

Implementation and evaluation of a self-monitoring approximation algorithm for 3-Hitting-Set

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vorgelegt von

Khoa Le

Musterstraße

Musterstadt

Matrikelnummer

Musterstadt, im [Monat] [Jahr]

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1 Programming Language

We chose the *Go* programming language. It is a statically typed, compiled language, with a C-like syntax. It is using a garbage collector to handle memory management, which at first seems off-putting for an application like this. Since Go's GC is very efficient, we are not worried about that fact. Go also provides great developer tooling without the need for additional third party tools. The builtin profiler pprof can visualize captured performance profiles as flame graphs or standard graphs, which makes it easy to spot performance bottlenecks.

2 Datastructures

2.1 Vertex

```
type Vertex struct {
    id int32
    data any
}
```

The **Vertex** datatype has two fields. The field **id** is an arbitrary identifier and **data** serves as a placeholder for actual data associated with the vertex.

2.2 Edge

```
type Edge struct {
    v map[int32]bool
}
```

The **Edge** datatype has one field. The field **v** is a map with keys of type **int32** and values of type **bool**. When working with the endpoints of an edge, we are usually not interested in the associated values, since we never mutate the edges. This simulates a *Set* datatype while allowing faster access times than simple arrays/slices.

2.3 Hypergraph

```
type HyperGraph struct {
    Vertices      map[int32]Vertex
    Edges         map[int32]Edge
    edgeCounter   int32
    IncMap        map[int32]map[int32]bool
    AdjCount      map[int32]map[int32]int32
}
```

The **HyperGraph** datatype has five fields. Both fields **Vertices** and **Edges** are maps with keys of type **int32** and values of type **Vertex** and **Edge** respectively. We chose this **Set**-like datastructure over lists again because of faster access times, but also operations that remove edges/vertices are built-in to the map type. The field **edgeCounter** is an internal counter used to assign ids to added edges. The field **IncMap** is a map of maps, essentially storing the hypergraph as a sparse incidence matrix. We will also derive vertex degrees from this map. And at last the **AdjCount** map, which will associate every vertex $v \in V$ all vertices adjacent to v . Additionally this map will also store the amount of times such a vertex is adjacent to v .

3 Misc. Algorithms/Utilities

3.1 Edge Hashing

Time Complexity: $n + n \cdot \log(n)$, where n denotes the size of `arr`.

We start by sorting `arr` with a QUICK-SORT-Algorithm. We then join the elements of `arr` with the delimiter `"|"`, returning a string of the form `"|id0|id1|\dots|idn|"`. Whenever we refer to *the hash of an edge* we refer to the output of this function, using the endpoints of the edge as the `arr` argument.

3.2 Compute Subsets of Size s

Given an array `arr` and an integer s , compute all subsets of `arr` of size s .

Algorithm 1: An algorithm to compute all subsets of size s

Input: An array `arr`, the subset size s and a list `subsets`.

Output:

```
1 data ← []
2 n ← |arr|
3 last ← s - 1
4 FnRecursive(0, 0)
5 func FnRecursive(i, next):
6     for j ← next to n do
7         data[i] ← arr[j]
8         if i = last then
9             subsets.push(data)
10        else
11            FnRecursive(i + 1, j + 1)
```

Lists in Go are not very memory efficient, but since we exclusively call this function with `arr` representing the vertices in an edge, the value is usually fixed at 3. The raised memory problems occur at values of $n > 10000$, thus justifying the continued usage of lists. For the case where we have to compute a lot of subsets, we provide a slightly different version of this function. Instead of passing in the `subsets` list, we pass in a callback function that shall be called whenever we find a subset, using the found subset as an argument.

3.3 Two-Sum

Given an array of integers and an integer target t , return indices of the two numbers such that they add up to t .

Time Complexity: n , where n denotes the size of `items`.

Algorithm 2: An algorithm for the Two-Sum problem

Input: An array of integers `arr`, a target value t

Output: Two indices a, b , such that $arr[a] + arr[b] = t$, a boolean indicating if a solution was found

```
1 lookup ← map[N]N
2 for i ← 0 to len(arr) do
3     if lookup[t - arr[i]] exists then return (i, lookup[t - arr[i]]), true
4
5     else
6         lookup[arr[i]] ← i
7 return nil, false
```

We start by creating a map called *lookup*. Iterating over the elements of *arr*, we check if the entry *lookup[t - arr[i]]* exists.

- If the entry exists, we return a pair $(i, \text{lookup}[t - \text{arr}[i]])$ and the boolean value *true* since we found a solution.
- If the entry does not exist, we add a new entry to the *lookup* map using *arr[i]* as key and *i* as value.

If no solution was found, we return *nil* and the boolean value *false*.

This algorithm is an ingredient for the implementation of one of the reduction rules, specifically the approximative vertex domination rule. The actual implementation accepts a map instead of an array as its first parameter. We also implemented a version that finds all solutions, which accepts an additional callback function as a parameter. Instead of returning the solution, we call the callback function with the solution as the first argument. This version is an ingredient for the implementation of the approximative double vertex domination. We refer to this version as Two-SumAll inside the algorithm listings.

4 Hypergraph Models

4.1 First Testing Model

```
func GenerateTestGraph(n int32, m int32, tinyEdges bool) *HyperGraph
```

Let us explain the arguments first:

- **n** are the number of vertices the graph will have
- **m** is the amount of edges the graph will at most have
- **tinyEdges** when **false** indicates that we do not want to generate edges of size 1.

We use a very naive approach for generating (pseudo-)random graphs. We first create an empty Hypergraph struct and add **n** many vertices to that graph. We then compute a random **float32** value **r** in the half-open interval $[0.0, 1.0)$. This value will be used to determine the size of an edge *e*. The edges are distributed based on their size as follows:

$$\text{size}(\mathbf{r}) = \begin{cases} 1 & \mathbf{r} < 0.01 \\ 2 & 0.01 \leq \mathbf{r} < 0.60 \\ 3 & \text{else} \end{cases}$$

The result of *size(r)* is stored in a variable **d**. We then randomly pick vertices in the half-open interval $[0, \mathbf{n})$, until we have picked **d** many distinct vertices. If an edge with these endpoints does not exist, we add it to our graph.

That results in the graph having at most **m** edges and not exactly **m**, since we did not want to artificially saturate the graph with edges. One could also look into generating random bipartite graphs that translate back to a hypergraph with the desired vertex and edge numbers.

4.2 Preferential Attachment Hypergraph Model

In the Preferential Attachment Model, one will add edges to an existing graph, with a probability proportional to the degree of the endpoints of that edge. This edge will either contain a newly added vertex, or will be comprised of vertices already part of the graph.

We will use an implementation by Antelmi et al. as reference [1], which is part of their work on *SimpleHypergraphs.jl* [2], a hypergraph software library written in the Julia language. The implementation is based on a preferential attachment model proposed by Avin et al. in [3].

```
func GeneratePrefAttachmentGraph(n int, p float64, maxEdgesize int32)
```

- **n** is the amount of vertices the graph will have

- p is the probability of adding a new vertex to the graph
- `maxEdgesize` is the maximum size of a generated edge

5 Reduction Rules

The usual signature of a reduction rule looks as follows:

```
func NameRule(g HyperGraph, c map[int32]bool) int32
```

We take both a `HyperGraph` struct `g` and a `Set` `c` as arguments and mutate them. We then return the number of rule executions. We prioritize time complexity over memory complexity when implementing rules, which does not equate to ignoring memory complexity completely.

5.1 Executions

The proposed rules are meant to be applied exhaustively. A one to one implementation of a rule will only find one of the structures the rule is targeting. Calling such a rule implementation exhaustively will take polynomial time, but will be very inefficient in regards to memory writes and execution time. It is therefore necessary to design the algorithms for the rules with the aspect of exhaustive application in mind.

The general outline of an algorithm will look as follows,

1. Construct auxiliary data structures that are used to find parts of the graph, for which the rule can be applied.
2. As long as we can apply rules do:

Iterate over the auxiliary data structure, or the vertices/edges themselves.

- Identify targets of the rule.
- Mutate the graph according to the rule.
- Mutate the auxiliary data structure according to the rule.

This way we minimize memory writes by reusing existing data structures we built in step 1, but also save on execution time, since we mutate and iterate the auxiliary data structure at the same time.

5.2 Algorithms

The algorithm descriptions occasionally omit implementation details. We do this to try and keep these descriptions as concise as possible. The real implementations correspond to the pseudocode listings. Proofs are provided in the case, where the algorithm does not simply follow the definition of a rule.

We further introduce a `map` data structure in our pseudocode with syntax `map[A]B`, which describes a mapping from A into B . We use square brackets to indicate access and mutation of the mapping, e.g. $\gamma[0] \leftarrow 1$. The `map` type also exposes a primitive function with the signature `delete(γ, x)`, which simply means that we want to delete the entry with key x from our map. When iterating over a map in a `for`-loop we destructure the entries into a $(key, value)$ -pair, e.g. `for ($_, v$) \in map do`. Note that unused values are omitted with the underscore symbol.

