Solutions of Worksheet # 6

(B) Laplace Transformation

(I) Find the Laplace transform of the following functions:

1. $t^3 + 3t^2 + 4t + 3$

Sol. $\mathcal{L}\left\{t^3 + 3t^2 + 4t + 3\right\} = \frac{3!}{s^4} + 3\left(\frac{2!}{s^3}\right) + 4\left(\frac{1}{s^2}\right) + \frac{3}{s} = \frac{6}{s^4} + \frac{6}{s^3} + \frac{4}{s^2} + \frac{3}{s}.$

2. $4\sin 3t + 5\cos 3t$

Sol. $\mathcal{L}\left\{4\sin 3t + 5\cos 3t\right\} = 4\left(\frac{3}{s^2+9}\right) + 5\left(\frac{s}{s^2+9}\right) = \frac{5s+12}{s^2+9}.$

3. $4t + \sin^2 3t$

Sol. $\mathcal{L}\left\{4t + \sin^2 3t\right\} = \mathcal{L}\left\{4t + \frac{1}{2} - \frac{1}{2}\cos 6t\right\} = \frac{4}{s^2} + \frac{1}{2s} - \frac{s}{2(s^2+36)}.$

4. $\sin(5t+2)$

Sol. $\because \sin(5t+2) = \cos(2)\sin(5t) + \sin(2)\cos(5t) = -0.416\sin 5t + 0.909\cos 5t.$

$\therefore \mathcal{L}\left\{\sin(5t+2)\right\} = -0.416\,\mathcal{L}\left\{\sin 5t\right\} + 0.909\,\mathcal{L}\left\{\cos 5t\right\}$

$$= -0.416\left(\frac{5}{s^2+25}\right) + 0.909\left(\frac{s}{s^2+25}\right) = \frac{0.909s - 2.081}{s^2+25}.$$

5. $\sin 4t \cos 2t$

Sol. $\because \sin 4t \cos 2t = \frac{1}{2}\left[\sin 6t + \sin 2t\right]$

$\therefore \mathcal{L}\left\{\sin 4t \cos 2t\right\} = \frac{1}{2} L\left\{\sin 6t + \sin 2t\right\} = \frac{1}{2}\left[\frac{6}{s^2+36} + \frac{2}{s^2+4}\right] = \frac{3}{s^2+36} + \frac{1}{s^2+4}$

6. $\cos 3t \cos t$

Sol. $\because \cos 3t \cos t = \frac{1}{2}\left[\cos 4t + \cos 2t\right].$

$$\therefore \mathcal{L}\{\cos 3t \cos t\} = \frac{1}{2}\mathcal{L}\{\cos 4t + \cos 2t\} = \frac{1}{2}\left[\frac{s}{s^2+16} + \frac{s}{s^2+4}\right].$$

7. $t^2 e^{-2t} + \cos 4t + 5e^{2t} \sinh 2t$

Sol. $\mathcal{L}\left\{e^{-2t}\right\} = \dfrac{1}{s+2} \Rightarrow \mathcal{L}\left\{t e^{-2t}\right\} = -\dfrac{d}{ds}\left(\dfrac{1}{s+2}\right) = \dfrac{1}{(s+2)^2}$

$$\Rightarrow \mathcal{L}\left\{t^2 e^{-2t}\right\} = \mathcal{L}\left\{t\left(t e^{-2t}\right)\right\} = -\frac{d}{ds}\left(\frac{1}{(s+2)^2}\right) = \frac{2}{(s+2)^3}.$$

$$\mathcal{L}\left\{5e^{2t}\sinh 2t\right\} = \mathcal{L}\left\{5e^{2t}\left(\frac{e^{2t}-e^{-2t}}{2}\right)\right\} = \frac{5}{2}\mathcal{L}\left\{e^{4t}-1\right\} = \frac{5}{2}\left[\frac{1}{s-4} - \frac{1}{s}\right]$$

$$= \frac{5}{2s-8} - \frac{5}{2s}.$$

$$\therefore \mathcal{L}\left\{t^2 e^{-2t} + \cos 4t + 5e^{2t}\sinh 2t\right\} = \frac{2}{(s+2)^3} + \frac{s}{s^2+16} + \frac{5}{2s-8} - \frac{5}{2s}.$$

