Work Sheet # 1 – Solution

1. Given the matrices

$$A = \begin{bmatrix} 2 & 3 & -1 \\ 1 & 0 & 3 \\ -3 & -1 & -2 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 2 & 4 \\ 5 & -3 & 0 \end{bmatrix}, C = \begin{bmatrix} -1 & 2 \\ 3 & 0 \\ 4 & -1 \end{bmatrix}, \text{find}$$

(if possible)

(a) A-2B

Sol. $A - 2B = \begin{bmatrix} 2 & 3 & -1 \\ 1 & 0 & 3 \\ -3 & -1 & -2 \end{bmatrix} - \begin{bmatrix} 2 & 0 & -2 \\ 4 & 4 & 8 \\ 10 & -6 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 3 & 1 \\ -3 & -4 & -5 \\ -13 & 5 & -2 \end{bmatrix}$

(b) AB

Sol.

$$AB = \begin{bmatrix} 2 & 3 & -1 \\ 1 & 0 & 3 \\ -3 & -1 & -2 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 2 & 2 & 4 \\ 5 & -3 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 9 & 10 \\ 16 & -9 & -1 \\ -15 & 4 & -1 \end{bmatrix}$$

(c) AC

Sol.

$$AC = \begin{bmatrix} 2 & 3 & -1 \\ 1 & 0 & 3 \\ -3 & -1 & -2 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 3 & 0 \\ 4 & -1 \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ 11 & -1 \\ -8 & -4 \end{bmatrix}$$

(d) CB

Sol.

$$CB = \begin{bmatrix} -1 & 2 \\ 3 & 0 \\ 4 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 & -1 \\ 2 & 2 & 4 \\ 5 & -3 & 0 \end{bmatrix}$$ = Not possible. The dimension of C is 3x2 and dimension of matrix B is 3x3 so number of columns for the matrix C not equal to the number of rows for the matrix B

(e) BAC

Sol.

$$BAC = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 2 & 4 \\ 5 & -3 & 0 \end{bmatrix}\begin{bmatrix} 2 & 3 & -1 \\ 1 & 0 & 3 \\ -3 & -1 & -2 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ 3 & 0 \\ 4 & -1 \end{bmatrix}$$
$$= \begin{bmatrix} 5 & 4 & 1 \\ -6 & 2 & -4 \\ 7 & 15 & -14 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ 3 & 0 \\ 4 & -1 \end{bmatrix} =$$
$$\begin{bmatrix} 1 & 0 & -1 \\ 2 & 2 & 4 \\ 5 & -3 & 0 \end{bmatrix}\begin{bmatrix} 3 & 5 \\ 11 & -1 \\ -8 & -4 \end{bmatrix} = \begin{bmatrix} 11 & 9 \\ -4 & -8 \\ -18 & 28 \end{bmatrix}$$

(f) A^2

Sol.

$$A^2 = \begin{bmatrix} 2 & 3 & -1 \\ 1 & 0 & 3 \\ -3 & -1 & -2 \end{bmatrix}\begin{bmatrix} 2 & 3 & -1 \\ 1 & 0 & 3 \\ -3 & -1 & -2 \end{bmatrix} = \begin{bmatrix} 10 & 7 & 9 \\ -7 & 0 & -7 \\ -1 & -7 & 4 \end{bmatrix}$$

(g) $B + C$

Sol.

$$B + C = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 2 & 4 \\ 5 & -3 & 0 \end{bmatrix} + \begin{bmatrix} -1 & 2 \\ 3 & 0 \\ 4 & -1 \end{bmatrix}$$ Not possible. The dimension of C is 3x2 and dimension of matrix B is 3x3 Not possible. The dimension of C not equal to the dimension of matrix B.

(h) C^2

Sol.

$$C^2 = \begin{bmatrix} -1 & 2 \\ 3 & 0 \\ 4 & -1 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ 3 & 0 \\ 4 & -1 \end{bmatrix}$$ =Not possible. The numbers of columns for the matrix C are not equal to the number of rows for the matrix C.

