# Theory of Computation

## CFG and PDA Equivalence

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**Some of this lecture slides are taken from previous course of prof. Dr.Abdulhussein Mohsin Abdullah**

**Lemma:**

If a **PDA** recognizes a language, then it is context-free.

**Proof idea:**

**Given PDA P = (Q, Σ, Γ, δ, q, F)**

**Construct a CFG G = (V, Σ, R, S) such that L(G) = L(P)**

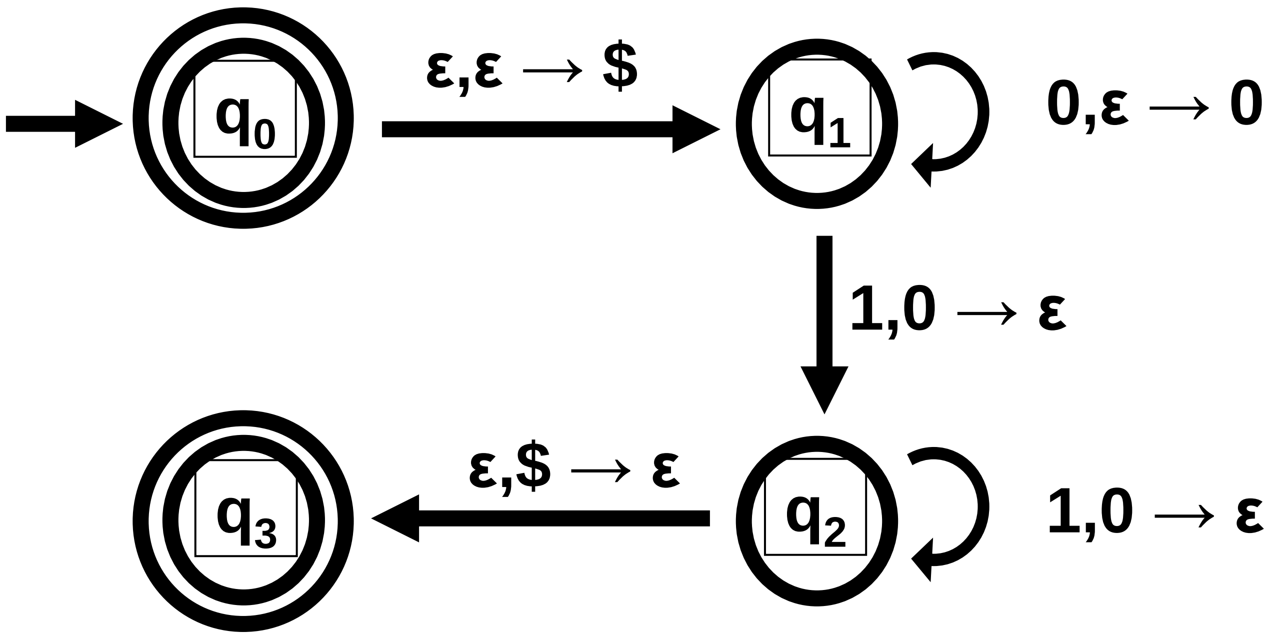
First, simplify P to have the following form:

