Chapter 3

Problem Solving Paradigms

If all you have is a hammer, everything looks like a nail
— **Abraham Maslow, 1962**

3.1 Overview and Motivation

In this chapter, we discuss *four* problem solving paradigms commonly used to attack problems in programming contests, namely Complete Search (a.k.a Brute Force), Divide and Conquer, the Greedy approach, and Dynamic Programming. All competitive programmers, including IOI and ICPC contestants, need to master these problem solving paradigms (and more) in order to be able to attack a given problem with the appropriate 'tool'. Hammering *every* problem with Brute Force solutions will not enable anyone to perform well in contests. To illustrate, we discuss four simple tasks below involving an array `A` containing $n \leq 10K$ small integers $\leq 100K$ *(e.g.* `A` $= \{10, 7, 3, 5, 8, 2, 9\}$, $n = 7$*)* to give an overview of what happens if we attempt every problem with Brute Force as our sole paradigm.

1. Find the largest and the smallest element of `A`. *(10 and 2 for the given example).*
2. Find the k^{th} smallest element in `A`. *(if $k = 2$, the answer is 3 for the given example).*
3. Find the largest gap g such that $x, y \in$ `A` and $g = |x - y|$. *(8 for the given example).*
4. Find the longest increasing subsequence of `A`. *($\{3, 5, 8, 9\}$ for the given example).*

The answer for the first task is simple: Try each element of `A` and check if it is the current largest (or smallest) element seen so far. This is an $O(n)$ **Complete Search** solution.

The second task is a little harder. We can use the solution above to find the smallest value and replace it with a large value (e.g. $1M$) to 'delete' it. We can then proceed to find the smallest value again (the second smallest value in the original array) and replace it with $1M$. Repeating this process k times, we will find the k^{th} smallest value. This works, but if $k = \frac{n}{2}$ (the median), this Complete Search solution runs in $O(\frac{n}{2} \times n) = O(n^2)$. Instead, we can sort the array `A` in $O(n \log n)$, returning the answer simply as `A[k-1]`. However, a better solution for a small number of queries is the expected $O(n)$ solution shown in Section 9.29. The $O(n \log n)$ and $O(n)$ solutions above are **Divide and Conquer** solutions.

For the third task, we can similarly consider all possible two integers x and y in `A`, checking if the gap between them is the largest for each pair. This Complete Search approach runs in $O(n^2)$. It works, but is slow and inefficient. We can prove that g can be obtained by finding the difference between the smallest and largest elements of `A`. These two integers can be found with the solution of the first task in $O(n)$. No other combination of two integers in `A` can produce a larger gap. This is a **Greedy** solution.

For the fourth task, trying all $O(2^n)$ possible subsequences to find the longest increasing one is not feasible for all $n \leq 10K$. In Section 3.5.2, we discuss a simple $O(n^2)$ **Dynamic Programming** solution and also the faster $O(n \log k)$ Greedy solution for this task.

Here is some advice for this chapter: Please do not just memorize the solutions for each problem discussed, but instead remember and internalize the thought process and problem solving strategies used. Good problem solving skills are more important than memorized solutions for well-known Computer Science problems when dealing with (often creative and novel) contest problems.

3.2 Complete Search

The Complete Search technique, also known as brute force or recursive backtracking, is a method for solving a problem by traversing the entire (or part of the) search space to obtain the required solution. During the search, we are allowed to prune (that is, choose not to explore) parts of the search space if we have determined that these parts have no possibility of containing the required solution.

In programming contests, a contestant *should* develop a Complete Search solution when there is clearly no other algorithm available (e.g. the task of enumerating *all* permutations of $\{0, 1, 2, \ldots, N-1\}$ clearly requires $O(N!)$ operations) or when better algorithms exist, but are *overkill* as the input size happens to be small (e.g. the problem of answering Range Minimum Queries as in Section 2.4.3 but on static arrays with $N \leq 100$ is solvable with an $O(N)$ loop for each query).

In ICPC, Complete Search should be the first solution considered as it is usually easy to come up with such a solution and to code/debug it. Remember the 'KISS' principle: Keep It Short and Simple. A *bug-free* Complete Search solution should *never* receive the Wrong Answer (WA) response in programming contests as it explores the *entire* search space. However, many programming problems do have better-than-Complete-Search solutions as illustrated in the Section 3.1. Thus a Complete Search solution may receive a Time Limit Exceeded (TLE) verdict. With proper analysis, you can determine the likely outcome (TLE versus AC) before attempting to code anything (Table 1.4 in Section 1.2.3 is a good starting point). If a Complete Search is likely to pass the time limit, then go ahead and implement one. This will then give you more time to work on harder problems in which Complete Search will be too slow.

In IOI, you will usually need better problem solving techniques as Complete Search solutions are usually only rewarded with very small fractions of the total score in the subtask scoring schemes. Nevertheless, Complete Search should be used when you cannot come up with a better solution—it will at least enable you to score some marks.

Sometimes, running Complete Search on *small instances* of a challenging problem can help us to understand its structure through patterns in the output (it is possible to *visualize* the pattern for some problems) that can be exploited to design a faster algorithm. Some combinatorics problems in Section 5.4 can be solved this way. Then, the Complete Search solution can also act as a verifier for *small instances*, providing an additional check for the faster but non-trivial algorithm that you develop.

After reading this section, you may have the impression that Complete Search only works for 'easy problems' and it is usually not the intended solution for 'harder problems'. This is not entirely true. There exist hard problems that are only solvable with creative Complete Search algorithms. We have reserved those problems for Section 8.2.

In the next two sections, we give several *(easier)* examples of this simple yet possibly challenging paradigm. In Section 3.2.1, we give examples that are implemented *iteratively*. In Section 3.2.2, we give examples on solutions that are implemented *recursively* (with backtracking). Finally, in Section 3.2.3, we provide a few tips to give your solution, especially your Complete Search solution, a better chance to pass the required Time Limit.

3.2.1 Iterative Complete Search

Iterative Complete Search (Two Nested Loops: UVa 725 - Division)

Abridged problem statement: Find and display all pairs of 5-digit numbers that collectively use the digits 0 through 9 once each, such that the first number divided by the second is equal to an integer `N`, where $2 \leq N \leq 79$. That is, `abcde / fghij = N`, where each letter represents a different digit. The first digit of one of the numbers is allowed to be zero, e.g. for `N = 62`, we have `79546 / 01283 = 62; 94736 / 01528 = 62`.

Quick analysis shows that `fghij` can only range from `01234` to `98765` which is at most $\approx 100K$ possibilities. An even better bound for `fghij` is the range `01234` to `98765 / N`, which has at most $\approx 50K$ possibilities for $N = 2$ and becomes smaller with increasing N. For each attempted `fghij`, we can get `abcde` from `fghij * N` and then check if all 10 digits are different. This is a doubly-nested loop with a time complexity of at most $\approx 50K \times 10 = 500K$ operations per test case. This is small. Thus, an iterative Complete Search is feasible. The main part of the code is shown below (we use a fancy bit manipulation trick shown in Section 2.2 to determine digit uniqueness):

```
for (int fghij = 1234; fghij <= 98765 / N; fghij++) {
  int abcde = fghij * N;  // this way, abcde and fghij are at most 5 digits
  int tmp, used = (fghij < 10000); // if digit f=0, then we have to flag it
  tmp = abcde; while (tmp) { used |= 1 << (tmp % 10); tmp /= 10; }
  tmp = fghij; while (tmp) { used |= 1 << (tmp % 10); tmp /= 10; }
  if (used == (1<<10) - 1)              // if all digits are used, print it
    printf("%0.5d / %0.5d = %d\n", abcde, fghij, N);
}
```

Iterative Complete Search (Many Nested Loops: UVa 441 - Lotto)

In programming contests, problems that are solvable with a *single* loop are usually considered *easy*. Problems which require doubly-nested iterations like UVa 725 - Division above are more challenging but they are not necessarily considered difficult. Competitive programmers must be comfortable writing code with *more than two* nested loops.

Let's take a look at UVa 441 which can be summarized as follows: Given $6 < k < 13$ integers, enumerate all possible subsets of size 6 of these integers in sorted order.

Since the size of the required subset is always 6 and the output has to be sorted lexicographically (the input is already sorted), the easiest solution is to use *six* nested loops as shown below. Note that even in the largest test case when $k = 12$, these six nested loops will only produce ${}_{12}C_6 = 924$ lines of output. This is small.

```
for (int i = 0; i < k; i++)                          // input: k sorted integers
  scanf("%d", &S[i]);
for (int a = 0    ; a < k - 5; a++)                          // six nested loops!
  for (int b = a + 1; b < k - 4; b++)
    for (int c = b + 1; c < k - 3; c++)
      for (int d = c + 1; d < k - 2; d++)
        for (int e = d + 1; e < k - 1; e++)
          for (int f = e + 1; f < k    ; f++)
            printf("%d %d %d %d %d %d\n",S[a],S[b],S[c],S[d],S[e],S[f]);
```

Iterative Complete Search (Loops + Pruning: UVa 11565 - Simple Equations)

Abridged problem statement: Given three integers A, B, and C ($1 \leq A, B, C \leq 10000$), find three other distinct integers x, y, and z such that $x + y + z = A$, $x \times y \times z = B$, and $x^2 + y^2 + z^2 = C$.

The third equation $x^2 + y^2 + z^2 = C$ is a good starting point. Assuming that C has the largest value of 10000 and y and z are one and two (x, y, z have to be distinct), then the possible range of values for x is $[-100 \ldots 100]$. We can use the same reasoning to get a similar range for y and z. We can then write the following triply-nested iterative solution below that requires $201 \times 201 \times 201 \approx 8M$ operations per test case.

```
bool sol = false; int x, y, z;
for (x = -100; x <= 100; x++)
  for (y = -100; y <= 100; y++)
    for (z = -100; z <= 100; z++)
      if (y != x && z != x && z != y &&      // all three must be different
          x + y + z == A && x * y * z == B && x * x + y * y + z * z == C) {
        if (!sol) printf("%d %d %d\n", x, y, z);
        sol = true; }
```

Notice the way a short circuit AND was used to speed up the solution by enforcing a *lightweight* check on whether `x`, `y`, and `z` are all different *before* we check the three formulas. The code shown above already passes the required time limit for this problem, but we can do better. We can also use the second equation $x \times y \times z = B$ and assume that $x = y = z$ to obtain $x \times x \times x < B$ or $x < \sqrt[3]{B}$. The new range of x is $[-22 \ldots 22]$. We can also prune the search space by using `if` statements to execute only some of the (inner) loops, or use `break` and/or `continue` statements to stop/skip loops. The code shown below is now much faster than the code shown above (there are a few other optimizations required to solve the extreme version of this problem: UVa 11571 - Simple Equations - Extreme!!):

```
bool sol = false; int x, y, z;
for (x = -22; x <= 22 && !sol; x++) if (x * x <= C)
  for (y = -100; y <= 100 && !sol; y++) if (y != x && x * x + y * y <= C)
    for (z = -100; z <= 100 && !sol; z++)
      if (z != x && z != y &&
          x + y + z == A && x * y * z == B && x * x + y * y + z * z == C) {
        printf("%d %d %d\n", x, y, z);
        sol = true; }
```

Iterative Complete Search (Permutations: UVa 11742 - Social Constraints)

Abridged problem statement: There are $0 < n \leq 8$ movie goers. They will sit in the front row in n consecutive open seats. There are $0 \leq m \leq 20$ seating constraints among them, i.e. movie goer **a** and movie goer **b** must be at most (or at least) **c** seats apart. The question is simple: How many possible seating arrangements are there?

The key part to solve this problem is in realizing that we have to explore **all** permutations (seating arrangements). Once we realize this fact, we can derive this simple $O(m \times n!)$ 'filtering' solution. We set `counter = 0` and then try all possible $n!$ permutations. We increase the `counter` by 1 if the current permutation satisfies all m constraints. When all $n!$ permutations have been examined, we output the final value of `counter`. As the maximum

n is 8 and maximum m is 20, the largest test case will still only require $20 \times 8! = 806400$ operations—a perfectly viable solution.

If you have never written an algorithm to generate all permutations of a set of numbers (see **Exercise 1.2.3**, task 7), you may still be unsure about how to proceed. The simple C++ solution is shown below.

```
#include <algorithm>             // next_permutation is inside this C++ STL
// the main routine
int i, n = 8, p[8] = {0, 1, 2, 3, 4, 5, 6, 7};     // the first permutation
do {   // try all possible O(n!) permutations, the largest input 8! = 40320
  ...              // check the given social constraint based on 'p' in O(m)
}                           // the overall time complexity is thus O(m * n!)
while (next_permutation(p, p + n));   // this is inside C++ STL <algorithm>
```

Iterative Complete Search (Subsets: UVa 12455 - Bars)

Abridged problem statement[1]: Given a list `l` containing $1 \leq n \leq 20$ integers, is there a subset of list `l` that sums to another given integer X?

We can try all 2^n possible subsets of integers, sum the selected integers for each subset in $O(n)$, and see if the sum of the selected integers equals to X. The overall time complexity is thus $O(n \times 2^n)$. For the largest test case when $n = 20$, this is just $20 \times 2^{20} \approx 21M$. This is 'large' but still viable (for reason described below).

If you have never written an algorithm to generate all subsets of a set of numbers (see **Exercise 1.2.3**, task 8), you may still be unsure how to proceed. An easy solution is to use the *binary representation* of integers from 0 to $2^n - 1$ to describe all possible subsets. If you are not familiar with bit manipulation techniques, see Section 2.2. The solution can be written in simple C/C++ shown below (also works in Java). Since bit manipulation operations are (very) fast, the required $21M$ operations for the largest test case are still doable in under a second. Note: A faster implementation is possible (see Section 8.2.1).

```
// the main routine, variable 'i' (the bitmask) has been declared earlier
for (i = 0; i < (1 << n); i++) {                  // for each subset, O(2^n)
  sum = 0;
  for (int j = 0; j < n; j++)                       // check membership, O(n)
    if (i & (1 << j))        // test if bit 'j' is turned on in subset 'i'?
      sum += l[j];                                    // if yes, process 'j'
  if (sum == X) break;                  // the answer is found: bitmask 'i'
}
```

Exercise 3.2.1.1: For the solution of UVa 725, why is it better to iterate through `fghij` and not through `abcde`?

Exercise 3.2.1.2: Does a 10! algorithm that permutes `abcdefghij` work for UVa 725?

Exercise 3.2.1.3*: Java does *not* have a built-in `next_permutation` function yet. If you are a Java user, write your own recursive backtracking routine to generate all permutations! This is similar to the recursive backtracking for the 8-Queens problem.

Exercise 3.2.1.4*: How would you solve UVa 12455 if $1 \leq n \leq \underline{30}$ and each integer can be as big as 1000000000? Hint: See Section 8.2.4.

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[1]This is also known as the 'Subset Sum' problem, see Section 3.5.3.

3.2.2 Recursive Complete Search

Simple Backtracking: UVa 750 - 8 Queens Chess Problem

Abridged problem statement: In chess (with an 8×8 board), it is possible to place eight queens on the board such that no two queens attack each other. Determine *all* such possible arrangements given the position of one of the queens (i.e. coordinate (a, b) must contain a queen). Output the possibilities in lexicographical (sorted) order.

The most naïve solution is to enumerate all combinations of 8 different cells out of the $8 \times 8 = 64$ possible cells in a chess board and see if the 8 queens can be placed at these positions without conflicts. However, there are ${}_{64}C_8 \approx 4B$ such possibilities—this idea is not even worth trying.

A better but still naïve solution is to realize that each queen can only occupy one column, so we can put exactly one queen in each column. There are only $8^8 \approx 17M$ possibilities now, down from $4B$. This is still a 'borderline'-passing solution for this problem. If we write a Complete Search like this, we are likely to receive the Time Limit Exceeded (TLE) verdict especially if there are multiple test cases. We can still apply the few more easy optimizations described below to further reduce the search space.

We know that no two queens can share the same column *or the same row*. Using this, we can further simplify the original problem to the problem of finding valid *permutations* of 8! row positions. The value of `row[i]` describes the row position of the queen in column `i`. Example: `row = {1, 3, 5, 7, 2, 0, 6, 4}` as in Figure 3.1 is one of the solutions for this problem; `row[0] = 1` implies that the queen in column 0 is placed in row 1, and so on (the index starts from 0 in this example). Modeled this way, the search space goes *down* from $8^8 \approx 17M$ to $8! \approx 40K$. This solution is already fast enough, but we can still do more.

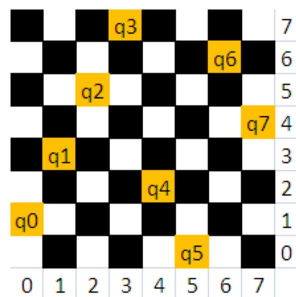


Figure 3.1: 8-Queens

We also know that no two queens can share any of the two diagonal lines. Let queen A be at `(i, j)` and queen B be at `(k, l)`. They attack each other iff `abs(i-k) == abs(j-l)`. This formula means that the vertical and horizontal distances between these two queens are equal, i.e. queen A and B lie on one of each other's two diagonal lines.

A *recursive backtracking* solution places the queens one by one in columns 0 to 7, observing all the constraints above. Finally, if a candidate solution is found, check if at least one of the queens satisfies the input constraints, i.e. `row[b] == a`. This *sub* (i.e. lower than) $O(n!)$ solution will obtain an AC verdict.

We provide our implementation below. If you have never written a recursive backtracking solution before, please scrutinize it and perhaps re-code it in your own coding style.

```
#include <cstdlib>                            // we use the int version of 'abs'
#include <cstdio>
#include <cstring>
using namespace std;

int row[8], TC, a, b, lineCounter;                  // ok to use global variables

bool place(int r, int c) {
  for (int prev = 0; prev < c; prev++)     // check previously placed queens
    if (row[prev] == r || (abs(row[prev] - r) == abs(prev - c)))
      return false;         // share same row or same diagonal -> infeasible
  return true; }
```

```
void backtrack(int c) {
  if (c == 8 && row[b] == a) {         // candidate sol, (a, b) has 1 queen
    printf("%2d      %d", ++lineCounter, row[0] + 1);
    for (int j = 1; j < 8; j++) printf(" %d", row[j] + 1);
    printf("\n"); }
  for (int r = 0; r < 8; r++)                           // try all possible row
    if (place(r, c)) {         // if can place a queen at this col and row
      row[c] = r; backtrack(c + 1);      // put this queen here and recurse
}   }

int main() {
  scanf("%d", &TC);
  while (TC--) {
    scanf("%d %d", &a, &b); a--; b--;         // switch to 0-based indexing
    memset(row, 0, sizeof row); lineCounter = 0;
    printf("SOLN       COLUMN\n");
    printf(" #      1 2 3 4 5 6 7 8\n\n");
    backtrack(0);          // generate all possible 8! candidate solutions
    if (TC) printf("\n");
} } // return 0;
```

Source code: `ch3_01_UVa750.cpp/java`

More Challenging Backtracking: UVa 11195 - Another n-Queen Problem

Abridged problem statement: Given an $n \times n$ chessboard ($3 < n < \mathbf{15}$) where some of the cells are bad (queens cannot be placed on those bad cells), how many ways can you place n queens in the chessboard so that no two queens attack each other? Note: Bad cells *cannot* be used to block queens' attack.

The recursive backtracking code that we have presented above is *not* fast enough for $n = 14$ and no bad cells, the worst possible test case for this problem. The *sub*-$O(n!)$ solution presented earlier is still OK for $n = 8$ but not for $n = 14$. We have to do better.

