1.   **A digital communication link carries binary-coded words representing samples of an input signal,**

$$x(t) = 3cos600\pi t + 2cos1800\pi t$$

The link is operated at 10,000 bits/s and each input sample is quantized into 1024 different voltage levels.

   i.   What is the sampling frequency & folding frequency?
   ii.   What is the Nyquist rate for the signal x(t) ?
   iii.   What are the frequencies in the resulting discrete time signal x[n] ?
   iv.   What is the resolution Δ ?

Solution: ➔

As the link is operated at 10,000 bits/s & each input sample is quantized into 1024 different voltage levels the each sampled value is represented by $log_2 1024 = 10$ bits/sample

   i.   Then maximum sampling frequency, $F_s = \dfrac{10,000\ bits/sec}{10\ bits/sample} = 1000\ samples/sec$

   Folding frequency (is the maximum frequency that can be represented uniquely by sampled signal), $\dfrac{F_s}{2} = 500\ samples/sec$.

   ii. $x(t) = 3cos600\pi t + 2cos1800\pi t$

   Here, $F_1 = 300$ Hz and $F_2 = 900$ Hz. Thus $F_{max} = 900$ Hz.

   The Nyquist rate, $F_N = 2Fmax = 1800$ Hz.

   iii.   For $F_s = 1000$ Hz,

$$x[n] \cong x(nT) = x\left(\frac{n}{F_s}\right) = 3cos2\pi\left(\frac{300}{1000}\right)n + 2cos2\pi\left(\frac{900}{1000}\right)n$$

$$= 3cos2\pi\left(\frac{3}{10}\right)n + 2cos2\pi\left(\frac{9}{10}\right)n$$

$$= 3cos2\pi\left(\frac{3}{10}\right)n + 2cos2\pi\left(1 - \frac{1}{10}\right)n$$

$$= 3cos2\pi\left(\frac{3}{10}\right)n + 2cos2\pi\left(\frac{1}{10}\right)n$$

$$\therefore f_1 = \frac{3}{10}\ \&\ f_2 = \frac{1}{10}$$

   iv.   ADC resolution = 10 bits

   Voltage resolution, $\Delta = \dfrac{x_{max} - x_{min}}{L-1} = \dfrac{5 - (-5)}{1024 - 1} = 9.76\ mV$.

2. Check for linearity, time invariant and stability of following system:
$$y[n] = x[n^2 - 3]u[n] + x[n]u[n]$$

Solution ➔

<u>For Linearity:</u>

$$y_1[n] = x_1[n^2 - 3]u[n] + x_1[n]u[n]$$

$$y_2[n] = x_2[n^2 - 3]u[n] + x_2[n]u[n]$$

$$let\ x_3[n] = a_1x_1[n] + a_2x_2[n]$$

$$\therefore y_3[n] = x_3[n^2 - 3]u[n] + x_3[n]u[n]$$

$$= \{a_1x_1[n^2 - 3] + a_2x_2[n^2 - 3]\}u[n] + \{a_1x_1[n] + a_2x_2[n]\}u[n]$$

$$= a_1\{x_1[n^2 - 3]u[n] + x_1[n]u[n]\} + a_2\{x_2[n^2 - 3]u[n] + x_2[n]u[n]\}$$

$$= a_1y_1[n] + a_2y_2[n]$$

Since $y_3[n] = a_1y_1[n] + a_2y_2[n]$, the system is linear.

<u>For Time invariant:</u>

Response of delayed input, $y(n, k) = x[n^2 - 3 - k]u[n] + x_1[n - k]u[n]$

Delayed response, $y[n - k] = x[(n - k)^2 - 3]u[n - k] + x[n - k]u[n - k]$

Since $y(n, k) \neq y[n - k]$, the system is time variant.

<u>For Stabilty:</u>

Bounded Input condition: $|x[n]| \leq M_x < \infty$

For bounded output,

$$|y[n]| = |x[n^2 - 3]u[n] + x[n]u[n]|$$

$$\leq |x[n^2 - 3]u[n]| + |x[n]u[n]|$$

$$\leq M_x + M_x \leq M_y < \infty$$

Here, the output is bounded, therefore the system is stable.

3. **The impulse response h[n] of an LTI system is known to be zero, except in the interval $N_0 \leq n \leq N_1$. The input x[n] is known to be zero, except in the interval $N_2 \leq n \leq N_3$. As a result, the output is constrained to be zero, except in some interval $N_4 \leq n \leq N_5$. Determine N_4 and N_5 in terms of $N_0, N_1, N_2,$ and N_3.**

Solution: -➔

For an LTI system the output is obtained from the convolution of the input with impulse response of the system:

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

Since h[n] ≠ 0 for $N_0 \le n \le N_1$

$$y[n] = \sum_{k=N_0}^{N_1} h[k]x[n-k]$$

The input x[n] ≠ 0 for $N_2 \le n \le N_3$, so

x[n – k] ≠ 0 for $N_2 \le (n-k) \le N_3$

Note the minimum value of (n – k) is N_2. Thus the lower bound on n, which occurs for $k = N_0$ is

$N_4 = N_0 + N_2$

Using similar argument

$N_5 = N_1 + N_3$ $N_0 + N_2 \le n \le N_1 + N_3$

4. Find the response of the system to the input signal $x[n] = u[n] - u[n-5]$. The impulse response of the system is given by $x[n] = \alpha^n u[n]$.

Solution:

Step 1: Plot x[n] and h[n] in terms of x[k] and h[k].

Step 2: Fold x[k] or h[k], here x[k] to obtain x[-k].

Step 3: Shift the folded sequence to obtain x[n-k].

Step 4: Multiply h[k] and x[n-k].

Step 5: Sum all the values in multiplication.

Step 6: Repeat step 3 to 5 for all possible values of n.

