Work Sheet # 4 Solution

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1. **Find a vector $\vec{u}$ that has the same direction as $\vec{v} = (1, 1, 1)$**

Solution:

Given that $\vec{u}$ has the same direction as $\vec{v} = (1,1,1)$, $\vec{u}$ will be in the form

$$\vec{u} = t\vec{v} = \boxed{(\boldsymbol{t}, \boldsymbol{t}, \boldsymbol{t})} \qquad , such\ that\ t > 0.$$

where t is a parameter denoting the ratio between the lengths of $\vec{u}$ and $\vec{v}$.

Note that: To determine the value of *t,* we need the length of $\vec{u}$ which is not known (because the end point is not specified). So the value of *t,* in our problem, is any number of your choice.

Let $t = 2$ then $\vec{u} = (2,2,2)$

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2. **Given the vectors $\vec{u} = (-3, 1, 2)$, $\vec{v} = (4, 0, -8)$ and $\vec{w} = (6, -1, -4)$. Find the vector $\vec{r}$ satisfying $2\vec{u} - \vec{v} + 2\vec{w} = 7\vec{r} + \vec{w}$**

Solution:

$$2\vec{u} - \vec{v} + 2\vec{w} = 7\vec{r} + \vec{w} \ \Rightarrow \vec{r} = \frac{1}{7}(2\vec{u} - \vec{v} + \vec{w})$$

$$\therefore\ \vec{r} = \frac{1}{7}(2(-3,1,2) - (4,0,-8) + (6,-1,-4)) = \boxed{\left(-\frac{\mathbf{4}}{\mathbf{7}}, \frac{\mathbf{1}}{\mathbf{7}}, \frac{\mathbf{8}}{\mathbf{7}}\right).}$$

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3. **Given the vectors $\vec{u} = (-2, 1, -1)$, *and* $\vec{v} = (0, 1, 1)$ and $\vec{w} = (1, -5, -4)$, find**

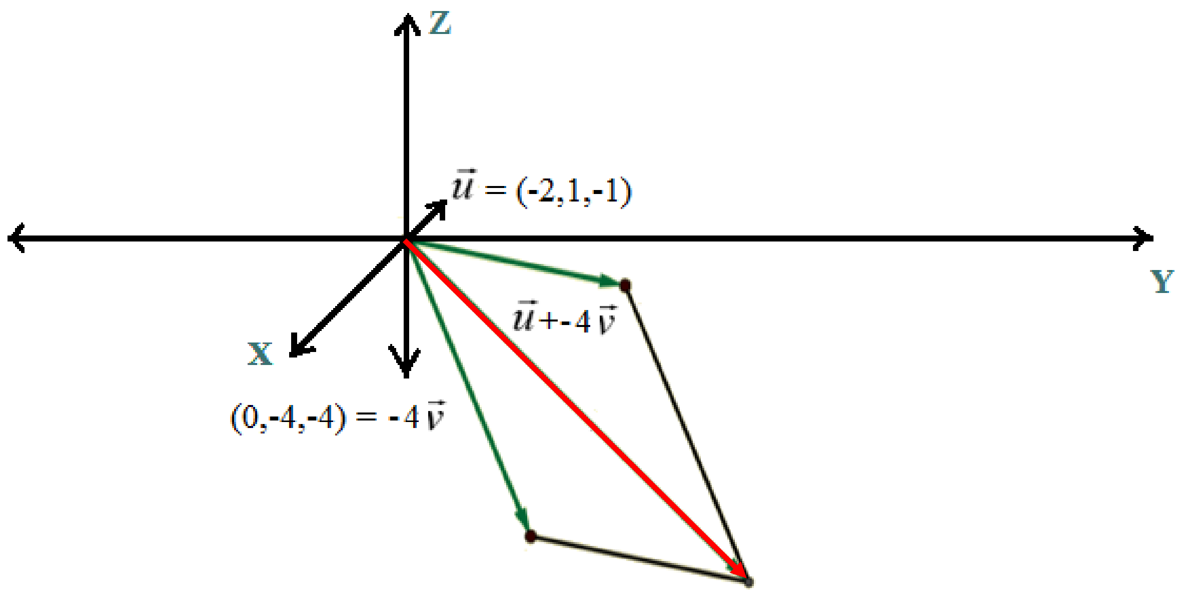
   (a) $\vec{u} - 4\vec{v}$ (represent it geometrically using the parallelogram and the triangle methods)

   Solution:

Construct $\vec{u} - 4\vec{v}$ by first draw the vector $\vec{v}$ pointing in the direction opposite to $\vec{u}$ (***negative of*** $\vec{v}$)and four times as long.

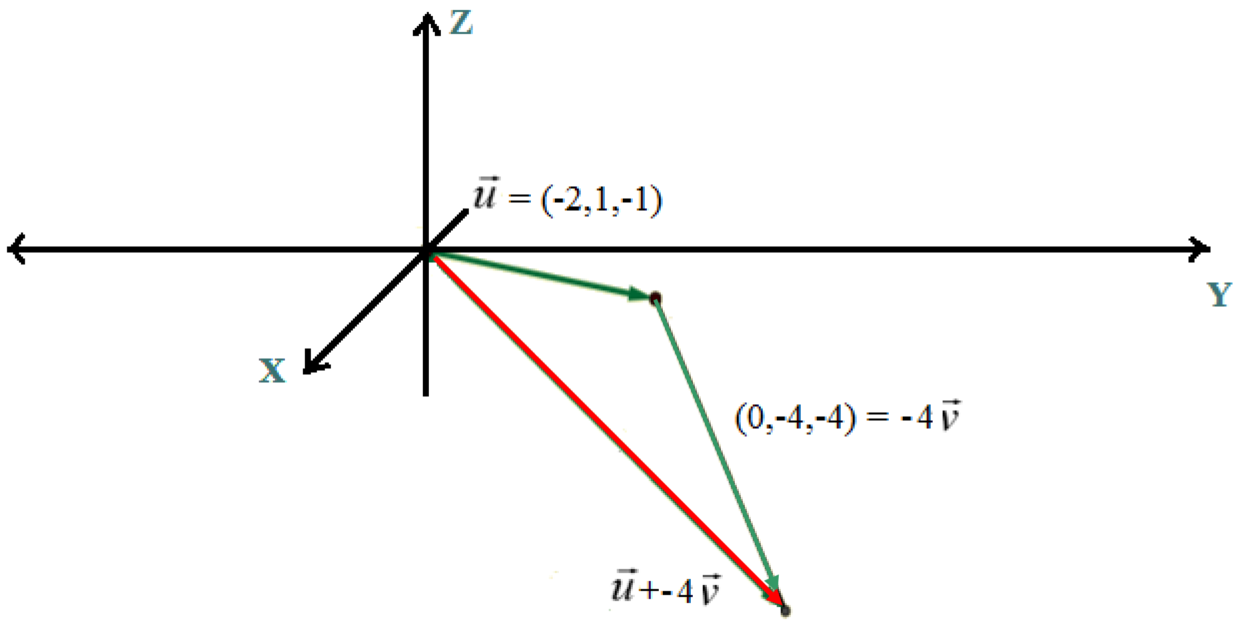
The parallelogram method:

1. Place both vectors $\vec{u}$ and $-4\vec{v}$ at the same initial point.
2. Complete the parallelogram.
3. The diagonal of the parallelogram is the resultant vector $\vec{u} + (-4\vec{v})$.



The triangle method (Head-to-Tail Method):

1. Place the vectors with the head of the $\vec{u}$ connected to the tail of $-4\vec{v}$
2. The resultant $\vec{u}+(-4\vec{v})$ vector is formed by connecting the tail of the first vector to the head of the last vector.



$(\boldsymbol{b})$ **A unit vector in the direction of $2\vec{v}-\vec{u}$**

Solution:

Computing the resultant vector,

$2\vec{v}-\vec{u} = 2(0,1,1)-(-2,1,-1) = (2,1,3).$

check it's magnitude $\rightarrow |2\vec{v}-\vec{u}| = \sqrt{4+1+9} = \sqrt{14} \neq 1$

So, the unit resultant vector will be $\dfrac{2\vec{v}-\vec{u}}{|2\vec{v}-\vec{u}|} = \dfrac{(2,1,3)}{\sqrt{4+1+9}} = \boxed{\left(\dfrac{2}{\sqrt{14}},\dfrac{1}{\sqrt{14}},\dfrac{3}{\sqrt{14}}\right)}$

And it act as the unit vector in the direction of $2\vec{v}-\vec{u}$.

