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WorkSheet # 3 – Solution

1. Find the eigenvalues and eigenvectors for the following matrices:

$(a) A = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$ $\quad\quad$ $(b)\ A = \begin{bmatrix} 3 & -1 & 1 \\ 0 & -5 & -12 \\ -2 & 1 & 0 \end{bmatrix}$

Solution:

$$(\boldsymbol{a}) \boldsymbol{A} = \begin{bmatrix} \mathbf{4} & \mathbf{0} & \mathbf{1} \\ -\mathbf{2} & \mathbf{1} & \mathbf{0} \\ -\mathbf{2} & \mathbf{0} & \mathbf{1} \end{bmatrix}$$

The characteristic equation:

$$|\lambda I - A| = \begin{bmatrix} \lambda - 4 & 0 & -1 \\ 2 & \lambda - 1 & 0 \\ 2 & 0 & \lambda - 1 \end{bmatrix}$$

$\det(|\lambda I - A|) = (\lambda - 4)(\lambda - 1)^2 + 2(\lambda - 1) = (\lambda - 1)\,[(\lambda - 4)(\lambda - 1) + 2]$

$\det(|\lambda I - A|) = 0 \Rightarrow (\lambda - 1)\,[(\lambda - 4)(\lambda - 1) + 2] = 0$

$\Rightarrow (\lambda - 1)\,(\lambda - 2)\,(\lambda - 3) = 0$

$\lambda_1 = 1, \lambda_2 = 2\ \&\ \lambda_3 = 3$ (Eigenvalues)

To find the corresponding eigenvectors:

At $\boldsymbol{\lambda_1} = \mathbf{1}$ $\rightarrow$ $\quad (\lambda_1 I - A)\boldsymbol{x} = \mathbf{0} \Rightarrow \left[\begin{array}{ccc|c} -3 & 0 & -1 & 0 \\ 2 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -\frac{2}{3} & 0 \end{array}\right]$

So the solution is $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ t \\ 0 \end{bmatrix} = t \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \ t \neq 0$

The eigenvector $v_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

At $\boldsymbol{\lambda_2 = 2}$ $\rightarrow$ $(\lambda_2 I - A)\boldsymbol{x} = \mathbf{0} \Rightarrow \left[\begin{array}{ccc|c} -2 & 0 & -1 & 0 \\ 2 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \end{array}\right] \dot{\rightarrow} \left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

The eigenvector $v_2 = \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 1 \end{bmatrix}$

At $\boldsymbol{\lambda_3 = 3}$ $\rightarrow$ $(\lambda_3 I - A)\boldsymbol{x} = \mathbf{0} \Rightarrow \left[\begin{array}{ccc|c} -1 & 0 & -1 & 0 \\ 2 & 2 & 0 & 0 \\ 2 & 0 & 2 & 0 \end{array}\right] \dot{\rightarrow} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

The eigenvector $v_3 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

$(\boldsymbol{b})\boldsymbol{A} = \begin{bmatrix} \mathbf{4} & \mathbf{0} & \mathbf{1} \\ \mathbf{-2} & \mathbf{1} & \mathbf{0} \\ \mathbf{-2} & \mathbf{0} & \mathbf{1} \end{bmatrix}$

The characteristic equation $|\lambda I - A| = \begin{bmatrix} \lambda - 3 & 1 & -1 \\ 0 & \lambda + 5 & 12 \\ 2 & -1 & \lambda \end{bmatrix}$

Using the second row:

$\det(|\lambda I - A|) = (\lambda + 5)[\lambda^2 - 3\lambda + 2] + 12(\lambda - 1) = (\lambda - 1)\ [(\lambda + 5)(\lambda - 1) + 12]$

$\det(|\lambda I - A|) = 0 \Rightarrow (\lambda - 1)\ [(\lambda + 5)(\lambda - 1) + 12] = 0$