The major issue with the previous n-queens code is that it is quite slow when checking whether the position of a new queen is valid since we compare the new queen's position with the previous `c-1` queens' positions (see function `bool place(int r, int c)`). It is better to store the same information with three boolean arrays (we use `bitset`s for now):

```
bitset<30> rw, ld, rd;      // for the largest n = 14, we have 27 diagonals
```

Initially all n rows (`rw`), $2 \times n - 1$ left diagonals (`ld`), and $2 \times n - 1$ right diagonals (`rd`) are unused (these three `bitset`s are all set to `false`). When a queen is placed at cell `(r, c)`, we flag `rw[r] = true` to disallow this row from being used again. Furthermore, all `(a, b)` where `abs(r - a) = abs(c - b)` also cannot be used anymore. There are two possibilities after removing the `abs` function: `r - c = a - b` and `r + c = a + b`. Note that `r + c` and `r - c` represent indices for the two diagonal axes. As `r - c` can be negative, we add an *offset* of `n - 1` to both sides of the equation so that `r - c + n - 1 = a - b + n - 1`. If a queen is placed on cell `(r, c)`, we flag `ld[r - c + n - 1] = true` and `rd[r + c] = true` to disallow these two diagonals from being used again. With these additional data structures and the additional problem-specific constraint in UVa 11195 (`board[r][c]` cannot be a bad cell), we can extend our code to become:

```
void backtrack(int c) {
  if (c == n) { ans++; return; }                              // a solution
  for (int r = 0; r < n; r++)                       // try all possible row
    if (board[r][c] != '*' && !rw[r] && !ld[r - c + n - 1] && !rd[r + c]) {
      rw[r] = ld[r - c + n - 1] = rd[r + c] = true;             // flag off
      backtrack(c + 1);
      rw[r] = ld[r - c + n - 1] = rd[r + c] = false;             // restore
}   }
```

Visualization: `www.comp.nus.edu.sg/∼stevenha/visualization/recursion.html`

Exercise 3.2.2.1: The code shown for UVa 750 can be further optimized by pruning the search when '`row[b] != a`' *earlier* during the recursion (not only when c == 8). Modify it!

Exercise 3.2.2.2*: Unfortunately, the updated solution presented using `bitsets`: `rw`, `ld`, and `rd` will still obtain a TLE for UVa 11195 - Another n-Queen Problem. We need to further speed up the solution using bitmask techniques and another way of using the left and right diagonal constraints. This solution will be discussed in Section 8.2.1. For now, use the (non Accepted) idea presented here for UVa 11195 to speed up the code for UVa 750 and two more similar problems: UVa 167 and 11085!

3.2.3 Tips

The biggest gamble in writing a Complete Search solution is whether it will or will not be able to pass the time limit. If the time limit is 10 seconds (online judges do not usually use large time limits for efficient judging) and your program currently runs in $\approx$ 10 seconds on several (can be more than one) test cases with the largest input size as specified in the problem description, yet your code is still judged to be TLE, you may want to tweak the 'critical code'[2] in your program instead of re-solving the problem with a faster algorithm which may not be easy to design.

Here are some tips that you may want to consider when designing your Complete Search solution for a certain problem to give it a higher chance of passing the Time Limit. Writing a good Complete Search solution is an art in itself.

Tip 1: Filtering versus Generating

Programs that examine lots of (if not all) candidate solutions and choose the ones that are correct (or remove the incorrect ones) are called 'filters', e.g. the naïve 8-queens solver with ${}_{64}C_8$ and 8^8 time complexity, the iterative solution for UVa 725 and UVa 11742, etc. Usually 'filter' programs are written iteratively.

Programs that gradually build the solutions and immediately prune invalid partial solutions are called 'generators', e.g. the improved recursive 8-queens solver with its *sub*-$O(n!)$ complexity plus diagonal checks. Usually, 'generator' programs are easier to implement when written recursively as it gives us greater flexibility for pruning the search space.

Generally, filters are easier to code but run slower, given that it is usually far more difficult to prune more of the search space iteratively. Do the math (complexity analysis) to see if a filter is good enough or if you need to create a generator.

[2]It is said that every program spends most of its time in only about 10% of its code—the critical code.

Tip 2: Prune Infeasible/Inferior Search Space Early

When generating solutions using recursive backtracking (see the tip no 1 above), we may encounter a partial solution that will never lead to a full solution. We can prune the search there and explore other parts of the search space. Example: The diagonal check in the 8-queens solution above. Suppose we have placed a queen at `row[0] = 2`. Placing the next queen at `row[1] = 1` or `row[1] = 3` will cause a diagonal conflict and placing the next queen at `row[1] = 2` will cause a row conflict. Continuing from any of these infeasible partial solutions will never lead to a valid solution. Thus we can prune these partial solutions at this juncture and concentrate only on the other valid positions: `row[1] = {0, 4, 5, 6, 7}`, thus reducing the overall runtime. As a rule of thumb, the earlier you can prune the search space, the better.

In other problems, we may be able to compute the 'potential worth' of a partial (and still valid) solution. If the potential worth is inferior to the worth of the current best found valid solution so far, we can prune the search there.

Tip 3: Utilize Symmetries

Some problems have symmetries and we should try to exploit symmetries to reduce execution time! In the 8-queens problem, there are 92 solutions but there are only 12 unique (or fundamental/canonical) solutions as there are rotational and line symmetries in the problem. You can utilize this fact by only generating the 12 unique solutions and, if needed, generate the whole 92 by rotating and reflecting these 12 unique solutions. Example: `row = {7-1, 7-3, 7-5, 7-7, 7-2, 7-0, 7-6, 7-4} = {6, 4, 2, 0, 5, 7, 1, 3}` is the horizontal reflection of the configuration in Figure 3.1.

However, we have to remark that it is true that sometimes considering symmetries can actually complicate the code. In competitive programming, this is usually not the best way (we want shorter code to minimize bugs). If the gain obtained by dealing with symmetry is not significant in solving the problem, just ignore this tip.

Tip 4: Pre-Computation a.k.a. Pre-Calculation

Sometimes it is helpful to generate tables or other data structures that accelerate the lookup of a result prior to the execution of the program itself. This is called Pre-Computation, in which one trades memory/space for time. However, this technique can rarely be used for recent programming contest problems.

For example, since we know that there are only 92 solutions in the standard 8-queens chess problem, we can create a 2D array `int solution[92][8]` and then fill it with all 92 valid permutations of the 8-queens row positions! That is, we can create a generator program (which takes some time to run) to fill this 2D array `solution`. Afterwards, we can write *another* program to simply and quickly print the correct permutations within the 92 pre-calculated configurations that satisfy the problem constraints.

Tip 5: Try Solving the Problem Backwards

Some contest problems look far easier when they are solved 'backwards' [53] (from a *less obvious* angle) than when they are solved using a frontal attack (from the more obvious angle). Be prepared to attempt unconventional approaches to problems.

This tip is best illustrated using an example: UVa 10360 - Rat Attack: Imagine a 2D array (up to 1024×1024) containing rats. There are $n \leq 20000$ rats spread across the cells. Determine which cell `(x, y)` should be gas-bombed so that the number of rats killed in a square box `(x-d, y-d)` to `(x+d, y+d)` is maximized. The value d is the power of the gas-bomb ($d \leq 50$), see Figure 3.2.

An immediate solution is to attack this problem in the most obvious fashion possible: bomb each of the 1024^2 cells and select the most effective location. For each bombed cell `(x, y)`, we can perform an $O(d^2)$ scan to count the number of rats killed within the square-bombing radius. For the worst case, when the array has size 1024^2 and $d = 50$, this takes $1024^2 \times 50^2 = 2621M$ operations. TLE[3]!

Another option is to attack this problem backwards: Create an array `int killed[1024][1024]`. For each rat population at coordinate `(x, y)`, add it to `killed[i][j]`, where $|i-x| \leq d$ and $|j-y| \leq d$. This is because if a bomb was placed at `(i, j)`, the rats at coordinate `(x, y)` will be killed. This pre-processing takes $O(n \times d^2)$ operations. Then, to determine the most optimal bombing position, we can simply find the coordinate of the highest entry in array `killed`, which can be done in 1024^2 operations. This approach only requires $20000 \times 50^2 + 1024^2 = 51M$ operations for the worst test case ($n = 20000, d = 50$), ≈ 51 times faster than the frontal attack! This is an AC solution.

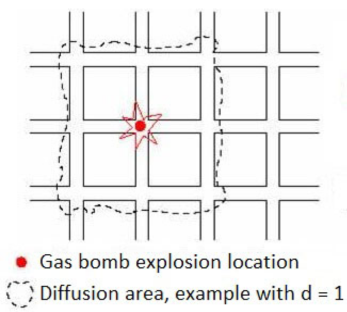


Figure 3.2: UVa 10360 [47]

Tip 6: Optimizing Your Source Code

There are many tricks that you can use to optimize your code. Understanding computer hardware and how it is organized, especially the I/O, memory, and cache behavior, can help you design better code. Some examples (not exhaustive) are shown below:

1. A biased opinion: Use C++ instead of Java. An algorithm implemented using C++ usually runs faster than the one implemented in Java in many online judges, including UVa [47]. Some programming contests give Java users extra time to account for the difference in performance.

2. For C/C++ users, use the faster C-style `scanf/printf` rather than `cin/cout`. For Java users, use the faster `BufferedReader/BufferedWriter` classes as follows:

```
BufferedReader br = new BufferedReader(                    // speedup
  new InputStreamReader(System.in));
// Note: String splitting and/or input parsing is needed afterwards

PrintWriter pr = new PrintWriter(new BufferedWriter(        // speedup
  new OutputStreamWriter(System.out)));
// PrintWriter allows us to use the pr.printf() function
// do not forget to call pr.close() before exiting your Java program
```

3. Use the *expected* $O(n \log n)$ but cache-friendly quicksort in C++ STL `algorithm::sort` (part of 'introsort') rather than the true $O(n \log n)$ but non cache-friendly heapsort (its root-to-leaf/leaf-to-root operations span a wide range of indices—lots of cache misses).

4. Access a 2D array in a row major fashion (row by row) rather than in a column major fashion—multidimensional arrays are stored in a row-major order in memory.

[3]Although 2013 CPU can compute $\approx 100M$ operations in a few seconds, $2621M$ operations will still take too long in a contest environment.

5. Bit manipulation on the built-in integer data types (up to the 64-bit integer) is more efficient than index manipulation in an array of booleans (see bitmask in Section 2.2). If we need more than 64 bits, use the C++ STL `bitset` rather than `vector<bool>` (e.g. for Sieve of Eratosthenes in Section 5.5.1).

6. Use lower level data structures/types at all times if you do not need the extra functionality in the higher level (or larger) ones. For example, use an `array` with a slightly larger size than the maximum size of input instead of using resizable `vector`s. Also, use 32-bit `int`s instead of 64-bit `long long`s as the 32-bit `int` is faster in most 32-bit online judge systems.

7. For Java, use the faster `ArrayList` (and `StringBuilder`) rather than `Vector` (and `StringBuffer`). Java `Vector`s and `StringBuffer`s are *thread safe* but this feature is not needed in competitive programming. Note: In this book, we will stick with `Vector`s to avoid confusing bilingual C++ and Java readers who use both the C++ STL `vector` and Java `Vector`.

8. Declare most data structures (especially the bulky ones, e.g. large arrays) once by placing them in global scope. Allocate enough memory to deal with the largest input of the problem. This way, we do not have to pass the data structures around as function arguments. For problems with multiple test cases, simply clear/reset the contents of the data structure before dealing with each test case.

9. When you have the option to write your code either iteratively or recursively, choose the iterative version. Example: The iterative C++ STL `next_permutation` and iterative subset generation techniques using bitmask shown in Section 3.2.1 are (far) faster than if you write similar routines recursively (mainly due to overheads in function calls).

10. Array access in (nested) loops can be slow. If you have an array `A` and you frequently access the value of `A[i]` (without changing it) in (nested) loops, it may be beneficial to use a local variable `temp = A[i]` and works with `temp` instead.

11. In C/C++, *appropriate* usage of macros or inline functions can reduce runtime.

12. For C++ users: Using C-style character arrays will yield faster execution than when using the C++ STL string. For Java users: Be careful with `String` manipulation as Java `String` objects are immutable. Operations on Java `String`s can thus be very slow. Use Java `StringBuilder` instead.

Browse the Internet or relevant books (e.g. [69]) to find (much) more information on how to speed up your code. Practice this 'code hacking skill' by choosing a harder problem in UVa online judge where the runtime of the best solution is not 0.000s. Submit several variants of your Accepted solution and check the runtime differences. Adopt hacking modification that consistently gives you faster runtime.

Tip 7: Use Better Data Structures & Algorithms :)

No kidding. Using better data structures and algorithms will always outperform any optimizations mentioned in Tips 1-6 above. If you are sure that you have written your fastest Complete Search code, but it is still judged as TLE, abandon the Complete Search approach.

Remarks About Complete Search in Programming Contests

The main source of the 'Complete Search' material in this chapter is the USACO training gateway [48]. We have adopted the name 'Complete Search' rather than 'Brute-Force' (with its negative connotations) as we believe that some Complete Search solutions can be clever and fast. We feel that the term 'clever Brute-Force' is also a little self-contradictory.

If a problem is solvable by Complete Search, it will also be clear when to use the iterative or recursive backtracking approaches. Iterative approaches are used when one can derive the different states *easily* with some formula relative to a certain *counter* and (almost) all states have to be checked, e.g. scanning all the indices of an array, enumerating (almost) all possible subsets of a small set, generating (almost) all permutations, etc. Recursive Backtracking is used when it is hard to derive the different states with a simple index and/or one also wants to (heavily) prune the search space, e.g. the 8-queens chess problem. If the search space of a problem that is solvable with Complete Search is large, then recursive backtracking approaches that allow early pruning of infeasible sections of the search space are usually used. Pruning in iterative Complete Searches is not impossible but usually difficult.

The best way to improve your Complete Search skills is to solve more Complete Search problems. We have provided a list of such problems, separated into several categories below. Please attempt as many as possible, especially those that are highlighted with the **must try *** indicator. Later in Section 3.5, readers will encounter further examples of recursive backtracking, but with the addition of the 'memoization' technique.

Note that we will discuss some *more* advanced search techniques later in Section 8.2, e.g. using bit manipulation in recursive backtracking, harder state-space search, Meet in the Middle, A* Search, Depth Limited Search (DLS), Iterative Deepening Search (IDS), and Iterative Deepening A* (IDA*).

Programming Exercises solvable using Complete Search:

- Iterative (One Loop, Linear Scan)
  1. UVa 00102 - Ecological Bin Packing (just try all 6 possible combinations)
  2. UVa 00256 - Quirksome Squares (brute force, math, pre-calculate-able)
  3. **UVa 00927 - Integer Sequence from ... *** (use sum of arithmetic series)
  4. **UVa 01237 - Expert Enough *** (LA 4142, Jakarta08, input is small)
  5. **UVa 10976 - Fractions Again ? *** (total solutions is asked upfront; therefore do brute force twice)
  6. UVa 11001 - Necklace (brute force math, maximize function)
  7. UVa 11078 - Open Credit System (one linear scan)
- Iterative (Two Nested Loops)
  1. UVa 00105 - The Skyline Problem (height map, sweep left-right)
  2. UVa 00347 - Run, Run, Runaround ... (simulate the process)
  3. UVa 00471 - Magic Numbers (somewhat similar to UVa 725)
  4. UVa 00617 - Nonstop Travel (try all integer speeds from 30 to 60 mph)
  5. UVa 00725 - Division (elaborated in this section)
  6. **UVa 01260 - Sales *** (LA 4843, Daejeon10, check all)
  7. UVa 10041 - Vito's Family (try all possible location of Vito's House)
  8. **UVa 10487 - Closest Sums *** (sort and then do $O(n^2)$ pairings)

9. *UVa 10730 - Antiarithmetic?* (2 nested loops with pruning can pass possibly pass the weaker test cases; note that this brute force solution is too slow for the larger test data generated in the solution of UVa 11129)
10. **UVa 11242 - Tour de France *** (plus sorting)
11. *UVa 12488 - Start Grid* (2 nested loops; simulate overtaking process)
12. *UVa 12583 - Memory Overflow* (2 nested loops; be careful of overcounting)

- Iterative (Three Or More Nested Loops, Easier)
  1. UVa 00154 - Recycling (3 nested loops)
  2. UVa 00188 - Perfect Hash (3 nested loops, until the answer is found)
  3. **UVa 00441 - Lotto *** (6 nested loops)
  4. UVa 00626 - Ecosystem (3 nested loops)
  5. UVa 00703 - Triple Ties: The ... (3 nested loops)
  6. **UVa 00735 - Dart-a-Mania *** (3 nested loops, then count)
  7. **UVa 10102 - The Path in the ... *** (4 nested loops will do, we do not need BFS; get max of minimum Manhattan distance from a '1' to a '3'.)
  8. UVa 10502 - Counting Rectangles (6 nested loops, rectangle, not too hard)
  9. UVa 10662 - The Wedding (3 nested loops)
  10. UVa 10908 - Largest Square (4 nested loops, square, not too hard)
  11. UVa 11059 - Maximum Product (3 nested loops, input is small)
  12. *UVa 11975 - Tele-loto* (3 nested loops, simulate the game as asked)
  13. *UVa 12498 - Ant's Shopping Mall* (3 nested loops)
  14. *UVa 12515 - Movie Police* (3 nested loops)

- Iterative (Three-or-More Nested Loops, Harder)
  1. UVa 00253 - Cube painting (try all, similar problem in UVa 11959)
  2. UVa 00296 - Safebreaker (try all 10000 possible codes, 4 nested loops, use similar solution as 'Master-Mind' game)
  3. UVa 00386 - Perfect Cubes (4 nested loops with pruning)
  4. UVa 10125 - Sumsets (sort; 4 nested loops; plus binary search)
  5. UVa 10177 - (2/3/4)-D Sqr/Rects/... (2/3/4 nested loops, precalculate)
  6. UVa 10360 - Rat Attack (also solvable using 1024^2 DP max sum)
  7. UVa 10365 - Blocks (use 3 nested loops with pruning)
  8. *UVa 10483 - The Sum Equals ...* (2 nested loops for a, b, derive c from a, b; there are 354 answers for range [0.01 .. 255.99]; similar with UVa 11236)
  9. **UVa 10660 - Citizen attention ... *** (7 nested loops, Manhattan distance)
  10. UVa 10973 - Triangle Counting (3 nested loops with pruning)
  11. UVa 11108 - Tautology (5 nested loops, try all $2^5 = 32$ values with pruning)
  12. ***UVa 11236 - Grocery Store **** (3 nested loops for a, b, c; derive d from a, b, c; check if you have 949 lines of output)
  13. UVa 11342 - Three-square (pre-calculate squared values from 0^2 to 224^2, use 3 nested loops to generate the answers; use `map` to avoid duplicates)
  14. *UVa 11548 - Blackboard Bonanza* (4 nested loops, string, pruning)
  15. **UVa 11565 - Simple Equations *** (3 nested loops with pruning)
  16. UVa 11804 - Argentina (5 nested loops)
  17. UVa 11959 - Dice (try all possible dice positions, compare with the 2nd one)
  Also see Mathematical Simulation in Section 5.2