$(\boldsymbol{c})$ **The dot product of $\vec{u}-4\vec{v}$ and of $2\vec{v}-\vec{u}$**

Solution:

$(\vec{u}-4\vec{v}).(2\vec{v}-\vec{u}) = \vec{u}.2\vec{v}-|\vec{u}|^2-8|\vec{v}|^2+4\vec{v}.\vec{u} = 2\,\vec{u}.\vec{v}-|\vec{u}|^2-8|\vec{v}|^2+4\,\vec{v}.\vec{u}$

$\xRightarrow{\because\ \vec{u}.\vec{v}=\vec{v}.\vec{u}\ \&\ \vec{u}.\vec{u}=||\vec{u}||^2} 6\,\vec{u}.\vec{v}-|\vec{u}|^2-8|\vec{v}|^2 = 6\big((-2)(0)+(1)(1)+(-1)(1)\big)-\left(\sqrt{4+1+1}\right)^2-8\left(\sqrt{0+1+1}\right)^2$

$= \boxed{-22}.$

Another method

$\vec{u} - 4\vec{v} = (-2,1,-1) - 4\,(0,1,1) = (-2,-3,-5)$.

$2\vec{v} - \vec{u} = 2\,(0,1,1) - (-2,1,-1) = (2,1,3)$.

The dot product ,

$(\vec{u} - 4\vec{v}) \,.\, (2\vec{v} - \vec{u}) = (-2,-3,-5).(2,1,3) = (-2)(2) + (-3)(1) + (-5)(3) = \boxed{-22\,.}$

(d) $(2\vec{u} + \vec{v}).\vec{w}$

Solution:

$(2\vec{u} + \vec{v})\,.\,\vec{w} \xRightarrow{distributing\ the\ \vec{w}} 2\vec{u}.\vec{w} + \vec{v}.\vec{w}$

$= 2\,[(-2)(1) + (1)(-5) + (-1)(-4)] \;+\; [(0)(1) + (1)(-5) + (1)(-4)]$

$= \boxed{-7\,.}$

Another method

$2\vec{u} + \vec{v} = 2(-2,1,-1) + (0,1,1) = (4,3,-1)$.

Then, $(2\vec{u} + \vec{v})\,.\,\vec{w} = (4,3,-1)\,.\,(1,-5,-4) = 4 - 15 + 4 = \boxed{-7.}$

(e) The cross product of $\vec{u} - 4\vec{v}$ and $2\vec{v} - \vec{u}$

Solution:

$$(\vec{u} - 4\vec{v}) \times (2\vec{v} - \vec{u}) = \vec{u} \times 2\vec{v} - \vec{u} \times \vec{u} - 8(\vec{v} \times \vec{v}) + 4(\vec{v} \times \vec{u})$$

$$\because \quad \vec{u} \times \vec{u} = \vec{v} \times \vec{v} = 0 \quad \& \quad \vec{u} \times \vec{v} = -\,\vec{v} \times \vec{u}$$

Therefore,

$$(\vec{u} - 4\vec{v}) \times (2\vec{v} - \vec{u}) = -\,2\,\vec{v} \times \vec{u} + 4(\vec{v} \times \vec{u}) = 2\,\vec{v} \times \vec{u} = 2\begin{vmatrix} i & j & k \\ 0 & 1 & 1 \\ -2 & 1 & -1 \end{vmatrix}$$

$$= 2(-2\,,-2\,,2) = \boxed{(-4,-4,4)\,.}$$

Another method

$\vec{u} - 4\vec{v} = < -2,1,-1 > -4 < 0,1,1 > = < -2,-3,-5 >$.

$2\vec{v} - \vec{u} = 2 < 0,1,1 > - < -2,1,-1 > = < 2,1,3 >$.

Then, the cross product $(\vec{u} - 4\vec{v}) \times (2\vec{v} - \vec{u}) = \begin{vmatrix} i & j & k \\ -2 & -3 & -5 \\ 2 & 1 & 3 \end{vmatrix}$

$$= (-3 * 3 - (-5 * 1)\,, -[(-2 * 3) - (-5 * 2)]\,, -2 * 1 - (-3 * 2)) = \boxed{(-4,-4,4).}$$

(f) A vector which is orthogonal to both $2\vec{u}+\vec{v}$ and $\vec{v}-2\vec{w}$

Solution:

$2\vec{u}+\vec{v} = 2(-2\,,1\,,-1)+(0\,,1\,,1) \;= (-4\,,3\,,-1)$.

$\vec{v}-2\vec{w} = (0,1\,,1)-2(1\,,-5\,,-4) \;= (-2\,,11\,,9)$.

Then, the vector orthogonal to both $2\vec{u}+\vec{v}$ and $\vec{v}-2\vec{w}$ will be

$$\vec{n} = (2\vec{u}+\vec{v})\times(\vec{v}-2\vec{w}) = \begin{vmatrix} i & j & k \\ -4 & 3 & -1 \\ -2 & 11 & 9 \end{vmatrix} = i[27 - - 11] - j[-36-2] + k[-44 - - 6]$$

$$\therefore\ \vec{n} = \boxed{(\mathbf{38,38,-38}).}$$

(g) The angle between $\vec{u}$ *and* $\vec{v}-2\vec{w}$

Solution:

The angle between $\vec{u}=(-2,1,-1)$ and $\vec{v}-2\vec{w} \;= (-2,11,9)$ *is calculated using*

$$cos\theta = \frac{\vec{u}.(\vec{v}-2\vec{w})}{||\vec{u}||\;||\vec{v}-2\vec{w}||} = \frac{(-2\,,1\,,-1)\,.\,(-2\,,11\,,\;9)}{\sqrt{4+1+1}\,\sqrt{4+121+81}} = \frac{3}{\sqrt{309}}$$

$$\therefore \theta = \cos^{-1}\frac{\mathbf{3}}{\sqrt{309}} \approx 1.4 \qquad (or\,\boxed{\mathbf{80°})\,.}$$

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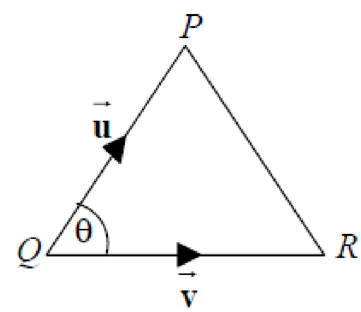
4. **Let the points P=(3,-2,-1), Q=(1,5,4), R=(2,0,-6) be the vertices of a triangle. Find**

(a) The angle $P\hat{Q}R$

Solution:



From the sketch, one can say that

$\vec{u} = \overrightarrow{QP} = P - Q = (2, -7, -5)$

$\vec{v} = \overrightarrow{QR} = R - Q = (1, -5, -10)$

So the required angle will be calculated using

$$cos\theta = \frac{\vec{u}.\vec{v}}{||\vec{u}||\ ||\vec{v}||} = \frac{(2,-7,-5).(1,-5,-10)}{\sqrt{4+49+25}\sqrt{1+25+100}} = \frac{87}{\sqrt{9829}}$$

$$\therefore \theta = \cos^{-1}\frac{87}{\sqrt{9829}} \approx 0.5 \qquad (or\ \boxed{\mathbf{28.654°}}).$$

(b) The Scalar projection of 2 $\overrightarrow{QR}$ onto $\overrightarrow{QP}$

Solution:

$$|\mathrm{Proj}_{\vec{u}}\vec{v}| = \left|2\vec{v}.\frac{\vec{u}}{||\vec{u}||}\right| = \left|(2,-10,-20).\frac{(2,-7,-5)}{\sqrt{4+49+25}}\right| = \boxed{\frac{\mathbf{174}}{\sqrt{\mathbf{78}}}\ unit.}$$

(c) The area of the triangle

Solution:

$$\vec{u} \times \vec{v} = \begin{vmatrix} i & j & k \\ 2 & -7 & -5 \\ 1 & -5 & -10 \end{vmatrix} = (45, 15, -3)$$

$$\therefore \text{area of triangle} = \frac{||\vec{u}\times\vec{v}||}{\mathbf{2}} = \boxed{\frac{\mathbf{3\sqrt{251}}}{\mathbf{2}}\ unit.}$$

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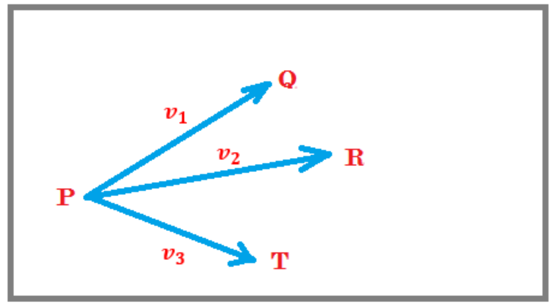
5. **Show that the points P = (3, -3, 2), Q = (1, 0, 1), R = (1, 1, 0) , T = (0 , 1 , 1)are located in the same plane and find a perpendicular vector to such plane.**