From figure

$y[n] = 0 \ for \ n < 0$

$y[0] = h[k]x[-k] = 1$

$y[1] = h[k]x[1-k] = 1 + \alpha$

$y[2] = h[k]x[2-k] = 1 + \alpha + \alpha^2$

$y[3] = h[k]x[3-k] = 1 + \alpha + \alpha^2 + \alpha^3$

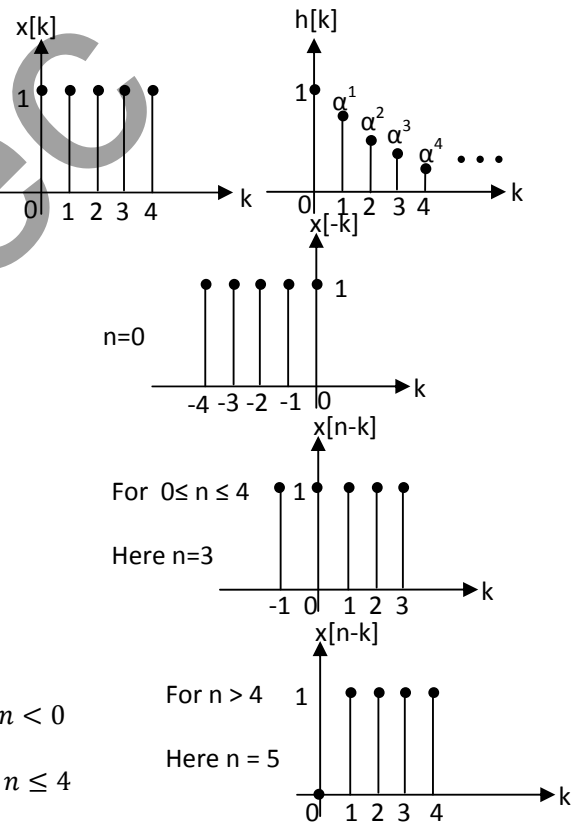$y[4] = h[k]x[4-k] = 1 + \alpha + \alpha^2 + \alpha^3 + \alpha^4$

$y[5] = h[k]x[5-k] = \alpha + \alpha^2 + \alpha^3 + \alpha^4 + \alpha^5$

$y[6] = h[k]x[6-k] = \alpha^2 + \alpha^3 + \alpha^4 + \alpha^5 + \alpha^6$

And so on..

Therefore,

$$y[n] = \begin{cases} 0, & for \ n < 0 \\ \sum_{k=0}^{n} \alpha^k, & 0 \le n \le 4 \\ \sum_{k=n-4}^{n} \alpha^k, & for \ n > 4 \end{cases}$$

5. Prove that "a LTI system is stable it its impulse response is absolutely summable". [PU Fall 2011 2(a)]

Solution: →

Stable System: An arbitrary relaxed system is BIBO stable if and only if its output sequence y[n] is bounded for every bounded input x[n].

For an LTI system:

Suppose we have an LTI system,

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

Taking absolute values on both sides,

$$|y[n]| = \left| \sum_{k=-\infty}^{\infty} h[k]x[n-k] \right|$$

$$\leq \sum_{k=-\infty}^{\infty} |h[k]||x[n-k]|$$

If the input is bounded, there exists a finite number M_x such that $|x[n]| \leq M_x$

$$|y[n]| \leq M_x \sum_{k=-\infty}^{\infty} |h[n]|$$

The output is bounded if the impulse response of the system satisfies the condition

$$S_h \equiv \sum_{k=-\infty}^{\infty} |h[n]| < \infty$$

"*A linear time-invariant system is stable if its impulse response is absolutely summable*".

6. Find z-transform of $x[n] = a^n(cos\omega_0 n)u[n]$.
Solution →

$$x[n] = a^n(cos\omega_0 n)u[n] = \frac{1}{2}a^n e^{j\omega_0 n}u[n] + \frac{1}{2}a^n e^{-j\omega_0 n}u[n]$$

We can write the above expression as $x[n] = v[n] + v^*[n]$

$$where \ v[n] = \frac{1}{2}\alpha^n u[n] \ with \ \alpha = ae^{j\omega_0}$$

The z-transform V(z) of v[n] by linearity and scaling in z-domain properties:

$$V(z) = \frac{1}{2} \cdot \frac{1}{1 - \alpha z^{-1}} = \frac{1}{2} \cdot \frac{1}{1 - a e^{j\omega_0} z^{-1}}, |z| > |\alpha| = a$$

From conjugate of complex sequence property,

$$V^*(z^*) = \frac{1}{2} \cdot \frac{1}{1 - \alpha^* z^{-1}} = \frac{1}{2} \cdot \frac{1}{1 - a e^{-j\omega_0} z^{-1}}, |z| > |\alpha| = a$$

By linearity property of the z-transform, we obtain

$$X(z) = V(z) + V^*(z^*)$$

$$= \frac{1}{2}\left(\frac{1}{1 - a e^{j\omega_0} z^{-1}} + \frac{1}{1 - a e^{-j\omega_0} z^{-1}}\right) = \frac{1 - (a\cos\omega_0)z^{-1}}{1 - (2a\cos\omega_0)z^{-1} + a^2 z^{-2}} \quad |z| > a$$

7. **If y[n] = {1, 1, 2, -1, 3} is output and impulse response is h[n] = {1, 2, 3} find discrete-time signal x[n].**
   Solution: →
   Taking z-transform,

   $$Y(z) = 1 + z^{-1} + 2z^{-2} - z^{-3} + 3z^{-4}$$
   $$H(z) = 1 + 2z^{-1} + 3z^{-2}$$
   $$\therefore X(z) = \frac{Y(z)}{H(z)} = \frac{1 + z^{-1} + 2z^{-2} - z^{-3} + 3z^{-4}}{1 + 2z^{-1} + 3z^{-2}}$$

   By division method:

   $$1 + 2z^{-1} + 3z^{-2} | 1 + z^{-1} + 2z^{-2} - z^{-3} + 3z^{-4} | 1 - z^{-1} + z^{-2}$$
   $$|\underline{1 + 2z^{-1} + 3z^{-2}} \qquad\qquad |$$
   $$|\; \overline{-z^{-1} - z^{-2} - z^{-3} + 3z^{-4}}|$$
   $$|-z^{-1} - 2z^{-2} - 3z^{-3} \qquad |$$
   $$|\; \overline{z^{-2} + 2z^{-3} + 3z^{-4}}|$$
   $$|\underline{z^{-2} + 2z^{-3} + 3z^{-4}}|$$

   Therefore X(z) is given by,

   $$X(z) = 1 - z^{-1} + z^{-2}$$

   Taking inverse z-transfrom, we get,

   $$x[n] = \{1, -1, 1\}.$$

8. **Determine the inverse z-transform of**

   $$H(z) = \frac{1 + 2z^{-1} + z^{-2}}{1 + 4z^{-1} + 4z^{-2}}$$

   For all possible ROCs. [PU 2009 Spring, 3(a)].
   Solution: →

   We have,

   $$H(z) = \frac{1 + 2z^{-1} + z^{-2}}{1 + 4z^{-1} + 4z^{-2}} = \frac{1 + 2z^{-1} + z^{-2}}{(1 + 2z^{-1})^2}$$

   $$= \frac{1}{(1 + 2z^{-1})^2} + \frac{2z^{-1}}{(1 + 2z^{-1})^2} + \frac{z^{-2}}{(1 + 2z^{-1})^2}$$

$$= -0.5z \times \frac{-2z^{-1}}{(1-(-2)z^{-1})^2} + (-1) \times \frac{-2z^{-1}}{(1-(-2)z^{-1})^2} + (-0.5)z^{-1} \times \frac{-2z^{-1}}{(1-(-2)z^{-1})^2}$$

