Work Sheet # 2 Solution

1. Determine whether each one of the following matrices is in the row reduced form or not:

a) $\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 7 \end{bmatrix}$

Sol.

Yes. It is in the reduced row echelon form.

b) $\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$

Sol.
No. It is in the row echelon form.

c) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 4 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

Sol.
No. It isn't in the reduced row echelon form because the pivot of the second column not equal to 1.

d) $\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Sol.
Yes. It is in the reduced row echelon form.

2. Use Gauss elimination method to solve each of the following linear systems:

a) **2x + y - z = 2**
x- y + 2z = 3
x + y - z = 1
3x - y + 3z = 7

Sol.

Write down the **augmented matrix** of the linear system

$$\left[\begin{array}{ccc|c} 2 & 1 & -1 & 2 \\ 1 & -1 & 2 & 3 \\ 1 & 1 & -1 & 1 \\ 3 & -1 & 3 & 7 \end{array}\right]$$

Then we will use some elementary row operations to find the row echelon form

$$\left[\begin{array}{ccc|c} 2 & 1 & -1 & 2 \\ 1 & -1 & 2 & 3 \\ 1 & 1 & -1 & 1 \\ 3 & -1 & 3 & 7 \end{array}\right]$$

$$\xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & -1 & 2 & 3 \\ 2 & 1 & -1 & 2 \\ 1 & 1 & -1 & 1 \\ 3 & -1 & 3 & 7 \end{array}\right] \xrightarrow[-3R_1+R_4 \to R_4]{\substack{-2R_1+R_2 \to R_2 \\ -R_1+R_3 \to R_3}} \left[\begin{array}{ccc|c} 1 & -1 & 2 & 3 \\ 0 & 3 & -5 & -4 \\ 0 & 2 & -3 & -2 \\ 0 & 2 & -3 & -2 \end{array}\right] \xrightarrow{R_3 - R_4 \to R_4}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 3 \\ 0 & 3 & -5 & -4 \\ 0 & 2 & -3 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{\frac{1}{3}R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & -1 & 2 & 3 \\ 0 & 1 & \frac{-5}{3} & \frac{-4}{3} \\ 0 & 2 & -3 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{-2R_2 + R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & -1 & 2 & 3 \\ 0 & 1 & \frac{-5}{3} & \frac{-4}{3} \\ 0 & 0 & \frac{1}{3} & \frac{2}{3} \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{3R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & -1 & 2 & 3 \\ 0 & 1 & \frac{-5}{3} & \frac{-4}{3} \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The last matrix is in the row echelon form. So re-write the equations as in the following:

$z = 2$

$y - \frac{5}{3}z = \frac{-4}{3} \Longrightarrow y = 2$

$x - y + 2z = 3 \Longrightarrow x = 1$

So the solution is $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$

b)

2x + 2y + z = 2
x - 2y + 2z = 1
3x - 6y + 6z = 3

Sol.

Write down the **augmented matrix** of the linear system

$$\left[\begin{array}{ccc|c} 2 & 2 & 1 & 2 \\ 1 & -2 & 2 & 1 \\ 3 & -6 & 6 & 3 \end{array}\right]$$

Then we will use some elementary row operations to find the row echelon form

$$\left[\begin{array}{ccc|c} 2 & 2 & 1 & 2 \\ 1 & -2 & 2 & 1 \\ 3 & -6 & 6 & 3 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & -2 & 2 & 1 \\ 2 & 2 & 1 & 2 \\ 3 & -6 & 6 & 3 \end{array}\right] \xrightarrow[-3R_1+R_3 \to R_3]{-2R_1+R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & -2 & 2 & 1 \\ 0 & 6 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{\frac{1}{6}R_2 \to R_2}$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 2 & 1 \\ 0 & 1 & \frac{-1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

By rewriting the equations:

$y - \frac{1}{2}z = 0$

$x - 2y + 2z = 1$

It is clear from the previous equations that we have an infinite number of solutions. To solve these equations

Let $z = t$ *where t is free real parameter so* $x = 1 - t$ *and* $y = \frac{t}{2}$

So the solution is $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 - t \\ \frac{t}{2} \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ \frac{1}{2} \\ 1 \end{bmatrix}$

c)

x + y + z = 6
2 x + 3 y + 4 z = 19
4 x + 5 y + 6 z = 30

Sol.

Write down the **augmented matrix** of the linear system

$$\left[\begin{array}{ccc|c} 1 & 1 & 6 & 6 \\ 2 & 3 & 4 & 19 \\ 4 & 5 & 6 & 30 \end{array}\right]$$

Then we will use some elementary row operations to find the row echelon form

$$\left[\begin{array}{ccc|c} 1 & 1 & 6 & 6 \\ 2 & 3 & 4 & 19 \\ 4 & 5 & 6 & 30 \end{array}\right] \xrightarrow[-4R_1+R_3\to R_3]{-2R_1+R_2\to R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 7 \\ 0 & 1 & 2 & 6 \end{array}\right] \xrightarrow{-R_2+R_3\to R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 7 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

From the last row, the system has no solution

Since The rank of the augmented matrix > the rank of the coefficient matrix
Then we have No solution

d)

x + 2y – z + w = 3
3x + 7y + z + 3w = 4
4x + 10y + 4z + 4w= 2
2x + 3y - 6z +2w = 11

Sol.

Write down the **augmented matrix** of the linear system

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 3 \\ 3 & 7 & 1 & 3 & 4 \\ 4 & 10 & 4 & 4 & 2 \\ 2 & 3 & -6 & 2 & 11 \end{array}\right]$$

Then we will use some elementary row operations to find the row echelon form

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 3 \\ 3 & 7 & 1 & 3 & 4 \\ 4 & 10 & 4 & 4 & 2 \\ 2 & 3 & -6 & 2 & 11 \end{array}\right] \xrightarrow[\substack{-4R_1+R_3\to R_3 \\ -2R_1+R_4\to R_4}]{-3R_1+R_2\to R_2} \left[\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 3 \\ 0 & 1 & 4 & 0 & -5 \\ 0 & 2 & 8 & 0 & -10 \\ 0 & -1 & -4 & 0 & 5 \end{array}\right] \xrightarrow[R_2+R_4\to R_4]{-2R_2+R_3\to R_3}$$

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 3 \\ 0 & 1 & 4 & 0 & -5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

By rewriting the equations:
$y - 4z = -5$
$x - 2y - z + w = 3$

It is clear from the previous equations that we have an infinite number of solutions.
To solve these equations
Let $z = t$ *and* $w = r$ *where* t *and* r *are free real parameters so*
$x = 13 + 9t - r$ *and* $y = -5 - 4t$

So the solution is $\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 13+9t-r \\ -5-4t \\ t \\ r \end{bmatrix} = \begin{bmatrix} 13 \\ -5 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 9 \\ -4 \\ 1 \\ 0 \end{bmatrix} + r\begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

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3. Find the rank for each one of the following matrices:

(a) $\begin{bmatrix} 1 & 1 & 1 & 3 \\ 2 & 3 & 3 & 7 \\ 4 & -3 & -2 & 3 \end{bmatrix}$

Sol.

Then we will use some elementary row operations to find the **row** echelon **form**

$$\begin{bmatrix} 1 & 1 & 1 & 3 \\ 2 & 3 & 3 & 7 \\ 4 & -3 & -2 & 3 \end{bmatrix} \xrightarrow[-4R_1+R_3\to R_3]{-2R_1+R_2\to R_2} \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & -7 & -6 & -9 \end{bmatrix}$$

$$\xrightarrow{7R_2+R_3\to R_3} \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

Then the last matrix is in the **row echelon form** so the rank is equal to 3.

(b) $\begin{bmatrix} 1 & -3 & 4 & 12 \\ 2 & -5 & 3 & 13 \\ 1 & -4 & 9 & 23 \end{bmatrix}$

Sol.

Then we will use some elementary row operations to find the **row echelon form**

$$\begin{bmatrix} 1 & -3 & 4 & 12 \\ 2 & -5 & 3 & 13 \\ 1 & -4 & 9 & 23 \end{bmatrix} \xrightarrow[-R_1+R_3\to R_3]{-2R_1+R_2\to R_2} \begin{bmatrix} 1 & -3 & 4 & 12 \\ 0 & 1 & -5 & -11 \\ 0 & -1 & 5 & 11 \end{bmatrix} \xrightarrow{R_2+R_3\to R_3} \begin{bmatrix} 1 & -3 & 4 & 12 \\ 0 & 1 & -5 & -11 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The last matrix is in the **row echelon form** so the rank is equal to 2.

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4. Determine whether each one of the following linear systems has a trivial or non-trivial solution:

a)

x + 2y + 3z = 0
x - 3y - 2z = 0
2x + y - 2z = 0

Sol.

