The 13th Asian Control Conference (ASCC 2022)

Tutorial: Homomorphic Encryption and Its Application to Feedback Control

ECLib: Encrypted Control Library for Python

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ECLib

Encrypted control requires the knowledge of control theory and cryptography.

For control engineers, studying cryptography is challenging.

ECLib (**E**ncrypted **C**ontrol **Lib**rary)

- **Python** library for numerical simulation of encrypted control
- **Open source software**
- Easy to use for researchers and students

https://github.com/KaoruTeranishi/EncryptedControl

Why Python?

In control society, MATLAB is a popular tool for design and analysis of control systems.
Unfortunately, MATLAB does not support **bignum arithmetic**.

Cryptography is typically based on very big integers, and so MATLAB is not suitable for encrypted control.

Python

- The most popular programming language in 2021 [1]
- Built-in support for bignums
- Many OSS packages for developing codes
- **Interactive coding and computing**

[1] https://spectrum.ieee.org/top-programming-languages/

Jupyter Notebook

Jupyter notebook is a web application for interactive computing.

https://jupyter.org

Jupyter notebook documents can contain codes, markdown texts, equations, and plots.
(≒ MATLAB live editor)

Visual Studio Code supports Jupyter Notebook.

Goal of This Presentation

This presentation is a brief introduction to Python and ECLib to develop numerical simulations of encrypted control.

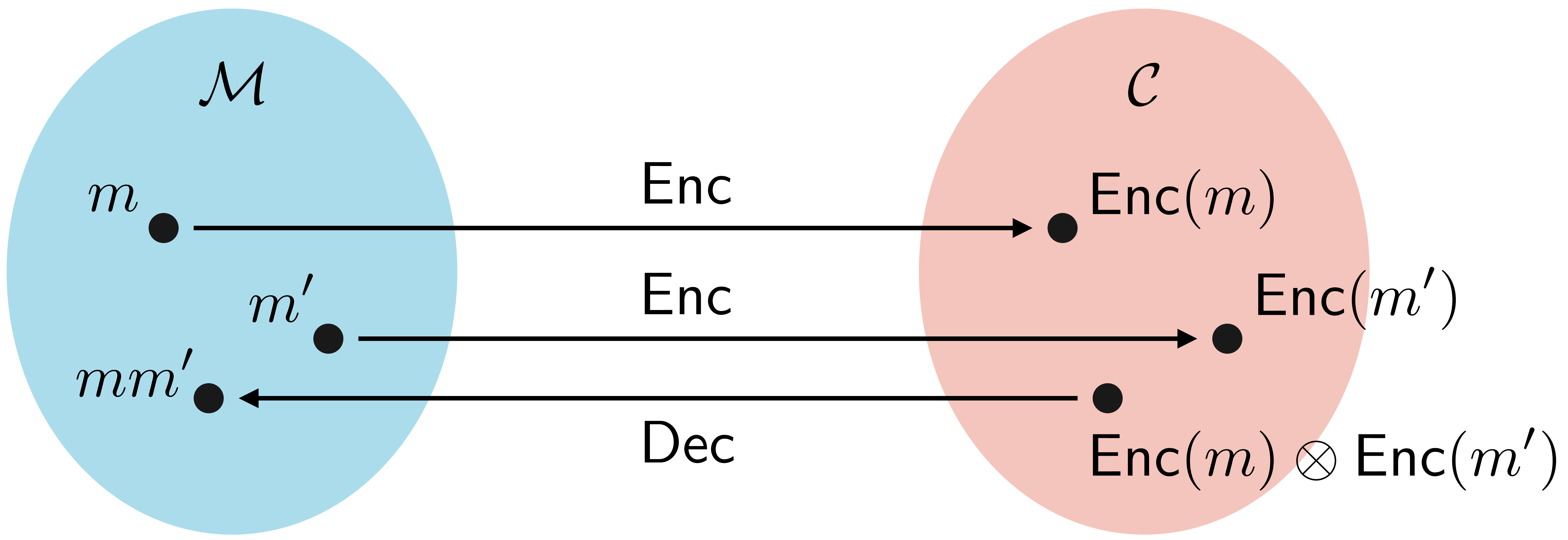
The usage of ECLib will be explained through two applications.

- ElGamal encryption
- Encrypted PI control

Example 1: ElGamal encryption

Algorithms

$$\begin{aligned}
\mathsf{KeyGen} &: 1^\lambda \mapsto (\mathsf{pk}, \mathsf{sk}) = ((\mathbb{G}, g, p, g, h), s) \\
\mathsf{Enc} &: (\mathsf{pk}, m) \mapsto c = (c_1, c_2) = (g^r \bmod p, mh^r \bmod p) \\
\mathsf{Dec} &: (\mathsf{sk}, c) \mapsto m = {c_1}^{-s} c_2 \bmod p \\
\mathsf{Eval} &: (c, c') \mapsto (c_1 c_1' \bmod p, c_2 c_2' \bmod p)
\end{aligned}$$