(i) $3A^T - 2B^T$

Sol.

$$3A^T - 2B^T = 3\begin{bmatrix} 2 & 3 & -1 \\ 1 & 0 & 3 \\ -3 & -1 & -2 \end{bmatrix}^T - 2\begin{bmatrix} 1 & 0 & -1 \\ 2 & 2 & 4 \\ 5 & -3 & 0 \end{bmatrix}^T$$
$$= 3\begin{bmatrix} 2 & 1 & -3 \\ 3 & 0 & -1 \\ -1 & 3 & -2 \end{bmatrix} - 2\begin{bmatrix} 1 & 2 & 5 \\ 0 & 2 & -3 \\ -1 & 4 & 0 \end{bmatrix} = \begin{bmatrix} 4 & -1 & -19 \\ 9 & -4 & 3 \\ -1 & 1 & -6 \end{bmatrix}$$

(j) $tr(A - B)$

Sol.

$$tr(A - B) = tr\left(\begin{bmatrix} 2 & 3 & -1 \\ 1 & 0 & 3 \\ -3 & -1 & -2 \end{bmatrix} - \begin{bmatrix} 1 & 0 & -1 \\ 2 & 2 & 4 \\ 5 & -3 & 0 \end{bmatrix}\right)$$
$$= tr\left(\begin{bmatrix} 1 & 3 & 0 \\ -1 & -2 & -1 \\ -8 & 2 & -2 \end{bmatrix}\right) = 1 + (-2) + (-2)$$
$$= -3$$

(k) $tr(AB)$

Sol.

$$tr(AB) = tr\left(\begin{bmatrix} 2 & 3 & -1 \\ 1 & 0 & 3 \\ -3 & -1 & -2 \end{bmatrix}\begin{bmatrix} 1 & 0 & -1 \\ 2 & 2 & 4 \\ 5 & -3 & 0 \end{bmatrix}\right)$$
$$= tr\left(\begin{bmatrix} 3 & 9 & 10 \\ 16 & -9 & -1 \\ -15 & 4 & -1 \end{bmatrix}\right) = 3 + (-9) + (-1)$$
$$= -7$$

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2. Given that $3A - 2B = \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}, -4A + B = \begin{bmatrix} -1 & 2 \\ -4 & 3 \end{bmatrix}$, find A, B

Sol.

$$3A - 2B + 2(-4A + B) = \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix} + 2\begin{bmatrix} -1 & 2 \\ -4 & 3 \end{bmatrix} \rightarrow -5A$$
$$= \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix} + 2\begin{bmatrix} -1 & 2 \\ -4 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 5 \\ -9 & 5 \end{bmatrix} \rightarrow$$
$$A = \begin{bmatrix} 0 & -1 \\ 9/5 & -1 \end{bmatrix}$$

$$-4A + B = \begin{bmatrix} -1 & 2 \\ -4 & 3 \end{bmatrix} \rightarrow B = \begin{bmatrix} -1 & 2 \\ -4 & 3 \end{bmatrix} + 4A$$
$$= \begin{bmatrix} -1 & 2 \\ -4 & 3 \end{bmatrix} + 4\begin{bmatrix} 0 & -1 \\ \frac{9}{5} & -1 \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ 16/5 & -1 \end{bmatrix}$$

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3. Solve the following matrix equation for a, b, c and d.

$$\begin{bmatrix} a-b & b+c \\ 3d+c & 2a-4d \end{bmatrix} = \begin{bmatrix} 8 & 1 \\ 7 & 6 \end{bmatrix}$$

Sol.

$a - b = 8 \;\&\; b + c = 1 \Longrightarrow \mathrm{a} + \mathrm{c} = 9$
$3d + c = 7 \;\&\; 2a - 4d = 6 \Longrightarrow 3\mathrm{a} + 2\mathrm{c} = 23$
Solve the last two equations together to get

$$a = 5, c = 4, \mathrm{b} = -3, \mathrm{d} = 1$$

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4. If A and B are two matrices Under what conditions, if any, is

a) $(AB)^2 = A^2B^2$?

Sol.

L.H.S = $(AB)^2$ = (AB)(AB) = ABAB
R.H.S = A^2B^2 = (AA) (BB) = AABB
L.H.S = R.H.S $\Longrightarrow$ ABAB = AABB $\Longrightarrow$ BA = AB
So the condition is BA = AB (Commute matrices)

b) $(A+B)^2 = A^2 + 2AB + B^2$

Sol.

L.H.S = $(A+B)^2$ = (A+B) (A+B) = $A^2 + AB + BA + B^2$
R.H.S = $A^2 + 2AB + B^2$
So the condition is BA = AB (Commute matrices)

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5. For any three matrices A, B, C, show that if AB = AC, doesn't necessarily imply that B = C.

Sol.

$$let\ A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, C = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$$

$$then\ \ AB = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ 4 & 6 \end{bmatrix}$$
$$and\ \ AC = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ 4 & 6 \end{bmatrix}$$

So AB = AC doesn't imply that B=C.