Write down the <u>**augmented matrix**</u> of the linear system

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 1 & -3 & -2 & 0 \\ 2 & 1 & -2 & 0 \end{array}\right]$$

Then we will use some elementary row operations to find the row echelon form

$$\left[\begin{array}{ccc|c}1 & 2 & 3 & 0\\ 1 & -3 & -2 & 0\\ 2 & 1 & -2 & 0\end{array}\right] \xrightarrow[-2R_1+R_3\to R_3]{-R_1+R_2\to R_2} \left[\begin{array}{ccc|c}1 & 2 & 3 & 0\\ 0 & -5 & -5 & 0\\ 0 & -3 & -8 & 0\end{array}\right] \xrightarrow{-\frac{1}{5}R_2\to R_2}$$

$$\left[\begin{array}{ccc|c}1 & 2 & 3 & 0\\ 0 & 1 & 1 & 0\\ 0 & -3 & -8 & 0\end{array}\right] \xrightarrow{3R_2+R_3\to R_3}$$

$$\left[\begin{array}{ccc|c}1 & 2 & 3 & 0\\ 0 & 1 & 1 & 0\\ 0 & 0 & 1 & 0\end{array}\right]$$

The last matrix is in the row echelon form and its **rank = no. of unknowns =3**, so the solution is **trivial**.

b)

x + y + z + w = 0

x + w = 0

x + 2y + z = 0

Sol.

Write down the **augmented matrix** of the linear system

$$\left[\begin{array}{cccc|c}1 & 1 & 1 & 1 & 0\\ 1 & 0 & 0 & 1 & 0\\ 1 & 2 & 1 & 0 & 0\end{array}\right]$$

Then we will use some elementary row operations to find the row reduced form

$$\left[\begin{array}{cccc|c}1 & 1 & 1 & 1 & 0\\ 1 & 0 & 0 & 1 & 0\\ 1 & 2 & 1 & 0 & 0\end{array}\right] \xrightarrow[-R_1+R_3\to R_3]{-R_1+R_2\to R_2} \left[\begin{array}{cccc|c}1 & 1 & 1 & 1 & 0\\ 0 & -1 & -1 & 0 & 0\\ 0 & 1 & 0 & -1 & 0\end{array}\right] \xrightarrow[-R_2]{R_2+R_3\to R_3} \left[\begin{array}{cccc|c}1 & 1 & 1 & 1 & 0\\ 0 & 1 & 1 & 0 & 0\\ 0 & 0 & 1 & 1 & 0\end{array}\right]$$

The last matrix is in the row echelon form and its rank < number of unknowns, so we have an infinite number of solutions.

To write the solution, rewrite the equations from the last matrix as in the following:

$-z - \ w = -4$

$-y - z = -2$

$x + y + z + w = 2$

Let $w = t \ so \ z = 4 - t, y = -2 + t, x = -t$

$which \ is \ a \ nontrivial \ solution \ at \ t \neq 0$

5. Find the value of a for which each one of the following linear systems has a unique solution, an infinite number of solutions or no solution.

(a) $ax + y + z = 2a$
$x + ay + z = -1$
$x + y + 2z = -1$

Sol.

At the beginning, we will rearrange the equations to be in the following form

$$z + y + ax = 2a$$
$$2z + y + x = -1$$
$$z + ay + x = -1$$

Then we will write down the **augmented matrix** of the linear system

$$\left[\begin{array}{ccc|c} 1 & 1 & a & 2a \\ 2 & 1 & 1 & -1 \\ 1 & a & 1 & -1 \end{array}\right]$$

Then we will use some elementary row operations to find the **row echelon form**

$$\left[\begin{array}{ccc|c} 1 & 1 & a & 2a \\ 2 & 1 & 1 & -1 \\ 1 & a & 1 & -1 \end{array}\right] \xrightarrow[-R_1+R_3\to R_3]{-2R_1+R_2\to R_2} \left[\begin{array}{ccc|c} 1 & 1 & a & 2a \\ 0 & -1 & 1-2a & -1-4a \\ 0 & a-1 & 1-a & -1-2a \end{array}\right]$$
$$\xrightarrow{-R_2\to R_2} \left[\begin{array}{ccc|c} 1 & 1 & a & 2a \\ 0 & -1 & -1+2a & 1+4a \\ 0 & a-1 & 1-a & -1-2a \end{array}\right] \xrightarrow{(1-a)R_2+R_3\to R_3} \left[\begin{array}{ccc|c} 1 & 1 & a & 2a \\ 0 & -1 & -1+2a & 1+4a \\ 0 & 0 & (1-a)(2a) & a(1-4a) \end{array}\right]$$

No solution: Since that will happen if $(1-a)(2a) = 0 \Rightarrow a = 0\ 0r\ a = 1\ and\ at\ the\ same\ time\ a(1-4a) \neq 0 \Rightarrow a \neq 0\ or\ a \neq \frac{1}{4}.then\ at\ a = 1\ the\ sytem\ has\ no\ solution.$

Infinite: Since that will happen if $(1-a)(2a) = 0$
$$\Rightarrow a = 0\ or\ a = 1\ and\ at\ the\ same\ time\ a(1-4a) = 0$$
$$\Rightarrow a = 0\ or\ a = \frac{1}{4}.then\ at$$
$$a = 0\ the\ sytem\ has\ infinite\ number\ of\ solutions.$$

Unique: the system has a unique solution at $a \in \mathbb{R} - \{0,1\}$.

(b) $x + y - z = 2$
$x + 2y + z = 3$
$x + y + (a^2 - 5)z = a$

Sol.

We will write down the **augmented matrix** of the linear system

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 1 & 2 & 1 & 3 \\ 1 & 1 & a^2-5 & a \end{array}\right]$$

Then we will use some elementary row operations to find the **row echelon form**

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 1 & 2 & 1 & 3 \\ 1 & 1 & a^2-5 & a \end{array}\right] \xrightarrow[-R_1+R_3\to R_3]{-R_1+R_2\to R_2} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & a^2-4 & a-2 \end{array}\right]$$

No solution: Since that will happen if $a^2 - 4 = 0$
$\Rightarrow a = -2\ 0r\ a = 2\ and\ at\ the\ same\ time\ a - 2 \neq 0$
$\Rightarrow a \neq 2\ .Then\ at\ a = -2\ the\ sytem\ has\ no\ solution.$

Infinite: Since that will happen if $a^2 - 4 = 0$
$\Rightarrow a = -2\ 0r\ a = 2\ and\ at\ the\ same\ time\ a - 2 = 0$
$\Rightarrow a = 2\ .Then\ at$
$a = 2\ the\ sytem\ has\ infinite\ number\ of\ solutions.$

Unique: the system has a unique solution at $a \in \mathbb{R} - \{-2,2\}$.

(c) $x + y + z = 2$
$2x + 3y + 2z = 5$
$x + y + (a^2 - 1)z = a + 1$

Sol.

We will write down the **augmented matrix** of the linear system

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 2 & 3 & 2 & 5 \\ 1 & 1 & a^2-1 & a+1 \end{array}\right]$$

Then we will use some elementary row operations to find the **row echelon form**

$$\left[\begin{array}{ccc|c}1 & 1 & 1 & 2\\ 2 & 3 & 2 & 5\\ 1 & 1 & a^2-1 & a+1\end{array}\right]\xrightarrow[-R_1+R_3\to R_3]{-2R_1+R_2\to R_2}\left[\begin{array}{ccc|c}1 & 1 & 1 & 2\\ 0 & 1 & 0 & 1\\ 0 & 0 & a^2-2 & a-1\end{array}\right]$$

No solution*:* Since that will happen if $a^2-2=0 \Rightarrow a=-\sqrt{2}\ 0r\ a=\sqrt{2}\ \ and\ at\ the\ same\ time\ a-1\neq 0 \Rightarrow a\neq 1. Then\ at\ a=\pm\sqrt{2}\ \ the\ sytem\ has\ no\ solution.$

Infinite*:* Since that will happen if $\mathrm{a}^2-2=0 \Rightarrow \mathrm{a}=-\sqrt{2}\ \mathrm{0r\ a}=\sqrt{2}$ and at the same time $\mathrm{a}-1=0\Rightarrow \mathrm{a}=1$. which make contradiction then there are no values for a at which the system has infinite number of solutions.

Unique*:* the system has a unique solution at $a\in\mathbb{R}-\{-\sqrt{2},\sqrt{2}\}$.

