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Asymptotic Expansions for Integrals

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Preface

The asymptotic theory of integrals is an important subject of applied mathematics and physics. Although it is an old topic, with origins dating back to Laplace, new methods, applications and formulations continue to appear in the literature. The purpose of this thesis is to review some of the classical procedures for obtaining asymptotic expansions of integrals, especially focusing on Laplace-type integrals. As a contribution to the topic, we give a new method for computing the coefficients of these asymptotic series with several illustrating examples. This method is a generalization of the one given in my paper about the Stirling Coefficients (*J. Integer Seqs.* **13** (6), Article 10.6.6, pp. 5 (2010)).

Chapter 1 covers the asymptotic theory of real Laplace-type integrals. The computation of the coefficients appearing in the asymptotic expansions are described completely in this chapter. In Chapter 2, we discuss two methods from the asymptotic theory of complex Laplace-type integrals: the Method of Steepest Descents and Perron's Method. Each chapter contains examples to demonstrate the application of the results. In Appendix A, we collect the basic properties of some combinatorial quantities we use in this thesis.

Budapest, Hungary, May 23, 2012 *Gergő Nemes*

Table of Contents

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Chapter 1

Real Laplace-type integrals

In this chapter, we investigate the asymptotic behavior of certain parametric integrals. The origins of the method date back to Pierre-Simon de Laplace (1749 – 1827), who studied the estimation of integrals arising in probability theory of the form

$$I_n = \int_a^b \mathrm{e}^{-nf(x)} g(x)\,\mathrm{d}x \quad (n \to +\infty)\,. \tag{1.1}$$

Here the functions f and g are real continuous functions defined on the real (finite or infinite) interval $[a,b]$. Hereinafter, we call integrals of the type (1.1), *Laplace-type integrals*. Laplace made the observation that the major contribution to the integral I_n should come from the neighborhood of the point where f attains its smallest value. If f has its minimum value only at the point x_0 in (a,b) where $f'(x_0) = 0$ and $f''(x_0) > 0$, then Laplace's result is

$$\int_a^b \mathrm{e}^{-nf(x)} g(x)\,\mathrm{d}x \sim g(x_0)\,\mathrm{e}^{-nf(x_0)} \sqrt{\frac{2\pi}{nf''(x_0)}}.$$

The sign $\sim$ is used to mean that the quotient of the left-hand side by the right-hand side approaches 1 as $n \to +\infty$. This formula is now known as *Laplace's approximation*. A heuristic proof of this formula may proceed as follows. First, we replace f and g by the leading terms in their Taylor series expansions around $x = x_0$, and then extend the integration limits to $-\infty$ and $+\infty$. Hence,

$$\begin{aligned}
\int_a^b \mathrm{e}^{-nf(x)} g(x)\,\mathrm{d}x &\approx \int_a^b \mathrm{e}^{-n\left(f(x_0) + \frac{f''(x_0)}{2}(x-x_0)^2\right)} g(x_0)\,\mathrm{d}x \\
&\approx g(x_0)\,\mathrm{e}^{-nf(x_0)} \int_{-\infty}^{+\infty} \mathrm{e}^{-n\frac{f''(x_0)}{2}(x-x_0)^2} \mathrm{d}x \\
&= g(x_0)\,\mathrm{e}^{-nf(x_0)} \sqrt{\frac{2\pi}{nf''(x_0)}}.
\end{aligned}$$

We divided this chapter into two parts. In Section 1.1, we deal with the fundamental theorem of the topic, called Watson's Lemma. Section 1.2 presents a general theorem on asymptotic expansions of Laplace-type integrals, and investigates several of their properties.

1.1 Fundamental concepts and results

This section is divided into two parts. In the first part, we collect all the basic concepts which we will use hereinafter. In the second part we state and prove an important theorem of the topic, known as Watson's Lemma.

1.1.1 Asymptotic expansions

Let f and g be two continuous complex functions defined on a subset H of the complex plane. Let z_0 be a limit point of H.

Definition 1.1.1. *We write $f(z) = \mathcal{O}(g(z))$ ("f is big-oh g"), as $z \to z_0$, to mean that there is a constant $C > 0$ and a neighborhood U of z_0 such that $|f(z)| \leq C\,|g(z)|$ for all $z \in U \cap H$.*

Definition 1.1.2. *We write $f(z) = o(g(z))$ ("f is little-oh g"), as $z \to z_0$, to mean that for every $\varepsilon > 0$, there exists a neighborhood U_ε of z_0 such that $|f(z)| \leq \varepsilon\,|g(z)|$ for all $z \in U_\varepsilon \cap H$.*

To define the asymptotic expansion of a function we need the concept of an asymptotic sequence.

Definition 1.1.3. *Let $\{\varphi_n\}_{n\geq 0}$ be a sequence of continuous complex functions defined on a subset H of the complex plane. Let z_0 be a limit point of H. We say that $\{\varphi_n\}_{n\geq 0}$ is an asymptotic sequence as $z \to z_0$ in H if, for all $n \geq 0$, $\varphi_{n+1}(z) = o(\varphi_n(z))$, as $z \to z_0$.*

Definition 1.1.4. *If $\{\varphi_n\}_{n\geq 0}$ is an asymptotic sequence as $z \to z_0$, we say that*

$$\sum_{n=1}^{\infty} a_n \varphi_n(z),$$

where the a_n's are constants, is an asymptotic expansion of the function f if for each $N \geq 0$

$$f(z) = \sum_{n=1}^{N} a_n \varphi_n(z) + o(\varphi_N(z)) \ \ \textit{as } z \to z_0. \tag{1.2}$$

If a function possesses an asymptotic expansion, we write

$$f(z) \sim \sum_{n=1}^{\infty} a_n \varphi_n(z) \ \ \textit{as } z \to z_0.$$

Note that (1.2) may be written as

$$f(z) = \sum_{n=1}^{N-1} a_n \varphi_n(z) + \mathcal{O}(\varphi_N(z)),$$

which implies that the error is of the same order of magnitude as the first term omitted. Usually, we use $\varphi_n(z) = z^{-\rho_n}$ with $0 < \mathfrak{Re}\,(\rho_0) < \mathfrak{Re}\,(\rho_1) < \mathfrak{Re}\,(\rho_2) < \ldots$, as $z \to \infty$ in some sector of the complex plane.

1.1.2 Watson's Lemma

Consider the integral

$$I(\lambda) = \int_0^{+\infty} \frac{\mathrm{e}^{-\lambda x}}{1+x} \mathrm{d}x$$

where $\lambda > 0$. Using the well-known identity

$$\frac{1}{1+x} = \sum_{k=0}^{N-1} (-1)^k x^k + (-1)^N \frac{x^N}{1+x} \quad (N \geq 0),$$

term-by-term integration gives

$$I(\lambda) = \sum_{k=0}^{N-1} (-1)^k \frac{k!}{\lambda^{k+1}} + (-1)^N \int_0^{+\infty} \frac{x^N}{1+x} \mathrm{e}^{-\lambda x} \mathrm{d}x,$$

where

$$\left| (-1)^N \int_0^{+\infty} \frac{x^N}{1+x} \mathrm{e}^{-\lambda x} \mathrm{d}x \right| = \int_0^{+\infty} \frac{x^N}{1+x} \mathrm{e}^{-\lambda x} \mathrm{d}x \leq \int_0^{+\infty} x^N \mathrm{e}^{-\lambda x} \mathrm{d}x = \frac{N!}{\lambda^{N+1}}.$$

And hence, by definition,

$$I(\lambda) \sim \sum_{k=0}^{\infty} (-1)^k \frac{k!}{\lambda^{k+1}}, \text{ as } \lambda \to +\infty.$$

This is a special case of a much more general result, now known as Watson's Lemma. The theorem is due to George Neville Watson (1886 – 1965). It is a significant result in the theory of asymptotic expansions of Laplace-type integrals. In view of its importance, the proof of the result is reproduced below.

Theorem 1.1.1 (Watson's Lemma)**.** *Let $f(x)$ be a complex valued function of a real variable x such that*

(i) f is continuous on $(0, \infty)$;

(ii) as $x \to 0^+$,

$$f(x) \sim \sum_{k=0}^{\infty} a_k x^{\rho_k - 1}$$

with $0 < \mathfrak{Re}(\rho_0) < \mathfrak{Re}(\rho_1) < \mathfrak{Re}(\rho_2) < \ldots$, $\lim_{k\to+\infty} \mathfrak{Re}(\rho_k) = +\infty$; and

(iii) for some fixed $c > 0$, $f(x) = \mathcal{O}(\mathrm{e}^{cx})$ as $x \to +\infty$.

Then we have

$$\int_0^{+\infty} \mathrm{e}^{-\lambda x} f(x)\, \mathrm{d}x \sim \sum_{k=0}^{\infty} \frac{a_k \Gamma(\rho_k)}{\lambda^{\rho_k}},$$

as $\lambda \to +\infty$.

Proof. By the conditions (i), (ii) and (iii); the integral

$$\int_0^{+\infty} \mathrm{e}^{-\lambda x} f(x)\,\mathrm{d}x$$

converges for $\lambda > c$. Conditions (ii) and (iii) imply that

$$\left| f(x) - \sum_{k=0}^{N-1} a_k x^{\rho_k - 1} \right| \le K_N \mathrm{e}^{cx} \left| x^{\rho_N - 1} \right| \quad \text{for } x > 0,$$

for every $N \ge 0$ with some constant K_N. Thus we have

$$\left| \int_0^{+\infty} \mathrm{e}^{-\lambda x} f(x)\,\mathrm{d}x - \sum_{k=0}^{N-1} a_k \int_0^{+\infty} \mathrm{e}^{-\lambda x} x^{\rho_k - 1} \mathrm{d}x \right| \le K_N \int_0^{+\infty} \mathrm{e}^{-(\lambda - c)x} \left| x^{\rho_N - 1} \right| \mathrm{d}x.$$

Note that we have for $\lambda > 0$

$$\int_0^{+\infty} \mathrm{e}^{-\lambda x} x^{\rho_k - 1} \mathrm{d}x = \frac{1}{\lambda^{\rho_k}} \int_0^{+\infty} \mathrm{e}^{-t} t^{\rho_k - 1} \mathrm{d}t = \frac{\Gamma(\rho_n)}{\lambda^{\rho_k}}.$$

Hence we have

$$\begin{aligned} \left| \int_0^{+\infty} \mathrm{e}^{-\lambda x} f(x)\,\mathrm{d}x - \sum_{k=0}^{N-1} \frac{a_k \Gamma(\rho_n)}{\lambda^{\rho_k}} \right| &\le K_N \int_0^{+\infty} \mathrm{e}^{-(\lambda - c)x} \left| x^{\rho_N - 1} \right| \mathrm{d}x \\ &= K_N \frac{\Gamma(\mathfrak{Re}(\rho_N))}{\left| (\lambda - c)^{\rho_N} \right|}, \end{aligned}$$

that is

$$\int_0^{+\infty} \mathrm{e}^{-\lambda x} f(x)\,\mathrm{d}x = \sum_{k=0}^{N-1} \frac{a_k \Gamma(\rho_n)}{\lambda^{\rho_k}} + \mathcal{O}\left(\frac{1}{\lambda^{\rho_N}} \right)$$

as $\lambda \to +\infty$, which proves Watson's Lemma. ∎

1.2 Laplace's Method

In this section, we revisit the classical *Laplace Method*. In the first subsection, we prove the fundamental theorem on asymptotic expansion of Laplace-type integrals, an extension of the formula mentioned in the introduction of the chapter. When the functions appearing in the integral are holomorphic, the coefficients of the asymptotic expansion can be given explicitly by *Perron's Formula*. In the second part, we extend this formula to the case when the power series of the functions are not necessarily convergent but asymptotic. In the third subsection, we reproduce the proof of the CFWW Formula which gives another explicit form of the coefficients in the asymptotic expansion. The fourth part of the section is about a new and simpler explicit formula for these coefficients. Finally, in the fifth subsection, we give three examples to demonstrate the application of the results.

1.2.1 Asymptotic expansion of Laplace-type integrals

Laplace's Method is one of the best-known techniques developing asymptotic expansions for integrals. Here we present an extension of the formula mentioned in the introduction of this chapter. Consider the integral of the form

$$I(\lambda) = \int_a^b \mathrm{e}^{-\lambda f(x)} g(x)\,\mathrm{d}x,$$

where (a, b) is a real (finite or infinite) interval, λ is a large positive parameter and the functions f and g are continuous. We observe that by subdividing the range of integration at the minima and maxima of f, and by reversing the sign of x whenever necessary, we may assume, without loss of generality, that f has only one minimum in $[a, b]$ which occurs at $x = a$. Next, we assume that, as $x \to a^+$,

$$f(x) \sim f(a) + \sum_{k=0}^{\infty} a_k (x-a)^{k+\alpha}, \tag{1.3}$$

and

$$g(x) \sim \sum_{k=0}^{\infty} b_k (x-a)^{k+\beta-1} \tag{1.4}$$

with $\alpha > 0$, $\mathfrak{Re}(\beta) > 0$; and that the expansion of f can be term wise differentiated, that is,

$$f'(x) \sim \sum_{k=0}^{\infty} a_k (k+\alpha)(x-a)^{k+\alpha-1} \tag{1.5}$$

as $x \to a^+$. Also, we suppose, without loss of generality, that $a_0 \neq 0$ and $b_0 \neq 0$. The following theorem is due to Arthur Erdélyi (1908 – 1977) (see, *e.g.*, [8, p. 85], [14, p. 57]).

Theorem 1.2.1. *For the integral*

$$I(\lambda) = \int_a^b \mathrm{e}^{-\lambda f(x)} g(x)\,\mathrm{d}x,$$

we assume that

(i) $f(x) > f(a)$ for all $x \in (a, b)$, and for every $\delta > 0$ the infimum of $f(x) - f(a)$ in $[a + \delta, b)$ is positive;

(ii) $f'(x)$ and $g(x)$ are continuous in a neighborhood of $x = a$, expect possibly at a;

(iii) the expansions (1.3), (1.4) and (1.5) hold; and

(iv) the integral $I(\lambda)$ converges absolutely for all sufficiently large λ.

Then

$$I(\lambda) \sim \mathrm{e}^{-\lambda f(a)} \sum_{n=0}^{\infty} \Gamma\left(\frac{n+\beta}{\alpha}\right) \frac{c_n}{\lambda^{(n+\beta)/\alpha}}, \tag{1.6}$$

as $\lambda \to +\infty$, where the coefficients c_n are expressible in terms of a_n and b_n.

