix} \qquad T(\mathbf{e}_2) = \begin{bmatrix} 3 \\ 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} \qquad T(\mathbf{e}_3) = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \\ 2 \end{bmatrix}$$

Let's row reduce the matrix of T to test linear independence.

$$\begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 0 & 1 \\ 3 & 1 & 2 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & 0 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & \boxed{1} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

so the set of vectors $\{T(\mathbf{e}_1), T(\mathbf{e}_2), T(\mathbf{e}_3)\}$ is linearly independent.