Digital Signal Analysis and Processing

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Digital Filter Structures

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- FIR and IIR systems
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FIR Systems

- Any system that has impulse response of finite length.

- A causal FIR system has impulse response of the form:

$$h[n] = 0 \qquad for \quad M-1 < n < 0$$

- The output convolution sum is a finite sum:

$$y[n] = \sum_{k=0}^{M-1} h[k]x[n-k]$$

- The difference equation for such system is:

$$y[n] = \sum_{k=0}^{M-1} b_k x[n-k]$$

- Thus, system function has the form:

$$Y(z) = \sum_{k=0}^{M-1} h[k]z^{k} = \sum_{k=0}^{M-1} b_k z^{-k}$$

FIR Systems

- Output for FIR system may be computed as weighted sum of input samples only. Thus non-recursive realizations using only forward paths are possible for FIR systems.

- However recursive realizations also exists.

- Also called all zero systems.

IIR Systems

- An IIR system has impulse response of infinite length. So, h[n] does not decay outside a finite interval. e.g., $h[n] = u[n]$
- The output convolution sum is: $y[n] = \sum_{k=0}^{\infty} h[k]x[n-k]$
- Clearly if output is computed only in terms of input sample values, it will require infinite number of mathematical operations.
-

IIR Systems

- Hence output is expressed recursively using past outputs as in the difference equation for such system is:

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

This is the difference equation for Nth order IIR system

- The system function is thus:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}}$$

where the coefficients have been normalized by a_0

Why consider different structures?

- Computation Complexity:

  -Number of arithmetic operations required to compute the output sample values.

- Memory Requirements:

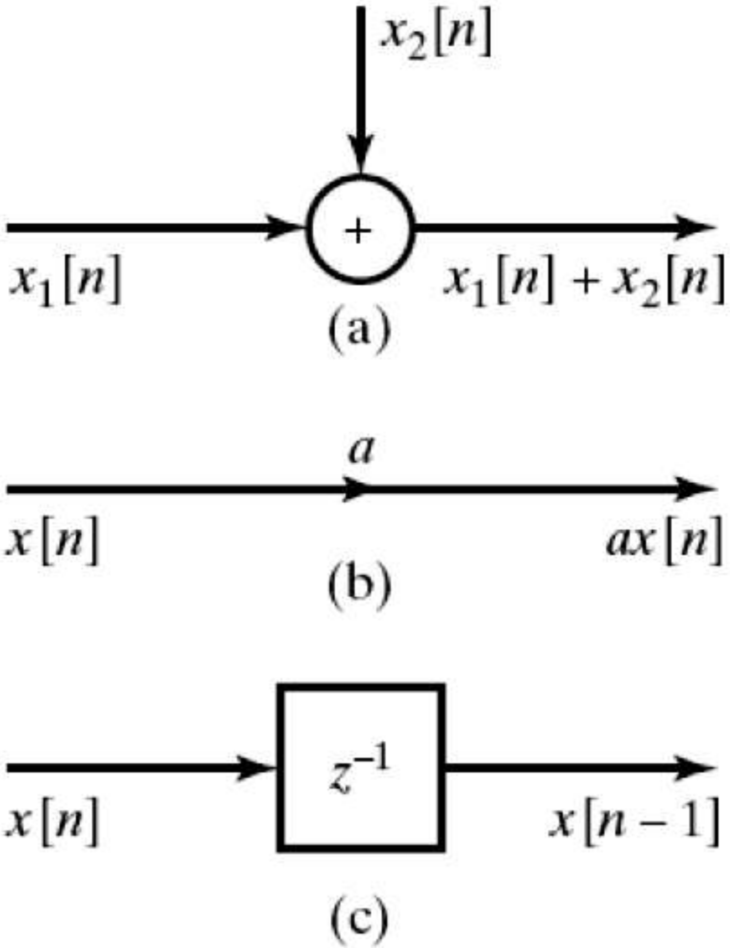
  -Number of memory elements required to store system coefficients, past inputs and outputs and any other intermediate computed values.

- Finite Word Length Effects:

  -Effect of finite word length on the system performance

- Other Considerations: Pipelining, parallel computing

- Basic blocks representing basic operations involved are:



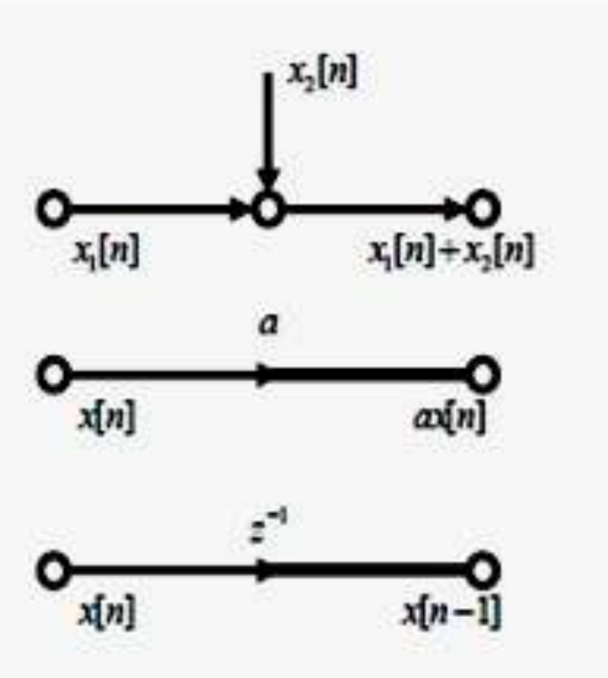
Adders for summation

Multipliers for scaling

Delay elements for storage

Signal Flow Graph Representation

- Network of directed branches connected at nodes.
- Signal at node is equal to sum of signals coming from all branches connected to the node.
- Input at source node and output at sink node.



An Example

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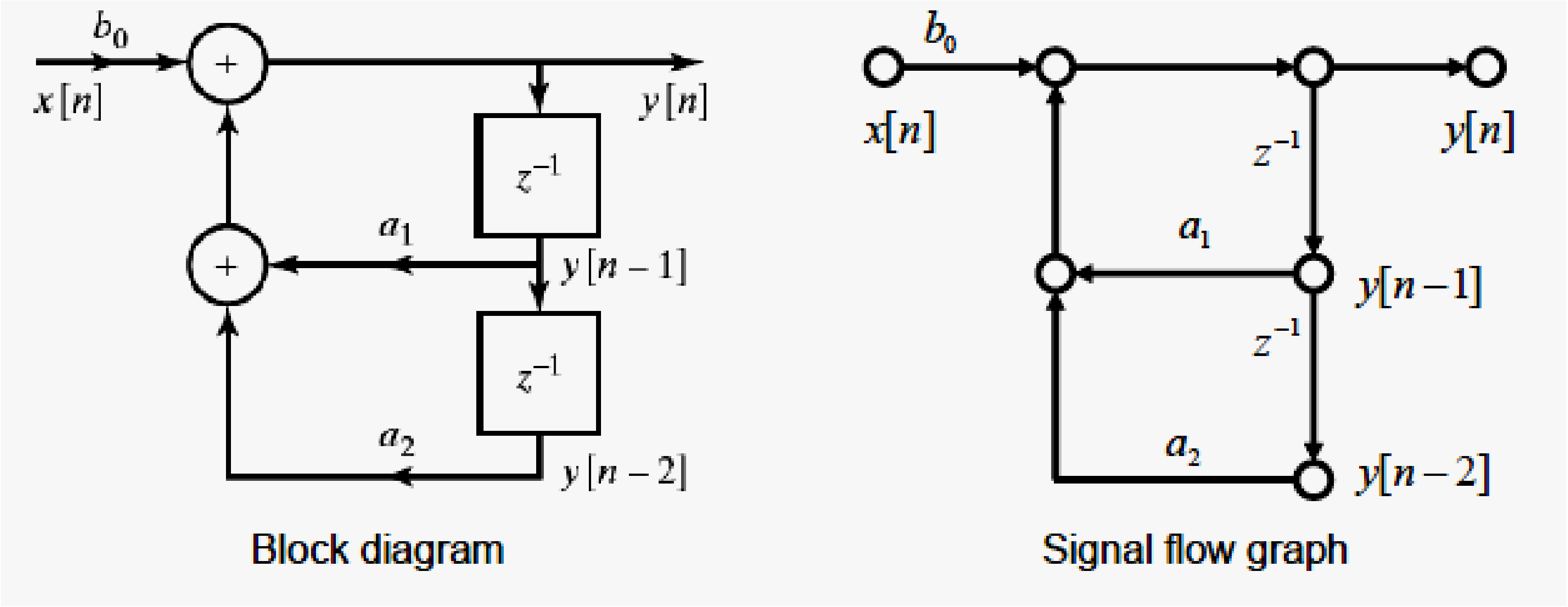
$$y[n] = a_1 y[n-1] + a_2 y[n-2] + b_0 x[n]$$



Structures For FIR Systems

- Direct Form structure

  -Direct implementation of non recursive difference equation or convolution sum

- Cascade structure

  -Larger system realized as cascade of several small order systems.

- Frequency sampled structure

  -Realization based on samples of frequency response rather than impulse response.