- Iterative (Fancy Techniques)
  1. UVa 00140 - Bandwidth (max n is just 8, use `next_permutation`; the algorithm inside `next_permutation` is iterative)
  2. *UVa 00234 - Switching Channels* (use `next_permutation`, simulation)
  3. UVa 00435 - Block Voting (only 2^{20} possible coalition combinations)
  4. UVa 00639 - Don't Get Rooked (generate 2^{16} combinations and prune)
  5. ***UVa 01047 - Zones **** (LA 3278, WorldFinals Shanghai05, notice that $n \leq 20$ so that we can try all possible subsets of towers to be taken; then apply inclusion-exclusion principle to avoid overcounting)
  6. *UVa 01064 - Network* (LA 3808, WorldFinals Tokyo07, permutation of up to 5 messages, simulation, mind the word 'consecutive')
  7. UVa 11205 - The Broken Pedometer (try all 2^{15} bitmask)
  8. UVa 11412 - Dig the Holes (`next_permutation`, find one possibility from 6!)
  9. **UVa 11553 - Grid Game *** (solve by trying all n! permutations; you can also use DP + bitmask, see Section 8.3.1, but it is overkill)
  10. UVa 11742 - Social Constraints (discussed in this section)
  11. *UVa 12249 - Overlapping Scenes* (LA 4994, KualaLumpur10, try all permutations, a bit of string matching)
  12. *UVa 12346 - Water Gate Management* (LA 5723, Phuket11, try all 2^n combinations, pick the best one)
  13. *UVa 12348 - Fun Coloring* (LA 5725, Phuket11, try all 2^n combinations)
  14. *UVa 12406 - Help Dexter* (try all 2^p possible bitmasks, change '0's to '2's)
  15. ***UVa 12455 - Bars **** (discussed in this section)
- Recursive Backtracking (Easy)
  1. UVa 00167 - The Sultan Successor (8-queens chess problem)
  2. UVa 00380 - Call Forwarding (simple backtracking, but we have to work with strings, see Section 6.2)
  3. UVa 00539 - The Settlers ... (longest simple path in a *small* general graph)
  4. **UVa 00624 - CD *** (input size is small, backtracking is enough)
  5. UVa 00628 - Passwords (backtracking, follow the rules in description)
  6. UVa 00677 - All Walks of length "n" ... (print all solutions with backtracking)
  7. UVa 00729 - The Hamming Distance ... (generate all possible bit strings)
  8. UVa 00750 - 8 Queens Chess Problem (discussed in this section with sample source code)
  9. UVa 10276 - Hanoi Tower Troubles Again (insert a number one by one)
  10. UVa 10344 - 23 Out of 5 (rearrange the 5 operands and the 3 operators)
  11. UVa 10452 - Marcus, help (at each pos, Indy can go forth/left/right; try all)
  12. **UVa 10576 - Y2K Accounting Bug *** (generate all, prune, take max)
  13. **UVa 11085 - Back to the 8-Queens *** (see UVa 750, pre-calculation)
- Recursive Backtracking (Medium)
  1. UVa 00222 - Budget Travel (looks like a DP problem, but the state cannot be memoized as 'tank' is floating-point; fortunately, the input is not large)
  2. *UVa 00301 - Transportation* (2^{22} with pruning is possible)
  3. UVa 00331 - Mapping the Swaps ($n \leq 5$...)
  4. UVa 00487 - Boggle Blitz (use `map` to store the generated words)
  5. **UVa 00524 - Prime Ring Problem *** (also see Section 5.5.1)

6. UVa 00571 - Jugs (solution can be suboptimal, add flag to avoid cycling)
7. **UVa 00574 - Sum It Up *** (print all solutions with backtracking)
8. UVa 00598 - Bundling Newspaper (print all solutions with backtracking)
9. *UVa 00775 - Hamiltonian Cycle* (backtracking suffices because the search space cannot be that big; in a dense graph, it is more likely to have a Hamiltonian cycle, so we can prune early; we do NOT have to find the best one like in TSP problem)
10. *UVa 10001 - Garden of Eden* (the upperbound of 2^{32} is scary but with efficient pruning, we can pass the time limit as the test case is not extreme)
11. *UVa 10063 - Knuth's Permutation* (do as asked)
12. *UVa 10460 - Find the Permuted String* (similar nature with UVa 10063)
13. UVa 10475 - Help the Leaders (generate and prune; try all)
14. **UVa 10503 - The dominoes solitaire *** (max 13 spaces only)
15. *UVa 10506 - Ouroboros* (any valid solution is AC; generate all possible next digit (up to base 10/digit [0..9]); check if it is still a valid Ouroboros sequence)
16. *UVa 10950 - Bad Code* (sort the input; run backtracking; the output should be sorted; only display the first 100 sorted output)
17. UVa 11201 - The Problem with the ... (backtracking involving strings)
18. *UVa 11961 - DNA* (there are at most 4^{10} possible DNA strings; moreover, the mutation power is at most $K \leq 5$ so the search space is much smaller; sort the output and then remove duplicates)

- Recursive Backtracking (Harder)

1. *UVa 00129 - Krypton Factor* (backtracking, string processing check, a bit of output formatting)
2. UVa 00165 - Stamps (requires some DP too; can be pre-calculated)
3. **UVa 00193 - Graph Coloring *** (Max Independent Set, input is small)
4. UVa 00208 - Firetruck (backtracking with some pruning)
5. **UVa 00416 - LED Test *** (backtrack, try all)
6. UVa 00433 - Bank (Not Quite O.C.R.) (similar to UVa 416)
7. UVa 00565 - Pizza Anyone? (backtracking with lots of pruning)
8. *UVa 00861 - Little Bishops* (backtracking with pruning as in 8-queens recursive backtracking solution; then pre-calculate the results)
9. UVa 00868 - Numerical maze (try row 1 to N; 4 ways; some constraints)
10. ***UVa 01262 - Password **** (LA 4845, Daejeon10, sort the columns in the two 6×5 grids first so that we can process common passwords in lexicographic order; backtracking; important: skip two similar passwords)
11. UVa 10094 - Place the Guards (this problem is like the n-queens chess problem, but must find/use the pattern!)
12. *UVa 10128 - Queue* (backtracking with pruning; try up to all N! (13!) permutations that satisfy the requirement; then pre-calculate the results)
13. UVa 10582 - ASCII Labyrinth (simplify complex input first; then backtrack)
14. *UVa 11090 - Going in Cycle* (minimum mean weight cycle problem; solvable with backtracking with important pruning when current running mean is greater than the best found mean weight cycle cost)

3.3 Divide and Conquer

Divide and Conquer (abbreviated as D&C) is a problem-solving paradigm in which a problem is made *simpler* by 'dividing' it into smaller parts and then conquering each part. The steps:

1. Divide the original problem into *sub*-problems—usually by half or nearly half,
2. Find (sub)-solutions for each of these sub-problems—which are now easier,
3. If needed, combine the sub-solutions to get a complete solution for the main problem.

We have seen examples of the D&C paradigm in the previous sections of this book: Various sorting algorithms (e.g. Quick Sort, Merge Sort, Heap Sort) and Binary Search in Section 2.2 utilize this paradigm. The way data is organized in Binary Search Tree, Heap, Segment Tree, and Fenwick Tree in Section 2.3, 2.4.3, and 2.4.4 also relies upon the D&C paradigm.

3.3.1 Interesting Usages of Binary Search

In this section, we discuss the D&C paradigm in the well-known Binary Search algorithm. We classify Binary Search as a 'Divide' and Conquer algorithm although one reference [40] suggests that it should be actually classified as 'Decrease (by-half)' and Conquer as it does not actually 'combine' the result. We highlight this algorithm because many contestants know it, but not many are aware that it can be used in many other non-obvious ways.

Binary Search: The Ordinary Usage

Recall that the *canonical* usage of Binary Search is searching for an item in a *static sorted array*. We check the middle of the sorted array to determine if it contains what we are looking for. If it is or there are no more items to consider, stop. Otherwise, we can decide whether the answer is to the left or right of the middle element and continue searching. As the size of search space is halved (in a binary fashion) after each check, the complexity of this algorithm is $O(\log n)$. In Section 2.2, we have seen that there are built-in library routines for this algorithm, e.g. the C++ STL `algorithm::lower_bound` (and the Java `Collections.binarySearch`).

This is *not* the only way to use binary search. The pre-requisite for performing a binary search—a *static sorted sequence (array or vector)*—can also be found in other uncommon data structures such as in the root-to-leaf path of a tree (not necessarily binary nor complete) that satisfies the *min heap* property. This variant is discussed below.

Binary Search on Uncommon Data Structures

This original problem is titled 'My Ancestor' and was used in the Thailand ICPC National Contest 2009. Abridged problem description: Given a weighted (family) tree of up to $N \leq 80K$ vertices with a special trait: *Vertex values are increasing from root to leaves*. Find the *ancestor* vertex closest to the root from a starting vertex v that has weight at least P. There are up to $Q \leq 20K$ such *offline* queries. Examine Figure 3.3 (left). If $P = 4$, then the answer is the vertex labeled with 'B' with value 5 as it is the ancestor of vertex v that is closest to root 'A' and has a value of ≥ 4. If $P = 7$, then the answer is 'C', with value 7. If $P \geq 9$, there is no answer.

The naïve solution is to perform a linear $O(N)$ scan per query: Starting from the given vertex v, we move up the (family) tree until we reach the first vertex whose direct parent has value $< P$ or until we reach the root. If this vertex has value $\geq P$ and it is not vertex v

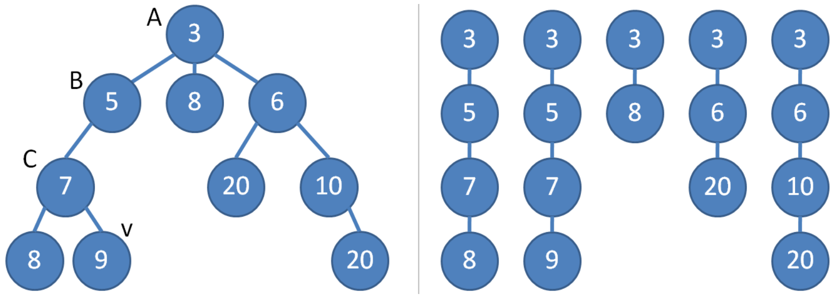


Figure 3.3: My Ancestor (all 5 root-to-leaf paths are sorted)

itself, we have found the solution. As there are Q queries, this approach runs in $O(QN)$ (the input tree can be a sorted linked list, or rope, of length N) and will get a TLE as $N \leq 80K$ and $Q \leq 20K$.

A better solution is to store all the $20K$ queries (we do not have to answer them immediately). Traverse the tree *just once* starting from the root using the $O(N)$ preorder tree traversal algorithm (Section 4.7.2). This preorder tree traversal is slightly modified to remember the partial root-to-current-vertex sequence as it executes. The array is always sorted because the vertices along the root-to-current-vertex path have increasing weights, see Figure 3.3 (right). The preorder tree traversal on the tree shown in Figure 3.3 (left) produces the following partial root-to-current-vertex sorted array: {{3}, {3, 5}, {3, 5, 7}, {3, 5, 7, 8}, backtrack, {3, 5, 7, 9}, backtrack, backtrack, backtrack, {3, 8}, backtrack, {3, 6}, {3, 6, 20}, backtrack, {3, 6, 10}, and finally {3, 6, 10, 20}, backtrack, backtrack, backtrack (done)}.

During the preorder traversal, when we land on a queried vertex, we can perform a $O(\log N)$ **binary search** (to be precise: `lower_bound`) on the partial root-to-current-vertex weight array to obtain the ancestor closest to the root with a value of at least P, recording these solutions. Finally, we can perform a simple $O(Q)$ iteration to output the results. The overall time complexity of this approach is $O(Q \log N)$, which is now manageable given the input bounds.

Bisection Method

We have discussed the applications of Binary Searches in finding items in static sorted sequences. However, the binary search **principle**[4] can also be used to find the root of a function that may be difficult to compute directly.

Example: You buy a car with loan and now want to pay the loan in monthly installments of d dollars for m months. Suppose the value of the car is originally v dollars and the bank charges an interest rate of $i\%$ for any unpaid loan at the end of each month. What is the amount of money d that you must pay per month (to 2 digits after the decimal point)?

Suppose $d = 576.19$, $m = 2$, $v = 1000$, and $i = 10\%$. After one month, your debt becomes $1000 \times (1.1) - 576.19 = 523.81$. After two months, your debt becomes $523.81 \times (1.1) - 576.19 \approx 0$. If we are only given $m = 2$, $v = 1000$, and $i = 10\%$, how would we determine that $d = 576.19$? In other words, find the root d such that the debt payment function $f(d, m, v, i) \approx 0$.

An *easy* way to solve this root finding problem is to use the bisection method. We pick a reasonable range as a starting point. We want to fix d within the range `[a..b]` where

[4]We use the term 'binary search principle' to refer to the D&C approach of halving the range of possible answers. The 'binary search algorithm' (finding index of an item in a sorted array), the 'bisection method' (finding the root of a function), and 'binary search the answer' (discussed in the next subsection) are all instances of this principle.

$a = 0.01$ as we have to pay at least one cent and $b = (1 + i\%) \times v$ as the earliest we can complete the payment is $m = 1$ if we pay exactly $(1 + i\%) \times v$ dollars after one month. In this example, $b = (1 + 0.1) \times 1000 = 1100.00$ dollars. For the bisection method to work[5], we must ensure that the function values of the two extreme points in the initial Real range `[a..b]`, i.e. $f(a)$ and $f(b)$ have opposite signs (this is true for the computed `a` and `b` above).

a	**b**	**d** $= \frac{a+b}{2}$	**status:** $f(d, m, v, i)$	**action**
0.01	1100.00	550.005	undershoot by 54.9895	increase d
550.005	1100.00	825.0025	overshoot by 522.50525	decrease d
550.005	825.0025	687.50375	overshoot by 233.757875	decrease d
550.005	687.50375	618.754375	overshoot by 89.384187	decrease d
550.005	618.754375	584.379688	overshoot by 17.197344	decrease d
550.005	584.379688	567.192344	undershoot by 18.896078	increase d
567.192344	584.379688	575.786016	undershoot by 0.849366	increase d
...	...	...	a **few** iterations later ...	...
...	...	576.190476	stop; error is now less than ϵ	answer = 576.19

Table 3.1: Running Bisection Method on the Example Function

Notice that bisection method only requires $O(\log_2((b - a)/\epsilon))$ iterations to get an answer that is good enough (the error is smaller than the threshold error ϵ that we can tolerate). In this example, bisection method only takes $\log_2 1099.99/\epsilon$ tries. Using a small ϵ = `1e-9`, this yields only ≈ 40 iterations. Even if we use a smaller ϵ = `1e-15`, we will still only need ≈ 60 tries. Notice that the number of tries is *small*. The bisection method is much more efficient compared to exhaustively evaluating each possible value of $d =$ `[0.01..1100.00]`$/\epsilon$ for this example function. Note: The bisection method can be written with a loop that tries the values of $d \approx 40$ to 60 times (see our implementation in the 'binary search the answer' discussion below).

Binary Search the Answer

The abridged version of UVa 11935 - Through the Desert is as follows: Imagine that you are an explorer trying to cross a desert. You use a jeep with a 'large enough' fuel tank – initially full. You encounter a series of events throughout your journey such as 'drive (that consumes fuel)', 'experience gas leak (further reduces the amount of fuel left)', 'encounter gas station (allowing you to refuel to the original capacity of your jeep's fuel tank)', 'encounter mechanic (fixes all leaks)', or 'reach goal (done)'. You need to determine the *smallest possible* fuel tank capacity for your jeep to be able to reach the goal. The answer must be precise to three digits after decimal point.

If we know the jeep's fuel tank capacity, then this problem is just a simulation problem. From the start, we can simulate each event in order and determine if the goal can be reached without running out of fuel. The problem is that we do not know the jeep's fuel tank capacity—this is the value that we are looking for.

From the problem description, we can compute that the range of possible answers is between `[0.000..10000.000]`, with 3 digits of precision. However, there are $10M$ such possibilities. Trying each value sequentially will get us a TLE verdict.

Fortunately, this problem has a property that we can exploit. Suppose that the correct answer is X. Setting your jeep's fuel tank capacity to any value between `[0.000..X-0.001]`

[5]Note that the requirements for the bisection method (which uses the binary search principle) are slightly different from the binary search algorithm which needs a sorted array.

will *not* bring your jeep safely to the goal event. On the other hand, setting your jeep fuel tank volume to any value between [X..10000.000] will bring your jeep safely to the goal event, usually with some fuel left. This property allows us to binary search the answer X! We can use the following code to obtain the solution for this problem.

```
#define EPS 1e-9  // this value is adjustable; 1e-9 is usually small enough
bool can(double f) {              // details of this simulation is omitted
  // return true if the jeep can reach goal state with fuel tank capacity f
  // return false otherwise
}

// inside int main()
  // binary search the answer, then simulate
  double lo = 0.0, hi = 10000.0, mid = 0.0, ans = 0.0;
  while (fabs(hi - lo) > EPS) {         // when the answer is not found yet
    mid = (lo + hi) / 2.0;                            // try the middle value
    if (can(mid)) { ans = mid; hi = mid; } // save the value, then continue
    else          lo = mid;
  }

  printf("%.3lf\n", ans);     // after the loop is over, we have the answer
```

Note that some programmers choose to use a constant number of refinement iterations instead of allowing the number of iterations to vary dynamically to avoid precision errors when testing `fabs(hi - lo) > EPS` and thus being trapped in an infinite loop. The only changes required to implement this approach are shown below. The other parts of the code are the same as above.

```
  double lo = 0.0, hi = 10000.0, mid = 0.0, ans = 0.0;
  for (int i = 0; i < 50; i++) {    // log_2 ((10000.0 - 0.0) / 1e-9) ~= 43
    mid = (lo + hi) / 2.0;     // looping 50 times should be precise enough
    if (can(mid)) { ans = mid; hi = mid; }
    else          lo = mid;
  }
```

Exercise 3.3.1.1: There is an alternative solution for UVa 11935 that does not use 'binary search the answer' technique. Can you spot it?

Exercise 3.3.1.2*: The example shown here involves binary-searching the answer where the answer is a floating point number. Modify the code to solve 'binary search the answer' problems where the answer lies in an *integer range*!

Remarks About Divide and Conquer in Programming Contests

The Divide and Conquer paradigm is usually utilized through popular algorithms that rely on it: Binary Search and its variants, Merge/Quick/Heap Sort, and data structures: Binary Search Tree, Heap, Segment Tree, Fenwick Tree, etc. However—based on our experience, we reckon that the most commonly used form of the Divide and Conquer paradigm in programming contests is the Binary Search principle. If you want to do well in programming contests, please spend time practicing the various ways to apply it.

Once you are more familiar with the 'Binary Search the Answer' technique discussed in this section, please explore Section 8.4.1 for a few more programming exercises that use this technique with *other algorithm* that we will discuss in the latter parts of this book.

We notice that there are not that many D&C problems outside of our binary search categorization. Most D&C solutions are 'geometry-related' or 'problem specific', and thus cannot be discussed in detail in this book. However, we will encounter some of them in Section 8.4.1 (binary search the answer plus geometry formulas), Section 9.14 (Inversion Index), Section 9.21 (Matrix Power), and Section 9.29 (Selection Problem).