The system have poles at $p_1 = -2$ and $p_2 = -2$. Therefore the possible ROCs are $|z|>2$ and $|z|<2$.

a. For ROC $|z| > 2$ (Causal Signal)

$$\therefore h[n] = -0.5(n+1)(-2)^{n+1}u[n+1] - n(-2)^n u[n] - 0.5(n-1)(-2)^{n-1}u[n-1]$$

b. For ROC $|z| < 2$ (Anticausal signal)
$$\therefore h[n] = 0.5(n+1)(-2)^{n+1}u[-n-2] + n(-2)^n u[-n-1]$$
$$+ 0.5(n-1)(-2)^{n-1}u[-n]$$

Note: We have used shifting property of z-transform and following z-transform table:

$$na^n u[n] \xleftrightarrow{z} \frac{az^{-1}}{(1-az^{-1})^2}, \qquad |z| > |a|$$

$$-na^n u[-n-1] \xleftrightarrow{z} \frac{az^{-1}}{(1-az^{-1})^2}, \qquad |z| < |a|$$

After using shifting property,

$$(n-k)a^{n-k}u[n-k] \xleftrightarrow{z} z^{-k} \times \frac{az^{-1}}{(1-az^{-1})^2}, \qquad |z| > |a|$$

$$-(n-k)a^{n-k}u[-(n-k)-1] \xleftrightarrow{z} z^{-k} \times \frac{az^{-1}}{(1-az^{-1})^2}, \qquad |z| < |a|$$

9. **Given the system y[n] = 3y[n-1] – 0.1y[n-2] + x[n]**
    a. **Find the system function, H(z).**
    b. **Is the system IIR or FIR?**
    c. **Plot the pole-zero diagram.**
    d. **Plot the magnitude response. [PU Fall 2012 5(a)]**

Solution: →

a. Taking z-transform of both sides, we get

$$Y(z) = 3z^{-1}Y(z) - 0.1z^{-2}Y(z) + X(z)$$

$$\Rightarrow Y(z)\{1 - 3z^{-1} + 0.1z^{-2}\} = X(z)$$

$$\therefore H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 - 3z^{-1} + 0.1z^{-2}}$$

This is the required System function of given system.

b. As the system contains poles the given system is IIR. (If the system contain poles then system is IIR)

c. Also,

$$H(z) = \frac{1}{1 - 3z^{-1} + 0.1z^{-2}} = \frac{1}{(1 - 2.9663z^{-1})(1 - 0.0337z^{-1})}$$

The system contains poles at $p_1 = 2.966$ and $p_2 = 0.034$, and zeros are at $z_1 = z_2 = 0$, the pole-zero diagram is shown below:

d. For the frequency response, we will take $z = e^{j\omega}$

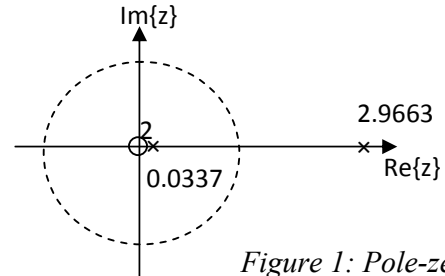$$\therefore H(e^{j\omega}) = \frac{1}{(1 - 2.9663e^{-j\omega})(1 - 0.0337e^{-j\omega})}$$

Figure 1: Pole-zero plot

Magnitude is given by

$$|H(e^{j\omega})| = \left|\frac{1}{(1 - 2.9663e^{-j\omega})(1 - 0.0337e^{-j\omega})}\right| = \frac{1}{|1 - 2.9663e^{-j\omega}||1 - 0.0337e^{-j\omega}|}$$

$$= \frac{1}{\sqrt{(1 - 2.9663cos\omega)^2 + (2.9663sin\omega)^2}.\sqrt{(1 - 0.0337cos\omega)^2 + (0.0337sin\omega)^2}}$$

For the magnitude response, we can

Find the ω Vs. |H(e^jω)| as,

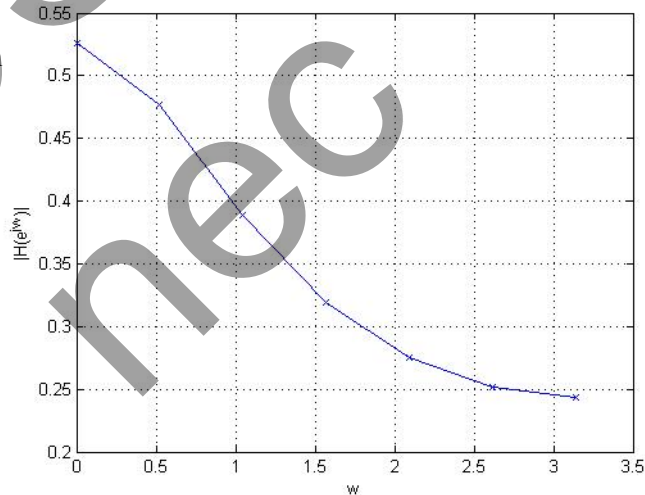| Ω	**|H(e^jω)|**
0	0.5263
π/6	0.4770
2π/6	0.3890
3π/6	0.3193
4π/6	0.2751
5π/6	0.2514
Π	0.2493