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6. For any nxn matrices A, B, show that

(a) $A + A^T$ is symmetric matrix, and $A - A^T$ is skew symmetric

Sol.

$(A + A^T)^T = A^T + (A^T)^T = A^T + A = A + A^T$ So $(A + A^T)^T$ is a symmetric matrix.

$(A - A^T)^T = A^T - (A^T)^T = A^T - A = -(A - A^T)$ So $(A - A^T)^T$ is a skew symmetric matrix.

(b) AA^T *and* A^TA Are symmetric matrices.

Sol.

$(AA^T)^T = (A^T)^T(A)^T = AA^T$ So AA^T is a symmetric matrix.

$(A^TA)^T = (A)^T(A^T)^T = A^TA$ So A^TA is a symmetric matrix.

(c) $(AB)^{-1} = B^{-1}A^{-1}$, A, B are non-singular

Sol.

We need to show that $(AB)(B^{-1}A^{-1}) = I$

$(AB)(B^{-1}A^{-1}) = (A)(BB^{-1})(A^{-1}) = (A)(I)(A^{-1}) = (A)A^{-1} = I$ So $(AB)^{-1} = B^{-1}A^{-1}$

(d) $(A^T)^{-1} = (A^{-1})^T$, A is non-singular

Sol.

We need to show that $(A^T)((A^{-1})^T) = I$

$(A^T)((A^{-1})^T) = (A^{-1}A)^T = (I)^T = I$ So $(A^T)^{-1} = (A^{-1})^T$

7. Express the matrix A as a sum of symmetric and skew-symmetric matrices:

a) $\boldsymbol{A} = \begin{bmatrix} \mathbf{8} & \mathbf{1} \\ \mathbf{7} & \mathbf{6} \end{bmatrix}$

Sol.

$$\boldsymbol{A} = \begin{bmatrix} \mathbf{8} & \mathbf{1} \\ \mathbf{7} & \mathbf{6} \end{bmatrix} \Longrightarrow \mathrm{A}^T = \begin{bmatrix} \mathbf{8} & \mathbf{7} \\ \mathbf{1} & \mathbf{6} \end{bmatrix}$$

The symmetric matrix will be:

$$\frac{\boldsymbol{A} + \mathrm{A}^T}{\mathbf{2}} = \begin{bmatrix} \mathbf{8} & \mathbf{4} \\ \mathbf{4} & \mathbf{6} \end{bmatrix}$$

The skew symmetric will be:

$$\frac{\boldsymbol{A} - \mathrm{A}^T}{\mathbf{2}} = \begin{bmatrix} \mathbf{0} & \mathbf{-3} \\ \mathbf{3} & \mathbf{0} \end{bmatrix}$$

$$\boldsymbol{A} = \begin{bmatrix} \mathbf{8} & \mathbf{4} \\ \mathbf{4} & \mathbf{6} \end{bmatrix} + \begin{bmatrix} \mathbf{0} & \mathbf{-3} \\ \mathbf{3} & \mathbf{0} \end{bmatrix}$$

b) $\boldsymbol{B} = \begin{bmatrix} \mathbf{2} & \mathbf{3} & \mathbf{-1} \\ \mathbf{1} & \mathbf{0} & \mathbf{3} \\ \mathbf{-3} & \mathbf{-1} & \mathbf{-2} \end{bmatrix}$

Sol.

$$\boldsymbol{B} = \begin{bmatrix} \mathbf{2} & \mathbf{3} & \mathbf{-1} \\ \mathbf{1} & \mathbf{0} & \mathbf{3} \\ \mathbf{-3} & \mathbf{-1} & \mathbf{-2} \end{bmatrix} \Longrightarrow \mathrm{B}^T = \begin{bmatrix} \mathbf{2} & \mathbf{1} & \mathbf{-3} \\ \mathbf{3} & \mathbf{0} & \mathbf{-1} \\ \mathbf{-1} & \mathbf{3} & \mathbf{-2} \end{bmatrix}$$

The symmetric matrix will be:

$$\frac{\boldsymbol{B} + \mathrm{B}^T}{\mathbf{2}} = \begin{bmatrix} \mathbf{2} & \mathbf{2} & \mathbf{-2} \\ \mathbf{2} & \mathbf{0} & \mathbf{1} \\ \mathbf{-2} & \mathbf{1} & \mathbf{-2} \end{bmatrix}$$