5.2.1 Tiny/Small Edge Rule

- tiny edges: Delete all hyperedges of size one and place the corresponding vertices into the hitting set.
- small edges: If e is a hyperedge of size two, i.e., $e = \{x, y\}$, then put both x and y into the hitting set.

$O(|E|^2)$ **Algorithm.** We iterate over all edges of the graph. If the current edge e is of size t , put e into the partial hitting set and remove e and all edges adjacent to vertices in e from the graph.

Algorithm 3: Algorithm for exhaustive application of Tiny/Small Edge Rule

Input: A hypergraph $G = (V, E)$, a set C , an integer t denoting the size of the edges to be removed

Output: An integer denoting the number of rule applications.

```
1  $rem \leftarrow \emptyset$ 
2  $exec \leftarrow 0$ 
3 for  $e \in E$  do
4   if  $|e| = t$  then
5      $exec \leftarrow exec + 1$ 
6     for  $v \in e$  do
7        $C \leftarrow C \cup \{v\}$ 
8        $V \leftarrow V \setminus \{v\}$ 
9       for  $f$  incident to  $v$  do
10         $E \leftarrow E \setminus \{f\}$ 
11 return  $exec$ 
```

5.2.2 Edge Domination Rule

- (hyper)edge domination: A hyperedge e is *dominated* by another hyperedge f if $f \subset e$. In that case, delete e .

$O(|E|)$ **Algorithm.** We partition our set of edges into two disjoint sets sub and dom . The set dom will contain edges that could possibly be dominated. The set sub will contain hashes of edges e that could dominate another edge. We then iterate over the set dom and compute every strict subset of the current edge f . For each of these subsets, we test if the hash of the subset is present in our set sub . If it is then f is dominated by another edge.

The exact time complexity is as follows:

$$T = |E| \cdot d \cdot \log(d) + (|E| \cdot (d + 2^d + (2^d \cdot d \cdot \log(d))))$$

Specifically applied to $d = 3$, this results in a time complexity of:

$$\begin{aligned} T &= |E| \cdot 3 \cdot \log(3) + (|E| \cdot (11 + 24 \cdot \log(3))) \\ &= |E| \cdot (3 \cdot \log(3) + (11 + 24 \cdot \log(3))) \end{aligned}$$

Lemma 5.1 *Algorithm 4 finds all edges of G that are dominated, iff G has no size one edges.*

Proof. Let e be a dominated edge. Since there are no size one edges, edges with size two cannot be dominated. Thus e has to be of size three. Then simply removing e will not create or eliminate an edge domination situation. It is therefore sufficient to only check size three edges for the domination condition. \square

This also allows us to parallelize the main part of the algorithm, where we check each edge in our dom set. We can achieve a speedup of approximately 2 on a six-core CPU on a pseudo-random graph with one million vertices and two million edges.

5.2.3 Vertex Domination Rule

- A vertex x is dominated by a vertex y if, whenever x belongs to some hyperedge e , then y also belongs to e . Then, we can simply delete x from the vertex set and from all edges it belongs to.

Algorithm 4: Algorithm for exhaustive application of Edge Domination Rule

Input: A hypergraph $G = (V, E)$ without size one edges, a set C

Output: An integer denoting the number of rule applications.

```
1  $sub \leftarrow \emptyset$ 
2  $dom \leftarrow \emptyset$ 
3  $exec \leftarrow 0$ 
4 for  $e \in E$  do
5   if  $|e| = 2$  then
6      $sub \leftarrow sub \cup \{hash(e)\}$ 
7   else
8      $dom \leftarrow dom \cup \{e\}$ 
9 if  $|sub| = 0$  then
10  return  $exec$ 
11 for  $e \in dom$  do
12    $subsets \leftarrow \text{getSubsetsRec}(e, 2)$ 
13   for  $f \in subsets$  do
14     if  $hash(f) \in sub$  then
15        $E \leftarrow E \setminus \{e\}$ 
16        $exec \leftarrow exec + 1$ 
17       break
18 return  $exec$ 
```

$O(|V|^2 \cdot |E|)$ **Algorithm** A vertex v is dominated, if one of the entries in $AdjCount[v]$ is equal to $deg(v)$. In that case we remove v from all edges and our vertex set.

5.2.4 Approximative Vertex Domination Rule

- approximative vertex domination: Assume there is a hyperedge $e = \{x, y, z\}$ such that, whenever x belongs to some hyperedge h , then y or z also belong to h . Then, we put y and z together into the hitting set that we produce.

$O(|V|^2 \cdot |E|)$ **Algorithm.** The additional factor $|E|$ looks scary at first, but will only occur in the worst case, if there exists a vertex v , that is incident to all edges in G .

We start by iterating over the $AdjCount$ map of the graph, referring to the current value in the iteration as $AdjCount[v]$. We then use the TWO-SUM-Algorithm to compute and return the first pair (a, b) in $AdjCount[v]$, s.t. for (a, b) holds,

$$AdjCount[v][a] + AdjCount[v][b] = deg(v) + 1$$

If such a pair exists, then we conclude that for every edge f such that $v \in f$, it holds that, either $a \in f$ or $b \in f$.

Lemma 5.2 *The outlined procedure above is correct, under the assumption that the underlying graph does not contain any duplicate edges.*

Proof. Let G be a hypergraph. We first remove all edges of size one with the *Tiny Edge Rule*. We then construct our map $AdjCount$. Let v be an entry in $AdjCount$ and $sol = (a, b)$ the result of calling our TWO-SUM implementation on $AdjCount[v]$ with a target sum of $n = deg(v) + 1$.

Proposition. If sol is non-empty, then the edge $\{v, a, b\}$ exists.

Algorithm 5: Algorithm for exhaustive application of Vertex Domination Rule

Input: A hypergraph $G = (V, E)$ **Output:** An integer denoting the number of rule applications.

```
1  $exec \leftarrow 0$ 
2  $outer \leftarrow true$ 
3 while  $outer$  do
4    $outer \leftarrow false$ 
5   for  $v \in V$  do
6      $dom \leftarrow false$ 
7     for  $(\_, val) \in AdjCount[v]$  do
8       if  $val = deg(v)$  then
9          $dom = true$ 
10        break
11     if  $dom$  then
12        $outer = true$ 
13       for  $e$  incident to  $v$  do
14          $e \leftarrow e \setminus \{v\}$ ;
15          $V \leftarrow V \setminus \{v\}$ 
16          $exec \leftarrow exec + 1$ 
17 return  $exec$ 
```

Let $sol = (a, b)$ be the solution obtained by calling our TWO-SUM algorithm on $AdjCount[v]$ with a target sum of $n = deg(v) + 1$. For the sake of contradiction let us assume that the edge $\{v, a, b\}$ does not exist. Since our graph does not contain duplicate edges and does not contain $\{v, a, b\}$, there exist $deg(v) + 1$ many edges that contain either $\{a, v\}$ or $\{b, v\}$. This however contradicts that there only exist $deg(v)$ many edges containing v . Therefore it must be, that the assumption that $\{v, a, b\}$ does not exist, is false.

Since $\{v, a, b\}$ exists, a and b can only occur $n - 2 = deg(v) - 1$ times in other edges containing v . Since duplicate edges of $\{v, a, b\}$ can not exist, we know that every other edge containing v also contains a or b , but not both simultaneously. \square

We then add the two vertices in the solution sol to our partial solution c .

Idea: The initial idea for this algorithm involved the usage of a complete incidence matrix, where edges are identified by the rows and the vertices are identified by the columns. To check the *Domination Condition* for a vertex v , the algorithm would select all edges/columns that contain v and then add up the columns. Now let n be the amount of edges containing v . If there exist two entries in the resulting column that have a combined value of $n + 1$, then the rule applies for v under the assumption that there are no duplicate edges. This would result in an algorithm with a worse time complexity of $|V| + |V|^2 \cdot |E|$.

5.2.5 Approximative Double Vertex Domination Rule

- approximative double vertex domination: Assume there is a hyperedge $e = \{x, y, a\}$ and another vertex b such that, whenever x or y belong to some hyperedge h , then a or b also belong to h . Then, we put a and b together into the hitting set that we produce.

$\mathcal{O}(|V|^2 + |E|)$ **Algorithm.** We start by creating a map called $tsHashes$, which maps subsets of E to subsets of vertices. We then iterate over all vertices in V . For the current vertex x , compute all TWO-SUM solutions with input array/map $AdjCount[x]$ and target $deg(x)$. If there exists a solution $sol = \{z_0, z_1\}$, such that $tsHashes[sol] \neq \emptyset$, then for all $y \in tsHashes[sol]$, $y \neq x$ construct two edges $\{x, y, z_0\}$ and $\{x, y, z_1\}$. If one of these two edges exists in E , then we found a approximative double vertex domination situation. If $tsHashes[sol] = \emptyset$, map $tsHashes[sol]$ to $tsHashes[sol] \cup \{x\}$.