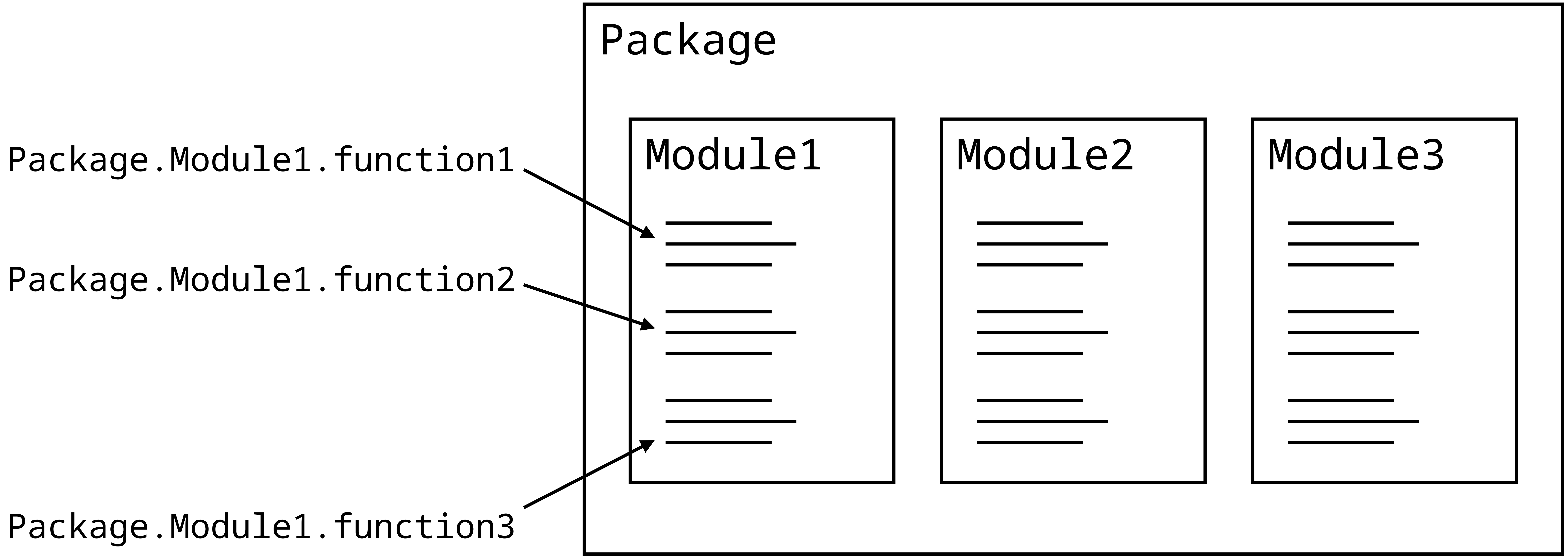
Import Packages


```python
from eclib.elgamal import *
```

Import all functions from `elgamal` module in `eclib` package.

Module: Script file consisting of some commands. Module can be used from external codes.

Package: A way structuring multiple modules.



Package Installation

We can use many OSS packages for developing codes.

NumPy SciPy matplotlib

pandas ECLib CVXPY

If you want to install ECLib, run `pip` command on your terminal.

```
$ pip install eclib
```

pip

- De facto standard of package management system in Python.
- The system installs packages from online repository, Python Package Index (PyPI).
- The system can also update and uninstall packages.

Key Generation


```
3 key_length = 20
4 params, pk, sk = keygen(key_length)
5
6 print(f'\u03BB = {key_length}')
7 print(f'p = {params.p}')
8 print(f'q = {params.q}')
9 print(f'g = {params.g}')
10 print(f'h = {pk}')
11 print(f's = {sk}')
```

Result:

```
λ = 20
p = 1049963
q = 524981
g = 3
h = 33790
s = 110369
```

Line 3: Assign `20` to `key_length`.

Line 4: Generate public and secret keys.

`params`: public parameters p, q, g

`pk`: public key h

`sk`: secret key s

Each public parameter can be referred by the dot operator.

Encryption and Decryption (1/2)


```
13 delta = 0.01
14 x1 = 1.23
15 m1 = encode(params, x1, delta)
16 c1 = encrypt(params, pk, m1)
17 n1 = decrypt(params, sk, c1)
18 y1 = decode(params, n1, delta)
19
20 print(f'x1 = {x1}')
21 print(f'm1 = Ecd(x1) = {m1}')
22 print(f'c1 = Enc(m1) = {c1}')
23 print(f'n1 = Dec(c1) = {n1}')
24 print(f'y1 = Dcd(n1) = {y1}')
```

Result:

```
x1 = 1.23
m1 = Ecd(x1) = 122
c1 = Enc(m1) = [40345, 874487]
n1 = Dec(c1) = 122
y1 = Dcd(n1) = 1.22
```

Line 15: Encode `x1 = 1.23` using `delta = 0.01`.

$$m_1 = \mathsf{Ecd}_\Delta(x_1),\ x_1 = 1.23,\ \Delta = 0.01$$

Line 16: Encrypt `m1`.

$$c_1 = \mathsf{Enc}(m_1)$$

Line 17: Decrypt `c1`.

$$n_1 = \mathsf{Dec}(c_1)$$

Line 18: Decode `n1` using `delta`.

$$y_1 = \mathsf{Dcd}_\Delta(n_1)$$

Encryption and Decryption (2/2)


```python
26 x2 = -4.56
27 c2 = enc(params, pk, x2, delta)
28 y2 = dec(params, sk, c2, delta)
29
30 print(f'x2 = {x2}')
31 print(f'c2 = Enc(Ecd(x2)) = {c2}')
32 print(f'y2 = Dcd(Dec(c2)) = {y2}')
```

Result:

```
x2 = -4.56
c2 = Enc(Ecd(x2)) = [424381, 935668]
y2 = Dcd(Dec(c2)) = -4.57
```

Line 27: We can encrypt float numbers by enc instead of using encode and encrypt.

$$c_2 = \mathrm{Enc}(\mathrm{Ecd}_\Delta(x_2)),\ x_2 = -4.56$$

Line 28: Similarly, we can use dec for decryption.

$$y_2 = \mathrm{Dcd}_\Delta(\mathrm{Dec}(c_2))$$

Homomorphic Computation


```python
34 x3 = x1 * x2
35 c3 = mult(params, c1, c2)
36 y3 = dec(params, sk, c3, delta ** 2)
37
38 print(f'x3 = x1 * x2 = {x3}')
39 print(f'c3 = Mult(c1, c2) = {c3}')
40 print(f'y3 = Dcd(Dec(c3)) = {y3}')
```

Result:

```
x3 = x1 * x2 = -5.6088
c3 = Mult(c1, c2) = [954767, 686157]
y3 = Dcd(Dec(c3)) = -5.5754
```

Line 34: Assign x1 multiplied by x2 to x3.

