Finding the Best Discriminator :

What is the condition for getting the best discriminator ?

Recall : The training criteria for a discriminator is to maximize the loss function $V(D,G)$.

$$\therefore \quad D_G^* = \underset{D}{\operatorname{argmax}} \; V(D,G)$$

This can be read as the value of D (for a fixed G) that will maximize the function $V(D,G)$.

$$D_G^* = \underset{D}{\operatorname{argmax}} \left\{ E_{x \sim P_{data}(x)} [\log(D(x))] + E_{z \sim P_z(z)} [\log(1 - D(G(z)))] \right\}$$

$$D_G^* = \underset{D}{\operatorname{argmax}} \left\{ \int_{-\infty}^{+\infty} P_{data}(x) \cdot \log(D(x))\, dx + \int_{-\infty}^{+\infty} P_z(z) \cdot \log(1 - D(G(z)))\, dz \right\}$$

!?

$$= \underset{D}{\operatorname{argmax}} \left\{ \int_{-\infty}^{+\infty} P_{data}(x) \cdot \log(D(x))\, dx + \int_{-\infty}^{+\infty} P_g(x) \cdot \log(1 - D(x))\, dx \right\} \quad - 5$$

Explaination :

The probability density function (pdf) of a Random Variable X is, say $f_X(x)$. Then the pdf of a function of X, say $g(X)$ is given by :

$$P_Y(y) = P(Y = y) = \frac{P_X(x_1)}{|g'(x_1)|} + \frac{P_X(x_2)}{|g'(x_2)|} + \ldots$$

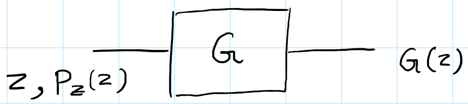
where $x_1, x_2, \ldots$ are the solutions of $y = g(x)$. Since we are assuming just one solution, thus :

$$P_Y(y) = \frac{P_X(x_1)}{|g'(x_1)|}$$

$$P_y(y) = \frac{P_x(x_1)}{|g'(x_1)|}$$

Note: This means that we are assuming Y to be invertible in nature having a unique solution x_1.

$$\therefore \; P_g(G(z)) = \frac{P_z(z_1)}{|G'(z_1)|} = P_z(z_1)\left|\frac{dz_1}{d(G(z_1))}\right|$$



Substituting z_1 by $G^{-1}(X)$ and $G(z_1)$ by X where X represents the generated images.

$$P_g(X) = P_z(G^{-1}(X))\left|\frac{d(G^{-1}(X))}{dX}\right| \approx P_z(G^{-1}(X))\,\frac{dG^{-1}(X)}{dX}$$

This just means that the distr. is coming out of the generator

From eqn. 5 :

$$\int_{-\infty}^{+\infty} P_z(z)\,.\,\log(1 - D(G(z)))\,dz = \int_{-\infty}^{+\infty} P_g(x)\,.\,\log(1 - D(x))\,dx$$

$$LHS = \int_{-\infty}^{+\infty} P_z(G^{-1}(X))\,\log(1 - D(X))\,dG^{-1}(X) \qquad (G(X) = z)$$

$$LHS = \int_{-\infty}^{+\infty} P_z(G^{-1}(X))\,\log(1 - D(X))\,\frac{dG^{-1}(X)}{dX}\,dX$$

$$LHS = \int_{-\infty}^{+\infty} \underbrace{P_z(G^{-1}(X))\,\frac{dG^{-1}(X)}{dX}}_{P_g(X)}\,.\,\log(1 - D(X))\,dX$$

$$LHS = \int_{-\infty}^{+\infty} P_g(X)\,\log(1 - D(X))\,dX = RHS$$

Phew...

$$D^* = \operatorname{argmax}\left\{\int p(x)\,\log(D(x))\,dx + \int_{}^{+\infty} p(x)\,\log(1 - D(x))\,dx\right\}$$

$$D_G^* = \underset{D}{\arg\max} \left\{ \int_{-\infty}^{+\infty} p_{data}(x) \cdot \log(D(x))\, dx + \int_{-\infty}^{+\infty} P_g(x) \cdot \log(1 - D(x))\, dx \right\}$$

Now we can easily digest this formula.

Maxima Calculation :

$$\max \left\{ \int_{-\infty}^{+\infty} p_{data}(x) \cdot \log(D(x))\, dx + \int_{-\infty}^{+\infty} P_g(x) \cdot \log(1 - D(x))\, dx \right\}$$ will occur

when $$\frac{d}{dD(X)} \left[p_{data}(X) \cdot \log(D(X))\, dX + P_g(X) \cdot \log(1 - D(X))\, dX \right] = 0$$

$$\Rightarrow \left[\frac{P_{data}(X)}{D(X)} - \frac{P_g(X)}{1 - D(X)} \right] = 0$$

$$\Rightarrow P_{data}(X) \{1 - D(X)\} = D(X) \{P_g(X)\}$$

$$\Rightarrow P_{data}(X) = D(X) \{P_{data}(X) + P_g(X)\}$$

$$\Rightarrow \boxed{D^*(X) = \frac{P_{data}(X)}{P_{data}(X) + P_g(X)}}$$

Optimal Discriminator

Double Check that this maximizes $V(G,D)$ with Double Derivative Test :

$$V''(G,D) = \frac{d}{dD(X)} \left[\frac{P_{data}(X)}{D(X)} - \frac{P_g(X)}{1 - D(X)} \right]$$

$$V''(G,D) = \frac{-P_{data}(X)}{(D(X))^2} - \frac{P_g(X)}{(1-D(X))^2} < 0$$

Confirms test for maxima

This is because p.d.f. of a R.V. can never be negative

Thus $-P_X(x) \leqslant 0 \quad \forall \quad x \in \mathbb{R}$.

Also, $x^2 \geqslant 0 \quad \forall \quad x \in \mathbb{R}$

Thus, the optimal discriminator is given by:

$$\boxed{D^*(x) = \frac{P_{data}(x)}{P_{data}(x) + P_g(x)}}$$