- Lattice structure

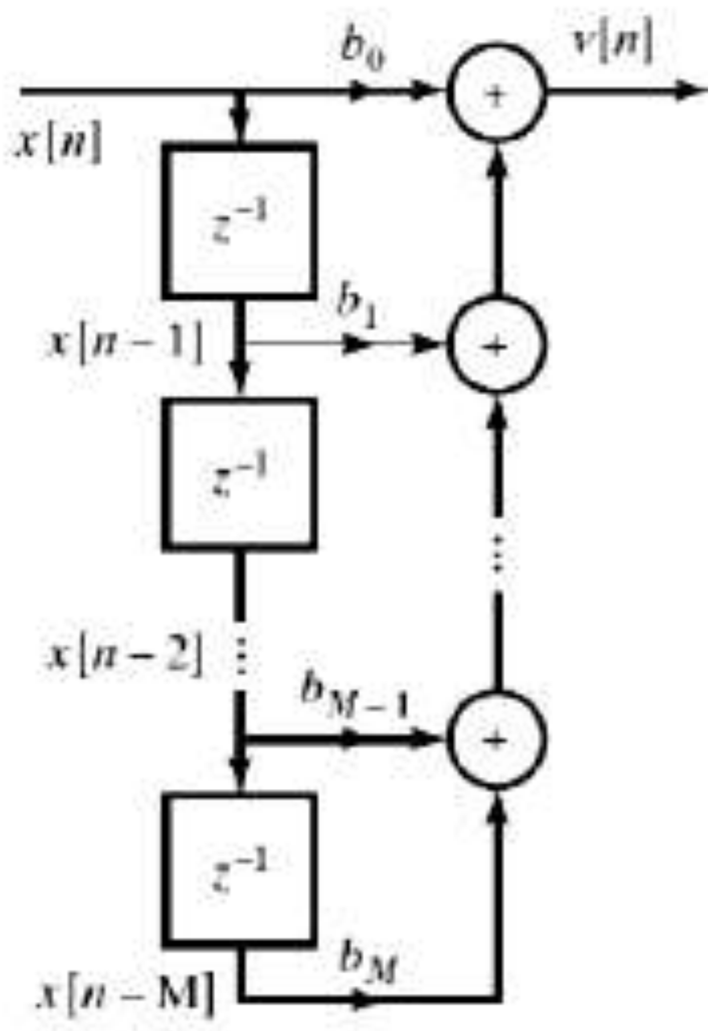
  -Realization of larger order systems by repeated regular lattice stages

- Transposed Structures:

Several other realizations other than these are also possible.

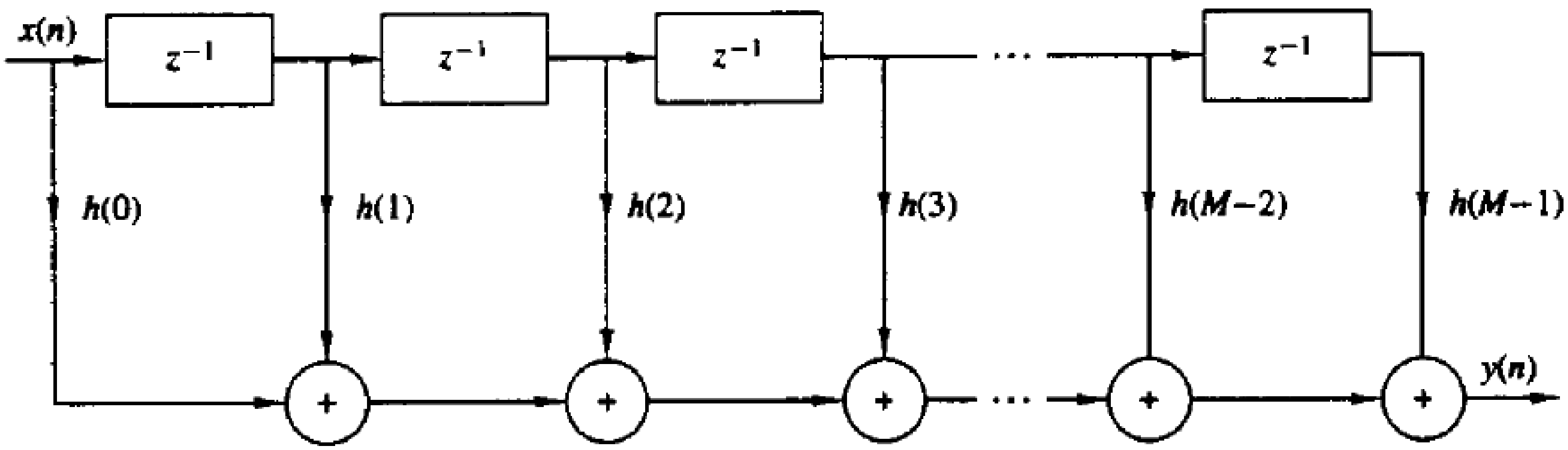
Direct Form Structure

- $$y[n] = \sum_{k=0}^{M-1} b_k x[n-k] = \sum_{k=0}^{M-1} h[k] x[n-k]$$
- Direct Form structure are direct implementation of the difference equation or convolution sum.
- Coefficients used to represent system are the same as the values of impulse response of the system.
- A FIR system of length M requires M multiplications, M-1 memory locations and M-1 additions.