Line 35: Compute homomorphic multiplication.

$$c_3 = c_1 \otimes c_2$$

Line 36: Decrypt and decode c3 using `delta` squared because `delta` is accumulated every homomorphic multiplication.

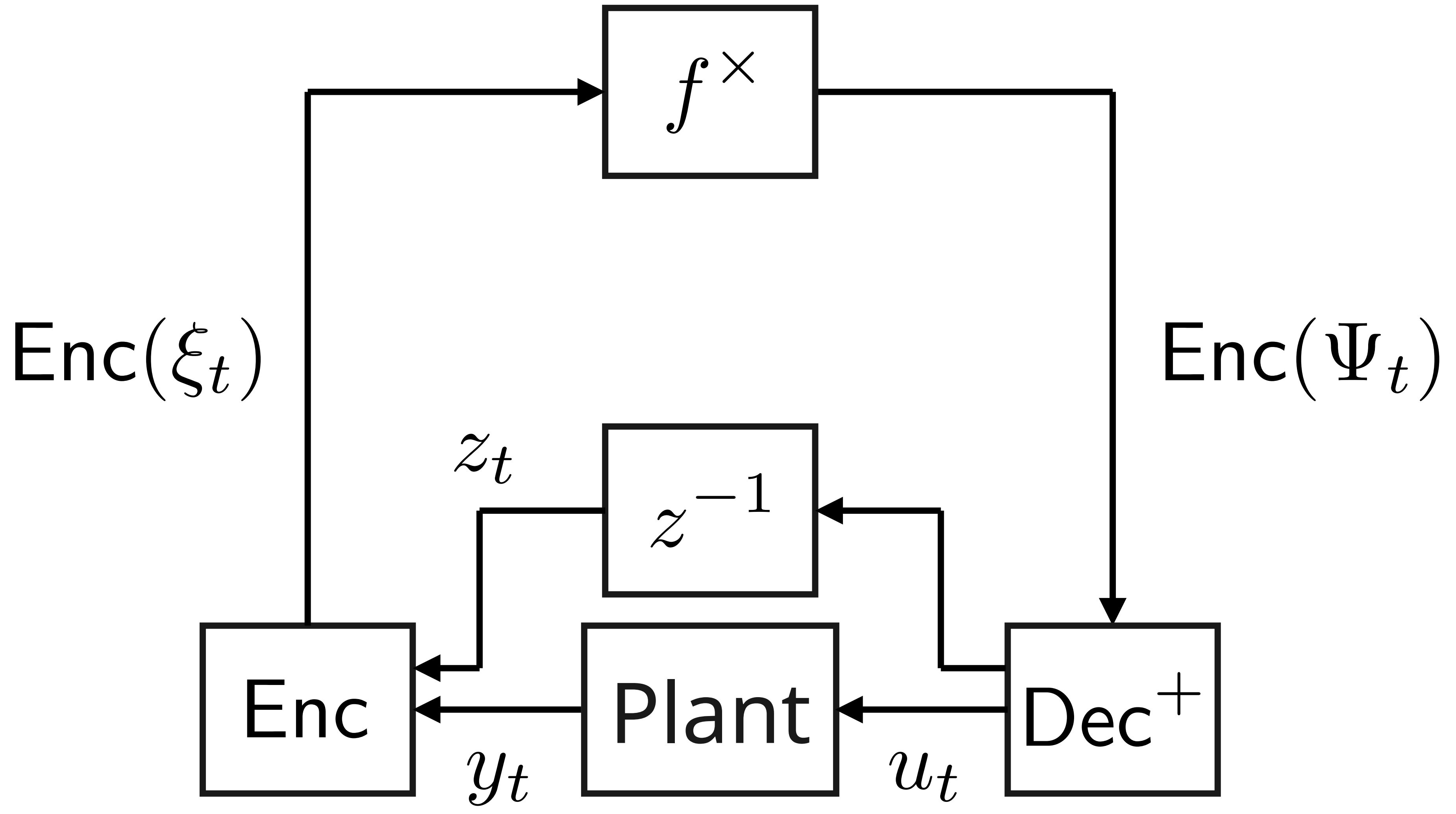
$$y_3 = \mathsf{Dcd}_{\Delta^2}(\mathsf{Dec}(c_3))$$