Solution:

the three vectors are:

$$v_1 = \overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = -2i + 3j - k$$
$$v_2 = \overrightarrow{PR} = -2i + 4j - 2k$$
$$v_3 = \overrightarrow{PT} = -3j + 4j - k$$

In order to show that the 4 points are coplanar we have to check the scalar triple product:

$$\mathbf{v_1 \cdot (v_2 \times v_3)}$$

(i.e. which is equivalent to $v_2 \cdot (v_3 \times v_1) = v_3 \cdot (v_1 \times v_2)$; refer to lecture 13 slide 3)

$$\vec{v_2} \times \vec{v_3} = \begin{vmatrix} i & j & k \\ -2 & 4 & -2 \\ -3 & 4 & -1 \end{vmatrix} = 4i + 4j + 4k = i + j + k\,.$$

And the scalar triple product is equal to:

$$v_1 \cdot (v_2 \times v_3) = (-2i + 3j - k) \cdot (i + j + k)$$
$$= -2 + 3 - 1 = \boxed{0.}$$

Or calculate the scalar triple product directly;

$$\vec{v_1} \,.\, (\vec{v_2} \times \vec{v_3}) = \begin{vmatrix} -2 & 3 & -1 \\ -2 & 4 & -2 \\ -3 & 4 & -1 \end{vmatrix} = \vec{v_1} \times (\vec{v_2} \,.\, \vec{v_3}) = \boxed{0.}$$

The resulting 0 indicates that the points are coplanar and that no volume can be determined between them.

✓ For the perpendicular vector to such plane,

$\overrightarrow{PQ} = Q - P = (-2\,,3\,,-1)$ ***and*** $\overrightarrow{PR} = R - P = (-2\,,4\,,-2)$

$$\text{The normal vector will be } \vec{n} = \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} i & j & k \\ -2 & 3 & -1 \\ -2 & 4 & -2 \end{vmatrix} = \boxed{(\mathbf{-2, -2, -2}).}$$

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6. Prove the Pythagoras' theorem:

"For all vectors **u** and **v** in $\mathbb{R}^3$, $||\boldsymbol{u} + \boldsymbol{v}||^2 = ||\boldsymbol{u}||^2 + ||\boldsymbol{v}||^2$ if and only if **u** and **v** are orthogonal"

Solution:

The statement to prove: $||\boldsymbol{u} + \boldsymbol{v}||^2 = ||\boldsymbol{u}||^2 + ||\boldsymbol{v}||^2 \iff$ **u** and **v** are orthogonal

(i) Starting with the statement's right hand side, Assume $\boldsymbol{u}$ & $\boldsymbol{v}$ are perpendicular if and only if $\boldsymbol{u} \,.\, \boldsymbol{v} = \boldsymbol{v} \,.\, \boldsymbol{u} = 0$. Now,

$||\boldsymbol{u} + \boldsymbol{v}||^2 = (\boldsymbol{u} + \boldsymbol{v}).(\boldsymbol{u} + \boldsymbol{v})$ by definition

$= (\boldsymbol{u}.\boldsymbol{u}) + (\boldsymbol{u}.\boldsymbol{v}) + (\boldsymbol{v}.\boldsymbol{u}) + (\boldsymbol{v}.\boldsymbol{v})$ by distributivity

$= (\boldsymbol{u}.\boldsymbol{u}) + (\boldsymbol{v}.\boldsymbol{v}) + 2(\boldsymbol{u}\,.\boldsymbol{v})$ by symmetry

$= ||\boldsymbol{u}||^2 + ||\boldsymbol{v}||^2 + 2||\boldsymbol{u}||||\boldsymbol{v}|| \cos\theta$ by definition

$\textit{were } \theta = \frac{\pi}{2},$ *is the angle between* $\boldsymbol{u}$ *and* $\boldsymbol{v}$.

$= ||\boldsymbol{u}||^2 + ||\boldsymbol{v}||^2$

(ii) Starting with the statement's left hand side

Assume $||\boldsymbol{u} + \boldsymbol{v}||^2 = ||\boldsymbol{u}||^2 + ||\boldsymbol{v}||^2$

applying the definition, distributivity and symmetry properties

$$\rightarrow (\boldsymbol{u}.\boldsymbol{u}) + (\boldsymbol{u}.\boldsymbol{v}) + (\boldsymbol{v}.\boldsymbol{u}) + (\boldsymbol{v}.\boldsymbol{v}) = ||\boldsymbol{u}||^2 + ||\boldsymbol{v}||^2 + 2(\boldsymbol{u}\,.\boldsymbol{v}).$$

Since, $||\boldsymbol{u}||^2 + ||\boldsymbol{v}||^2 + 2(\boldsymbol{u}.\boldsymbol{v}) = ||\boldsymbol{u}||^2 + ||\boldsymbol{v}||^2 \rightarrow \boldsymbol{u}.\boldsymbol{v} = 0.$ Hence, **u** and **v** are orthogonal.

Therefore, $||\boldsymbol{u} + \boldsymbol{v}||^2 = ||\boldsymbol{u}||^2 + ||\boldsymbol{v}||^2$ if and only if **u** and **v** are orthogonal

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7. Consider a set of vectors in 2D {**e**1 , **e**2} given by $\boldsymbol{e_1} = \frac{1}{\sqrt{2}}(i - j)$ and $\boldsymbol{e_2} = \frac{1}{\sqrt{2}}(i + j)$

(a) Show that they form an orthonormal set.

(We say that they form another basis vectors for the 2D space i.e. any vector in 2D can be expanded in terms of **e**1 , **e**2)

Solution

i.e. Orthonormal = orthogonal + normalized

$$\boldsymbol{e_1}.\boldsymbol{e_2} = \frac{1}{\sqrt{2}}(\boldsymbol{i} - \boldsymbol{j}).\frac{1}{\sqrt{2}}(\boldsymbol{i} + \boldsymbol{j}) = \frac{1}{2} + \left(-\frac{1}{2}\right) = 0 \ \rightarrow orthogonal$$

$$||\boldsymbol{e_1}|| = \sqrt{\left(\frac{\mathbf{1}}{\sqrt{\mathbf{2}}}\right)^2 + \left(\frac{\mathbf{-1}}{\sqrt{\mathbf{2}}}\right)^2} = 1 \quad \text{and} \quad ||\boldsymbol{e_2}|| = \sqrt{\left(\frac{\mathbf{1}}{\sqrt{\mathbf{2}}}\right)^2 + \left(\frac{\mathbf{1}}{\sqrt{\mathbf{2}}}\right)^2} = 1 \rightarrow unit\ length\ "normalized"$$

(b) Find the components of the vector $\vec{u} = 4\boldsymbol{i} - 2\boldsymbol{j}$ with respect to the new set {**e**1 , **e**2}

Solution

Let $\vec{A} = (A_1\vec{e_1} + A_2\vec{e_2})$,

Then the first component is $\vec{A}.\vec{e_1} = A_1 \rightarrow \ (4\boldsymbol{i} - 2\boldsymbol{j}).\frac{1}{\sqrt{2}}(\boldsymbol{i} - \boldsymbol{j}) = \frac{4}{\sqrt{2}} + \frac{2}{\sqrt{2}} = \frac{6}{\sqrt{2}} = 3\sqrt{2}$

And the second component is $\vec{A}.\vec{e_2} = A_2 \rightarrow \ (4\boldsymbol{i} - 2\boldsymbol{j}).\frac{1}{\sqrt{2}}(\boldsymbol{i} + \boldsymbol{j}) = \frac{4}{\sqrt{2}} - \frac{2}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2}$

Thus, $\vec{A} = \boxed{(3\sqrt{2}\,\vec{\boldsymbol{e_1}} + \sqrt{2}\,\vec{\boldsymbol{e_2}})}$.

Another method

Let (a, b) satisfies $(4, -2) \ = \ a(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}) + \ b(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$

Equating components;

$$\left.\begin{aligned} \frac{1}{\sqrt{2}}a + \frac{1}{\sqrt{2}}b &= 4 \\ \frac{-1}{\sqrt{2}}a + \frac{1}{\sqrt{2}}b &= -2 \end{aligned}\right\} \qquad a = 3\sqrt{2} \ \text{ and } b = \sqrt{2}$$

Thus, $\vec{A} = \boxed{(3\sqrt{2}\,\vec{\boldsymbol{e_1}} + \sqrt{2}\,\vec{\boldsymbol{e_2}})}$.

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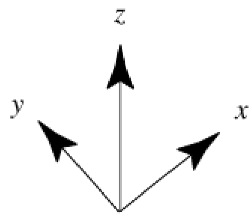
8. Show that the basic unit vectors $\{\mathbf{i}\,,\mathbf{j}\,,\mathbf{k}\}$ in 3D form right-handed system.



right-handed coordinate system

A three-dimensional coordinate system in which the axes satisfy the right-hand rule.

