Quasiprobability distributions of bright "banana" states

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Problem statement

We start with the following Hamiltonian:

$$\hat{H}_{Kerr} = \hbar\omega\hat{a}^\dagger\hat{a} + \hbar\gamma\hat{a}^{\dagger 2}\hat{a}^2$$

And let's study the evolution of the coherent state $|\alpha\rangle$:

$$|\psi\rangle = e^{-\frac{i}{\hbar}\hat{H}_{Kerr}\tau}|\alpha\rangle \sim \frac{e^{-\frac{|\alpha|^2}{2}}}{\pi^{1/4}}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}e^{i\Gamma n^2}|n\rangle$$

The main parametrs have the following orders:

$$\begin{cases}\gamma\tau = |\Gamma| \sim 10^{-6} \\ |\alpha|^2 \sim 10^{+6}\end{cases}$$

The state $|\psi\rangle$ we study using Husimi and Wigner functions:

$$Q(\beta) = \frac{|\langle\psi|\beta\rangle|^2}{\pi} = \frac{e^{-|\alpha|^2-|\beta|^2}}{\pi}\left|\sum_{n=0}^{\infty}\frac{(\alpha\beta^*)^n\, e^{-i\Gamma n(n-1)}}{n!}\right|^2$$

$$W(\beta) = \frac{1}{\pi}\int_{-\infty}^{\infty} dy\, \psi^*(\operatorname{Re}\beta + y)\psi(\operatorname{Re}\beta - y)e^{2ip\operatorname{Im}\beta}$$

They have the following connection via Fourier transform:

$$C_s(z) = \mathcal{F}\{W\}(z) = e^{\frac{|z|^2}{2}}\mathcal{F}\{Q\}(z) = e^{\frac{|z|^2}{2}}C_a(z)$$

The main problem is to calculate this quasi-probabilistic functions in a reasonable time.

Table of nonlinearities

Material	Q	γ, Hz	$\Gamma \cdot 10^{-6}$
Al_2O_3	2×10^9	0.06	24
CaF_2	3×10^{11}	0.4	24000
MgF_2	6×10^9	0.03 (e, o)	36
Quartz	5×10^9	0.1	100
Fused silica	9×10^9	0.08	144
$LiNbO_3$	10^9	0.26 (o)	52
Si_3N_4	8×10^7	0.39	6
Si	10^9	0.5	100

The best performance archived for cubically nonlinear media in microresonators. $\Gamma = \gamma\tau$ is defined by archived quality with non-linearity determined by the material.

Basic idea

Instead of studing row which appears in Husimi function let's study function F that has less arguments:

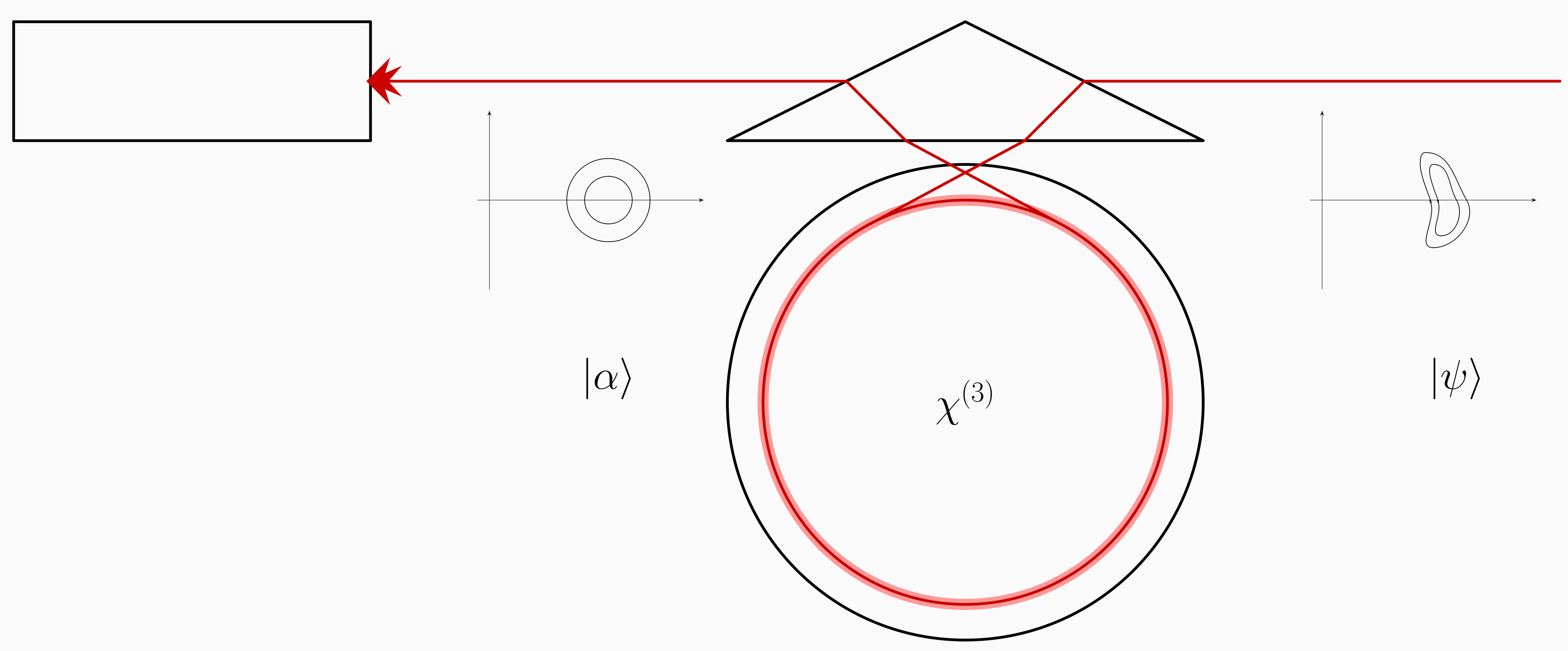
$$F(A, e^{i\Gamma}) = \sum_{n=0}^{\infty}\frac{A^n e^{i\Gamma n^2}}{n!}, \tag{1}$$

