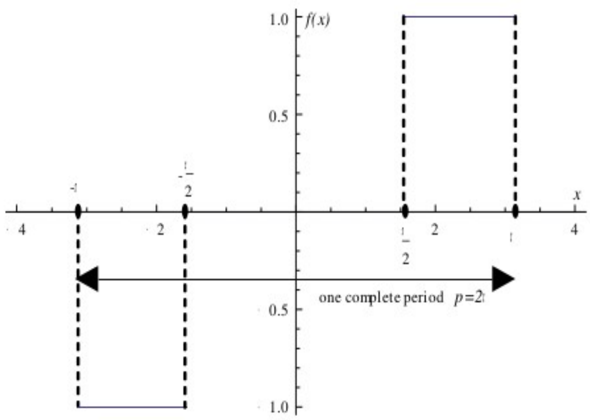
Solutions of Worksheet # 5
Fourier Series

Find the Fourier series for each one of the following functions:

1. $$f(x)=\begin{cases}-1 & -\pi<x<-\dfrac{\pi}{2}\\ 0 & -\dfrac{\pi}{2}<x<\dfrac{\pi}{2}\\ 1 & \dfrac{\pi}{2}<x<\pi\end{cases},\qquad f(x+2\pi)=f(x)$$

Sol. It is clear that $f(x)$ is odd so



$$a_0=a_n=0.$$

$$\therefore b_n=\frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx$$

$$\Rightarrow b_n=\frac{2}{\pi}\int_{\pi/2}^{\pi}\sin(nx)\,dx$$

$$\Rightarrow b_n=-\frac{2}{n\pi}\left[\cos(nx)\right]_{x=\pi/2}^{x=\pi}$$

$$\Rightarrow b_n=-\frac{2}{n\pi}\left[\cos(nx)\right]_{x=\pi/2}^{x=\pi}=-\frac{2}{n\pi}\left[(-1)^n-\cos\left(\frac{n\pi}{2}\right)\right].$$

Hence, the Fourier series of $f(x)$ is

$$f(x)=\sum_{n=1}^{\infty}\frac{2}{n\pi}\left[\cos\left(\frac{n\pi}{2}\right)-(-1)^n\right]\sin(nx).$$

2. $f(x) = x + |x|, \quad -\pi < x < \pi, \quad f(x+2\pi) = f(x)$

Sol. It is clear that $f(x)$ is neither odd nor even, so

Hint:

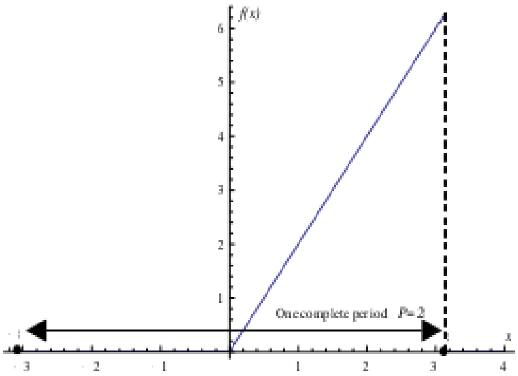
$$x + |x| = \begin{cases} 2x & x \geq 0 \\ 0 & x < 0 \end{cases}$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{0}^{\pi} 2x\,dx = \frac{1}{\pi}\left[x^2\right]_{x=0}^{x=\pi} = \pi.$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx = \frac{2}{\pi}\int_{0}^{\pi} x\cos(nx)\,dx.$$

$$\therefore a_n = \frac{2}{\pi}\left[\frac{x}{n}\sin(nx) + \frac{1}{n^2}\cos(nx)\right]_{x=0}^{x=\pi}.$$

$$\therefore a_n = \frac{2}{\pi}\left[\frac{(-1)^n}{n^2} - \frac{1}{n^2}\right] = \frac{2}{\pi n^2}[(-1)^n - 1].$$



d/dx	$\int dx$
x (+)	$\cos nx$
1 (-)	$\frac{1}{n}\sin nx$
0	$-\frac{1}{n^2}\cos nx$

d/dx	$\int dx$
x (+)	$\sin nx$
1 (-)	$-\frac{1}{n}\cos nx$
0	$-\frac{1}{n^2}\sin nx$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx = \frac{2}{\pi}\int_{0}^{\pi} x\sin(nx)\,dx.$$

$$\therefore b_n = \frac{2}{\pi}\left[-\frac{x}{n}\cos(nx) + \frac{1}{n^2}\sin(nx)\right]_{x=0}^{x=\pi}.$$

