Comp6211e: Optimization for Machine Learning

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Lecture 6: Nesterov's Acceleration Method

Convex Optimization

L - smooth

λ - strongly convex

L = max eigvalue of Hessian matrix

λ = min

In this lecture, we consider the general unconstrained convex optimization problem:

$$\min_{x \in \mathbb{R}^d} f(x).$$

We consider the the following form of recursion:

shrinking factor (usually < 1).

$$
\begin{aligned}
x_t &= x_{t-1} + p_{t-1} \\
p_t &= -\alpha_t \nabla f(x_t) + \beta_t p_{t-1}.
\end{aligned}
$$

how fast to forget previous gradient.

learning rate (usually $\frac{1}{L}$).

We may refer to this class of methods as momentum methods, and it can be employed for general unconstrained optimization problems.

Reformulation

CG

$$\begin{cases} x_t = x_{t-1} + p_{t-1} \\ p_t = -\alpha_t \nabla f(x_t) + \beta_t p_{t-1} \end{cases}$$

$\Longrightarrow p_{t-1} = x_t - x_{t-1}$

$\Rightarrow p_t = -\alpha_t \nabla f(x_t) + \beta_t (x_t - x_{t-1})$

$\Downarrow$

$x_t = \underbrace{x_{t-1}}_{} - \alpha_{t-1} \nabla f(x_{t-1}) + \underbrace{\beta_{t-1}(x_{t-1} - x_{t-2})}_{}$

$y_t = x_{t-1} + \beta_t (x_{t-1} - x_{t-2})$

Equivalence

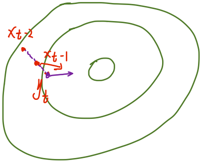
Heavy Ball

$$\begin{cases} y_t = x_{t-1} + \beta_t (x_{t-1} - x_{t-2}) \\ x_t = y_t - \alpha_t \nabla f(x_{t-1}) \end{cases}$$

The momentum (Heavy Ball) method can be written as the following form:

$\beta_t = \beta = 1 - \frac{1}{\sqrt{\kappa}}$

$$y_t = x_{t-1} + \beta_t (x_{t-1} - x_{t-2})$$
$$x_t = y_t - \alpha_t \nabla f(x_{t-1}).$$



more aggressive using Nesterov's

Nesterov's Method

① Stochastic (CG not applicable)

② Non-smooth (CG not good in general).

Nesterov modified this equation as follows:

$$\begin{aligned} y_t &= x_{t-1} + \beta_t (x_{t-1} - x_{t-2}) \\ x_t &= y_t - \alpha_t \nabla f(y_t). \end{aligned}$$

With this modification, one can prove the global convergence of the resulting algorithm for convex functions.

Algorithm

Algorithm 1: Nesterov's Acceleration Method

Input: $f(x)$, x_0, $\alpha_1, \beta_1, \alpha_2, \beta_2, \ldots$

Output: x_T

Let $x_{-1} = x_0$

for $t = 1, \ldots, T$ **do**

  Let $y_t = x_{t-1} + \beta_t(x_{t-1} - x_{t-2})$

  Let $x_t = y_t - \alpha_t \nabla f(y_t)$

Return: x_T

Equivalent Formulation

Formula for CG : $p_t = -\alpha_t \nabla f(x_t) + \beta_t p_{t-1}$

$x_t = x_{t-1} + p_t$.

Note that equivalently, we may write Nesterov's method as follows, with a different choice of parameters.

$$
\begin{aligned}
y_t =& x_{t-1} + \beta_t p_{t-1} \quad \cdots \text{new.} \\
p_t =& \beta_t p_{t-1} - \alpha_t \nabla f(y_t) \\
x_t =& x_{t-1} + p_t .
\end{aligned}
$$

average of historical gradients.

This can be compared to the heavy ball formulation.

Motivation: GD versus CG

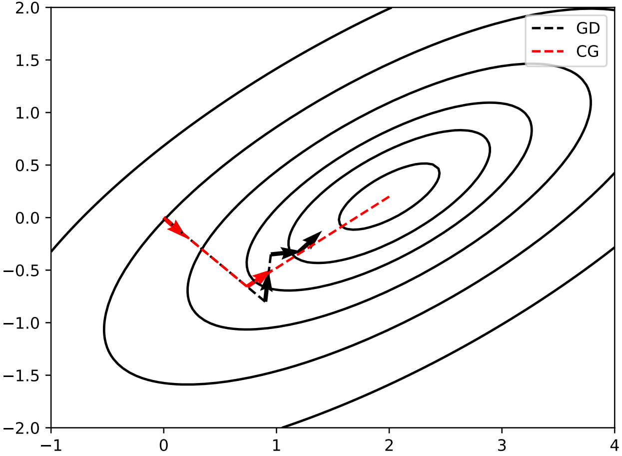


Figure: Gradient Descent and CG

Motivation: GD versus Accelerated Methods

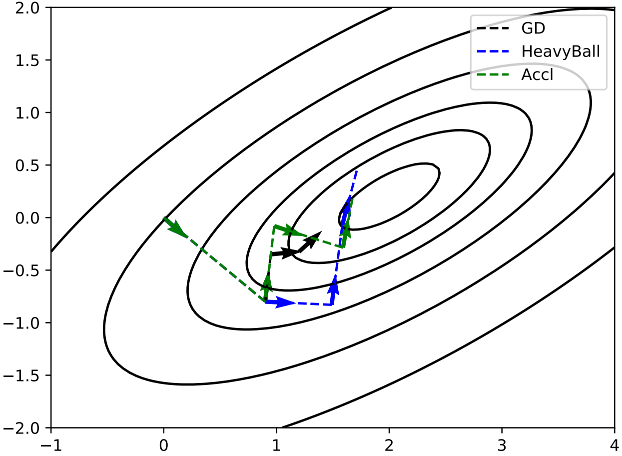
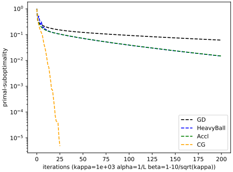
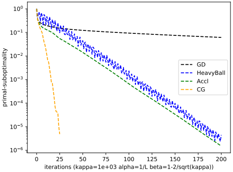