Programming Exercises solvable using Divide and Conquer:

- Binary Search
  1. UVa 00679 - Dropping Balls (binary search; bit manipulation solutions exist)
  2. UVa 00957 - Popes (complete search + binary search: `upper_bound`)
  3. UVa 10077 - The Stern-Brocot ... (binary search)
  4. UVa 10474 - Where is the Marble? (simple: use `sort` and then `lower_bound`)
  5. ***UVa 10567 - Helping Fill Bates **** (store increasing indices of each char of 'S' in 52 vectors; for each query, binary search for the position of the char in the correct vector)
  6. UVa 10611 - Playboy Chimp (binary search)
  7. UVa 10706 - Number Sequence (binary search + some mathematical insights)
  8. UVa 10742 - New Rule in Euphomia (use sieve; binary search)
  9. **UVa 11057 - Exact Sum *** (sort, for price `p[i]`, check if price `(M - p[i])` exists with binary search)
  10. UVa 11621 - Small Factors (generate numbers with factor 2 and/or 3, `sort`, `upper_bound`)
  11. *UVa 11701 - Cantor* (a kind of ternary search)
  12. UVa 11876 - N + NOD (N) (`[lower|upper]_bound` on sorted sequence N)
  13. ***UVa 12192 - Grapevine **** (the input array has special sorted properties; use `lower_bound` to speed up the search)
  14. Thailand ICPC National Contest 2009 - My Ancestor (author: Felix Halim)
- Bisection Method or Binary Search the Answer
  1. **UVa 10341 - Solve It *** (bisection method discussed in this section; for alternative solutions, see `http://www.algorithmist.com/index.php/UVa_10341`)
  2. **UVa 11413 - Fill the ... *** (binary search the answer + simulation)
  3. UVa 11881 - Internal Rate of Return (bisection method)
  4. UVa 11935 - Through the Desert (binary search the answer + simulation)
  5. ***UVa 12032 - The Monkey ... **** (binary search the answer + simulation)
  6. *UVa 12190 - Electric Bill* (binary search the answer + algebra)
  7. IOI 2010 - Quality of Living (binary search the answer)

  Also see: Divide & Conquer for Geometry Problems (see Section 8.4.1)
- Other Divide & Conquer Problems
  1. ***UVa 00183 - Bit Maps **** (simple exercise of Divide and Conquer)
  2. IOI 2011 - Race (D&C; whether the solution path uses a vertex or not)

  Also see: Data Structures with Divide & Conquer flavor (see Section 2.3)

3.4 Greedy

An algorithm is said to be greedy if it makes the locally optimal choice at each step with the hope of eventually reaching the globally optimal solution. In some cases, greedy works—the solution is short and runs efficiently. For *many* others, however, it does not. As discussed in other typical Computer Science textbooks, e.g. [7, 38], a problem must exhibit these two properties in order for a greedy algorithm to work:

1. It has optimal sub-structures.
   Optimal solution to the problem contains optimal solutions to the sub-problems.
2. It has the greedy property (difficult to prove in time-critical contest environment!).
   If we make a choice that seems like the best at the moment and proceed to solve the remaining subproblem, we reach the optimal solution. We will never have to reconsider our previous choices.

3.4.1 Examples

Coin Change - The Greedy Version

Problem description: Given a target amount V cents and a list of denominations of n coins, i.e. we have `coinValue[i]` (in cents) for coin types `i` $\in$ `[0..n-1]`, what is the minimum number of coins that we must use to represent amount V? Assume that we have an unlimited supply of coins of any type. Example: If $n = 4$, `coinValue = {25, 10, 5, 1}` cents[6], and we want to represent $V = 42$ cents, we can use this Greedy algorithm: Select the largest coin denomination which is not greater than the remaining amount, i.e. 42-$\underline{25}$ = 17 $\rightarrow$ 17-$\underline{10}$ = 7 $\rightarrow$ 7-$\underline{5}$ = 2 $\rightarrow$ 2-$\underline{1}$ = 1 $\rightarrow$ 1-$\underline{1}$ = 0, a total of 5 coins. This is optimal.

The problem above has the two ingredients required for a successful greedy algorithm:

1. It has optimal sub-structures.
   We have seen that in our quest to represent 42 cents, we used 25+10+5+1+1.
   This is an optimal 5-coin solution to the original problem!
   Optimal solutions to sub-problem are contained within the 5-coin solution, i.e.
   a. To represent 17 cents, we can use 10+5+1+1 (part of the solution for 42 cents),
   b. To represent 7 cents, we can use 5+1+1 (also part of the solution for 42 cents), etc
2. It has the greedy property: Given every amount V, we can greedily subtract from it the largest coin denomination which is not greater than this amount V. It can be proven (not shown here for brevity) that using any other strategies will not lead to an optimal solution, at least for this set of coin denominations.

However, this greedy algorithm does *not* work for *all* sets of coin denominations. Take for example {4, 3, 1} cents. To make 6 cents with that set, a greedy algorithm would choose 3 coins {4, 1, 1} instead of the optimal solution that uses 2 coins {3, 3}. The general version of this problem is revisited later in Section 3.5.2 (Dynamic Programming).

UVa 410 - Station Balance (Load Balancing)

Given $1 \leq C \leq 5$ chambers which can store 0, 1, or 2 specimens, $1 \leq S \leq 2C$ specimens and a list M of the masses of the S specimens, determine which chamber should store each specimen in order to minimize 'imbalance'. See Figure 3.4 for a visual explanation[7].

[6] The presence of the 1-cent coin ensures that we can always make every value.

[7] Since $C \leq 5$ and $S \leq 10$, we can actually use a Complete Search solution for this problem. However, this problem is simpler to solve using the Greedy algorithm.

$A = (\sum_{j=1}^{S} M_j)/C$, i.e. A is the average of the total mass in each of the C chambers.

Imbalance $= \sum_{i=1}^{C} |X_i - A|$, i.e. the sum of differences between the total mass in each chamber w.r.t. A where X_i is the total mass of specimens in chamber i.

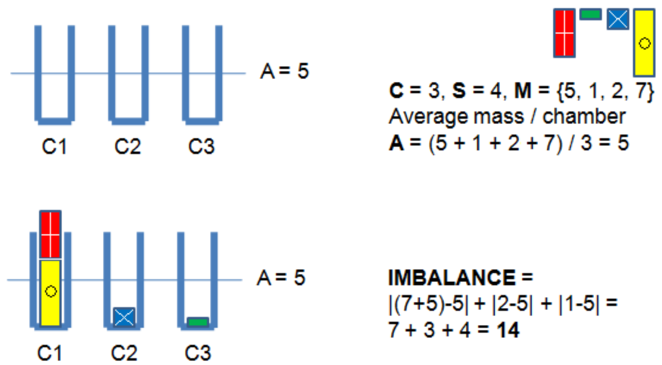


Figure 3.4: Visualization of UVa 410 - Station Balance

This problem can be solved using a greedy algorithm, but to arrive at that solution, we have to make several observations.

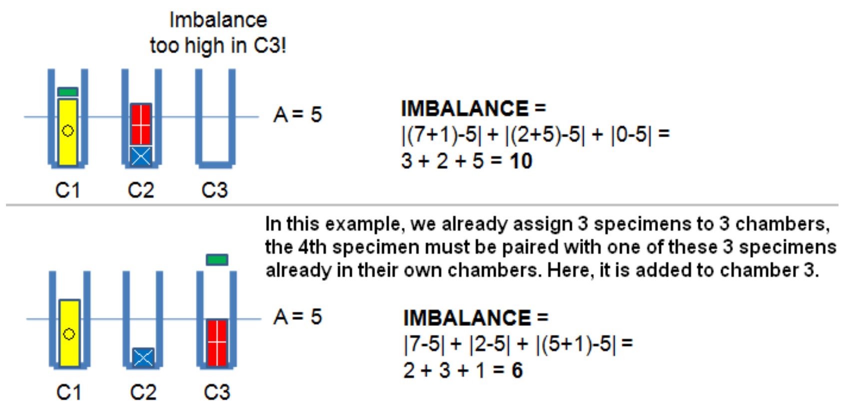


Figure 3.5: UVa 410 - Observations

Observation 1: If there exists an empty chamber, it is usually beneficial and never worse to move one specimen from a chamber with two specimens to the empty chamber! Otherwise, the empty chamber contributes more to the imbalance as shown in Figure 3.5, top.

Observation 2: If $S > C$, then $S - C$ specimens must be paired with a chamber already containing other specimens—the Pigeonhole principle! See Figure 3.5, bottom.

The key insight is that the solution to this problem can be simplified with sorting: if $S < 2C$, add $2C - S$ dummy specimens with mass 0. For example, $C = 3$, $S = 4$, $M = \{5, 1, 2, 7\} \rightarrow C = 3, S = 6, M = \{5, 1, 2, 7, 0, 0\}$. Then, sort the specimens on their mass such that $M_1 \leq M_2 \leq \ldots \leq M_{2C-1} \leq M_{2C}$. In this example, $M = \{5, 1, 2, 7, 0, 0\} \rightarrow \{0, 0, 1, 2, 5, 7\}$. By adding dummy specimens and then sorting them, a greedy strategy becomes 'apparent':

- Pair the specimens with masses $M_1 \& M_{2C}$ and put them in chamber 1, then
- Pair the specimens with masses $M_2 \& M_{2C-1}$ and put them in chamber 2, and so on ...

This greedy algorithm—known as *load balancing*—works! See Figure 3.6.

It is hard to impart the techniques used in deriving this greedy solution. Finding greedy solutions is an art, just as finding good Complete Search solutions requires creativity. A tip that arises from this example: If there is no obvious greedy strategy, try *sorting* the data or introducing some tweak and see if a greedy strategy emerges.

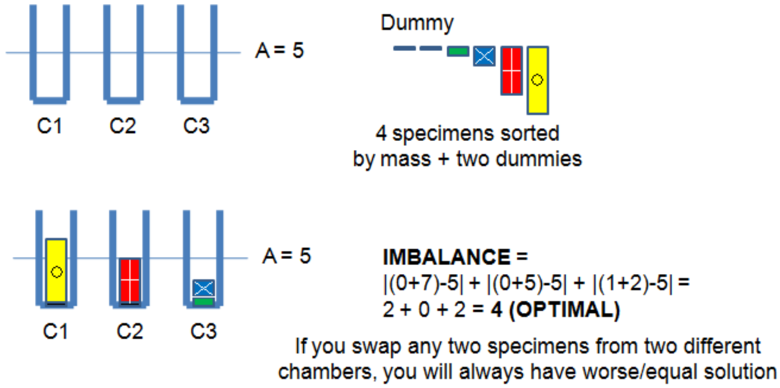


Figure 3.6: UVa 410 - Greedy Solution

UVa 10382 - Watering Grass (Interval Covering)

Problem description: n sprinklers are installed in a horizontal strip of grass L meters long and W meters wide. Each sprinkler is centered vertically in the strip. For each sprinkler, we are given its position as the distance from the left end of the center line and its radius of operation. What is the minimum number of sprinklers that should be turned on in order to water the entire strip of grass? Constraint: $n \leq 10000$. For an illustration of the problem, see Figure 3.7—left side. The answer for this test case is 6 sprinklers (those labeled with {A, B, D, E, F, H}). There are 2 unused sprinklers: {C, G}.

We cannot solve this problem with a brute force strategy that tries all possible subsets of sprinklers to be turned on since the number of sprinklers can go up to 10000. It is definitely infeasible to try all 2^{10000} possible subsets of sprinklers.

This problem is actually a variant of the well known greedy problem called the *interval covering* problem. However, it includes a simple geometric twist. The original interval covering problem deals with intervals. This problem deals with sprinklers that have circles of influence in a horizontal area rather than simple intervals. We first have to transform the problem to resemble the standard interval covering problem.

See Figure 3.7—right side. We can convert these circles and horizontal strips into intervals. We can compute `dx = sqrt(`R^2 `-` $(W/2)^2$`)`. Suppose a circle is centered at `(x, y)`. The interval represented by this circle is `[x-dx..x+dx]`. To see why this works, notice that the additional circle segment beyond `dx` away from x does not completely cover the strip in the horizontal region it spans. If you have difficulties with this geometric transformation, see Section 7.2.4 which discusses basic operations involving a *right triangle*.

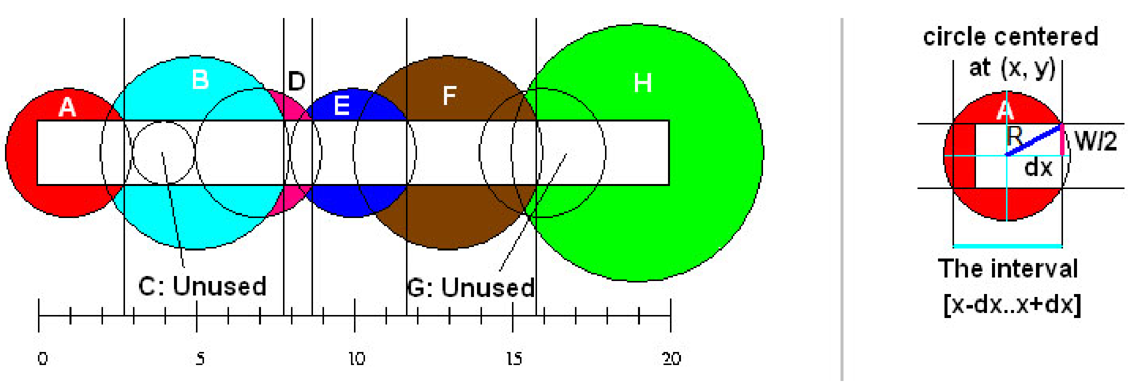


Figure 3.7: UVa 10382 - Watering Grass

Now that we have transformed the original problem into the interval covering problem, we can use the following Greedy algorithm. First, the Greedy algorithm sorts the intervals by *increasing* left endpoint and by *decreasing* right endpoint if ties arise. Then, the Greedy algorithm processes the intervals one at a time. It takes the interval that covers 'as far right as possible' and yet still produces uninterrupted coverage from the leftmost side to the rightmost side of the horizontal strip of grass. It ignores intervals that are already completely covered by other (previous) intervals.

For the test case shown in Figure 3.7—left side, this Greedy algorithm first sorts the intervals to obtain the sequence {A, B, C, D, E, F, G, H}. Then it processes them one by one. First, it takes 'A' (it has to), takes 'B' (connected to interval 'A'), ignores 'C' (as it is embedded inside interval 'B'), takes 'D' (it has to, as intervals 'B' and 'E' are not connected if 'D' is not used), takes 'E', takes 'F', ignores 'G' (as taking 'G' is not 'as far right as possible' and does not reach the rightmost side of the grass strip), takes 'H' (as it connects with interval 'F' and covers more to the right than interval of 'G' does, going beyond the rightmost end of the grass strip). In total, we select 6 sprinklers: {A, B, D, E, F, H}. This is the minimum possible number of sprinklers for this test case.

UVa 11292 - Dragon of Loowater (Sort the Input First)

Problem description: There are n dragon heads and m knights ($1 \leq n, m \leq 20000$). Each dragon head has a *diameter* and each knight has a *height*. A dragon head with diameter **D** can be chopped off by a knight with height **H** if **D** $\leq$ **H**. A knight can only chop off one dragon head. Given a list of diameters of the dragon heads and a list of heights of the knights, is it possible to chop off all the dragon heads? If yes, what is the minimum total height of the knights used to chop off the dragons' heads?

There are several ways to solve this problem, but we will illustrate one that is probably the easiest. This problem is a bipartite matching problem (this will be discussed in more detail in Section 4.7.4), in the sense that we are required to match (pair) certain knights to dragon heads in a maximal fashion. However, this problem can be solved greedily: Each dragon head should be chopped by a knight with the shortest height that is at least as tall as the diameter of the dragon's head. However, the input is given in an arbitrary order. If we sort both the list of dragon head diameters and knight heights in $O(n \log n + m \log m)$, we can use the following $O(min(n, m))$ scan to determine the answer. This is yet another example where sorting the input can help produce the required greedy strategy.

```
gold = d = k = 0; // array dragon+knight are sorted in non decreasing order
while (d < n && k < m) {              // still have dragon heads or knights
  while (dragon[d] > knight[k] && k < m) k++;   // find the required knight
  if (k == m) break;      // no knight can kill this dragon head, doomed :S
  gold += knight[k];                       // the king pay this amount of gold
  d++; k++;                               // next dragon head and knight please
}

if (d == n) printf("%d\n", gold);            // all dragon heads are chopped
else        printf("Loowater is doomed!\n");
```

Exercise 3.4.1.1*: Which of the following sets of coins (all in cents) are solvable using the greedy 'coin change' algorithm discussed in this section? If the greedy algorithm fails on a certain set of coin denominations, determine the smallest counter example V cents on which it fails to be optimal. See [51] for more details about finding such counter examples.

1. $S_1 = \{10, 7, 5, 4, 1\}$
2. $S_2 = \{64, 32, 16, 8, 4, 2, 1\}$
3. $S_3 = \{13, 11, 7, 5, 3, 2, 1\}$
4. $S_4 = \{7, 6, 5, 4, 3, 2, 1\}$
5. $S_5 = \{21, 17, 11, 10, 1\}$

Remarks About Greedy Algorithm in Programming Contests

In this section, we have discussed three classical problems solvable with Greedy algorithms: Coin Change (the special case), Load Balancing, and Interval Covering. For these classical problems, it is helpful to memorize their solutions (for this case, ignore that we have said earlier in the chapter about not relying too much on memorization). We have also discussed an important problem solving strategy usually applicable to greedy problems: Sorting the input data to elucidate hidden greedy strategies.

There are two other classical examples of Greedy algorithms in this book, e.g. Kruskal's (and Prim's) algorithm for the Minimum Spanning Tree (MST) problem (see Section 4.3) and Dijkstra's algorithm for the Single-Source Shortest Paths (SSSP) problem (see Section 4.4.3). There are many more known Greedy algorithms that we have chosen not to discuss in this book as they are too 'problem specific' and rarely appear in programming contests, e.g. Huffman Codes [7, 38], Fractional Knapsack [7, 38], some Job Scheduling problems, etc.

However, today's programming contests (both ICPC and IOI) rarely involve the purely canonical versions of these classical problems. Using Greedy algorithms to attack a 'non classical' problem is usually risky. A Greedy algorithm will normally not encounter the TLE response as it is often lightweight, but instead tends to obtain WA verdicts. Proving that a certain 'non-classical' problem has optimal sub-structure and greedy property during contest time may be difficult or time consuming, so a competitive programmer should usually use this rule of thumb:

If the input size is 'small enough' to accommodate the time complexity of either Complete Search or Dynamic Programming approaches (see Section 3.5), then use these approaches as both will ensure a correct answer. *Only* use a Greedy algorithm if the input size given in the problem statement are too large even for the best Complete Search or DP algorithm.

Having said that, it is increasingly true that problem authors try to set the input bounds of problems that allow for Greedy strategies to be in an ambiguous range so that contestants *cannot* use the input size to quickly determine the required algorithm!

We have to remark that it is quite challenging to come up with new 'non-classical' Greedy problems. Therefore, the number of such novel Greedy problems used in competitive programming is lower than that of Complete Search or Dynamic Programming problems.