Example 2: Encrypted PI Control

Plant ($T_s = 0.1$ s)

$$x_{t+1} = \begin{bmatrix} 0.35 & -0.16 \\ 0.13 & 0.98 \end{bmatrix} x_t + \begin{bmatrix} 0.031 \\ 0.004 \end{bmatrix} u_t$$

$$y_t = \begin{bmatrix} 0 & 1 \end{bmatrix} x_t$$



Dec$^+ \coloneqq f^+ \circ$ Dec

PI controller ($K_p = 15.34$, $K_i = 15.62$)

$$w_{t+1} = w_t + \begin{bmatrix} T_s & -T_s \end{bmatrix} \begin{bmatrix} r_t \\ y_t \end{bmatrix}$$

$$u_t = K_i w_t + \begin{bmatrix} K_p & -K_p \end{bmatrix} \begin{bmatrix} r_t \\ y_t \end{bmatrix}$$

$$\iff \begin{bmatrix} w_{t+1} \\ u_t \end{bmatrix} = \begin{bmatrix} 1 & T_s & -T_s \\ K_i & K_p & -K_p \end{bmatrix} \begin{bmatrix} w_t \\ r_t \\ y_t \end{bmatrix}$$

$$\psi_t = \begin{bmatrix} w_{t+1} \\ u_t \end{bmatrix}, \ \Phi = \begin{bmatrix} 1 & T_s & -T_s \\ K_i & K_p & -K_p \end{bmatrix} = \begin{bmatrix} 1 & 0.1 & -0.1 \\ 15.62 & 15.34 & -15.34 \end{bmatrix}, \ \xi_t = \begin{bmatrix} w_t \\ r_t \\ y_t \end{bmatrix}$$

Import Packages


```python
1 from eclib.elgamal import *
2 from eclib.colors import *
3 import eclib.figsetup
4 import numpy as np
5 import numpy.linalg as la
6 from control.matlab import *
7 import matplotlib.pyplot as plt
```

Import modules in ECLib, NumPy, Python-control, and matplotlib.

Line 4: Import numpy module as the short name, np. After this line, we can refer numpy module as np.

Lines 5, 7: Import `linalg` and `pyplot` modules from `control` and `matplotlib` packages as `la` and `plt`, respectively.

Simulation Settings (1/2)


```
 9 # sampling time
10 Ts = 0.1
11
12 # simulation setting
13 simulation_time = 10
14 t = np.linspace(0, simulation_time - Ts, int(simulation_time / Ts))
```

In this simulation, a sampling time is 0.1 s, and the total simulation time is 10 s.

Line 14: Generate an array including sampling points.

`linspace(start, stop, num)` returns an array consisting of num elements, which are equally spaced in the interval from `start` to `stop`.

Simulation Settings (2/2)

Example:

```
a = np.linspace(0, 10, 5)
print(f'a = {a}')
```

Result:

```
a = [ 0.    2.5   5.    7.5 10. ]
```

```
t = np.linspace(0, simulation_time - Ts, int(simulation_time / Ts))
```

`t` is an array consisting of `int(simulation_time / Ts)` elements in the interval from `0` to `simulation_time - Ts`.

Here, `simulation_time` is 10, and Ts is 0.1. Hence, `t` is an array,

`t = [0, 0.1, 0.2, 0.3, ... 9.7, 9.8, 9.9]`

Plant

```
16 # plant (continuous time)
17 A = np.array([[-10, -2.5],
18               [  2,  0  ]])
19 B = np.array([[0.5],
20               [0  ]])
21 C = np.array([0, 1])
22 D = np.array(0)
23
24 # plant (discrete time)
25 sys = c2d(ss(A, B, C, D), Ts)
26 A = sys.A
27 B = sys.B
28 C = sys.C
29 D = sys.D
30
31 # dimension
32 n = A.shape[0]
33 m = B.shape[1]
34 l = C.shape[0]
```

Lines 17-22: Define plant parameters.

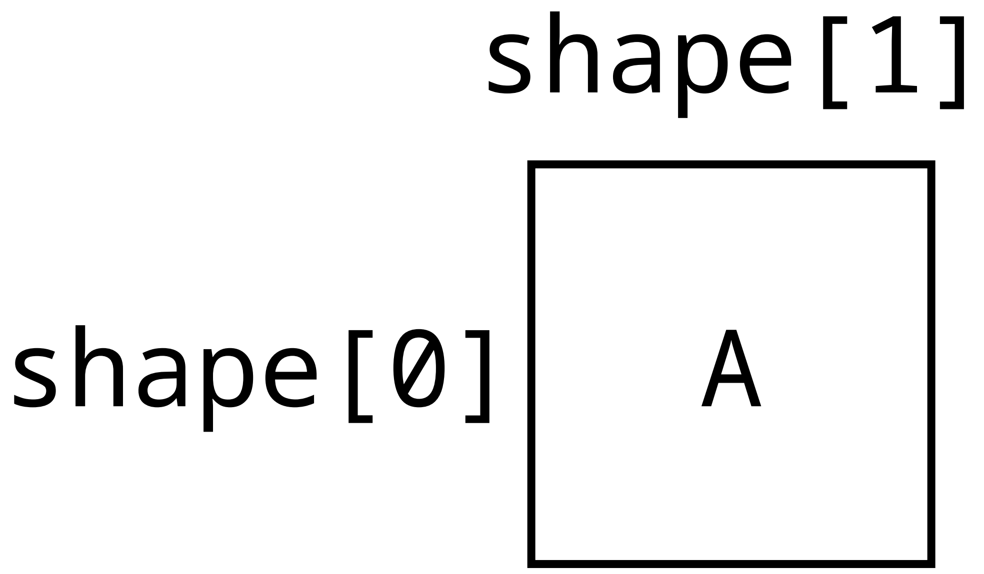
$$\dot{x}(\tau) = \begin{bmatrix} -10 & -2.5 \\ 2 & 0 \end{bmatrix} x(\tau) + \begin{bmatrix} 0.5 \\ 0 \end{bmatrix} u(\tau)$$

$$y(\tau) = \begin{bmatrix} 0 & 1 \end{bmatrix} x(\tau)$$

$$A = \begin{bmatrix} -10 & -2.5 \\ 2 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0.5 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 \end{bmatrix}, \quad D = 0$$

Lines 25-29: Discretize plant parameters with `Ts`.

Lines 32-34: Assign signal dimensions to `n`, `m`, and `l`.