Figure 2: Magnitude Response

10. Find fft of the signal x[n] = {1, 2, 5, 4, -3, 0, 4, 1}.

$Solution: \longrightarrow$ $Here, W_8^0 = 1,$ $W_8^1 = e^{-j\frac{2\pi}{8}} = \cos\frac{2\pi}{8} - jsin\frac{2\pi}{8} = 0.707 - j0.707$
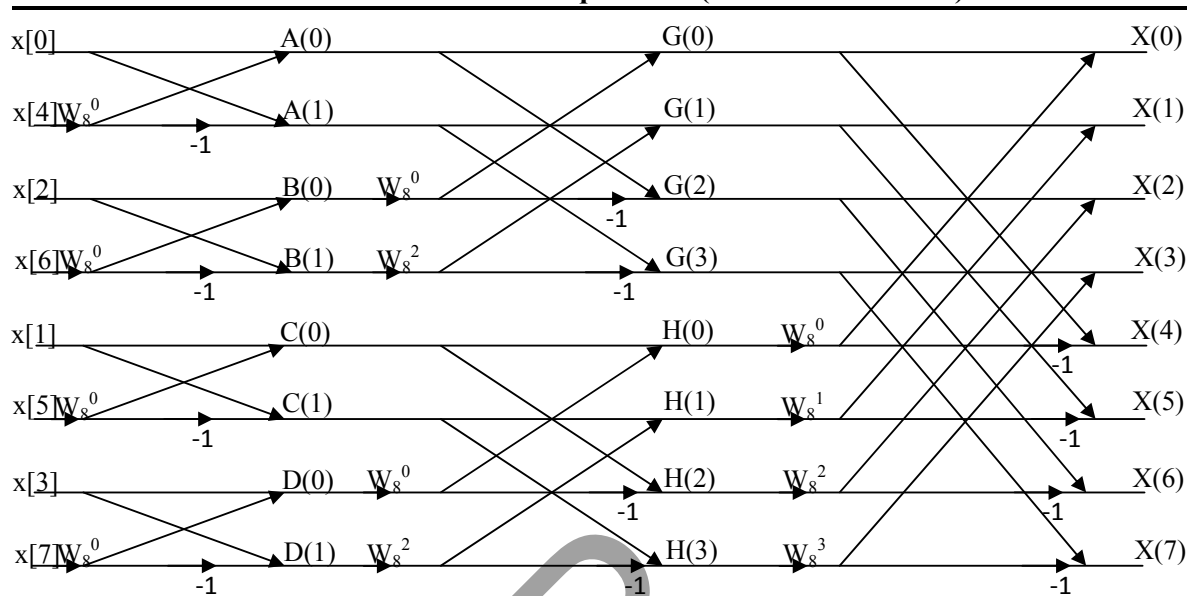
$$W_8^2 = e^{-\left(\frac{j2\pi}{8}\right)\times 2} = \cos\frac{\pi}{2} - jsin\frac{\pi}{2} = -j$$

$$W_8^3 = e^{-j\left(\frac{2\pi}{8}\right)\times 3} = \cos\frac{3\pi}{4} - jsin\frac{3\pi}{4} = 0.707 - j0.707$$

Output of first Stage:

$$A(0) = x[0] + W_8^0 x[4] = 1 + 1 \times -3 = -2, \qquad A(1) = x[0] - W_8^0.x[4] = 1 - 1 \times -3 = 4$$

$$B(0) = x[2] + W_8^0.x[6] = 5 + 1 \times 4 = 9, \qquad B(1) = x[2] - W_8^0.x[6] = 5 - 1 \times 4 = 1$$

$$C(0) = x[1] + W_8^0.x[5] = 2 + 1 \times 0 = 2, \qquad C(1) = x[1] - W_8^0.x[5] = 2 - 1 \times 0 = 2$$

$$D(0) = x[3] + W_8^0.x[7] = 4 + 1 \times 1 = 5, \qquad D(1) = x[3] - W_8^0.x[7] = 4 - 1 \times 1 = 3$$

Output of second stage:

$$G(0) = A(0) + W_8^0.B(0) = -2 + 1 \times 9 = 7, \qquad H(0) = C(0) + W_8^0.D(0) = 2 + 1 \times 5 = 7$$

$$G(1) = A(1) + W_8^2.B(1) = 4 + (-j) \times 1 = 4 - j,$$
$$H(1) = C(1) + W_8^2.D(1) = 2 + (-j) \times 3 = 2 - j3$$

$$G(2) = A(0) - W_8^0.B(0) = -2 - 1 \times 9 = -11, \qquad H(2) = A(0) - W_8^0.B(0) = 2 - 1 \times 5 = -3$$

$$G(3) = A(1) - W_8^2.B(1) = 4 - (-j) \times 1 = 4 + j,$$
$$H(3) = C(1) - W_8^2.D(1) = 2 - (-j) \times 3 = 2 + j3$$

Output of third stage:

$$X(0) = G(0) + W_8^0.H(0) = 7 + 1 \times 7 = 14$$

$$X(1) = G(1) + W_8^1.H(1) = (4 - j) + (0.707 - j0.707) \times (2 - j3) = 3.293 - j4.535$$

$$X(2) = G(2) + W_8^2.H(2) = -11 + (-j) \times (-3) = -11 + j3$$

$$X(3) = G(3) + W_8^2.H(3) = (4 + j) + (-0.707 - j0.707) \times (2 + j3) = 4.707 - j2.535$$

$$X(4) = G(0) - W_8^0.H(0) = 7 - 1 \times 7 = 0$$

$$X(5) = G(1) - W_8^1.H(1) = (4 - j) - (0.707 - j0.707) \times (2 - j3) = 4.707 + j2.535$$

$$X(6) = G(2) - W_8^2 . H(2) = -11 - (-j) \times (-3) = -11 - j3$$

$$X(7) = G(3) - W_8^2 . H(3) = (4 + j) - (-0.707 - j0.707) \times (2 + j3) = 3.293 + j4.535$$

$$\therefore X(k) = \{14, 3.293 - j4.535, -11 + j3, 4.707 - j2.535, 0, 4.707 + j2.535, -11 - j3, 3.293 + j4.535\}$$

11. Draw the lattice structure of the following system:
$$H(z) = \frac{1 + 3z^{-1} + 3z^{-2} + z^{-3}}{2 - 1.8z^{-1} + 1.28z^{-2} - 1.152z^{-3}} . Is \ this \ system \ stable?$$

Solution: ➔

$$Given, H(z) = \frac{C_3(z)}{2A_3(z)} = \frac{1 + 3z^{-1} + 3z^{-2} + z^{-3}}{2(1 - 0.9z^{-1} + 0.64z^{-2} - 0.576z^{-3})}$$

Note: We have to make $\alpha_m(0) = 1$ for mathematical convenience.