The first three coefficients c_n are given explicitly by

$$c_0 = \frac{b_0}{\alpha a_0^{\beta/\alpha}}, \qquad c_1 = \frac{1}{a_0^{(\beta+1)/\alpha}} \left(\frac{b_1}{\alpha} - \frac{(\beta+1)\, a_1 b_0}{\alpha^2 a_0} \right),$$

and

$$c_2 = \frac{1}{a_0^{(\beta+2)/\alpha}} \left(\frac{b_2}{\alpha} - \frac{(\beta+2)\, a_1 b_1}{\alpha^2 a_0} + \left((\beta+\alpha+2)\, a_1^2 - 2\alpha a_0 a_2 \right) \frac{(\beta+1)\, b_0}{2\alpha^2 a_0^2} \right).$$

Proof. By conditions (ii) and (iii) there exists a number $c \in (a,b)$ such that $f'(x)$ and $g(x)$ are continuous in $(a,c]$, and $f'(x)$ is also positive there. Let $T = f(c) - f(a)$, and introduce the variable t as

$$t = f(x) - f(a).$$

Since $f(x)$ is increasing in (a,c), we can write

$$\mathrm{e}^{\lambda f(a)} \int_a^c \mathrm{e}^{-\lambda f(x)} g\,(x)\,\mathrm{d}x = \int_0^T \mathrm{e}^{-\lambda t} h\,(t)\,\mathrm{d}t \tag{1.7}$$

with $h(t)$ being the continuous function in $(0,T]$ given by

$$h(t) = g(x)\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{g(x)}{f'(x)}. \tag{1.8}$$

By assumption

$$t \sim \sum_{k=0}^{\infty} a_k\,(x-a)^{k+\alpha} \quad \text{as } x \to a^+,$$

and thus, by series reversion we obtain

$$x - a \sim \sum_{k=1}^{\infty} d_k t^{k/\alpha} \quad \text{as } t \to 0^+.$$

Substituting this into (1.8) yields

$$h\,(t) \sim \sum_{k=0}^{\infty} c_k t^{(k+\beta)/\alpha - 1}$$

as $t \to 0^+$, where the coefficients c_k are expressible in terms of a_k and b_k. We now apply Watson's Lemma to the integral on the right-hand side of (1.7) to obtain

$$\int_a^c \mathrm{e}^{-\lambda f(x)} g\,(x)\,\mathrm{d}x \sim \mathrm{e}^{-\lambda f(a)} \sum_{n=0}^{\infty} \Gamma\left(\frac{n+\beta}{\alpha}\right) \frac{c_n}{\lambda^{(n+\beta)/\alpha}},$$

as $\lambda \to +\infty$. All that remains is to show that the integral on the remaining range (c,b) is negligible. Define

$$\varepsilon \stackrel{\text{def}}{=} \inf_{c \le x < b} \left(f\,(x) - f\,(a) \right).$$

This is positive by condition (i). Let λ_0 be a value of λ for which $I(\lambda)$ is absolutely convergent. Assume that $\lambda \geq \lambda_0$, then

$$\begin{aligned}\lambda\left(f\left(x\right)-f\left(a\right)\right) &= \left(\lambda-\lambda_0\right)\left(f\left(x\right)-f\left(a\right)\right)+\lambda_0\left(f\left(x\right)-f\left(a\right)\right)\\ &\geq \left(\lambda-\lambda_0\right)\varepsilon+\lambda_0\left(f\left(x\right)-f\left(a\right)\right)\end{aligned}$$

and

$$\left|\mathrm{e}^{\lambda f(a)}\int_c^b \mathrm{e}^{-\lambda f(x)}g\left(x\right)\mathrm{d}x\right| \leq K\mathrm{e}^{-\varepsilon\lambda},$$

where

$$K=\mathrm{e}^{\lambda_0(\varepsilon+f(a))}\int_c^b \mathrm{e}^{-\lambda_0 f(x)}\left|g\left(x\right)\right|\mathrm{d}x$$

is a constant. ■

1.2.2 Generalization of Perron's Formula

In this subsection, we derive an explicit formula for the coefficients c_n appearing in the asymptotic expansion (1.6). The notations are the same as in the previous subsection.

Theorem 1.2.2. *The coefficients c_n appearing in* (1.6) *are given explicitly by*

$$\begin{aligned}c_n &= \frac{1}{\alpha n!}\left[\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left\{G\left(x\right)\left(\frac{\left(x-a\right)^\alpha}{f\left(x\right)-f\left(a\right)}\right)^{(n+\beta)/\alpha}\right\}\right]_{x=a}\\ &= \frac{1}{\alpha}\sum_{k=0}^{n}\frac{b_{n-k}}{k!}\left[\frac{\mathrm{d}^k}{\mathrm{d}x^k}\left(\frac{\left(x-a\right)^\alpha}{f\left(x\right)-f\left(a\right)}\right)^{(n+\beta)/\alpha}\right]_{x=a},\end{aligned} \tag{1.9}$$

where $G\left(x\right)$ is the (formal) power series $\sum_{k=0}^\infty b_k\left(x-a\right)^k$ and $f(x)$ should be identified with its (formal) expansion (1.3).

Remark. When $\left(f\left(x\right)-f\left(a\right)\right)\left(x-a\right)^{-\alpha}$ and $g\left(x\right)\left(x-a\right)^{1-\beta}$ are holomorphic functions, the representation (1.9) is known as Perron's Formula [10] [14, p. 103] (see also Section 2.2). But here we show that this formula holds also when the expansions (1.3), (1.4) are merely asymptotic. An alternative proof of essentially the same formula has been given previously by Wojdylo [13].

Proof. Let $\ell > 0$ be the index of the first nonvanishing coefficient in the expansion (1.3) of f apart from a_0. Define

$$f_\ell\left(x\right) \stackrel{\text{def}}{=} \frac{f\left(x\right)-f\left(a\right)-a_0\left(x-a\right)^\alpha}{\left(x-a\right)^{\alpha+\ell}} \sim \sum_{k=0}^\infty a_{k+\ell}\left(x-a\right)^k \quad \text{as } x\to a^+.$$

Then

$$\begin{aligned}I\left(\lambda\right) = \int_a^b \mathrm{e}^{-\lambda f(x)}g\left(x\right)\mathrm{d}x &= \mathrm{e}^{-\lambda f(a)}\int_a^b \mathrm{e}^{-a_0\lambda(x-a)^\alpha}\underbrace{\mathrm{e}^{-\lambda(x-a)^{\alpha+\ell}f_\ell(x)}g\left(x\right)}_{h(\lambda,x)}\mathrm{d}x\\ &= \mathrm{e}^{-\lambda f(a)}\int_a^b \mathrm{e}^{-a_0\lambda(x-a)^\alpha}h\left(\lambda,x\right)\mathrm{d}x\\ &= \frac{\mathrm{e}^{-\lambda f(a)}}{\lambda^{1/\alpha}}\int_0^{\lambda^{1/\alpha}(b-a)}\mathrm{e}^{-a_0t^\alpha}h\left(\lambda,\lambda^{-1/\alpha}t+a\right)\mathrm{d}t.\end{aligned} \tag{1.10}$$

Set $s = \lambda^{-1/\alpha} t$; then the function $h\left(\lambda, \lambda^{-1/\alpha} t + a\right)$ reads

$$h(\lambda, s + a) = \mathrm{e}^{-t^\alpha s^\ell f_\ell(s+a)} g(s + a).$$

We have

$$\mathrm{e}^{-t^\alpha s^\ell f_\ell(s+a)} \sim \sum_{k=0}^{\infty} \frac{1}{k!} \left[\frac{\mathrm{d}^k}{\mathrm{d}w^k} \mathrm{e}^{-t^\alpha w^\ell f_\ell(w+a)} \right]_{w=0} s^k \quad \text{as } s \to 0^+$$

and hence,

$$h(\lambda, s + a) \sim \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \frac{b_{n-k}}{k!} \left[\frac{\mathrm{d}^k}{\mathrm{d}w^k} \mathrm{e}^{-t^\alpha w^\ell f_\ell(w+a)} \right]_{w=0} \right) s^{n+\beta-1} \quad \text{as } s \to 0^+.$$

Introducing this expansion in the last integral of (1.10) we find

$$\begin{aligned}
I(\lambda) &= \frac{\mathrm{e}^{-\lambda f(a)}}{\lambda^{1/\alpha}} \int_0^{\lambda^{1/\alpha}(b-a)} \mathrm{e}^{-a_0 t^\alpha} h\left(\lambda, \lambda^{-1/\alpha} t + a\right) \mathrm{d}t \\
&\sim \frac{\mathrm{e}^{-\lambda f(a)}}{\lambda^{1/\alpha}} \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \frac{b_{n-k}}{k!} \left[\frac{\mathrm{d}^k}{\mathrm{d}w^k} \left(\int_0^{\lambda^{1/\alpha}(b-a)} \mathrm{e}^{-\left(a_0 + w^\ell f_\ell(w+a)\right) t^\alpha} s^{n+\beta-1} \mathrm{d}t \right) \right]_{w=0} \right) \\
&= \mathrm{e}^{-\lambda f(a)} \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \frac{b_{n-k}}{k!} \left[\frac{\mathrm{d}^k}{\mathrm{d}w^k} \left(\int_0^{\lambda^{1/\alpha}(b-a)} \mathrm{e}^{-\left(a_0 + w^\ell f_\ell(w+a)\right) t^\alpha} t^{n+\beta-1} \mathrm{d}t \right) \right]_{w=0} \right) \frac{1}{\lambda^{(n+\beta)/\alpha}} \\
&\sim \mathrm{e}^{-\lambda f(a)} \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \frac{b_{n-k}}{k!} \left[\frac{\mathrm{d}^k}{\mathrm{d}w^k} \left(\int_0^{+\infty} \mathrm{e}^{-\left(a_0 + w^\ell f_\ell(w+a)\right) t^\alpha} t^{n+\beta-1} \mathrm{d}t \right) \right]_{w=0} \right) \frac{1}{\lambda^{(n+\beta)/\alpha}} \\
&= \mathrm{e}^{-\lambda f(a)} \sum_{n=0}^{\infty} \left(\frac{\Gamma\left(\frac{n+\beta}{\alpha}\right)}{\alpha} \sum_{k=0}^{n} \frac{b_{n-k}}{k!} \left[\frac{\mathrm{d}^k}{\mathrm{d}w^k} \left(\frac{1}{\left(a_0 + w^\ell f_\ell(w+a)\right)^{(n+\beta)/\alpha}} \right) \right]_{w=0} \right) \frac{1}{\lambda^{(n+\beta)/\alpha}}
\end{aligned}$$

and using the definition of $f_\ell(w + a)$:

$$\begin{aligned}
I(\lambda) &\sim \mathrm{e}^{-\lambda f(a)} \sum_{n=0}^{\infty} \left(\frac{\Gamma\left(\frac{n+\beta}{\alpha}\right)}{\alpha} \sum_{k=0}^{n} \frac{b_{n-k}}{k!} \left[\frac{\mathrm{d}^k}{\mathrm{d}w^k} \left(\frac{w^\alpha}{f(w+a) - f(a)} \right)^{(n+\beta)/\alpha} \right]_{w=0} \right) \frac{1}{\lambda^{(n+\beta)/\alpha}} \\
&= \mathrm{e}^{-\lambda f(a)} \sum_{n=0}^{\infty} \left(\frac{\Gamma\left(\frac{n+\beta}{\alpha}\right)}{\alpha} \sum_{k=0}^{n} \frac{b_{n-k}}{k!} \left[\frac{\mathrm{d}^k}{\mathrm{d}x^k} \left(\frac{(x-a)^\alpha}{f(x) - f(a)} \right)^{(n+\beta)/\alpha} \right]_{x=a} \right) \frac{1}{\lambda^{(n+\beta)/\alpha}}
\end{aligned}$$

as $\lambda \to +\infty$. Formula (1.9) now follows from the asymptotic expansion (1.6) and the uniqueness theorem on asymptotic series. ∎

1.2.3 The CFWW Formula

In the previous subsection, we introduced formula (1.9) for the coefficients c_n appearing in the asymptotic expansion of certain Laplace-type integrals. In this subsection, we shall extract the higher order derivatives in formula (1.9) by means of the Partial Ordinary Bell Polynomials (see Appendix A.1). Using the series expansion (1.3) of $f(x)$ we find

$$\begin{aligned}
c_n &= \frac{1}{\alpha}\sum_{k=0}^{n}\frac{b_{n-k}}{k!}\left[\frac{\mathrm{d}^k}{\mathrm{d}x^k}\left(\frac{(x-a)^\alpha}{f(x)-f(a)}\right)^{(n+\beta)/\alpha}\right]_{x=a}\\
&= \frac{1}{\alpha a_0^{(n+\beta)/\alpha}}\sum_{k=0}^{n}\frac{b_{n-k}}{k!}\left[\frac{\mathrm{d}^k}{\mathrm{d}x^k}\left(\frac{a_0(x-a)^\alpha}{f(x)-f(a)}\right)^{(n+\beta)/\alpha}\right]_{x=a}\\
&= \frac{1}{\alpha a_0^{(n+\beta)/\alpha}}\sum_{k=0}^{n}\frac{b_{n-k}}{k!}\left[\frac{\mathrm{d}^k}{\mathrm{d}x^k}\left(\frac{f(x)-f(a)}{a_0(x-a)^\alpha}\right)^{-(n+\beta)/\alpha}\right]_{x=a}\\
&= \frac{1}{\alpha a_0^{(n+\beta)/\alpha}}\sum_{k=0}^{n}\frac{b_{n-k}}{k!}\left[\frac{\mathrm{d}^k}{\mathrm{d}x^k}\left(1+\sum_{k=1}^{\infty}\frac{a_k}{a_0}(x-a)^k\right)^{-(n+\beta)/\alpha}\right]_{x=a}.
\end{aligned}$$

From the definition of the Ordinary Potential Polynomials and the Partial Ordinary Bell Polynomials we obtain

$$\begin{aligned}
c_n &= \frac{1}{\alpha a_0^{(n+\beta)/\alpha}}\sum_{k=0}^{n} b_{n-k}\mathsf{A}_{-(n+\beta)/\alpha,k}\left(\frac{a_1}{a_0},\frac{a_2}{a_0},\ldots,\frac{a_k}{a_0}\right)\\
&= \frac{1}{\alpha a_0^{(n+\beta)/\alpha}}\sum_{k=0}^{n} b_{n-k}\sum_{j=0}^{k}\binom{-\frac{n+\beta}{\alpha}}{j}\frac{1}{a_0^j}\mathsf{B}_{k,j}(a_1,a_2,\ldots,a_{k-j+1})\\
&= \frac{1}{\alpha\Gamma\left(\frac{n+\beta}{\alpha}\right)}\sum_{k=0}^{n} b_{n-k}\sum_{j=0}^{k}\frac{(-1)^j}{a_0^{(n+\beta)/\alpha+j}}\frac{\mathsf{B}_{k,j}(a_1,a_2,\ldots,a_{k-j+1})}{j!}\Gamma\left(\frac{n+\beta}{\alpha}+j\right).
\end{aligned} \tag{1.11}$$

This is the *Campbell–Fröman–Walles–Wojdylo Formula* (from now on the CFWW Formula) [1, 12, 13]. Sometimes it is possible to find the Partial Ordinary Bell Polynomials explicitly, in other cases the recurrence (A.4) could be used.