The skew symmetric will be:

$$\frac{\boldsymbol{B} - \mathrm{B}^T}{\mathbf{2}} = \begin{bmatrix} \mathbf{0} & \mathbf{1} & \mathbf{1} \\ \mathbf{-1} & \mathbf{0} & \mathbf{2} \\ \mathbf{-1} & \mathbf{-2} & \mathbf{0} \end{bmatrix}$$

$$\boldsymbol{B} = \begin{bmatrix} \mathbf{2} & \mathbf{2} & \mathbf{-2} \\ \mathbf{2} & \mathbf{0} & \mathbf{1} \\ \mathbf{-2} & \mathbf{1} & \mathbf{-2} \end{bmatrix} + \begin{bmatrix} \mathbf{0} & \mathbf{1} & \mathbf{1} \\ \mathbf{-1} & \mathbf{0} & \mathbf{2} \\ \mathbf{-1} & \mathbf{-2} & \mathbf{0} \end{bmatrix}$$

8. Let S be n X n symmetric matrix, and A be n X n anti-symmetric

(skew symmetric) matrix. Show that tr (SA) = 0.

Sol.

$$Tr(SA) = \sum_{i=1}^{n} (SA)_{ii}$$
$$= \sum_{i=1}^{n}\sum_{j=1}^{n} S_{ij}A_{ji}$$
$$= \sum_{i=1}^{n}\sum_{j=1}^{n} S_{ji}(-A_{ij})$$

$$= -\sum_{i=1}^{n}\sum_{j=1}^{n} S_{ji}(A_{ij})$$
$$= \sum_{j=1}^{n} (SA)_{jj}$$
$$= -Tr(SA)$$

So $Tr(SA) = 0$

Another Sol.

$$Tr(SA) = Tr(SA)^T$$
$$Tr(SA)^T = Tr\ (A^T S^T) = -Tr(SA)$$
$$Tr(SA) = -Tr(SA)$$
$$m = -m \implies \text{m must} = 0$$
$$so\ Tr(SA) = 0$$

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9. If A is an invertible matrix show that

(a) The inverse is unique.

Sol.

Suppose that we have more than one inverse for A so let matrix A has 2 inverse matrices B and C where $AB = BA = I$ and $AC = CA = I$

So $B = BI = B(AC) = (BA)C = IC = C$

So B must be equal to C so there is only one unique inverse matrix for A.

(b) $(kA)^{-1} = \frac{1}{k}(A)^{-1}$

Sol.

We need to show that $(kA)\frac{1}{k}(A)^{-1} = I$

$(kA)\frac{1}{k}(A)^{-1} = k\frac{1}{k}(A)(A^{-1}) = I$ So $(kA)^{-1} = \frac{1}{k}(A)^{-1}$

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10. If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $with$ $\det(A) \neq 0. Show\ that\ B = \frac{1}{\det(A)}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} isthe\ inverse\ of\ A.$

Sol.

We need to show that $AB = I$

$$\det(A) = ad - bc \ \rightarrow B = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \rightarrow AB$$
$$= \begin{bmatrix} a & b \\ c & d \end{bmatrix}\left(\frac{1}{ad - bc}\right)\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$
$$= \left(\frac{1}{ad - bc}\right)\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$
$$= \left(\frac{1}{ad - bc}\right)\begin{bmatrix} ad - bc & -ba + ba \\ cd - cd & ad - bc \end{bmatrix}$$
$$= \begin{bmatrix} \frac{ad - bc}{ad - bc} & 0 \\ 0 & \frac{ad - bc}{ad - bc} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

So $AB = I, so\ B\ is\ the\ inverse\ of\ A$

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11. In each of the following cases, find a 4x4 matrix $[a_{ij}]$ that satisfies the stated condition.

(a) $a_{ij} = i + j$

Sol.

$$\boldsymbol{a_{ij} = i + j} \Rightarrow [a_{ij}] = \begin{bmatrix} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \\ 5 & 6 & 7 & 8 \end{bmatrix}$$

(b) $a_{ij} = 0\ if\ i \neq j$

Sol.

$$\boldsymbol{a_{ij} = 0 \text{ if } i \neq j} \Rightarrow [a_{ij}] = \begin{bmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{22} & 0 & 0 \\ 0 & 0 & a_{33} & 0 \\ 0 & 0 & 0 & a_{44} \end{bmatrix}$$

(c) $a_{ij} = 0\ if\ i > j$

Sol.