First consider, $A_3(z) = 1 - 0.9z^{-1} + 0.64z^{-2} - 0.576z^{-3}$

$$B_3(z) = -0.576 + 0.64z^{-1} - 0.9z^{-2} + z^{-3} . (reverse \ polynomial \ of \ A_3(z))$$

$$We \ know, k_m = \alpha_m(m), therefore \ k_3 = -0.576$$

$$Also, A_{m-1}(z) = \frac{A_m(z) - k_m B_m(z)}{1 - k_m^2}$$

$$therefore, A_2(z) = \frac{A_3(z) - k_3 B_3(z)}{(1 - k_3^2)}$$

$$= \frac{1 - 0.9\,z^{-1} + 0.64z^{-2} - 0.576z^{-3} - 0.576(-0.576 + 0.64z^{-1} - 0.9z^{-2} + z^{-3})}{1 - 0.576^2}$$

$$= 1 - 0.795z^{-1} + 0.182z^{-2}$$

$$\therefore k_2 = 0.182$$

Similarly,

$$A_1(z) = \frac{A_2(z) - k_2 B_2(z)}{(1 - k_2^2)} = 1 - 0.672z^{-1}$$

$$\therefore k_2 = 0.182$$

Now consider the numerator part $C_3(z)$

$$C_3(z) = c_3(0) + c_3(1)z^{-1} + c_3(2)z^{-2} + c_3(3)z^{-3} = 1 + 3z^{-1} + 3z^{-2} + z^{-3}$$

$$We \ know, v_m = c_m(m). Therefore \ v_3 = c_3(3) = 1$$

$$Now \ we \ have, C_{m-1}(z) = C_m(z) - v_m B_m(z)$$

$$\therefore C_2(z) = C_3(z) - v_3 B_3(z) \quad (Note: B_3(z) \ is \ from \ lattice \ part)$$

$$C_3(z) = 1 + 3z^{-1} + 3z^{-2} + z^{-3} - 1(-0.576 + 0.64z^{-1} - 0.9z^{-2} + z^{-3})$$

$$= 1.576 + 2.36z^{-1} + 3.9z^{-2}$$

$$\therefore v_2 = c_2(2) = 3.9$$

Similarly, $C_1(z) = C_2(z) - v_2 B_2(z) = 0.866 + 5.46z^{-1} ==> \therefore v_1 = c_1(1) = 5.46$

And, $C_0(z) = C_1(z) - v_1 B_1(z) = 4.535 ==> \therefore v_0 = c_0(0) = C_0(Z) = 4.535$

For Stability, $|k_m| < 1$. Therefore the system is stable.

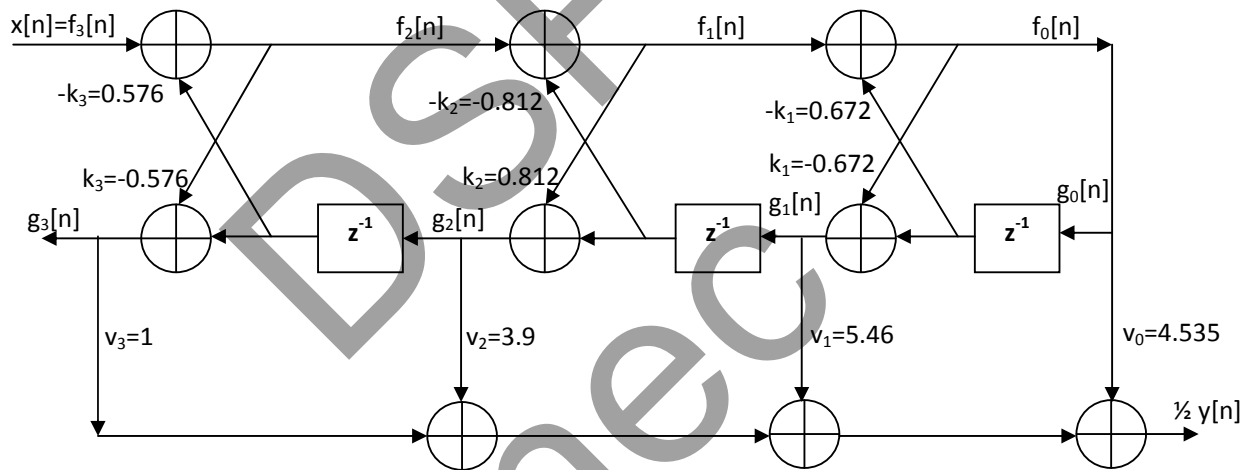The lattice ladder structure is shown below:

Fig: Lattice-ladder structure of given question.

12. Design a lowpass Butterworth filter to meet the following specifications.
   Passband gain = 0.89
   Passband frequency edge = 30 Hz
   Attenuation = 0.20
   Stopband edge = 75 Hz
   (Use impulse invariance and Bilinear Transformation method to convert the filter from analog to digital).

Solution:→

We have $\Omega_p = 2\pi \times 30 = 60\pi$ rad/s   (F_p = 30 Hz)

                        $\Omega_s = 2\pi \times 75 = 150\pi$ rad/s   (F_s = 75 Hz)

$\alpha_{max} = -20\log_{10}(0.89) = 1.0122 \ dB$

$\alpha_{min} = -20 \log_{10}(0.20) = 13.9794 \, dB$

$\varepsilon^2 = 10^{0.1\alpha_{max}} - 1 = 10^{0.10122} - 1 = 0.2625 ==> \varepsilon = 0.5123$

$\delta^2 = 10^{0.1\alpha_{min}} - 1 = 10^{1.39794} - 1 = 24 ==> \delta = 4.899$

$$\therefore N = \frac{\log_{10}\left(\frac{\delta}{\varepsilon}\right)}{\log_{10}\left(\frac{\Omega_s}{\Omega_p}\right)} = \frac{\log_{10}\left(\frac{4.899}{0.5123}\right)}{\log_{10}\left(\frac{150\pi}{60\pi}\right)} = \frac{0.9806}{0.3979} = 2.4644$$

Since the number of poles must be an integer, we round up to N = 3.

Matching the frequency response exactly at stopband produces,

$$\Omega_c = \frac{\Omega_s}{\delta^{\frac{1}{N}}} = \frac{150\pi}{4.899^{\frac{1}{3}}} = 277.4632$$

[*Note: If we were instead to match the frequency response at passband, we would obtain*

$$\Omega_c = \frac{\Omega_p}{\varepsilon^{\frac{1}{N}}} = \frac{60\pi}{0.5123^{\frac{1}{3}}} = 235.5734$$

In principle, any value of the critical frequency that satisfies $\frac{\Omega_p}{\varepsilon^{\frac{1}{N}}} \leq \Omega_c \leq \frac{\Omega_s}{\delta^{\frac{1}{N}}}$ would be valid.]