$\Rightarrow (\lambda - 1)\ (\lambda + 2)\ (\lambda + 1) = 0$

$\lambda_1 = 1, \lambda_2 = -2\ \&\ \lambda_3 = -1$ (Eigenvalues)

To find the corresponding eigenvectors:

At $\boldsymbol{\lambda_1 = 1} \rightarrow$ $\quad (\lambda_1 I - A)\boldsymbol{x} = \mathbf{0} \Rightarrow \left[\begin{array}{ccc|c} -2 & 1 & -1 & 0 \\ 0 & 6 & 12 & 0 \\ 2 & -1 & 1 & 0 \end{array}\right] \xrightarrow{by\ ERO} \left[\begin{array}{ccc|c} 1 & 0 & \frac{3}{2} & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

So the solution is $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -\frac{3}{2}t \\ -2t \\ t \end{bmatrix} = t\begin{bmatrix} -\frac{3}{2} \\ -2 \\ 1 \end{bmatrix}, \ t \neq 0$

The eigenvector $v_1 = \begin{bmatrix} -\frac{3}{2} \\ -2 \\ 1 \end{bmatrix}$

At $\boldsymbol{\lambda_2 = -2} \rightarrow$ $\quad (\lambda_2 I - A)\boldsymbol{x} = \mathbf{0} \Rightarrow \left[\begin{array}{ccc|c} -5 & 1 & -1 & 0 \\ 0 & -3 & 12 & 0 \\ 2 & -1 & -2 & 0 \end{array}\right] \dot{\rightarrow} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

The eigenvector $v_2 = \begin{bmatrix} -1 \\ -4 \\ 1 \end{bmatrix}$

At $\boldsymbol{\lambda_3 = -1} \rightarrow$ $\quad (\lambda_3 I - A)\boldsymbol{x} = \mathbf{0} \Rightarrow \left[\begin{array}{ccc|c} -4 & 1 & -1 & 0 \\ 0 & 4 & 12 & 0 \\ 2 & -1 & -1 & 0 \end{array}\right] \dot{\rightarrow} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

The eigenvector $v_3 = \begin{bmatrix} -1 \\ -3 \\ 1 \end{bmatrix}$

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2. **Validate the Cayley-Hamilton theorem for the given matrices, and then use it to evaluate $\boldsymbol{A^4}$ and $\boldsymbol{A^{-1}}$ for each matrix.**

$(\boldsymbol{a})\ \boldsymbol{A} = \begin{bmatrix} \mathbf{2} & \mathbf{3} \\ \mathbf{3} & \mathbf{-6} \end{bmatrix}$ $\qquad\qquad$ $(\boldsymbol{b})\ A = \begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{-1} \\ \mathbf{1} & \mathbf{0} & \mathbf{1} \\ \mathbf{4} & \mathbf{-4} & \mathbf{5} \end{bmatrix}$

Solution:

$(\boldsymbol{a})\ \boldsymbol{A} = \begin{bmatrix} \mathbf{2} & \mathbf{3} \\ \mathbf{3} & \mathbf{-6} \end{bmatrix}$

Characteristic polynomial: $\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 3 \\ 3 & -6-\lambda \end{vmatrix} = \lambda^2 + 4\lambda - 21$

Characteristic equation: $\quad \lambda^2 + 4\lambda - 21 = 0$

To check the cayley − Hamilton validation substitute in the following function

$A^2 + 4A - 21I$

$$A^2 + 4A - 21I = \begin{bmatrix} 13 & -12 \\ -12 & 45 \end{bmatrix} + 4\begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix} - 21\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

So it satisfies the equation $\boldsymbol{A^2 + 4A - 21I = 0}$

To get A^4:-

$A^2 = -4A + 21I$

For $A^3 \rightarrow AA^2 = A(-4A + 21I)$

$$\rightarrow A^3 = -4A^2 + 21A = -4(-4A + 21I) + 21A = 37A - 84I$$

For $A^4 \rightarrow AA^3 = A(37A - 84I)$

$$\rightarrow A^4 = 37A^2 - 84A = 37(-4A + 21I) - 84A = -232A + 777I$$

$$A^4 = -232\begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix} + 777\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 312 & -696 \\ -696 & 2169 \end{bmatrix}$$