Controller


```python
36 # controller
37 Kp = 15.34
38 Ki = 15.62
39
40 Phi = np.array([[ 1, Ts, -Ts],
41                 [Ki, Kp, -Kp]])
```

Line 40: Define controller parameter matrix.

$$w_{t+1} = w_t + \begin{bmatrix} T_s & -T_s \end{bmatrix} \begin{bmatrix} r_t \\ y_t \end{bmatrix}$$

$$u_t = K_i w_t + \begin{bmatrix} K_p & -K_p \end{bmatrix} \begin{bmatrix} r_t \\ y_t \end{bmatrix}$$

$$\iff \quad \begin{bmatrix} w_{t+1} \\ u_t \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & T_s & -T_s \\ K_i & K_p & -K_p \end{bmatrix}}_{\texttt{Phi}} \begin{bmatrix} w_t \\ r_t \\ y_t \end{bmatrix}$$

Controller Encryption

```
43 # cryptosystem
44 key_length = 256
45 params, pk, sk = keygen(key_length)
46
47 # scaling parameter
48 delta = 0.0001
49
50 # controller encryption
51 Phi_enc = enc(params, pk, Phi, delta)
```

Line 51: Encrypt controller parameter matrix.

enc can be used for a scalar, vector, and matrix.

The dimensions of return value are decided automatically.

```
Phi_enc =
                                   [c1, c2]

[ enc(params,pk,Phi[0,0],delta)  enc(params,pk,Phi[0,1],delta)  enc(params,pk,Phi[0,2],delta) ]
[ enc(params,pk,Phi[1,0],delta)  enc(params,pk,Phi[1,1],delta)  enc(params,pk,Phi[1,2],delta) ]
```

Signals (1/3)


```python
# state
x = np.zeros([len(t) + 1, n])
x_ = np.zeros([len(t) + 1, n])

# input
u = np.zeros([len(t), m])
u_ = np.zeros([len(t), m])

# output
y = np.zeros([len(t), l])
y_ = np.zeros([len(t), l])

# reference
r = np.zeros([len(t), l])
r_ = np.zeros([len(t), l])

# controller state
w = np.zeros([len(t) + 1, l])
w_ = np.zeros([len(t) + 1, l])
```

Define arrays for saving signal data.

x, u, y, `r`, and `w` are used for unencrypted control.

Variables with underscore are used for encrypted control.

`np.zeros(shape)` returns an array consisting of zeros. shape is an array of integers.

`len(t)` is a length of `t`.

Example:

```python
a = np.zeros([2, 3])
print(a)
```

```python
b = len(np.zeros(5))
print(b)
```

Result:

```
[[0. 0. 0.]
 [0. 0. 0.]]
```

```
5
```

Signals (2/3)


```
73 # controller input
74 xi = np.zeros([len(t), 3 * l])
75 xi_ = np.zeros([len(t), 3 * l])
76 xi_enc = [[[0, 0] for j in range(3 * l)] for i in range(len(t))]
77
78 # controller output
79 psi = np.zeros([len(t), l + m])
80 psi_ = np.zeros([len(t), l + m])
81 psi_enc = [[[[0, 0] for k in range(3 * l)] for j in range(l + m)] for i in range(len(t))]
```

Line 76: Define 2-dimensional array of which elements are `[0, 0]` by using list comprehension.

`range(stop)` represents an immutable sequence numbers from `0` to `stop` - 1.

Example:

```
a = range(5)
print(list(a))
```

Result:

```
[0, 1, 2, 3, 4]
```

Signals (3/3)

List comprehension is a way to create a various lists.

