Problems

Example 1. Evaluate $\mathcal{L}^{-1}\left\{\frac{5}{s+3}\right\}$.

$$\mathcal{L}^{-1}\left\{\frac{5}{s+3}\right\} = 5\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\} = 5e^{-3t}.$$

Example 2. Evaluate $\mathcal{L}^{-1}\left\{\frac{2}{s^2+16}\right\}$.

$$\mathcal{L}^{-1}\left\{\frac{2}{s^2+16}\right\} = \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{4}{s^2+16}\right\} = \frac{1}{2}\sin 4t.$$

Example 3. Evaluate $\mathcal{L}^{-1}\left\{\frac{s+1}{s^2+1}\right\}$.

$$\mathcal{L}^{-1}\left\{\frac{s+1}{s^2+1}\right\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = \cos t + \sin t.$$

Example 4. Evaluate $\mathcal{L}^{-1}\left\{\frac{1}{s^4}\right\}$.

$$\mathcal{L}^{-1}\left\{\frac{1}{s^4}\right\} = \frac{1}{3!}\mathcal{L}^{-1}\left\{\frac{3!}{s^4}\right\} = \frac{1}{6}t^3.$$

In order to find inverse transforms we very often have to perform a partial fraction decomposition. Here are some typical examples:

Example 5. Evaluate $\mathcal{L}^{-1}\left\{\frac{s^2-10s-25}{s^3-25s}\right\}$.

$$\frac{s^2-10s-25}{s^3-25s} = \frac{s^2-10s-25}{s(s^2-25)} = \frac{s^2-10s-25}{s(s-5)(s+5)}\ .$$

Write the last fraction as $\frac{s^2-10s-25}{s(s-5)(s+5)} = \frac{A}{s} + \frac{B}{s-5} + \frac{C}{s+5}$. We need to find A, B and C. Multiply both sides of the equation by $s(s-5)(s+5)$ and we obtain

$$s^2 - 10s - 25 = A(s-5)(s+5) + Bs(s+5) + Cs(s-5).$$

The denominator is zero when $s = 0,\ +5,\ -5$, so put these values into the above equation.

$$\begin{array}{lll} \underline{s=0} & -25 = A(-5)(5) & \therefore A = 1. \\ \underline{s=5} & 25-50-25 = B(5)(10) & \therefore B = -1. \\ \underline{s=-5} & 25+50-25 = C(-5)(-10) & \therefore C = 1. \end{array}$$

Hence

$$\begin{aligned}
\frac{s^2-10s-25}{s(s-5)(s+5)} &= \frac{1}{s}-\frac{1}{s-5}+\frac{1}{s+5}\,. \\
\therefore \mathcal{L}^{-1}\left\{\frac{s^2-10s-25}{s(s-5)(s+5)}\right\} &= \mathcal{L}^{-1}\left\{\frac{1}{s}\right\}-\mathcal{L}^{-1}\left\{\frac{1}{s-5}\right\}+\mathcal{L}^{-1}\left\{\frac{1}{s+5}\right\} \\
&= 1-e^{5t}+e^{-5t}.
\end{aligned}$$

Example 6. Evaluate $\mathcal{L}^{-1}\left\{\dfrac{2s-1}{s^3(s+1)}\right\}$.

Write $$\frac{2s-1}{s^3(s+1)}=\frac{A}{s}+\frac{B}{s^2}+\frac{C}{s^3}+\frac{D}{s+1}\,.$$

Multiply both sides of the equation by $s^3(s+1)$ to get

$$2s-1=As^2(s+1)+Bs(s+1)+C(s+1)+Ds^3.$$

$$\begin{aligned}
&\text{Put } s=0 && : -1=C && \therefore C=-1 \\
&\text{Put } s=-1 && : -3=-D && \therefore D=3,
\end{aligned}$$

$$\text{i.e., } 2s-1=A(s^3+s^2)+B(s^2+s)-s-1+3s^3.$$

$$\begin{aligned}
&s^3 \text{ terms} && : && A+3=0 && \therefore A=-3 \\
&s^2 \text{ terms} && : && A+B=0 && \therefore B=3 \\
&s \text{ terms} && : && B-1=2 && \therefore B=3.
\end{aligned}$$

$$\therefore \frac{2s-1}{s^3(s+1)}=-\frac{3}{s}+\frac{3}{s^2}-\frac{1}{s^3}+\frac{3}{s+1}\,.$$

$$\begin{aligned} \therefore \mathcal{L}^{-1}\left\{\frac{2s-1}{s^3(s+1)}\right\} &= -3\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + 3\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\} + 3\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} \\ &= -3 + 3t - \frac{1}{2}t^2 + 3e^{-t}. \end{aligned}$$

Example 7. Evaluate $\mathcal{L}^{-1}\left\{\frac{s+2}{(s+1)(s^2+4)}\right\}$.

