

Lab Course

Scientific Computing

Worksheet 2

distributed: 16.11.2017

due: 26.11.2017, midnight, submission on Moodle

oral examination: 28.11.2017 (exact slots will be announced)

We examine the following ordinary differential equation describing the dynamics of the population of a certain species:

$$\dot{p} = \left(1 - \frac{p}{10}\right) \cdot p \quad (1)$$

with initial condition

$$p(0) = 1. \quad (2)$$

The analytical solution is given by

$$p(t) = \frac{10}{1 + 9e^{-t}}.$$

We use this rather simple equation with a known exact solution to examine the properties of different numerical methods.

a) Use `matlab` to plot the function $p(t)$ in a graph.

- b) Consider a general initial value problem

$$\dot{y} = f(y), \quad y(0) = y_0.$$

Implement the following explicit numerical methods

- 1) explicit Euler method,
- 2) method of Heun,
- 3) Runge-Kutta method (fourth order)

as a `matlab` function depending on the right hand side $f(y)$, the initial value y_0 , the timestep size δt and the end time t_{end} . The output of the function shall be a vector containing all computed approximate values for y .

- c) For each of the three methods implemented, compute approximate solutions for equation (1) with initial conditions (2), end time $t_{end} = 5$, and with time steps $\delta t = 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}$. For each case, compute the approximation error

$$E = \sqrt{\frac{\delta t}{t_{end}} \sum_k (p_k - p_{k,exact})^2},$$

where p_k denotes the approximation, $p_{exact,k}$ the exact solution at $t = \delta t \cdot k$.

Plot your solutions in one graph per method (together with the given solution from a)) and write down the errors in the tables below.

- d) For each of the three methods, determine the factor by which the error is reduced if the step size δt is halved. Write down the results in the table below.
- e) In general, we do not know the exact solution of an equation we have to solve numerically (otherwise, we would not have to use a numerical method, in fact ;)). To anyhow guess the accuracy of a method, we can use the difference between our best approximation (the one with the smallest time step δt) and the other approximations:

$$\tilde{E} = \sqrt{\frac{\delta t}{t_{end}} \sum_k (p_k - p_{k,best})^2},$$

where p_k denotes the approximation with time step δt and $p_{best,k}$ the best approximation at $t = \delta t \cdot k$.

Compute \tilde{E} for all time steps and methods used, write down the results in the tables below and compare them to the exact error.

explicit Euler method ($q = 1$)				
δt				
error				
error red.				
error app.				

method of Heun ($q = 2$)				
δt				
error				
error red.				
error app.				

Runge-Kutta method ($q = 4$)				
δt				
error				
error red.				
error app.				

Questions:

1) By which factor is the error reduced for each halving of δt if you apply a

- first order ($O(\delta t)$),
- second order ($O(\delta t^2)$),
- third order ($O(\delta t^3)$),
- fourth order ($O(\delta t^4)$)

method.

2) For which integer q can you conclude that the error of the

- a) explicit Euler method,
- b) method of Heun,
- c) Runge-Kutta method (fourth order)

behaves like $O(\delta t^q)$?

3) Is a higher order method always more accurate than a lower order method (for the same stepsize δt)?

- 4) Assume you have to compute the solution up to a certain prescribed accuracy limit and that you see that you can do with less time steps if you use the Runge-Kutta-method than if you use Euler or the method of Heun. Can you conclude in this case that the Runge-Kutta method is the most efficient one of the three alternatives?