Algorithm 6: Algorithm for exhaustive application of Approximative Vertex Domination Rule

Input: A hypergraph $G = (V, E)$, a set C

Output: An integer denoting the number of rule applications.

```
1  $exec \leftarrow 0$ 
2  $outer \leftarrow true$ 
3 while  $outer$  do
4    $outer \leftarrow false$ 
5   for  $(v, count) \in AdjCount$  do
6      $sol, ex \leftarrow TwoSum(count, deg(v) + 1)$ 
7     if not  $ex$  then
8       continue
9      $outer \leftarrow true$ 
10     $exec \leftarrow exec + 1$ 
11    for  $w \in sol$  do
12       $C \leftarrow C \cup \{w\}$ 
13       $V \leftarrow V \setminus \{w\}$ 
14      for  $e$  incident to  $w$  do
15         $E \leftarrow E \setminus \{e\}$ 
16 return  $exec$ 
```

Lemma 5.3 *Algorithm 7 is correct.*

Proof. Let $G = (V, E)$ be a hypergraph. Let x be the current vertex in the algorithm's iteration over V . If $T = TwoSumAll(AdjCount[x], deg(x))$ is empty, then x will not be able to trigger a approximative double vertex domination situation. If T is not empty, then we know that there exist sets $\{a, b\}$, such that all edges incident to x either contain a or b . If $tsHashes[\{a, b\}]$ is empty, then there are currently no other vertices for which $\{a, b\}$ is a TWO-SUM solution. In that case we add x to $tsHashes[\{a, b\}]$. Otherwise there exist vertices y , such that $\{a, b\}$ is a TWO-SUM solution for y . All edges incident to x and y either contain a or b . If for one of these vertices y there exists a size three edge $\{x, y, a\}$, or without loss of generality $\{x, y, b\}$, then we found a approximative double vertex domination situation at $\{x, y, a\}$ or $\{x, y, b\}$ respectively. If none of these edges exist, then x could still trigger another approximative double vertex situation with another vertex. Thus we need to add x to $tsHashes[\{a, b\}]$. \square

5.2.6 Small Triangle Rule

- small triangle situation: Assume there are three small hyperedges $e = \{y, z\}$, $f = \{x, y\}$, $g = \{x, z\}$. This describes a triangle situation (e, f, g) . Then, we put $\{x, y, z\}$ together into the hitting set, and we can even choose another hyperedge of size three to worsen the ratio.

$O(|E| + |V|^2)$ **Algorithm** We start by constructing an adjacency list **adjList** for all edges of size two. We then iterate over the entries of the list. For the current entry $adjList[v]$ we compute all subsets of size two of the entry. If there exists a subset s such that $s \in E$, then we found a small triangle situation. If we find a triangle situation we put the corresponding vertices in our partial solution and alter the adjacency list to reflect these changes. We do this by iterating over all vertices that are adjacent to the triangle. For every vertex w of these vertices we delete all vertices of the triangle from the entry $adjList[w]$.

This last step will introduce the quadratic complexity, since in the worst case, for a vertex v in a triangle, there could exist $|V|$ many size two edges that contain v . This worst case occurs very rarely, which justifies using this quadratic algorithm. We could alternatively move the last step of the algorithm outside of the loop, and wrap both procedures with an outer loop which breaks if we don't find any more triangles. This simulates calling the rule exhaustively, while achieving a linear time complexity.

Algorithm 7: Algorithm for exhaustive application of Approximative Double Vertex Domination Rule

Input: A hypergraph $G = (V, E)$, a set C

Output: An integer denoting the number of rule applications.

```

1  $exec \leftarrow 0$ 
2  $outer \leftarrow true$ 
3 for  $outer$  do
4    $outer \leftarrow false$ 
5    $tsHashes \leftarrow \text{map}[2^V]2^V$ 
6   for  $x \in V$  do
7     for  $sol \in \text{TwoSumAll}(\text{AdjCount}[x], \text{deg}(x))$  do
8        $\{z_0, z_1\} \leftarrow sol$ 
9       if  $tsHashes[sol] \neq \emptyset$  then
10         for  $y \in tsHashes[sol]$  do
11           if  $y = x$  then
12             continue
13            $f_0 \leftarrow \{x, y, z_0\}$ 
14            $f_1 \leftarrow \{x, y, z_1\}$ 
15            $found \leftarrow false$ 
16           for  $e$  incident to  $y$  do
17             if  $e = f_0$  or  $e = f_1$  then
18                $found \leftarrow true$ 
19               break
20           if  $found$  then
21              $exec \leftarrow exec + 1$ 
22              $outer \leftarrow true$ 
23              $C \leftarrow C \cup sol$ 
24             for  $a \in sol$  do
25               for  $e$  incident to  $a$  do
26                  $E \leftarrow E \setminus \{e\}$ 
27             break
28            $tsHashes[sol] \leftarrow tsHashes[sol] \cup \{x\}$ 
29       else
30          $tsHashes[sol] \leftarrow \{x\}$ 
31 return  $exec$ 

```

Algorithm 8: : Algorithm for exhaustive application of Small Triangle Rule

Input: A hypergraph $G = (V, E)$, a set C

Output: An integer denoting the number of rule applications.

```
1  $adjList \leftarrow \text{map}[V]2^V$ 
2  $rem \leftarrow \emptyset$ 
3  $exec \leftarrow 0$ 
4 for  $e \in E$  do
5   if  $|e| \neq 2$  then
6     continue
7    $\{x, y\} \leftarrow e$ 
8    $adjList[x] \leftarrow adjList[x] \cup \{y\}$ 
9    $adjList[y] \leftarrow adjList[y] \cup \{x\}$ 
10 for  $(z, val) \in adjList$  do
11   if  $|val| < 2$  then
12     continue
13    $subsets \leftarrow \text{getSubsetsRec}(val, 2)$ 
14   for  $s \in subsets$  do
15      $\{x, y\} \leftarrow s$ 
16     if  $y \in adjList[x]$  or  $x \in adjList[y]$  then
17        $exec \leftarrow exec + 1$ 
18        $C \leftarrow C \cup \{x, y, z\}$ 
19        $rem \leftarrow rem \cup \{x, y, z\}$ 
20       for  $u \in \{x, y, z\}$  do
21         for  $v \in adjList[u]$  do
22            $adjList[v] \leftarrow adjList[v] \setminus \{u\}$ 
23            $delete(adjList, u)$ 
24       break;
25 for  $v \in rem$  do
26    $V \leftarrow V \setminus \{v\}$ 
27   for  $e$  incident to  $v$  do
28      $E \leftarrow E \setminus \{e\}$ 
29 return  $exec$ 
```

5.2.7 Extended Triangle Rule

- Assume that the hypergraph contains a small edge $e = \{y, z\}$. Moreover, there are hyperedges f, g such that $e \cap f = \{y\}$, $e \cap g = \{z\}$, $f \cup g = \{v, x, y, z\}$ and $|f| = 3$. Then, put all of $f \cup g$ into the hitting set.

Algorithm 9: Algorithm for exhaustive application of Extended Triangle Rule

Input: A hypergraph $G = (V, E)$, a set C

Output: An integer denoting the number of rule applications.

```

1  $exec \leftarrow 0$ 
2  $outer \leftarrow true$ 
3 for  $outer$  do
4    $outer \leftarrow false$ 
5   for  $e \in E$  do
6     if  $|e| \neq 2$  then
7       continue
8     for  $a \in e$  do
9        $y \leftarrow a$ 
10       $\{z\} \leftarrow e \setminus \{y\}$ 
11       $f_0 \leftarrow nil$ 
12      incy:
13      for  $f$  incident to  $y$  do
14        if  $|f| \neq 3$  or  $z \in f$  then
15          continue
16        for  $g$  incident to  $z$  do
17           $cond \leftarrow true$ 
18          for  $b \in g$  do
19            if  $b = z$  then
20              continue
21            if  $b \notin f$  then
22               $cond \leftarrow false$ 
23            break
24          if  $cond$  then
25             $f_0 \leftarrow f$ 
26            break  $incy$ 
27      if  $f_0 \neq nil$  then
28         $outer \leftarrow true$ 
29         $exec \leftarrow exec + 1$ 
30         $C \leftarrow C \cup f \cup \{z\}$ 
31         $V \leftarrow V \setminus f \cup \{z\}$ 
32        for  $h$  incident to  $f \cup \{z\}$  do
33           $E \leftarrow E \setminus \{h\}$ 
34 return  $exec$ 

```

5.2.8 Small Edge Degree 2 Rule

- small edge degree 2: Let v be a vertex of degree 2, and let the two hyperedges containing v be $e = \{x, v\}$ and $f = \{v, y, z\}$. Then we can select a hyperedge g that contains one of the neighbors of v in f but not x , for example $g = \{u, w, z\}$ (when $y = w$ is possible as a special case) or $g = \{u, z\}$. We put x, u and z and w (when existing) into the hitting set.