To get A^{-1}:-

$I = \frac{1}{21}(A^2 + 4A)$

For $A^{-1} \rightarrow A^{-1}I = A^{-1}\frac{1}{21}(A^2 + 4A) \rightarrow A^{-1} = \frac{1}{21}(A + 4I)$

$$A^{-1} = \frac{1}{21}\left(\begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix} + 4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = \frac{1}{21}\begin{bmatrix} 6 & 3 \\ 3 & -2 \end{bmatrix}$$

$$(\boldsymbol{b})\ A = \begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{-1} \\ \mathbf{1} & \mathbf{0} & \mathbf{1} \\ \mathbf{4} & \mathbf{-4} & \mathbf{5} \end{bmatrix}$$

Characteristic polynomial:

$$\det(A-\lambda I)=\begin{vmatrix}1-\lambda & 2 & -1\\ 1 & -\lambda & 1\\ 4 & -4 & 5-\lambda\end{vmatrix}=(1-\lambda)\begin{vmatrix}-\lambda & 1\\ -4 & 5-\lambda\end{vmatrix}-2\begin{vmatrix}1 & 1\\ 4 & 5-\lambda\end{vmatrix}-1\begin{vmatrix}1 & -\lambda\\ 4 & -4\end{vmatrix}$$

$$=\lambda^3-6\lambda^2+11\lambda-6$$

Characteristic equation:

$\lambda^3-6\lambda^2+11\lambda-6=0$

$To\ check\ the\ cayley-Hamilton\ validation\ substitute\ in\ the\ following\ function$

$A^3-6A^2+11A-6I$

$$A^3-6A^2+11A-6I$$

$$\begin{bmatrix}-11 & 14 & -13\\ 19 & -6 & 13\\ 76 & -28 & 53\end{bmatrix}-6\begin{bmatrix}-1 & 6 & -4\\ 5 & -2 & 4\\ 20 & -12 & 17\end{bmatrix}+11\begin{bmatrix}1 & 2 & -1\\ 1 & 0 & 1\\ 4 & -4 & 5\end{bmatrix}-6\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

So it satisfies the equation $\boldsymbol{A^3-6A^2+11A-6I=0}$

To get A^4:-

$$\boxed{A^3=6A^2-11A+6I}$$

For A^4 $\rightarrow AA^3=A(6A^2-11A+6I)$

$\rightarrow A^4=6A^3-11A^2+6A=6(6A^2-11A+6I)-11A^2+6A$

$=25A^2-60A+36I$

$$A^4=25A^2-60A+36I$$

$$A^4=25\begin{bmatrix}-1 & 6 & -4\\ 5 & -2 & 4\\ 20 & -12 & 17\end{bmatrix}-60\begin{bmatrix}1 & 2 & -1\\ 1 & 0 & 1\\ 4 & -4 & 5\end{bmatrix}+36\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}=\begin{bmatrix}-49 & 30 & -40\\ 65 & -14 & 40\\ 26 & -60 & 161\end{bmatrix}$$

To get A^{-1}:-

$$\boxed{I = \frac{1}{6}(A^3 - 6A^2 + 11A)}$$

For A^{-1} $\rightarrow A^{-1}I = A^{-1}\frac{1}{6}(A^3 - 6A^2 + 11A) \rightarrow A^{-1} = \frac{1}{6}(A^2 - 6A^1 + 11I)$

$$A^{-1} = \frac{1}{6}(A^2 - 6A^1 + 11I)$$

$$A^{-1} = \frac{1}{6}\left(\begin{bmatrix} -1 & 6 & -4 \\ 5 & -2 & 4 \\ 20 & -12 & 17 \end{bmatrix} - 6\begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{bmatrix} + 11\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} \frac{2}{3} & -1 & \frac{1}{3} \\ \frac{-1}{6} & \frac{3}{2} & \frac{-1}{3} \\ \frac{-2}{3} & 2 & \frac{-1}{3} \end{bmatrix}$$