1.2.4 Another method

In the previous subsection, we showed an explicit formula for the coefficients c_n appearing in the asymptotic expansion of Laplace-type integrals. This formula, that we called the CFWW Formula, followed from the generalization of Perron's Formula (1.9) using the Partial Ordinary Bell Polynomials. In this subsection, we give a new explicit formula using Lagrange Interpolation and the Ordinary Potential Polynomials. We write formula (1.9) in the form

$$c_n = \frac{1}{\alpha a_0^{(n+\beta)/\alpha}}\sum_{k=0}^{n}\frac{b_{n-k}}{k!}\left[\frac{\mathrm{d}^k}{\mathrm{d}x^k}\left(\frac{a_0(x-a)^\alpha}{f(x)-f(a)}\right)^{(n+\beta)/\alpha}\right]_{x=a}. \tag{1.12}$$

Define the sequence of functions $(F_k(z))_{k\geq 0}$ by the generating function

$$\left(\frac{a_0(x-a)^\alpha}{f(x)-f(a)}\right)^z = \sum_{k=0}^{\infty}F_k(z)(x-a)^k.$$

Now, equation (1.12) can be written as

$$c_n = \frac{1}{\alpha a_0^{(n+\beta)/\alpha}} \sum_{k=0}^{n} b_{n-k} F_k \left(\frac{n+\beta}{\alpha}\right). \tag{1.13}$$

Since

$$\begin{aligned}\left(\frac{a_0 (x-a)^\alpha}{f(x) - f(a)}\right)^z &= \left(1 + \sum_{k=1}^{\infty} \frac{a_k}{a_0} (x-a)^k\right)^{-z} \\ &= \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} \binom{-z}{j} \frac{1}{a_0^j} \mathsf{B}_{k,j} (a_1, a_2, \ldots, a_{k-j+1})\right) (x-a)^k,\end{aligned}$$

the F_k is a polynomial of degree at most k. By Lagrange Interpolation

$$F_k(z) = \frac{\Gamma(z+k+1)}{k!\Gamma(z)} \sum_{j=0}^{k} (-1)^j \binom{k}{j} \frac{F_k(-j)}{z+j}, \tag{1.14}$$

where the $F_k(-j)$'s can be computed as follows

$$\begin{aligned}\sum_{k=0}^{\infty} F_k(-j)(x-a)^k = \left(\frac{f(x) - f(a)}{a_0 (x-a)^\alpha}\right)^j &= \left(1 + \sum_{k=1}^{\infty} \frac{a_k}{a_0} (x-a)^k\right)^j \\ &= \sum_{k=0}^{\infty} \mathsf{A}_{j,k} \left(\frac{a_1}{a_0}, \frac{a_2}{a_0}, \ldots, \frac{a_k}{a_0}\right) (x-a)^k,\end{aligned}$$

that is

$$F_k(-j) = \mathsf{A}_{j,k} \left(\frac{a_1}{a_0}, \frac{a_2}{a_0}, \ldots, \frac{a_k}{a_0}\right). \tag{1.15}$$

From (1.14) and (1.15) we deduce

$$F_k\left(\frac{n+\beta}{\alpha}\right) = \frac{\Gamma\left(\frac{n+\beta}{\alpha} + k + 1\right)}{k!\Gamma\left(\frac{n+\beta}{\alpha}\right)} \sum_{j=0}^{k} \frac{(-1)^j}{\frac{n+\beta}{\alpha} + j} \binom{k}{j} \mathsf{A}_{j,k} \left(\frac{a_1}{a_0}, \frac{a_2}{a_0}, \ldots, \frac{a_k}{a_0}\right).$$

Plugging this into (1.13) yields

Theorem 1.2.3. *The coefficients c_n appearing in (1.6) are given explicitly by*

$$c_n = \frac{1}{\alpha\Gamma\left(\frac{n+\beta}{\alpha}\right)} \sum_{k=0}^{n} \frac{\Gamma\left(\frac{n+\beta}{\alpha} + k + 1\right) b_{n-k}}{k! a_0^{(n+\beta)/\alpha}} \sum_{j=0}^{k} \frac{(-1)^j}{\frac{n+\beta}{\alpha} + j} \binom{k}{j} \mathsf{A}_{j,k} \left(\frac{a_1}{a_0}, \frac{a_2}{a_0}, \ldots, \frac{a_k}{a_0}\right). \tag{1.16}$$

As we will see in the following subsection, it is possible, in several cases, to derive an explicit formula for the Ordinary Potential Polynomials due to the somewhat simple generating function

$$\left(1 + \sum_{k=1}^{\infty} \frac{a_k}{a_0} (x-a)^k\right)^j = \left(\frac{f(x) - f(a)}{a_0 (x-a)^\alpha}\right)^j = \sum_{k=0}^{\infty} \mathsf{A}_{j,k} \left(\frac{a_1}{a_0}, \frac{a_2}{a_0}, \ldots, \frac{a_k}{a_0}\right) (x-a)^k,$$

whereas in the case of the CFWW formula, the corresponding Partial Ordinary Bell Polynomials have more complicated generating functions, such as

$$\begin{aligned}\exp\left(y\sum_{k=1}^{\infty}a_k\left(x-a\right)^k\right) &= \exp\left(y\frac{f\left(x\right)-f\left(a\right)-a_0\left(x-a\right)^{\alpha}}{\left(x-a\right)^{\alpha}}\right)\\ &= \sum_{k=0}^{\infty}\left(\sum_{j=0}^{k}\frac{\mathsf{B}_{k,j}\left(a_1,a_2,\ldots,a_{k-j+1}\right)}{j!}y^j\right)\left(x-a\right)^k.\end{aligned}$$

1.2.5 Examples

Example 1.2.1. As a first example we take the Gamma Function

$$\Gamma\left(\lambda+1\right)=\int_0^{+\infty}\mathrm{e}^{-t}t^{\lambda}\mathrm{d}t \tag{1.17}$$

for $\lambda>0$. If we put $t=\lambda(1+x)$, we obtain

$$\Gamma\left(\lambda+1\right)=\lambda^{\lambda+1}\mathrm{e}^{-\lambda}\int_{-1}^{+\infty}\mathrm{e}^{-\lambda(x-\log(1+x))}\mathrm{d}x$$

and hence,

$$\frac{\Gamma\left(\lambda\right)}{\lambda^{\lambda}\mathrm{e}^{-\lambda}}=\int_0^{+\infty}\mathrm{e}^{-\lambda(x-\log(1+x))}\mathrm{d}x+\int_0^{1}\mathrm{e}^{-\lambda(-x-\log(1-x))}\mathrm{d}x.$$

Using Theorem 1.2.1 with $f\left(x\right)=x-\log\left(1+x\right)$ (and $f\left(x\right)=-x-\log\left(1-x\right)$), $g(x)\equiv1$, $\alpha=2$ and $\beta=1$; one finds that

$$\int_0^{+\infty}\mathrm{e}^{-\lambda(x-\log(1+x))}\mathrm{d}x\sim\sum_{n=0}^{\infty}\Gamma\left(\frac{n+1}{2}\right)\frac{c_n}{\lambda^{(n+1)/2}}$$

and

$$\int_0^{1}\mathrm{e}^{-\lambda(-x-\log(1-x))}\mathrm{d}x\sim\sum_{n=0}^{\infty}\Gamma\left(\frac{n+1}{2}\right)\frac{\left(-1\right)^n c_n}{\lambda^{(n+1)/2}},$$

as $\lambda\to+\infty$, where, by Theorem 1.2.2,

$$c_n=\frac{1}{2n!}\left[\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(\frac{x^2}{x-\log\left(1+x\right)}\right)^{(n+1)/2}\right]_{x=0}.$$

Finally,

$$\Gamma\left(\lambda\right)\sim\sqrt{2\pi}\lambda^{\lambda-1/2}\mathrm{e}^{-\lambda}\sum_{n=0}^{\infty}\frac{\gamma_n}{\lambda^n}, \tag{1.18}$$

as $\lambda\to+\infty$, where

$$\begin{aligned}\gamma_n &= \sqrt{\frac{2}{\pi}}\Gamma\left(n+\frac{1}{2}\right)c_{2n} \qquad (1.19)\\ &= \frac{1}{2^n n!}\left[\frac{\mathrm{d}^{2n}}{\mathrm{d}x^{2n}}\left(\frac{1}{2}\frac{x^2}{x-\log\left(1+x\right)}\right)^{n+1/2}\right]_{x=0}\end{aligned}$$

are the so-called *Stirling Coefficients*. The first few are $\gamma_0 = 1$ and

$$\gamma_1 = \frac{1}{12}, \quad \gamma_2 = \frac{1}{288}, \quad \gamma_3 = -\frac{139}{51840}, \quad \gamma_4 = -\frac{571}{2488320}.$$

We shall give a more explicit formula for the γ_n's using the CFWW Formula (1.11). Since

$$f(x) = x - \log(1+x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k+2} x^{k+2}, \quad g(x) \equiv 1,$$

we have to compute the Partial Ordinary Bell Polynomials $\mathsf{B}_{k,j}\left(-\frac{1}{3}, \frac{1}{4}, \ldots, \frac{(-1)^{k-j+1}}{k-j+3}\right)$. We have the exponential generating function

$$\exp\left(y \sum_{k=1}^{\infty} \frac{(-1)^k}{k+2} x^k\right) = \exp\left(\frac{y}{x^2}\left(x - \frac{x^2}{2} - \log(1+x)\right)\right)$$
$$= \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} \mathsf{B}_{k,j}\left(-\frac{1}{3}, \frac{1}{4}, \ldots, \frac{(-1)^{k-j+1}}{k-j+3}\right) \frac{y^j}{j!}\right) x^k.$$

From the generating functions of the *Hermite Polynomials* and the *Stirling Numbers of the First Kind* we have

$$\exp\left(t\left(x - \frac{x^2}{2}\right)\right) = \sum_{k=0}^{\infty} \left(\sum_{j=0}^{\lfloor k/2 \rfloor} \frac{(-1)^j}{j!\,(k-2j)!2^j} t^{k-j}\right) x^k$$

and

$$\exp(-t \log(1+x)) = \sum_{k=0}^{\infty} \left((-1)^k \sum_{j=0}^{k} s(k,j)\, t^j\right) \frac{x^k}{k!}.$$

Performing the Cauchy product of the series and rearranging the terms in the coefficients yields

$$\exp\left(t\left(x - \frac{x^2}{2} - \log(1+x)\right)\right)$$
$$= \sum_{k=0}^{\infty} \left(\sum_{j=0}^{\lfloor k/3 \rfloor} \sum_{i=0}^{k-j} \frac{(-1)^i}{2^i i!} \sum_{r=0}^{j-i} \frac{(-1)^{k+r} s(j-i-r, k-2i-r)}{r!\,(k-2i-r)!} t^j\right) x^k.$$

Plugging $t = \frac{y}{x^2}$ gives

$$\exp\left(\frac{y}{x^2}\left(x - \frac{x^2}{2} - \log(1+x)\right)\right)$$
$$= \sum_{k=0}^{\infty} \left(\sum_{j=0}^{\lfloor k/3 \rfloor} \sum_{i=0}^{k-j} \frac{(-1)^i}{2^i i!} \sum_{r=0}^{j-i} \frac{(-1)^{k+r} s(j-i-r, k-2i-r)}{r!\,(k-2i-r)!} \frac{y^j}{x^{2j}}\right) x^k$$
$$= \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} \sum_{i=0}^{j} \frac{(-1)^i}{2^i i!} \sum_{r=0}^{j-i} \frac{(-1)^r s(k+2j-2i-r, j-i-r)}{r!\,(k+2j-2i-r)!} y^j\right) x^k,$$

that is

$$\mathsf{B}_{k,j}\left(-\frac{1}{3},\frac{1}{4},\ldots,\frac{(-1)^{k-j+1}}{k-j+3}\right) = j!\sum_{i=0}^{j}\frac{(-1)^i}{2^i i!}\sum_{r=0}^{j-i}\frac{(-1)^r s(k+2j-2i-r,j-i-r)}{r!\,(k+2j-2i-r)!}.$$

Finally, formula (1.11) produces

$$c_n=\sum_{j=0}^{n}\frac{2^{(n-1)/2+j}\Gamma\left(\frac{n+1}{2}+j\right)}{\Gamma\left(\frac{n+1}{2}\right)}\sum_{i=0}^{j}\frac{1}{2^i i!}\sum_{r=0}^{j-i}\frac{(-1)^{j+i+r}s(n+2j-2i-r,j-i-r)}{r!\,(n+2j-2i-r)!}.$$

From this and the expression (1.19) we deduce

$$\gamma_n=\sum_{j=0}^{2n}\frac{2^{n+j}\Gamma\left(n+j+\frac{1}{2}\right)}{\sqrt{\pi}}\sum_{i=0}^{j}\frac{1}{2^i i!}\sum_{r=0}^{j-i}\frac{(-1)^{j+i+r}s(2n+2j-2i-r,j-i-r)}{r!\,(2n+2j-2i-r)!}. \tag{1.20}$$

This is the result of López, Pagola and Sinusía [6].

Now we show how our new method gives a much simpler result than (1.20) with less calculation. To use formula (1.16) we have to compute the Ordinary Potential Polynomials $\mathsf{A}_{j,k}\left(-\frac{2}{3},\frac{1}{2},\ldots,(-1)^k\frac{2}{k+2}\right)$. This time the generating function is

$$\left(1+\sum_{k=1}^{\infty}(-1)^k\frac{2}{k+2}x^k\right)^j=\left(2\frac{x-\log(1+x)}{x^2}\right)^j = \sum_{k=0}^{\infty}\mathsf{A}_{j,k}\left(-\frac{2}{3},\frac{1}{2},\ldots,(-1)^k\frac{2}{k+2}\right)x^k.$$

Using the generating function of the Stirling Numbers of the First Kind yields

$$\begin{aligned}(x-\log(1+x))^j &= \sum_{i=0}^{j}\binom{j}{i}x^{j-i}(-\log(1+x))^i\\ &=\sum_{i=0}^{j}\binom{j}{i}x^{j-i}\sum_{k=0}^{\infty}(-1)^k i!s(k,i)\frac{x^k}{k!}\\ &=\sum_{k=0}^{\infty}\sum_{i=0}^{j}\binom{j}{i}(-1)^k i!s(k,i)\frac{x^{k+j-i}}{k!}\\ &=\sum_{k=0}^{\infty}\left(\sum_{i=0}^{j}\binom{j}{i}(-1)^{k-j+i}i!\frac{s(k-j+i,i)}{(k-j+i)!}\right)x^k,\end{aligned}$$

which gives

$$\begin{aligned}\left(2\frac{x-\log(1+x)}{x^2}\right)^j &= \sum_{k=0}^{\infty}\left(2^j\sum_{i=0}^{j}\binom{j}{i}(-1)^{k-j+i}i!\frac{s(k-j+i,i)}{(k-j+i)!}\right)x^{k-2j}\\ &=\sum_{k=0}^{\infty}\left(2^j\sum_{i=0}^{j}\binom{j}{i}(-1)^{k-j+i}i!\frac{s(k+j+i,i)}{(k+j+i)!}\right)x^k,\end{aligned}$$

that is

$$\mathsf{A}_{j,k}\left(-\frac{2}{3},\frac{1}{2},\ldots,(-1)^k\frac{2}{k+2}\right)=2^j\sum_{i=0}^{j}\binom{j}{i}(-1)^{k-j+i}\,i!\frac{s\,(k+j+i,i)}{(k+j+i)!}.$$

Finally, by formula (1.16) we find

$$\begin{aligned}c_n&=\frac{\Gamma\left(\frac{3n+1}{2}+1\right)}{\Gamma\left(\frac{n+1}{2}\right)}\sum_{j=0}^{n}\frac{2^{n/2+j-1/2}}{\left(\frac{n+1}{2}+j\right)(n-j)!}\sum_{i=0}^{j}\frac{(-1)^{n+i}\,s\,(n+j+i,i)}{(j-i)!\,(n+j+i)!}\\&=\frac{\Gamma\left(\frac{3n}{2}+\frac{3}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)}\sum_{j=0}^{n}\frac{2^{n/2+j+1/2}}{(n+2j+1)\,(n-j)!}\sum_{i=0}^{j}\frac{(-1)^{n+j-i}\,s\,(n+2j-i,j-i)}{i!\,(n+2j-i)!}.\end{aligned}$$