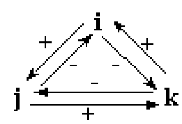
Solution

$\vec{\imath} \times \vec{\jmath} = \|\vec{\imath}\|\ \|\vec{\jmath}\|\ \sin\theta\ \hat{n} = \sqrt{1+0+0}\ \sqrt{0+1+0}\ \sin 90° \,(0\,,0\,,1)\ = \vec{k}\,.$

$\hat{n}$ is the unit vector, that is orthogonal (normal) to both $\vec{\boldsymbol{\imath}}$ and $\vec{\boldsymbol{\jmath}}$.

But there are two possible vectors that could be – (one on either side of the plane formed by the two vectors), so we choose $\hat{n}$ to be the one which makes $(\vec{\imath}\,,\ \vec{\jmath}\,,\ \vec{n})$ a right handed triad.

Similarly ;



$\vec{\boldsymbol{\jmath}} \times \vec{\boldsymbol{k}} = (\,1\,,0\,,0\,) = \vec{\boldsymbol{\imath}}$

$\vec{\boldsymbol{k}} \times \vec{\boldsymbol{\imath}} = (\,0\,,1\,,0\,)\ = \vec{\boldsymbol{\jmath}}$

and,

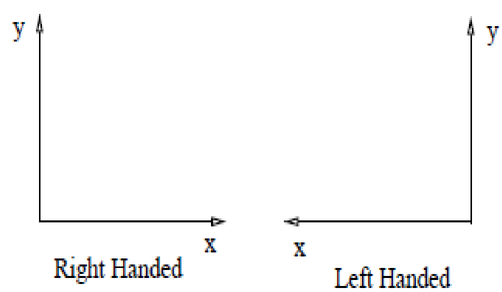
$\vec{\boldsymbol{\jmath}}\times\vec{\boldsymbol{\imath}} = -\vec{\boldsymbol{k}} \qquad \vec{\boldsymbol{\imath}}\times\vec{\boldsymbol{\imath}} = 0$

$\vec{\boldsymbol{k}}\times\vec{\boldsymbol{\jmath}} = -\vec{\boldsymbol{\imath}} \qquad \vec{\boldsymbol{\jmath}}\times\vec{\boldsymbol{\jmath}} = 0$

$\vec{\boldsymbol{\imath}}\times\vec{\boldsymbol{k}} = -\vec{\boldsymbol{\jmath}} \qquad \vec{\boldsymbol{k}}\times\vec{\boldsymbol{k}} = 0$

Refer to lecture 12 slide 23.

Remark: The same classification occurs in $\mathbb{R}^2$:



9. In 3D, show that the following sets of **ordered** vectors {$\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$} are orthonormal and form right handed systems

(a) $\boldsymbol{e_1} = \cos\theta\, \boldsymbol{i} + \sin\theta\, \boldsymbol{j}$, $\quad \boldsymbol{e_2} = -\sin\theta\, \boldsymbol{i} + \cos\theta\, \boldsymbol{j}$, $\quad \boldsymbol{e_3} = \boldsymbol{k}$; $\quad \boldsymbol{0 \le \theta \le 2\pi}$

Solution

$$\left.\begin{aligned}
\boldsymbol{e_1}.\boldsymbol{e_2} &= (\cos\theta\, \boldsymbol{i} + \sin\theta\, \boldsymbol{j}).(-\sin\theta\, \boldsymbol{i} + \cos\theta\, \boldsymbol{j}) = -\sin\theta\cos\theta + \sin\theta\cos\theta = 0\\
\boldsymbol{e_2}.\boldsymbol{e_3} &= (-\sin\theta\, \boldsymbol{i} + \cos\theta\, \boldsymbol{j}).(\boldsymbol{k}) = 0 + 0 + 0 = 0\\
\boldsymbol{e_1}.\boldsymbol{e_3} &= (\cos\theta\, \boldsymbol{i} + \sin\theta\, \boldsymbol{j}).(\boldsymbol{k}) = 0 + 0 + 0 = 0
\end{aligned}\right\} \text{Mutually orthogonal}$$

$$\left.\begin{aligned}
||\boldsymbol{e_1}|| &= \sqrt{(\cos\theta)^2 + (\sin\theta)^2 + (0)^2} = 1 \qquad ||\boldsymbol{e_2}|| = \sqrt{(-\sin\theta)^2 + (\cos\theta)^2 + (0)^2} = 1\\
||\boldsymbol{e_3}|| &= \sqrt{(0)^2 + (0)^2 + (1)^2} = 1
\end{aligned}\right\} \textbf{Normalized}$$

Therefore, **Orthonormal**.

Regarding the right handed system;

$$\boldsymbol{e_1} \times \boldsymbol{e_2} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \end{vmatrix} = (0, 0, \cos^2\theta - (-\sin^2\theta)) = \boldsymbol{k} = \boldsymbol{e_3}$$

It is sufficient for the orientation of the **ordered triplet vectors set** to be right-handed system that

$$\mathbf{e_1} \times \mathbf{e_2} = \mathbf{e_3}\,.$$

consequently;

$$\boldsymbol{e_2} \times \boldsymbol{e_3} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = (\cos\theta, \sin\theta, 0) = \boldsymbol{e_1}$$

$$\boldsymbol{e_3} \times \boldsymbol{e_1} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 0 & 0 & 1 \\ \cos\theta & \sin\theta & 0 \end{vmatrix} = (-\sin\theta, \cos\theta, 0) = \boldsymbol{e_2}$$

Therefore, the set of vectors form **right handed system**.

Another solution for showing the right handed system

The ordered triplet {e_1, e_2, e_3} of vectors in R^3 is called right-handed if $\det(\boldsymbol{e_1}, \boldsymbol{e_2}, \boldsymbol{e_3}) > 0$, where ($e_1$, e_2, e_3) is the square matrix formed by taking the three vectors as columns. (Note that the determinant is necessarily nonzero since it was shown the vectors to be orthogonal and nonzero.) *Moreover, similarly the triplet is left-handed if the determinant is negative.*
Some intuitive properties are easy to observe from this point of view: If you interchange any two vectors in the multiplet, the handedness is **reversed**. If you permute the vectors cyclically, handedness is **preserved**.

$$\begin{vmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1 > 0.$$

Therefore, The ordered triplet {e_1, e_2, e_3} of vectors in R^3 is called right-handed

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(b) $e_1 = \frac{\sqrt{3}}{4}\boldsymbol{i} + \frac{1}{4}\boldsymbol{j} + \frac{\sqrt{3}}{2}\boldsymbol{k}$, $\quad e_2 = \frac{3}{4}\boldsymbol{i} + \frac{\sqrt{3}}{4}\boldsymbol{j} - \frac{1}{2}\boldsymbol{k}$, $\quad e_3 = -\frac{1}{2}\boldsymbol{i} + \frac{\sqrt{3}}{2}\boldsymbol{j}$

Solution

$$\boldsymbol{e_1}.\boldsymbol{e_2} = \left(\frac{\sqrt{3}}{4}\boldsymbol{i} + \frac{1}{4}\boldsymbol{j} + \frac{\sqrt{3}}{2}\boldsymbol{k}\right).\left(\frac{3}{4}\boldsymbol{i} + \frac{\sqrt{3}}{4}\boldsymbol{j} - \frac{1}{2}\boldsymbol{k}\right) = \frac{3\sqrt{3}}{16} + \frac{\sqrt{3}}{16} - \frac{\sqrt{3}}{4} = 0$$

$$\boldsymbol{e_2}.\boldsymbol{e_3} = \left(\frac{3}{4}\boldsymbol{i} + \frac{\sqrt{3}}{4}\boldsymbol{j} - \frac{1}{2}\boldsymbol{k}\right).\left(-\frac{1}{2}\boldsymbol{i} + \frac{\sqrt{3}}{2}\boldsymbol{j}\right) = -\frac{3}{8} + \frac{3}{8} + 0 = 0$$

$$\boldsymbol{e_1}.\boldsymbol{e_3} = \left(\frac{\sqrt{3}}{4}\boldsymbol{i} + \frac{1}{4}\boldsymbol{j} + \frac{\sqrt{3}}{2}\boldsymbol{k}\right).\left(-\frac{1}{2}\boldsymbol{i} + \frac{\sqrt{3}}{2}\boldsymbol{j}\right) = \frac{-\sqrt{3}}{8} + \frac{\sqrt{3}}{8} + 0 = 0$$

Mutually orthogonal

$$||\boldsymbol{e_1}|| = \sqrt{\left(\frac{\sqrt{3}}{4}\right)^2 + \left(\frac{1}{4}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = 1 \qquad ||\boldsymbol{e_2}|| = \sqrt{\left(\frac{3}{4}\right)^2 + \left(\frac{\sqrt{3}}{4}\right)^2 + \left(-\frac{1}{2}\right)^2} = 1$$

$$||\boldsymbol{e_3}|| = \sqrt{\left(-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 + (0)^2} = 1$$

Normalized

Therefore, **Orthonormal**.