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6. Solve each of the following systems of nonlinear equations:

(a) $\begin{cases} x^2+y^2+z^2 &= 6\\ x^2-y^2+2z^2 &= 2\\ 2x^2+y^2-z^2 &= 3\end{cases}$

(b) $\begin{cases} \frac{1}{x}+\frac{2}{y}-\frac{4}{z} &= 1\\ \frac{2}{x}+\frac{3}{y}+\frac{8}{z} &= 0\\ -\frac{1}{x}+\frac{9}{y}+\frac{10}{z} &= 5\end{cases}$

Solution:

(a) First of all, we have to rename the unknowns to be linear as follows:

Let $x^2=X, y^2=Y,$ and $z^2=Z$. So, the given system will be:

$$\begin{cases} x^2+y^2+z^2 &= 6\\ x^2-y^2+2z^2 &= 2\\ 2x^2+y^2-z^2 &= 3\end{cases}\Longrightarrow\begin{cases} X+Y+Z &= 6\\ X-Y+2Z &= 2\\ 2X+Y-Z &= 3\end{cases}$$

By using Gauss Jordan elimination method:

$$\xRightarrow[R_3-2R_1\to R_3]{R_2-R_1\to R_2}\left[\begin{array}{ccc|c}1 & 1 & 1 & 6\\ 0 & -2 & 1 & -4\\ 0 & -1 & -3 & -9\end{array}\right]\xRightarrow{\frac{R_2}{-2}\to R_2}\left[\begin{array}{ccc|c}1 & 1 & 1 & 6\\ 0 & 1 & -\frac{1}{2} & 2\\ 0 & -1 & -3 & -9\end{array}\right]$$

$$\xRightarrow{R_3+R_2\to R_3}\left[\begin{array}{ccc|c}1 & 1 & 1 & 6\\ 0 & 1 & -\frac{1}{2} & 2\\ 0 & 0 & -\frac{7}{2} & -7\end{array}\right]$$

$$\xRightarrow{\frac{R_3}{-7/2}\to R_3}\left[\begin{array}{ccc|c}1 & 1 & 1 & 6\\ 0 & 1 & -\frac{1}{2} & 2\\ 0 & 0 & 1 & 2\end{array}\right]\xRightarrow{R_1-R_3\to R_1}\left[\begin{array}{ccc|c}1 & 1 & 0 & 4\\ 0 & 1 & -\frac{1}{2} & 2\\ 0 & 0 & 1 & 2\end{array}\right]\xRightarrow{R_2+\frac{1}{2}R_3\to R_2}\left[\begin{array}{ccc|c}1 & 1 & 0 & 4\\ 0 & 1 & 0 & 3\\ 0 & 0 & 1 & 2\end{array}\right]$$

$$\xRightarrow{R_1-R_2\to R_1}\left[\begin{array}{ccc|c}1 & 0 & 0 & 1\\ 0 & 1 & 0 & 3\\ 0 & 0 & 1 & 2\end{array}\right]$$

From the last matrix, one can note that:

$$X = 1, Y = 3, \text{and } Z = 2$$

So, the solutions of the given unknowns are:

$$x^2 = 1 \Longrightarrow x = \pm 1, \qquad y^2 = 3 \Longrightarrow y = \pm\sqrt{3}, \qquad \text{and } z^2 = 2 \Longrightarrow z = \pm\sqrt{2}$$

(b) First of all, we have to rename the unknowns to be linear as follows:

Let $\frac{1}{x} = X, \frac{1}{y} = Y, \text{and} \frac{1}{z} = Z$. So, the given system will be:

$$\begin{cases} \dfrac{1}{x} + \dfrac{2}{y} - \dfrac{4}{z} & = 1 \\ \dfrac{2}{x} + \dfrac{3}{y} + \dfrac{8}{z} & = 0 \\ -\dfrac{1}{x} + \dfrac{9}{y} + \dfrac{10}{z} & = 5 \end{cases} \Longrightarrow \begin{cases} X + 2Y - 4Z & = 1 \\ 2X + 3Y + 8Z & = 0 \\ -X + 9Y + 10Z & = 5 \end{cases}$$

By using Gauss Jordan elimination method:

$$\xrightarrow[R_3+R_1 \to R_3]{R_2-2R_1 \to R_2} \left[\begin{array}{ccc|c} 1 & 2 & -4 & 1 \\ 0 & -1 & 16 & -2 \\ 0 & 11 & 6 & 6 \end{array}\right] \xrightarrow{R_2/-1 \to R_2} \left[\begin{array}{ccc|c} 1 & 2 & -4 & 1 \\ 0 & 1 & -16 & 2 \\ 0 & 11 & 6 & 6 \end{array}\right]$$

$$\xrightarrow{R_3-11R_2 \to R_3} \left[\begin{array}{ccc|c} 1 & 2 & -4 & 1 \\ 0 & 1 & -16 & 2 \\ 0 & 0 & 182 & -16 \end{array}\right]$$

$$\xrightarrow{R_3/182 \to R_3} \left[\begin{array}{ccc|c} 1 & 2 & -4 & 1 \\ 0 & 1 & -16 & 2 \\ 0 & 0 & 1 & -\frac{8}{91} \end{array}\right] \xrightarrow{R_1+4R_3 \to R_1} \left[\begin{array}{ccc|c} 1 & 2 & 0 & \frac{59}{91} \\ 0 & 1 & -16 & 2 \\ 0 & 0 & 1 & -\frac{8}{91} \end{array}\right]$$

$$\xrightarrow{R_2+16R_3 \to R_2} \left[\begin{array}{ccc|c} 1 & 2 & 0 & \frac{59}{91} \\ 0 & 1 & 0 & \frac{54}{91} \\ 0 & 0 & 1 & -\frac{8}{91} \end{array}\right]$$

$$\xrightarrow{R_1-2R_2 \to R_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -\frac{7}{13} \\ 0 & 1 & 0 & \frac{54}{91} \\ 0 & 0 & 1 & -\frac{8}{91} \end{array}\right]$$

From the last matrix, one can note that:

$$X = -\frac{7}{13}, Y = \frac{54}{91}, \text{and } Z = -\frac{8}{91}$$

So, the solutions of the given unknowns are:

$$\frac{1}{x} = -\frac{7}{13} \Rightarrow x = -\frac{13}{7}, \qquad \frac{1}{y} = \frac{54}{91} \Rightarrow y = \frac{91}{54},$$

$$\text{and } \frac{1}{z} = -\frac{8}{91} \Rightarrow z = -\frac{91}{8}$$

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7. <u>In each of the following cases, find the matrix X:</u>

(a) $\begin{bmatrix} 1 & -1 & 1 \\ 2 & 3 & 0 \\ 0 & 2 & -1 \end{bmatrix} X = \begin{bmatrix} 2 & -1 & 5 & 7 & 8 \\ 4 & 0 & -3 & 0 & 1 \\ 3 & 5 & -7 & 2 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} -2 & 0 & 1 \\ 0 & -1 & -4 \\ 1 & 1 & -1 \end{bmatrix} X = \begin{bmatrix} 4 & 3 & 2 & 1 \\ 6 & 7 & 8 & 9 \\ 1 & 3 & 7 & 9 \end{bmatrix}$

<u>Solution:</u>

(a) the given system can be written as : $A_{3\times3}X_{3\times5} = B_{3\times5}$

Let $X = [x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5]$, $B = [b_1 \quad b_2 \quad b_3 \quad b_4 \quad b_5]$

$$AX = [Ax_1 \quad Ax_2 \quad Ax_3 \quad Ax_4 \quad Ax_5] = B$$

Solve simultaneously $Ax_1 = b_1, Ax_2 = b_2, Ax_3 = b_3, Ax_4 = b_4, Ax_5 = b_5$

By using Gauss Jordan elimination method in parallel as follows:

$$\xrightarrow{R_2-2R_1 \to R_2} \left[\begin{array}{ccc|ccccc} 1 & -1 & 1 & 2 & -1 & 5 & 7 & 8 \\ 0 & 5 & -2 & 0 & 2 & -13 & -14 & -15 \\ 0 & 2 & -1 & 3 & 5 & -7 & 2 & 1 \end{array}\right]$$

$$\xrightarrow{R_2/5 \to R_2} \left[\begin{array}{ccc|ccccc} 1 & -1 & 1 & 2 & -1 & 5 & 7 & 8 \\ 0 & 1 & -\frac{2}{5} & 0 & \frac{2}{5} & -\frac{13}{5} & -\frac{14}{5} & -3 \\ 0 & 2 & -1 & 3 & 5 & -7 & 2 & 1 \end{array}\right]$$

$$\xrightarrow{R_3-2R_2 \to R_3} \left[\begin{array}{ccc|ccccc} 1 & -1 & 1 & 2 & -1 & 5 & 7 & 8 \\ 0 & 1 & -\frac{2}{5} & 0 & \frac{2}{5} & -\frac{13}{5} & -\frac{14}{5} & -3 \\ 0 & 0 & -\frac{1}{5} & 3 & \frac{21}{5} & -\frac{9}{5} & \frac{38}{5} & 7 \end{array}\right]$$

$$\xrightarrow{R_3/(-1/5) \to R_3} \left[\begin{array}{ccc|ccccc} 1 & -1 & 1 & 2 & -1 & 5 & 7 & 8 \\ 0 & 1 & -\frac{2}{5} & 0 & \frac{2}{5} & -\frac{13}{5} & -\frac{14}{5} & -3 \\ 0 & 0 & 1 & -15 & -21 & 9 & -38 & -35 \end{array}\right]$$

$$\xrightarrow[R_1+R_2 \to R_1]{\substack{R_1-R_3 \to R_1 \\ R_2+\frac{2}{5}R_3 \to R_2}} \left[\begin{array}{ccc|ccccc} 1 & 0 & 0 & 11 & 12 & -3 & 27 & 26 \\ 0 & 1 & 0 & -6 & -8 & 1 & -18 & -17 \\ 0 & 0 & 1 & -15 & -21 & 9 & -38 & -35 \end{array}\right]$$