From this and the expression (1.19) it follows that

$$\begin{aligned}\gamma_n&=\sum_{j=0}^{2n}\frac{2^{n+j+1}\Gamma\left(3n+\frac{3}{2}\right)}{\sqrt{\pi}\,(2n+2j+1)\,(2n-j)!}\sum_{i=0}^{j}\frac{(-1)^{j-i}\,s\,(2n+2j-i,j-i)}{i!\,(2n+2j-i)!}\\&=\sum_{j=n}^{3n}\frac{(-1)^n\,2^{j+1}\Gamma\left(3n+\frac{3}{2}\right)}{\sqrt{\pi}\,(2j+1)\,(3n-j)!}\sum_{i=0}^{j-n}\frac{(-1)^{j-i}\,s\,(2j-i,j-n-i)}{i!\,(2j-i)!},\end{aligned}$$

which is much simpler than (1.20). ♣

Remark. The substitution $x=\log(t/\lambda)$ in (1.17) leads to the form

$$\frac{\Gamma\left(\lambda\right)}{\lambda^{\lambda}\mathrm{e}^{-\lambda}}=\int_0^{+\infty}\mathrm{e}^{-\lambda(\mathrm{e}^x-x-1)}\mathrm{d}x+\int_0^{+\infty}\mathrm{e}^{-\lambda\left(\mathrm{e}^{-x}+x-1\right)}\mathrm{d}x.$$

Similar procedures to that described above give (1.18) with

$$\begin{aligned}\gamma_n&=\frac{1}{2^n n!}\left[\frac{\mathrm{d}^{2n}}{\mathrm{d}x^{2n}}\left(\frac{1}{2}\frac{x^2}{\mathrm{e}^x-x-1}\right)^{n+1/2}\right]_{x=0}\\&=\sum_{j=0}^{2n}\frac{2^{n+j}\Gamma\left(n+j+\frac{1}{2}\right)}{\sqrt{\pi}}\sum_{i=0}^{j}\frac{1}{2^i i!}\sum_{r=0}^{j-i}\frac{(-1)^{j+i+r}\,S\,(2n+2j-2i-r,j-i-r)}{r!\,(2n+2j-2i-r)!}\\&=\sum_{j=n}^{3n}\frac{(-1)^n\,2^{j+1}\Gamma\left(3n+\frac{3}{2}\right)}{\sqrt{\pi}\,(2j+1)\,(3n-j)!}\sum_{i=0}^{j-n}\frac{(-1)^{j-i}\,S\,(2j-i,j-n-i)}{i!\,(2j-i)!},\end{aligned}$$

using formula (1.9), (1.11) and (1.16) respectively. Here $S(n,k)$ denotes the *Stirling Numbers of the Second Kind*.

Example 1.2.2. Our second example is the family of the *Modified Bessel Functions of the Second Kind* K_ν. Suppose that $\nu,t>0$, then we have the integral representation

$$K_\nu\left(\nu t\right)=\frac{1}{2}\int_{-\infty}^{+\infty}\mathrm{e}^{-\nu(t\cosh s-s)}\mathrm{d}s.$$

The substitution $s = \sinh^{-1}(1/t) + x$ gives

$$K_\nu(\nu t) = \frac{1}{2}\left(\frac{\sqrt{t^2+1}+1}{t}\right)^\nu \int_{-\infty}^{+\infty} \mathrm{e}^{-\nu\left(\sqrt{t^2+1}\cosh x + \sinh x - x\right)} \mathrm{d}x$$
$$= \frac{1}{2}\left(\frac{\sqrt{t^2+1}+1}{t}\right)^\nu \mathrm{e}^{-\nu\sqrt{t^2+1}} \int_{-\infty}^{+\infty} \mathrm{e}^{-\nu\left(\sqrt{t^2+1}(\cosh x - 1) + \sinh x - x\right)} \mathrm{d}x,$$

and by splitting up the integral

$$2\left(\frac{\sqrt{t^2+1}+1}{t}\right)^{-\nu} \mathrm{e}^{\nu\sqrt{t^2+1}} K_\nu(\nu t)$$
$$= \int_0^{+\infty} \mathrm{e}^{-\nu\left(\sqrt{t^2+1}(\cosh x - 1) + \sinh x - x\right)} \mathrm{d}x + \int_0^{+\infty} \mathrm{e}^{-\nu\left(\sqrt{t^2+1}(\cosh x - 1) - \sinh x + x\right)} \mathrm{d}x.$$

Using Theorem 1.2.1 with $f(x) = \sqrt{t^2+1}(\cosh x - 1) + \sinh x - x$ (and $f(x) = \sqrt{t^2+1}(\cosh x - 1) - \sinh x + x$), $g(x) \equiv 1$, $\alpha = 2$ and $\beta = 1$; one finds that

$$\int_0^{+\infty} \mathrm{e}^{-\nu\left(\sqrt{t^2+1}(\cosh x - 1) + \sinh x - x\right)} \mathrm{d}x \sim \sum_{n=0}^{\infty} \Gamma\left(\frac{n+1}{2}\right) \frac{c_n}{\nu^{(n+1)/2}}$$

and

$$\int_0^{+\infty} \mathrm{e}^{-\nu\left(\sqrt{t^2+1}(\cosh x - 1) - \sinh x + x\right)} \mathrm{d}x \sim \sum_{n=0}^{\infty} \Gamma\left(\frac{n+1}{2}\right) \frac{(-1)^n c_n}{\nu^{(n+1)/2}},$$

as $\nu \to +\infty$, where, by Theorem 1.2.2,

$$c_n = \frac{1}{2n!}\left[\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(\frac{x^2}{\sqrt{t^2+1}(\cosh x - 1) - \sinh x + x}\right)^{(n+1)/2}\right]_{x=0}.$$

Finally,

$$K_\nu(\nu t) \sim \left(\frac{\sqrt{t^2+1}+1}{t}\right)^\nu \mathrm{e}^{-\nu\sqrt{t^2+1}} \sqrt{\frac{\pi}{2\nu\sqrt{t^2+1}}} \sum_{n=0}^{\infty} \frac{(-1)^n U_n(\tau)}{\nu^n}, \tag{1.21}$$

as $\nu \to +\infty$, where

$$U_n(\tau) = (-1)^n \sqrt{\frac{2}{\pi}\sqrt{t^2+1}}\,\Gamma\left(n + \frac{1}{2}\right) c_{2n} \tag{1.22}$$
$$= (-1)^n \frac{(t^2+1)^{1/4}}{2^{2n+1/2} n!}\left[\frac{\mathrm{d}^{2n}}{\mathrm{d}x^{2n}}\left(\frac{x^2}{\sqrt{t^2+1}(\cosh x - 1) - \sinh x + x}\right)^{n+1/2}\right]_{x=0}$$
$$= \frac{(-1)^n}{2^{2n+1/2} n! \tau^{1/2}}\left[\frac{\mathrm{d}^{2n}}{\mathrm{d}x^{2n}}\left(\frac{x^2}{\tau^{-1}(\cosh x - 1) - \sinh x + x}\right)^{n+1/2}\right]_{x=0},$$

and $\tau = (t^2+1)^{-1/2}$. It is known that the $U_n(\tau)$'s are polynomials in τ of degree $3n$. The first few are given by $U_0(\tau) = 1$ and

$$U_1(\tau) = -\frac{5}{24}\tau^3 + \frac{1}{8}\tau,$$

$$U_2(\tau) = \frac{385}{1152}\tau^6 - \frac{77}{192}\tau^4 + \frac{9}{128}\tau^2,$$
$$U_3(\tau) = -\frac{85085}{82944}\tau^9 + \frac{17017}{9216}\tau^7 - \frac{4563}{5120}\tau^5 + \frac{75}{1024}\tau^3.$$

Thus, the expansion (1.21) holds uniformly for $t > 0$. Since

$$\sqrt{t^2+1}\,(\cosh x - 1) - \sinh x + x = \sum_{k=1}^{\infty} \frac{1}{\tau\,(2k)!}x^{2k} - \sum_{k=1}^{\infty} \frac{1}{(2k+1)!}x^{2k+1},$$

we have

$$a_{2k} = \frac{1}{\tau\,(2k+2)!},\; a_{2k+1} = -\frac{1}{(2k+3)!} \quad \text{for } k \geq 0.$$

From the CFWW Formula (1.11) we obtain

$$c_n = \frac{1}{\Gamma\left(\frac{n+1}{2}\right)} \sum_{j=0}^{n} \frac{(-1)^j\,(2\tau)^{(n+1)/2+j}}{j!} \mathsf{B}_{n,j}\left(-\frac{1}{6}, \frac{1}{24\tau}, \ldots, a_{n-j+1}\right) \Gamma\left(\frac{n+1}{2}+j\right),$$

from which it follows by (1.22) that

$$U_n(\tau) = \sum_{j=0}^{2n} (-1)^{n+j} \frac{2^{n+j+1}\tau^{n+j}}{j!\sqrt{\pi}} \Gamma\left(n+j+\frac{1}{2}\right) \mathsf{B}_{2n,j}.$$

Here $\mathsf{B}_{n,j} = \mathsf{B}_{n,j}\left(-\frac{1}{6}, \frac{1}{24\tau}, \ldots, a_{n-j+1}\right)$, and from the recurrence of the Partial Ordinary Bell Polynomials, $\mathsf{B}_{n,0} = 0$, $\mathsf{B}_{n,1} = a_n$ and

$$\mathsf{B}_{n,j} = -\frac{1}{6}\mathsf{B}_{n-1,j-1} + \frac{1}{24\tau}\mathsf{B}_{n-2,j-1} + \cdots + a_{n-j+1}\mathsf{B}_{j-1,j-1}.$$

In this case, it is quite complicated to give any explicit formula for the Partial Ordinary Bell Polynomials. Nevertheless, formula (1.16) enables us to derive a fairly explicit expression for the polynomials $U_n(\tau)$. To use formula (1.16) we have to compute the Ordinary Potential Polynomials $\mathsf{A}_{j,k}\left(-\frac{\tau}{3}, \frac{1}{12}, \ldots, 2\tau a_k\right)$. The generating function is

$$\left(2\tau\frac{\tau^{-1}(\cosh x - 1) - \sinh x + x}{x^2}\right)^j = \sum_{k=0}^{\infty} \mathsf{A}_{j,k}\left(-\frac{\tau}{3}, \frac{1}{12}, \ldots, 2\tau a_k\right) x^k.$$

Algebraic manipulation and the generating function of the 1-associated Stirling Numbers of the Second Kind (see expression (A.5)) yields

$$\begin{aligned}
&\left(\tau^{-1}(\cosh x - 1) - \sinh x + x\right)^j \\
&= \left((\mathrm{e}^x - x - 1)\left(\frac{1}{2\tau} - \frac{1}{2}\right) + (\mathrm{e}^{-x} + x - 1)\left(\frac{1}{2\tau} + \frac{1}{2}\right)\right)^j \\
&= \sum_{i=0}^{j} \binom{j}{i} \left(\frac{1}{2\tau} - \frac{1}{2}\right)^i \left(\frac{1}{2\tau} + \frac{1}{2}\right)^{j-i} (\mathrm{e}^x - x - 1)^i \left(\mathrm{e}^{-x} + x - 1\right)^{j-i} \\
&= j! \sum_{i=0}^{j} \left(\frac{1}{2\tau} - \frac{1}{2}\right)^i \left(\frac{1}{2\tau} + \frac{1}{2}\right)^{j-i} \sum_{k=0}^{\infty} S_1(k,i)\frac{x^k}{k!} \sum_{m=0}^{\infty} (-1)^m S_1(m, j-i)\frac{x^m}{m!} \\
&= j! \sum_{k=0}^{\infty} \left(\sum_{i=0}^{j} \sum_{m=2i}^{k-2j+2i} (-1)^{k-m} \left(\frac{1}{2\tau} - \frac{1}{2}\right)^i \left(\frac{1}{2\tau} + \frac{1}{2}\right)^{j-i} \frac{S_1(m,i)\,S_1(k-m, j-i)}{m!\,(k-m)!}\right) x^k,
\end{aligned}$$

which gives

$$
\begin{aligned}
&\left(2\tau\frac{\tau^{-1}(\cosh x-1)-\sinh x+x}{x^2}\right)^j\\
&= j!\sum_{k=0}^{\infty}\left(\sum_{i=0}^{j}\sum_{m=2i}^{k-2j+2i}(-1)^{k-m}(1-\tau)^i(1+\tau)^{j-i}\frac{S_1(m,i)\,S_1(k-m,j-i)}{m!\,(k-m)!}\right)x^{k-2j}\\
&= j!\sum_{k=0}^{\infty}\left(\sum_{i=0}^{j}\sum_{m=2i}^{k+2i}(-1)^{k-m}(1-\tau)^i(1+\tau)^{j-i}\frac{S_1(m,i)\,S_1(k+2j-m,j-i)}{m!\,(k+2j-m)!}\right)x^{k},
\end{aligned}
$$

that is

$$
\begin{aligned}
&\mathsf{A}_{j,k}\left(-\frac{\tau}{3},\frac{1}{12},\ldots,2\tau a_k\right)\\
&\qquad = j!\sum_{i=0}^{j}\sum_{m=2i}^{k+2i}(-1)^{k-m}(1-\tau)^i(1+\tau)^{j-i}\frac{S_1(m,i)\,S_1(k+2j-m,j-i)}{m!\,(k+2j-m)!}.
\end{aligned}
$$

Finally, from formula (1.16), we find

$$
\begin{aligned}
c_n &= (2\tau)^{(n+1)/2}\frac{\Gamma\left(\frac{3n+1}{2}+1\right)}{2\Gamma\left(\frac{n+1}{2}\right)}\sum_{j=0}^{n}\frac{(-1)^j}{\left(\frac{n+1}{2}+j\right)}\frac{\mathsf{A}_{j,n}\left(-\frac{\tau}{3},\frac{1}{12},\ldots,2\tau a_n\right)}{j!\,(n-j)!}\\
&= (2\tau)^{(n+1)/2}\frac{\Gamma\left(\frac{3n}{2}+\frac{3}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)}\sum_{j=0}^{n}\frac{(-1)^j}{(n+2j+1)}\frac{\mathsf{A}_{j,n}\left(-\frac{\tau}{3},\frac{1}{12},\ldots,2\tau a_n\right)}{j!\,(n-j)!},
\end{aligned}
$$

and thus by (1.22),

$$
U_n(\tau) = (-1)^n\frac{2(2\tau)^n\,\Gamma\left(3n+\frac{3}{2}\right)}{\sqrt{\pi}}\sum_{j=0}^{2n}(-1)^j\frac{\sum_{i=0}^{j}u_n(i,j)(1-\tau)^i(1+\tau)^{j-i}}{(2n+2j+1)\,(2n-j)!},
$$

where

$$
u_n(i,j) \overset{\text{def}}{=} \sum_{m=2i}^{2n+2i}(-1)^m\frac{S_1(m,i)\,S_1(2n+2j-m,j-i)}{m!\,(2n+2j-m)!}.
$$

♣

Remark. It can be shown that the *Modified Bessel Functions of the First Kind* I_ν have a similar asymptotic expansion:

$$
I_\nu(\nu t) \sim \left(\frac{t}{\sqrt{t^2+1}+1}\right)^\nu\frac{\mathrm{e}^{\nu\sqrt{t^2+1}}}{\sqrt{2\pi\nu\sqrt{t^2+1}}}\sum_{n=0}^{\infty}\frac{U_n(\tau)}{\nu^n},
$$

as $\nu\to+\infty$, uniformly for $t>0$. The coefficients $U_n(\tau)$ are the same as above. It is known that a recurrence for the polynomials $U_n(\tau)$ is given as follows

$$
U_{n+1}(\tau) = \frac{1}{2}\tau^2\left(1-\tau^2\right)U_n'(\tau) - \frac{1}{8}\int_0^\tau\left(5x^2-1\right)U_n(x)\,\mathrm{d}x,
$$

with $U_0(\tau)=1$ [8, p. 376].