8. $t\cos^2 t$

Sol. $\mathcal{L}\left\{\cos^2 t\right\} = \dfrac{1}{2}\mathcal{L}\{1+\cos 2t\} = \dfrac{1}{2}\left[\dfrac{1}{s} + \dfrac{s}{s^2+4}\right].$

$$\therefore \mathcal{L}\left\{t\cos^2 t\right\} = -\frac{1}{2}\frac{d}{ds}\left[\frac{1}{s} + \frac{s}{s^2+4}\right] = -\frac{1}{2}\left[-\frac{1}{s^2} + \frac{(s^2+4)(1) - s(2s)}{(s^2+4)^2}\right]$$

$$= \frac{1}{2s^2} + \frac{s^2-4}{2(s^2+4)^2}.$$

9. $\sinh t \cosh t + t\cosh 4t + e^{1-3t}$

Sol. $\mathcal{L}\{\sinh t \cosh t\} = \mathcal{L}\left\{\left(\dfrac{e^t - e^{-t}}{2}\right)\left(\dfrac{e^t + e^{-t}}{2}\right)\right\} = \dfrac{1}{4}\mathcal{L}\left\{e^{2t} - e^{-2t}\right\}.$

$$\therefore \mathcal{L}\{\sinh t \cosh t\} = \frac{1}{4}\left[\frac{1}{s-2} - \frac{1}{s+2}\right] = \frac{1}{4s-8} - \frac{1}{4s+8}.$$

$$\because \mathcal{L}\{\cosh 4t\} = \frac{s}{s^2-16}.$$

$$\therefore \mathcal{L}\{t\cosh t\} = -\frac{d}{ds}\left(\frac{s}{s^2-16}\right) = -\frac{(s^2-16)(1)-s(2s)}{(s^2-16)^2} = \frac{s^2+16}{(s^2-16)^2}.$$

$$\because \mathcal{L}\left\{e^{1-3t}\right\} = e\,L\left\{e^{-3t}\right\} = \frac{e}{s+3}.$$

$$\therefore \mathcal{L}\left\{\sinh t\ \cosh t + t\cosh 4t + e^{1-3t}\right\} = \frac{1}{4s-8} - \frac{1}{4s+8} + \frac{s^2+16}{(s^2-16)^2} + \frac{e}{s+3}.$$

10. $t\sin 6t$

Sol. $\mathcal{L}\{t\sin 6t\} = -\frac{d}{ds}\left(\frac{6}{s^2+36}\right) = \frac{12s}{(s^2+36)^2}$.

11. $t^2\cos 4t$

Sol. $\because \mathcal{L}\{t\cos 4t\} = -\frac{d}{ds}\left(\frac{s}{s^2+16}\right) = -\frac{(s^2+16)(1)-s(2s)}{(s^2+16)^2} = \frac{s^2-16}{(s^2+16)^2}$.

$$\therefore \mathcal{L}\left\{t^2\cos 4t\right\} = -\frac{d}{ds}\left(\frac{s^2-16}{(s^2+16)^2}\right) = -\frac{(s^2+16)^2(2s)-(s^2-16)(4s)(s^2+16)}{(s^2+16)^4}$$

$$= \frac{2s^3-96s}{(s^2+16)^3}.$$

12. $f(t)=t^2, \quad 0<t<1$

Sol. $F(s)=\int\limits_0^\infty f(t)\,e^{-st}\,dt=\int\limits_0^1 t^2 e^{-st}\,dt.$

d/dt	$\int dt$
t^2 (+)	e^{-st}
$2t$ (-)	$-e^{-st}/s$
2 (+)	e^{-st}/s^2
0	$-e^{-st}/s^3$

$$\therefore F(s)=\left[-\frac{t^2e^{-st}}{s}-\frac{2t\,e^{-st}}{s^2}-\frac{2e^{-st}}{s^3}\right]_{t=0}^{t=1}=\frac{2}{s^3}-\frac{e^{-s}}{s}-\frac{2e^{-s}}{s^2}-\frac{2e^{-s}}{s^3}.$$