Example:

```python
a = [x ** 2 for x in range(5)]
print(f'a = {a}')
```

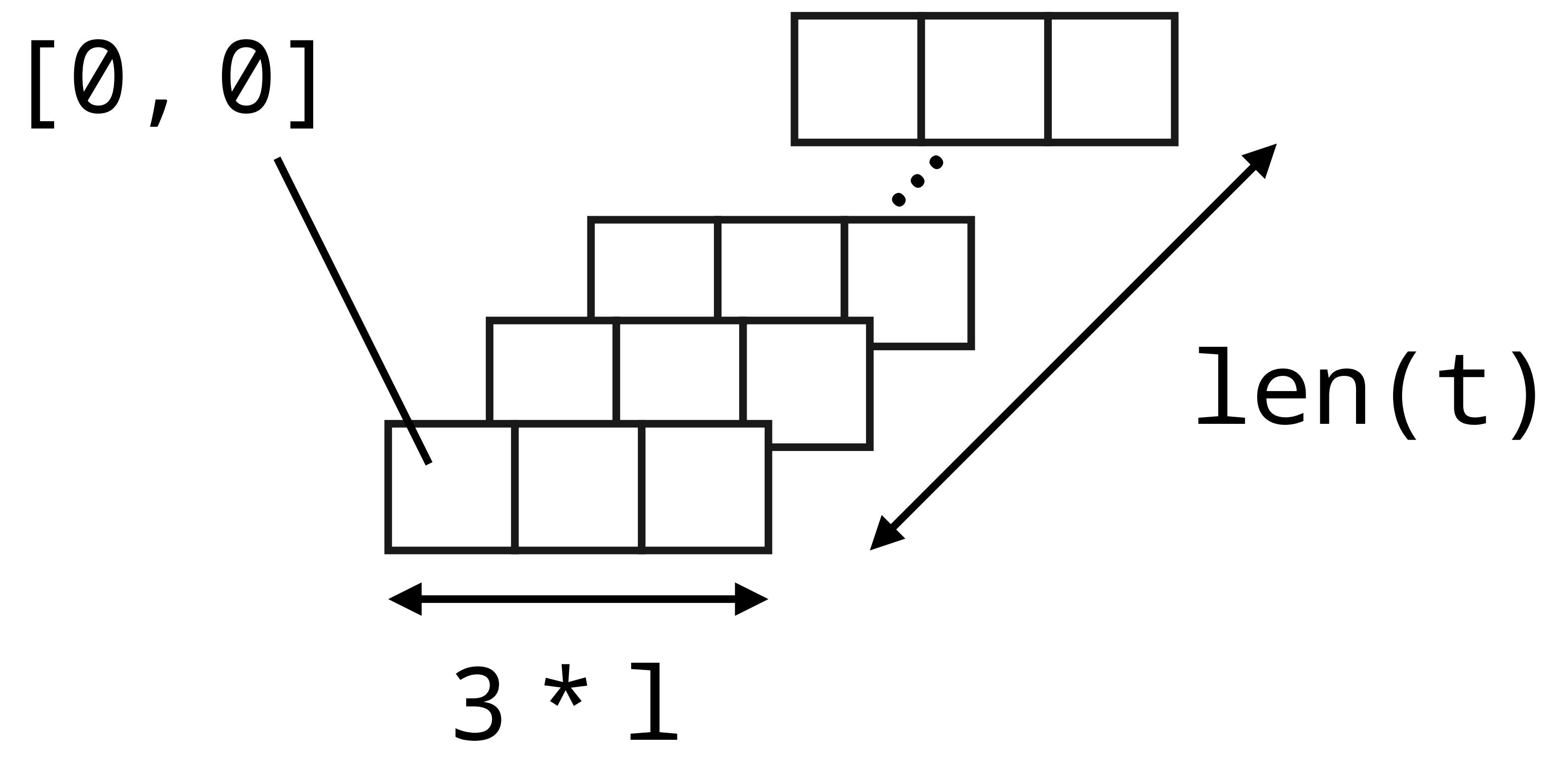
Result:

```
a = [0, 1, 4, 9, 16]
```

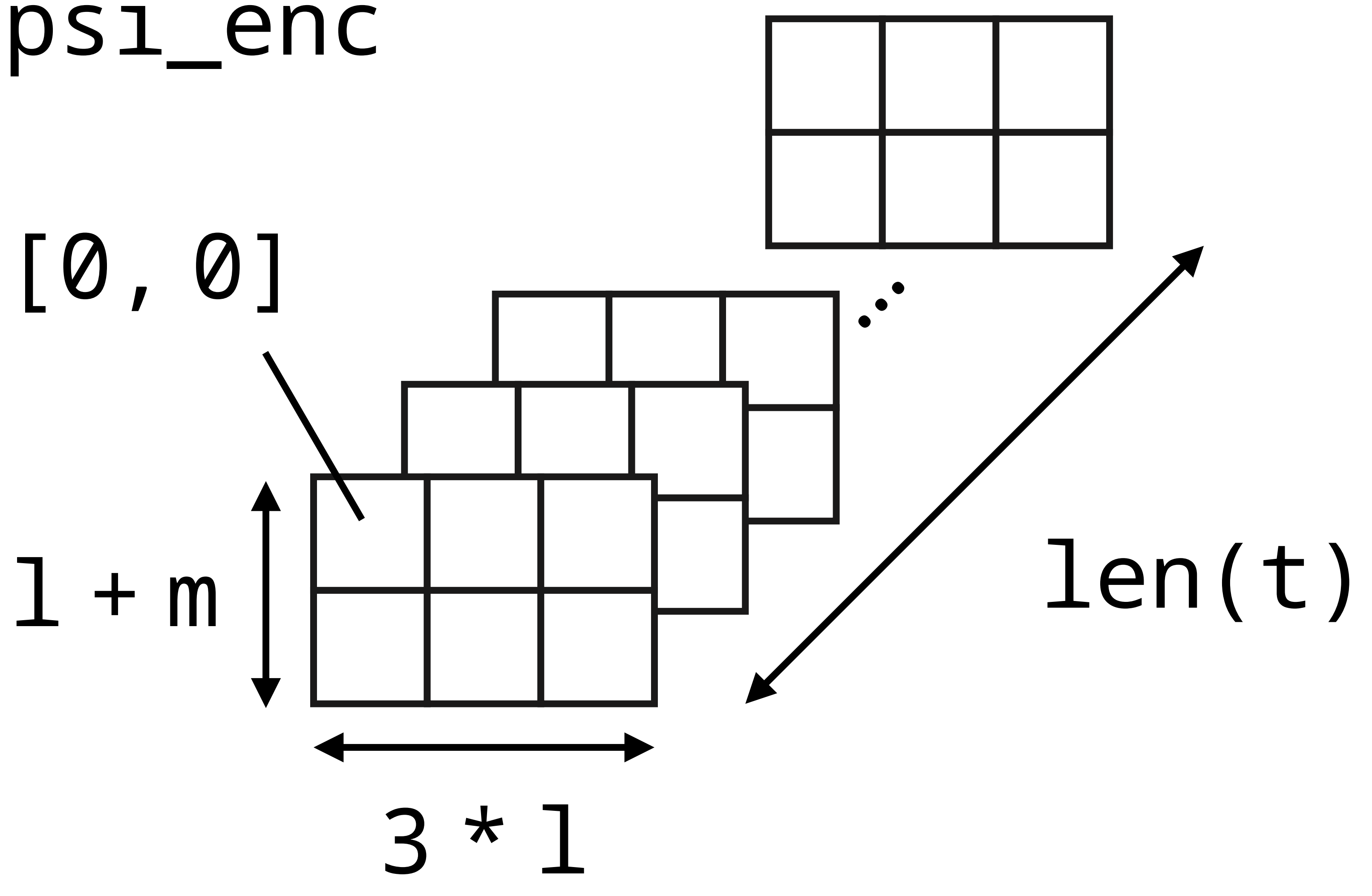
```python
xi_enc = [[[0, 0] for j in range(3 * l)] for i in range(len(t))]
```

```python
psi_enc = [[[[0, 0] for k in range(3 * l)] for j in range(l + m)] for i in range(len(t))]
```