Programming Exercises solvable using Greedy
(most hints are omitted to keep the problems challenging):

- Classical, Usually Easier
  1. UVa 00410 - Station Balance (discussed in this section, load balancing)
  2. UVa 01193 - Radar Installation (LA 2519, Beijing02, interval covering)
  3. UVa 10020 - Minimal Coverage (interval covering)
  4. UVa 10382 - Watering Grass (discussed in this section, interval covering)
  5. ***UVa 11264 - Coin Collector **** (coin change variant)

6. **UVa 11389 - The Bus Driver Problem *** (load balancing)
7. *UVa 12321 - Gas Station* (interval covering)
8. ***UVa 12405 - Scarecrow **** (simpler interval covering problem)
9. IOI 2011 - Elephants (optimized greedy solution can be used up to subtask 3, but the harder subtasks 4 and 5 must be solved using efficient data structure)

- Involving Sorting (Or The Input Is Already Sorted)

1. UVa 10026 - Shoemaker's Problem
2. *UVa 10037 - Bridge*
3. UVa 10249 - The Grand Dinner
4. UVa 10670 - Work Reduction
5. UVa 10763 - Foreign Exchange
6. UVa 10785 - The Mad Numerologist
7. ***UVa 11100 - The Trip, 2007 ****
8. UVa 11103 - WFF'N Proof
9. *UVa 11269 - Setting Problems*
10. **UVa 11292 - Dragon of Loowater ***
11. UVa 11369 - Shopaholic
12. UVa 11729 - Commando War
13. UVa 11900 - Boiled Eggs
14. ***UVa 12210 - A Match Making Problem ****
15. *UVa 12485 - Perfect Choir*

- Non Classical, Usually Harder

1. UVa 00311 - Packets
2. *UVa 00668 - Parliament*
3. UVa 10152 - ShellSort
4. UVa 10340 - All in All
5. UVa 10440 - Ferry Loading II
6. UVa 10602 - Editor Nottobad
7. **UVa 10656 - Maximum Sum (II) ***
8. UVa 10672 - Marbles on a tree
9. UVa 10700 - Camel Trading
10. UVa 10714 - Ants
11. **UVa 10718 - Bit Mask ***
12. *UVa 10982 - Troublemakers*
13. UVa 11054 - Wine Trading in Gergovia
14. **UVa 11157 - Dynamic Frog ***
15. *UVa 11230 - Annoying painting tool*
16. *UVa 11240 - Antimonotonicity*
17. *UVa 11335 - Discrete Pursuit*
18. UVa 11520 - Fill the Square
19. UVa 11532 - Simple Adjacency ...
20. UVa 11567 - Moliu Number Generator
21. *UVa 12482 - Short Story Competition*

3.5 Dynamic Programming

Dynamic Programming (from now on abbreviated as DP) is perhaps the most challenging problem-solving technique among the four paradigms discussed in this chapter. Thus, make sure that you have mastered the material mentioned in the previous chapters/sections before reading this section. Also, prepare to see lots of recursion and recurrence relations!

The key skills that you have to develop in order to master DP are the abilities to determine the problem *state*s and to determine the relationships or *transitions* between current problems and their sub-problems. We have used these skills earlier in recursive backtracking (see Section 3.2.2). In fact, DP problems with small input size constraints may already be solvable with recursive backtracking.

If you are new to DP technique, you can start by assuming that (the 'top-down') DP is a kind of 'intelligent' or 'faster' recursive backtracking. In this section, we will explain the reasons why DP is often faster than recursive backtracking for problems amenable to it.

DP is primarily used to solve *optimization* problems and *counting* problems. If you encounter a problem that says "minimize this" or "maximize that" or "count the ways to do that", then there is a (high) chance that it is a DP problem. Most DP problems in programming contests only ask for the optimal/total value and not the optimal solution itself, which often makes the problem easier to solve by removing the need to backtrack and produce the solution. However, some harder DP problems also require the optimal solution to be returned in some fashion. We will continually refine our understanding of Dynamic Programming in this section.

3.5.1 DP Illustration

We will illustrate the concept of Dynamic Programming with an example problem: UVa 11450 - Wedding Shopping. The abridged problem statement: Given different options for each garment (e.g. 3 shirt models, 2 belt models, 4 shoe models, ...) and a certain *limited* budget, our task is to *buy one model of each garment.* We cannot spend more money than the given budget, but we want to spend *the maximum possible* amount.

The input consists of two integers $1 \leq M \leq 200$ and $1 \leq C \leq 20$, where M is the budget and C is the number of garments that you have to buy, followed by some information about the C garments. For the garment `g` $\in$ `[0..`C`-1]`, we will receive an integer $1 \leq K \leq 20$ which indicates the number of different models there are for that garment `g`, followed by K integers indicating the price of each model $\in$ `[1..`K`]` of that garment `g`.

The output is one integer that indicates the maximum amount of money we can spend purchasing one of each garment *without exceeding the budget.* If there is no solution due to the small budget given to us, then simply print "`no solution`".

Suppose we have the following test case A with $M = 20$, $C = 3$:
Price of the 3 models of garment `g = 0` $\rightarrow$ 6 4 <u>8</u> // the prices are not sorted in the input
Price of the 2 models of garment `g = 1` $\rightarrow$ 5 <u>10</u>
Price of the 4 models of garment `g = 2` $\rightarrow$ <u>1</u> 5 3 5

For this test case, the answer is 19, which *may* result from buying the <u>underlined</u> items (8+10+1). This is not unique, as solutions (6+10+3) and (4+10+5) are also optimal.

However, suppose we have this test case B with $M =$ **9 (limited budget)**, $C = 3$:
Price of the 3 models of garment `g = 0` $\rightarrow$ 6 4 8
Price of the 2 models of garment `g = 1` $\rightarrow$ 5 10
Price of the 4 models of garment `g = 2` $\rightarrow$ 1 5 3 5

The answer is then "`no solution`" because even if we buy all the cheapest models for each garment, the total price (4+5+1) = 10 still exceeds our given budget $M = 9$.

In order for us to appreciate the usefulness of Dynamic Programming in solving the above-mentioned problem, let's explore how far the *other* approaches discussed earlier will get us in this particular problem.

Approach 1: Greedy (Wrong Answer)

Since we want to maximize the budget spent, one greedy idea (there are other greedy approaches—which are also WA) is to take the most expensive model for each garment `g` which still fits our budget. For example in test case A above, we can choose the most expensive model 3 of garment `g = 0` with price 8 (`money` is now `20-8 = 12`), then choose the most expensive model 2 of garment `g = 1` with price 10 (`money = 12-10 = 2`), and finally for the last garment `g = 2`, we can only choose model 1 with price 1 as the `money` we have left does not allow us to buy the other models with price 3 or 5. This greedy strategy 'works' for test cases A and B above and produce the same optimal solution (8+10+1) = 19 and "`no solution`", respectively. It also runs very fast[8]: $20 + 20 + \ldots + 20$ for a total of 20 times = 400 operations in the worst case. However, this greedy strategy does not work for many other test cases, such as this *counter-example* below (test case C):

Test case C with $M = 12$, $C = 3$:
3 models of garment `g = 0` → 6 <u>4</u> **8**
2 models of garment `g = 1` → <u>5</u> 10
4 models of garment `g = 2` → 1 5 <u>3</u> 5

The Greedy strategy selects model 3 of garment `g = 0` with price **8** (`money = 12-8 = 4`), causing us to not have enough money to buy any model in garment `g = 1`, thus incorrectly reporting "`no solution`". One optimal solution is <u>4+5+3</u> = 12, which uses up all of our budget. The optimal solution is not unique as 6+5+1 = 12 also depletes the budget.

Approach 2: Divide and Conquer (Wrong Answer)

This problem is not solvable using the Divide and Conquer paradigm. This is because the sub-problems (explained in the Complete Search sub-section below) are not independent. Therefore, we cannot solve them separately with the Divide and Conquer approach.

Approach 3: Complete Search (Time Limit Exceeded)

Next, let's see if Complete Search (recursive backtracking) can solve this problem. One way to use recursive backtracking in this problem is to write a function `shop(money, g)` with two parameters: The current `money` that we have and the current garment `g` that we are dealing with. The pair `(money, g)` is the *state* of this problem. Note that the order of parameters does not matter, e.g. `(g, money)` is also a perfectly valid state. Later in Section 3.5.3, we will see more discussion on how to select appropriate states for a problem.

We start with `money = M` and garment `g = 0`. Then, we try all possible models in garment `g = 0` (a maximum of 20 models). If model i is chosen, we subtract model i's price from `money`, then repeat the process in a recursive fashion with garment `g = 1` (which can also have up to 20 models), etc. We stop when the model for the last garment `g = C-1` has been chosen. If `money < 0` before we choose a model from garment `g = C-1`, we can prune the infeasible solution. Among all valid combinations, we can then pick the one that results in the smallest non-negative `money`. This maximizes the money spent, which is (`M - money`).

[8]We do not need to sort the prices just to find the model with the maximum price as there are only up to $K \leq 20$ models. An $O(K)$ scan is enough.

We can formally define these Complete Search recurrences (transitions) as follows:

1. If `money < 0` (i.e. money goes negative),
`shop(money, g) =` $-\infty$ (in practice, we can just return a large negative value)

2. If a model from the last garment has been bought, that is, `g = C`,
`shop(money, g) = M - money` (this is the actual money that we spent)

3. In general case, $\forall$ `model` $\in$ `[1..K]` of current garment `g`,
`shop(money, g) = max(shop(money - price[g][model], g + 1))`
We want to maximize this value (Recall that the invalid ones have large negative value)

This solution works correctly, but it is **very slow**! Let's analyze the worst case time complexity. In the largest test case, garment `g = 0` has up to 20 models; garment `g = 1` *also* has up to 20 models and all garments including the last garment `g = 19` ***also*** have up to 20 models. Therefore, this Complete Search runs in $20 \times 20 \times \ldots \times 20$ operations in the worst case, i.e. 20^{20} = a **very large** number. If we can *only* come up with this Complete Search solution, we cannot solve this problem.

Approach 4: Top-Down DP (Accepted)

To solve this problem, we have to use the DP concept as this problem satisfies the two prerequisites for DP to be applicable:

1. This problem has optimal sub-structures[9].
   This is illustrated in the third Complete Search recurrence above: The solution for the sub-problem is part of the solution of the original problem. In other words, if we select model i for garment $g = 0$, for our final selection to be optimal, our choice for garments $g = 1$ and above must also be the optimal choice for a reduced budget of $M - price$, where *price* refers to the price of model i.
2. This problem has overlapping sub-problems.
   This is the key characteristic of DP! The search space of this problem is *not* as big as the rough 20^{20} bound obtained earlier because **many** sub-problems are *overlapping*!

Let's verify if this problem indeed has overlapping sub-problems. Suppose that there are 2 models in a certain garment `g` with the *same* price `p`. Then, a Complete Search will move to the **same** sub-problem `shop(money - p, g + 1)` after picking *either* model! This situation will also occur if some combination of `money` and chosen model's price causes $\texttt{money}_1$ - $\texttt{p}_1$ = $\texttt{money}_2$ - $\texttt{p}_2$ at the same garment `g`. This will—in a Complete Search solution—cause the same sub-problem to be computed *more than once*, an inefficient state of affairs!

So, how many *distinct* sub-problems (a.k.a. **states** in DP terminology) are there in this problem? Only $201 \times 20 = 4020$. There are only 201 possible values for `money` (0 to 200 inclusive) and 20 possible values for the garment `g` (0 to 19 inclusive). Each sub-problem just needs to be computed *once*. If we can ensure this, we can solve this problem *much faster*.

The implementation of this DP solution is surprisingly simple. If we already have the recursive backtracking solution (see the recurrences—a.k.a. **transitions** in DP terminology—shown in the Complete Search approach above), we can implement the **top-down** DP by adding these two additional steps:

1. Initialize[10] a DP 'memo' table with dummy values that are not used in the problem, e.g. '-1'. The DP table should have dimensions corresponding to the problem states.

[9] Optimal sub-structures are also required for Greedy algorithms to work, but this problem lacks the 'greedy property', making it unsolvable with the Greedy algorithm.

[10] For C/C++ users, the `memset` function in <`cstring`> is a good tool to perform this step.

2. At the start of the recursive function, check if this state has been computed before.
   (a) If it has, simply return the value from the DP memo table, $O(1)$.
   (This the origin of the term 'memoization'.)
   (b) If it has not been computed, perform the computation as per normal (only once) and then store the computed value in the DP memo table so that *further calls* to this sub-problem (state) return immediately.

Analyzing a basic[11] DP solution is easy. If it has M distinct states, then it requires $O(M)$ memory space. If computing one state (the complexity of the DP transition) requires $O(k)$ steps, then the overall time complexity is $O(kM)$. This UVa 11450 problem has $M = 201 \times 20 = 4020$ and $k = 20$ (as we have to iterate through at most 20 models for each garment g). Thus, the time complexity is at most $4020 \times 20 = 80400$ operations per test case, a very manageable calculation.

We display our code below for illustration, especially for those who have never coded a top-down DP algorithm before. Scrutinize this code and verify that it is indeed very similar to the recursive backtracking code that you have seen in Section 3.2.

```
/* UVa 11450 - Wedding Shopping - Top Down */
// assume that the necessary library files have been included
// this code is similar to recursive backtracking code
// parts of the code specific to top-down DP are commented with: 'TOP-DOWN'

int M, C, price[25][25];                 // price[g (<= 20)][model (<= 20)]
int memo[210][25];    // TOP-DOWN: dp table memo[money (<= 200)][g (<= 20)]
int shop(int money, int g) {
  if (money < 0) return -1000000000;     // fail, return a large -ve number
  if (g == C) return M - money;        // we have bought last garment, done
  // if the line below is commented, top-down DP will become backtracking!!
  if (memo[money][g] != -1) return memo[money][g]; // TOP-DOWN: memoization
  int ans = -1;   // start with a -ve number as all prices are non negative
  for (int model = 1; model <= price[g][0]; model++)      // try all models
    ans = max(ans, shop(money - price[g][model], g + 1));
  return memo[money][g] = ans; }     // TOP-DOWN: memoize ans and return it

int main() {            // easy to code if you are already familiar with it
  int i, j, TC, score;
  scanf("%d", &TC);
  while (TC--) {
    scanf("%d %d", &M, &C);
    for (i = 0; i < C; i++) {
      scanf("%d", &price[i][0]);                  // store K in price[i][0]
      for (j = 1; j <= price[i][0]; j++) scanf("%d", &price[i][j]);
    }
    memset(memo, -1, sizeof memo);    // TOP-DOWN: initialize DP memo table
    score = shop(M, 0);                            // start the top-down DP
    if (score < 0) printf("no solution\n");
    else           printf("%d\n", score);
} } // return 0;
```

[11]Basic means "without fancy optimizations that we will see later in this section and in Section 8.3".

We want to take this opportunity to illustrate another style used in implementing DP solutions (only applicable for C/C++ users). Instead of frequently addressing a certain cell in the memo table, we can use a local *reference* variable to store the memory address of the required cell in the memo table as shown below. The two coding styles are not very different, and it is up to you to decide which style you prefer.

```
int shop(int money, int g) {
  if (money < 0) return -1000000000; // order of >1 base cases is important
  if (g == C) return M - money;  // money can't be <0 if we reach this line
  int &ans = memo[money][g];                  // remember the memory address
  if (ans != -1) return ans;
  for (int model = 1; model <= price[g][0]; model++)
    ans = max(ans, shop(money - price[g][model], g + 1));
  return ans;                // ans (or memo[money][g]) is directly updated
}
```

Source code: `ch3_02_UVa11450_td.cpp/java`

Approach 5: Bottom-Up DP (Accepted)

There is another way to implement a DP solution often referred to as the **bottom-up** DP. This is actually the 'true form' of DP as DP was originally known as the 'tabular method' (computation technique involving a table). The *basic* steps to build bottom-up DP solution are as follows:

1. Determine the required set of parameters that uniquely describe the problem (the state). This step is similar to what we have discussed in recursive backtracking and top-down DP earlier.

2. If there are N parameters required to represent the states, prepare an N dimensional DP table, with one entry per state. This is equivalent to the memo table in top-down DP. However, there are differences. In bottom-up DP, we only need to initialize some cells of the DP table with known initial values (the base cases). Recall that in top-down DP, we initialize the memo table completely with dummy values (usually -1) to indicate that we have not yet computed the values.

3. Now, with the base-case cells/states in the DP table already filled, determine the cells/states that can be filled next (the transitions). Repeat this process until the DP table is complete. For the bottom-up DP, this part is usually accomplished through iterations, using loops (more details about this later).

For UVa 11450, we can write the bottom-up DP as follow: We describe the state of a subproblem with two parameters: The current garment `g` and the current `money`. This state formulation is essentially equivalent to the state in the top-down DP above, except that we have reversed the order to make `g` the first parameter (thus the values of `g` are the row indices of the DP table so that we can take advantage of cache-friendly row-major traversal in a 2D array, see the speed-up tips in Section 3.2.3). Then, we initialize a 2D table (boolean matrix) `reachable[g][money]` of size 20×201. Initially, only cells/states reachable by buying any of the models of the first garment `g = 0` are set to true (in the first row). Let's use test case A above as example. In Figure 3.8, top, the only columns '20-6 = 14', '20-4 = 16', and '20-8 = 12' in row 0 are initially set to true.

money =>

g =>	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1	0	1	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1	0	1	0	0	0	0
1	0	0	1	0	1	0	1	1	0	1	0	1	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1	0	1	0	0	0	0
1	0	0	1	0	1	0	1	1	0	1	0	1	0	0	0	0	0	0	0	0	0
2	0	1	1	1	1	1	1	0	1	0	1	0	0	0	0	0	0	0	0	0	0

Figure 3.8: Bottom-Up DP (columns 21 to 200 are not shown)

Now, we loop from the second garment `g = 1` (second row) to the last garment `g = C-1 = 3-1 = 2` (third and last row) in row-major order (row by row). If `reachable[g-1][money]` is true, then the next state `reachable[g][money-p]` where `p` is the price of a model of current garment `g` is also reachable as long as the second parameter (`money`) is not negative. See Figure 3.8, middle, where `reachable[0][16]` propagates to `reachable[1][16-5]` and `reachable[1][16-10]` when the model with price 5 and 10 in garment `g = 1` is bought, respectively; `reachable[0][12]` propagates to `reachable[1][12-10]` when the model with price 10 in garment `g = 1` is bought, etc. We repeat this table filling process row by row until we are done with the last row[12].

Finally, the answer can be found in the last row when `g = C-1`. Find the state in that row that is both nearest to index 0 and reachable. In Figure 3.8, bottom, the cell `reachable[2][1]` provides the answer. This means that we can reach state (`money = 1`) by buying some combination of the various garment models. The required final answer is actually `M - money`, or in this case, `20-1 = 19`. The answer is "`no solution`" if there is no state in the last row that is reachable (where `reachable[C-1][money]` is set to true). We provide our implementation below for comparison with the top-down version.

```
/* UVa 11450 - Wedding Shopping - Bottom Up */
// assume that the necessary library files have been included

int main() {
  int g, money, k, TC, M, C;
  int price[25][25];                        // price[g (<= 20)][model (<= 20)]
  bool reachable[25][210];    // reachable table[g (<= 20)][money (<= 200)]

  scanf("%d", &TC);
  while (TC--) {
    scanf("%d %d", &M, &C);
    for (g = 0; g < C; g++) {
      scanf("%d", &price[g][0]);                    // we store K in price[g][0]
      for (money = 1; money <= price[g][0]; money++)
        scanf("%d", &price[g][money]);
    }
```

[12]Later in Section 4.7.1, we will discuss DP as a traversal of an (implicit) DAG. To avoid unnecessary 'backtracking' along this DAG, we have to visit the vertices in their topological order (see Section 4.2.5). The order in which we fill the DP table is a topological ordering of the underlying implicit DAG.

```
    memset(reachable, false, sizeof reachable);         // clear everything
    for (g = 1; g <= price[0][0]; g++)       // initial values (base cases)
      if (M - price[0][g] >= 0)      // to prevent array index out of bound
        reachable[0][M - price[0][g]] = true;  // using first garment g = 0

    for (g = 1; g < C; g++)                        // for each remaining garment
      for (money = 0; money < M; money++) if (reachable[g-1][money])
        for (k = 1; k <= price[g][0]; k++) if (money - price[g][k] >= 0)
          reachable[g][money - price[g][k]] = true;   // also reachable now

    for (money = 0; money <= M && !reachable[C - 1][money]; money++);

    if (money == M + 1) printf("no solution\n");  // last row has no on bit
    else                printf("%d\n", M - money);
  }
} // return 0;
```

Source code: `ch3_03_UVa11450_bu.cpp/java`

There is an advantage for writing DP solutions in the bottom-up fashion. For problems where we only need the last row of the DP table (or, more generally, the last updated slice of all the states) to determine the solution—including this problem, we can optimize the *memory usage* of our DP solution by sacrificing one dimension in our DP table. For harder DP problems with tight memory requirements, this 'space saving trick' may prove to be useful, though the overall time complexity does not change.