For N = 3 and $\Omega_c = 1$ (Normalized Transfer function)

$$H(s) = \frac{1}{(s+1)(s^2+s+1)}$$

(This can be obtained from formula as well as table given)

This transfer function is normalized for $\Omega_c = 1$. However $\Omega_c = 277.46$, we need to denormalised H(s) by $\Omega_c = 277.46$ rad/s

$$H(s)|\, s = {}^s/\Omega_c = \frac{\Omega_c^3}{(s+\Omega_c)(s^2+\Omega_c s+\Omega_c^2)} = \frac{2.136 \times 10^7}{(s+277.46)(s^2+277.46s+76984)}$$

  i.   Using Impulse invariance method:

$$H(s) = \frac{2.136 \times 10^7}{(s+277.46)(s^2+277.46s+76984)} = \frac{A}{s+277.46} + \frac{Bs+c}{s^2+277.46s+76984}$$

Solving we get, A = 277.46, B = -277.46 and c = 0.

$$\therefore H(s) = \frac{277.46}{s + 277.46} - \frac{277.46s}{s^2 + 277.46s + 76984}$$

$$= \frac{277.46}{s + 277.46} - \frac{277.46s}{s^2 + 2 \times s \times 138.73s + 138.73^2 + 230.29^2}$$

$$= \frac{277.46}{s + 277.46} - \frac{277.46(s + 138.73) - 38492}{(s + 138.73)^2 + 230.29^2}$$

$$= \frac{277.46}{s + 277.46} - \frac{277.46(s + 138.73)}{(s + 138.73)^2 + 230.29^2} + \frac{\frac{38492}{230.29} * 230.29}{(s + 138.73)^2 + 230.29^2}$$

Using mapping method:

$$H(z) = \frac{277.46}{1 - e^{-277.46T}z^{-1}} - 277.46 \times \frac{1 - e^{-138.73T}\cos(230.29T)\,z^{-1}}{1 - 2e^{-138.73T}\cos(230.29T)\,z^{-1} + e^{-277.46T}z^{-2}}$$

$$+ 167.14 \times \frac{e^{-138.73T}\sin(230.29T)\,z^{-1}}{1 - 2e^{-138.73T}\cos(230.29T)\,z^{-1} + e^{-277.46T}z^{-2}}$$

Analog Domain H(s)	Digital Domain H(z)
$\dfrac{1}{s - p_k}$	$\dfrac{1}{1 - e^{p_k T}z^{-1}}$
$\dfrac{s + a}{(s + a)^2 + b^2}$	$\dfrac{1 - e^{-aT}\cos(bT)z^{-1}}{1 - 2e^{-aT}\cos(bT)z^{-1} + e^{-2aT}z^{-2}}$
$\dfrac{b}{(s + a)^2 + b^2}$	$\dfrac{e^{-aT}\sin(bT)z^{-1}}{1 - 2e^{-aT}\cos(bT)z^{-1} + e^{-2aT}z^{-2}}$

ii.   Using Bilinear Transformation Method:

$$s = \frac{2}{T}\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right).$$

$$H(s) = \frac{2.136 \times 10^7}{(s + 277.46)(s^2 + 277.46s + 76984)}$$

$$\therefore H(z) = \frac{2.136 \times 10^7}{\left(\frac{2}{T}\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right) + 277.46\right)\left(\left(\frac{2}{T}\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right)\right)^2 + 277.46\frac{2}{T}\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right) + 76984\right)}$$

13. Design a digital lowpass Butterworth filter to meet the following specifications.
- **Passband cutoff frequency: $\omega_p = 0.15\pi$**
- **Stopband cutoff frequency: $\omega_s = 0.35\pi$**
- **Passband ripple: $-3dB \leq |H(e^{j\omega})| \leq 0dB, \ |\omega| \leq \omega_p$**
- **Stopband ripple: $|H(e^{j\omega}) \leq -20dB, \ \omega_s \leq |\omega| \leq \pi$**

Use impuse invariance method.

Solution:

Given,

$$\alpha_{max} = 3dB, \ \ \alpha_{min} = 20dB, \omega_p = 0.15\pi, \omega_s = 0.35\pi$$

$\varepsilon^2 = 10^{0.1\alpha_{max}} - 1 = 10^{0.3} - 1 = 0.995 ==> \varepsilon = 0.998$

$\delta^2 = 10^{0.1\alpha_{min}} - 1 = 10^2 - 1 = 99 ==> \delta = 9.95$

Using the impulse invariance design procedure, we have noted that the relation between frequency in the continuous-time and discrete-time domains is ω = ΩT, where T is merely a design parameter. Leaving T as an arbitrary constant for now, we obtain

$$\Omega_p = \frac{\omega_p}{T} = \frac{0.47124}{T}, \ \ and \ \Omega_s = \frac{\omega_s}{T} = \frac{1.0996}{T}$$

$$\therefore N = \frac{\log_{10}\left(\frac{\delta}{\varepsilon}\right)}{\log_{10}\left(\frac{\Omega_s}{\Omega_p}\right)} = \frac{\log_{10}\left(\frac{9.95}{0.998}\right)}{\log_{10}\left(\frac{1.0996}{0.47124}\right)} = \frac{0.998}{0.369} = 2.712$$

Since the number of poles must be an integer, we round up to N = 3.

Matching the frequency response exactly at passband produces,

$$\Omega_c = \frac{\Omega_p}{\varepsilon^{\frac{1}{N}}} = \frac{0.4716}{T}$$

For N = 3 and Ω_c = 1 (Normalized Transfer function)

$$H(s) = \frac{1}{(s+1)(s^2+s+1)}$$

(This can be obtained from formula as well as table given)

This transfer function is normalized for Ω_c = 1. However Ω_c = 0.4716/T, we need to denormalise H(s) by Ω_c = 0.4716/T

$$H(s)|\ s = {}^s/_{\Omega_c} = \frac{\Omega_c{}^3}{(s+\Omega_c)(s^2+\Omega_c s+\Omega_c{}^2)} = \frac{\left(\frac{0.4716}{T}\right)^3}{\left(s+\frac{0.4716}{T}\right)\left(s^2+\frac{0.4716}{T}s+\left(\frac{0.4716}{T}\right)^2\right)}$$

Using partial fractions we can rewrite the system function of the continuous-time prototype filter as

$$H(s) = \sum_{k=1}^{N} \frac{A_k}{s - s_k},$$

$\ \ \ \ \ \ \ \ \ where \ the \ parameters \ A_k are \ the \ continuous \ time \ residues \ of \ the \ poles \ s_k.$

Using the MATLAB routine *residue* we find the residues for the three poles, producing the transfer function

$$H(s) = \frac{0.4719}{s + \dfrac{0.4716}{T}} \pm \frac{0.236 - 0.136j}{s - \dfrac{-0.236 + 0.408j}{T}} + \frac{-0.236 + 0.136j}{s - \dfrac{-0.236 - 0.408j}{T}}$$