Regarding the right handed system;

$$e_1 \times e_2 = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\sqrt{3}}{4} & \frac{1}{4} & \frac{\sqrt{3}}{2} \\ \frac{3}{4} & \frac{\sqrt{3}}{4} & -\frac{1}{2} \end{vmatrix} = (-\frac{1}{2}, \frac{\sqrt{3}}{2}, 0) \ = e_3$$

It is suffuccient for the **ordered triplets of vectors** to be right − handed system that $\mathbf{e_1} \times \mathbf{e_2} = \mathbf{e_3}$.

consequently;

$$e_2 \times e_3 = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{3}{4} & \frac{\sqrt{3}}{4} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \end{vmatrix} = (\frac{\sqrt{3}}{4}, \frac{1}{4}, \frac{\sqrt{3}}{2}) \ = e_1$$

$$e_3 \times e_1 = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{4} & \frac{1}{4} & \frac{\sqrt{3}}{2} \end{vmatrix} = (\frac{3}{4}, \frac{\sqrt{3}}{4}, -\frac{1}{2}) \ = e_2$$

Therefore, the set of vectors form **right handed system**.

Another solution for showing the right handed system

The ordered triplet {$\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$} of vectors in R^3 is called right-handed if $\det(\boldsymbol{e_1}, \boldsymbol{e_2}, \boldsymbol{e_3}) > 0$, where ($\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$) is the square matrix formed by taking the three vectors as columns. (Note that the determinant is necessarily nonzero since it was shown the vectors to be orthogonal and nonzero.)

Moreover, similarly the triplet is left-handed if the determinant is negative.

Some intuitive properties are easy to observe from this point of view: If you interchange any two vectors in the multiplet, the handedness is **reversed**. If you permute the vectors cyclically, handedness is **preserved**.

$$\begin{vmatrix} \frac{\sqrt{3}}{4} & \frac{3}{4} & -\frac{1}{2} \\ \frac{1}{4} & \frac{\sqrt{3}}{4} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \end{vmatrix} \approx 1 > 0.$$

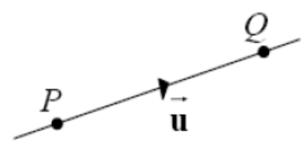
Therefore, The ordered triplet {$\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$} of vectors in R^3 is called right-handed

10. Find the parametric and symmetric equations of the line:

(a) **Passing through the two points $\boldsymbol{P(1,-2,4)}$ *and* $\boldsymbol{Q(5,-3,2)}$.**

Solution

The directional vector will be $\vec{u} = Q - P = (4,-1,-2)$



By choosing the P, the parametric equation of the line will be

$$\boxed{\boldsymbol{x = 1 + 4t\,,\ \ y = -2 - t\,,\quad z = 4 - 2t\,.}}$$

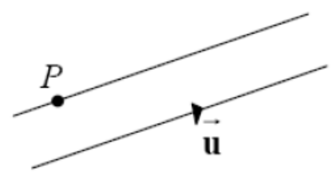
The line equation can be rearranged in the symmetric form

$$\boxed{\boldsymbol{\frac{x-1}{4} = \frac{y+2}{-1}\,,\frac{z-4}{-2} = t\,.}}$$

(b) **Passing through $\boldsymbol{P(1,1,3)}$ *and parallel to the line* :** $\frac{x-3}{2} = \frac{y}{-3} = z + 7$.

Solution

The directional vector of the given line is $\vec{u} = < 2,-3,1 >$



Which can be used as a directional vector for the required line (because the two lines are parallel). The parametric equation of the required line will be

$$\boxed{\boldsymbol{x = 1 + 2t\,,\qquad y = 1 - 3t\,,\qquad z = 3 + t\,.}}$$

The line equation can be rearranged in the symmetric form

$$\boxed{\boldsymbol{\frac{x-1}{2} = \frac{y-1}{-3}\,,\frac{z-3}{1} = t\,.}}$$

(c) Can you find the intersection of the line in part (b) with $\boldsymbol{xy, yz,}$ *and* $\boldsymbol{xz}$ *planes*.

Solution

$\underline{xy - plane\ means\ that\ z = 0.}$ Substitute this value in the line equation to get $t = -3$.

This means that the line intersect the xy-plane at the point $(-5, 10, 0)$.

$\underline{yz - plane\ means\ that\ x = 0.}$ Substitute this value in the line equation to get $t = -0.5$.

This means that the line intersect the yz-plane at the point $\left(0,\frac{5}{2},\frac{5}{2}\right)$.

$\underline{xz - plane\ means\ that\ y = 0.}$ Substitute this value in the line equation to get $t = \frac{1}{3}$.

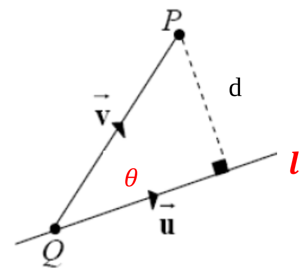
This means that the line intersect the yz-plane at the point $\left(\frac{5}{3}, 0, \frac{5}{2}\right)$.

---

11. Find the shortest distance from the point $P(2, 1, 3)$ to the line l: $x = 2 + 2t$, $y = 1 + 6t$, $z = 3t$. Then find the equation of the line through P and orthogonal to l.

Solution

To get the shortest distance (d)



From the line $\boldsymbol{l}$: $\boldsymbol{x = 2 + 2t}$, $\boldsymbol{y = 1 + 6t}$, $\boldsymbol{z = 3t}$, we can get a point Q(2,1,0) and a directional vector $\vec{u} = (2,6,3)$ on it.

From the sketch, we can get the vector

$\vec{v} = \overrightarrow{QP} = \overrightarrow{OP} - \overrightarrow{OQ} = (0,0,3)$

then,

$$\sin\theta = \frac{d}{|\overrightarrow{QP}|} = \frac{d}{|\vec{v}|} = \frac{|\vec{u} \times \vec{v}|}{|\vec{u}|\,|\vec{v}|},$$

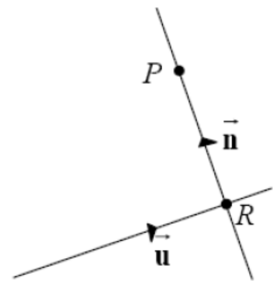
Therefore, $\boxed{d = \frac{|\vec{u} \times \vec{v}|}{|\vec{u}|}}$

and, $\vec{u} \times \vec{v} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 2 & 6 & 3 \\ 0 & 0 & 3 \end{vmatrix} = (18, -6, 0)$

$$d = \frac{|\vec{u} \times \vec{v}|}{|\vec{u}|} = \frac{\sqrt{324 + 36 + 0}}{\sqrt{4 + 36 + 9}} = \boxed{\frac{6\sqrt{10}}{7}}.$$

to get the line

Let R to be the point of intersection of the required and the given lines. Given that R is on the line $l: x = 2 + 2t\,, y = 1 + 6t\,, z = 3t,$ so its coordinates will be $R(2 + 2t\,, 1 + 6t\,, 3t)$.



$\vec{n} = \overrightarrow{OP} - \overrightarrow{OR} = (-2t, -6t, 3 - 3t)$

$$\because \vec{u} \perp \vec{n} \quad \Rightarrow \vec{u}\,.\vec{n} = 0 \ \Rightarrow (2,6,3).(-2t, -6t, 3 - 3t) = 0$$

$$\therefore -4t - 36t + 9 - 9t = 0 \quad \Rightarrow t = \frac{9}{49} \Rightarrow \vec{n} = \left(-\frac{18}{49}, -\frac{54}{49}, \frac{120}{49}\right).$$

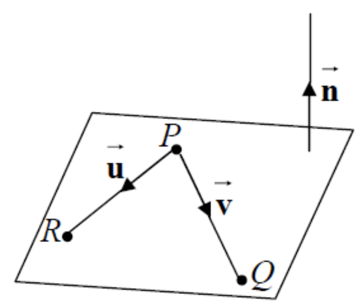
So the equation of the line passing through the point P whose directional vector is $\vec{n}$ and perpendicular to the line l will be $\boxed{\boldsymbol{x = 2 - \frac{18}{49}s\,, \quad y = 1 - \frac{54}{49}s\,, \quad z = 3 + \frac{120}{49}s}}$.