Write
$$\frac{s+2}{(s+1)(s^2+4)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+4}\,.$$

Multiply both sides by $(s+1)(s^2+4)$ to get

$$s+2 = A(s^2+4) + (Bs+C)(s+1).$$

$$\begin{array}{lll} \text{Put } s=-1: & 1 = 5A & \therefore A = \frac{1}{5} \\ s^2 \text{ terms} & 0 = A + B & \therefore B = -\frac{1}{5} \\ s \text{ terms} & 1 = B + C & \therefore C = \frac{6}{5}\,. \end{array}$$

$$\begin{aligned} \therefore \frac{s+2}{(s+1)(s^2+4)} &= \frac{\frac{1}{5}}{s+1} + \frac{-\frac{1}{5}s+\frac{6}{5}}{s^2+4}\,. \\ \therefore \mathcal{L}^{-1}\left\{\frac{s+2}{(s+1)(s^2+4)}\right\} &= \frac{1}{5}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} - \frac{1}{5}\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} + \frac{3}{5}\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\}. \\ &= \frac{1}{5}e^{-t} - \frac{1}{5}\cos 2t + \frac{3}{5}\sin 2t. \end{aligned}$$

Problem Set 1.4

Find $f(t)$ if $\mathcal{L}\{f(t)\}$ is given by

1. $\dfrac{s+12}{s^2+4s}$
2. $\dfrac{s-3}{s^2-1}$
3. $\dfrac{3s}{s^2+2s-8}$
4. $\dfrac{2s^2+5s-1}{s^3-s}$
5. $\dfrac{s+1}{s^3(s-1)(s+2)}$
6. $\dfrac{3s^2-2s-1}{(s-3)(s^2+1)}$

Use partial fractions (if necessary) and the First Shifting Theorem to find the inverse Laplace transforms of the following functions:

1. $\dfrac{10-4s}{(s-2)^2}$ 2. $\dfrac{s^2+s-2}{(s+1)^3}$ 3. $\dfrac{s^3-7s^2+14s-9}{(s-1)^2(s-2)^3}$

4. $\dfrac{s^2-6s+7}{(s^2-4s+5)s}$ 5. $\dfrac{2s-1}{s^2(s+1)^3}$ 6. $\dfrac{3!}{(s-2)^4}$

Find the Laplace transforms of the following functions:

7. $\mathcal{L}\{te^{8t}\}$ 8. $\mathcal{L}\{t^7e^{-5t}\}$ 9. $\mathcal{L}\{e^{-2t}\cos 4t\}$

10. $\mathcal{L}\{e^{3t}\sinh t\}$ 11. $\mathcal{L}\left\{\dfrac{\sin 2t}{e^t}\right\}$ 12. $\mathcal{L}\{e^{2t}\cos^2 2t\}$

13. $\mathcal{L}\{e^{3t}(t+2)^2\}$ 14. $\mathcal{L}\{t^{1/2}(e^t+e^{-2t})\}$

Evaluate the following

1. $\mathcal{L}\{(t-1)u_1(t)\}$ 2. $\mathcal{L}\{e^{2-t}u_2(t)\}$ 3. $\mathcal{L}\{(3t+1)u_3(t)\}$

4. $\mathcal{L}\{(t-1)^3e^{t-1}u_1(t)\}$ 5. $\mathcal{L}\{te^{t-5}u_5(t)\}$ 6. $\mathcal{L}\{\cos t\cdot u_{2\pi}(t)\}$

7. $\mathcal{L}^{-1}\left\{\dfrac{e^{-2s}}{s^3}\right\}$ 8. $\mathcal{L}^{-1}\left\{\dfrac{(1+e^{-2s})^2}{s+2}\right\}$ 9. $\mathcal{L}^{-1}\left\{\dfrac{e^{-\pi s}}{s^2+1}\right\}$

10. $\mathcal{L}^{-1}\left\{\dfrac{se^{-\pi s/2}}{s^2+4}\right\}$ 11. $\mathcal{L}^{-1}\left\{\dfrac{e^{-s}}{s(s+1)}\right\}$ 12. $\mathcal{L}^{-1}\left\{\dfrac{e^{-2s}}{s^2(s-1)}\right\}$

13. $\mathcal{L}^{-1}\left\{\dfrac{1-e^{-s}}{s^2}\right\}$ 14. $\mathcal{L}^{-1}\left\{\dfrac{2}{s}-\dfrac{3e^{-s}}{s^2}+\dfrac{5e^{-2s}}{s^2}\right\}$

In Problems 15 - 20 write each function in terms of unit step functions and find the Laplace transform of the given function.

15. $f(t) = \begin{cases} 2, & 0 \le t < 3 \\ -2, & t \ge 3 \end{cases}$

16. $f(t) = \begin{cases} 1, & 0 \le t < 4 \\ 0, & 4 \le t < 5 \\ 1, & t \ge 5 \end{cases}$

17. $f(t) = \begin{cases} 0, & 0 \le t < 1 \\ t^2, & t \ge 1 \end{cases}$

18. $f(t) = \begin{cases} 0, & 0 \le t < \frac{3\pi}{2} \\ \sin t, & t \ge \frac{3\pi}{2} \end{cases}$

19. $f(t) = \begin{cases} t, & 0 \le t < 2 \\ 0, & t \ge 2 \end{cases}$

20. $f(t)$ is the staircase function,

i.e., see graph below.