Example 1.2.3. As a last example we take the *Legendre Polynomials* P_m. Suppose that $t > 1$, then we have the integral representation

$$P_m(t) = \frac{1}{\pi}\int_0^\pi \left(t + \cos x\sqrt{t^2-1}\right)^m \mathrm{d}x.$$

The substitution $t = \cosh\theta$ ($\theta > 0$) and a little algebraic manipulation gives

$$P_m(\cosh\theta) = \frac{\mathrm{e}^{m\theta}}{\pi}\int_0^\pi \mathrm{e}^{-m\left(-\log\left(1-\sin^2\left(\frac{x}{2}\right)\left(1-\mathrm{e}^{-2\theta}\right)\right)\right)}\mathrm{d}x.$$

Using Theorem 1.2.1 with $f(x) = -\log\left(1-\sin^2\left(\frac{x}{2}\right)\left(1-\mathrm{e}^{-2\theta}\right)\right)$, $g(x) \equiv 1$, $\alpha = 2$ and $\beta = 1$; one finds that

$$\int_0^\pi \mathrm{e}^{-m\left(-\log\left(1-\sin^2\left(\frac{x}{2}\right)\left(1-\mathrm{e}^{-2\theta}\right)\right)\right)}\mathrm{d}x \sim \sum_{n=0}^\infty \Gamma\left(\frac{n+1}{2}\right)\frac{c_n}{m^{(n+1)/2}}$$

as $m \to +\infty$, where, by Theorem 1.2.2,

$$c_n = \frac{1}{2n!}\left[\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(\frac{x^2}{-\log\left(1-\sin^2\left(\frac{x}{2}\right)\left(1-\mathrm{e}^{-2\theta}\right)\right)}\right)^{(n+1)/2}\right]_{x=0}.$$

It is not hard to see that $c_n = (-1)^n c_n$, thus

$$P_m(\cosh\theta) \sim \frac{\mathrm{e}^{(m+1)\theta}}{\sqrt{\pi m\left(\mathrm{e}^{2\theta}-1\right)}}\sum_{n=0}^\infty \frac{\rho_n(\theta)}{m^n}, \tag{1.23}$$

as $m \to +\infty$, where

$$\begin{aligned}\rho_n(\theta) &= \frac{1}{\sqrt{\pi}}\sqrt{1-\mathrm{e}^{-2\theta}}\,\Gamma\left(n+\frac{1}{2}\right)c_{2n} \\ &= \frac{\sqrt{1-\mathrm{e}^{-2\theta}}}{2^{2n+1}n!}\left[\frac{\mathrm{d}^{2n}}{\mathrm{d}x^{2n}}\left(\frac{x^2}{-\log\left(1-\sin^2\left(\frac{x}{2}\right)\left(1-\mathrm{e}^{-2\theta}\right)\right)}\right)^{n+1/2}\right]_{x=0}.\end{aligned} \tag{1.24}$$

The first few are $\rho_0(\theta) = 1$ and

$$\rho_1(\theta) = -\frac{\mathrm{e}^{2\theta}-3}{8\left(\mathrm{e}^{2\theta}-1\right)},\ \rho_2(\theta) = \frac{\mathrm{e}^{4\theta}+10\mathrm{e}^{2\theta}+25}{128\left(\mathrm{e}^{2\theta}-1\right)^2},\ \rho_3(\theta) = \frac{5\mathrm{e}^{6\theta}+35\mathrm{e}^{4\theta}+455\mathrm{e}^{2\theta}+105}{1024\left(\mathrm{e}^{2\theta}-1\right)^3}.$$

Since

$$\begin{aligned}&-\log\left(1-\sin^2\left(\frac{x}{2}\right)\left(1-\mathrm{e}^{-2\theta}\right)\right) = \sum_{k=1}^\infty \frac{\left(1-\mathrm{e}^{-2\theta}\right)^k}{k}\sin^{2k}\left(\frac{x}{2}\right) \\ &= \sum_{k=1}^\infty \frac{\left(1-\mathrm{e}^{-2\theta}\right)^k}{k}\sum_{j=0}^\infty\left(\frac{(-1)^{k+j}}{2^{2k-1}(2k+2j)!}\sum_{i=1}^k(-1)^i\binom{2k}{k-i}i^{2j+2k}\right)\left(x^2\right)^{k+j} \\ &= \sum_{k=1}^\infty\left(\sum_{j=0}^{k-1}\frac{(-1)^k\left(1-\mathrm{e}^{-2\theta}\right)^{j+1}}{2^{2j+1}(j+1)(2k)!}\sum_{i=1}^{j+1}(-1)^i\binom{2j+2}{j-i+1}i^{2k}\right)x^{2k},\end{aligned}$$

we have

$$a_{2k} = \sum_{j=0}^{k} \frac{(-1)^{k+1}\left(1-\mathrm{e}^{-2\theta}\right)^{j+1}}{2^{2j+1}(j+1)(2k+2)!} \sum_{i=1}^{j+1} (-1)^i \binom{2j+2}{j-i+1} i^{2k+2}$$

and $a_{2k+1} = 0$ for $k \geq 0$. From the CFWW Formula (1.11) we obtain

$$c_n = \frac{1}{2\Gamma\left(\frac{n+1}{2}\right)} \sum_{j=0}^{n} \frac{(-1)^j\, 4^{(n+1)/2+j}}{j!\left(1-\mathrm{e}^{-2\theta}\right)^{(n+1)/2+j}} \Gamma\left(\frac{n+1}{2}+j\right) \mathsf{B}_{n,j},$$

with $\mathsf{B}_{n,j} = \mathsf{B}_{n,j}\left(0, \frac{3\mathrm{e}^{-4\theta}-4\mathrm{e}^{-2\theta}+1}{96}, \ldots, a_{n-j+1}\right)$. From this and (1.24) it follows that

$$\rho_n(\theta) = \sum_{j=0}^{2n} (-1)^j \frac{4^{n+j}}{j!\sqrt{\pi}\left(1-\mathrm{e}^{-2\theta}\right)^{n+j}} \Gamma\left(n+j+\frac{1}{2}\right) \mathsf{B}_{2n,j}.$$

By the recurrence of the Partial Ordinary Bell Polynomials, $\mathsf{B}_{2n,0} = 0$, $\mathsf{B}_{2n,1} = a_{2n}$ and

$$\mathsf{B}_{2n,2j} = \frac{3\mathrm{e}^{-4\theta}-4\mathrm{e}^{-2\theta}+1}{96} \mathsf{B}_{2n-2,2j-1} + \cdots + a_{2n-2j}\mathsf{B}_{2j,2j},$$

$$\mathsf{B}_{2n,2j+1} = \frac{3\mathrm{e}^{-4\theta}-4\mathrm{e}^{-2\theta}+1}{96} \mathsf{B}_{2n-2,2j} + \cdots + a_{2n-2j}\mathsf{B}_{2j,2j}$$

for $j \geq 0$. If one thinks of the generating function of these Partial Ordinary Bell Polynomials, it is clear that giving an explicit formula for them would be quite complicated. On the other hand, formula (1.16) enables us to give an explicit expression for the coefficients $\rho_n(\theta)$ in terms of the Stirling Numbers of the First Kind. To use formula (1.16) we have to compute the Ordinary Potential Polynomials $\mathsf{A}_{j,k}\left(0, \frac{1-3\mathrm{e}^{-2\theta}}{24}, \ldots, \frac{4a_k}{1-\mathrm{e}^{-2\theta}}\right)$. The generating function is

$$\left(4\frac{-\log\left(1-\sin^2\left(\frac{x}{2}\right)\left(1-\mathrm{e}^{-2\theta}\right)\right)}{x^2\left(1-\mathrm{e}^{-2\theta}\right)}\right)^j = \sum_{k=0}^{\infty} \mathsf{A}_{j,k}\left(0, \frac{1-3\mathrm{e}^{-2\theta}}{24}, \ldots, \frac{4a_k}{1-\mathrm{e}^{-2\theta}}\right) x^k.$$

To compute the corresponding Ordinary Potential Polynomials, note that

$$\left(-\log\left(1-\sin^2\left(\frac{x}{2}\right)\left(1-\mathrm{e}^{-2\theta}\right)\right)\right)^j = \sum_{k=1}^{\infty} j!s(k,j) \frac{\left(1-\mathrm{e}^{-2\theta}\right)^k}{k!} \sin^{2k}\left(\frac{x}{2}\right)$$

$$= \sum_{k=1}^{\infty} j!s(k,j) \frac{\left(1-\mathrm{e}^{-2\theta}\right)^k}{k!} \sum_{m=0}^{\infty} \left(\frac{(-1)^{k+m}}{2^{2k-1}(2k+2m)!} \sum_{i=1}^{k} (-1)^i \binom{2k}{k-i} i^{2k+2m}\right) \left(x^2\right)^{k+m}$$

$$= \sum_{k=1}^{\infty} \left(\sum_{m=0}^{k} \frac{(-1)^k j!s(m,j)\left(1-\mathrm{e}^{-2\theta}\right)^m}{2^{2m-1}m!(2k)!} \sum_{i=1}^{m} (-1)^i \binom{2m}{m-i} i^{2k}\right) x^{2k},$$

for $j \geq 1$, which gives

$$\mathsf{A}_{j,2k}\left(0, \frac{1-3\mathrm{e}^{-2\theta}}{24}, \ldots, \frac{4a_{2k}}{1-\mathrm{e}^{-2\theta}}\right)$$
$$= \sum_{m=0}^{k+j} \frac{(-1)^{k+j}\, 2^{2j-2m+1} j!s(m,j)}{\left(1-\mathrm{e}^{-2\theta}\right)^{j-m} m!(2k+2j)!} \sum_{i=1}^{m} (-1)^i \binom{2m}{m-i} i^{2k+2j},$$

for $k, j \geq 1$. Finally, formula (1.16) yields

$$c_n = \frac{1}{2\Gamma\left(\frac{n+1}{2}\right)} \frac{2^{n+1}\Gamma\left(\frac{n+1}{2}+n+1\right)}{n!\left(1-\mathrm{e}^{-2\theta}\right)^{(n+1)/2}} \sum_{j=0}^{n} \frac{(-1)^j}{\left(\frac{n+1}{2}+j\right)} \binom{n}{j} \mathsf{A}_{j,n}$$
$$= \frac{1}{\Gamma\left(\frac{n+1}{2}\right)} \frac{2^{n+1}\Gamma\left(\frac{3n}{2}+\frac{3}{2}\right)}{\left(1-\mathrm{e}^{-2\theta}\right)^{(n+1)/2}} \sum_{j=0}^{n} \frac{(-1)^j}{(n+2j+1)} \frac{\mathsf{A}_{j,n}}{(n-j)!j!},$$

for $n \geq 1$ with $\mathsf{A}_{j,n} = \mathsf{A}_{j,n}\left(0, \frac{1-3\mathrm{e}^{-2\theta}}{24}, \ldots, \frac{4a_n}{1-\mathrm{e}^{-2\theta}}\right)$. From this and the expression (1.24) we deduce

$$\rho_n(\theta) = \frac{\Gamma\left(3n+\frac{3}{2}\right)}{\sqrt{\pi}} \sum_{j=0}^{2n} \frac{(-1)^n}{(2n-j)!} \sum_{m=0}^{n+j} \frac{2^{2n+2j-2m+2} s(m,j)}{\left(1-\mathrm{e}^{-2\theta}\right)^{n+j-m} m!\,(2n+2j+1)!} \sum_{i=1}^{m} (-1)^i \binom{2m}{m-i} i^{2n+2j}$$
$$= \frac{\Gamma\left(3n+\frac{3}{2}\right)}{\sqrt{\pi}} \sum_{j=n}^{3n} \frac{(-1)^n}{(3n-j)!} \sum_{m=0}^{j} \frac{4^{j-m+1} s(m,j-n)}{\left(1-\mathrm{e}^{-2\theta}\right)^{j-m} m!\,(2j+1)!} \sum_{i=1}^{m} (-1)^i \binom{2m}{m-i} i^{2j}, \tag{1.25}$$

for $n \geq 1$, with $\rho_0(\theta) = 1$. ♣

Remark. Another possible way to derive the asymptotic expansion (1.23) is to use the generating function

$$\frac{1}{\sqrt{1-2x\cosh\theta+x^2}} = \sum_{m=0}^{\infty} P_m(\cosh\theta)\, x^m,\ \ \theta > 0.$$

A simple algebraic manipulation yields

$$\frac{1}{\sqrt{1-z}} \frac{\mathrm{e}^{\theta}}{\sqrt{\mathrm{e}^{2\theta}-z}} = \sum_{m=0}^{\infty} \mathrm{e}^{-m\theta} P_m(\cosh\theta)\, z^m.$$

This generating function has an algebraic singularity at $z=1$, the full asymptotic expansion can now be carried out by *Darboux's Method* (see, *e.g.*, [8, p. 309], [14, p. 116]). The coefficients $\rho_n(\theta)$ this time take the form

$$\rho_n(\theta) = \sum_{k=0}^{n} \frac{(-1)^k (2n-4k+1)}{2^{4n-4k}(2n-2k+1)\left(\mathrm{e}^{2\theta}-1\right)^{n-k}} \binom{2n-2k}{n-k}^2 \binom{n-k+1/2}{k} B_k^{(n-k+1/2)},$$

where $B_n^{(\mu)}$ denote the Generalized Bernoulli Numbers. Using the explicit formula [5]

$$B_k^{(\mu)} = \sum_{j=0}^{k} \frac{\binom{k+\mu}{k-j}\binom{k-\mu}{k+j}}{\binom{k-\mu}{k}} S(k+j,j),$$

we get the following expression involving the Stirling Numbers of the Second Kind:

$$\rho_n(\theta) = \sum_{k=0}^{n} \frac{(-1)^k (2n-4k+1)}{2^{4n-4k}\left(\mathrm{e}^{2\theta}-1\right)^{n-k}} \binom{2n-2k}{n-k}^2 \binom{n+1/2}{2k} \sum_{j=0}^{k} \binom{2k}{k-j} \frac{(-1)^j S(k+j,j)}{2n-2k+2j+1},$$

which is even simpler than (1.25).

Chapter 2

Complex Laplace-type integrals

In this chapter, we investigate the asymptotic behavior of Laplace-type integrals with complex parameter. The integrals under consideration are of the form

$$I(\lambda) = \int_{\mathscr{C}} \mathrm{e}^{\lambda f(z)} g(z)\, \mathrm{d}z, \tag{2.1}$$

where the path of integration $\mathscr{C}$ is a contour in the complex plane and λ is a large real parameter. The functions f and g are independent of λ and holomorphic in a domain containing the path of integration.

Chapter 2 is organized as follows. In the first section, we revisit the Method of Steepest Descents, a well-known procedure in the asymptotic theory of complex Laplace-type integrals. In the second part, we present Perron's Method that avoids the computation of the path of steepest descent. Finally, in the third section, we give three examples to demonstrate the application of these methods.

2.1 The Method of Steepest Descents

Consider the integral (2.1) with the assumptions we made. The basic idea of the method is to deform the contour of integration $\mathscr{C}$ into a new path of integration $\mathscr{C}'$ so that the following conditions hold:

(i) $\mathscr{C}'$ passes through one or more zeros of f';

(ii) the function $\mathfrak{Im}(f)$ is constant on $\mathscr{C}'$.