xi_enc



psi_enc



Simulation of Unencrypted Control (1/2)


```
83 # unencrypted control
84 for k in range(len(t)):
85     # reference
86     r[k] = 1
87     # sensor measurement
88     y[k] = C @ x[k]
89     # controller input
90     xi[k,0:l] = w[k]
91     xi[k,l:2*l] = r[k]
92     xi[k,2*l:3*l] = y[k]
93     # controller computation
94     psi[k] = Phi @ xi[k]
95     # controller output
96     w[k+1] = psi[k,0:l]
97     u[k] = psi[k,l:l+m]
98     # plant update
99     x[k+1] = A @ x[k] + B @ u[k]
```

Line 84: For loop from k = `0` to k = `len(t) - 1`.

Line 88: Compute plant output. `@` is the operator for matrix product.

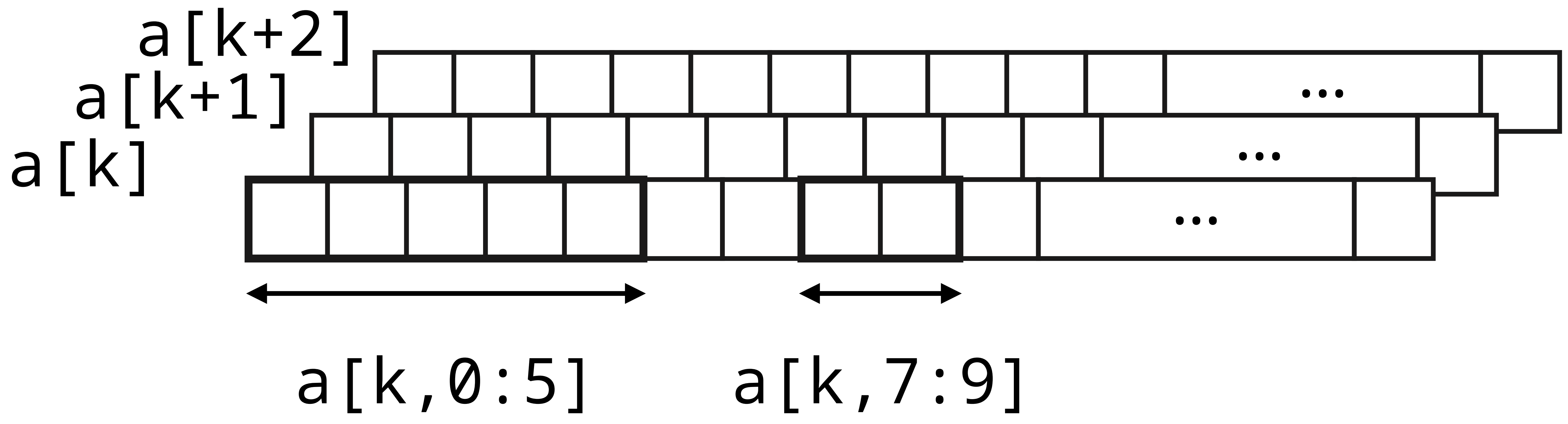
$$y_k = Cx_k$$

Lines 90-92: Construct controller input.

$$\xi_k = \begin{bmatrix} w_k^\top & r_k^\top & y_k^\top \end{bmatrix}^\top$$

Colon is an operator for slicing an array.

Example (2-dimensional array):



Simulation of Unencrypted Control (2/2)


```
83 # unencrypted control
84 for k in range(len(t)):
85     # reference
86     r[k] = 1
87     # sensor measurement
88     y[k] = C @ x[k]
89     # controller input
90     xi[k,0:l] = w[k]
91     xi[k,l:2*l] = r[k]
92     xi[k,2*l:3*l] = y[k]
93     # controller computation
94     psi[k] = Phi @ xi[k]
95     # controller output
96     w[k+1] = psi[k,0:l]
97     u[k] = psi[k,l:l+m]
98     # plant update
99     x[k+1] = A @ x[k] + B @ u[k]
```

Line 94: Compute controller output.

$$\psi_k = \Phi \xi_k$$

Lines 96-97: Decompose controller output.

$$\begin{bmatrix} w_{k+1} \\ u_k \end{bmatrix} = \psi_k$$

Line 99: Update plant state.

Simulation of Encrypted Control

```python
101 # encrypted control
102 for k in range(len(t)):
103     # reference
104     r_[k] = 1
105     # sensor measurement
106     y_[k] = C @ x_[k]
107     # controller input
108     xi_[k,0:l] = w_[k]
109     xi_[k,l:2*l] = r_[k]
110     xi_[k,2*l:3*l] = y_[k]
111     xi_enc[k] = enc(params, pk, xi_[k], delta)
112     # encrypted controller computation
113     psi_enc[k] = mult(params, Phi_enc, xi_enc[k])
114     # controller output
115     psi_[k] = dec_add(params, sk, psi_enc[k], delta ** 2)
116     w_[k+1] = psi_[k,0:l]
117     u_[k] = psi_[k,l:l+m]
118     # plant update
119     x_[k+1] = A @ x_[k] + B @ u_[k]
```