$O(|V| \cdot |E|)$ **Algorithm** We start by iterating over all vertices in V . If the vertex v in the current iteration is of degree two, check if there is a size two edge e and a size three edge f incident to v . If these two edges exist, save $e \setminus \{v\}$ in a variable x . Then iterate over the vertices w in $f \setminus v$ and check if there exists an edge h incident to w , such that $h \neq f$ and $x \not\subseteq h$. If such an edge exists, then we found a small edge degree 2 situation. We then put both x and all vertices of h into the partial hitting set.

Algorithm 10: Algorithm for exhaustive application of Small Edge Degree 2 Rule

Input: A hypergraph $G = (V, E)$, a set C

Output: An integer denoting the number of rule applications.

```

1 exec  $\leftarrow$  0
2 outer  $\leftarrow$  true
3 for outer do
4   for  $v \in V$  do
5     if  $\deg(v) \neq 2$  then
6        $\mid$  continue
7      $s2, s3 \leftarrow \text{nil}$ 
8     for  $e$  incident to  $v$  do
9       if  $|e| = 3$  then
10         $\mid$   $s3 \leftarrow e$ 
11        else if  $|e| = 2$  then
12           $\mid$   $s2 \leftarrow e$ 
13     if  $s2 = \text{nil}$  or  $s3 = \text{nil}$  then
14        $\mid$  continue
15      $\{x, \_ \} \leftarrow s2$ 
16     found  $\leftarrow$  false
17     rem  $\leftarrow$  nil
18     for  $w \in s3 \setminus \{v\}$  do
19       for  $f$  incident to  $w$  do
20         if  $x \in f$  or  $s3 = f$  then
21            $\mid$  continue
22         else
23            $\mid$  found  $\leftarrow$  true
24            $\mid$  rem  $\leftarrow$   $f$ 
25            $\mid$  break
26     if found then
27        $\mid$  break
28     if found then
29        $\mid$  outer  $\leftarrow$  true
30        $\mid$  exec  $\leftarrow$  exec + 1
31        $\mid$  for  $a \in \{x\} \cup \text{rem}$  do
32          $\mid$   $C \leftarrow C \cup \{a\}$ 
33          $\mid$  for  $h$  incident to  $a$  do
34            $\mid$   $E \leftarrow E \setminus \{h\}$ 

```

5.3 Self-Monitoring

Each of the reduction rule functions returns a `int32` value, which indicates the number of rule executions.

We store the ratios for each rule in a map of the form:

```

var ratios = map[string]pkg.IntTuple{
    "kRuleName": {A:1, B:1},
}

```

Where A denotes the amount of vertices put into the partial solution by a single rule execution. And B denotes the amount of vertices present in an optimal solution. We can then use these values to calculate the estimated approximation factor as follows:

```

execs := ApplyRules(g, c)

var nom float64 = 0
var denom float64 = 0

for key, val := range execs {
    nom += float64(ratios[key].A * val)
    denom += float64(ratios[key].B * val)
}

```

We conducted some preliminary testing on graphs with 10, 100, 1000 and 10000 vertices. We calculated the estimated approximation factor for these graphs. We did this for ratios $r = \frac{|E|}{|V|}$ of 1 to 20.

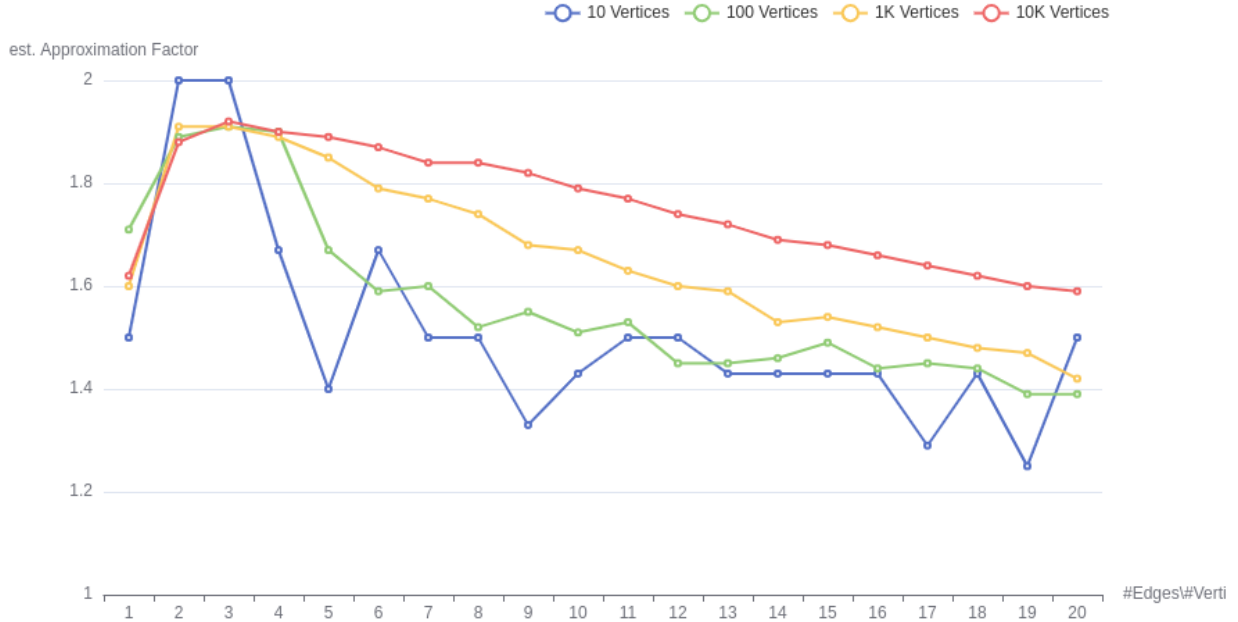


Figure 1: Estimated Approximation Factor for Graphs with 10, 100, 1K, 10K Vertices for $r \in [1, 20]$

We also looked at the number of rule executions for $r = 1$ and $r = 10$.

Note that the reduction rules alone are sufficient to compute a Hitting-Set for our graphs. The est. ratio is quite low, since we do not need to put whole size 3 edges into our hitting set. This is due to the way our random-graph model works. This will not work for all graph classes, namely 3-uniform graphs.

We can observe a large performance hit, once our graph is near 3-uniform. We present some ideas that could potentially speedup the execution for these graphs.

Factor-3 Rule Targeting. The Factor-3 rule will select the to be removed edge at random. We could alternatively choose an edge e , s.t. the removal of e will allow the execution of the Vertex Domination Rule.