13. $g(t)=\begin{cases}1 & 0\le t<1\\ t & 1\le t<2\\ 0 & otherwise\end{cases}$

Sol. $G(s)=\int\limits_0^\infty g(t)e^{-st}\,dt=\int\limits_0^1 e^{-st}\,dt+\int\limits_1^2 t\,e^{-st}\,dt.$

d/dt	$\int dt$
t (+)	e^{-st}
1 (-)	$-e^{-st}/s$
0	e^{-st}/s^2

$$\therefore G(s)=\left[-\frac{e^{-st}}{s}\right]_{t=0}^{t=1}+\left[-\frac{t\,e^{-st}}{s}-\frac{e^{-st}}{s^2}\right]_{t=1}^{t=2}=\frac{1}{s}+\frac{e^{-s}}{s^2}-\frac{2e^{-2s}}{s}-\frac{e^{-2s}}{s^2}.$$

(II) Prove that $\lim_{t\to 0^+} f(t)$ is exist and find the Laplace transform of $f(t)$:

1. $f(t)=\dfrac{\sin 2t}{t}$ then use it to evaluate $\int_0^\infty \dfrac{e^{-3t}\sin 2t}{t}dt$.

Sol. $\lim_{t\to 0^+} f(t)=\lim_{t\to 0^+}\dfrac{\sin 2t}{t}=2$ (exists).

$$\therefore \mathcal{L}\left\{\frac{\sin 2t}{t}\right\}=\int_s^\infty \frac{2}{\omega^2+4}d\omega=\left[\tan^{-1}\left(\frac{\omega}{2}\right)\right]_{\omega=s}^{\omega=\infty}=\frac{\pi}{2}-\tan^{-1}\left(\frac{s}{2}\right). \qquad (1)$$

Also, from the Laplace definition, we can say that

$$\therefore \mathcal{L}\left\{\frac{\sin 2t}{t}\right\}=\int_0^\infty \frac{e^{-st}\sin 2t}{t}dt. \qquad (2)$$

From equations (1) and (2), we can infer that

$$\int_0^\infty \frac{e^{-st}\sin 2t}{t}dt=\frac{\pi}{2}-\tan^{-1}\left(\frac{s}{2}\right)\xrightarrow{s=3}\int_0^\infty \frac{e^{-3t}\sin 2t}{t}dt=\frac{\pi}{2}-\tan^{-1}\left(\frac{3}{2}\right).$$

$$\therefore \int_0^\infty \frac{e^{-3t}\sin 2t}{t}dt=0.1872\pi.$$

2. $f(t)=\dfrac{1-\cos t}{t}$ then use it to evaluate $\int_0^\infty \dfrac{e^{-t}(1-\cos t)}{t}dt$.

Sol. $\lim_{t\to 0^+} f(t)=\lim_{t\to 0^+}\dfrac{1-\cos t}{t}=\lim_{t\to 0^+}\dfrac{\sin t}{1}=0$ (exists).

$$\therefore \mathcal{L}\left\{\frac{1-\cos t}{t}\right\}=\int_s^\infty \left[\frac{1}{\omega}-\frac{\omega}{\omega^2+1}\right]d\omega=\left[\ln\omega-\frac{1}{2}\ln\left(\omega^2+1\right)\right]_{\omega=s}^{\omega=\infty}=\left[\ln\frac{\omega}{\sqrt{\omega^2+1}}\right]_{\omega=s}^{\omega=\infty}$$

$$\therefore \mathcal{L}\left\{\frac{1-\cos t}{t}\right\}=-\ln\frac{s}{\sqrt{s^2+1}}. \qquad (1)$$

Also, from the Laplace definition, we can say that

$$\therefore \mathcal{L}\left\{\frac{1-\cos t}{t}\right\}=\int_0^\infty \frac{e^{-st}(1-\cos t)}{t}dt. \qquad (2)$$

From equations (1) and (2), we can infer that

$$\int_0^\infty \frac{e^{-st}(1-\cos t)}{t}dt=-\ln\frac{s}{\sqrt{s^2+1}}\xrightarrow{s=1}\int_0^\infty \frac{e^{-t}(1-\cos t)}{t}dt=-\ln\left(\frac{1}{\sqrt{2}}\right).$$

$$\therefore \int_0^\infty \frac{e^{-t}(1-\cos t)}{t}dt=0.3466.$$

(III) Find the Laplace transform of the following functions:

1. $t^2e^{-2t}+e^{-3t}\cos 4t+5e^{2t}\sinh 2t$

Sol. $\mathcal{L}\left\{t^2e^{-2t}+e^{-3t}\cos 4t+5e^{2t}\sinh 2t\right\}=\dfrac{2!}{(s+2)^3}+\dfrac{s+3}{(s+3)^2+16}+\dfrac{10}{(s-2)^2-4}.$

2. $te^{2t}\cos 3t$

Sol. $\because \mathcal{L}\{t\cos 3t\}=-\dfrac{d}{ds}\left(\dfrac{s}{s^2+9}\right)=-\dfrac{(s^2+9)(1)-s(2s)}{(s^2+9)^2}=\dfrac{s^2-9}{(s^2+9)^2}.$

$\therefore \mathcal{L}\left\{te^{2t}\cos 3t\right\}=\dfrac{(s-2)^2-9}{\left[(s-2)^2+9\right]^2}.$

3. $3u(t)+5u(t-3)+4(t-2)u(t-2)$

Sol. $\mathcal{L}\{3u(t)+5u(t-3)+4(t-2)u(t-2)\}=\dfrac{3}{s}+\dfrac{5}{s}e^{-3s}+\dfrac{4}{s^2}e^{-2s}.$

4. $t^2u(t-2)$

Sol. $\because t^2u(t-2)=\left[(t-2)+2\right]^2u(t-2)=\left[(t-2)^2+4(t-2)+4\right]u(t-2).$

$\therefore \mathcal{L}\left\{t^2u(t-2)\right\}=\left[\dfrac{2!}{s^3}+\dfrac{4}{s^2}+\dfrac{4}{s}\right]e^{-2s}.$

5. $e^{-2t}u(t-3)$

Sol. $\because \mathcal{L}\{u(t-3)\}=\dfrac{1}{s}e^{-3s}.$

$\therefore \mathcal{L}\left\{e^{-2t}u(t-3)\right\}=\dfrac{1}{(s+2)}e^{-3(s+2)}.$

6. $\sin(2t)u(t-\pi)$

Sol. $\because \sin(2t)=\sin(2(t-\pi)+2\pi)=\cos(2\pi)\sin 2(t-\pi)+\sin(2\pi)\cos 2(t-\pi)$

$=\sin 2(t-\pi).$

$\therefore \mathcal{L}\{\sin(2t)u(t-\pi)\}=\mathcal{L}\{\sin 2(t-\pi)u(t-\pi)\}=\dfrac{2}{s^2+4}e^{-\pi s}.$

7. $5\delta(t)+3\delta(t-2)$

Sol. $\mathcal{L}\{5\delta(t)+3\delta(t-2)\}=5+3e^{-2s}.$

8. $e^{3t}\delta(t-2)$

Sol. $\because \mathcal{L}\{\delta(t-2)\}=e^{-2s}.$

$\therefore \mathcal{L}\left\{e^{3t}\delta(t-2)\right\}=e^{-2(s-3)}.$

(IV) Find the Laplace transform of the following functions:

1. $f(t)=\begin{cases} t & 0\le t<1 \\ 0 & 1<t<2 \end{cases}, \quad f(t)=f(t+2).$

Sol. $period \Rightarrow p=2$.

$$F(s)=\frac{1}{1-e^{-ps}}\int_0^p f(t)e^{-st}dt=\frac{1}{1-e^{-2s}}\int_0^2 f(t)e^{-st}dt=\frac{1}{1-e^{-2s}}\int_0^1 te^{-st}dt$$

$$\therefore F(s)=\frac{1}{1-e^{-2s}}\left[-\frac{te^{-st}}{s}-\frac{e^{-st}}{s^2}\right]_{t=0}^{t=1}=\frac{1}{1-e^{-2s}}\left[\frac{1}{s^2}-\frac{e^{-s}}{s}-\frac{e^{-s}}{s^2}\right]$$

d/dt		$\int dt$
t	(+)	e^{-st}
1	(-)	$-e^{-st}/s$
0		e^{-st}/s^2

2. $g(t)=\begin{cases} \sin t & 0\le t\le \pi \\ 0 & \pi<t\le 2\pi \end{cases}, \quad g(t)=g(t+2\pi)$

Sol. $period \Rightarrow p=2\pi$.