This function has relatively simple dependency on the z argument:

$$\left|F(A,e^{i\Gamma})\right| \approx \frac{1}{\sqrt{2|A\Gamma|}}\left(1 - \frac{1}{(4|A\Gamma|)^2} + \frac{5/2}{(4|A\Gamma|)^4} + \dots\right)\times \\ \times\exp\left(|A| - \frac{\left(\arg\left(Ae^{2i|A|\Gamma}\right) - \Gamma\right)^2}{8|A\Gamma^2|} + \dots\right) \tag{2}$$

pic with dependence

Scheme



Wigner calculation

$$O(1+|\alpha\Gamma|) \\ |\alpha|^2 \gtrsim \Gamma^{-2} \tag{3}$$

$$\Gamma = 2\pi\frac{k}{n} \tag{4}$$

We use the interesting fact:

$$\frac{dF}{dA}(A, e^{i\Gamma}) = e^{i\Gamma}F(Ae^{2i\Gamma}, e^{i\Gamma})$$

So we find an alternative repr for F for $\Gamma = 2\pi\frac{k}{n}$:

$$F(A, e^{i\Gamma}) = \frac{\sum_{j\in\mathbb{Z}_n}\exp\left(-i\Gamma j^2 + Ae^{2ij\Gamma}\right)}{\sum_{j\in\mathbb{Z}_n}\exp\left(-i\Gamma j^2\right)} \tag{5}$$

picture with some peaks

Then using characteristic functions connection and finding a lot of Gaussian integrals we find:

$$W(\beta) = \frac{2}{\pi}e^{|\alpha|^2-2(|\alpha|-|\beta|)^2}\sum_{m=0}^{\infty}\frac{\left(-|\alpha|^2\right)^m}{m!}\left|F(2\alpha\beta^* e^{i\Gamma(2m-1)}, e^{i\Gamma})e^{-2|\alpha\beta^*|}\right|^2$$

$$W(\beta) = 2e^{2|\beta|^2}\sum_{m=0}^{\infty}\frac{\left(-|\alpha|^2\right)^m}{m!}Q(2\beta e^{-i\Gamma 2m})$$