remove
Factor-3
Rule func-
tion para-
graph

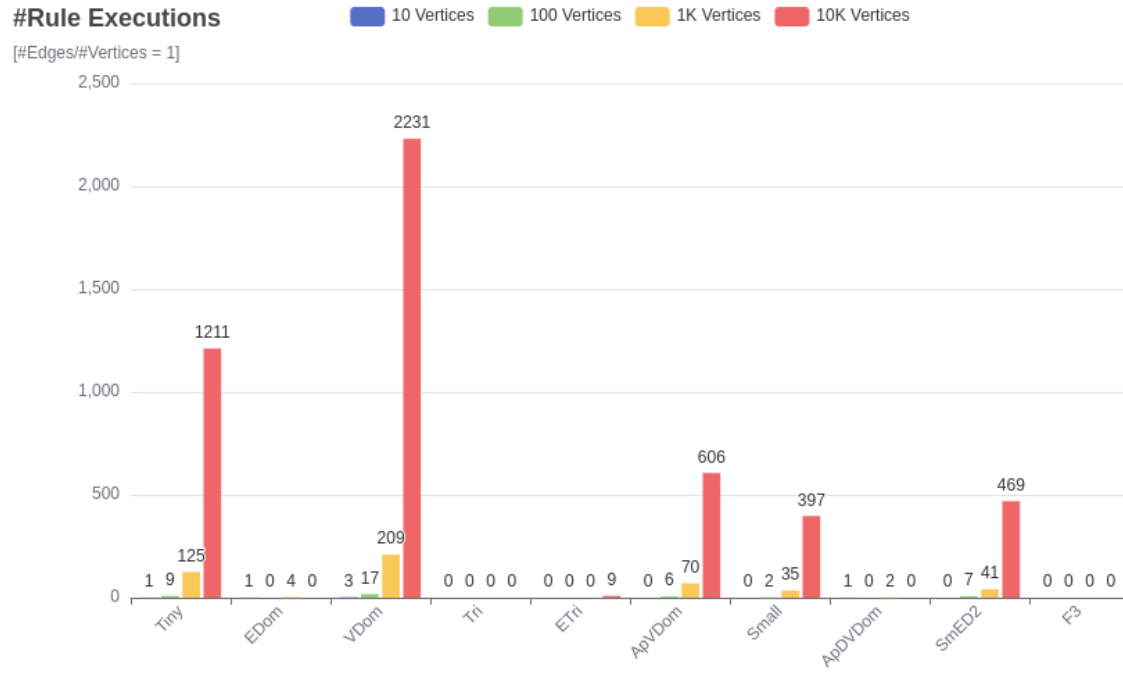


Figure 2: Number of Rule Executions, $r = 1$

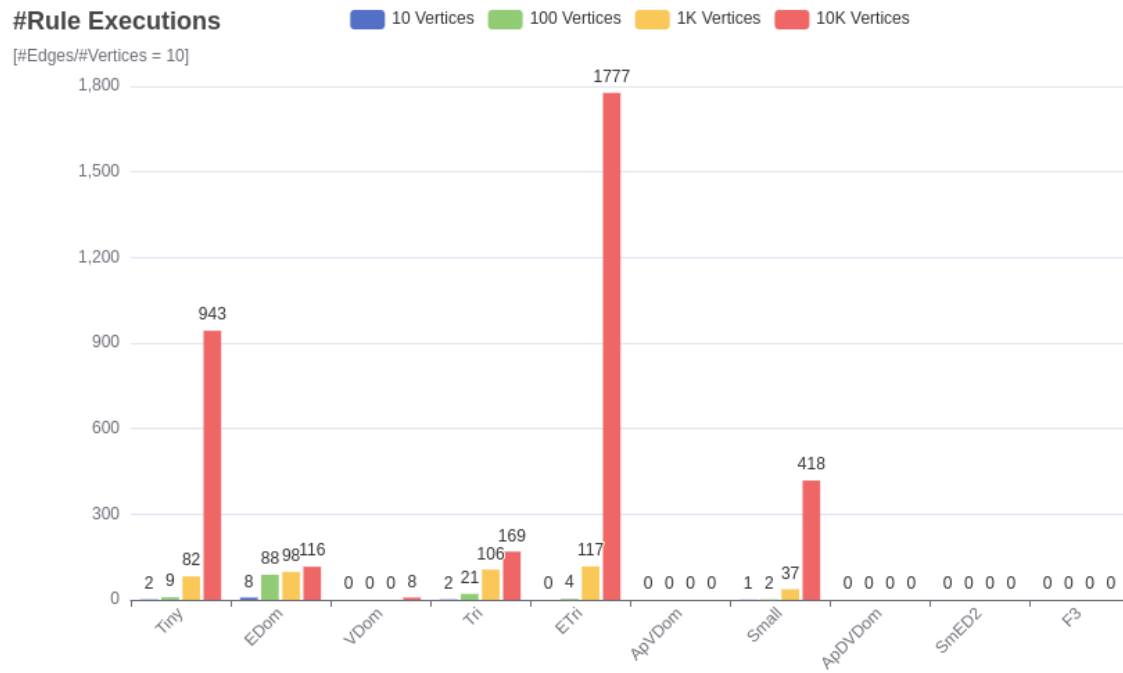


Figure 3: Number of Rule Executions, $r = 10$

This could potentially lower the number of Factor-3 Rule executions quite significantly.

6 Algorithms

6.1 Main Algorithm

It is crucial to apply the rules in a given order. Some rules expect that other rules can not be applied. As for the exact rules, one can save execution time, if a specific order is enforced when executing them. We use the order proposed in the original paper [4], also referred to as precedence of rule executions. We will just use the shorter term *precedence* going forward.

1. Exact Rules: vertex domination \rightarrow tiny edge \rightarrow edge domination
2. approximate (double) vertex domination rules
3. small edge degree 2 rule
4. small triangle rule
5. extended triangle rule
6. small edge rule

The main algorithm will apply all rules exhaustively according to the precedence, possibly mutating the input graph. If the graph has no more edges, then we are done. If not, put a size three edge into the partial hitting set and start over with the rule application. Note that the step of “applying all rules exhaustively” leaves room for interpretation and that the produced hitting sets are highly dependant on the actions taken during this step. We will refer to the procedure in this step as *rule strategy* or just *strategy* if the context allows it. We now present three strategies to apply the rules in the precedence.

The base strategy is quite simple. Execute a rule exhaustively and then do the same for the next rule in the precedence. Do this until no more rules can be applied. This is obviously not very optimal, since we do not check, whether a “better” rule can be applied again, before possibly executing a worse rule. We can however add some heuristics, that preserve the fast execution time and improve on effectiveness. Instead of executing the vertex domination \rightarrow tiny edge \rightarrow edge domination cascade only once, we execute it three times. By just doing this, we can recoup a lot of cascades we might have missed with the base strategy. Iteration values higher than three did not yield better results. We should also execute this cascade after rules, that are applied the most. Since these rules are the ones most likely to make the exact rules applicable. That would be the approximative (double) vertex domination rules. The second strategy is also derived from the same base strategy. The only modification is that, we start over with the vertex domination rule, whenever one of the rules was applied at least once. This should in theory lead to more exact rules being executed, while sacrificing a bit of execution time. The third strategy tries to maximize the exact rule executions. We start by applying the exact rules exhaustively. We then execute the other rules at most once, according to the precedence. If a rule has been applied, we execute the exact rules exhaustively and start over with the first non-exact rule in the precedence.

6.2 Incremental Frontier Algorithm

As experienced with the DBLP coauthor graph, there are some graph instances that do not work well with the prior algorithm. These instances do not admit a lot of rule executions after applying the fallback rule. Running the algorithm on the whole graph again, knowing that only small parts of the graph have changed is not very time efficient. The algorithm should only look at the part of the graph where the fallback rule, or in fact any rule, was applied. We therefore propose a new approach, which involves the usage of a *vertex frontier*. Such frontiers are common in search algorithms such as BFS, where each vertex in the frontier has the same distance to the root vertex. But let us explain the general structure of the algorithm first.

We first apply all reduction rules exhaustively for the entire graph G . If the graph has no more edges we are done. Else we try to find a size three edge e , which will trigger a vertex domination situation if removed. We choose a random size three edge if there is none. We then build up our initial frontier. We put all vertices into the frontier that are adjacent to a vertex in e , excluding vertices in e itself. Then remove all edges that are incident to vertices in e . Next the frontier will be expanded, by adding all edges incident to the frontier to

a new graph H . All vertices in H that were not part of the frontier will form the next frontier. The amount of expansion steps can be set by the user. The fields **AdjCount** and **IncMap** of G will be reused by H . Thus changes in H will be reflected in G and vice versa. The main loop will be explained next.

We apply the rules as usual, but only on H . The rules are modified, such that they also return the vertices adjacent to vertices in a modified/removed edge. If there are any, then H will be expanded at these vertices and the loop will be continued. If not, apply the targeted fallback rule, considering the edges in the original graph G and expand accordingly. This will be repeated until G has no more edges.

proof that
expansion
step of 1 is
sufficient ?

Algorithm 11: Incremental 3-approximation algorithm for 3-HS

Input: Hypergraph $G = (V, E)$, and a partial hitting set C

Output: A hitting set C

```

1 Apply all reduction rules exhaustively mutating  $G$  and  $C$ 
2  $l \leftarrow 2$ 
3 if  $|E| = 0$  then
4   return  $C$ 
5  $H \leftarrow$  empty hypergraph
   /*  $H$  will act as a mask over  $G$ , the rules will only be applied to vertices/edges in
    $H$  */
6 for  $|E| > 0$  do
7   Apply all reduction rules exhaustively on  $H$  mutating  $H, G$  and  $C$ 
8    $exp \leftarrow$  vertices of edges adjacent to removed or modified edge
9   if  $|exp| > 0$  then
10     $H \leftarrow \text{ExpandFrontier}(G, l, exp)$ 
11    continue
12    $e \leftarrow \text{F3TargetLowDegree}(G)$ 
13   if  $e = \emptyset$  then
14     continue
15    $H \leftarrow \text{ExpandFrontier}(G, l, e)$ 
16 return  $C$ 
17 func  $\text{ExpandFrontier}(G, l, exp)$ :
18    $H \leftarrow$  empty hypergraph
19   for  $i \leftarrow 0$  to  $l$  do
20      $next \leftarrow \emptyset$ 
21     for  $v \in exp$  do
22       for  $e \in G.E$  incident to  $v$  do
23         if  $e \notin H.E$  then
24            $H.E \leftarrow H.E \cup \{e\}$ 
25           for  $w \in e$  do
26             if  $w \notin H.V$  then
27                $H.V \leftarrow H.V \cup \{w\}$ 
28                $next \leftarrow next \cup \{w\}$ 
29   if  $|next| = 0$  then
30     break
31    $exp \leftarrow next$ 
32    $H.IncMap = G.IncMap$ 
33    $H.AdjCount = G.AdjCount$ 
34   return  $H$ 