$$\boldsymbol{a_{ij} = 0 \text{ if } i > j} \Rightarrow [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{bmatrix}$$

(d) $a_{ij} = 0\ if\ |i - j| < 1$

Sol.

$$\boldsymbol{a_{ij} = 0 \text{ if } |i - j| < 1} \Rightarrow [a_{ij}] = \begin{bmatrix} 0 & a_{12} & a_{13} & a_{14} \\ a_{21} & 0 & a_{23} & a_{24} \\ a_{31} & a_{32} & 0 & a_{34} \\ a_{41} & a_{42} & a_{43} & 0 \end{bmatrix}$$

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12. Let

$$A = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

Compute A^2 and A^3. What will A^n turn out to be?

Sol.

$$A^2 = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} = A$$

$$A^3 = A^2 A = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} = A$$

From the above matrices, one can conclude that:

$$A^n = \underbrace{A \times A \times A \times \cdots \times A}_{n-\text{times}}$$

$$= \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \cdots\cdots\cdots \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} = A$$

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13. Let

$$A = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

Compute A^2 and A^3. Then find A^n.

Sol.

$$A^2 = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = I$$

$$A^3 = A^2 A = IA = A = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$A^{2n} = (A^2)^n = \underbrace{A^2 \times A^2 \times A^2 \times \cdots \times A^2}_{n-\text{times}} = \underbrace{I \times I \times I \times \cdots \times I}_{n-\text{times}} = I$$

$$\Rightarrow A^{2n+1} = A^{2n} A = IA = A$$

14.

(a) Assuming that all matrices are $n \times n$ and invertible, simplify

$$(AB)^{-1}(AC^{-1})(D^{-1}C^{-1})^{-1}\, D^{-1}$$

(b) Assuming that all matrices are $n \times n$ and invertible, solve for D where

$$C^T B^{-1} A^2 C^{-1} D A^{-2} C^{-2} = I_n$$

Sol.

(a)

$$(AB)^{-1}(AC^{-1})(D^{-1}C^{-1})^{-1}\, D^{-1} = B^{-1} \underbrace{A^{-1}A}_{=I_n} \underbrace{C^{-1}C}_{=I_n} \underbrace{DD^{-1}}_{=I_n} = B^{-1}$$

(b)

$$\boldsymbol{C^T B^{-1} A^2 C^{-1} D A^{-2} C^{-2} = I_n} \xrightarrow{\text{Multiply by } (C^T)^{-1} \text{ from the left}} B^{-1} A^2 C^{-1} D A^{-2} C^{-2}$$

$$= (C^T)^{-1} I_n$$

$$\xrightarrow{\text{Multiply by } B \text{ from the left}} A^2 C^{-1} D A^{-2} C^{-2} = B(C^T)^{-1} I_n$$

$$\xRightarrow{\text{Multiply by } A^{-2} \text{ from the left}} C^{-1}DA^{-2}C^{-2} = A^{-2}B(C^T)^{-1}I_n$$

$$\xRightarrow{\text{Multiply by } C \text{ from the left}} DA^{-2}C^{-2} = CA^{-2}B(C^T)^{-1}I_n$$

$$\xRightarrow{\text{Multiply by } C^2 \text{ from the right}} DA^{-2} = CA^{-2}B(C^T)^{-1}I_nC^2$$

$$\xRightarrow{\text{Multiply by } A^2 \text{ from the right}} \boxed{D = CA^{-2}B(C^T)^{-1}I_nC^2A^2}$$

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15. Let $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 1 & -2 & 0 \end{bmatrix}$. Calculate $A^3 - A$ then deduce that A is invertible and determine A^{-1}.

Sol.

$$A^3 - A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 1 & -2 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 1 & -2 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 1 & -2 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 1 & -2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} = 4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 4I$$

$$A^3 - A = 4I \Longrightarrow \frac{1}{4}[A^3 - A] = I \Longrightarrow A\left(\frac{1}{4}[A^2 - I]\right)$$

$$= I \text{ OR } \left(\frac{1}{4}[A^2 - I]\right)A = I$$

A is invertible with inverse of $\frac{1}{4}[A^2 - I]$

$$A^{-1} = \frac{1}{4}[A^2 - I] = \frac{1}{4}\left[\begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 1 & -2 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 1 & -2 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right]$$

$$= \begin{bmatrix} 0.5 & -1 & 0.5 \\ 0.25 & -0.5 & -0.25 \\ 0.25 & 0.5 & -0.25 \end{bmatrix}$$