(1) It has a unique accept state, $q_{acc}$

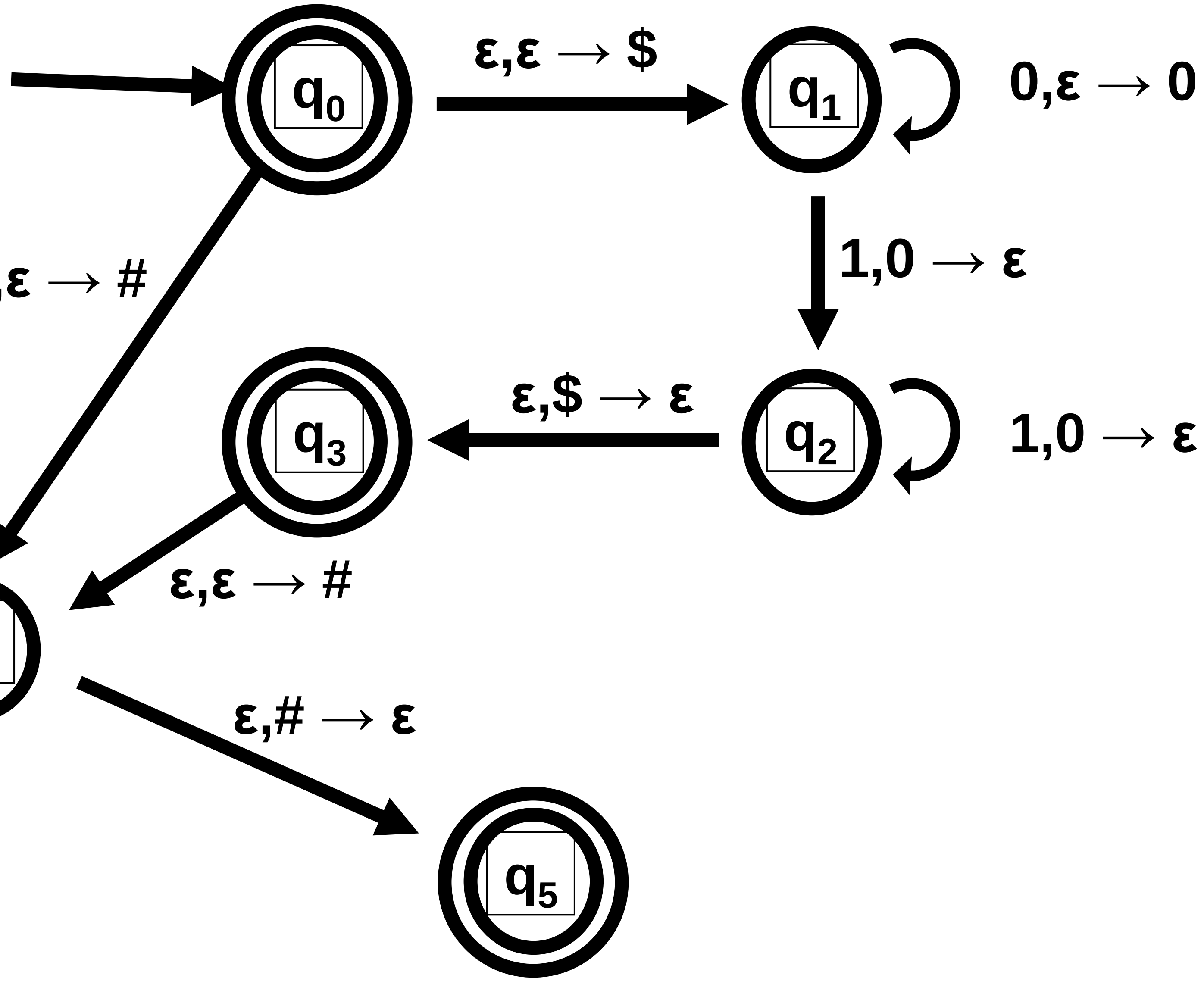
(2) It empties the stack before accepting

(3) Each transition either pushes a symbol or pops a symbol, but not both at the same time

# SIMPLIFY



# (1) It has a unique accept state, $q_{acc}$



Idea For Our Grammar G:
For every pair of states p and q in PDA P,

G will have a variable $A_{pq}$ whose production rules will generate all strings x that can take:

P from p with an empty stack
to q with an empty stack

$$V = \{A_{pq} \mid p,q \in Q\}$$

$$S = A_{q_0 q_{acc}}$$

**WANT: $A_{pq}$ generates all strings that take p with an empty stack to q with empty stack**

**Let x be such a string**

- **P's first move on x must be a push**
- **P's last move on x must be a pop**

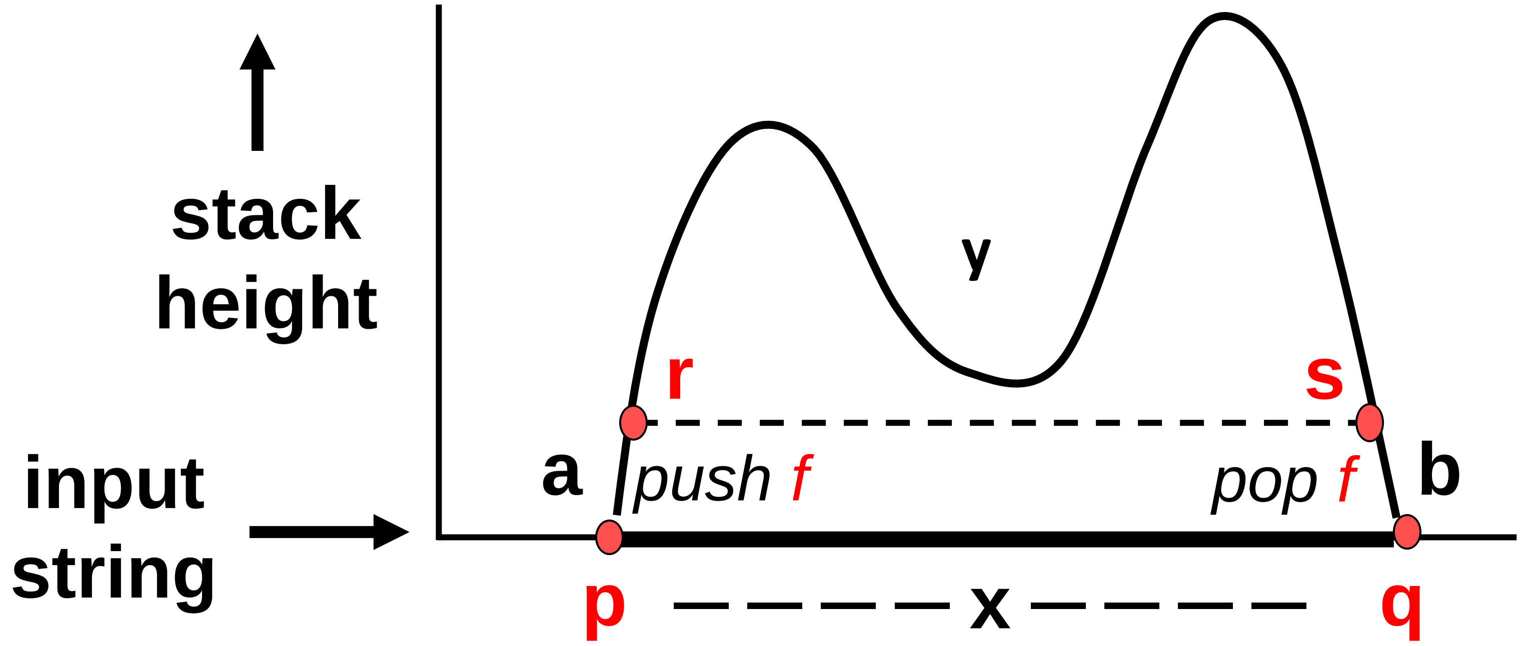
**Two possibilities:**

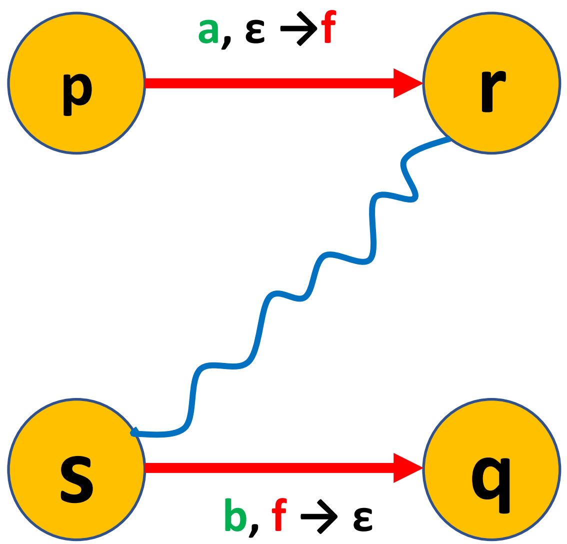
1. **The symbol popped at the end is exactly the one pushed at the beginning**