So, $X = [x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5] = \begin{bmatrix} 11 & 12 & -3 & 27 & 26 \\ -6 & -8 & 1 & -18 & -17 \\ -15 & -21 & 9 & -38 & -35 \end{bmatrix}$

(b) the given system can be written as : $A_{3\times3}X_{3\times4} = B_{3\times4}$

Let $X = [x_1 \quad x_2 \quad x_3 \quad x_4]$, $B = [b_1 \quad b_2 \quad b_3 \quad b_4]$

$$AX = [Ax_1 \quad Ax_2 \quad Ax_3 \quad Ax_4] = B$$

Solve simultaneously $Ax_1 = b_1, Ax_2 = b_2, Ax_3 = b_3, Ax_4 = b_4$

By using Gauss Jordan elimination method in parallel as follows:

$$\xrightarrow{R_1/-2 \to R_1} \left[\begin{array}{ccc|cccc} 1 & 0 & -\frac{1}{2} & -2 & -\frac{3}{2} & -1 & -\frac{1}{2} \\ 0 & -1 & -4 & 6 & 7 & 8 & 9 \\ 1 & 1 & -1 & 1 & 3 & 7 & 9 \end{array}\right]$$

$$\xrightarrow{R_3-R_1 \to R_3} \left[\begin{array}{ccc|cccc} 1 & 0 & -\frac{1}{2} & -2 & -\frac{3}{2} & -1 & -\frac{1}{2} \\ 0 & -1 & -4 & 6 & 7 & 8 & 9 \\ 0 & 1 & -\frac{1}{2} & 3 & \frac{9}{2} & 8 & \frac{19}{2} \end{array}\right]$$

$$\xrightarrow{R_2/-1 \to R_2} \left[\begin{array}{ccc|cccc} 1 & 0 & -\frac{1}{2} & -2 & -\frac{3}{2} & -1 & -\frac{1}{2} \\ 0 & 1 & 4 & -6 & -7 & -8 & -9 \\ 0 & 1 & -\frac{1}{2} & 3 & \frac{9}{2} & 8 & \frac{19}{2} \end{array}\right]$$

$$\xrightarrow{R_3-R_2 \to R_3} \left[\begin{array}{ccc|cccc} 1 & 0 & -\frac{1}{2} & -2 & -\frac{3}{2} & -1 & -\frac{1}{2} \\ 0 & 1 & 4 & -6 & -7 & -8 & -9 \\ 0 & 0 & -\frac{9}{2} & 9 & \frac{23}{2} & 16 & \frac{37}{2} \end{array}\right]$$

$$\xrightarrow{R_3/(-9/2) \to R_3} \left[\begin{array}{ccc|cccc} 1 & 0 & -\frac{1}{2} & -2 & -\frac{3}{2} & -1 & -\frac{1}{2} \\ 0 & 1 & 4 & -6 & -7 & -8 & -9 \\ 0 & 0 & 1 & -2 & -\frac{23}{9} & -\frac{32}{9} & -\frac{37}{9} \end{array}\right]$$

$$\xrightarrow[R_2-4R_3 \to R_2]{R_1+\frac{1}{2}R_3 \to R_1} \left[\begin{array}{ccc|cccc} 1 & 0 & 0 & -3 & -\frac{25}{9} & -\frac{25}{9} & -\frac{23}{9} \\ 0 & 1 & 0 & 2 & \frac{29}{9} & \frac{56}{9} & \frac{67}{9} \\ 0 & 0 & 1 & -2 & -\frac{23}{9} & -\frac{32}{9} & -\frac{37}{9} \end{array}\right]$$

$$\text{So,} \qquad X = [x_1 \quad x_2 \quad x_3 \quad x_4] = \begin{bmatrix} -3 & -\dfrac{25}{9} & -\dfrac{25}{9} & -\dfrac{23}{9} \\ 2 & \dfrac{29}{9} & \dfrac{56}{9} & \dfrac{67}{9} \\ -2 & -\dfrac{23}{9} & -\dfrac{32}{9} & -\dfrac{37}{9} \end{bmatrix}$$

8. Evaluate the determinant for each one of the following matrices

a) $\begin{bmatrix} \mathbf{3} & \mathbf{1} \\ \mathbf{7} & \mathbf{6} \end{bmatrix}$

Sol.

$$\Delta = \begin{vmatrix} 3 & 1 \\ 7 & 6 \end{vmatrix} = (3)(6) - (1)(7) = 11$$

$$b)\begin{bmatrix} 2 & 3 & -1 \\ 1 & 0 & 3 \\ -3 & -1 & -2 \end{bmatrix}$$

Sol.

$$\Delta = \begin{vmatrix} 2 & 3 & -1 \\ 1 & 0 & 3 \\ -3 & -1 & -2 \end{vmatrix} = 2[0+3] - 3[-2+9] - 1[-1]$$
$$= -14$$

$$c)\begin{bmatrix} 1 & 0 & 1 \\ 0 & 4 & 3 \\ 0 & 0 & 2 \end{bmatrix}$$

Sol.

It’s an upper triangle matrix so the $\Delta = \begin{vmatrix} 1 & 0 & 1 \\ 0 & 4 & 3 \\ 0 & 0 & 2 \end{vmatrix} = 1 * 4 * 2 =$ 8

$$d)\begin{bmatrix} -1 & -2 & 1 & 3 \\ 0 & 4 & 3 & -2 \\ 2 & 0 & -1 & 2 \\ 1 & 5 & -1 & 3 \end{bmatrix}$$

Sol.

We will use some properties of determinants to transform the determinant to the upper triangular matrix form without changing the determinant value

$$\Delta = \begin{vmatrix} -1 & -2 & 1 & 3 \\ 0 & 4 & 3 & -2 \\ 2 & 0 & -1 & 2 \\ 1 & 5 & -1 & 3 \end{vmatrix} \xrightarrow{R_1 \leftrightarrow R_4} \Delta$$
$$= -\begin{vmatrix} 1 & 5 & -1 & 3 \\ 0 & 4 & 3 & -2 \\ 2 & 0 & -1 & 2 \\ -1 & -2 & 1 & 3 \end{vmatrix} \xrightarrow[R_1+R_4 \to R_4]{-2R_1+R_3 \to R_3}$$

$$\Delta = \begin{vmatrix} 1 & 5 & -1 & 3 \\ 0 & 4 & 3 & -2 \\ 0 & -10 & 1 & -4 \\ 0 & 3 & 0 & 6 \end{vmatrix} \xrightarrow[-\frac{3}{4}R_2+R_4\to R_4]{2.5R_2+R_3\to R_3} \Delta$$

$$= \begin{vmatrix} 1 & 5 & -1 & 3 \\ 0 & 4 & 3 & -2 \\ 0 & 0 & 8.5 & -9 \\ 0 & 0 & \frac{-9}{4} & \frac{15}{2} \end{vmatrix} \xrightarrow{\frac{18}{68}R_3+R_4\to R_4}$$

$$\Delta = \begin{vmatrix} 1 & 5 & -1 & 3 \\ 0 & 4 & 3 & -2 \\ 0 & 0 & 8.5 & 2 \\ 0 & 0 & 0 & \frac{-87}{17} \end{vmatrix} = 1 * 4 * 8.5 * \frac{\mathbf{-87}}{\mathbf{17}} = \mathbf{-174}$$

---

9. Show that $\det(A^{-1}) = \frac{1}{\det(A)}$

Sol.

If A is an invertible nxn matrix, then $A(A^{-1}) = I$ and from determinant properties then
$det(A)\, det(A^{-1}) = de\,t(\,I) \Rightarrow det(AA^{-1}) = 1$ And since $det(A) \neq 0$, we can conclude that
$det(A^{-1}) = 1/\,det(A)$

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10. Given that the determinant $\begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix} = 9$, find the value of

(a) $\begin{vmatrix} -4a_1 & -4a_2 & -4a_3 & -4a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1/3 & d_2/3 & d_3/3 & d_4/3 \end{vmatrix}$

Sol.