Let's take a look again at Figure 3.8. We only need to store two rows, the current row we are processing and the previous row we have processed. To compute row 1, we only need to know the columns in row 0 that are set to true in `reachable`. To compute row 2, we similarly only need to know the columns in row 1 that are set to true in `reachable`. In general, to compute row g, we only need values from the previous row $g-1$. So, instead of storing a boolean matrix `reachable[g][money]` of size $\underline{20} \times 201$, we can simply store `reachable[2][money]` of size $\underline{2} \times 201$. We can use this programming trick to reference one row as the 'previous' row and another row as the 'current' row (e.g. `prev = 0, cur = 1`) and then swap them (e.g. now `prev = 1, cur = 0`) as we compute the bottom-up DP row by row. Note that for this problem, the memory savings are not significant. For harder DP problems, for example where there might be thousands of garment models instead of 20, this space saving trick can be important.

Top-Down versus Bottom-Up DP

Although both styles use 'tables', the way the bottom-up DP table is filled is different to that of the top-down DP *memo* table. In the top-down DP, the memo table entries are filled 'as needed' through the recursion itself. In the bottom-up DP, we used a correct 'DP table filling order' to compute the values such that the previous values needed to process the current cell have already been obtained. This table filling order is the topological order of the implicit DAG (this will be explained in more detail in Section 4.7.1) in the recurrence structure. For most DP problems, a topological order can be achieved simply with the proper sequencing of some (nested) loops.

For most DP problems, these two styles are equally good and the decision to use a particular DP style is a matter of preference. However, for harder DP problems, one of the styles can be better than the other. To help you understand which style that you should use when presented with a DP problem, please study the trade-offs between top-down and bottom-up DPs listed in Table 3.2.

Top-Down	**Bottom-Up**
Pros: 1. It is a natural transformation from the normal Complete Search recursion 2. Computes the sub-problems only when necessary (sometimes this is faster)	Pros: 1. Faster if many sub-problems are revisited as there is no overhead from recursive calls 2. Can save memory space with the 'space saving trick' technique
Cons: 1. Slower if many sub-problems are revisited due to function call overhead (this is not usually penalized in programming contests) 2. If there are M states, an $O(M)$ table size is required, which can lead to MLE for some harder problems (except if we use the trick in Section 8.3.4)	Cons: 1. For programmers who are inclined to recursion, this style may not be intuitive 2. If there are M states, bottom-up DP visits and fills the value of *all* these M states

Table 3.2: DP Decision Table

Displaying the Optimal Solution

Many DP problems request only for the value of the optimal solution (like the UVa 11450 above). However, many contestants are caught off-guard when they are also required to print the optimal solution. We are aware of two ways to do this.

The first way is mainly used in the bottom-up DP approach (which is still applicable for top-down DPs) where we store the predecessor information at each state. If there are more than one optimal predecessors and we have to output all optimal solutions, we can store those predecessors in a list. Once we have the optimal final state, we can do backtracking from the optimal final state and follow the optimal transition(s) recorded at each state until we reach one of the base cases. If the problem asks for all optimal solutions, this backtracking routine will print them all. However, most problem authors usually set additional output criteria so that the selected optimal solution is unique (for easier judging).

Example: See Figure 3.8, bottom. The optimal final state is `reachable[2][1]`. The predecessor of this optimal final state is state `reachable[1][2]`. We now backtrack to `reachable[1][2]`. Next, see Figure 3.8, middle. The predecessor of state `reachable[1][2]` is state `reachable[0][12]`. We then backtrack to `reachable[0][12]`. As this is already one of the initial base states (at the first row), we know that an optimal solution is: (20→12) = price 8, then (12→2) = price 10, then (2→1) = price 1. However, as mentioned earlier in the problem description, this problem may have several other optimal solutions, e.g. We can also follow the path: `reachable[2][1]` → `reachable[1][6]` → `reachable[0][16]` which represents another optimal solution: (20→16) = price 4, then (16→6) = price 10, then (6→1) = price 5.

The second way is applicable mainly to the top-down DP approach where we utilize the strength of recursion and memoization to do the same job. Using the top-down DP code shown in Approach 4 above, we will add another function `void print_shop(int money, int g)` that has the same structure as `int shop(int money, int g)` except that it uses the values stored in the memo table to reconstruct the solution. A sample implementation (that only prints out one optimal solution) is shown below:

```
void print_shop(int money, int g) {          // this function returns void
  if (money < 0 || g == C) return;                     // similar base cases
  for (int model = 1; model <= price[g][0]; model++) // which model is opt?
    if (shop(money - price[g][model], g + 1) == memo[money][g]) {
      printf("%d%c", price[g][model], g == C-1 ? '\n' : '-');   // this one
      print_shop(money - price[g][model], g + 1);  // recurse to this state
      break;                                     // do not visit other states
}   }
```

Exercise 3.5.1.1: To verify your understanding of UVa 11450 problem discussed in this section, determine what is the output for test case D below?

Test case D with $M = 25$, $C = 3$:
Price of the 3 models of garment `g = 0` $\rightarrow$ 6 4 8
Price of the 2 models of garment `g = 1` $\rightarrow$ 10 6
Price of the 4 models of garment `g = 2` $\rightarrow$ 7 3 1 5

Exercise 3.5.1.2: Is the following state formulation `shop(g, model)`, where `g` represents the current garment and `model` represents the current model, appropriate and exhaustive for UVa 11450 problem?

Exercise 3.5.1.3: Add the space saving trick to the bottom-up DP code in Approach 5!

3.5.2 Classical Examples

The problem UVa 11450 - Wedding Shopping above is a (relatively easy) non-classical DP problem, where we had to come up with the correct DP states and transitions *by ourself*. However, there are many other *classical* problems with efficient DP solutions, i.e. their DP states and transitions are *well-known*. Therefore, such classical DP problems and their solutions should be mastered by every contestant who wishes to do well in ICPC or IOI! In this section, we list down six classical DP problems and their solutions. Note: Once you understand the basic form of these DP solutions, try solving the programming exercises that enumerate their *variants*.

1. Max 1D Range Sum

Abridged problem statement of UVa 507 - Jill Rides Again: Given an integer array `A` containing $n \leq 20K$ non-zero integers, determine the maximum (1D) range sum of `A`. In other words, find the maximum Range Sum Query (RSQ) between two indices `i` and `j` in `[0..n-1]`, that is: `A[i] + A[i+1] + A[i+2] +...+ A[j]` (also see Section 2.4.3 and 2.4.4).

A Complete Search algorithm that tries all possible $O(n^2)$ pairs of `i` and `j`, computes the required `RSQ(i, j)` in $O(n)$, and finally picks the maximum one runs in an overall time complexity of $O(n^3)$. With n up to $20K$, this is a TLE solution.

In Section 2.4.4, we have discussed the following DP strategy: Pre-process array `A` by computing `A[i] += A[i-1]` $\forall$`i` $\in$ `[1..n-1]` so that `A[i]` contains the sum of integers in subarray `A[0..i]`. We can now compute `RSQ(i, j)` in $O(1)$: `RSQ(0, j) = A[j]` and `RSQ(i, j) = A[j] - A[i-1]` $\forall$`i > 0`. With this, the Complete Search algorithm above can be made to run in $O(n^2)$. For n up to $20K$, this is still a TLE approach. However, this technique is still useful in other cases (see the usage of this 1D Range Sum in Section 8.4.2).

There is an even better algorithm for this problem. The main part of Jay Kadane's $O(n)$ (can be viewed as a greedy or DP) algorithm to solve this problem is shown below.

```
// inside int main()
  int n = 9, A[] = { 4, -5, 4, -3, 4, 4, -4, 4, -5 };   // a sample array A
  int sum = 0, ans = 0;          // important, ans must be initialized to 0
  for (int i = 0; i < n; i++) {                        // linear scan, O(n)
    sum += A[i];                       // we greedily extend this running sum
    ans = max(ans, sum);                   // we keep the maximum RSQ overall
    if (sum < 0) sum = 0;                     // but we reset the running sum
  }                                                  // if it ever dips below 0
  printf("Max 1D Range Sum = %d\n", ans);
```

Source code: `ch3_04_Max1DRangeSum.cpp/java`

The key idea of Kadane's algorithm is to keep a running sum of the integers seen so far and greedily reset that to 0 if the running sum dips below 0. This is because re-starting from 0 is always better than continuing from a negative running sum. Kadane's algorithm is the required algorithm to solve this UVa 507 problem as $n \leq 20K$.

Note that we can also view this Kadane's algorithm as a DP solution. At each step, we have two choices: We can either leverage the previously accumulated maximum sum, or begin a new range. The DP variable `dp(i)` thus represents the maximum sum of a range of integers that ends with element `A[i]`. Thus, the final answer is the maximum over all the values of `dp(i)` where i ∈ `[0..n-1]`. If zero-length ranges are allowed, then 0 must also be considered as a possible answer. The implementation above is essentially an efficient version that utilizes the space saving trick discussed earlier.

2. Max 2D Range Sum

Abridged problem statement of UVa 108 - Maximum Sum: Given an $n \times n$ $(1 \leq n \leq 100)$ square matrix of integers `A` where each integer ranges from `[-127..127]`, find a sub-matrix of `A` with the maximum sum. For example: The 4×4 matrix $(n = 4)$ in Table 3.3.A below has a 3×2 sub-matrix on the lower-left with maximum sum of 9 + 2 - 4 + 1 - 1 + 8 = 15.

A					B					C				
	0	-2	-7	0		0	-2	-9	-9		0	-2	-9	-9
	9	2	-6	2		9	9	-4	2		9	9	-4	2
	-4	1	-4	1		5	6	-11	-8		5	6	-11	-8
	-1	8	0	-2		4	13	-4	-3		4	13	-4	-3

Table 3.3: UVa 108 - Maximum Sum

Attacking this problem naïvely using a Complete Search as shown below does not work as it runs in $O(n^6)$. For the largest test case with $n = 100$, an $O(n^6)$ algorithm is too slow.

```
maxSubRect = -127*100*100;     // the lowest possible value for this problem
for (int i = 0; i < n; i++) for (int j = 0; j < n; j++) // start coordinate
  for (int k = i; k < n; k++) for (int l = j; l < n; l++) {     // end coord
    subRect = 0;                           // sum the items in this sub-rectangle
    for (int a = i; a <= k; a++) for (int b = j; b <= l; b++)
      subRect += A[a][b];
    maxSubRect = max(maxSubRect, subRect); }          // the answer is here
```

The solution for the Max 1D Range Sum in the previous subsection can be extended to two (or more) dimensions as long as the inclusion-exclusion principle is properly applied. The only difference is that while we dealt with overlapping sub-ranges in Max 1D Range Sum, we will deal with overlapping sub-matrices in Max 2D Range Sum. We can turn the $n \times n$ input matrix into an $n \times n$ *cumulative sum matrix* where `A[i][j]` no longer contains its own value, but the sum of all items within sub-matrix `(0, 0)` to `(i, j)`. This can be done simultaneously while reading the input and still runs in $O(n^2)$. The code shown below turns the input square matrix (see Table 3.3.A) into a cumulative sum matrix (see Table 3.3.B).

```
scanf("%d", &n);                        // the dimension of input square matrix
for (int i = 0; i < n; i++) for (int j = 0; j < n; j++) {
  scanf("%d", &A[i][j]);
  if (i > 0) A[i][j] += A[i - 1][j];              // if possible, add from top
  if (j > 0) A[i][j] += A[i][j - 1];             // if possible, add from left
  if (i > 0 && j > 0) A[i][j] -= A[i - 1][j - 1];     // avoid double count
}                                            // inclusion-exclusion principle
```

With the sum matrix, we can answer the sum of any sub-matrix `(i, j)` to `(k, l)` in $O(1)$ using the code below. For example, let's compute the sum of `(1, 2)` to `(3, 3)`. We split the sum into 4 parts and compute `A[3][3] - A[0][3] - A[3][1] + A[0][1] = -3 - 13 - (-9) + (-2) = -9` as highlighted in Table 3.3.C. With this $O(1)$ DP formulation, the Max 2D Range Sum problem can now be solved in $O(n^4)$. For the largest test case of UVa 108 with $n = 100$, this is still fast enough.

```
maxSubRect = -127*100*100;    // the lowest possible value for this problem
for (int i = 0; i < n; i++) for (int j = 0; j < n; j++) // start coordinate
  for (int k = i; k < n; k++) for (int l = j; l < n; l++) {    // end coord
    subRect = A[k][l];      // sum of all items from (0, 0) to (k, l): O(1)
    if (i > 0) subRect -= A[i - 1][l];                                // O(1)
    if (j > 0) subRect -= A[k][j - 1];                                // O(1)
    if (i > 0 && j > 0) subRect += A[i - 1][j - 1];                   // O(1)
    maxSubRect = max(maxSubRect, subRect); }          // the answer is here
```

Source code: `ch3_05_UVa108.cpp/java`

From these two examples—the Max 1D and 2D Range Sum Problems—we can see that not every range problem requires a Segment Tree or a Fenwick Tree as discussed in Section 2.4.3 or 2.4.4. Static-input range-related problems are often solvable with DP techniques. It is also worth mentioning that the solution for a range problem is very natural to produce with bottom-up DP techniques as the operand is already a 1D or a 2D array. We can still write the recursive top-down solution for a range problem, but the solution is not as natural.

3. Longest Increasing Subsequence (LIS)

Given a sequence {`A[0]`, `A[1]`,..., `A[n-1]`}, determine its Longest Increasing Subsequence (LIS)[13]. Note that these 'subsequences' are not necessarily contiguous. Example: $n = 8$, $A = \{\underline{-7}, 10, 9, \underline{2, 3, 8}, 8, 1\}$. The length-4 LIS is {-7, 2, 3, 8}.

[13]There are other variants of this problem, including the Longest *Decreasing* Subsequence and Longest *Non Increasing/Decreasing* Subsequence. The increasing subsequences can be modeled as a Directed Acyclic Graph (DAG) and finding the LIS is equivalent to finding the Longest Paths in the DAG (see Section 4.7.1).

Index	0	1	2	3	4	5	6	7
A	-7	10	9	2	3	8	8	1
LIS(i)	1	2	2	2	3	4	4	2

Figure 3.9: Longest Increasing Subsequence

As mentioned in Section 3.1, a naïve Complete Search that enumerates all possible subsequences to find the longest increasing one is too slow as there are $O(2^n)$ possible subsequences. Instead of trying all possible subsequences, we can consider the problem with a different approach. We can write the state of this problem with just one parameter: `i`. Let `LIS(i)` be the LIS ending at index `i`. We know that `LIS(0) = 1` as the first number in `A` is itself a subsequence. For `i` $\geq$ `1`, `LIS(i)` is slightly more complex. We need to find the index `j` such that `j` $<$ `i` and `A[j]` $<$ `A[i]` and `LIS(j)` is the largest. Once we have found this index `j`, we know that `LIS(i) = 1 + LIS(j)`. We can write this recurrence formally as:

1. `LIS(0) = 1` // the base case
2. `LIS(i) = max(LIS(j) + 1)`, $\forall$`j` $\in$ `[0..i-1]` and `A[j]` $<$ `A[i]` // the recursive case, one more than the previous best solution ending at `j` for all `j` $<$ `i`.

The answer is the largest value of `LIS(k)` $\forall$`k` $\in$ `[0..n-1]`.

Now let's see how this algorithm works (also see Figure 3.9):

- `LIS(0)` is 1, the first number in `A` = {-7}, the base case.
- `LIS(1)` is 2, as we can extend `LIS(0)` = {-7} with {10} to form {-7, 10} of length 2.
  The best `j` for `i = 1` is `j = 0`.
- `LIS(2)` is 2, as we can extend `LIS(0)` = {-7} with {9} to form {-7, 9} of length 2.
  We cannot extend `LIS(1)` = {-7, 10} with {9} as it is non increasing.
  The best `j` for `i = 2` is `j = 0`.
- `LIS(3)` is 2, as we can extend `LIS(0)` = {-7} with {2} to form {-7, 2} of length 2.
  We cannot extend `LIS(1)` = {-7, 10} with {2} as it is non-increasing.
  We also cannot extend `LIS(2)` = {-7, 9} with {2} as it is also non-increasing.
  The best `j` for `i = 3` is `j = 0`.
- `LIS(4)` is 3, as we can extend `LIS(3)` = {-7, 2} with {3} to form {-7, 2, 3}.
  This is the best choice among the possibilities.
  The best `j` for `i = 4` is `j = 3`.
- `LIS(5)` is 4, as we can extend `LIS(4)` = {-7, 2, 3} with {8} to form {-7, 2, 3, 8}.
  This is the best choice among the possibilities.
  The best `j` for `i = 5` is `j = 4`.
- `LIS(6)` is 4, as we can extend `LIS(4)` = {-7, 2, 3} with {8} to form {-7, 2, 3, 8}.
  This is the best choice among the possibilities.
  The best `j` for `i = 6` is `j = 4`.
- `LIS(7)` is 2, as we can extend `LIS(0)` = {-7} with {1} to form {-7, 1}.
  This is the best choice among the possibilities.
  The best `j` for `i = 7` is `j = 0`.
- The answers lie at `LIS(5)` or `LIS(6)`; both values (LIS lengths) are 4.
  See that the index `k` where `LIS(k)` is the highest can be anywhere in `[0..n-1]`.

There are clearly many overlapping sub-problems in LIS problem because to compute `LIS(i)`, we need to compute `LIS(j)` $\forall$`j` $\in$ `[0..i-1]`. However, there are only n distinct states, the indices of the LIS ending at index `i`, $\forall$`i` $\in$ `[0..n-1]`. As we need to compute each state with an $O(n)$ loop, this DP algorithm runs in $O(n^2)$.

If needed, the LIS solution(s) can be reconstructed by storing the predecessor information (the arrows in Figure 3.9) and tracing the arrows from index `k` that contain the highest value of `LIS(k)`. For example, `LIS(5)` is the optimal final state. Check Figure 3.9. We can trace the arrows as follow: `LIS(5)` $\rightarrow$ `LIS(4)` $\rightarrow$ `LIS(3)` $\rightarrow$ `LIS(0)`, so the optimal solution (read backwards) is index {0, 3, 4, 5} or {-7, 2, 3, 8}.

The LIS problem can also be solved using the *output-sensitive* $O(n \log k)$ greedy + D&C algorithm (where k is the length of the LIS) instead of $O(n^2)$ by maintaining an array that is *always sorted* and therefore amenable to binary search. Let array `L` be an array such that `L(i)` represents the smallest ending value of all length-`i` LISs found so far. Though this definition is slightly complicated, it is easy to see that it is always ordered—`L(i-1)` will always be smaller than `L(i)` as the second-last element of any LIS (of length-i) is smaller than its last element. As such, we can binary search array `L` to determine the longest possible subsequence we can create by appending the current element `A[i]`—simply find the index of the last element in `L` that is less than `A[i]`. Using the same example, we will update array `L` step by step using this algorithm:

- Initially, at `A[0] = -7`, we have `L =` {`-7`}.
- We can insert `A[1] = 10` at `L[1]` so that we have a length-2 LIS, `L =` {`-7, 10`}.
- For `A[2] = 9`, we replace `L[1]` so that we have a 'better' length-2 LIS ending: `L =` {`-7, 9`}.
  This is a *greedy* strategy. By storing the LIS with smaller ending value, we maximize our ability to further extend the LIS with future values.
- For `A[3] = 2`, we replace `L[1]` to get an 'even better' length-2 LIS ending: `L =` {`-7, 2`}.
- We insert `A[4] = 3` at `L[2]` so that we have a longer LIS, `L =` {`-7, 2, 3`}.
- We insert `A[5] = 8` at `L[3]` so that we have a longer LIS, `L =` {`-7, 2, 3, 8`}.
- For `A[6] = 8`, nothing changes as `L[3] = 8`.
  `L =` {`-7, 2, 3, 8`} remains unchanged.
- For `A[7] = 1`, we improve `L[1]` so that `L =` {`-7, 1, 3, 8`}.
  This illustrates how the array `L` is *not* the LIS of `A`. This step is important as there can be longer subsequences *in the future* that may extend the length-2 subsequence at `L[1] = 1`. For example, try this test case: `A =` {`-7, 10, 9, 2, 3, 8, 8, 1, 2, 3, 4`}. The length of LIS for this test case is 5.
- The answer is the largest length of the sorted array `L` at the end of the process.