Figure: Gradient Descent, Heavy Ball, and Acceleration



(a) $\beta \approx 0.7$

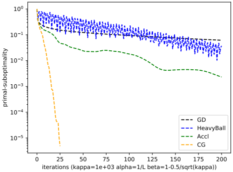
$\beta = 1 - \frac{10}{\sqrt{\kappa}}$



(b) $\beta \approx 0.94$

$\beta = 1 - \frac{2}{\sqrt{\kappa}}$

β is like convergence rate

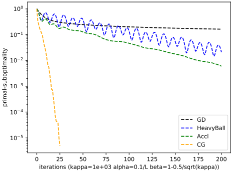


(c) $\beta \approx 0.98$

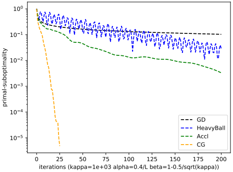
$\beta = 1 - \frac{1}{2\sqrt{\kappa}}$

Figure: Convergence Comparisons with Fixed α

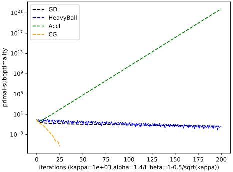
Sensitivity to α



(a) $\alpha = 0.1/L$



(b) $\alpha \approx 0.4/L$



(c) $\alpha \approx 1.4/L$

Figure: Convergence Comparisons with Fixed β

Convergence Analysis

Theorem

Assume $f(x)$ is L-smooth and λ-strongly convex. Let $\eta \leq 1/L$ and $\theta = \sqrt{\eta\lambda}$. Let $\alpha_t = \eta \leq 1/L$ and $\beta_t = \beta = (1-\theta)/(1+\theta)$. Then

$$f(x_t) \leq f(x_*) + (1-\theta)^t \left[f(x_0) - f(x_*) + \frac{\lambda}{2}\|z - x_0\|_2^2 \right]$$

.

$\theta = \frac{1}{\sqrt{K}}$

Convergence rate $1 - \frac{1}{\sqrt{K}}$

Proof: Estimation Sequence

Definition

A pair of sequences $\{(\phi_t(x), \lambda_t \geq 0\}$ is called an estimation sequence of function $f(x)$, if for any $x \in \mathbb{R}^d$ and all $t \geq 0$:

$$\phi_t(x) \leq (1 - \lambda_t) f(x) + \lambda_t \phi_0(x).$$

Convergence Analysis with Estimation Sequence

If for an estimation sequence, we have the following property (upper bound of $f(x_t)$)

$$f(x_t) \leq \phi_t(v_t) = \min_z \phi_t(z).$$

then

$$f(x_t) \leq (1 - \lambda_t) f(x_*) + \lambda_t \phi_0(x_*).$$

Estimation Sequence Lemma

Lemma

Let $x^+ = y - \eta \nabla f(y)$*. We define*

$$\phi(z; y) = f(x^+) - \frac{1}{2\eta}\|x^+ - y\|_2^2 + \frac{1}{\eta}(y - x^+)^\top (z - x^+) + \frac{\lambda}{2}\|z - y\|_2^2.$$

Then the following inequality holds:

$$\phi(z; y) \le f(z).$$

Therefore if we define recursively

$$\phi_t(z) = (1-\theta)\phi_{t-1}(z) + \theta\phi(z; y_t)$$

with

$$\phi_0(z) = f(x_0) + \frac{\lambda}{2}\|z - x_0\|_2^2,$$

then $\{\phi_t, (1-\theta)^t\}$ *is an estimation sequence.*

Proof of Second part

Since the following hold trivially at $t = 0$:

$$\phi_0(z) \le (1 - (1-\theta)^0) f(x) + (1-\theta)^0 \phi_0(x).$$

and thus we can assume by induction that at $t - 1$:
$\phi_{t-1}(x) \le (1 - (1-\theta)^{t-1}) f(x) + (1-\theta)^t \phi_0(x)$. Then

$$\begin{aligned}
\phi_t(x) =& (1-\theta)\phi_{t-1}(x) + \theta\phi(x; y_t) \\
\le& (1-\theta)[(1 - (1-\theta)^{t-1}) f(x) + (1-\theta)^{t-1}\phi_0(x)] + \theta f(x) \\
=& (1 - (1-\theta)^t) f(x) + (1-\theta)^t \phi_0(x).
\end{aligned}$$

Upper Bound

Lemma

We have

$$f(x_t) \le \phi_t(v_t) = \min_z \phi_t(z).$$

Summary

We have studied accelerated first order methods with momentum terms and tuning parameters α and β.

$\alpha = O(\frac{1}{L})$

$\beta = 1 - O(\frac{1}{\sqrt{\kappa}})$

} empirical setting.

- α is like learning rate;
- β is decaying term for the aggregated gradients.

In practice, careful tuning of α and β are important.

- Right setting of β can significantly improve convergence, but inappropriate setting can lead to oscillation
- The sensitivity to α is less severe with appropriate $\beta > 0$, because the gradients are aggregated.