From the properties of the determinants, $det(cA) = c^n det(A)$

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix} \xrightarrow[(\frac{1}{3})R_4 \to R_4]{-4R_1 \to R_1} \begin{vmatrix} -4a_1 & -4a_2 & -4a_3 & -4a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1/3 & d_2/3 & d_3/3 & d_4/3 \end{vmatrix}$$

So $\begin{vmatrix} -4a_1 & -4a_2 & -4a_3 & -4a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1/3 & d_2/3 & d_3/3 & d_4/3 \end{vmatrix} = (-4)\left(\frac{1}{3}\right)\det(A) =$
$-4\left(\frac{1}{3}\right)(9) = -12$

(b) $\begin{vmatrix} d_1 & d_2 & d_3 & d_4 \\ c_1 & c_2 & c_3 & c_4 \\ b_1 & b_2 & b_3 & b_4 \\ a_1 & a_2 & a_3 & a_4 \end{vmatrix}$

Sol.

From the properties of the determinants, if matrix B results from interchanging any 2 rows or columns in matrix A then $det(B) = -det(A)$

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix} = -\begin{vmatrix} d_1 & d_2 & d_3 & d_4 \\ c_1 & c_2 & c_3 & c_4 \\ b_1 & b_2 & b_3 & b_4 \\ a_1 & a_2 & a_3 & a_4 \end{vmatrix}$$

So $\begin{vmatrix} d_1 & d_2 & d_3 & d_4 \\ c_1 & c_2 & c_3 & c_4 \\ b_1 & b_2 & b_3 & b_4 \\ a_1 & a_2 & a_3 & a_4 \end{vmatrix} = -(-9) = 9$

(c) $\begin{vmatrix} a_1 + c_1 & a_2 + c_2 & a_3 + c_3 & a_4 + c_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix}$

Sol.

From the properties of the determinants, if matrix B results from adding a constant multiple from a row to another row or column in matrix A then $det(B) = de\,t(\,A)$

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix} \xrightarrow{R_3+R_1\to R_1} \begin{vmatrix} a_1+c_1 & a_2+c_2 & a_3+c_3 & a_4+c_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix}$$

$$\text{So} \begin{vmatrix} a_1+c_1 & a_2+c_2 & a_3+c_3 & a_4+c_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix} = 9$$

(d) $\begin{vmatrix} -2a_1 & -2a_2 & -2a_3 & -2a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1-b_1 & d_2-b_2 & d_3-b_3 & d_4-b_4 \end{vmatrix}$

Sol.

From the properties of the determinants, if matrix B results from adding a constant multiple from a row to another row or column in matrix A then $det(B) = de\,t(\,A)$ and $det(cA) = c^n de\,t(\,A)$

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix} \xrightarrow[-2(R_1)\to R_1]{-R_2+R_4\to R_4} \begin{vmatrix} -2a_1 & -2a_2 & -2a_3 & -2a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1-b_1 & d_2-b_2 & d_3-b_3 & d_4-b_4 \end{vmatrix}$$

$$\text{So} \begin{vmatrix} -2a_1 & -2a_2 & -2a_3 & -2a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1-b_1 & d_2-b_2 & d_3-b_3 & d_4-b_4 \end{vmatrix} = -2(9) = -18$$

---

11. Given the matrix A= $\begin{bmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{bmatrix}$ use the properties of determinants to show that |A| = (b - a) (c - a) (c - b)

Sol.

We will use some row elementary operations to simplify the determinant without changing the determinant value

$$\Delta = \begin{vmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{a} & \mathbf{b} & \mathbf{c} \\ \mathbf{a^2} & \mathbf{b^2} & \mathbf{c^2} \end{vmatrix} \xrightarrow[-c_1+c_3\to c_3]{-c_1+c_2\to c_2} \Delta = \begin{vmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{a} & \mathbf{b-a} & \mathbf{c-a} \\ \mathbf{a^2} & \mathbf{b^2-a^2} & \mathbf{c^2-a^2} \end{vmatrix}$$

$$\begin{aligned} \Delta = (\mathbf{b-a})(\mathbf{c^2-a^2}) - (\boldsymbol{c-a})(\mathbf{b^2-a^2}) \\ = (\boldsymbol{b-a})(\boldsymbol{c-a})[(\boldsymbol{c+a}) - (\boldsymbol{b+a})] \\ = (\boldsymbol{b-a})(\boldsymbol{c-a})(\boldsymbol{c-b}) \end{aligned}$$

---

12. Solve the following equation in x, $\begin{bmatrix} 1 & x & x^2 \\ 1 & 1 & 1 \\ 1 & -3 & 9 \end{bmatrix} = 0$

Sol.

$$\Delta = \begin{vmatrix} \mathbf{1} & \mathbf{x} & \mathbf{x^2} \\ \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{-3} & \mathbf{9} \end{vmatrix} = (12) - x(8) + \mathbf{x^2}(\mathbf{-4}) = \mathbf{0}$$

$$\Rightarrow \mathbf{x^2 + 2\boldsymbol{x} - 3 = 0}$$

$$\Rightarrow (x+3)(x-1) = 0 \Rightarrow x = 1 \textit{ or } x = -3$$

---

13. Find A^{-1} if possible, using two different methods

(a) $\begin{bmatrix} 5 & 0 & 1 \\ 3 & 1 & 2 \\ -3 & -1 & -2 \end{bmatrix}$

Sol.

Det (A) $=5[-2+2] - (0)[-6+6] + 1[-3+3] = 0 - 0 + 0 = 0$ so matrix A does not have an inverse matrix. (Non –Invertible matrix)

(b) $\begin{bmatrix} 2 & -1 & 3 \\ 1 & 0 & 4 \\ 2 & -3 & 1 \end{bmatrix}$

Sol.

Det (A) $=2[0+12] - (-1)[1-8] + 3[-3-0] = 24 - 7 - 9 = 8 \neq 0$ so matrix A has an inverse matrix.

First method by Gauss Jordan method:

$$\left[\begin{array}{ccc|ccc} 2 & -1 & 3 & 1 & 0 & 0 \\ 1 & 0 & 4 & 0 & 1 & 0 \\ 2 & -3 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 4 & 0 & 1 & 0 \\ 2 & -1 & 3 & 1 & 0 & 0 \\ 2 & -3 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[-2(R_1)+R_2 \rightarrow R_2]{-2(R_1)+R_3 \rightarrow R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 4 & 0 & 1 & 0 \\ 0 & -1 & -5 & 1 & -2 & 0 \\ 0 & -3 & -7 & 0 & -2 & 1 \end{array}\right]$$

$$\xrightarrow{-(R_2) \rightarrow R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 4 & 0 & 1 & 0 \\ 0 & 1 & 5 & -1 & 2 & 0 \\ 0 & -3 & -7 & 0 & -2 & 1 \end{array}\right]$$

$$\xrightarrow{3(R_2)+R_3 \rightarrow R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 4 & 0 & 1 & 0 \\ 0 & 1 & 5 & -1 & 2 & 0 \\ 0 & 0 & 8 & -3 & 4 & 1 \end{array}\right]$$

$$\xrightarrow{\left(\frac{1}{8}\right)(R_3) \rightarrow R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 4 & 0 & 1 & 0 \\ 0 & 1 & 5 & -1 & 2 & 0 \\ 0 & 0 & 1 & -\frac{3}{8} & \frac{1}{2} & \frac{1}{8} \end{array}\right]$$

$$\xrightarrow[-5(R_3)+R_2 \rightarrow R_2]{-4(R_3)+R_1 \rightarrow R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{3}{2} & -1 & -\frac{1}{2} \\ 0 & 1 & 0 & \frac{7}{8} & -\frac{1}{2} & -\frac{5}{8} \\ 0 & 0 & 1 & -\frac{3}{8} & \frac{1}{2} & \frac{1}{8} \end{array}\right]$$

So $A^{-1} = \begin{bmatrix} \frac{3}{2} & -1 & -\frac{1}{2} \\ \frac{7}{8} & -\frac{1}{2} & -\frac{5}{8} \\ -\frac{3}{8} & \frac{1}{2} & \frac{1}{8} \end{bmatrix} = \frac{1}{8}\begin{bmatrix} 12 & -8 & -4 \\ 7 & -4 & -5 \\ -3 & 4 & 1 \end{bmatrix}$

Second method by the adjoint matrix:

$$Adj(A) = [cofactors]^T = \begin{bmatrix} 12 & 7 & -3 \\ -8 & -4 & 4 \\ -4 & -5 & 1 \end{bmatrix}^T = \begin{bmatrix} 12 & -8 & -4 \\ 7 & -4 & -5 \\ -3 & 4 & 1 \end{bmatrix}$$

So $A^{-1} = \frac{adj(A)}{|A|} = \frac{1}{8}\begin{bmatrix} 12 & -8 & -4 \\ 7 & -4 & -5 \\ -3 & 4 & 1 \end{bmatrix} = \frac{1}{8}\begin{bmatrix} 12 & -8 & -4 \\ 7 & -4 & -5 \\ -3 & 4 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ -1 & -2 & 1 \end{bmatrix}$

Sol.