---

12. Find an equation of the plane passing through $\boldsymbol{P(2,1,1), Q(0,2,3)}$ *and* $\boldsymbol{R(1,0,1)}$.

Solution

From the sketch



$\vec{u} = \overrightarrow{PR} = \overrightarrow{OR} - \overrightarrow{OP} = < -1, -1, 0 > and\ \vec{v} = \overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = (-2,1,2)$

$$\therefore \vec{n} = \vec{u} \times \vec{v} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ -1 & -1 & 0 \\ -2 & 1 & 2 \end{vmatrix} = (-2,2,-3)$$

By choosing the point $\boldsymbol{P}(2,1,1)$ and using the normal $\vec{n}$ The equation of the plane will be

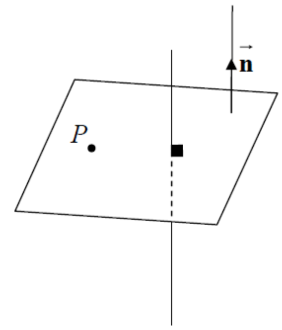
$$-2(x-2) + 2(y-1) - 3(z-1) = 0 \xrightarrow{\textbf{the general equation of a plane}} \boxed{\boldsymbol{-2x + 2y - 3z = -5}}$$

---

13. Find an equation of the plane passing through $P(7,5,3)$ *and perpendicular to the line* $x = 5 - 3t, y = t, z = -2t$.

Solution

Given that the line is perpendicular to the required plane. Hence, the normal vector of the plane will be the directional vector of the line



i.e. $\vec{n} = (-3, 1, -2)$

By using the point $P(7,5,3)$ and the normal $\vec{n}$, the equation of the plane will be

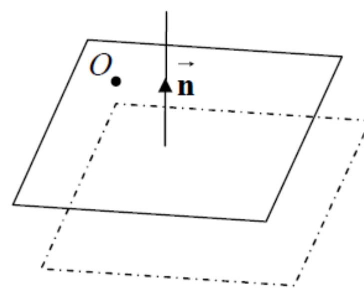
$$-3(x-7) + (y-5) - 2(z-3) = 0 \xrightarrow{\text{the general equation of a plane}} \boxed{-3x + y - 2z = -22}.$$

---

14. Find an equation of the plane through the origin that is parallel to the plane $2x - 5y + 3z = 9$.

Solution

The normal vector of the required plane will be the same as the one of the given plane so



$$\vec{n} = (2, -5, 3)$$

By using the point $O(0,0,0)$ and the normal $\vec{n}$, the equation of the plane will be

$$2(x-0) - 5(y-0) + 3(z-0) = 0 \xrightarrow{\text{the general equation of a plane}} \boxed{2x - 5y + 3z = 0}$$

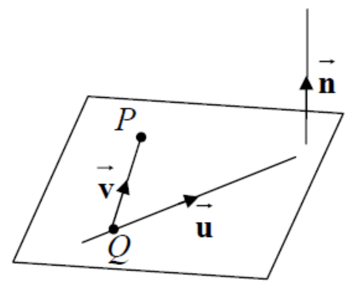
---

15. Find the equation of the plane containing the point $P(2, 0, 3)$ and the line

$$\boldsymbol{x = 1 + 2t, \quad y = 3 - 3t, \qquad z = t}$$

Solution

To find the plane equation one needs a ***point*** *and a* ***normal vector***
Taking the given point P and construct the normal using $\vec{u}$ and $\vec{v}$ from the sketch.



$$x = 1 + 2t, \quad y = 3 - 3t, \quad z = t \quad \Rightarrow \quad Q(1, 3, 0) \;\; \vec{u} = (2, -3, 1)$$

From the sketch

$$\vec{v} = \overrightarrow{QP} = \overrightarrow{OP} - \overrightarrow{OQ} = (1, -3, 3)$$

$$\therefore \vec{n} = \vec{u} \times \vec{v} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 2 & -3 & 1 \\ 1 & -3 & 3 \end{vmatrix} = (-6, -5, -3)$$

By using the point $P(2,0,3)$and the normal $\vec{n}$, the equation of the plane will be

$$-6(x-2) - 5(y-0) - 3(z-3) = 0 \xRightarrow{\text{the general equation of a plane}} \boxed{\boldsymbol{6x + 5y + 3z = 21}}.$$

---

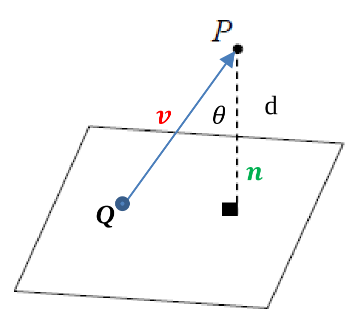
16. Find the shortest distance between the point $P(2, -1, 3)$ and the plane $2x - 3y + 5z - 6 = 0$.

Solution

The distance from $P$to the plane, is simply the length of the projection of $\vec{v}$ onto the unit normal vector $\vec{n}$. This distance is simply.

$$d = |\vec{v}| \cos\theta = \frac{|\overrightarrow{QP} \,.\, \vec{n}|}{|\vec{n}|} = \frac{|(x_1 - x_2\,, y_1 - y_2\,,\; z_1 - z_2)\,.\;(a, b, c)|}{\sqrt{(a)^2 + (b)^2 + (c)^2}}$$

$$d = \frac{|a(x_1 - x_2) + b(y_1 - y_2) + c(z_1 - z_2)|}{\sqrt{(a)^2 + (b)^2 + (c)^2}}$$



Recall that we can also write the equation for the plane as

$ax + by + cz = D$, such that D=$(a\,x_2 + by_2 + cz_2)$

As one can see that the distance didn't depend on the point Q. We'll substitute into the above formula, to arrive at the following expression for the distance from P=(x_1 ,y_1 ,z_1) to the plane $ax + by + cz = D$:

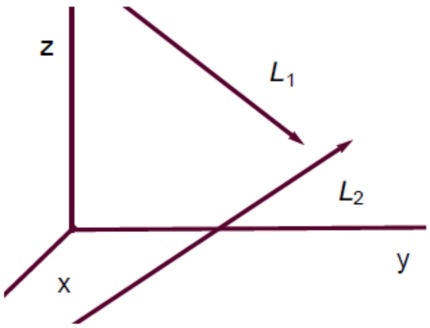
$$\boxed{d = \frac{|a(x_1) + b(y_1) + c(z_1) - D\,|}{\sqrt{(a)^2 + (b)^2 + (c)^2}}} = \frac{|2(2) - 3(-1) + 5(3) - 6|}{\sqrt{(2)^2 + (-3)^2 + (5)^2}} = \boxed{\frac{\boldsymbol{16}}{\sqrt{\boldsymbol{38}}}\; \boldsymbol{unit}}.$$

17. Find the point of intersection of the lines:

$\boldsymbol{l1: x = 2t + 1\,, y = 3t + 2\,, z = 4t + 3}$, **and** $\boldsymbol{l2: x = s + 2\,, y = 2s + 4\,, z = -4s - 1}$,

and then find the plane determined by these lines.

Solution



In 3 dimensions, two lines intersect if and only if there are distinct real numbers t and s.

Satisfying $x_1 = x_2$, $y_1 = y_2$, and $z_1 = z_2$.

Construct the system of equations:

> - Any two pair of equations can be converted to a linear system of equations, solving them to find t and s.
>
> *Notice : "If there is a solution. There's no guarantee that the two lines will intersect! Unless the 3 equations are satisfied completely."*

$l1 :\ x_1 = 2t + 1\,,\ y_1 = 3t + 2\,,\ z_1 = 4t + 3$

$l2 :\ x_2 = s + 2\,,\ y_2 = 2s + 4\,,\ z_2 = -4s - 1$

Choosing the $x'sand\ y's\ equations$;

(i) assuming $x_1 = x_2 \Rightarrow 2t + 1 = s + 2 \Rightarrow s = 2t - 1$,
(ii) assuming $y_1 = y_2 \Rightarrow 3t + 2 = 2s + 4 \Rightarrow 3t + 2 = 2(2t - 1) + 4 \Rightarrow \boldsymbol{t = 0\,, and\ s = -1}$

Check if the t = 0 and $s = -1$ *satisfies* $z_1 = z_2$ *equation*

$\Longrightarrow\ 4t + 3\ = -4s - 1\ \Longrightarrow$ ***<u>3 = 3</u>*** *correct!* ✓

Therefore, the two lines will intersect.