The corresponding discrete-time filter has the transfer function

$$H(z) = \sum_{k=1}^{N} \frac{A_k}{1 - e^{s_k T} z^{-1}}$$

Note that the poles of the continuous-time filter, s_k are all of the form of a (generally complex) constant divided by the parameter T. Since s_k is multiplied by T wherever it appears in the equation for H(z) above, the specific value chosen for T has no effect at all on the discrete-time filter that results from the design process. Hence we normally let T = 1 for simplicity. This produces the transfer function

$$H(z) = \frac{0.4716}{1 - e^{-0.4716} z^{-1}} + \frac{-0.236 - 0.136j}{1 - e^{(-0.236 + 0.408j)} z^{-1}} + \frac{-0.236 + 0.136j}{1 - e^{(-0.236 - 0.408j)} z^{-1}}$$

Combining we get,

$$H(z) = \frac{0.4716}{1 - 0.624 z^{-1}} + \frac{-0.472 + 0.341 z^{-1}}{1 - 1.45 z^{-1} + 0.624 z^{-2}}$$

14. **The following specifications are given for a Chebyshev lowpass filter: α_{max} = 0.5 dB, α_{min} = 22 dB, Ω_p = 1 rad/s, Ω_s = 2.33 rad/s. Find the order of the filter, N and the transfer function. If the same specifications are to be satisfied by using a Butterworth filter, find the order N.**
    Solution:

$$\varepsilon^2 = 10^{0.1\alpha_{max}} - 1 = 10^{0.05} - 1 = 0.0233 ==> \varepsilon = 0.153$$
$$and \; \delta^2 = 10^{0.1\alpha_{min}} - 1 = 10^{2.2} - 1 = 157.49 ==> \delta = 12.55$$

The order of filter using Chebysev filter is

$$N = \frac{cosh^{-1}\left(\delta/\varepsilon\right)}{cosh^{-1}\left(\Omega_s/\Omega_p\right)} = \frac{cosh^{-1}\left(\dfrac{12.55}{0.153}\right)}{cosh^{-1}\left(\dfrac{2.33}{1}\right)} = \frac{5.1}{1.49} = 3.42 \cong 4$$

If the same specifications are to be satisfied by using a Butterworth filter, then

$$N = \frac{\log_{10}\left(\dfrac{\delta}{\varepsilon}\right)}{\log_{10}\left(\dfrac{\Omega_s}{\Omega_p}\right)} = \frac{\log_{10}\left(\dfrac{12.55}{0.153}\right)}{\log_{10}\left(\dfrac{2.33}{1}\right)} = \frac{1.91}{0.367} = 5.12 \cong 6$$

15. **Convert the single-pole lowpass Butterworth filter with system function**

$$H(z) = \frac{0.245(1 + z^{-1})}{1 - 0.509 z^{-1}}$$

into a bandpass filter with upper and lower cutoff frequencies ω_u and ω_l, respectively. The lowpass filter has -3 dB bandwidth, ω_p = 0.2π.

Solution: →

The desired transformation is

$$z^{-1} \rightarrow -\frac{z^{-2} - a_1 z^{-1} + a_2}{a_2 z^{-2} - a_1 z^{-1} + 1}$$

Where,

$$a_1 = \frac{2\alpha K}{K+1}; \; a_2 = \frac{K-1}{K+1}$$

$$\alpha = \frac{\cos[\,(\omega_u + \omega_l)/2)]}{\cos[\,(\omega_u - \omega_l)/2)]} \; ; K = \cot \frac{\omega_u - \omega_l}{2} \tan \frac{\omega_p}{2}$$

Substitution into H(z) yields,

$$H(z) = \frac{0.245 \left(1 - \dfrac{z^{-2} - a_1 z^{-1} + a_2}{a_2 z^{-2} - a_1 z^{-1} + 1}\right)}{1 + 0.509 \left(\dfrac{z^{-2} - a_1 z^{-1} + a_2}{a_2 z^{-2} - a_1 z^{-1} + 1}\right)}$$

$$= \frac{0.245(1 - a_2)(1 - z^{-2})}{(1 + 0.509 a_2) - 1.509 a_1 z^{-1} + (a_2 + 0.509)z^{-2}}$$

Note that the resulting filter has zeros at z = ±1 and a pair of poles that depend on the choice of ω_u and ω_l.

For example, suppose that ω_u = 3π/5 and ω_l = 2π/5. Since ω_p = 0.2π, we find that K = 1, a_2 = 0, and a_1= 0. Then

$$H(z) = \frac{0.245(1 - z^{-2})}{1 + 0.509 z^{-2}}$$

This filter has poles at z = ±j0.713 and hence resonates at ω = π/2.

16. Design an FIR filter with 9 coefficients for the following specifications:
- Passband edge frequency = 0.5 KHz
- Sampling frequency = 2 KHz
- Use suitable window in design.

Solution:

Given,

M = 9 F_c = 0.5 KHz F_s = 2 KHz

We know,

ω_c = 2πf = 2π F_c/F_s = 0.5π

The desired frequency response is given by

$$H_d(\omega) = \begin{cases} e^{-j\omega\tau} & for \ |\omega| \le \omega_c \\ 0 & elsewhere \end{cases}$$

Taking inverse discrete-time Fourier transform to obtain desired unit sample response, we get,

$$h_d(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(\omega) \, e^{j\omega n} d\omega = \frac{1}{2\pi} \int_{-0.5\pi}^{-0.5\pi} e^{-j\omega\tau} \, e^{j\omega n} d\omega$$

Solving above equation we get,

$$h_d(n) = \begin{cases} \dfrac{\sin 0.5\pi(n-\tau)}{\pi(n-\tau)}, & for \ n \ne \tau \\ \dfrac{1}{2}, & for \ n = \tau \end{cases}$$

We know for symmetric FIR filter, the conditions are

h[n] = h[M − 1 − n] & $\tau = \frac{M-1}{2}$

for M = 9, the value of τ = 4

$$\therefore h_d(n) = \begin{cases} \dfrac{\sin 0.5\pi(n-4)}{\pi(n-\tau)}, & for \ n \ne 4 \\ \dfrac{1}{2}, & for \ n = 4 \end{cases}$$

As minimum stopband attenuation condition is not given, we can take any window for the design. So, taking Hanning window for our design,

Then the required unit sample is given by,

h[n] = h_d[n].w[n]

Taking Hanning window in design, the required unit sample of given filter is given in table below:

n	h_d[n]	$w[n] = 0.5\left(1 - \cos\dfrac{2\pi n}{M-1}\right)$	h[n] = h_d[n].w[n]
0	0	0	0
1	-0.106	0.146	0.015
2	0	0.5	0
3	0.318	0.854	0.269
4	0.5	1	0.5
5	0.318	0.854	0.269
6	0	0.5	0
7	-0.106	0.146	0.015
8	0	0	0