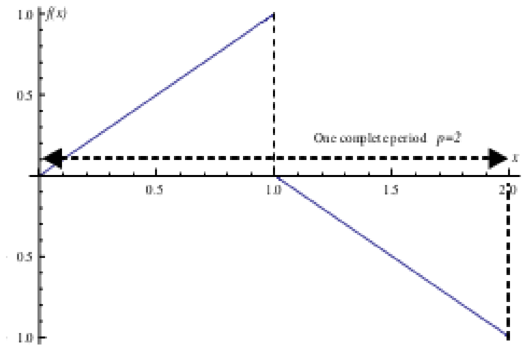
$$\therefore b_n = \frac{2}{\pi}\left[-\frac{\pi}{n}(-1)^n\right] = \frac{2}{n}(-1)^{n+1}.$$

Hence, the Fourier series of $f(x)$ is

$$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty}\left(\frac{2}{n^2\pi}[(-1)^n - 1]\cos(nx) + \frac{2}{n}(-1)^{n+1}\sin(nx)\right).$$

3. $f(x) = \begin{cases} x & 0 < x < 1 \\ 1 - x & 1 < x < 2 \end{cases}, \quad f(x+2) = f(x)$

Sol. It is clear that $f(x)$ is neither odd nor even, so



$$a_0 = \int_0^2 f(x)\,dx = \int_0^1 x\,dx + \int_1^2 (1-x)\,dx.$$

$$\therefore a_0 = \frac{1}{2}\left[x^2\right]_{x=0}^{x=1} + \left[x - \frac{x^2}{2}\right]_{x=1}^{x=2} = 0.$$

$$a_n = \int_0^2 f(x)\cos(n\pi x)\,dx.$$

$$\therefore a_n = \int_0^1 x\cos(n\pi x)\,dx + \int_1^2 (1-x)\cos(n\pi x)\,dx.$$

d/dx	$\int dx$
x (+)	$\cos n\pi x$
1 (-)	$\frac{1}{n\pi}\sin n\pi x$
0	$-\frac{1}{n^2\pi^2}\cos n\pi x$

d/dx	$\int dx$
$1-x$ (+)	$\cos n\pi x$
-1 (-)	$\frac{1}{n\pi}\sin n\pi x$
0	$-\frac{1}{n^2\pi^2}\cos n\pi x$

$$\therefore a_n = \left[\frac{x}{n\pi}\sin(n\pi x) + \frac{1}{n^2\pi^2}\cos(n\pi x)\right]_{x=0}^{x=1} + \left[\frac{1-x}{n\pi}\sin(n\pi x) - \frac{1}{n^2\pi^2}\cos(n\pi x)\right]_{x=1}^{x=2}.$$

$$\therefore a_n = \left[\frac{(-1)^n - 1}{n^2\pi^2}\right] + \left[\frac{(-1)^n - 1}{n^2\pi^2}\right] = \frac{2}{n^2\pi^2}[(-1)^n - 1].$$

$$b_n = \int_0^2 f(x)\sin(n\pi x)\,dx.$$

$$\therefore b_n = \int_0^1 x\sin(n\pi x)\,dx + \int_1^2 (1-x)\sin(n\pi x)\,dx.$$

d/dx	$\int dx$
x (+)	$\sin n\pi x$
1 (-)	$-\frac{1}{n\pi}\cos n\pi x$
0	$-\frac{1}{n^2\pi^2}\sin n\pi x$

d/dx	$\int dx$
$1-x$ (+)	$\sin n\pi x$
-1 (-)	$-\frac{1}{n\pi}\cos n\pi x$
0	$-\frac{1}{n^2\pi^2}\sin n\pi x$

$$\therefore b_n = \left[-\frac{x}{n\pi}\cos(n\pi x) + \frac{1}{n^2\pi^2}\sin(n\pi x) \right]_{x=0}^{x=1} + \left[\frac{x-1}{n\pi}\cos(n\pi x) - \frac{1}{n^2\pi^2}\sin(n\pi x) \right]_{x=1}^{x=2}.$$