$substitute\ t = 0\ in\ l1 \to\ the\ point\ is\ (1,\ 2,\ 3)$

$substitute\ s = -1\ in\ l2 \to the\ same\ point$

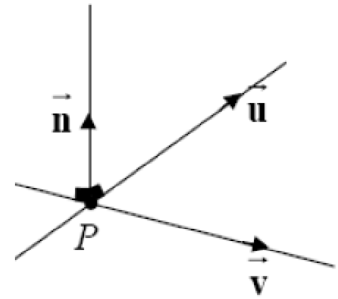
So the two lines are intersected at the point $P(1,2,3)$. ✓

<u>To get the plane determined by these lines;</u>

$l1\ directional\ vector: \vec{u} = (2,3,4)$

$l2\ directional\ vector: \vec{v} = (1,2,-4)$

Form the sketch $\vec{n} = \vec{u} \times \vec{v} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 2 & 3 & 4 \\ 1 & 2 & -4 \end{vmatrix} = (-20, 12, 1)$



By using the **point** of intersection as it lies on the plane as well, and the normal $\vec{n}$, the equation of the plane will be

$$-20(x-1) + 12(y-2) + (z-3) = 0 \xrightarrow{\text{the general equation of a plane}} \boxed{\boldsymbol{-20x + 12y + z = 7}}$$

---

18. Determine whether the following lines, are parallel, intersect or are skew.

***If they intersect, find the normal line to both of them and passing the point of intersection.**

****If they are skew, find the equations of the parallel planes containing them.**

*****If they are parallel find the shortest distance between them.**

(a) $\boldsymbol{l1}: \boldsymbol{x} = \mathbf{3} + \mathbf{2}\boldsymbol{t}\, , \boldsymbol{y} = -\mathbf{1} + \mathbf{4}\boldsymbol{t}\, , \boldsymbol{z} = \mathbf{2} - \boldsymbol{t}$.

(b) $\boldsymbol{l2}: \boldsymbol{x} = \mathbf{1} + \mathbf{4}\boldsymbol{s}\, , \boldsymbol{y} = \mathbf{1} + \mathbf{2}\boldsymbol{s}\, , \boldsymbol{z} = -\mathbf{3} + \mathbf{4}\boldsymbol{s}$.

(c) $\boldsymbol{l3}: \boldsymbol{x} = \mathbf{3} + \mathbf{2}\boldsymbol{r}\, , \boldsymbol{y} = \mathbf{2} + \boldsymbol{r}\, , \boldsymbol{z} = -\mathbf{2} + \mathbf{2}\boldsymbol{r}$.

Solution

$l1 \; directional \; vector \; \vec{t} = (2,4,-1)$

$l2 \; directional \; vector \; \vec{u} = (4,2,4)$

$l3 \; directional \; vector \; \vec{v} = (2,1,2)$

For <u>*l1 and l2*</u>

(I) It is clear that $\vec{t}$ is not multiple of $\vec{u}$ so $l1$ is **<u>not parallel</u>** to $l2$.

(II) Solve the equation $l1 \; : \; x = 3 + 2t \, , \; \; y = -1 + 4t \, , \; \; z = 2 - t$ with the equation $l2 : \; x = 1 + 4s \, , \; \; y = 1 + 2s \, , \; \; z = -3 + 4s$ by:

a. assuming $x = x \Rightarrow 2t + 3 = 4s + 1 \Rightarrow t = 2s - 1$,

β. assuming y= $y \Rightarrow 4t - 1 = 2s + 1 \Rightarrow \; 4(2s - 1) - 1 = 2s + 1 \Rightarrow \boldsymbol{s = 1 \, , and \; t = 1}$

Check if the t = 1 and $s = 1$ *satisfies* $z = \; z$ *equation*

$\Rightarrow \; 2 - t \; \; = -3 + 4s \; \Rightarrow$ ***<u>1 = 1</u>*** *correct!* ✓

Therefore, the two lines will intersect.

$substitute \; t = 1 \; in \; \; l1 \rightarrow \; \; the \; point \; is \; (\, 5, \qquad 3, \qquad 1)$

$substitute \; s = 1 \; in \; \; l2 \rightarrow the \; same \; point$

So the two lines are intersected at the point $P(5 \, , \; 3 \; , \; 1)$. ✓

<u>To get the normal line to both of them and passing the point of intersection</u>

$$\vec{n} = \vec{t} \times \vec{u} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 2 & 4 & -1 \\ 4 & 2 & 4 \end{vmatrix} = (18, -12, -12)$$

The parametric equation of the line passing through the point (5, 3, 1) with the directional vector $\vec{n}$ will be

$$\boxed{\boldsymbol{l: x = 5 + 18w\,,\ \ y = 3 - 12w\ ,\ \ \ z = 1 - 12w\,.}}$$

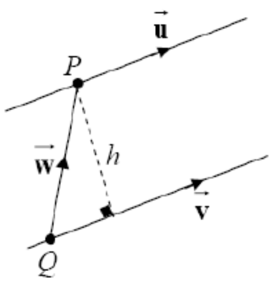
For $\underline{l2 \ and \ l3}$

$\because\ \vec{u} = \ 2\vec{v} \rightarrow so\ the\ vector\ \vec{u}\ is\ multiple\ of\ \vec{v}\ which\ infer\ that\ \vec{u}\ //\ v\ and\ consequently,$

$l2\ //\ l3\,.$

<u>To get the shortest distance (h)</u>

From the sketch, we can get the vector



$\vec{w} = \overrightarrow{QP} = (\,-2, -1, -1)$

$$So, \vec{w} \times \vec{v} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ -2 & -1 & -1 \\ 2 & 1 & 2 \end{vmatrix} = (-20, 12, 1).$$

The shortest distance will be

$$h = \frac{||\vec{w} \times \vec{v}||}{||\vec{v}||} = \frac{\sqrt{(-20)^2 + (12)^2 + (1)^2}}{\sqrt{(2)^2 + (1)^2 + (2)^2}} = \boxed{\boldsymbol{\frac{\sqrt{545}}{3}\ unit.}}$$

For $l1$ and $l3$

It is clear that $\vec{t}$ is not multiple of $\vec{v}$ so $l1$ is **not parallel** to $l3$.   (I)

Solve the equation $\boldsymbol{l1}: \; x = 3 + 2t \,, \;\; y = -1 + 4t \,, \;\; z = 2 - t$ with the equation

$\boldsymbol{l3}: x = 3 + 2r \,, y = 2 + r \,, z = -2 + 2r$ by:

α. assuming $x = x \Rightarrow 2t + 3 = 2r + 3 \Rightarrow t = r$,

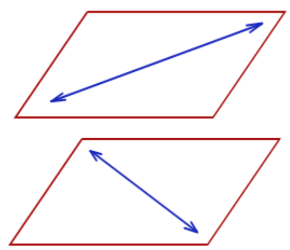
β. assuming y $= y \Rightarrow 4t - 1 = 2 + r \Rightarrow \; 4(r) - 1 = 2 + r \Rightarrow \boldsymbol{r = 1 \,, and\ t = 1}$

Check if the $r = 1$ *and* $t = 1$ *satisfies* $z = z$ *equation*

$\Rightarrow \; 2 - t \;\; = -2 + 2r \;\; \Rightarrow \; \underline{1 = 0}$ *False statement!*

Therefore, the two lines will **not intersect.**   (II)

From (I) and (II), the two lines are not parallel nor intersecting so they are **skew**.



Refer to Lecture 13 slide 10

To find the equations of the parallel planes containing them

$l1: x = 3 + 2t \,, y = -1 + 4t \,, z = 2 - t \; \Rightarrow$ Plane 1: $\vec{w} = (2, 4, -1)$ and $P1 = (3, -1, 2)$

$l3: x = 3 + 2r \,, y = 2 + r \,, z = -2 + 2r \Rightarrow$ Plane 3: $\vec{m} = (2, 1, 2)$ and $P2 = (3, 2, -2)$

parallel planes have the same normal

$$\vec{n} = \vec{w} \times \vec{m} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 2 & 4 & -1 \\ 2 & 1 & 2 \end{vmatrix} = (9, -6, -6)$$

$therefore\ plane\ 1: 9(x - 3) - 6(y + 1) - 6(z - 2) = 0 \rightarrow \boxed{\boldsymbol{9x - 6y - 6z = 21},}$

$plane\ 3: 9(x - 3) - 6(y - 2) - 6(z + 2) = 0 \rightarrow \boxed{\boldsymbol{9x - 6y - 6z = 27}.}$

19. **Determine whether the following planes, are parallel or intersecting. If they are intersecting, find the parametric equations of their line of intersection and the acute angle of intersection. If they are parallel, find the shortest distance between them**

(a) $\mathbf{3x - 2y + z = 4}$ and $\mathbf{6x - 4y + 3z = 7}$

Solution

<u>To check if the two planes are parallel;</u>

$\mathbf{3x - 2y + z = 4} \Rightarrow \vec{n}_1 = (3, -2, 1)$

$\mathbf{6x - 4y + 3z = 7} \Rightarrow \vec{n}_2 = (6, -4, 3)$ **"equation 2"**

$$\boldsymbol{It\ is\ clear\ that}\ \vec{n}_1\ is\ not\ a\ multiple\ of\ \vec{n}_2\ so\ the\ two\ planes\ are\ not\ parallel$$
$$then\ they\ are\ (\boldsymbol{intersecting}).$$

<u>To get the parametric equation of their line of intersection</u>

Solving the 2 planes equations;

Multiply the first plane equation by 2 → $6x - 4y + 2z = 8$ **"equation 3"**

then subtract **equation 3** from **equation 2** → (x & y will be eliminated) and the equation of the line of intersection is $l : z = -1$.