The choice of a path with $\mathfrak{Im}(f) = \text{constant}$ has two major advantages. It removes the rapid oscillations of the integrand and on such paths $\mathfrak{Re}(f)$ changes the most rapidly (see, *e.g.*, [9, p. 5]). Thus, the dominant contribution will arise from a neighborhood of the point where $\mathfrak{Re}(f)$ is the greatest.

In order to obtain a geometrical interpretation of the method and the new path of integration $\mathscr{C}'$, we introduce the notations

$$f(z) = u(x, y) + \mathrm{i}v(x, y)$$

with $z = x + \mathrm{i}y$, and the functions u and v are real. Suppose that $z_0 = x_0 + \mathrm{i}y_0$ is a zero of f'. It follows easily that (x_0, y_0) is a critical point of u, and since u is harmonic, it must be a saddle point of u. For this reason, we call z_0 a *saddle point* of f.

Consider the surface F in the (x, y, u) space defined by $u = u(x, y)$. The shape of the surface F can also be represented on the (x, y) plane by drawing the level curves on which u is constant. From the Cauchy–Riemann equations it follows that the families of curves corresponding to constant values of $u(x, y)$ and $v(x, y)$ are orthogonal at all their points of intersection [9, p. 6]. The regions where $u(x, y) > u(x_0, y_0)$ are called *hills* and those where $u(x, y) < u(x_0, y_0)$ are called *valleys*. The level curve through the saddle, $u(x, y) = u(x_0, y_0)$, separates the neighborhood of the saddle point (x_0, y_0) into a series of hills and valleys.

Suppose that z_0 is a saddle point of order $m-1$, $m \geq 2$, *i.e.*,

$$f'(z_0) = f''(z_0) = \cdots = f^{(m-1)}(z_0) = 0,$$

with $f^{(m)}(z_0) = a\mathrm{e}^{\mathrm{i}\varphi}$, $a > 0$. Then, if $z = z_0 + r\mathrm{e}^{\mathrm{i}\theta}$, $r > 0$, we have

$$f(z) = f(z_0) + \frac{r^m}{m!} a\mathrm{e}^{\mathrm{i}(m\theta+\varphi)} + \cdots$$

and hence, near the saddle point z_0 the level curves and steepest paths are roughly the same as

$$u(x, y) = u(x_0, y_0) + \frac{r^m}{m!} a\cos(m\theta + \varphi),$$
$$v(x, y) = v(x_0, y_0) + \frac{r^m}{m!} a\sin(m\theta + \varphi).$$

The directions of the level curves where u is constant, are given by the solutions of the equation $\cos(m\theta + \varphi) = 0$, *i.e.*,

$$\theta = -\frac{\varphi}{m} + \left(k + \frac{1}{2}\right)\frac{\pi}{m}, \quad k = 0, 1, \ldots, 2m-1.$$

Similarly, the directions of the steepest paths satisfy $\sin(m\theta + \varphi) = 0$, *i.e.*,

$$\theta = -\frac{\varphi}{m} + k\frac{\pi}{m}, \quad k = 0, 1, \ldots, 2m-1.$$

Therefore, there are $2m$ equally spaced steepest directions from z_0: m directions of *steepest descent* and m directions of *steepest ascent*. In the neighborhood of z_0, the level curves $u = u(x_0, y_0)$ form the boundaries of m valleys surrounding the saddle point, in which $\cos(m\theta + \varphi) < 0$, and m hills on which $\cos(m\theta + \varphi) > 0$. The valleys and hills are situated respectively entirely below and above the saddle point, and each has angular width equal to π/m.

Now suppose that the path of integration $\mathscr{C}$ in (2.1) is a steepest path through the saddle point z_0 of order $m-1$. On this path we have

$$f(z) = f(z_0) - \tau,$$

where τ is non-negative and monotonically increasing as one progresses down the path of steepest descent. Then (2.1) becomes

$$I(\lambda) = \mathrm{e}^{\lambda f(z_0)} \int_{z_0}^{T} \mathrm{e}^{-\lambda\tau} g(z)\,\mathrm{d}z = \mathrm{e}^{\lambda f(z_0)} \int_{0}^{T'} \mathrm{e}^{-\lambda\tau} g(z) \frac{\mathrm{d}z}{\mathrm{d}\tau}\mathrm{d}\tau, \tag{2.2}$$

where T denotes some point on the steepest descent path and $T' > 0$ is the map of T in the τ-plane. Since, for large positive λ, the factor $\mathrm{e}^{-\lambda\tau}$ decays rapidly, the main contribution comes from the neighborhood of $\tau = 0$. Thus, we can apply Watson's Lemma to the integral in the right-hand side of (2.2). To do this, we first require a series expansion for $g(z)\frac{\mathrm{d}z}{\mathrm{d}\tau}$ in ascending powers of τ. Near z_0,

$$f(z) = f(z_0) - \sum_{k=0}^{\infty} a_k (z - z_0)^{m+k}, \quad a_0 \neq 0,$$

that is

$$\tau = \sum_{k=0}^{\infty} a_k (z - z_0)^{m+k}$$

as $z \to z_0$. If we put $\tau = \upsilon^m$, then

$$\upsilon = a_0^{1/m} (z - z_0) \varphi(z),$$

where φ is analytic around z_0 with $\varphi(z_0) \neq 0$ and $a_0^{1/m}$ takes its principal value. Hence, υ is a single-valued analytic function of z in a neighborhood of z_0 and that $\upsilon'(z_0) \neq 0$. Therefore, by the inverse function theorem (see [2, p. 121])

$$z - z_0 = \sum_{k=1}^{\infty} \alpha_k \upsilon^k = \sum_{k=1}^{\infty} \alpha_k \tau^{k/m},$$

where

$$\alpha_1 = \frac{1}{a_0^{1/m}}, \quad \alpha_2 = -\frac{a_1}{m a_0^{1+2/m}}, \quad \alpha_3 = \frac{(m+3) a_1^2 - 2m a_0 a_2}{2m^2 a_0^{2+3/m}}, \ldots .$$

Furthermore, for small τ, $g(z)\frac{\mathrm{d}z}{\mathrm{d}\tau}$ has a convergent expansion of the form

$$g(z)\frac{\mathrm{d}z}{\mathrm{d}\tau} = \sum_{n=0}^{\infty} c_n \tau^{(n-m+1)/m}. \tag{2.3}$$

The first three coefficients c_n are given explicitly by

$$c_0 = \frac{b_0}{m a_0^{1/m}}, \qquad c_1 = \frac{1}{a_0^{2/m}} \left(\frac{b_1}{m} - \frac{2a_1 b_0}{m^2 a_0} \right),$$

and

$$c_2 = \frac{1}{a_0^{3/m}} \left(\frac{b_2}{m} - \frac{3a_1 b_1}{m^2 a_0} + \left((m+3) a_1^2 - 2m a_0 a_2 \right) \frac{b_0}{m^2 a_0^2} \right).$$

Here $n! b_n = g^{(n)}(z_0)$. We are now in a position to derive the asymptotic expansion of the integral (2.2) taken from the saddle point z_0 of order $m-1$ down one of the m valleys. Depending on the path $\mathscr{C}$, we choose one of the m valleys, labelled by l, say. We replace τ by $\tau e^{2\pi i l}$ in the expansion (2.3) and substitute it into the integral (2.2). The asymptotic expansion of $I(\lambda)$ can now be obtained directly from Watson's Lemma:

$$I(\lambda) \sim e^{\lambda f(z_0)} \sum_{n=0}^{\infty} \Gamma\left(\frac{n+1}{m} \right) \frac{c_n e^{2\pi i l(n+1)/m}}{\lambda^{(n+1)/m}}, \tag{2.4}$$

as $\lambda \to +\infty$. To obtain an explicit formula for the coefficients c_n we observe that with $\tau = \upsilon^m$, so that $\frac{\mathrm{d}z}{\mathrm{d}\tau} = \frac{\upsilon^{1-m}}{m}\frac{\mathrm{d}z}{\mathrm{d}\upsilon}$, we find from (2.3)

$$g(z)\frac{\mathrm{d}z}{\mathrm{d}\upsilon} = m\sum_{n=0}^{\infty} c_n \upsilon^n.$$

Cauchy's formula then shows that

$$c_n = \frac{1}{2\pi \mathrm{i} m}\int_{\gamma} g(z)\frac{\mathrm{d}z}{\mathrm{d}\upsilon}\frac{\mathrm{d}\upsilon}{\upsilon^{n+1}} = \frac{1}{2\pi \mathrm{i} m}\int_{\gamma'} \frac{g(z)}{\left(f(z_0) - f(z)\right)^{(n+1)/m}}\mathrm{d}z,$$

with an appropriate branch of $(f(z_0) - f(z))^{1/m}$. Here γ and γ' are simple closed contours with positive orientation that enclose the points $\upsilon = 0$ and $z = z_0$, respectively. This is *Dingle's Formula* [3, p. 119]. It follows that

$$\begin{aligned} c_n &= \frac{1}{mn!}\left[\frac{\mathrm{d}^n}{\mathrm{d}z^n}\left\{g(z)\left(\frac{(z-z_0)^m}{f(z_0)-f(z)}\right)^{(n+1)/m}\right\}\right]_{z=z_0} \\ &= \frac{1}{m}\sum_{k=0}^{n}\frac{b_{n-k}}{k!}\left[\frac{\mathrm{d}^k}{\mathrm{d}z^k}\left(\frac{(z-z_0)^m}{f(z_0)-f(z)}\right)^{(n+1)/m}\right]_{z=z_0}, \end{aligned} \tag{2.5}$$

which is Perron's Formula. In the next subsection we derive this formula in more general conditions.

Remark. It can be shown that the asymptotic expansion (2.4) is also valid when

$$|\arg\lambda + \arg a_0 + m\omega - 2\pi l| \le \frac{\pi}{2} - \delta < \frac{\pi}{2}$$

and $|\lambda| \to +\infty$, where $0 < \delta \le \frac{\pi}{2}$ is fixed and

$$\omega = \lim_{\mathscr{C}\ni z\to a} \arg(z-a)$$

is the angle of slope of $\mathscr{C}$ at a.

2.2 Perron's Method

In the previous subsection, we presented a method, namely, the Method of Steepest Descents, to derive asymptotic expansions for integrals of the form

$$I(\lambda) = \int_{\mathscr{C}} \mathrm{e}^{\lambda f(z)} g(z)\,\mathrm{d}z. \tag{2.6}$$

However, in many specific cases, the construction of steepest descent path can be extremely complicated. In this subsection, we shall describe a method due to Perron that avoids the computation of the path of steepest descent. Our presentation of Perron's Method follows closely that given by Wong [14, p. 103].

In this subsection we allow λ to be complex. Let a be the starting point, and b be the endpoint of the continuous curve $\mathscr{C}$ (b can by finite or infinite). We impose the following four conditions:

(i) The path $\mathscr{C}$ lies in the sector

$$|\arg(\lambda(f(a) - f(z)))| \leq \frac{\pi}{2} - \delta,$$

where δ is a fixed positive number.

(ii) For each point $c \neq a$ in $\mathscr{C}$, there exists a number $\eta = \eta(c)$, such that $|f(a) - f(z)| \geq \eta > 0$ for all z on the portion of $\mathscr{C}$ joining c to b.

(iii) In a neighborhood of a,

$$f(a) - f(z) = \sum_{k=0}^{\infty} a_k (z-a)^{k+\alpha}, \tag{2.7}$$

where $a_0 \neq 0$ and $\alpha > 0$. Since α is not necessarily a positive integer, f is need not be analytic at a. We make $(z-a)^{\alpha}$ single-valued by introducing a cut in the z-plane from a to infinity along a convenient radial line.

(iv) The contour $\mathscr{C}$ must not cross this cut. Furthermore, suppose that there exists a point $c' \neq a$ on $\mathscr{C}$ such that for any c on $\mathscr{C}$ with $|ac| < |ac'|$, the portion of $\mathscr{C}$ from a to c can be deformed into the straight line $\arg(z-a) = \arg(c-a)$.

Let

$$\omega = \lim_{\mathscr{C} \ni z \to a} \arg(z-a)$$

be the angle of slope of $\mathscr{C}$ at a. In order to condition (i) holds, the set through which $|\lambda| \to +\infty$ must be contained in the sector

$$\left(2l - \frac{1}{2}\right)\pi + \delta \leq \arg\lambda + \arg a_0 + \alpha\omega \leq \left(2l + \frac{1}{2}\right)\pi - \delta \tag{2.8}$$

for some fixed δ in $0 < \delta \leq \frac{\pi}{2}$, where l is a fixed integer.

Theorem 2.2.1. *Consider the integral* (2.6)*, and assume that it exists absolutely for every fixed* λ *satisfying the inequalities in* (2.8)*. Furthermore, assume that in a neighborhood of* a*,*

$$g(z) = \sum_{k=0}^{\infty} b_k (z-a)^{k+\beta-1}, \quad b_0 \neq 0,$$

with some fixed complex number β*,* $\mathfrak{Re}(\beta) > 0$*. Then, under the conditions (i) to (iv), we have*

$$I(\lambda) \sim \mathrm{e}^{\lambda f(a)} \sum_{n=0}^{\infty} \Gamma\left(\frac{n+\beta}{\alpha}\right) \frac{c_n \mathrm{e}^{2\pi l \mathrm{i}(n+\beta)/\alpha}}{\lambda^{(n+\beta)/\alpha}},$$

as $|\lambda| \to +\infty$*, uniformly with respect to* $\arg\lambda$ *confined to the sector* (2.8)*. The coefficients* c_n *are given by*

$$\begin{aligned} c_n &= \frac{1}{\alpha a_0^{(n+\beta)/\alpha} n!} \left[\frac{\mathrm{d}^n}{\mathrm{d}z^n} \left\{ G(z) \left(\frac{a_0 (z-a)^{\alpha}}{f(a) - f(z)}\right)^{(n+\beta)/\alpha} \right\}\right]_{z=a} \\ &= \frac{1}{\alpha a_0^{(n+\beta)/\alpha}} \sum_{k=0}^{n} \frac{b_{n-k}}{k!} \left[\frac{\mathrm{d}^k}{\mathrm{d}z^k} \left(\frac{a_0 (z-a)^{\alpha}}{f(a) - f(z)}\right)^{(n+\beta)/\alpha}\right]_{z=a}, \end{aligned} \tag{2.9}$$

where $G(z) \stackrel{\mathrm{def}}{=} \sum_{k=0}^{\infty} b_k (z-a)^k$.