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3. Use Cayley-Hamilton theorem, to compute the indicated power of the matrix.

$$\begin{bmatrix} \mathbf{-4} & \mathbf{6} \\ \mathbf{-3} & \mathbf{5} \end{bmatrix}^{9}, \quad \begin{bmatrix} \mathbf{0} & \mathbf{3} \\ \mathbf{1} & \mathbf{2} \end{bmatrix}^{k}, \quad \begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{0} & \mathbf{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{-1} \end{bmatrix}^{2002}, \quad \begin{bmatrix} \mathbf{2} & \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} & \mathbf{2} \end{bmatrix}^{-5}$$

Note:
Cayley-Hamilton theorem: Every square matrix satisfies its characteristic equation.

Solution:

$\boldsymbol{A} = \begin{bmatrix} \mathbf{-4} & \mathbf{6} \\ \mathbf{-3} & \mathbf{5} \end{bmatrix}$ **find $\boldsymbol{A^9}$**

Characteristic polynomial:

$$\det(A - \lambda I) = \begin{vmatrix} -4-\lambda & 6 \\ -3 & 5-\lambda \end{vmatrix} = (5-\lambda)(-4-\lambda) + 18 = \lambda^2 - \lambda - 2$$

Characteristic equation:

$\lambda^2 - \lambda - 2 = 0 \quad \Rightarrow \; A^2 - A - 2I = 0$

$A^2 - A - 2I = 0 \;\; \rightarrow \; A^2 = A + 2I$

For $A^3 \rightarrow AA^2 = A(A + 2I) \; \rightarrow A^3 = A^2 + 2A = A + 2I + 2A = 3A + 2I$

For $A^4 \rightarrow AA^3 = A(3A + 2I) \; \rightarrow A^4 = 3A^2 + 2A = 3(A + 2I) + 2A = 5A + 6I$

For $A^5 \rightarrow AA^4 = A(5A + 6I) \rightarrow A^5 = 5A^2 + 6A = 5(A + 2I) + 6A = 11A + 10I$

For $A^6 \rightarrow AA^5 = A(11A + 10I) \rightarrow A^6 = 11A^2 + 10A = 11(A + 2I) + 10A = 21A + 22I$

For $A^7 \rightarrow AA^6 = A(21A + 22I) \rightarrow A^7 = 21A^2 + 22A = 21(A + 2I) + 22A = 43A + 42I$

For $A^8 \rightarrow AA^7 = A(43A + 42I) \rightarrow A^8 = 43A^2 + 42A = 43(A + 2I) + 42A = 85A + 86I$

For $A^9 \rightarrow AA^8 = A(85A + 86I) \rightarrow A^9 = 85A^2 + 86A = 85(A + 2I) + 86A = 171A + 170I$

$A^9 = 171A + 170I$

$$A^9 = 171\begin{bmatrix} -4 & 6 \\ -3 & 5 \end{bmatrix} + 170\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -514 & 1026 \\ -513 & 1025 \end{bmatrix}$$

Another solution

Characteristic polynomial:

$$\det(A - \lambda I) = \begin{vmatrix} -4-\lambda & 6 \\ -3 & 5-\lambda \end{vmatrix} = (5-\lambda)(-4-\lambda) + 18 = \lambda^2 - \lambda - 2$$

Characteristic equation:

$\lambda^2 - \lambda - 2 = 0 \quad \Rightarrow \; A^2 - A - 2I = 0$

$A^2 - A - 2I = 0 \;\rightarrow\; A^2 = A + 2I \;\rightarrow\; A^m = k_1 A + k_0 I \;\rightarrow\; \lambda^m = k_1\lambda + k_0$