$$G(s)=\frac{1}{1-e^{-ps}}\int_0^p g(t)e^{-st}dt=\frac{1}{1-e^{-2\pi s}}\int_0^{2\pi} g(t)e^{-st}dt=\frac{1}{1-e^{-2\pi s}}\int_0^{\pi} e^{-st}\sin t\,dt.$$

$$\because I=\int e^{-st}\sin t\,dt=e^{-st}(-\cos t)-\int -se^{-st}(-\cos t)\,dt$$

$$=-e^{-st}\cos t-s\int e^{-st}\cos t\,dt$$

$$=-e^{-st}\cos t-s\left[e^{-st}(\sin t)-\int -se^{-st}(\sin t)\,dt\right]$$

$$=-e^{-st}\cos t-se^{-st}\sin t-s^2\int e^{-st}\sin t\,dt$$

$$=-e^{-st}\cos t-se^{-st}\sin t-s^2 I$$

$$\therefore I+s^2I=-e^{-st}\cos t-se^{-st}\sin t$$

$$\Rightarrow I=\int e^{-st}\sin t\,dt=-\frac{e^{-st}}{1+s^2}\left[\cos t+s\sin t\right]+C.$$

$$\therefore G(s)=\frac{1}{1-e^{-2\pi s}}\int_0^{\pi} e^{-st}\sin t\,dt=\frac{1}{(1+s^2)(1-e^{-2\pi s})}\left[-e^{-st}\cos t-se^{-st}\sin t\right]_{t=0}^{t=\pi}$$

$$=\frac{1+e^{-\pi s}}{(1+s^2)(1-e^{-2\pi s})}.$$

3. $h(t)=|\sin kt|$

Sol. Given that $\sin t$ is a periodic function with period 2π then $\sin kt$ is a periodic function with period $\dfrac{2\pi}{k}$.

So the function $h(t)=|\sin kt|$ is a periodic one with period $p=\dfrac{\pi}{k}$.

$$H(s)=\frac{1}{1-e^{-ps}}\int_0^p h(t)e^{-st}dt$$
$$=\frac{1}{1-e^{-\pi s/k}}\int_0^{\pi/k} e^{-st}\sin kt\,dt.$$

Hint:

$$|\sin t|=\begin{cases}\sin t & 0\le t\le\pi\\ -\sin t & -\pi<t<0\end{cases}.$$

$$|\sin kt|=\begin{cases}\sin kt & 0\le t\le\pi/k\\ -\sin kt & -\pi/k<t<0\end{cases}.$$

$$\because I=\int e^{-st}\sin kt\,dt=e^{-st}\left(-\frac{1}{k}\cos kt\right)-\int -s\,e^{-st}\left(-\frac{1}{k}\cos kt\right)dt$$
$$=-\frac{1}{k}e^{-st}\cos kt-\frac{s}{k}\int e^{-st}\cos kt\,dt$$
$$=-\frac{1}{k}e^{-st}\cos kt-\frac{s}{k}\left[e^{-st}\left(\frac{1}{k}\sin kt\right)-\int -s\,e^{-st}\left(\frac{1}{k}\sin kt\right)dt\right]$$
$$=-\frac{1}{k}e^{-st}\cos kt-\frac{s}{k^2}e^{-st}\sin kt-\frac{s^2}{k^2}\int e^{-st}\sin kt\,dt$$
$$=-\frac{1}{k}e^{-st}\cos kt-\frac{s}{k^2}e^{-st}\sin kt-\frac{s^2}{k^2}I$$
$$\therefore I+\frac{s^2}{k^2}I=-\frac{1}{k}e^{-st}\cos kt-\frac{s}{k^2}e^{-st}\sin kt$$
$$\Rightarrow I=\int e^{-st}\sin kt\,dt=-\frac{k\,e^{-st}}{k^2+s^2}\left[\cos kt+\frac{s}{k}\sin kt\right]+C.$$

$$\therefore H(s)=\frac{1}{1-e^{-\pi s/k}}\int_0^{\pi/k} e^{-st}\sin kt\,dt$$
$$=\frac{k}{(k^2+s^2)(1-e^{-\pi s/k})}\left[-e^{-st}\cos kt-\frac{s}{k}e^{-st}\sin kt\right]_{t=0}^{t=\pi/k}$$
$$=\frac{k(1+e^{-\pi s/k})}{(k^2+s^2)(1-e^{-\pi s/k})}.$$