Proof. Let F be the function

$$F(z) \stackrel{\text{def}}{=} -\sum_{k=1}^{\infty} \frac{a_k}{a_0}(z-a)^k$$

which is analytic at $z = a$ and satisfies

$$f(z) = f(a) - a_0 (z-a)^{\alpha} (1 - F(z)) \tag{2.10}$$

in a neighborhood of a. Since the function $G(z)\exp(wF(z))$ is, for each fixed w, analytic at $z = a$, we have

$$G(z)\exp(wF(z)) = \sum_{k=0}^{\infty} P_k(w)(z-a)^k, \quad |z-a| < \rho,$$

for sufficiently small ρ, where $P_k(w)$ is a polynomial in w whose degree does not exceed k, and

$$P_k(w) = \frac{1}{k!}\left[\frac{\mathrm{d}^k}{\mathrm{d}z^k}(G(z)\exp(wF(z)))\right]_{z=a}. \tag{2.11}$$

Using (2.10), (2.11) can be written as

$$\mathrm{e}^{-w} P_k(w) = \frac{1}{k!}\left[\frac{\mathrm{d}^k}{\mathrm{d}z^k}\left(G(z)\exp\left(w\frac{f(z)-f(a)}{a_0(z-a)^{\alpha}}\right)\right)\right]_{z=a}. \tag{2.12}$$

By Cauchy's inequality

$$|P_k(w)| \le \frac{1}{r^k}\max_{|z-a|=r}|G(z)\exp(wF(z))|,$$

for $0 < r < \rho$. Since F vanishes at a, for every $0 < r < \rho$,

$$|P_k(w)| \le \frac{1}{r^k} M_1 \exp(M_2 |w| r),$$

where M_1 and M_2 are fixed numbers. Thus, for any fixed $N > 0$,

$$G(z)\exp(wF(z)) = \sum_{k=0}^{N} P_k(w)(z-a)^k + R_N, \tag{2.13}$$

where, for $|z-a| \le r < \rho$,

$$|R_N| \le M_3 |z-a|^{N+1}\exp(M_2|w|r),$$

where M_3 is a fixed number depending on N. Now, we put $w = \lambda a_0(z-a)^{\alpha}$ in (2.13). It follows that

$$g(z)\exp(\lambda f(z) - \lambda f(a)) = \sum_{k=0}^{N} \mathrm{e}^{-w} P_k(w)(z-a)^{k+\beta-1} + \mathrm{e}^{-w}(z-a)^{\beta-1} R_N \tag{2.14}$$

and, for $|z-a| \le r < \rho$,

$$\left|\mathrm{e}^{-w}(z-a)^{\beta-1} R_N\right| \le M_3 \left|(z-a)^{N+\beta}\right| \exp(-\mathfrak{Re}(w) + M_2|w|r). \tag{2.15}$$

We write the integral (2.6) in the form

$$I(\lambda) = \int_a^c e^{\lambda f(z)} g(z)\,dz + \int_c^b e^{\lambda f(z)} g(z)\,dz, \tag{2.16}$$

where c is a point on $\mathscr{C}$ such that the portion of $\mathscr{C}$ form a to c can be deformed into the radial line $\arg(z-a) = \arg(c-a)$ (see condition (iv)). Furthermore, we require $|c-a| < r$ so that the expansion (2.14)-(2.15) holds for all z on the new path of integration from a to c. Inserting (2.14) in the first integral on the right of (2.16) yields

$$\int_a^c e^{\lambda f(z)} g(z)\,dz = e^{\lambda f(a)} \left[\sum_{k=0}^{N} I_k(\lambda) + E_N(\lambda) \right], \tag{2.17}$$

where

$$I_k(\lambda) \stackrel{\text{def}}{=} \int_a^c e^{-w} P_k(w)\,(z-a)^{k+\beta-1}\,dz \tag{2.18}$$

and

$$E_N(\lambda) \stackrel{\text{def}}{=} \int_a^c e^{-w}\,(z-a)^{\beta-1}\,R_N dz.$$

By choosing c closer to a if necessary, we also have from (2.8)

$$\left(2l - \frac{1}{2}\right)\pi + \delta_0 \le \arg\lambda + \arg a_0 + \alpha\arg(c-a) \le \left(2l + \frac{1}{2}\right)\pi - \delta_0$$

for some $0 < \delta_0 < \delta$. This implies that

$$\left(2l - \frac{1}{2}\right)\pi + \delta_0 \le \arg w \le \left(2l + \frac{1}{2}\right)\pi - \delta_0, \tag{2.19}$$

and thus $\mathfrak{Re}(w) \ge |w|\sin\delta_0$. Thus we obtain from (2.15)

$$\left|e^{-w}\,(z-a)^{\beta-1}\,R_N\right| \le M_3 \left|(z-a)^{N+\beta}\right| \exp\left(-|w|\sin\delta_0 + M_2\,|w|\,r\right)$$

for the points on the path of integration. If we choose r such that $r < \sin\delta_0/M_2$, then there will exist a constant $M_4 > 0$ such that for all z with $\arg(z-a) = \arg(c-a)$ and $|z-a| \le r < \rho$,

$$\left|e^{-w}\,(z-a)^{\beta-1}\,R_N\right| \le M_3 \left|(z-a)^{N+\beta}\right| \exp\left(-M_4\,|w|\right),$$

and therefore,

$$|E_N(\lambda)| \le M_3 \int_a^c \left|(z-a)^{N+\beta}\right| \exp\left(-M_4\,|\lambda a_0\,(z-a)^{\alpha}|\right) |dz|.$$

If we let $\theta = \arg(c-a)$ and $z - a = \rho e^{i\theta}$, then

$$|E_N(\lambda)| \le M_3 \int_0^{+\infty} \left|\rho^{\beta+N}\right| \exp\left(-M_4\,|\lambda a_0|\,\rho^{\alpha}\right) d\rho = \mathcal{O}\left(\lambda^{-(\beta+N+1)/\alpha}\right). \tag{2.20}$$

We now turn our attention to the integrals $I_k(\lambda)$ in (2.18). It is not hard to show that
$$\int_c^{a+\infty e^{i\theta}} e^{-w} P_k(w)(z-a)^{k+\beta-1}\,dz = \mathcal{O}\left(e^{-\varepsilon|\lambda|}\right)$$
for some $\varepsilon > 0$. Next we note, from (2.19), that $w = \lambda a_0 (z-a)^\alpha$ gives

$$z - a = w^{1/\alpha} e^{2\pi l i/\alpha} a_0^{-1/\alpha} \lambda^{-1/\alpha}$$

and
$$(z-a)^{k+\beta-1} = w^{(k+\beta-1)/\alpha} e^{2\pi l i(k+\beta-1)/\alpha} a_0^{-(k+\beta-1)/\alpha} \lambda^{-(k+\beta-1)/\alpha}.$$

Hence,

$$I_k(\lambda) = \frac{1}{\alpha} a_0^{-(k+\beta)/\alpha} \lambda^{-(k+\beta)/\alpha} e^{2\pi l i(k+\beta)/\alpha} \int_0^{\infty e^{i\theta'}} e^{-w} P_k(w)\, w^{(k+\beta)/\alpha - 1} dw + \mathcal{O}\left(e^{-\varepsilon|\lambda|}\right),$$

where, for fixed λ, $|\theta'| < \frac{\pi}{2}$. Now, we deform the path of integration into the positive real axis. Therefore

$$I_k(\lambda) = \frac{1}{\alpha} a_0^{-(k+\beta)/\alpha} \lambda^{-(k+\beta)/\alpha} e^{2\pi l i(k+\beta)/\alpha} \int_0^{+\infty} e^{-w} P_k(w)\, w^{(k+\beta)/\alpha - 1} dw + \mathcal{O}\left(e^{-\varepsilon|\lambda|}\right).$$

We now insert (2.12) into the integral, and then interchange the order of integration and differentiation. This leads to

$$I_k(\lambda) = \Gamma\left(\frac{k+\beta}{\alpha}\right) \frac{c_k e^{2\pi l i(k+\beta)/\alpha}}{\lambda^{(k+\beta)/\alpha}} + \mathcal{O}\left(e^{-\varepsilon|\lambda|}\right).$$

Using this and (2.20) in (2.17) yields

$$\int_a^c e^{\lambda f(z)} g(z)\,dz = e^{\lambda f(a)} \left(\sum_{k=0}^{N} \Gamma\left(\frac{k+\beta}{\alpha}\right) \frac{c_k e^{2\pi l i(k+\beta)/\alpha}}{\lambda^{(k+\beta)/\alpha}} + \mathcal{O}\left(\frac{1}{\lambda^{(N+\beta+1)/\alpha}}\right)\right). \tag{2.21}$$

What remains is to consider the second integral on the right-hand side of (2.16). We choose a point λ_0 that satisfies the inequalities in (2.8) and write

$$\int_c^b e^{\lambda f(z)} g(z)\,dz = \int_c^b e^{(\lambda-\lambda_0)f(z)} e^{\lambda_0 f(z)} g(z)\,dz,$$

whence
$$\left|\int_c^b e^{\lambda f(z)} g(z)\,dz\right| \le \max\left|e^{(\lambda-\lambda_0)f(z)}\right| \int_c^b \left|e^{\lambda_0 f(z)} g(z)\right| |dz|,$$
where the maximum is taken over all points on the portion of $\mathscr{C}$ joining c to b. By assumption, the integral on the right exist, thus

$$\left|\int_c^b e^{\lambda f(z)} g(z)\,dz\right| \le K \max\left|e^{(\lambda-\lambda_0)f(z)}\right|$$

for some constant $K > 0$. It follows that

$$\left|\int_c^b e^{\lambda f(z)} g(z)\,dz\right| \le \left(K e^{-\lambda_0 f(a)}\right) e^{\lambda f(a)} \max\left|e^{-(\lambda-\lambda_0)(f(a)-f(z))}\right|. \tag{2.22}$$

Then

$$\begin{aligned}
&\mathfrak{Re}\left(\left(\lambda-\lambda_0\right)\left(f\left(a\right)-f\left(z\right)\right)\right) \\
&= \left|f\left(a\right)-f\left(z\right)\right|\left\{\left|\lambda\right|\cos\left[\arg\left(\lambda\left(f\left(a\right)-f\left(z\right)\right)\right)\right]-\left|\lambda_0\right|\cos\left[\arg\left(\lambda_0\left(f\left(a\right)-f\left(z\right)\right)\right)\right]\right\} \\
&= \left|\lambda\right|\left|f\left(a\right)-f\left(z\right)\right|\left\{\cos\left[\arg\left(\lambda\left(f\left(a\right)-f\left(z\right)\right)\right)\right]+o\left(1\right)\right\},
\end{aligned}$$

as $|\lambda| \to +\infty$. Thus, by conditions (i) and (ii),

$$\mathfrak{Re}\left(\left(\lambda-\lambda_0\right)\left(f\left(a\right)-f\left(z\right)\right)\right) \geq \left|\lambda\right|\eta\left(\sin\delta+o\left(1\right)\right),$$

as $|\lambda| \to +\infty$ in the sector (2.8), and consequently,

$$\mathfrak{Re}\left(\left(\lambda-\lambda_0\right)\left(f\left(a\right)-f\left(z\right)\right)\right) \geq \varepsilon_1\left|\lambda\right|$$

for some $\varepsilon_1 > 0$, uniformly in $\arg\lambda$ for all λ satisfying (2.8) and uniformly in z for all z on $\mathscr{C}$ from c to b. This together with (2.22) implies that

$$\left|\int_c^b \mathrm{e}^{\lambda f(z)} g\left(z\right) \mathrm{d}z\right| = \mathcal{O}\left(\mathrm{e}^{\lambda f(a) - \varepsilon_1|\lambda|}\right). \tag{2.23}$$

The result now follows from (2.16), (2.21) and (2.23). ∎

Remark. From (2.7) and (2.9) it can be seen that the representations (1.11) and (1.16) hold in the complex case too.

2.3 Examples

Example 2.3.1. Our first example is the Reciprocal Gamma Function

$$\frac{1}{\Gamma\left(\lambda\right)} = \frac{1}{2\pi\mathrm{i}}\int_{\mathscr{H}} \mathrm{e}^t t^{-\lambda}\mathrm{d}t,$$

where the path of integration starts at $\infty\mathrm{e}^{-\pi\mathrm{i}}$, goes round the origin once and ends at $\infty\mathrm{e}^{\pi\mathrm{i}}$. Suppose that λ is real an positive, then the substitution $t = \lambda z$ gives

$$\frac{1}{\Gamma\left(\lambda\right)} = \frac{1}{2\pi\mathrm{i}\lambda^{\lambda-1}}\int_{\mathscr{H}} \mathrm{e}^{\lambda(z-\log z)}\mathrm{d}z,$$

with the same path as before. This formula holds also when $\mathfrak{Re}(\lambda) > 0$, provided that λ^λ means $\mathrm{e}^{\lambda\log\lambda}$ where $\log\lambda$ has its principal value. Using the notations of Section 2.1, $f\left(z\right) = z - \log z$ and $g(z) \equiv 1$. The function f has only one saddle point, namely $z_0 = 1$. The steepest paths through z_0 are described by the equation

$$y - \tan^{-1}\left(\frac{y}{x}\right) = 0.$$

The equation $y = 0$ represents the path of steepest ascent whereas the equation $x = y\cot y$ (or rather the branch through $(1, 0)$) gives the path of steepest descent. Then $\mathscr{H}$ can be deformed into $\mathscr{C} \stackrel{\text{def}}{=} \{(x, y) : x = y\cot y,\ |y| < \pi\}$. On the path of steepest descent, we have

$$f\left(z\right) - f\left(1\right) = -1 + z - \log z = -\tau. \tag{2.24}$$

As $z \to 1$,

$$f(z) - f(1) = \frac{(z-1)^2}{2} - \frac{(z-1)^3}{3} + \cdots,$$

thus

$$z_\pm = 1 + \sum_{n=0}^{\infty} (\pm 1)^n \alpha_n \tau^{n/2}, \tag{2.25}$$

$$\frac{\mathrm{d}z_\pm}{\mathrm{d}\tau} = \sum_{n=0}^{\infty} (\pm 1)^{n+1} c_n \tau^{(n-1)/2}, \quad c_n = \frac{n+1}{2}\alpha_{n+1}, \tag{2.26}$$

where $z_\pm$ denotes the upper and lower path of $\mathscr{C}$ through the saddle point $z_0 = 1$. Equation (2.24) defines a many valued function $z(\tau)$ of a complex variable τ with branch points at $\tau = 2\pi i n$ with $n \in \mathbb{Z} \setminus \{0\}$. Hence, the expansions (2.25)-(2.26) are converget in the disc $|\tau| < 2\pi$. Since

$$\frac{\mathrm{d}z_\pm}{\mathrm{d}\tau} = \frac{z_\pm}{1 - z_\pm}$$

is bounded when $\tau > 0$, Watson's Lemma is applicable and gives

$$\begin{aligned} \frac{1}{\Gamma(\lambda)} &= \frac{\mathrm{e}^\lambda}{2\pi \mathrm{i} \lambda^{\lambda-1}} \int_0^{+\infty} \mathrm{e}^{-\lambda\tau} \left(\frac{\mathrm{d}z_+}{\mathrm{d}\tau} - \frac{\mathrm{d}z_-}{\mathrm{d}\tau} \right) \mathrm{d}\tau \\ &\sim \mathrm{e}^\lambda \lambda^{-\lambda} \sqrt{\frac{\lambda}{2\pi}} \sum_{n=0}^{\infty} \sqrt{\frac{2}{\pi}} \Gamma\left(n + \frac{1}{2}\right) \frac{-\mathrm{i} c_{2n}}{\lambda^n}, \end{aligned}$$

as $\lambda \to +\infty$. From Perron's Formula (2.5) we find

$$c_n = \frac{1}{2n!} \left[\frac{\mathrm{d}^n}{\mathrm{d}z^n} \left(\frac{(z-1)^2}{1 - z + \log z} \right)^{(n+1)/2} \right]_{z=1} = \frac{\mathrm{i}^{n+1}}{2n!} \left[\frac{\mathrm{d}^n}{\mathrm{d}z^n} \left(\frac{z^2}{z - \log(1+z)} \right)^{(n+1)/2} \right]_{z=0},$$

and hence

$$\begin{aligned} \sqrt{\frac{2}{\pi}} \Gamma\left(n + \frac{1}{2}\right) (-\mathrm{i} c_{2n}) &= \sqrt{\frac{2}{\pi}} \Gamma\left(n + \frac{1}{2}\right) \frac{(-1)^n}{2(2n)!} \left[\frac{\mathrm{d}^{2n}}{\mathrm{d}z^{2n}} \left(\frac{z^2}{z - \log(1+z)} \right)^{n+1/2} \right]_{z=0} \\ &= (-1)^n \gamma_n. \end{aligned}$$

