Optimal Generator for GANs

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As derived earlier, the optimal Discriminator is given by :

$$\boxed{D^*(x) = \frac{P_{data}(x)}{P_{data}(x) + P_g(x)}}$$

Given the optimal discriminator, the Generator will try to minimize the loss function $V(D^*_G, G)$.

So, once the Discriminator has become good at classifying real and fake images, we would want our Generator to be able to fool this Discriminator (mathematically).

$$\therefore \quad G^* = \underset{G}{\operatorname{argmin}} \; V(D^*_G, G)$$

Using the loss function again :

$$G^* = \underset{G}{\operatorname{argmin}} \left\{ \int_{-\infty}^{+\infty} P_{data}(x) \cdot \log(D^*(x))\, dx + \int_{-\infty}^{+\infty} P_g(x) \cdot \log(1 - D^*(x))\, dx \right\}$$

$$G^* = \underset{G}{\operatorname{argmin}} \left\{ \int_{-\infty}^{+\infty} P_{data}(x) \log\left[\frac{P_{data}(x)}{P_{data}(x) + P_g(x)}\right] + P_g(x) \log\left[1 - \frac{P_{data}(x)}{P_{data}(x) + P_g(x)}\right] dx \right\}$$

$$G^* = \underset{G}{\operatorname{argmin}} \left\{ \int_{-\infty}^{+\infty} P_{data}(x) \log\left[\frac{P_{data}(x)}{P_{data}(x) + P_g(x)}\right] + P_g(x) \log\left[\frac{P_g(x)}{P_{data}(x) + P_g(x)}\right] dx \right\}$$

Add and subtract $(\log 2) P_{data}(x)$ and $(\log 2) P_g(x)$ from the integral.

$$G^* = \underset{G}{\operatorname{argmin}} \left\{ \int_{-\infty}^{+\infty} P_{data}(x) \left[\log(2) - \log(2)\right] + P_{data}(x) \log\left[\frac{P_{data}(x)}{P_{data}(x) + P_g(x)}\right] \right.$$

$$\left. + P_g(x) \left[\log(2) - \log(2)\right] + P_g(x) \log\left[\frac{P_g(x)}{P_{data}(x) + P_g(x)}\right] dx \right\}$$

$$+ P_g(x)\left[\log(2) - \log(2)\right] + P_g(x) \log\left[\frac{P_g(x)}{P_{data}(x) + P_g(x)}\right] dx \Bigg\}$$

$$G^* = \underset{G}{\operatorname{argmin}} \Bigg\{ \int_{-\infty}^{+\infty} -\log(2)\left[P_{data}(x) + P_g(x)\right] + P_{data}(x)\left(\log(2) + \log\left[\frac{P_{data}(x)}{P_{data}(x) + P_g(x)}\right]\right)$$

$$+ P_g(x)\left(\log(2) + \log\left[\frac{P_g(x)}{P_{data}(x) + P_g(x)}\right]\right) dx \Bigg\}$$

$$G^* = \underset{G}{\operatorname{argmin}} \Bigg\{ \int_{-\infty}^{+\infty} -\log(2)\left[P_{data}(x) + P_g(x)\right] dx$$

$$+ \int_{-\infty}^{+\infty} P_{data}(x)\left(\log(2) + \log\left[\frac{P_{data}(x)}{P_{data}(x) + P_g(x)}\right]\right) dx$$

$$+ \int_{-\infty}^{+\infty} P_g(x)\left(\log(2) + \log\left[\frac{P_g(x)}{P_{data}(x) + P_g(x)}\right]\right) dx \Bigg\}$$

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$$+ \int_{-\infty}^{+\infty} P_g(x)\left(\log\left[\frac{P_g(x)}{\frac{P_{data}(x) + P_g(x)}{2}}\right]\right) dx \Bigg\}$$

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$$+ \int^{+\infty} P_g(x)\left(\log\left[\frac{P_g(x)}{\dots}\right]\right) dx \Bigg\}$$

$$+ \int_{-\infty}^{+\infty} P_g(x) \left(\log \left[\frac{P_g(x)}{\frac{P_{data}(x) + P_g(x)}{2}} \right] \right) dx \Bigg\}$$

$$G^* = \underset{G}{\operatorname{argmin}} \Bigg\{ -2\log(2) + KL\left[P_{data}(x) \,\Big\|\, \left(\frac{P_{data}(x) + P_g(x)}{2} \right) \right] + KL\left[P_g(x) \,\Big\|\, \left(\frac{P_{data}(x) + P_g(x)}{2} \right) \right] \Bigg\}$$

Jensen Shanon Divergence

$$G^* = \underset{G}{\operatorname{argmin}} \Big\{ -\log(4) + 2\, JSD\left(P_{data}(x) \,\|\, P_g(x) \right) \Big\}$$

$$JSD = \frac{1}{2} \Big\{ KL(P \| M) + KL(Q \| M) \Big\}$$ where $M = \frac{P+Q}{2}$

Now, to minimize G^*, we will have to minimize JSD, this means :

min $JSD\left(P_{data}(x) \,\|\, P_g(x) \right)$

$$\min \frac{1}{2} \Bigg\{ KL\left(P_{data}(x) \,\Big\|\, \left(\frac{P_{data}(x) + P_g(x)}{2} \right) \right) + KL\left(P_g(x) \,\Big\|\, \left(\frac{P_{data}(x) + P_g(x)}{2} \right) \right) \Bigg\}$$

$$\min \Bigg\{ P_{data}(x) \underbrace{\left(\log \left[\frac{P_{data}(x)}{\frac{P_{data}(x) + P_g(x)}{2}} \right] \right)}_{\text{minimize this}} + P_g(x) \underbrace{\left(\log \left[\frac{P_g(x)}{\frac{P_{data}(x) + P_g(x)}{2}} \right] \right)}_{\text{minimize this}} \Bigg\}$$

$$\therefore \frac{P_{data}(x) + P_g(x)}{2} = P_g(x) = P_{data}(x)$$

This gives us :

$$\boxed{P_{data}(x) = P_g(x)}$$

At the optimum, $P_{data}(x) = P_g(x)$, this causes :

$$\boxed{\begin{aligned} G^* &= -\log(4) \\ D^* &= \frac{1}{2} \end{aligned}}$$