```

Algorithm 11 was designed around both rule strategy 1 and 2. It is not compatible with strategy 3, since the

number of rule executions

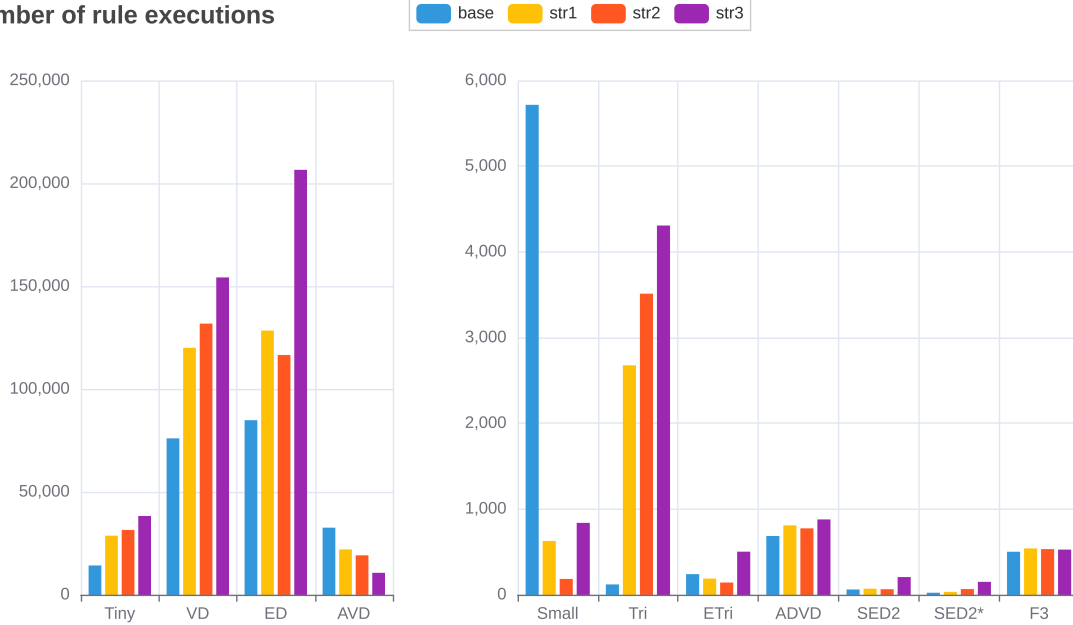


Figure 4: Number of rule executions for DBLP coauthor graph per rule strategy, $n = 100$

auxiliary graph could at some point be empty and thus $|exp| = 0$, with the original graph still admitting rule applications. In such a case, we have to apply the first applicable rule in the precedence and use the neighborhood of the removed vertices to rebuild the auxiliary graph.

7 Applications and Results

All tests were run on a machine with a AMD Ryzen 5 3600 3.6 GHz six-core, 12-thread CPU and 32 gigabytes of 3200 MT/s DDR4 RAM. The machine was running Ubuntu Server 22.04.04 LTS and the binaries were compiled with Go version 1.22.1. Measured execution times do not include the time it takes to load in a graph from disk or the time needed to transform the graph to a new problem instance. We provide results for TRIANGLE VERTEX DELETION and CLUSTER VERTEX DELETION problems using real world networks, as well as randomly generated hypergraphs.

7.1 Triangle Vertex Deletion

[PROBLEM DEFINITION]

TRIANGLE VERTEX DELETION can be reduced to the 3-HITTING SET problem. Triangles in the input graph will be represented as size 3 edges in a hypergraph, containing the vertices that make up the triangle.

7.1.1 DBLP Coauthor Graph

We tested the algorithms on a DBLP coauthor graph and ER graphs of varying densities. It was considered to use the complete DBLP coauthor graph, which was quite large. We ultimately decided to use a dataset of the largest connected component. The network is made available by the Stanford Network Analysis Project and part of a paper by J. Yang and J. Leskovec [5] about community detection in real-world networks. The network contains 2224385 triangles, resulting in a 3-HITTING SET instance of the same size. We now present the collected data during a run of the algorithm on this hypergraph, which can be found in table ??.

(a) base rule strategy					(b) rule strategy 1				
	ratio	$ C $	est. opt	time		ratio	$ C $	est. opt	time
mean	1.7594	115672	65745	17 sec	mean	1.5708	106339	67697	29 sec
std	0.0005	79	41	1 sec	std	0.0006	65	40	1 sec
min	1.7577	115469	65635	16 sec	min	1.5691	106183	67597	26 sec
median	1.7594	115677	65741	17 sec	median	1.5708	106342	67699	29 sec
max	1.7610	115998	65904	19 sec	max	1.5727	106500	67797	31 sec

(c) rule strategy 2					(d) rule strategy 3				
	ratio	$ C $	est. opt	time		ratio	$ C $	est. opt	time
mean	1.5437	105084	68071	104 sec	mean	1.4195	99456	70066	474 sec
std	0.0006	63	35	2 sec	std	0.0009	52	31	7 sec
min	1.5422	104922	67956	99 sec	min	1.4171	99323	69987	457 sec
median	1.5438	105084	68072	104 sec	median	1.4196	99450	70069	474 sec
max	1.5453	105261	68146	110 sec	max	1.4215	99575	70133	493 sec

Table 1: Results for TRIANGLE VERTEX DELETION on DBLP coauthor graph; $n = 100$

Rule strategy 1 has the fastest execution time with a mean of 29 seconds, but also the worst ratio with a mean of 1.5708. Rule strategy 3 has the best ratio with a mean of 1.4195, but also the worst execution time of about 8 minutes. We used the frontier technique in all of the tests, since we could observe a speedup of about 90, just using the base rule strategy. This speedup can be attributed to the fact, that the original coauthor graph exerts a community structure. These communities are preserved when converting it to a TRIANGLE VERTEX DELETION and thus 3-HITTING-SET instance. The expansion steps will only expand the auxiliary graph H inside these communities. This drastically reduces the amount of edges the algorithm has to process per iteration. One can see the opposite happen in really dense hypergraphs. Due to the high density, even a small expansion two neighborhoods deep will add all edges of G to H , eliminating possible gains. In fact, constant rebuilding of the auxiliary graph H will even worsen the execution time.

needs proof?

Figure 4 and ?? display the number of rule executions per strategy. There are several interesting observations to be made, for example the decrease of edge domination rule executions for strategy 2 compared to strategy 1. Quite frankly we do not know why this happens. Another observation can be made about the number of small edge rule executions. Strategy 2 does not apply the small edge rule at all, which is surprising. This is probably a result of only applying rules one at a time. The exact rules that are executed after each rule application will possibly destroy situations that are favourable for other rules. The number of fallback rule executions is also rather interesting. All strategies are executing this rule around 3000 times. This leads us to the hypothesis that, as long as the order in the precedence is obeyed, a “worse” strategy will not result in significantly more fallback rule executions.

explanation ?

7.1.2 Amazon Product Co-Purchasing Graph

To validate that our previous findings were not just due to the structure of the DBLP coauthor graph, we conduct the same tests with an unrelated network. We chose an Amazon product co-purchasing graph, which had a similar amount of vertices and edges compared to the DBLP coauthor graph. The resulting TRIANGLE-VERTEX-DELETION instance only contains 667129 edges and thus should be structurally distinct, compared to the instance obtained from the DBLP coauthor graph. As seen in table ??

add amazon graph section

interpretation of results

7.2 Cluster Vertex Deletion

interpretation of results

Triangle Vertex Deletion for ER Graphs

1000 vertices, Edge\Vertex ratio of 10-25, 100 graphs per EVR

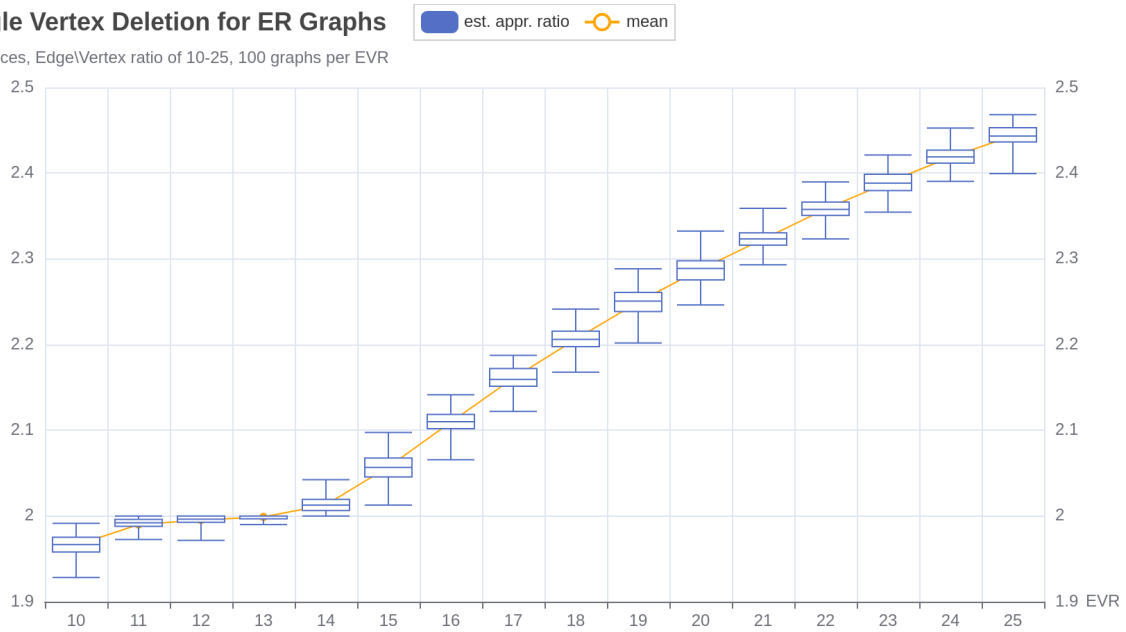


Figure 5: Triangle Vertex Deletion on ER graphs, est. appr. ratios

#Rule Executions

EVR=10 for input ER graph



Figure 6: Triangle Vertex Deletion on ER graphs, amount of rule executions for EVR=10