Source code: `ch3_06_LIS.cpp/java`

---

4. 0-1 Knapsack (Subset Sum)

Problem[14]: Given n items, each with its own value V_i and weight W_i, $\forall i \in$ `[0..n-1]`, and a maximum knapsack size S, compute the maximum value of the items that we can carry, if we can either[15] ignore or take a particular item (hence the term 0-1 for ignore/take).

[14]This problem is also known as the Subset Sum problem. It has a similar problem description: Given a set of integers and an integer S, is there a (non-empty) subset that has a sum equal to S?

[15]There are other variants of this problem, e.g. the Fractional Knapsack problem with Greedy solution.

Example: $n = 4$, $V = \{100, 70, 50, 10\}$, $W = \{10, 4, 6, 12\}$, $S = 12$.
If we select item 0 with weight 10 and value 100, we cannot take any other item. Not optimal.
If we select item 3 with weight 12 and value 10, we cannot take any other item. Not optimal.
If we select item 1 and 2, we have total weight 10 and total value 120. This is the maximum.

Solution: Use these Complete Search recurrences `val(id, remW)` where `id` is the index of the current item to be considered and `remW` is the remaining weight left in the knapsack:

1. `val(id, 0) = 0` // if `remW = 0`, we cannot take anything else
2. `val(n, remW) = 0` // if `id = n`, we have considered all items
3. if `W[id]` $>$ `remW`, we have no choice but to ignore this item
`val(id, remW) = val(id + 1, remW)`
4. if `W[id]` $\leq$ `remW`, we have two choices: ignore or take this item; we take the maximum
`val(id, remW) = max(val(id + 1, remW), V[id] + val(id + 1, remW - W[id]))`

The answer can be found by calling `value(0, S)`. Note the overlapping sub-problems in this 0-1 Knapsack problem. Example: After taking item 0 and ignoring item 1-2, we arrive at state `(3, 2)`—at the third item (`id = 3`) with two units of weight left (`remW = 2`). After ignoring item 0 and taking item 1-2, we also arrive at the same state `(3, 2)`. Although there are overlapping sub-problems, there are only $O(nS)$ possible distinct states (as `id` can vary between `[0..n-1]` and `remW` can vary between `[0..S]`)! We can compute each of these states in $O(1)$, thus the overall time complexity[16] of this DP solution is $O(nS)$.

Note: The top-down version of this DP solution is often faster than the bottom-up version. This is because not all states are actually visited, and hence the critical DP states involved are actually only a (very small) subset of the entire state space. Remember: The top-down DP only visits *the required states* whereas bottom-up DP visits *all distinct states*. Both versions are provided in our source code library.

Source code: `ch3_07_UVa10130.cpp/java`

5. Coin Change (CC) - The General Version

Problem: Given a target amount V cents and a list of denominations for n coins, i.e. we have `coinValue[i]` (in cents) for coin types `i` $\in$ `[0..n-1]`, what is the minimum number of coins that we must use to represent V? Assume that we have unlimited supply of coins of any type (also see Section 3.4.1).

Example 1: $V = 10$, $n = 2$, `coinValue` $= \{1, 5\}$; We can use:
A. Ten 1 cent coins $= 10 \times 1 = 10$; Total coins used $= 10$
B. One 5 cents coin + Five 1 cent coins $= 1 \times 5 + 5 \times 1 = 10$; Total coins used $= 6$
C. Two 5 cents coins $= 2 \times 5 = 10$; Total coins used $= 2 \rightarrow$ Optimal

We can use the Greedy algorithm if the coin denominations are suitable (see Section 3.4.1). Example 1 above is solvable with the Greedy algorithm. However, for general cases, we have to use DP. See Example 2 below:

Example 2: $V = 7$, $n = 4$, `coinValue` $= \{1, 3, 4, 5\}$
The Greedy approach will produce 3 coins as its result as 5+1+1 = 7, but the optimal solution is actually 2 coins (from 4+3)!

Solution: Use these Complete Search recurrence relations for `change(value)`, where `value` is the remaining amount of cents that we need to represent in coins:

[16] If S is large such that $NS >> 1M$, this DP solution is not feasible, even with the space saving trick!

1. `change(0) = 0` // we need 0 coins to produce 0 cents
2. `change(<` 0`) =` ∞ // in practice, we can return a large positive value
3. `change(value) = 1 + min(change(value - coinValue[i]))` $\forall$`i` $\in$ `[0..n-1]`

The answer can be found in the return value of `change(V)`.

<0	0	1	2	3	4	5	6	7	8	9	10
∞	0	1	2	3	4	1	2	3	4	5	2

V = 10, N = 2, coinValue= {1, 5}

Figure 3.10: Coin Change

Figure 4.2.3 shows that:
`change(0) = 0` and `change(<` 0`) =` ∞: These are the base cases.
`change(1) = 1`, from `1 + change(1-1)`, as `1 + change(1-5)` is infeasible (returns ∞).
`change(2) = 2`, from `1 + change(2-1)`, as `1 + change(2-5)` is also infeasible (returns ∞).
... same thing for `change(3)` and `change(4)`.
`change(5) = 1`, from `1 + change(5-5) = 1` coin, smaller than `1 + change(5-1) = 5` coins.
... and so on until `change(10)`.
The answer is in `change(V)`, which is `change(10) = 2` in this example.

We can see that there are a lot of overlapping sub-problems in this Coin Change problem (e.g. both `change(10)` and `change(6)` require the value of `change(5)`). However, there are only $O(V)$ possible distinct states (as `value` can vary between `[0..V]`)! As we need to try n types of coins per state, the overall time complexity of this DP solution is $O(nV)$.

A variant of this problem is to count *the number of possible (canonical) ways* to get value V cents using a list of denominations of n coins. For example 1 above, the answer is 3: {1+1+1+1+1 + 1+1+1+1+1, 5 + 1+1+1+1+1, 5 + 5}.

Solution: Use these Complete Search recurrence relation: `ways(type, value)`, where `value` is the same as above but we now have one more parameter `type` for the index of the coin type that we are currently considering. This second parameter `type` is important as this solution considers the coin types sequentially. Once we choose to ignore a certain coin type, we should not consider it again to avoid double-counting:

1. `ways(type, 0) = 1` // one way, use nothing
2. `ways(type, <0) = 0` // no way, we cannot reach negative value
3. `ways(n, value) = 0` // no way, we have considered all coin types $\in$ `[0..n-1]`
4. `ways(type, value) = ways(type + 1, value) +` // if we ignore this coin type,
`ways(type, value - coinValue[type])` // plus if we use this coin type

There are only $O(nV)$ possible distinct states. Since each state can be computed in $O(1)$, the overall time complexity[17] of this DP solution is $O(nV)$. The answer can be found by calling `ways(0, V)`. Note: If the coin values are not changed and you are given many queries with different `V`, then we can choose *not* to reset the memo table. Therefore, we run this $O(nV)$ algorithm once and just perform an $O(1)$ lookup for subsequent queries.

Source code (this coin change variant): `ch3_08_UVa674.cpp/java`

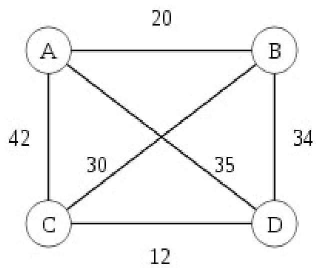
---

[17]If V is large such that $nV >> 1M$, this DP solution is not feasible even with the space saving trick!

6. Traveling Salesman Problem (TSP)

Problem: Given n cities and their pairwise distances in the form of a matrix `dist` of size $n \times n$, compute the cost of making a tour[18] that starts from any city s, goes through all the other $n-1$ cities *exactly once*, and finally returns to the starting city s.

Example: The graph shown in Figure 3.11 has $n = 4$ cities. Therefore, we have $4! = 24$ possible tours (permutations of 4 cities). One of the minimum tours is `A-B-C-D-A` with a cost of `20+30+12+35 = 97` (notice that there can be more than one optimal solution).



	A	B	C	D
A	0	20	42	35
B	20	0	30	34
C	42	30	0	12
D	35	34	12	0

Figure 3.11: A Complete Graph

A 'brute force' TSP solution (either iterative or recursive) that tries all $O((n-1)!)$ possible tours (fixing the first city to vertex A in order to take advantage of symmetry) is only effective when n is at most 12 as $11! \approx 40M$. When $n > 12$, such brute force solutions will get a TLE in programming contests. However, if there are multiple test cases, the limit for such 'brute force' TSP solution is probably just $n = 11$.

We can utilize DP for TSP since the computation of sub-tours is clearly overlapping, e.g. the tour $A - B - C -(n-3)$ *other cities that finally return to* A clearly overlaps the tour $A - C - B -$*the same* $(n-3)$ *other cities that also return to* A. If we can avoid re-computing the lengths of such sub-tours, we can save a lot of computation time. However, a distinct state in TSP depends on two parameters: The last city/vertex visited `pos` and something that we may have not seen before—a *subset* of visited cities.

There are many ways to represent a set. However, since we are going to pass this set information around as a parameter of a recursive function (if using top-down DP), the representation we use must be lightweight and efficient! In Section 2.2, we have presented a viable option for this usage: The *bitmask*. If we have n cities, we use a binary integer of length n. If bit i is '1' (on), we say that item (city) i is inside the set (it has been visited) and item i is not inside the set (and has not been visited) if the bit is instead '0' (off). For example: `mask`$= 18_{10} = 10010_2$ implies that items (cities) $\{1, 4\}$ are in[19] the set (and have been visited). Recall that to check if bit i is on or off, we can use `mask & (1 << i)`. To set bit i, we can use `mask |= (1 << i)`.

Solution: Use these Complete Search recurrence relations for `tsp(pos, mask)`:

1. `tsp(pos,` $2^n - 1$`) = dist[pos][0]` // all cities have been visited, return to starting city
// Note: `mask = (1 << n) - 1` or $2^n - 1$ implies that all n bits in `mask` are on.
2. `tsp(pos, mask) = min(dist[pos][nxt] + tsp(nxt, mask | (1 << nxt)))`
// $\forall$ `nxt` $\in$ `[0..n-1]`, `nxt != pos`, and `(mask & (1 << nxt))` is '0' (turned off)
// We basically tries all possible next cities that have not been visited before at each step.

There are only $O(n \times 2^n)$ distinct states because there are n cities and we remember up to 2^n other cities that have been visited in each tour. Each state can be computed in $O(n)$,

[18] Such a tour is called a Hamiltonian tour, which is a cycle in an undirected graph which visits each vertex exactly once and also returns to the starting vertex.

[19] Remember that in `mask`, indices starts from 0 and are counted from the right.

thus the overall time complexity of this DP solution is $O(2^n \times n^2)$. This allows us to solve up to[20] $n \approx 16$ as $16^2 \times 2^{16} \approx 17M$. This is not a huge improvement over the brute force solution but if the programming contest problem involving TSP has input size $11 \leq n \leq 16$, then DP is the solution, not brute force. The answer can be found by calling `tsp(0, 1)`: We start from city 0 (we can start from any vertex; but the simplest choice is vertex 0) and set `mask = 1` so that city 0 is never re-visited again.

Usually, DP TSP problems in programming contests require some kind of graph preprocessing to generate the distance matrix `dist` before running the DP solution. These variants are discussed in Section 8.4.3.

DP solutions that involve a (small) set of Booleans as one of the parameters are more well known as the DP with bitmask technique. More challenging DP problems involving this technique are discussed in Section 8.3 and 9.2.

Visualization: `www.comp.nus.edu.sg/~stevenha/visualization/rectree.html`

Source code: `ch3_09_UVa10496.cpp/java`

Exercise 3.5.2.1: The solution for the Max 2D Range Sum problem runs in $O(n^4)$. Actually, there exists an $O(n^3)$ solution that combines the DP solution for the Max Range 1D Sum problem on one dimension and uses the same idea as proposed by Kadane on the other dimension. Solve UVa 108 with an $O(n^3)$ solution!

Exercise 3.5.2.2: The solution for the `Range Minimum Query(i, j)` on 1D arrays in Section 2.4.3 uses Segment Tree. This is overkill if the given array is static and unchanged throughout all the queries. Use a DP technique to answer `RMQ(i, j)` in $O(n \log n)$ preprocessing and $O(1)$ per query.

Exercise 3.5.2.3: Solve the LIS problem using the $O(n \log k)$ solution and *also* reconstruct one of the LIS.

Exercise 3.5.2.4: Can we use an iterative Complete Search technique that tries all possible subsets of n items as discussed in Section 3.2.1 to solve the 0-1 Knapsack problem? What are the limitations, if any?

Exercise 3.5.2.5*: Suppose we add one more parameter to this classic 0-1 Knapsack problem. Let K_i denote the number of copies of item i for use in the problem. Example: $n = 2$, $V = \{100, 70\}$, $W = \{5, 4\}$, $K = \{2, 3\}$, $S = 17$ means that there are two copies of item 0 with weight 5 and value 100 and there are three copies of item 1 with weight 4 and value 70. The optimal solution for this example is to take one of item 0 and three of item 1, with a total weight of 17 and total value 310. Solve new variant of the problem assuming that $1 \leq n \leq 500$, $1 \leq S \leq 2000$, $n \leq \sum_{i=0}^{n-1} K_i \leq 40000$! Hint: Every integer can be written as a sum of powers of 2.

Exercise 3.5.2.6*: The DP TSP solution shown in this section can still be *slightly* enhanced to make it able to solve test case with $n = 17$ in contest environment. Show the required minor change to make this possible! Hint: Consider symmetry!

Exercise 3.5.2.7*: On top of the minor change asked in **Exercise 3.5.2.5***, what *other change(s)* is/are needed to have a DP TSP solution that is able to handle $n = 18$ (or even $n = 19$, but with much lesser number of test cases)?

---

[20]As programming contest problems usually require exact solutions, the DP-TSP solution presented here is already one of the best solutions. In real life, the TSP often needs to be solved for instances with thousands of cities. To solve larger problems like that, we have non-exact approaches like the ones presented in [26].

3.5.3 Non-Classical Examples

Although DP is the single most popular problem type with the highest frequency of appearance in recent programming contests, the classical DP problems in their *pure forms* usually never appear in modern ICPCs or IOIs again. We study them to understand DP, but we have to learn to solve many other non-classical DP problems (which may become classic in the near future) and develop our 'DP skills' in the process. In this subsection, we discuss two more non-classical examples, adding to the UVa 11450 - Wedding Shopping problem that we have discussed in detail earlier. We have also selected some easier non-classical DP problems as programming exercises. Once you have cleared most of these problems, you are welcome to explore the more challenging ones in the other sections in this book, e.g. Section 4.7.1, 5.4, 5.6, 6.5, 8.3, 9.2, 9.21, etc.

1. UVa 10943 - How do you add?

Abridged problem description: Given an integer n, how many ways can K non-negative integers less than or equal to n add up to n? Constraints: $1 \leq n, K \leq 100$. Example: For $n = 20$ and $K = 2$, there are 21 ways: $0 + 20$, $1 + 19$, $2 + 18$, $3 + 17$, $\ldots$, $20 + 0$.

Mathematically, the number of ways can be expressed as ${}^{(n+k-1)}C_{(k-1)}$ (see Section 5.4.2 about Binomial Coefficients). We will use this simple problem to re-illustrate Dynamic Programming principles that we have discussed in this section, especially the process of deriving appropriate states for a problem and deriving correct transitions from one state to another given the base case(s).

First, we have to determine the parameters of this problem to be selected to represent distinct states of this problem. There are only two parameters in this problem, n and K. Therefore, there are only 4 possible combinations:

1. If we do not choose any of them, we cannot represent a state. This option is ignored.
2. If we choose only n, then we do not know how many numbers $\leq n$ have been used.
3. If we choose only K, then we do not know the target sum n.
4. Therefore, the state of this problem should be represented by a pair (or tuple) (n, K). The order of chosen parameter(s) does not matter, i.e. the pair (K, n) is also OK.

Next, we have to determine the base case(s). It turns out that this problem is very easy when $K = 1$. Whatever n is, there is only *one way* to add exactly one number less than or equal to n to get n: Use n itself. There is no other base case for this problem.

For the general case, we have this recursive formulation which is not too difficult to derive: At state (n, K) where $K > 1$, we can split n into one number $X \in$ `[0..`n`]` and $n - X$, i.e. $n = X + (n - X)$. By doing this, we arrive at the subproblem $(n - X, K - 1)$, i.e. given a number $n - X$, how many ways can $K - 1$ numbers less than or equal to $n - X$ add up to $n - X$? We can then sum all these ways.

These ideas can be written as the following Complete Search recurrence `ways(n, K)`:

1. `ways(n, 1) = 1` // we can only use 1 number to add up to n, the number n itself
2. `ways(n, K) =` $\sum_{X=0}^{n}$ `ways(n - X, K - 1)` // sum all possible ways, recursively

This problem has overlapping sub-problems. For example, the test case $n = 1, K = 3$ has overlapping sub-problems: The state $(n = 0, K = 1)$ is reached twice (see Figure 4.39 in Section 4.7.1). However, there are only $n \times K$ possible states of (n, K). The cost of computing each state is $O(n)$. Thus, the overall time complexity is $O(n^2 \times K)$. As $1 \leq n, K \leq 100$, this is feasible. The answer can be found by calling `ways(n, K)`.

Note that this problem actually just needs the result modulo $1M$ (i.e. the last 6 digits of the answer). See Section 5.5.8 for a discussion on modulo arithmetic computation.

Source code: `ch3_10_UVa10943.cpp/java`

2. UVa 10003 - Cutting Sticks

Abridged problem statement: Given a stick of length $1 \leq l \leq 1000$ and $1 \leq n \leq 50$ cuts to be made to the stick (the cut coordinates, lying in the range `[0..`l`]`, are given). The cost of a cut is determined by the length of the stick to be cut. Your task is to find a cutting sequence so that the overall cost is minimized.

Example: $l = 100$, $n = 3$, and cut coordinates: `A` $= \{25, 50, 75\}$ (already sorted)

If we cut from left to right, then we will incur cost = 225.
1. First cut is at coordinate 25, total cost so far = 100;
2. Second cut is at coordinate 50, total cost so far = 100 + 75 = 175;
3. Third cut is at coordinate 75, final total cost = 175 + 50 = 225;

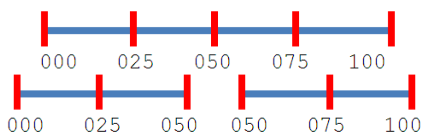


Figure 3.12: Cutting Sticks Illustration

However, the optimal answer is 200.
1. First cut is at coordinate 50, total cost so far = 100; (this cut is shown in Figure 3.12)
2. Second cut is at coordinate 25, total cost so far = 100 + 50 = 150;
3. Third cut is at coordinate 75, final total cost = 150 + 50 = 200;

How do we tackle this problem? An initial approach might be this Complete Search algorithm: Try all possible cutting points. Before that, we have to select an appropriate state definition for the problem: The (intermediate) sticks. We can describe a stick with its two endpoints: `left` and `right`. However, these two values can be very huge and this can complicate the solution later when we want to memoize their values. We can take advantage of the fact that there are only $n+1$ smaller sticks after cutting the original stick n times. The endpoints of each smaller stick can be described by 0, the cutting point coordinates, and l. Therefore, we will add two more coordinates so that `A` $= \{0$, the original `A`, and $l\}$ so that we can denote a stick by the indices of its endpoints in `A`.