Since

$$m = 9 \;\; \& \;\; \lambda_1 = -1\,, \lambda_2 = 2$$

For $\lambda_1 = -1 \;\rightarrow\; -1^9 = -k_1 + k_0 \qquad \rightarrow\; -1 = -k_1 + k_0 \quad \Rrightarrow eq1$

For $\lambda_1 = 2 \;\rightarrow\; 2^9 = 2k_1 + k_0 \qquad \rightarrow\; 512 = 2k_1 + k_0 \quad \Rrightarrow eq2$

By solving $eq1$ & $eq2$

$$\boldsymbol{k_1 = 171} \;\&\; \boldsymbol{k_0 = 170}$$

So

$$A^m = k_1 A + k_0 I \;\longrightarrow\; A^9 = 171A + 170I$$

$$A^9 = 171\begin{bmatrix}-4 & 6\\-3 & 5\end{bmatrix} + 170\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} = \begin{bmatrix}-514 & 1026\\-513 & 1025\end{bmatrix}$$

$\boldsymbol{A = \begin{bmatrix}0 & 3\\1 & 2\end{bmatrix}}$ **find $\boldsymbol{A^k}$**

Characteristic polynomial:

$$\det(A-\lambda I) = \begin{vmatrix}-\lambda & 3\\1 & 2-\lambda\end{vmatrix} = -\lambda(2-\lambda) - 3 = \lambda^2 - 2\lambda - 3$$

Characteristic equation:

$$\lambda^2 - 2\lambda - 3 = 0 \quad \Rightarrow \; A^2 - 2A - 3I = 0$$

$$A^2 - 2A - 3I = 0 \;\rightarrow\; A^2 = 2A + 3I \;\rightarrow\; A^m = k_1A + k_0I \;\rightarrow\; \lambda^m = k_1\lambda + k_0$$

Since

$$m = k \;\;\&\; \lambda_1 = -1\,, \lambda_2 = 3$$

For $\lambda_1 = -1 \qquad \rightarrow \; (-1)^k = -k_1 + k_0 \quad \Rrightarrow eq1$

For $\lambda_1 = 3 \qquad \rightarrow \; 3^k = 3k_1 + k_0 \quad \Rrightarrow eq2$

By solving $eq1$ & $eq2$

$$\boldsymbol{k_1 = \frac{(-1)^k - 3^k}{-4} \;\&\; k_0 = \frac{3(-1)^k + 3^k}{4}}$$

$\boldsymbol{A = \begin{bmatrix}1 & 1 & 1\\0 & -1 & 0\\0 & 0 & -1\end{bmatrix}}$ **find $\boldsymbol{A^{2002}}$**

Characteristic polynomial:

$$\det(A-\lambda I) = \begin{vmatrix}1-\lambda & 1 & 1\\0 & -1-\lambda & 0\\0 & 0 & -1-\lambda\end{vmatrix} = (-1-\lambda)^2(1-\lambda) = \lambda^3 + \lambda^2 - \lambda - 1$$

Characteristic equation:

$$\lambda^3 + \lambda^2 - \lambda - 1 = 0 \quad \Rightarrow \; A^3 + A^2 - A - I = 0$$

$$A^3 + A^2 - A - I = 0 \;\rightarrow\; A^3 = -A^2 + A + I$$

$$\boldsymbol{A^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I}$$

$$\boldsymbol{A^3 = -\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = A}$$

For A^4 $\rightarrow AA^3 = A(-A^2 + A + I) \rightarrow A^4 = -A^3 + A^2 + A = -A + A^2 + A = A^2$

We can notice that

$$\boldsymbol{A^{even} = I} \ \& \ \boldsymbol{A^{odd} = A}$$

So $\boldsymbol{A^{2002} = I}$

$\boldsymbol{A = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix}}$ **find** $\boldsymbol{A^{-5}}$