17. Design an FIR linear phase filter using Kaiser window to meet the following specifications.

$$0.99 \le |H(e^{jw})| \le 1.01, for \ 0 \le |w| \le 0.19\pi$$

$$|H(e^{jw})| \leq 0.01 , \qquad for \; 0.21\pi \leq |w| \leq \pi$$

Solution:

Step I: Given data

The given specifications may be rewritten as under

$1-0.01 \leq |H(e^{jw})| \leq 1+0.01$, for $0 \leq \omega \leq 0.19\pi$

$\qquad\qquad |H(e^{jw})| \leq 0.01$, for $0.21\pi \leq \omega \leq \pi$

On comparing the above given specifications with magnitude response of filter,

$\delta_1 = 0.01$ $\delta_2 = 0.01$

$\omega_p = 0.19\pi$ $\qquad \omega_s = 0.21\pi$

$\Delta\omega = \omega_s - \omega_p = 0.02\pi$

δ = minimum of δ_1 and δ_2 = 0.01

Now, $A = -20\log_{10}\delta = 40$

Step 2: To determine cutoff frequency ω_c

Cutoff frequency is given by

$\omega_c = (\omega_p + \omega_s)/2 = 0.2\pi$

Step 3: To obtain β and M

Here A = 40, which lies in the range of 21 to 51

Hence, β can be obtained as,

$\beta = 0.5842(A-21)^{0.4} + 0.07886(A-21) = 3.395$

and $\qquad M = \frac{A-8}{2.285\Delta\omega} = 222.88 \cong 223$

Step 4: To obtain expression of Kaiser window

The window will be defined as under

$$w[n] = \frac{I_0\left[\beta\left(\sqrt{1-\left(\left(n-\frac{M}{2}\right)\middle/\frac{M}{2}\right)^2}\right)\right]}{I_0(\beta)} \; for \; 0 \leq n \leq M$$

The function $I_0()$ in the above expression may be calculated with the help of following equation

$$I_0(x) = 1 + \sum_{r=1}^{\infty}\left[\frac{(x/2)^r}{r!}\right]^2$$

Since, it is an infinite series, maximum possible terms must be taken to reduce the error.

The ideal desired frequency response is given by

$$H_d(\omega) = \begin{cases} e^{-j\omega\tau\left(\frac{M-1}{2}\right)} & for \; |\omega| \leq \omega_c \\ 0 & elsewhere \end{cases}$$

Now, $h_d(n)$ may be obtained by taking inverse Fourier transform of above equation, i.e

$$h_d(n) = \frac{\sin\left[\omega_c\left(n - \frac{M}{2}\right)\right]}{\pi\left(n - \frac{M}{2}\right)}, \; for \; 0 \le n \le M,$$

Substituting for $\omega_c = 0.2\pi$ and M = 223,

$$h_d(n) = \frac{\sin[0.2\pi(n - 111.5)]}{\pi(n - 111.5)}, \; for \; 0 \le n \le 223,$$

The unit sample response of FIR filter may be obtained by windowing, i.e.

$$h[n] = h_d[n]\omega[n]$$

$$h(n) = \frac{\sin[0.2\pi(n - 111.5)]}{\pi(n - 111.5)} \frac{I_0\left[3.395\left(\sqrt{1 - \left(\frac{(n - 115.5)}{111.5}\right)^2}\right)\right]}{I_0(3.395)}, \; for \; 0 \le n \le 223,$$

This is the required relationship for unit sample response FIR filter using Kaiser Window.

18. Design a LP FIR filter using frequency sampling method having cutoff frequency of π/2 rad/samples. The filter should have linear phase & length of 17.
Solution: →

i) **To determine the desired frequency response:**
   The desired frequency response is $H_d(\omega)$. For the linear phase FIR lowpass filter, it is given as under:

$$H_d(\omega) = \begin{cases} e^{-j\omega\left(\frac{M-1}{2}\right)}, & for \; 0 \le \omega \le \omega_c \\ 0 \; , & \omega_c \le \omega \le \pi \end{cases}$$

The length of the filter is M = 17 and the cutoff frequency is $\omega_c = \pi/2$ rad/samples.

$$H_d(\omega) = \begin{cases} e^{-j\omega 8}, & for \; 0 \le \omega \le \dfrac{\pi}{2} \\ 0 \; , & for \; \dfrac{\pi}{2} \le \omega \le \pi \end{cases}$$

This is the desired frequency response of required lowpass filter.

ii) **To obtain $H_d(\omega)$**

$$H(k) = H_d(\omega)|_{\omega = \omega_k}$$

$$\omega_k = \frac{2\pi}{M}k, \; k = 0,1,2,3,\dots,M-1$$

$$H(k) = \begin{cases} e^{-j\frac{2\pi}{17}k8}, & for \; 0 \le \dfrac{2\pi}{17}k \le \dfrac{\pi}{2} \\ 0 \; , & for \; \dfrac{\pi}{2} \le \dfrac{2\pi}{17}k \le \pi \end{cases}$$

Now for $\frac{2\pi}{17}k = 0, k = 0$ & $\frac{2\pi}{17}k = \frac{\pi}{2}, k = 17/4$

$$H(k) = \begin{cases} e^{-j\frac{16\pi k}{17}}, & for\ 0 \le k \le \frac{17}{4} == 0 \le k \le 4.25 \approx 4 \\ 0\ , & for\ \frac{17}{4} \le k \le \frac{17}{2} == 4.25 \approx 5 \le k \le 8 \end{cases}$$

The above equation gives sampled version of $H_d(\omega)$ i.e. H(k)

iii) **To obtain h[n]**

For odd value of M, h[n] is given as under

$$h[n] = \frac{1}{M}\left[H(0) + 2\sum_{k=1}^{\frac{M-1}{2}} Re\left\{H(k)e^{\frac{j2\pi kn}{M}}\right\}\right]$$

$$= \frac{1}{17}\left[1 + 2\sum_{k=1}^{8} Re\left\{H(k)e^{\frac{j2\pi kn}{17}}\right\}\right] = \frac{1}{17}\left[1 + 2\sum_{k=1}^{4} Re\left\{e^{-j\frac{16\pi k}{17}}e^{\frac{j2\pi kn}{17}}\right\}\right]$$

Solving,

$$h[n] = \frac{1}{17}\left[1 + 2\sum_{k=1}^{4} \cos\left(\frac{2\pi k(8-n)}{17}\right)\right],\ n = 0,1,2,3,\ldots,16$$

This is the unit sample response of the FIR filter obtained using frequency sampling method.