Det (A) $=0[0+6]-(1)[1+3]+2[-2-0]=0-4-4=-8\neq 0$ so matrix A has an inverse matrix.

<u>First method by Gauss Jordan method:</u>

$$\left[\begin{array}{ccc|ccc} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ -1 & -2 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ -1 & -2 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{(R_1)+R_3 \to R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & -2 & 4 & 0 & 1 & 1 \end{array}\right]$$

$$\xrightarrow{2(R_2)+R_3 \to R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 8 & 2 & 1 & 1 \end{array}\right]$$

$$\xrightarrow{\left(\frac{1}{8}\right)(R_3) \to R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{1}{4} & \frac{1}{8} & \frac{1}{8} \end{array}\right]$$

$$\xrightarrow[-2(R_3)+R_2 \to R_2]{-3(R_3)+R_1 \to R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{3}{4} & \frac{5}{8} & -\frac{3}{8} \\ 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{4} & -\frac{1}{4} \\ 0 & 0 & 1 & \frac{1}{4} & \frac{1}{8} & \frac{1}{8} \end{array}\right]$$

$$\text{So } A^{-1} = \begin{bmatrix} -\frac{3}{4} & \frac{5}{8} & -\frac{3}{8} \\ \frac{1}{2} & -\frac{1}{4} & -\frac{1}{4} \\ \frac{1}{4} & \frac{1}{8} & \frac{1}{8} \end{bmatrix} = \frac{1}{8}\begin{bmatrix} 6 & -5 & 3 \\ -4 & 2 & 2 \\ -2 & -1 & -1 \end{bmatrix}$$

<u>Second method by the adjoint matrix:</u>

$$Adj(A) = [cofactors]^T = \begin{bmatrix} 6 & -4 & -2 \\ -5 & 2 & -1 \\ 3 & 2 & -1 \end{bmatrix}^T = \begin{bmatrix} 6 & -5 & 3 \\ -4 & 2 & 2 \\ -2 & -1 & -1 \end{bmatrix}$$

$$\text{So } A^{-1} = \frac{adj(A)}{|A|} = \frac{1}{8}\begin{bmatrix} 6 & -5 & 3 \\ -4 & 2 & 2 \\ -2 & -1 & -1 \end{bmatrix} = \frac{1}{8}\begin{bmatrix} 6 & -5 & 3 \\ -4 & 2 & 2 \\ -2 & -1 & -1 \end{bmatrix}$$

14. Solve each one of the following systems by the Cramer's method and the inverse matrix method:

a)

2x + 2y + z = 2
x + 2y - 2z = 1
3x - 6y + 6z = 3

Sol.

$$\mathrm{A} = \begin{bmatrix} \mathbf{2} & \mathbf{2} & \mathbf{1} \\ \mathbf{1} & \mathbf{2} & \mathbf{-2} \\ \mathbf{3} & \mathbf{-6} & \mathbf{6} \end{bmatrix} \text{ And } \mathrm{B} = \begin{bmatrix} \mathbf{2} \\ \mathbf{1} \\ \mathbf{3} \end{bmatrix}$$

$$\Delta = \begin{vmatrix} \mathbf{2} & \mathbf{2} & \mathbf{1} \\ \mathbf{1} & \mathbf{2} & \mathbf{-2} \\ \mathbf{3} & \mathbf{-6} & \mathbf{6} \end{vmatrix} = 2(0) - 2(12) + 1(-12) = -36$$

$$\Delta_1 = \begin{vmatrix} \mathbf{2} & \mathbf{2} & \mathbf{1} \\ \mathbf{1} & \mathbf{2} & \mathbf{-2} \\ \mathbf{3} & \mathbf{-6} & \mathbf{6} \end{vmatrix} = -36 \Longrightarrow x = \frac{\Delta_1}{\Delta} = 1$$

$$\Delta_2 = \begin{vmatrix} \mathbf{2} & \mathbf{2} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & \mathbf{-2} \\ \mathbf{3} & \mathbf{3} & \mathbf{6} \end{vmatrix} = 2(12) - 2(12) + 1(0) = 0 \Longrightarrow y = \frac{\Delta_2}{\Delta} = 0$$

$$\Delta_3 = \begin{vmatrix} \mathbf{2} & \mathbf{2} & \mathbf{2} \\ \mathbf{1} & \mathbf{2} & \mathbf{1} \\ \mathbf{3} & \mathbf{-6} & \mathbf{3} \end{vmatrix} = 2(12) - 2(0) + 2(-12) = 0 \Longrightarrow z = \frac{\Delta_3}{\Delta} = 0$$

<u>By the inverse matrix method</u>

$$adj(A) = [cofactors]^T = \begin{bmatrix} \mathbf{0} & \mathbf{12} & \mathbf{-12} \\ \mathbf{18} & \mathbf{9} & \mathbf{-18} \\ \mathbf{-6} & \mathbf{-5} & \mathbf{2} \end{bmatrix}^T$$

$$= \begin{bmatrix} \mathbf{0} & \mathbf{18} & \mathbf{-6} \\ \mathbf{12} & \mathbf{9} & \mathbf{-5} \\ \mathbf{-12} & \mathbf{-18} & \mathbf{2} \end{bmatrix}$$

$$A^{-1} = \frac{adj(A)}{|A|} = -\frac{1}{36} \begin{bmatrix} \mathbf{0} & \mathbf{18} & \mathbf{-6} \\ \mathbf{12} & \mathbf{9} & \mathbf{-5} \\ \mathbf{-12} & \mathbf{-18} & \mathbf{2} \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1}B = -\frac{1}{36}\begin{bmatrix} \mathbf{0} & \mathbf{18} & \mathbf{-6} \\ \mathbf{12} & \mathbf{9} & \mathbf{-5} \\ \mathbf{-12} & \mathbf{-18} & \mathbf{2} \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} \mathbf{1} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}$$

b)

x + y + z = 3
x + 3 y + 3 z = 7
4 x - 3 y - 2 z = 3

Sol.

$$\mathrm{A} = \begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{3} & \mathbf{3} \\ \mathbf{4} & \mathbf{-3} & \mathbf{-2} \end{bmatrix} \text{ And B} = \begin{bmatrix} 3 \\ 7 \\ 3 \end{bmatrix}$$

$$\Delta = \begin{vmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{3} & \mathbf{3} \\ \mathbf{4} & \mathbf{-3} & \mathbf{-2} \end{vmatrix} = 1(3) - 1(-14) + 1(-15) = 2$$

$$\Delta_1 = \begin{vmatrix} \mathbf{3} & \mathbf{1} & \mathbf{1} \\ \mathbf{7} & \mathbf{3} & \mathbf{3} \\ \mathbf{3} & \mathbf{-3} & \mathbf{-2} \end{vmatrix} = 2 \Longrightarrow x = \frac{\Delta_1}{\Delta} = 1$$

$$\Delta_2 = \begin{vmatrix} \mathbf{1} & \mathbf{3} & \mathbf{1} \\ \mathbf{1} & \mathbf{7} & \mathbf{3} \\ \mathbf{4} & \mathbf{3} & \mathbf{-2} \end{vmatrix} = -6 \Longrightarrow y = \frac{\Delta_2}{\Delta} = -3$$

$$\Delta_3 = \begin{vmatrix} \mathbf{1} & \mathbf{1} & \mathbf{3} \\ \mathbf{1} & \mathbf{3} & \mathbf{7} \\ \mathbf{4} & \mathbf{-3} & \mathbf{3} \end{vmatrix} = 10 \Longrightarrow z = \frac{\Delta_3}{\Delta} = 5$$

By the inverse matrix method

$$adj(A) = [cofactors]^T = \begin{bmatrix} \mathbf{3} & \mathbf{-14} & \mathbf{-15} \\ \mathbf{1} & \mathbf{-6} & \mathbf{-7} \\ \mathbf{0} & \mathbf{2} & \mathbf{2} \end{bmatrix}^T = \begin{bmatrix} \mathbf{3} & \mathbf{1} & \mathbf{0} \\ \mathbf{-14} & \mathbf{-6} & \mathbf{2} \\ \mathbf{-15} & \mathbf{-7} & \mathbf{2} \end{bmatrix}$$

$$A^{-1} = \frac{adj(A)}{|A|} = \frac{1}{2}\begin{bmatrix} \mathbf{3} & \mathbf{1} & \mathbf{0} \\ \mathbf{-14} & \mathbf{-6} & \mathbf{2} \\ \mathbf{-15} & \mathbf{-7} & \mathbf{2} \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1}B = \frac{1}{2}\begin{bmatrix} \mathbf{0} & \mathbf{18} & \mathbf{-6} \\ \mathbf{12} & \mathbf{9} & \mathbf{-5} \\ \mathbf{-12} & \mathbf{-18} & \mathbf{2} \end{bmatrix}\begin{bmatrix} 3 \\ 7 \\ 3 \end{bmatrix} = \begin{bmatrix} \mathbf{1} \\ \mathbf{-3} \\ \mathbf{5} \end{bmatrix}$$