2. **The symbol popped at the end is not the one pushed at the beginning**

x = ayb  takes p with empty stack to q with empty stack

1. The symbol f popped at the end is exactly the one pushed at the beginning



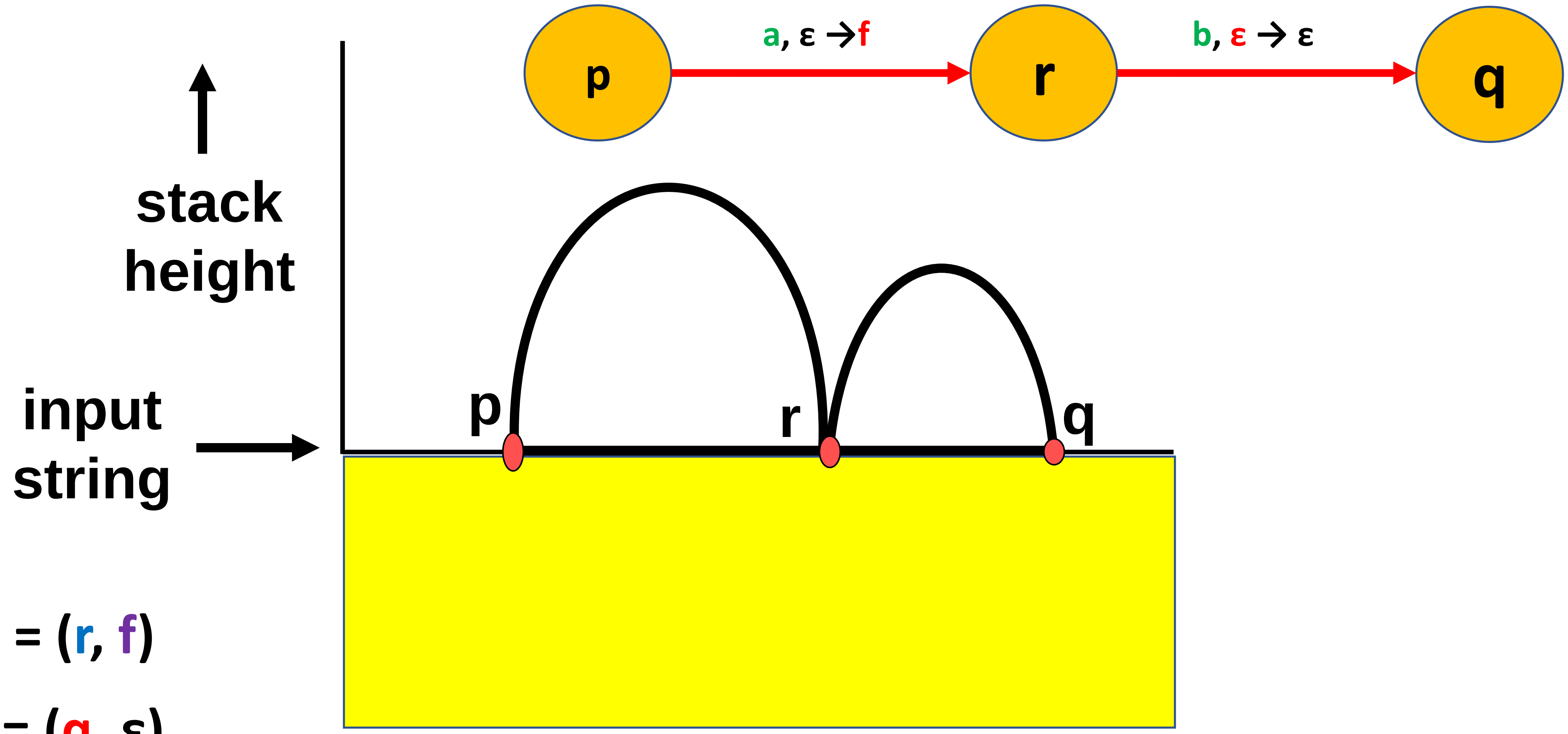




$\delta(p, a, \varepsilon) \rightarrow (r, f)$

$\delta(s, b, f) \rightarrow (q, \varepsilon)$

$A_{pq} \rightarrow aA_{rs}b$

## 2. The symbol popped at the end is not the one pushed at the beginning




$\delta(p, a, \varepsilon) = (r, f)$

$\delta(r, b, \varepsilon) = (q, \varepsilon)$

$$A_{pq} \rightarrow A_{pr}A_{rq}$$

**Formally:**

$V = \{A_{pq} \mid p, q \in Q\}$

$S = A_{q_0 q_{acc}}$

For every $p, q, r, s \in Q$, $f \in \Gamma$ and $a, b \in \Sigma_\varepsilon$

If $\delta(p, a, \varepsilon) = (r, f)$ and $\delta(s, b, f) = (q, \varepsilon)$

add the rule $A_{pq} \to aA_{rs}b$

For every $p, q, r \in Q$,

If $\delta(p, a, \varepsilon) = (r, f)$ and $\delta(r, b, \varepsilon) = (q, \varepsilon)$