#Rule Executions

rule executions

EVR=25 for input ER graph

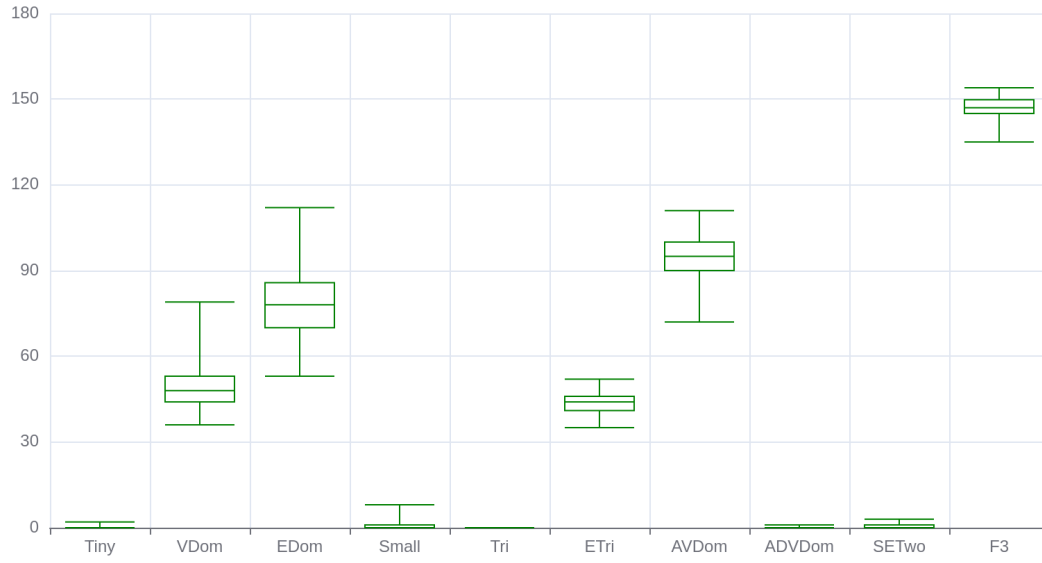


Figure 7: Triangle Vertex Deletion on ER graphs, amount of rule executions for EVR=25

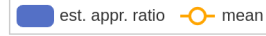
		Ratio	Tiny	VD	ED	Small	Tri	ETri	AVD	ADVD	SED2	F3	$ C $
stock	mean	1.7529	3.07	11.69	35.54	3.10	0.03	0	7.77	0.03	0.05	0	25.13
	std	0.1703	2	6.11	29.80	2.35	0.18	0.07	4.59	0.18	0.23	0.01	13.24
	min	1	0	0	0	0	0	0	0	0	0	0	2
	median	1.7778	3	11	29	3	0	0	7	0	0	0	23
	max	2.6667	13	37	198	16	2	4	25	3	4	2	62
no EDom and small	mean	1.7109	3.47	12.10	0	0	0	0	9.75	0	0	0	22.98
	std	0.1807	2.18	6.32	0	0	0.06	0	5.59	0.02	0	0.01	12.03
	min	1	0	0	0	0	0	0	0	0	0	0	2
	median	1.7391	3	11	0	0	0	0	9	0	0	0	21
	max	2.50	15	42	0	0	2	1	27	2	1	2	56

Table 2: Results for CLUSTER VERTEX DELETION on the Rome-Graph dataset, $n = 100$

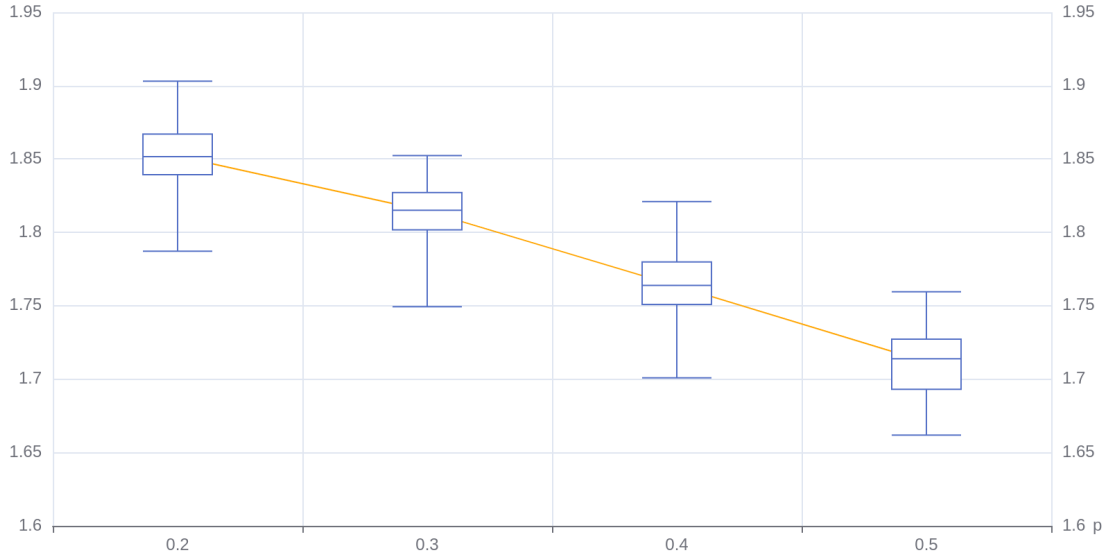
7.3 Comparison with LP rounding based Algorithms

7.4 Preferential Attachment Hypergraphs

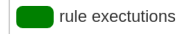
Preferential Attachment Graphs



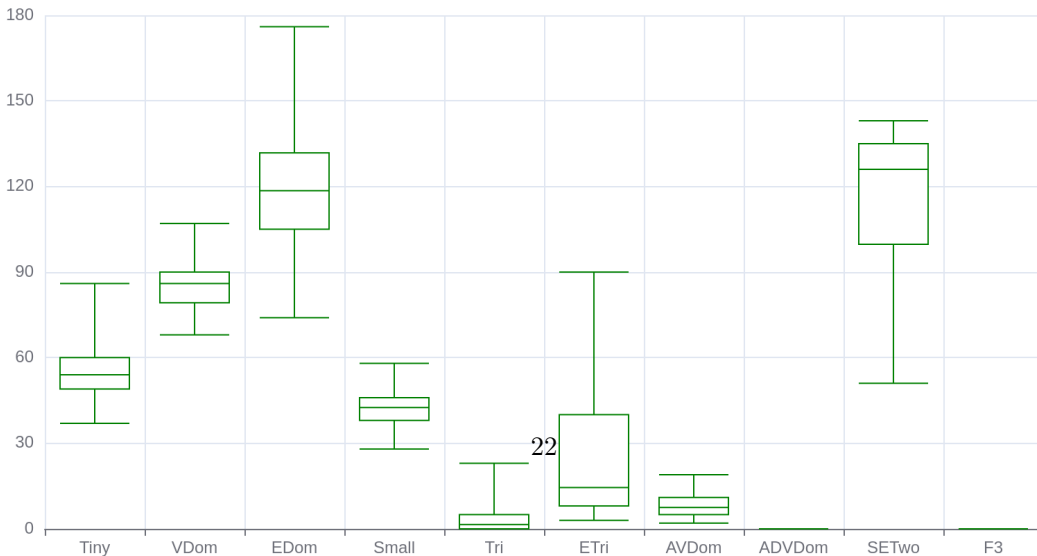
Probability $p=0.2$ until 0.5 in steps of 0.1, for adding a new vertex