We can then use these recurrences for `cut(left, right)`, where `left/right` are the left/right indices of the current stick w.r.t. `A`. Originally, the stick is described by `left = 0` and `right =` n`+1`, i.e. a stick with length `[0..`l`]`:

1. `cut(i-1, i) = 0`, $\forall$`i` $\in$ `[1..`n`+1]` // if `left + 1 = right` where `left` and `right` are the indices in `A`, then we have a stick segment that does not need to be divided further.

2. `cut(left, right) = min(cut(left, i) + cut(i, right) + (A[right]-A[left]))` $\forall$`i` $\in$ `[left+1..right-1]` // try all possible cutting points and pick the best.
The cost of a cut is the length of the current stick, captured in `(A[right]-A[left])`.
The answer can be found at `cut(0,` n`+1)`.

Now let's analyze the time complexity. Initially, we have n choices for the cutting points. Once we cut at a certain cutting point, we are left with $n - 1$ further choices of the second cutting point. This repeats until we are left with zero cutting points. Trying all possible cutting points this way leads to an $O(n!)$ algorithm, which is impossible for $1 \leq n \leq 50$.

However, this problem has overlapping sub-problems. For example, in Figure 3.12 above, cutting at index 2 (cutting point = 50) produces two states: `(0, 2)` and `(2, 4)`. The same state `(2, 4)` can also be reached by cutting at index 1 (cutting point 25) and then cutting at index 2 (cutting point 50). Thus, the search space is actually not that large. There are only $(n+2) \times (n+2)$ possible left/right indices or $O(n^2)$ distinct states and be memoized. The time to required to compute one state is $O(n)$. Thus, the overall time complexity (of the top-down DP) is $O(n^3)$. As $n \leq 50$, this is a feasible solution.

Source code: `ch3_11_UVa10003.cpp/java`

Exercise 3.5.3.1*: Almost all of the source code shown in this section (LIS, Coin Change, TSP, and UVa 10003 - Cutting Sticks) are written in a top-down DP fashion due to the preferences of the authors of this book. Rewrite them using the bottom-up DP approach.

Exercise 3.5.3.2*: Solve the Cutting Sticks problem in $O(n^2)$. Hint: Use Knuth-Yao DP Speedup by utilizing that the recurrence satisfies Quadrangle Inequality (see [2]).

Remarks About Dynamic Programming in Programming Contests

Basic (Greedy and) DP techniques techniques are always included in popular algorithm textbooks, e.g. Introduction to Algorithms [7], Algorithm Design [38] and Algorithm [8]. In this section, we have discussed six classical DP problems and their solutions. A brief summary is shown in Table 3.4. These classical DP problems, if they are to appear in a programming contest today, will likely occur only as part of bigger and harder problems.

	1D RSQ	2D RSQ	LIS	Knapsack	CC	TSP
State	`(i)`	`(i,j)`	`(i)`	`(id,remW)`	`(v)`	`(pos,mask)`
Space	$O(n)$	$O(n^2)$	$O(n)$	$O(nS)$	$O(V)$	$O(n2^n)$
Transition	subarray	submatrix	all $j < i$	take/ignore	all n coins	all n cities
Time	$O(1)$	$O(1)$	$O(n^2)$	$O(nS)$	$O(nV)$	$O(2^n n^2)$

Table 3.4: Summary of Classical DP Problems in this Section

To help keep up with the growing difficulty and creativity required in these techniques (especially the non-classical DP), we recommend that you also read the TopCoder algorithm tutorials [30] and attempt the more recent programming contest problems.

In this book, we will revisit DP again on several occasions: Floyd Warshall's DP algorithm (Section 4.5), DP on (implicit) DAG (Section 4.7.1), String Alignment (Edit Distance), Longest Common Subsequence (LCS), other DP on String algorithms (Section 6.5), More Advanced DP (Section 8.3), and several topics on DP in Chapter 9.

In the past (1990s), a contestant who is good at DP can become a 'king of programming contests' as DP problems were usually the 'decider problems'. Now, mastering DP is a *basic* requirement! You cannot do well in programming contests without this knowledge. However, we have to keep reminding the readers of this book not to claim that they know DP if they only memorize the solutions of the classical DP problems! Try to master the art of DP problem solving: Learn to determine the states (the DP table) that can uniquely and efficiently represent sub-problems and also how to fill up that table, either via top-down recursion or bottom-up iteration.

There is no better way to master these problem solving paradigms than solving real programming problems! Here, we list several examples. Once you are familiar with the examples shown in this section, study the newer DP problems that have begun to appear in recent programming contests.

Programming Exercises solvable using Dynamic Programming:

- Max 1D Range Sum
  1. UVa 00507 - Jill Rides Again (standard problem)
  2. **UVa 00787 - Maximum Sub ... *** (max 1D range *product*, be careful with 0, use Java BigInteger, see Section 5.3)
  3. **UVa 10684 - The Jackpot *** (standard problem; easily solvable with the given sample source code)
  4. ***UVa 10755 - Garbage Heap **** (combination of max 2D range sum in two of the three dimensions—see below—and max 1D range sum using Kadane's algorithm on the third dimension)
  See more examples in Section 8.4.
- Max 2D Range Sum
  1. **UVa 00108 - Maximum Sum *** (discussed in this section with sample source code)
  2. UVa 00836 - Largest Submatrix (convert '0' to -INF)
  3. UVa 00983 - Localized Summing for ... (max 2D range sum, get submatrix)
  4. UVa 10074 - Take the Land (standard problem)
  5. UVa 10667 - Largest Block (standard problem)
  6. **UVa 10827 - Maximum Sum on ... *** (copy $n \times n$ matrix into $n \times 2n$ matrix; then this problem becomes a standard problem again)
  7. ***UVa 11951 - Area **** (use long long; max 2D range sum; prune the search space whenever possible)
- Longest Increasing Subsequence (LIS)
  1. UVa 00111 - History Grading (be careful of the ranking system)
  2. UVa 00231 - Testing the Catcher (straight-forward)
  3. UVa 00437 - The Tower of Babylon (can be modeled as LIS)
  4. **UVa 00481 - What Goes Up? *** (use $O(n \log k)$ LIS; print solution; see our sample source code)
  5. UVa 00497 - Strategic Defense Initiative (solution must be printed)
  6. UVa 01196 - Tiling Up Blocks (LA 2815, Kaohsiung03; sort all the blocks in increasing `L[i]`, then we get the classical LIS problem)
  7. UVa 10131 - Is Bigger Smarter? (sort elephants based on decreasing IQ; LIS on increasing weight)
  8. UVa 10534 - Wavio Sequence (must use $O(n \log k)$ LIS twice)
  9. *UVa 11368 - Nested Dolls* (sort in one dimension, LIS in the other)
  10. **UVa 11456 - Trainsorting *** (max(LIS(i) + LDS(i) - 1), $\forall i \in [0 \ldots n\text{-}1]$)
  11. **UVa 11790 - Murcia's Skyline *** (combination of LIS+LDS, weighted)

- 0-1 Knapsack (Subset Sum)
  1. UVa 00562 - Dividing Coins (use a one dimensional table)
  2. UVa 00990 - Diving For Gold (print the solution)
  3. UVa 01213 - Sum of Different Primes (LA 3619, Yokohama06, extension of 0-1 Knapsack, use three parameters: (id, remN, remK) on top of (id, remN))
  4. UVa 10130 - SuperSale (discussed in this section with sample source code)
  5. UVa 10261 - Ferry Loading (s: current car, left, right)
  6. **UVa 10616 - Divisible Group Sum *** (input can be -ve, use long long)
  7. UVa 10664 - Luggage (Subset Sum)
  8. **UVa 10819 - Trouble of 13-Dots *** (0-1 knapsack with 'credit card' twist!)
  9. *UVa 11003 - Boxes* (try all max weight from 0 to $max(weight[i]+capacity[i])$, $\forall i \in [0..n\text{-}1]$; if a max weight is known, how many boxes can be stacked?)
  10. UVa 11341 - Term Strategy (s: id, h_learned, h_left; t: learn module 'id' by 1 hour or skip)
  11. ***UVa 11566 - Let's Yum Cha **** (English reading problem, actually just a knapsack variant: double each dim sum and add one parameter to check if we have bought too many dishes)
  12. UVa 11658 - Best Coalition (s: id, share; t: form/ignore coalition with id)
- Coin Change (CC)
  1. UVa 00147 - Dollars (similar to UVa 357 and UVa 674)
  2. UVa 00166 - Making Change (two coin change variants in one problem)
  3. **UVa 00357 - Let Me Count The Ways *** (similar to UVa 147/674)
  4. UVa 00674 - Coin Change (discussed in this section with sample source code)
  5. **UVa 10306 - e-Coins *** (variant: each coin has two components)
  6. UVa 10313 - Pay the Price (modified coin change + DP 1D range sum)
  7. UVa 11137 - Ingenuous Cubrency (use long long)
  8. **UVa 11517 - Exact Change *** (a variation to the coin change problem)
- Traveling Salesman Problem (TSP)
  1. **UVa 00216 - Getting in Line *** (TSP, still solvable with backtracking)
  2. **UVa 10496 - Collecting Beepers *** (discussed in this section with sample source code; actually, since $n \leq 11$, this problem is still solvable with recursive backtracking and sufficient pruning)
  3. **UVa 11284 - Shopping Trip *** (requires shortest paths pre-processing; TSP variant where we can go home early; we just need to tweak the DP TSP recurrence a bit: at each state, we have one more option: go home early)

  See more examples in Section 8.4.3 and Section 9.2.
- Non Classical (The Easier Ones)
  1. UVa 00116 - Unidirectional TSP (similar to UVa 10337)
  2. *UVa 00196 - Spreadsheet* (notice that the dependencies of cells are acyclic; we can therefore memoize the direct (or indirect) value of each cell)
  3. *UVa 01261 - String Popping* (LA 4844, Daejeon10, a simple backtracking problem; but we use a `set<string>` to prevent the same state (a substring) from being checked twice)
  4. UVa 10003 - Cutting Sticks (discussed in details in this section with sample source code)
  5. UVa 10036 - Divisibility (must use offset technique as value can be negative)

6. *UVa 10086 - Test the Rods* (s: idx, rem1, rem2; which site that we are now, up to 30 sites; remaining rods to be tested at NCPC; and remaining rods to be tested at BCEW; t: for each site, we split the rods, x rods to be tested at NCPC and $m[i] - x$ rods to be tested at BCEW; print the solution)
7. **UVa 10337 - Flight Planner *** (DP; shortest paths on DAG)
8. UVa 10400 - Game Show Math (backtracking with clever pruning is sufficient)
9. *UVa 10446 - The Marriage Interview* (edit the given recursive function a bit, add memoization)
10. UVa 10465 - Homer Simpson (one dimensional DP table)
11. *UVa 10520 - Determine it* (just write the given formula as a top-down DP with memoization)
12. *UVa 10688 - The Poor Giant* (note that the sample in the problem description is a bit wrong, it should be: 1+(1+3)+(1+3)+(1+3) = 1+4+4+4 = 13, beating 14; otherwise a simple DP)
13. **UVa 10721 - Bar Codes *** (s: n, k; t: try all from 1 to m)
14. UVa 10910 - Mark's Distribution (two dimensional DP table)
15. UVa 10912 - Simple Minded Hashing (s: len, last, sum; t: try next char)
16. **UVa 10943 - How do you add? *** (discussed in this section with sample source code; s: n, k; t: try all the possible splitting points; alternative solution is to use the closed form mathematical formula: $C(n + k - 1, k - 1)$ which also needs DP, see Section 5.4)
17. *UVa 10980 - Lowest Price in Town* (simple)
18. *UVa 11026 - A Grouping Problem* (DP, similar idea with binomial theorem in Section 5.4)
19. UVa 11407 - Squares (can be memoized)
20. UVa 11420 - Chest of Drawers (s: prev, id, numlck; lock/unlock this chest)
21. UVa 11450 - Wedding Shopping (discussed in details in this section with sample source code)
22. UVa 11703 - sqrt log sin (can be memoized)

- Other Classical DP Problems in this Book
  1. Floyd Warshall's for All-Pairs Shortest Paths problem (see Section 4.5)
  2. String Alignment (Edit Distance) (see Section 6.5)
  3. Longest Common Subsequence (see Section 6.5)
  4. Matrix Chain Multiplication (see Section 9.20)
  5. Max (Weighted) Independent Set (on tree, see Section 9.22)
- Also see Section 4.7.1, 5.4, 5.6, 6.5, 8.3, 8.4 and parts of Chapter 9 for *more* programming exercises related to Dynamic Programming.

3.6 Solution to Non-Starred Exercises

Exercise 3.2.1.1: This is to avoid the division operator so that we only work with integers! If we iterate through `abcde` instead, we may encounter a non-integer result when we compute `fghij = abcde / N`.

Exercise 3.2.1.2: It wil get an AC too as 10! $\approx$ 3 million, about the same as the algorithm presented in Section 3.2.1.

Exercise 3.2.2.1: Modify the backtrack function to resemble this code:

```
void backtrack(int c) {
  if (c == 8 && row[b] == a) {         // candidate sol, (a, b) has 1 queen
    printf("%2d      %d", ++lineCounter, row[0] + 1);
    for (int j = 1; j < 8; j++) printf(" %d", row[j] + 1);
    printf("\n"); }
  for (int r = 0; r < 8; r++)                          // try all possible row
    if (col == b && r != a) continue;                         // ADD THIS LINE
    if (place(r, c)) {          // if can place a queen at this col and row
      row[c] = r; backtrack(c + 1);      // put this queen here and recurse
}   }
```

Exercise 3.3.1.1: This problem can be solved without the 'binary search the answer' technique. Simulate the journey once. We just need to find the largest fuel requirement in the entire journey and make the fuel tank be sufficient for it.

Exercise 3.5.1.1: Garment $g = 0$, take the third model (cost 8); Garment $g = 1$, take the first model (cost 10); Garment $g = 2$, take the first model (cost 7); Money used = 25. Nothing left. Test case C is also solvable with Greedy algorithm.

Exercise 3.5.1.2: No, this state formulation does not work. We need to know how much money we have left at each sub-problem so that we can determine if we still have enough money to buy a certain model of the current garment.

Exercise 3.5.1.3: The modified bottom-up DP code is shown below:

```
#include <cstdio>
#include <cstring>
using namespace std;

int main() {
  int g, money, k, TC, M, C, cur;
  int price[25][25];
  bool reachable[2][210]; // reachable table[ONLY TWO ROWS][money (<= 200)]
  scanf("%d", &TC);
  while (TC--) {
    scanf("%d %d", &M, &C);
    for (g = 0; g < C; g++) {
      scanf("%d", &price[g][0]);
      for (money = 1; money <= price[g][0]; money++)
        scanf("%d", &price[g][money]);
    }
```

```
    memset(reachable, false, sizeof reachable);
    for (g = 1; g <= price[0][0]; g++)
      if (M - price[0][g] >= 0)
        reachable[0][M - price[0][g]] = true;

    cur = 1;                                       // we start with this row
    for (g = 1; g < C; g++) {
      memset(reachable[cur], false, sizeof reachable[cur]);    // reset row
      for (money = 0; money < M; money++) if (reachable[!cur][money])
        for (k = 1; k <= price[g][0]; k++) if (money - price[g][k] >= 0)
          reachable[cur][money - price[g][k]] = true;
      cur = !cur;                         // IMPORTANT TRICK: flip the two rows
    }

    for (money = 0; money <= M && !reachable[!cur][money]; money++);

    if (money == M + 1) printf("no solution\n");  // last row has no on bit
    else                printf("%d\n", M - money);
} } // return 0;
```

Exercise 3.5.2.1: The $O(n^3)$ solution for Max 2D Range Sum problem is shown below:

```
scanf("%d", &n);                        // the dimension of input square matrix
for (int i = 0; i < n; i++) for (int j = 0; j < n; j++) {
  scanf("%d", &A[i][j]);
  if (j > 0) A[i][j] += A[i][j - 1];       // only add columns of this row i
}

maxSubRect = -127*100*100;    // the lowest possible value for this problem
for (int l = 0; l < n; l++) for (int r = l; r < n; r++) {
  subRect = 0;
  for (int row = 0; row < n; row++) {
    // Max 1D Range Sum on columns of this row i
    if (l > 0) subRect += A[row][r] - A[row][l - 1];
    else       subRect += A[row][r];

    // Kadane's algorithm on rows
    if (subRect < 0) subRect = 0;     // greedy, restart if running sum < 0
    maxSubRect = max(maxSubRect, subRect);
} }
```

Exercise 3.5.2.2: The solution is given in Section 9.33.

Exercise 3.5.2.3: The solution is already written inside `ch3_06_LIS.cpp/java`.

Exercise 3.5.2.4: The iterative Complete Search solution to generate and check all possible subsets of size n runs in $O(n \times 2^n)$. This is OK for $n \leq 20$ but too slow when $n > 20$. The DP solution presented in Section 3.5.2 runs in $O(n \times S)$. If S is not that large, we can have a much larger n than just 20 items.

3.7 Chapter Notes

Many problems in ICPC or IOI require a combination (see Section 8.4) of these problem solving strategies. If we have to nominate only one chapter in this book that contestants have to really master, we would choose this one.

In Table 3.5, we compare the four problem solving techniques in their likely results for various problem types. In Table 3.5 and the list of programming exercises in this section, you will see that there are *many more* Complete Search and DP problems than D&C and Greedy problems. Therefore, we recommend that readers concentrate on improving their Complete Search and DP skills.

	BF Problem	D&C Problem	Greedy Problem	DP Problem
BF Solution	AC	TLE/AC	TLE/AC	TLE/AC
D&C Solution	WA	AC	WA	WA
Greedy Solution	WA	WA	AC	WA
DP Solution	MLE/TLE/AC	MLE/TLE/AC	MLE/TLE/AC	AC
Frequency	High	(Very) Low	Low	High

Table 3.5: Comparison of Problem Solving Techniques (Rule of Thumb only)

We will conclude this chapter by remarking that for some real-life problems, especially those that are classified as NP-hard [7], many of the approaches discussed in this section will not work. For example, the 0-1 Knapsack Problem which has an $O(nS)$ DP complexity is too slow if S is big; TSP which has a $O(2^n \times n^2)$ DP complexity is too slow if n is any larger than 18 (see **Exercise 3.5.2.7***). For such problems, we can resort to heuristics or local search techniques such as Tabu Search [26, 25], Genetic Algorithms, Ant-Colony Optimizations, Simulated Annealing, Beam Search, etc. However, all these heuristic-based searches are not in the IOI syllabus [20] and also not widely used in ICPC.

Statistics	First Edition	Second Edition	Third Edition
Number of Pages	32	32 (+0%)	52 (+63%)
Written Exercises	7	16 (+129%)	11+10*=21 (+31%)
Programming Exercises	109	194 (+78%)	245 (+26%)

The breakdown of the number of programming exercises from each section is shown below:

Section	Title	Appearance	% in Chapter	% in Book
3.2	**Complete Search**	112	45%	7%
3.3	Divide and Conquer	23	9%	1%
3.4	Greedy	45	18%	3%
3.5	**Dynamic Programming**	67	27%	4%