the work in this direction is in progress

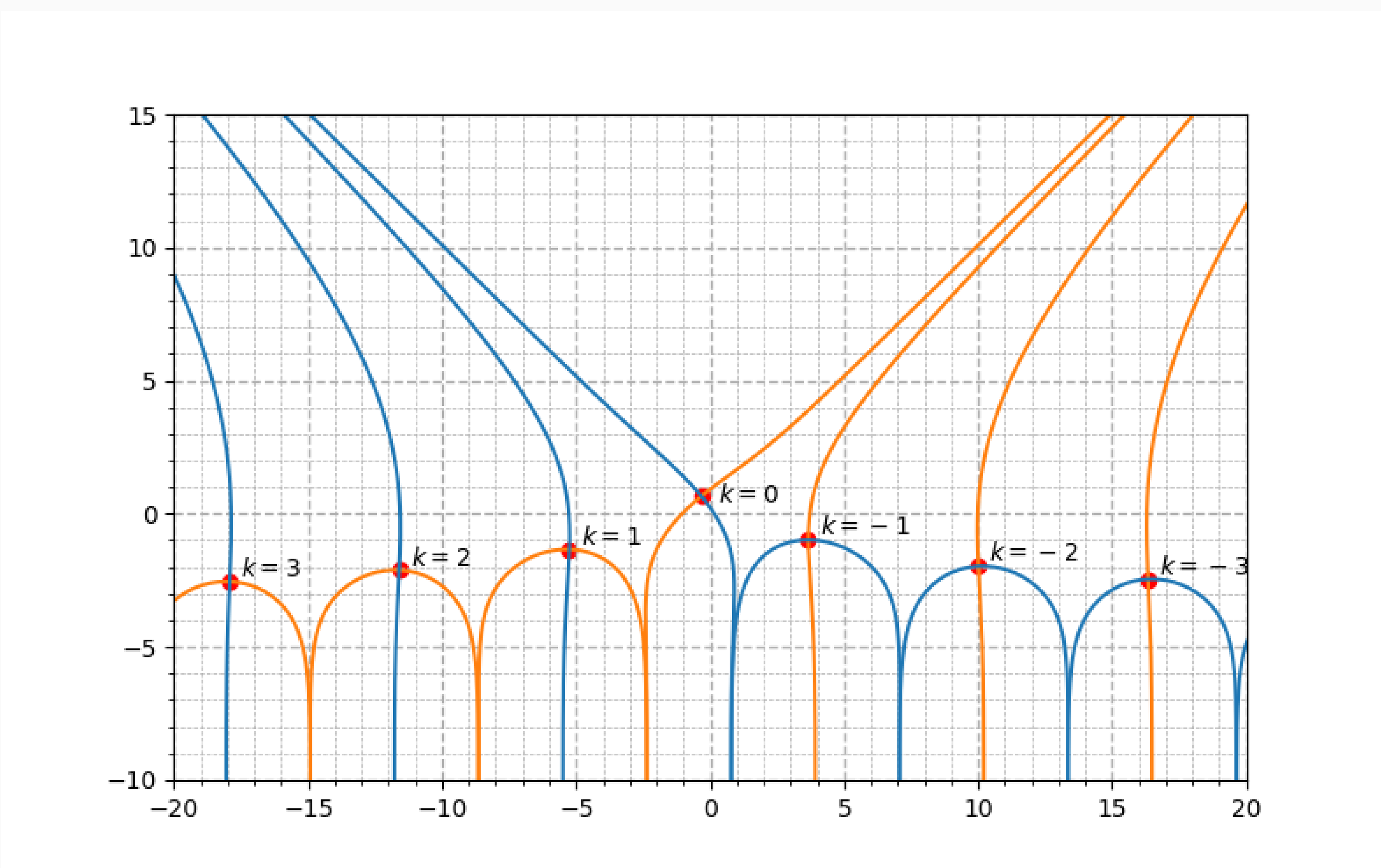
Husimi calculation

Firstly we rewrite the expression for F using integral:

$$\Gamma = \frac{\pi}{6}$$

$$\sum_{n=0}^{\infty}\frac{A^n e^{i\Gamma n^2}}{n!} = \frac{e^{i\frac{\pi}{4}\operatorname{sign}\Gamma}}{2\sqrt{\pi|\Gamma|}}\int_{-\infty}^{\infty}\exp\left(-\frac{z^2}{4\Gamma} + iAz\right)dz \tag{6}$$

then we deform the integration contour:



we use the method of steepest descent:

$$\int\exp(f(z))\,dz = \sum_k\int_{\gamma_k}\exp(f(z))\,dz \approx \sum_k \exp(f(z_k))\sqrt{\frac{2\pi}{-f''(z_k)}} \tag{7}$$

concluding, we find the asymptotic expr for F:

$$\sum_{n=0}^{\infty}\frac{A^n e^{i\Gamma n^2}}{n!} \approx e^{\frac{i\pi}{4}\operatorname{sign}\Gamma}\frac{\exp\left(\frac{-i+i(W_{\bar{k}}(-2iA\Gamma)+1)^2}{4\Gamma}\right)}{\sqrt{-i\operatorname{sign}\Gamma\left(1+W_{\bar{k}}(-2iA\Gamma)\right)}}$$

$$\bar{k} \approx -\operatorname{sign}\Gamma\left[\frac{2|A\Gamma| + |\arg A|}{2\pi}\right]$$

where we've hidden bulky remainder.