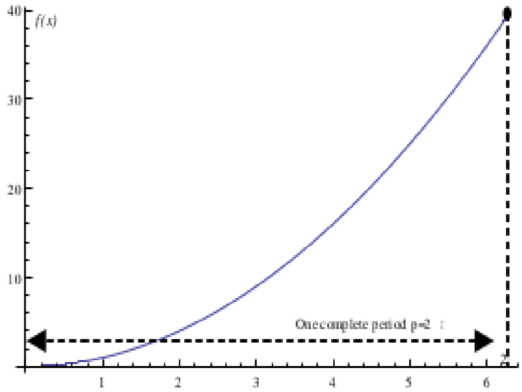
$$\therefore b_n = \left[\frac{(-1)^{n+1}}{n\pi} \right] + \left[\frac{1}{n\pi} \right] = \frac{1}{n\pi}[1+(-1)^{n+1}].$$

Hence, the Fourier series of $f(x)$ is

$$f(x) = \sum_{n=1}^{\infty} \left(\frac{2}{n^2\pi^2}[(-1)^n - 1]\cos(n\pi x) + \frac{1}{n\pi}[1+(-1)^{n+1}]\sin(n\pi x) \right).$$

4. $f(x) = x^2, \quad 0 < x < 2\pi, \quad f(x+2\pi) = f(x).$

Sol. It is clear that $f(x)$ is neither odd nor even, so



$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi} x^2 dx.$$

$$\therefore a_0 = \frac{1}{3\pi}\left[x^3\right]_{x=0}^{x=2\pi} = \frac{8\pi^2}{3}.$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos(nx)\,dx.$$

$$\therefore a_n = \frac{1}{\pi}\int_0^{2\pi} x^2\cos(nx)\,dx.$$

d/dx	$\int dx$
x^2 (+)	$\cos nx$
$2x$ (-)	$\frac{1}{n}\sin nx$
2 (+)	$-\frac{1}{n^2}\cos nx$
0	$-\frac{1}{n^3}\sin nx$

d/dx	$\int dx$
x^2 (+)	$\sin nx$
$2x$ (-)	$-\frac{1}{n}\cos nx$
2 (+)	$-\frac{1}{n^2}\sin nx$
0	$\frac{1}{n^3}\cos nx$

$$\therefore a_n = \frac{1}{\pi}\left[\frac{x^2}{n}\sin(nx) + \frac{2x}{n^2}\cos(nx) - \frac{2}{n^3}\sin(nx)\right]_{x=0}^{x=2\pi} = \frac{1}{\pi}\left[\frac{4\pi}{n^2}\right] = \frac{4}{n^2}.$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin(nx)\,dx.$$

$$\therefore b_n = \frac{1}{\pi}\int_0^{2\pi} x^2\sin(nx)\,dx.$$

$$\therefore b_n = \frac{1}{\pi}\left[-\frac{x^2}{n}\cos(nx) + \frac{2x}{n^2}\sin(nx) + \frac{2}{n^3}\cos(nx)\right]_{x=0}^{x=2\pi} = \frac{1}{\pi}\left[-\frac{4\pi^2}{n}\right] = -\frac{4\pi}{n}.$$

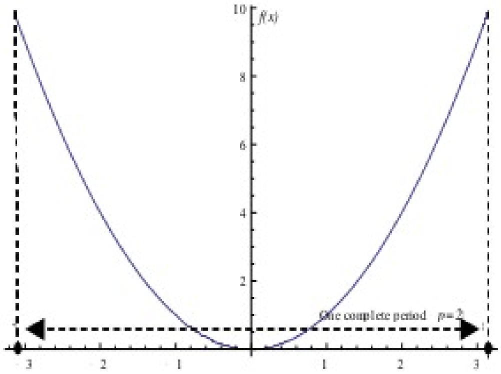
Hence, the Fourier series of $f(x)$ is

$$f(x) = \frac{4\pi^2}{3} + \sum_{n=1}^{\infty}\left(\frac{4}{n^2}\cos(nx) - \frac{4\pi}{n}\sin(nx)\right).$$

5. $f(x)=x^2, \quad -\pi \le x \le \pi, \quad f(x+2\pi)=f(x)$. Hence, deduce that

$\frac{\pi^2}{12}=1-\frac{1}{4}+\frac{1}{9}-\frac{1}{16}+\cdots$

Sol. It is clear that $f(x)$ is even, so $b_n=0$.