Line 111: Encryption of controller input

$$\xi_{\mathsf{Enc}} = \mathsf{Enc}(\mathsf{Ecd}_{\Delta}(\xi_k))$$

Line 113: Homomorphic computation

$$\Psi_{\mathsf{Enc}} = f^{\times}(\Phi_{\mathsf{Enc}}, \xi_{\mathsf{Enc}})$$

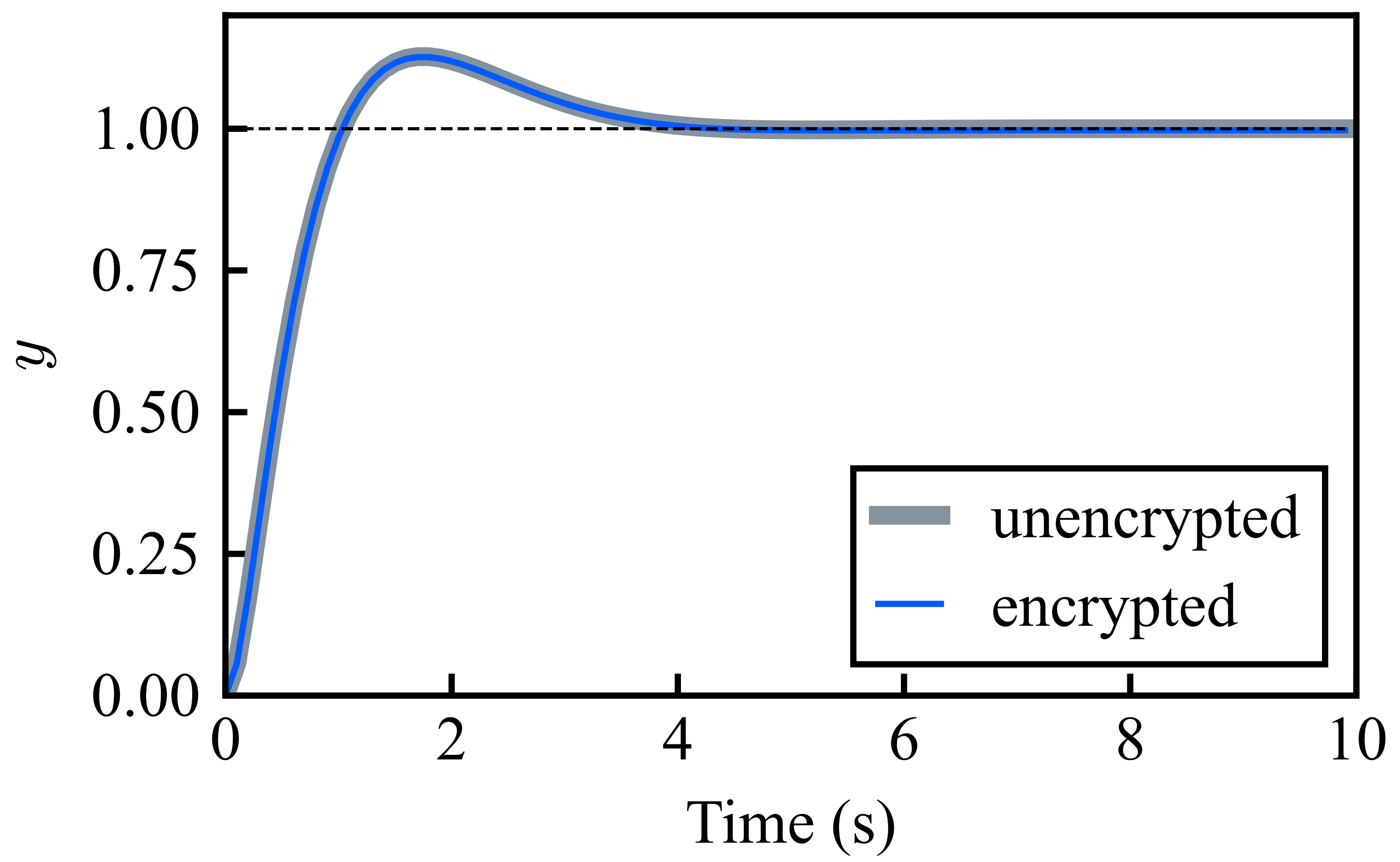
Line 115: Decryption of controller output and addition.

$$\psi_k = \mathsf{Dcd}_{\Delta^2}(\mathsf{Dec}^{+}(\Psi_{\mathsf{Enc}}))$$

Figures


```
130 plt.figure()
131 plt.plot(t, y, linestyle='-', color=gray, linewidth=3.0, label='unencrypted')
132 plt.plot(t, y_, linestyle='-', color=blue, linewidth=1.0, label='encrypted')
133 plt.plot(t, r, linestyle='--', color=black, linewidth=0.5)
134 plt.xlabel('Time (s)')
135 plt.ylabel(r'$y$')
136 plt.xlim(0, simulation_time)
137 plt.ylim(0, 1.2)
138 plt.legend(loc='lower right')
```

```
266 plt.show()
```



Line 130: Create a new figure.

Lines 131-133: Plot y, y_, and `r`.

Lines 134-137: Set labels and ranges of x- and y-axises.

Lines 138: Set a legend position.

Lines 266: Show the figure.

Conclusion

- ECLib is an OSS Python library for researchers and students.
- ECLib provides functions of homomorphic encryption to develop numerical simulation codes of encrypted control.
- Supported cryptosystems in ECLib are ElGamal encryption and Paillier encryption.
- We welcome your suggestions, bug reports, and contributions.
  https://github.com/KaoruTeranishi/EncryptedControl