#Rule Executions

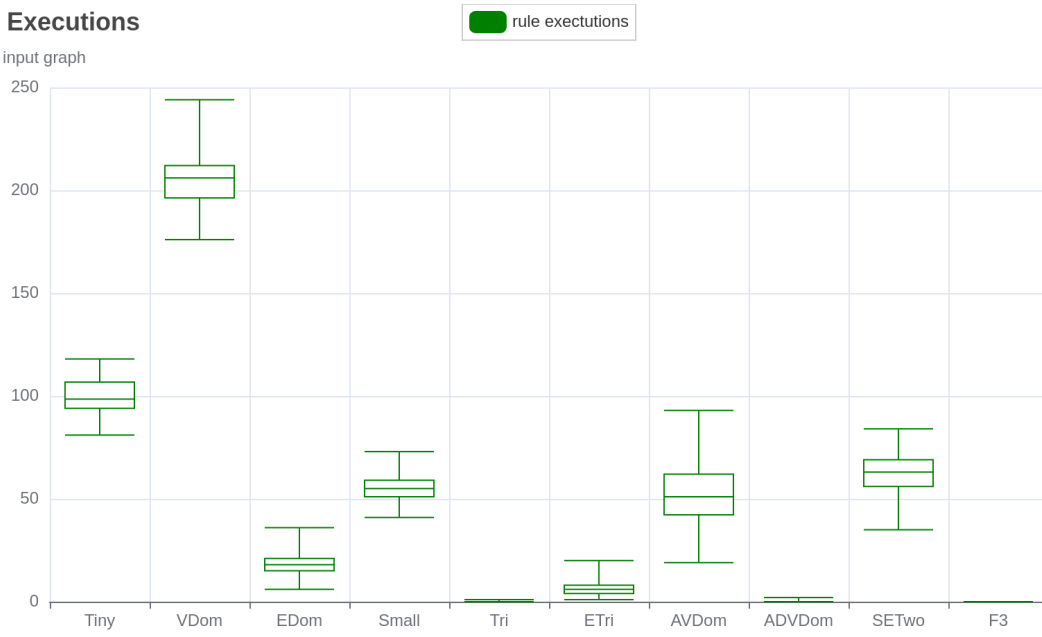


$p=0.2$ for input graph



#Rule Executions

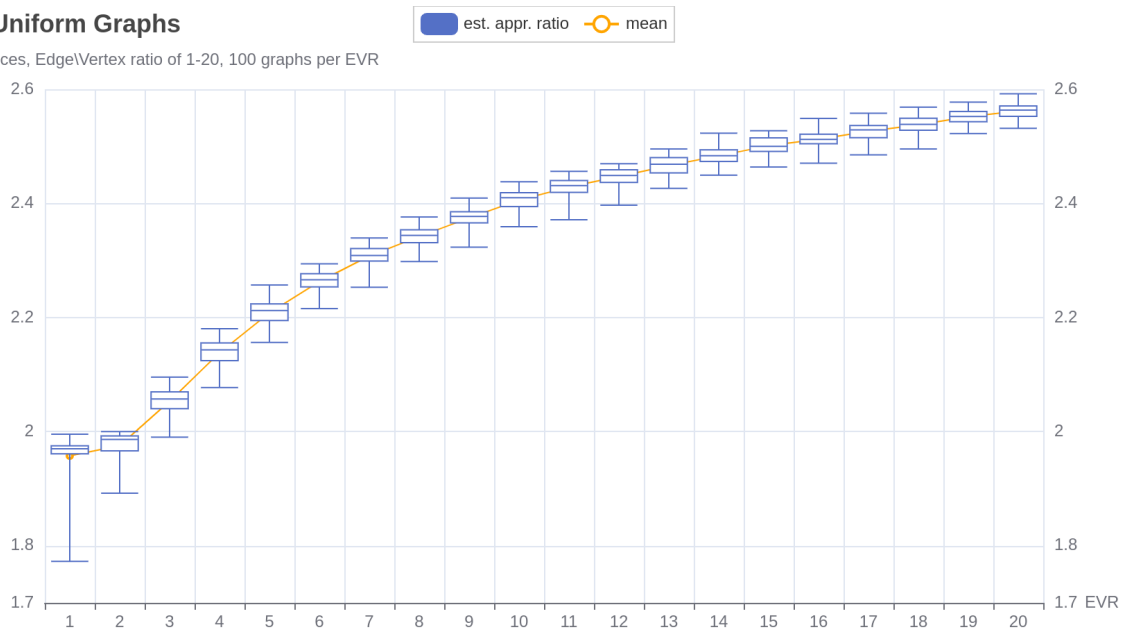
p=0.5 for input graph



7.5 3-Uniform ER Hypergraphs

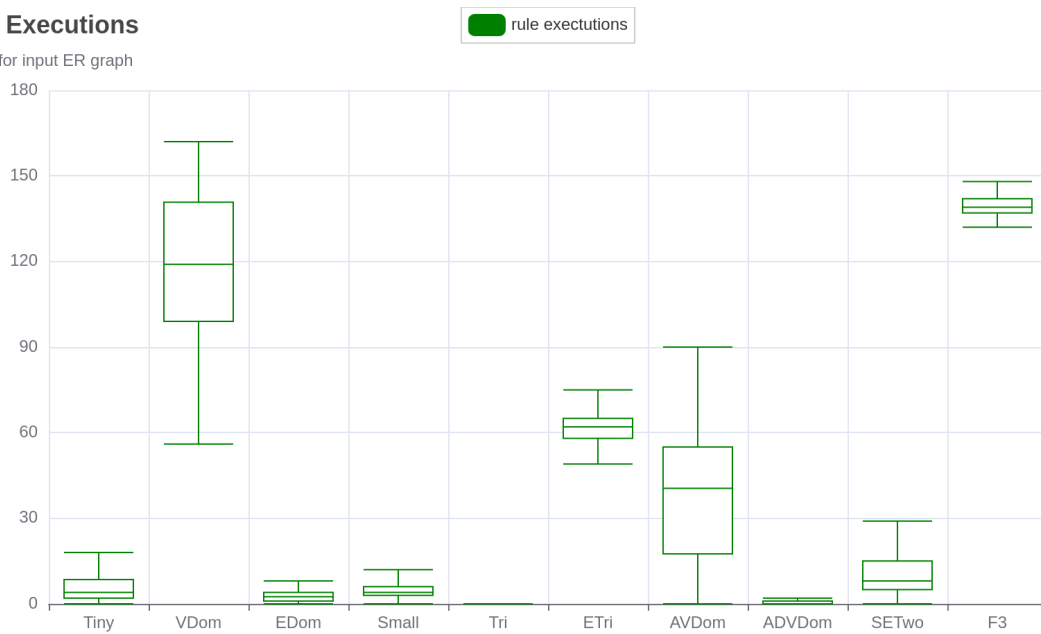
ER 3-Uniform Graphs

1000 vertices, Edge\Vertex ratio of 1-20, 100 graphs per EVR



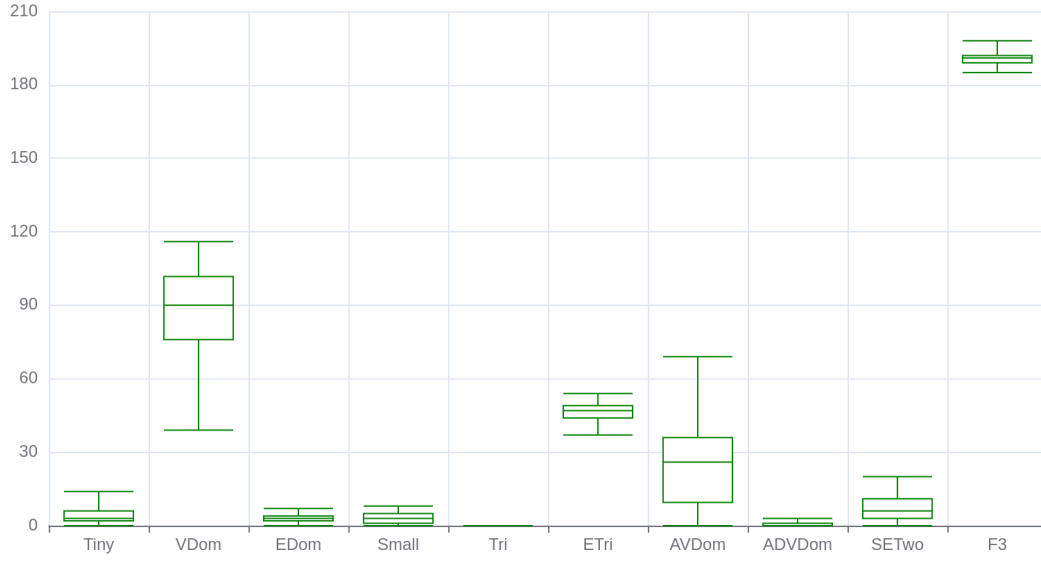
#Rule Executions

EVR=10 for input ER graph



#Rule Executions

EVR=20 for input ER graph



8 Testing

Every reduction rule is tested for their correctness with unit tests. We create small graphs in these tests, that contain structures, which the rules are targeting. We then test for the elements in the partial solution, amount of edges/vertices left in the graph and degree of the vertices left.

References

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- [2] C. Spagnuolo *et al.*, “Analyzing, exploring, and visualizing complex networks via hypergraphs using simplehypergraphs.jl,” *Internet Math.*, vol. 2020, 2020, doi: 10.24166/IM.01.2020.
- [3] C. Avin, Z. Lotker, and D. Peleg, “Random preferential attachment hypergraphs,” *CoRR*, vol. abs/1502.02401, 2015, Available: <http://arxiv.org/abs/1502.02401>
- [4] L. Brankovic and H. Fernau, “Parameterized approximation algorithms for hitting set,” in *Approximation and online algorithms*, R. Solis-Oba and G. Persiano, Eds., Berlin, Heidelberg: Springer Berlin Heidelberg, 2012, pp. 63–76.
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