15. Consider the matrices $A = \begin{bmatrix} 2 & 1 & 2 \\ 2 & 2 & -2 \\ 3 & 1 & 1 \end{bmatrix}$ **and** $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

(a) Show that the equation $A\mathbf{x} = \mathbf{x}$ can be rewritten as $(A - I_3)\mathbf{x} = \mathbf{0}$ and use this result to solve the equation $A\mathbf{x} = \mathbf{x}$

(b) Solve the equation $A\mathbf{x} = 4\mathbf{x}$

<u>Solution:</u>

(a) Show that the equation $A\mathbf{x} = \mathbf{x}$ can be rewritten as $(A - I_3)\mathbf{x} = \mathbf{0}$ and use this result to solve the equation $A\mathbf{x} = \mathbf{x}$

$$A\mathbf{x} = \mathbf{x} \Leftrightarrow A\mathbf{x} - \mathbf{x} = \mathbf{0} \Leftrightarrow A\mathbf{x} - I_3\mathbf{x} = \mathbf{0} \Leftrightarrow (A - I_3)\mathbf{x} = \mathbf{0}$$

(b) Solve the equation $A\mathbf{x} = 4\mathbf{x}$

$$A\mathbf{x} = 4\mathbf{x} \Longrightarrow (A - 4I_3)\mathbf{x} = \mathbf{0}$$

$$\xrightarrow{(A-4I_3)\mathbf{x}=\mathbf{0}} \begin{array}{l} -2x_1 + 2x_2 + 2x_3 = 0 \\ -2x_1 - 2x_2 - 2x_3 = 0 \\ -3x_1 + 2x_2 - 3x_3 = 0 \end{array} \xrightarrow{\text{Augmented matrix}} \left(\begin{array}{ccc|c} -2 & 1 & 2 & 0 \\ 2 & -2 & -2 & 0 \\ 3 & 1 & -3 & 0 \end{array}\right)$$

$$\xrightarrow{R_3+R_1 \to R_1} \left(\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 2 & -2 & -2 & 0 \\ 3 & 1 & -3 & 0 \end{array}\right)$$

$$\xrightarrow[-3R_1+R_3 \to R_3]{-2R_1+R_2 \to R_2} \left(\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & -5 & 0 & 0 \end{array}\right) \xrightarrow[-R_3/5 \to R_3]{-R_2/4 \to R_2} \left(\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{-R_2+R_3 \to R_3} \left(\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Longrightarrow \text{Echelon matrix}$$

$$\Longrightarrow \begin{cases} x_1 + 2x_2 - x_3 = 0 \\ x_1 + 2x_2 + x_3 = 0 \\ x_1 + 2x_2 + x_3 = 0 \end{cases} \xrightarrow{\text{Let } x_3 = t} \begin{cases} x_1 = t \\ x_2 = 0 \\ x_3 = t \end{cases} \Longrightarrow \boxed{\mathbf{x} = t\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad \text{where } t \in \mathbb{R}}$$

15. For each of the following matrices,
(i) decide whether the matrix is nonsingular and give its rank
(ii) if nonsingular find its inverse using the Gauss-Jordan method

(a) $\boldsymbol{D} = \begin{bmatrix} 2 & -4 & 0 & 0 \\ 1 & 2 & 12 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & -1 & -4 & -5 \end{bmatrix}$ **(b)** $\boldsymbol{E} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 1 & 3 & 5 & 0 \\ 1 & 3 & 5 & 7 \end{bmatrix}$ **(f)** $\boldsymbol{F} = \begin{bmatrix} -8 & 17 & 2 & 1/3 \\ 4 & 0 & 2/5 & -9 \\ 20 & 0 & 2 & -45 \\ -1 & 13 & 4 & 2 \end{bmatrix}$

Solution:

(a) $\boldsymbol{D} = \begin{bmatrix} \mathbf{2} & \mathbf{-4} & \mathbf{0} & \mathbf{0} \\ \mathbf{1} & \mathbf{2} & \mathbf{12} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{2} & \mathbf{0} \\ \mathbf{0} & \mathbf{-1} & \mathbf{-4} & \mathbf{-5} \end{bmatrix}$

$$\mathbf{i(i)}\ \begin{bmatrix} 2 & -4 & 0 & 0 \\ 1 & 2 & 12 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & -1 & -4 & -5 \end{bmatrix} \xRightarrow[R_1 \to R_2]{R_2 \to R_1} \begin{bmatrix} 1 & 2 & 12 & 0 \\ 2 & -4 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & -1 & -4 & -5 \end{bmatrix}$$

$$\xRightarrow{-2R_1 + R_2 \to R_2} \begin{bmatrix} 1 & 2 & 12 & 0 \\ 0 & -8 & -24 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & -1 & -4 & -5 \end{bmatrix}$$

$$\mathbf{i(i)}\ \xRightarrow{-R_2/8 \to R_2} \begin{bmatrix} 1 & 2 & 12 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & -1 & -4 & -5 \end{bmatrix} \xRightarrow{R_2 + R_4 \to R_4} \begin{bmatrix} 1 & 2 & 12 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & -1 & -5 \end{bmatrix}$$

$$\xRightarrow{R_3/2 \to R_3} \begin{bmatrix} 1 & 2 & 12 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & -5 \end{bmatrix}$$

$$\mathbf{i(i)}\ \xRightarrow{R_3 + R_4 \to R_4} \begin{bmatrix} 1 & 2 & 12 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -5 \end{bmatrix}$$

$$\mathbf{i(i)}\ \xRightarrow{\text{Echelon matrix}} \text{rank of } D = \text{Number of nonzero rows in echelon form}$$

$$= 4$$

$$\mathbf{i(i)}\ \xRightarrow{\text{rank of } D = \text{Number of rows} = 4} \boxed{D \text{ is nonsingular}} \Longrightarrow D \text{ is invertible}$$

$$\mathbf{(ii)} [D | I_4] \Longrightarrow \left[\begin{array}{cccc|cccc} 2 & -4 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 2 & 12 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & -4 & -5 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\xRightarrow[R_1 \to R_2]{R_2 \to R_1} \left[\begin{array}{cccc|cccc} 1 & 2 & 12 & 0 & 0 & 1 & 0 & 0 \\ 2 & -4 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & -4 & -5 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\mathbf{i(i)} \xRightarrow{-2R_1+R_2\to R_2} \left[\begin{array}{cccc|cccc} 1 & 2 & 12 & 0 & 0 & 1 & 0 & 0 \\ 0 & -8 & -24 & 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & -4 & -5 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\xRightarrow[R_2\to R_4]{-R_4\to R_2} \left[\begin{array}{cccc|cccc} 1 & 2 & 12 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 4 & 5 & 0 & 0 & 0 & -1 \\ 0 & 0 & 2 & 0 & 0 & 0 & 1 & 0 \\ 0 & -8 & -24 & 0 & 1 & -2 & 0 & 0 \end{array}\right]$$

$$\mathbf{i(i)} \xRightarrow[8R_2+R_4\to R_4]{-2R_2+R_1\to R_1} \left[\begin{array}{cccc|cccc} 1 & 0 & 4 & -10 & 0 & 1 & 0 & 2 \\ 0 & 1 & 4 & 5 & 0 & 0 & 0 & -1 \\ 0 & 0 & 2 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 8 & 40 & 1 & -2 & 0 & -8 \end{array}\right]$$

$$\xRightarrow{R_3/2\to R_3} \left[\begin{array}{cccc|cccc} 1 & 0 & 4 & -10 & 0 & 1 & 0 & 2 \\ 0 & 1 & 4 & 5 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 8 & 40 & 1 & -2 & 0 & -8 \end{array}\right]$$

$$\mathbf{i(i)} \xRightarrow{\substack{-4R_3+R_1\to R_1 \\ -4R_3+R_2\to R_2 \\ -8R_3+R_4\to R_4}} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & -10 & 0 & 1 & -2 & 2 \\ 0 & 1 & 0 & 5 & 0 & 0 & -2 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 40 & 1 & -2 & -4 & -8 \end{array}\right]$$