<u>Characteristic polynomial:</u>

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 0 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 0 & 2-\lambda \end{vmatrix} = (2-\lambda)\begin{vmatrix} 1-\lambda & 1 \\ 0 & 2-\lambda \end{vmatrix} + 1\begin{vmatrix} 1 & 1-\lambda \\ 1 & 0 \end{vmatrix}$$

$$= \lambda^3 - 5\lambda^2 + 7\lambda - 3$$

<u>Characteristic equation:</u>

$\lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0 \quad \Rightarrow \quad A^3 - 5A^2 + 7A - 3I = 0$

$A^3 - 5A^2 + 7A - 3I = 0 \quad \rightarrow \quad I = \frac{1}{3}(A^3 - 5A^2 + 7A)$

For A^{-1} $\rightarrow A^{-1}I = A^{-1}\frac{1}{3}(A^3 - 5A^2 + 7A) = \frac{1}{3}(A^{-1}A^3 - 5A^{-1}A^2 + 7A^{-1}A) = \frac{1}{3}(A^2 - 5A + 7I)$

$$A^{-1} = \frac{1}{3}(A^2 - 5A + 7I)$$

For $A^{-2} \rightarrow A^{-1}A^{-1} = \frac{1}{3}A^{-1}(A^2 - 5A + 7I) \rightarrow A^{-2} = \frac{1}{3}(A - 5I + 7A^{-1}) = \frac{1}{3}(A - 5I + \frac{7}{3}((A^2 - 5A + 7I)))$

$$A^{-2} = \frac{7}{9}A^2 - \frac{11}{9}A + \frac{34}{9}I$$

For $A^{-3} \quad \rightarrow A^{-1}A^{-2} = A^{-1}\left(\frac{7}{9}A^2 - \frac{11}{9}A + \frac{34}{9}I\right) \rightarrow A^{-3} = \frac{7}{9}A - \frac{11}{9}I + \frac{34}{9}A^{-1}$

$$= \frac{7}{9}A - \frac{11}{9}I + \frac{34}{27}\ (A^2 - 5A + 7I)$$

$$A^{-3} = \frac{34}{27}A^2 - \frac{149}{27}A + \frac{205}{27}I$$

For $A^{-4} \quad \rightarrow A^{-1}A^{-3} = A^{-1}\left(\frac{34}{27}A^2 - \frac{149}{27}A + \frac{205}{27}I\right) \rightarrow A^{-4} = \frac{34}{27}A - \frac{149}{27}I + \frac{205}{27}A^{-1}$

$$= \frac{34}{27}A - \frac{149}{27}I + \frac{205}{81}(A^2 - 5A + 7I)$$

$$A^{-4} = \frac{205}{81}A^2 - \frac{923}{81}A + \frac{988}{81}I$$

For $A^{-5} \quad \rightarrow A^{-1}A^{-4} = A^{-1}\left(\frac{205}{81}A^2 - \frac{923}{81}A + \frac{988}{81}I\right) \rightarrow A^{-5} = \frac{205}{81}A - \frac{923}{81}I + \frac{988}{81}A^{-1}$

$$= \frac{205}{81}A - \frac{923}{81}I + \frac{988}{81}(A^2 - 5A + 7I)$$

$$A^{-5} = \frac{988}{81}A^2 - \frac{4735}{81}A + \frac{5993}{81}I$$

$$A^2 = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix}\begin{bmatrix} 2 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 4 \\ 4 & 1 & 4 \\ 4 & 0 & 5 \end{bmatrix}$$

$$A^{-5} = \frac{988}{81}\begin{bmatrix} 5 & 0 & 4 \\ 4 & 1 & 4 \\ 4 & 0 & 5 \end{bmatrix} - \frac{4735}{81}\begin{bmatrix} 2 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix} + \frac{5993}{81}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1463}{81} & 0 & \frac{-29}{3} \\ \frac{-29}{3} & \frac{2246}{81} & \frac{-29}{3} \\ \frac{-29}{3} & 0 & \frac{1463}{81} \end{bmatrix}$$