$$a_0=\frac{2}{\pi}\int_0^{\pi} f(x)\,dx=\frac{2}{\pi}\int_0^{\pi} x^2 dx.$$

$$\therefore a_0=\frac{2}{3\pi}\left[x^3\right]_{x=0}^{x=\pi}=\frac{2\pi^2}{3}.$$

$$a_n=\frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)\,dx.$$

$$\therefore a_n=\frac{2}{\pi}\int_0^{\pi} x^2\cos(nx)\,dx.$$

d/dx	$\int dx$
x^2 (+)	$\cos nx$
$2x$ (-)	$\frac{1}{n}\sin nx$
2 (+)	$-\frac{1}{n^2}\cos nx$
0	$-\frac{1}{n^3}\sin nx$

$$\therefore a_n=\frac{2}{\pi}\left[\frac{x^2}{n}\sin(nx)+\frac{2x}{n^2}\cos(nx)-\frac{2}{n^3}\sin(nx)\right]_{x=0}^{x=\pi}=\frac{2}{\pi}\left[\frac{2\pi}{n^2}(-1)^n\right]=\frac{4}{n^2}(-1)^n.$$

Hence, the Fourier series of $f(x)$ is

$$f(x)=\frac{\pi^2}{3}+\sum_{n=1}^{\infty}\left(\frac{4}{n^2}(-1)^n\cos(nx)\right).$$

Let $x=0$ so that $f(0)=0$ and

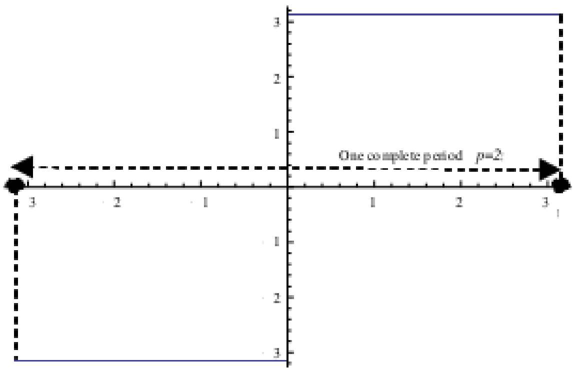
$$\frac{\pi^2}{3}+\sum_{n=1}^{\infty}\frac{4}{n^2}(-1)^n=0 \Rightarrow \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}=-\frac{\pi^2}{12}.$$

$$\therefore \left[-1+\frac{1}{4}-\frac{1}{9}+\frac{1}{16}-\cdots\right]=-\frac{\pi^2}{12}.$$

$$\therefore \left[1-\frac{1}{4}+\frac{1}{9}-\frac{1}{16}+\cdots\right]=\frac{\pi^2}{12}.$$

6. $f(x)=\begin{cases} -\pi & -\pi<x<0 \\ \pi & 0<x<\pi \end{cases}$, $f(x+2\pi)=f(x)$. Hence, find the sum of the series $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2n-1}$

Sol. It is clear that $f(x)$ is odd, so $a_0=a_n=0$.



$$b_n=\frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx.$$

$$\therefore b_n=\frac{2}{\pi}\int_0^{\pi} \pi\sin(nx)\,dx.$$

$$\therefore b_n=\frac{2}{\pi}\left[-\frac{\pi}{n}\cos(nx)\right]_{x=0}^{x=\pi}.$$

$$\therefore b_n=\frac{2}{n}[1-(-1)^n].$$

Hence, the Fourier series of $f(x)$ is

$$f(x)=\sum_{n=1}^{\infty}\left(\frac{2}{n}[1-(-1)^n]\sin(nx)\right).$$

Let $x=\pi/2$ so that $f(\pi/2)=\pi$ and

$$\sum_{n=1}^{\infty}\frac{2}{n}[1-(-1)^n]\sin\left(\frac{n\pi}{2}\right)=\pi \Rightarrow \sum_{n=1}^{\infty}\frac{[1-(-1)^n]}{n}\sin\left(\frac{n\pi}{2}\right)=\frac{\pi}{2}.$$

$$\therefore \left[2-\frac{2}{3}+\frac{2}{5}-\frac{2}{7}+\cdots\right]=\frac{\pi}{2} \Rightarrow \left[1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots\right]=\frac{\pi}{4}.$$

$$\therefore \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2n-1}=\frac{\pi}{4}.$$