Here γ_n denotes the Stirling Coefficients given in Example 1.2.1. Therefore,

$$\frac{1}{\Gamma(\lambda)} \sim \mathrm{e}^\lambda \lambda^{-\lambda} \sqrt{\frac{\lambda}{2\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n \gamma_n}{\lambda^n},$$

as $\lambda \to +\infty$. This expansion is also valid when $|\arg \lambda| \le \frac{\pi}{2} - \delta < \frac{\pi}{2}$, with fixed $0 < \delta \le \frac{\pi}{2}$. ♣

Example 2.3.2. As a second example we take the *Airy Function*

$$\mathrm{Ai}\left(\mu^2\right) = \frac{1}{2\pi \mathrm{i}} \int_{\mathscr{L}} \mathrm{e}^{\mu^2 t - \frac{t^3}{3}} \mathrm{d}t,$$

where $\mu > 0$ and the path $\mathscr{L}$ consists of two rays, one with end points $\infty e^{\frac{4}{3}\pi i}$ and 0, and the other with end points 0 and $\infty e^{\frac{2}{3}\pi i}$. The substitution $t = \mu z$ yields

$$\mathrm{Ai}\left(\mu^2\right) = \frac{\mu}{2\pi i} \int_{\mathscr{L}} e^{\mu^3\left(z-\frac{z^3}{3}\right)} dz,$$

where $\mathscr{L}$ can be taken as the same as in the original integral. Using the notations of Section 2.1, $f(z) = z - \frac{z^3}{3}$, $g(z) \equiv 1$ and $\lambda = \mu^3$. The saddle points of f are $z_0 = -1$ and $z_1 = 1$, we choose the former one. The steepest paths through z_0 are described by the equation

$$y\left(y^2 - 3x^2 + 3\right) = 0.$$

The equation $y = 0$ represents the path of steepest ascent whereas the equation $y^2 - 3x^2 + 3 = 0$ (left branch of a hyperbola) gives the path of steepest descent. It is clear that $\mathscr{L}$ can be deformed into $\mathscr{C} \overset{\text{def}}{=} \{(x, y) : y^2 - 3x^2 + 3 = 0,\ x < 0\}$. We put

$$f(z) - f(-1) = z - \frac{z^3}{3} + \frac{2}{3} = -\tau.$$

We have

$$f(z) - f(-1) = (z+1)^2 - \frac{(z+1)^3}{3}$$

as $z \to -1$, thus

$$z_{\pm} = -1 + \sum_{n=0}^{\infty} (\pm 1)^n \alpha_n \tau^{n/2}, \tag{2.27}$$

$$\frac{dz_{\pm}}{d\tau} = \sum_{n=0}^{\infty} (\pm 1)^{n+1} c_n \tau^{(n-1)/2}, \quad c_n = \frac{n+1}{2}\alpha_{n+1}, \tag{2.28}$$

where $z_{\pm}$ denotes the upper and lower path of $\mathscr{C}$ through the saddle point $z_0 = -1$. Since

$$\frac{dz_{\pm}}{d\tau} = \frac{1}{z_{\pm}^2 - 1},$$

and $\tau(1) = -\frac{4}{3}$, the expansions (2.27)-(2.28) are converget in the disc $|\tau| < \frac{4}{3}$. Watson's Lemma then gives

$$\begin{aligned} \mathrm{Ai}\left(\mu^2\right) &= \frac{\mu}{2\pi i} \int_0^{+\infty} e^{-\frac{2}{3}\mu^3 - \mu^3\tau} \left(\frac{dz_+}{d\tau} - \frac{dz_-}{d\tau}\right) d\tau \\ &\sim \frac{e^{-\frac{2}{3}\mu^3}}{2\sqrt{\pi\mu}} \sum_{n=0}^{\infty} \Gamma\left(n + \frac{1}{2}\right) \frac{-2ic_{2n}}{\sqrt{\pi}\mu^{3n}}, \end{aligned}$$

as $\mu \to +\infty$. From Perron's Formula (2.5) we find

$$\begin{aligned} c_n = \frac{1}{2n!}\left[\frac{d^n}{dz^n}\left(\frac{3}{z-2}\right)^{(n+1)/2}\right]_{z=-1} &= \frac{(-i)^{n+1}}{2n!}\left[\frac{d^n}{dz^n}\left(1 - \frac{z}{3}\right)^{-(n+1)/2}\right]_{z=0} \\ &= \frac{(-i)^{n+1}}{2} \frac{\Gamma\left(\frac{3n+1}{2}\right)}{3^n n! \Gamma\left(\frac{n+1}{2}\right)}, \end{aligned}$$

and thus

$$\mathrm{Ai}\left(\mu^2\right) \sim \frac{\mathrm{e}^{-\frac{2}{3}\mu^3}}{2\sqrt{\pi\mu}} \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma\left(3n+\frac{1}{2}\right)}{\sqrt{\pi}9^n\,(2n)!} \frac{1}{\mu^{3n}} \quad \text{as } \mu \to +\infty.$$

This expansion is also valid when $|\arg \mu^3| \le \frac{\pi}{2} - \delta < \frac{\pi}{2}$, with fixed $0 < \delta \le \frac{\pi}{2}$. The substitution $z = \mu^2$ leads to the final form

$$\mathrm{Ai}\left(z\right) \sim \frac{\mathrm{e}^{-\frac{2}{3}z^{3/2}}}{2\sqrt{\pi}z^{1/4}} \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma\left(3n+\frac{1}{2}\right)}{\sqrt{\pi}9^n\,(2n)!} \frac{1}{z^{3n/2}},$$

as $|z| \to +\infty$ in the sector $|\arg z| \le \frac{\pi}{3} - \varepsilon < \frac{\pi}{3}$, with fixed $0 < \varepsilon \le \frac{\pi}{3}$. ♣

Example 2.3.3. Our last example is the integral

$$I\left(\lambda,\nu\right) = \frac{1}{\sqrt{\pi}} \int_0^{+\infty} \mathrm{e}^{\lambda\left(2z-z^2\right)} z^\nu \mathrm{d}z, \quad \mathfrak{Re}(\nu) > -1.$$

We shall use Perron's Method to obtain the asymptotic expansion of $I\left(\lambda,\nu\right)$ for fixed ν and $|\lambda| \to +\infty$ in an appropriate sector of the complex plane. We split up the integral into two parts as follows

$$I\left(\lambda,\nu\right) = \frac{1}{\sqrt{\pi}} \int_0^{+\infty} \mathrm{e}^{\lambda\left(1-z^2\right)} \left(1+z\right)^\nu \mathrm{d}z + \frac{1}{\sqrt{\pi}} \int_0^{1} \mathrm{e}^{\lambda\left(1-z^2\right)} \left(1-z\right)^\nu \mathrm{d}z.$$

Using the notations of Section 2.2, $f\left(z\right) = 1 - z^2$, $g_+(z) = \left(1+z\right)^\nu$, $g_-(z) = \left(1-z\right)^\nu$. The only saddle point of f is $z_0 = 0$. We have $f\left(0\right) - f\left(z\right) = z^2$ and

$$g_\pm\left(z\right) = \sum_{k=0}^{\infty} \left(\pm 1\right)^k \binom{\nu}{k} z^k,$$

as $z \to 0$. We can apply Theorem 2.2.1 with $\alpha = 2$, $\beta = 1$:

$$\int_0^{+\infty} \mathrm{e}^{\lambda\left(1-z^2\right)} \left(1+z\right)^\nu \mathrm{d}z \sim \mathrm{e}^\lambda \sum_{n=0}^{\infty} \Gamma\left(\frac{n+1}{2}\right) \frac{c_n}{\lambda^{(n+1)/2}},$$

$$\int_0^{1} \mathrm{e}^{\lambda\left(1-z^2\right)} \left(1-z\right)^\nu \mathrm{d}z \sim \mathrm{e}^\lambda \sum_{n=0}^{\infty} \Gamma\left(\frac{n+1}{2}\right) \frac{\left(-1\right)^n c_n}{\lambda^{(n+1)/2}},$$

as $|\lambda| \to +\infty$ in the sector $|\arg \lambda| \le \frac{\pi}{2} - \delta < \frac{\pi}{2}$, with fixed $0 < \delta \le \frac{\pi}{2}$. Here

$$c_n = \frac{1}{2n!} \left[\frac{\mathrm{d}^n}{\mathrm{d}z^n} \left(1+z\right)^\nu\right]_{z=0} = \frac{1}{2}\binom{\nu}{n}.$$

After simplification, the final result is

$$I\left(\lambda,\nu\right) \sim \frac{\mathrm{e}^\lambda}{\sqrt{\lambda}} \sum_{n=0}^{\infty} \binom{\nu}{2n} \frac{(2n)!}{2^{2n} n!} \frac{1}{\lambda^n},$$

as $|\lambda| \to +\infty$ and $|\arg \lambda| \le \frac{\pi}{2} - \delta < \frac{\pi}{2}$, with fixed $0 < \delta \le \frac{\pi}{2}$. ♣

Appendix A

Combinatorial objects

A.1 Ordinary Potential Polynomials

Let $F(x) = 1 + \sum_{n=1}^{\infty} f_n x^n$ be a formal power series. For any complex number ρ, we define the *Ordinary Potential Polynomial* $\mathsf{A}_{\rho,n}(f_1, f_2, \ldots, f_n)$ (associated to F) by the generating function

$$(F(x))^{\rho} = \left(1 + \sum_{n=1}^{\infty} f_n x^n\right)^{\rho} = \sum_{n=0}^{\infty} \mathsf{A}_{\rho,n}(f_1, f_2, \ldots, f_n)\, x^n.$$

The first few are $\mathsf{A}_{\rho,0} = 1$, $\mathsf{A}_{\rho,1} = \rho f_1$, $\mathsf{A}_{\rho,2} = \rho f_2 + \binom{\rho}{2} f_1^2$, and in general

$$\mathsf{A}_{\rho,n}(f_1, f_2, \ldots, f_n) = \sum \binom{\rho}{k} \frac{k!}{k_1! k_2! \cdots k_n!} f_1^{k_1} f_2^{k_2} \cdots f_n^{k_n}, \tag{A.1}$$

where the sum extending over all sequences $k_1, k_2, \ldots, k_n$ of non-negative integers such that $k_1 + 2k_2 + \cdots + nk_n = n$ and $k_1 + k_2 + \cdots + k_n = k$. We write

$$\mathsf{A}_{\rho,n}(f_1, f_2, \ldots, f_n) = \sum_{k=1}^{n} \binom{\rho}{k} \mathsf{B}_{n,k}(f_1, f_2, \ldots, f_{n-k+1}),$$

where the $\mathsf{B}_{n,k}(f_1, f_2, \ldots, f_{n-k+1})$'s are called the *Partial Ordinary Bell Polynomials*. From (A.1) it follows that

$$\mathsf{B}_{n,k}(f_1, f_2, \ldots, f_{n-k+1}) = \sum \frac{k!}{k_1! k_2! \cdots k_{n-k+1}!} f_1^{k_1} f_2^{k_2} \cdots f_{n-k+1}^{k_{n-k+1}}. \tag{A.2}$$

Here the sum runs over all sequences $k_1, k_2, \ldots, k_n$ of non-negative integers such that $k_1 + 2k_2 + \cdots + (n-k+1)\, k_{n-k+1} = n$ and $k_1 + k_2 + \cdots + k_{n-k+1} = k$.

Since $(F(x))^{\rho} = (F(x))^{\rho-1} F(x)$, we have the recurrence

$$\mathsf{A}_{\rho,n}(f_1, f_2, \ldots, f_n) = \mathsf{A}_{\rho-1,n}(f_1, f_2, \ldots, f_{n-k}) + \sum_{k=1}^{n} f_k \mathsf{A}_{\rho-1,n-k}(f_1, f_2, \ldots, f_{n-k}).$$

Taking into account formula (A.2), one finds that

$$(F(x) - 1)^k = \left(\sum_{n=1}^{\infty} f_n x^n\right)^k = \sum_{n=k}^{\infty} \mathsf{B}_{n,k}(f_1, f_2, \ldots, f_{n-k+1})\, x^n, \tag{A.3}$$

and therefore

$$\mathsf{B}_{n,k+1}\left(f_1, f_2, \ldots, f_{n-k}\right) = \sum_{k=1}^{n-k} f_k \mathsf{B}_{n-k,k}\left(f_1, f_2, \ldots, f_{n-2k+1}\right). \tag{A.4}$$

If $G\left(x\right) = \sum_{n=0}^{\infty} g_n x^n$ is a formal power series, then by (A.3), we have

$$G\left(y\left(F\left(x\right)-1\right)\right) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} g_k \mathsf{B}_{n,k}\left(f_1, f_2, \ldots, f_{n-k+1}\right) y^k\right) x^n.$$

Specially,

$$\exp\left(y\sum_{n=1}^{\infty} f_n x^n\right) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} \frac{\mathsf{B}_{n,k}\left(f_1, f_2, \ldots, f_{n-k+1}\right)}{k!} y^k\right) x^n.$$

For more details see, *e.g.*, Riordan's book [11, p. 189].

A.2 The r-associated Stirling Numbers

For every nonnegative integers r and k we define the *r-associated Stirling Numbers of the First and Second Kind* by the generating functions

$$\frac{1}{k!}\left(-\log\left(1-x\right) - \sum_{m=1}^{r}\frac{x^m}{m}\right)^k = \sum_{n=(r+1)k}^{\infty} s_r\left(n,k\right)\frac{x^n}{n!},$$

$$\frac{1}{k!}\left(\mathrm{e}^x - \sum_{m=0}^{r}\frac{x^m}{m!}\right)^k = \sum_{n=(r+1)k}^{\infty} S_r\left(n,k\right)\frac{x^n}{n!}. \tag{A.5}$$

If $r=0$ then $s\left(n,k\right) \stackrel{\text{def}}{=} s_0\left(n,k\right)$ and $S\left(n,k\right) \stackrel{\text{def}}{=} S_0\left(n,k\right)$ are the Stirling Numbers of the First and Second Kind, respectively.

It follows that

$$\exp\left(y\left(-\log\left(1-x\right) - \sum_{m=1}^{r}\frac{x^m}{m}\right)\right) = \sum_{n=0}^{\infty}\left(\sum_{j=0}^{\lfloor n/r+1\rfloor} s_r\left(n,j\right) y^j\right)\frac{x^n}{n!},$$

$$\exp\left(y\left(\mathrm{e}^x - \sum_{m=0}^{r}\frac{x^m}{m!}\right)\right) = \sum_{n=0}^{\infty}\left(\sum_{j=0}^{\lfloor n/r+1\rfloor} S_r\left(n,j\right) y^j\right)\frac{x^n}{n!}.$$

It is known (see [4]) that

$$s_r\left(n,k\right) = n!\sum_{j=0}^{k}\frac{\left(-1\right)^j s_{r-1}\left(n-rj,k-j\right)}{r^j j!\left(n-rj\right)!},$$

$$S_r\left(n,k\right) = n!\sum_{j=0}^{k}\frac{\left(-1\right)^j S_{r-1}\left(n-rj,k-j\right)}{\left(r!\right)^j j!\left(n-rj\right)!}.$$

For further identities see, *e.g.*, Howard's paper [4].

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