To parametrize the last equation, assume that $x = t$ then substitute the values of $x\ and\ z$

in any of the planes equation to get $y = \frac{3}{2}t - \frac{5}{2}$

∴ $\boxed{\boldsymbol{l: x = t\ ,\ y = \frac{3}{2}t - \frac{5}{2}\ ,\ z = -1.}}$

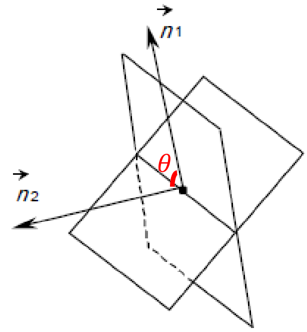
<u>and the acute angle of intersection</u> (θ)

From the sketch, the angle (θ) between the two intersected planes is equivalent to the angle between their vectors $\vec{n}_1$ and $\vec{n}_2$ so;

$$\cos\theta = \frac{\vec{n}_1.\vec{n}_2}{||\vec{n}_1||\,||\vec{n}_2||} = \frac{(3,-2,1).(6,-4,3)}{\sqrt{(3)^2+(-2)^2+(1)^2}\sqrt{(6)^2+(-4)^2+(3)^2}}$$

$$= \frac{29}{\sqrt{854}}$$

$$\therefore\ \theta = \cos^{-1}\left(\frac{29}{\sqrt{854}}\right) \approx 0.1237\quad \boxed{(\mathbf{7.09}^\circ).}$$



$$(\boldsymbol{b})\mathbf{2x-8y-6z-2=0}\ \text{ and }\ \mathbf{-x+4y+3z-5=0}$$

<u>Solution</u>

<u>To show that the two planes are parallel or intersecting</u>

$\mathbf{2x-8y-6z=2}\quad \Rightarrow \vec{n}_1 = (2,-8,-6)$

$\mathbf{-x+4y+3z=5} \Rightarrow \vec{n}_2 = (-1,4,3)$

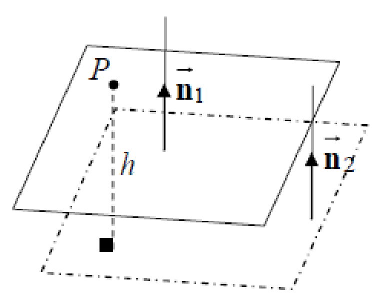
$\because \vec{n}_1 = (2,-8,-6) = -2(-1,4,3) = -2\,\vec{n}_2$

So the vector $\vec{n}_1$ is a multiple of the vector $\vec{n}_2$ which infer that $\vec{n}_1 // \vec{n}_2$ and consequently, the two planes are **parallel**.

<u>To get the shortest distance</u> (h)

Choose a point on the first plane by assuming $y = z = 0$ in the equation of this plane to get $x = 1$ so the point will be $P(1,0,0)$.

Now h is the shortest distance from the point $P(1,0,0)$ to the second plane $-x+4y+3z-5=0$



So,

$$h = \frac{|-(1)+4(0)+3(0)-5|}{\sqrt{(-1)^2+(4)^2+(3)^2}} = \boxed{\frac{\mathbf{6}}{\sqrt{\mathbf{26}}}\ \boldsymbol{unit}.}$$

$(c)\ \boldsymbol{x = 0}$ and $\boldsymbol{2x - y + z - 4 = 0}$

Solution

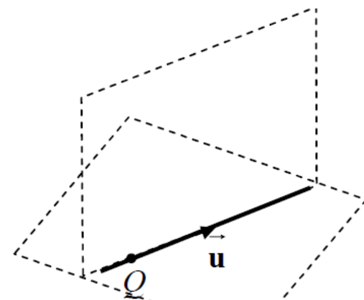
<u>To show that the two planes are parallel or intersecting</u>

$\boldsymbol{x = 0} \quad \Rightarrow \vec{n}_1 = (1, 0, 0)$

$\boldsymbol{2x - y + z = 4} \Rightarrow \vec{n}_2 = (2, -1, 1)$

It is clear that $\vec{n}_1$ is not a multiple of $\vec{n}_2$ so the two planes are not parallel (**intersecting**).

<u>To get the parametric equation of their line of intersection</u>



Substitute the first plane equation in the other plane equation to get the equation of the line of intersection of them

$l: z - y = 4$

To parametrize the last equation, assume that $y = t$ then $z = 4 + t$.

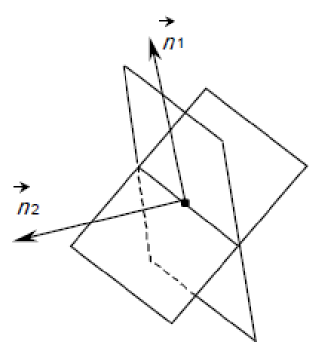
$\therefore \boxed{\boldsymbol{l: x = 0\ ,\ y = t\ \ ,\ \ z = 4 + t\,.}}$

<u>To get the acute angle of intersection</u> (θ)

From the sketch, the angle (θ) between the two intersected planes is equivalent to the angle between their vectors $\vec{n}_1$ and $\vec{n}_2$ so;

$$\cos\theta = \frac{\vec{n}_1 . \vec{n}_2}{||\vec{n}_1||\ ||\vec{n}_2||} = \frac{(1,0,0) . (2,-1,1)}{\sqrt{1}\sqrt{(2)^2 + (-1)^2 + (1)^2}} = \frac{2}{\sqrt{6}}$$

$\therefore\ \theta = \cos^{-1}\left(\frac{2}{\sqrt{6}}\right) \approx 0.6155\ \ \boxed{(\boldsymbol{35.26°})}.$



$(\boldsymbol{d})\ \boldsymbol{x+2y-2z=5}$ **and** $\boldsymbol{6x-3y+2z=8}$

Solution

To show that the two planes are parallel or intersecting

$\boldsymbol{x+2y-2z=5} \quad \Rightarrow \vec{n}_1 = (1,2,-2)$

$\boldsymbol{6x-3y+2z=8} \Rightarrow \vec{n}_2 = (6,-3,2)$

It is clear that $\vec{n}_1$ is not a multiple of $\vec{n}_2$ so the two planes are not parallel (**intersecting**).

To get the parametric equation of their line of intersection

Add the two plane equations $x+2y-2z=5$ and $6x-3y+2z=8$ together (z will be eliminated) to get the equation of the line of intersection of them

$l: 7x-y=13$

To parametrize the last equation, assume that $x=t$ so $y=-13+7t$ then substitute the values of x and y in anyone of the plane equations to get $= -\frac{31}{2}+\frac{15}{2}t$.

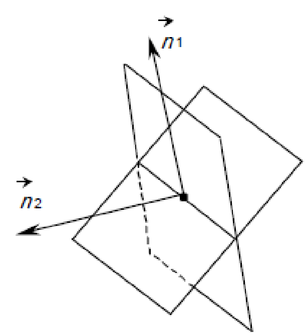
$$\therefore \boxed{\boldsymbol{l: x=t\ ,\ y=-13+7t\ ,\ z=-\frac{31}{2}+\frac{15}{2}t}.}$$

To get the acute angle of intersection (θ)

From the sketch, the angle (θ) between the two intersected planes is equivalent to the angle between their vectors $\vec{n}_1$ and $\vec{n}_2$ so;

$$\cos\theta = \frac{\vec{n}_1.\vec{n}_2}{||\vec{n}_1||\,||\vec{n}_2||} = \frac{(1,2,-2).(6,-3,2)}{\sqrt{1+4+4}\sqrt{(6)^2+(-3)^2+(2)^2}} = \frac{-4}{21}$$

$$\therefore\ \theta = \cos^{-1}\left(\frac{-4}{21}\right) \approx 1.763 \quad (100.98°).$$



Then the acute angle will be

$$\theta \approx 180° - 100.98° = \boxed{79.02°}$$

Viel Gluck! ☺