$$\mathbf{i(i)} \xRightarrow{R_4/40\to R_4} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & -10 & 0 & 1 & -2 & 2 \\ 0 & 1 & 0 & 5 & 0 & 0 & -2 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1 & 1/40 & -1/20 & -1/10 & -1/5 \end{array}\right]$$

$$\mathbf{i(i)} \xRightarrow[-5R_4+R_2\to R_2]{10R_4+R_1\to R_1} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1/4 & 1/2 & -3 & 0 \\ 0 & 1 & 0 & 0 & -1/8 & 1/4 & -3/2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1 & 1/40 & -1/20 & -1/10 & -1/5 \end{array}\right] \Rightarrow [I_4|D^{-1}]$$

$$\mathbf{i(i)} \Rightarrow D^{-1} = \begin{bmatrix} 1/4 & 1/2 & -3 & 0 \\ -1/8 & 1/4 & -3/2 & 0 \\ 0 & 0 & 1/2 & 0 \\ 1/40 & -1/20 & -1/10 & -1/5 \end{bmatrix}$$

$$\textbf{(b) } \boldsymbol{E} = \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{1} & \mathbf{3} & \mathbf{0} & \mathbf{0} \\ \mathbf{1} & \mathbf{3} & \mathbf{5} & \mathbf{0} \\ \mathbf{1} & \mathbf{3} & \mathbf{5} & \mathbf{7} \end{bmatrix}$$

$$\mathbf{i(i)}\ \begin{bmatrix}1&0&0&0\\1&3&0&0\\1&3&5&0\\1&3&5&7\end{bmatrix}\xrightarrow[-R_1+R_4\to R_4]{\substack{-R_1+R_2\to R_2\\-R_1+R_3\to R_3}}\begin{bmatrix}1&0&0&0\\0&3&0&0\\0&3&5&0\\0&3&5&7\end{bmatrix}\xrightarrow[-R_3+R_4\to R_4]{\substack{R_2/3\to R_2\\-R_2+R_3\to R_3}}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&5&0\\0&0&0&7\end{bmatrix}$$

$$\xrightarrow[R_4/7\to R_4]{R_3/5\to R_5}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$$

$$\mathbf{i(i)}\ \xRightarrow{\text{Echelon matrix}}\ \text{rank of } E = \text{Number of nonzero rows in echelon form}$$

$$= 4$$

$$\mathbf{(i)}\ \xRightarrow{\text{rank of } E=\text{Number of rows}=4}\ \boxed{E \text{ is nonsingular}} \Longrightarrow E \text{ is invertible}$$

$$\mathbf{(ii)}\ [E|I_4] \Longrightarrow \left[\begin{array}{cccc|cccc}1&0&0&0&1&0&0&0\\1&3&0&0&0&1&0&0\\1&3&5&0&0&0&1&0\\1&3&5&7&0&0&0&1\end{array}\right]$$

$$\xRightarrow[-R_1+R_4\to R_4]{\substack{-R_1+R_2\to R_2\\-R_1+R_3\to R_3}}\left[\begin{array}{cccc|cccc}1&0&0&0&1&0&0&0\\0&3&0&0&-1&1&0&0\\0&3&5&0&-1&0&1&0\\0&3&5&7&-1&0&0&1\end{array}\right]$$

$$\mathbf{(ii)}\ \xRightarrow[-R_3+R_4\to R_4]{\substack{R_2/3\to R_2\\-R_2+R_3\to R_3}}\left[\begin{array}{cccc|cccc}1&0&0&0&1&0&0&0\\0&1&0&0&-1/3&1/3&0&0\\0&0&5&0&0&-1&1&0\\0&0&0&7&0&0&-1&1\end{array}\right]$$

$$\mathbf{(ii)}\ \xRightarrow[R_4/7\to R_4]{R_3/5\to R_5}\left[\begin{array}{cccc|cccc}1&0&0&0&1&0&0&0\\0&1&0&0&-1/3&1/3&0&0\\0&0&1&0&0&-1/5&1/5&0\\0&0&0&1&0&0&-1/7&1/7\end{array}\right] \Longrightarrow [I_4|E^{-1}]$$

$$\mathbf{(ii)}\ \Longrightarrow E^{-1} = \begin{bmatrix}1&0&0&0\\-1/3&1/3&0&0\\0&-1/5&1/5&0\\0&0&-1/7&1/7\end{bmatrix}$$

$$\mathbf{(f)}\ \boldsymbol{F} = \begin{bmatrix}\mathbf{-8}&\mathbf{17}&\mathbf{2}&\mathbf{1/3}\\\mathbf{4}&\mathbf{0}&\mathbf{2/5}&\mathbf{-9}\\\mathbf{20}&\mathbf{0}&\mathbf{2}&\mathbf{-45}\\\mathbf{-1}&\mathbf{13}&\mathbf{4}&\mathbf{2}\end{bmatrix}$$

$$\begin{bmatrix} -8 & 17 & 2 & 1/3 \\ 4 & 0 & 2/5 & -9 \\ 20 & 0 & 2 & -45 \\ -1 & 13 & 4 & 2 \end{bmatrix} \xRightarrow[R_1 \to R_4]{-R_4 \to R_1} \begin{bmatrix} 1 & -13 & -4 & -2 \\ 4 & 0 & 2/5 & -9 \\ 20 & 0 & 2 & -45 \\ -8 & 17 & 2 & 1/3 \end{bmatrix}$$

$$\xRightarrow{\substack{-4R_1+R_2 \to R_2 \\ -20R_1+R_3 \to R_3 \\ 8R_1+R_4 \to R_4}} \begin{bmatrix} 1 & -13 & -4 & -2 \\ 0 & 52 & 82/5 & -1 \\ 0 & 260 & 82 & -5 \\ 0 & -87 & -30 & -47/3 \end{bmatrix}$$

$$\xRightarrow{R_2/52 \to R_2} \begin{bmatrix} 1 & -13 & -4 & -2 \\ 0 & 1 & 41/130 & -1/52 \\ 0 & 260 & 82 & -5 \\ 0 & -87 & -30 & -47/3 \end{bmatrix}$$

$$\xRightarrow{\substack{-260R_2+R_3 \to R_3 \\ 87R_2+R_4 \to R_4}} \begin{bmatrix} 1 & -13 & -4 & -2 \\ 0 & 1 & 41/130 & -1/52 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -333/130 & -2705/156 \end{bmatrix}$$

$$\xRightarrow{\substack{-130R_4/333 \to R_3 \\ R_3 \to R_4}} \begin{bmatrix} 1 & -13 & -4 & -2 \\ 0 & 1 & 41/130 & -1/52 \\ 0 & 0 & 1 & 13525/1998 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xRightarrow{\text{Echelon matrix}} \text{rank of } F = \text{Number of nonzero rows in echelon form } = 3$$

$$\xRightarrow{\text{rank of } F \neq \text{Number of rows} = 4} \boxed{F \text{ is singular}} \Longrightarrow F \text{ is not invertible}$$

---

16. Show that $A = \begin{bmatrix} 0 & a & 0 & 0 & 0 \\ b & 0 & c & 0 & 0 \\ 0 & d & 0 & e & 0 \\ 0 & 0 & f & 0 & g \\ 0 & 0 & 0 & h & 0 \end{bmatrix}$ is not invertible for any values of the entries.

Sol.

The condition for any matrix to be not invertible is that its determinant must be equal to zero. So its determinant will be calculated.

$$
\begin{aligned}
\det(A) &= \begin{vmatrix} 0 & a & 0 & 0 & 0 \\ b & 0 & c & 0 & 0 \\ 0 & d & 0 & e & 0 \\ 0 & 0 & f & 0 & g \\ 0 & 0 & 0 & h & 0 \end{vmatrix} \\
&= 0 - a \begin{vmatrix} b & c & 0 & 0 \\ 0 & 0 & e & 0 \\ 0 & f & 0 & g \\ 0 & 0 & h & 0 \end{vmatrix} + 0 - 0 + 0 \\
&= -a \begin{vmatrix} b & c & 0 & 0 \\ 0 & 0 & e & 0 \\ 0 & f & 0 & g \\ 0 & 0 & h & 0 \end{vmatrix} \\
&= -a(b) \begin{vmatrix} 0 & e & 0 \\ f & 0 & g \\ 0 & h & 0 \end{vmatrix} - 0 + 0 - 0 \\
&= -a(b) \begin{vmatrix} 0 & e & 0 \\ f & 0 & g \\ 0 & h & 0 \end{vmatrix} = -a(b) \left[0 - e \begin{vmatrix} f & g \\ 0 & 0 \end{vmatrix} + 0 \right] \\
&= -a(b)(-e)[0 - 0] = abe(0) = 0
\end{aligned}
$$

So for any values of entries the determinant of A will always be zero because the determinant of a matrix is unique. So matrix A will be not invertible (Singular) for any values of entries.