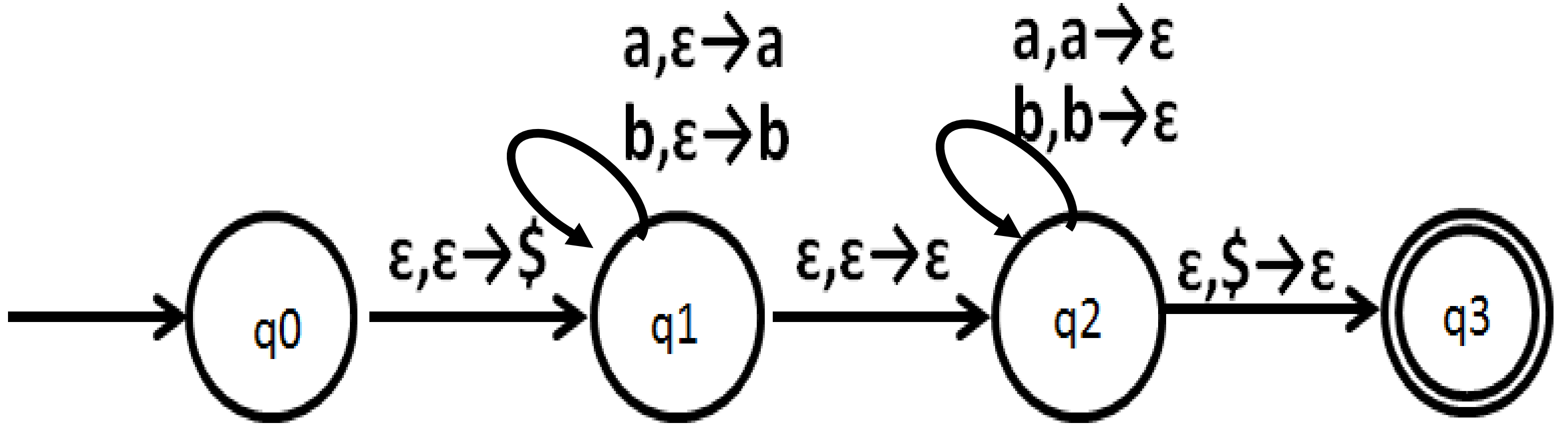
add the rule $A_{pq} \to A_{pr}A_{rq}$

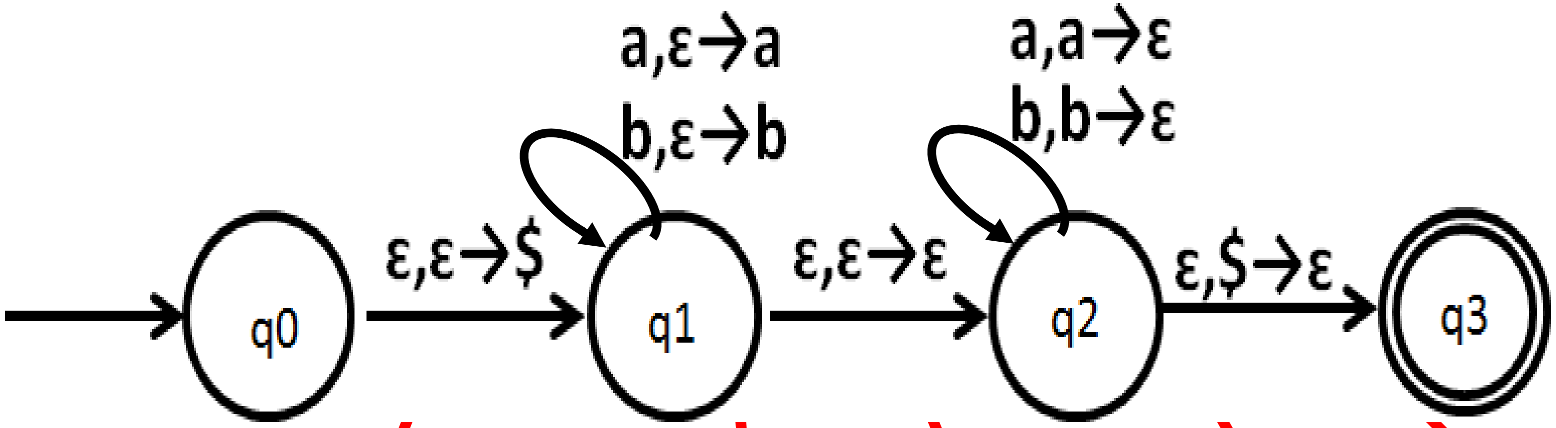
For every $p \in Q$,

add the rule $A_{pp} \to \varepsilon$

# Example

Write a CFG for the following PDA






1. $\delta(q0, \varepsilon, \varepsilon) = (q1,\$)$

2. $\delta(q1, a, \varepsilon) = (q1,a)$
3. $\delta(q1, b, \varepsilon) = (q1,b)$

4. $\delta(q1, \varepsilon, \varepsilon) = (q2, \varepsilon)$

5. $\delta(q2, a, a) = (q2, \varepsilon)$
6. $\delta(q2, b, b) = (q2, \varepsilon)$

7. $\delta(q2, \varepsilon, \$) = (q3, \varepsilon)$

1. $\delta(q0, \varepsilon, \varepsilon) = (q1,\$)$
2. $\delta(q1, a, \varepsilon) = (q1,a)$
3. $\delta(q1, b, \varepsilon) = (q1,b)$
4. $\delta(q1, \varepsilon, \varepsilon) = (q2, \varepsilon)$
5. $\delta(q2, a, a) = (q2, \varepsilon)$
6. $\delta(q2, b, b) = (q2, \varepsilon)$
7. $\delta(q2, \varepsilon, \$) = (q3, \varepsilon)$

$A_{00} \rightarrow \varepsilon$

$A_{11} \rightarrow \varepsilon$

$A_{22} \rightarrow \varepsilon$

$A_{33} \rightarrow \varepsilon$

| | |
|---|---|
| $A_{03} \rightarrow \varepsilon A_{12} \varepsilon$ | 1 , 7 |
| $A_{01} \rightarrow A_{01}A_{11}$ | 1 , 2 |
| $A_{01} \rightarrow A_{01}A_{11}$ | 1 , 3 |
| $A_{02} \rightarrow A_{01}A_{12}$ | 1 , 4 |
| | 1 , 5 |
| | 1 , 6 |
| $A_{11} \rightarrow A_{11}A_{11}$ | 2 , 3 |
| $A_{12} \rightarrow A_{11}A_{12}$ | 2 , 4 |
| $A_{12} \rightarrow aA_{12}\,a$ | 2 , 5 |
| | 2 , 6 |
| | 2 , 7 |

| | |
|---|---|
| $A_{12} \rightarrow A_{11}A_{12}$ | 3 , 4 |
| | 3 , 5 |
| $A_{12} \rightarrow bA_{12}\,b$ | 3 , 6 |
| | 3 , 7 |
| $A_{12} \rightarrow A_{12}A_{22}$ | 4 , 5 |
| $A_{12} \rightarrow A_{12}A_{22}$ | 4 , 6 |
| $A_{13} \rightarrow A_{12}A_{23}$ | 4 , 7 |
| $A_{22} \rightarrow A_{22}A_{22}$ | 5 ,6 |
| $A_{23} \rightarrow A_{22}A_{23}$ | 5 ,7 |
| $A_{23} \rightarrow A_{22}A_{23}$ | 6 ,7 |

$A_{00} \rightarrow \varepsilon$

$A_{11} \rightarrow \varepsilon$

$A_{22} \rightarrow \varepsilon$

$A_{33} \rightarrow \varepsilon$

$A_{03} \rightarrow \varepsilon A_{12} \varepsilon$ 1 , 7

$A_{01} \rightarrow A_{01}A_{11}$ 1 , 2

$A_{01} \rightarrow A_{01}A_{11}$ 1 , 3

$A_{02} \rightarrow A_{01}A_{12}$ 1 , 4

1 , 5

1 , 6

$A_{11} \rightarrow A_{11}A_{11}$ 2 , 3

$A_{12} \rightarrow A_{11}A_{12}$ 2 , 4

$A_{12} \rightarrow aA_{12}\,a$ 2 , 5

2 , 6

2 , 7

$A_{12} \rightarrow A_{11}A_{12}$ 3 , 4

3 , 5

$A_{12} \rightarrow bA_{12}\,b$ 3 , 6

3 , 7

$A_{12} \rightarrow A_{12}A_{22}$ 4 , 5

$A_{12} \rightarrow A_{12}A_{22}$ 4 , 6

$A_{13} \rightarrow A_{12}A_{23}$ 4 , 7

$A_{22} \rightarrow A_{22}A_{22}$ 5 ,6

$A_{23} \rightarrow A_{22}A_{23}$ 5 ,7

$A_{23} \rightarrow A_{22}A_{23}$ 6 ,7

$$A_{03} \rightarrow \varepsilon A_{12} \varepsilon$$

$$A_{12} \rightarrow a A_{12}\, a$$

$$A_{12} \rightarrow b A_{12}\, b$$

$A_{03} \rightarrow A_{12}$

$A_{12} \rightarrow aA_{12}a$

$A_{12} \rightarrow bA_{12}b$

$S \rightarrow aSa \mid bSb \mid \varepsilon$

$S \rightarrow aSa$

$S \rightarrow bSb$