Tapped Delay Line or Transversal System

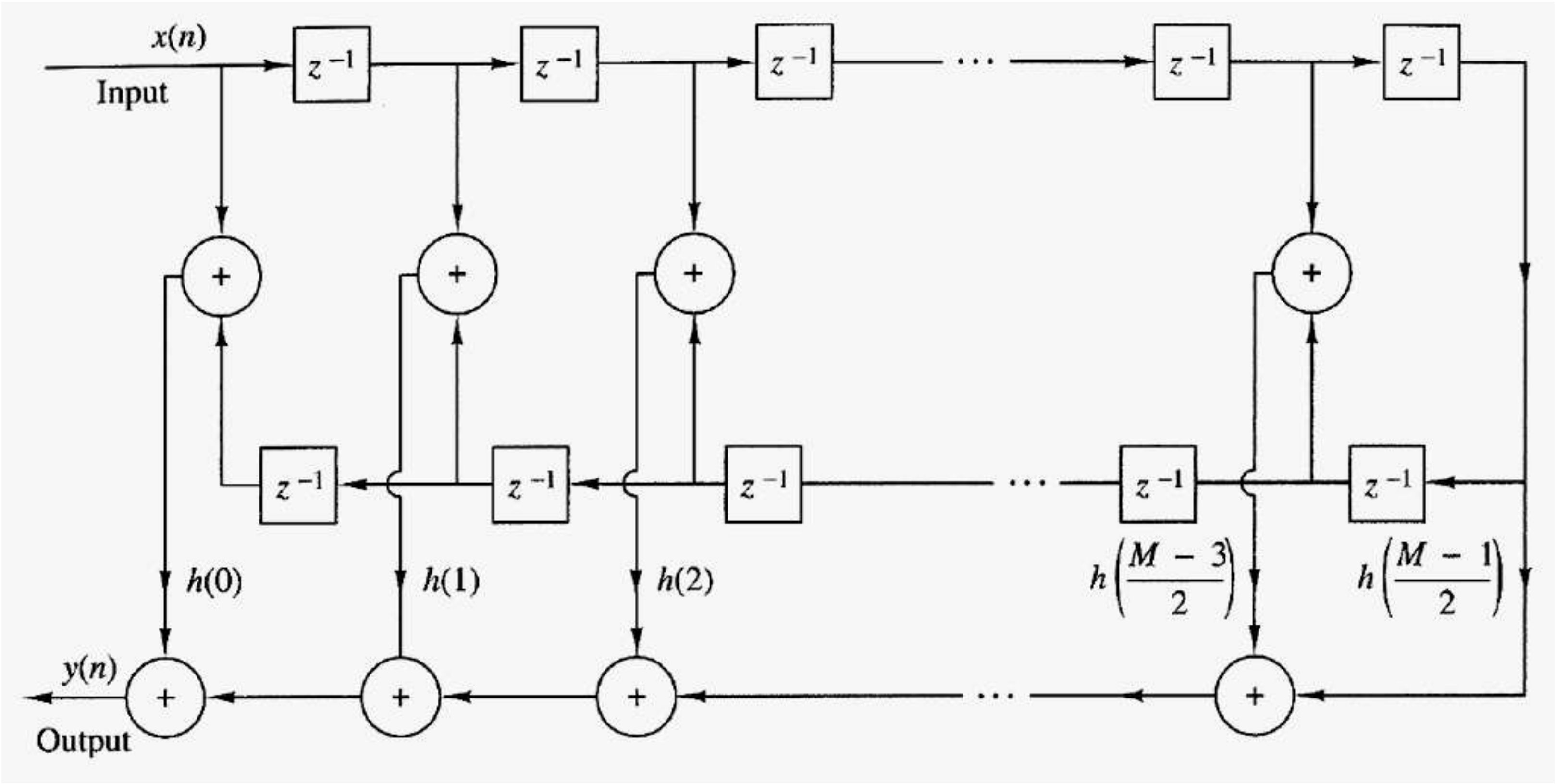
- Direct Form structure in fact computes output as weighted linear combination of past M-1 values of the input.
- So resembles a tapped delay line or a transversal system.



Symmetrical Impulse Response

- Linear phase FIR filters have symmetric or anti symmetric impulse response. $$h[n] = \pm h[M-1-n]$$

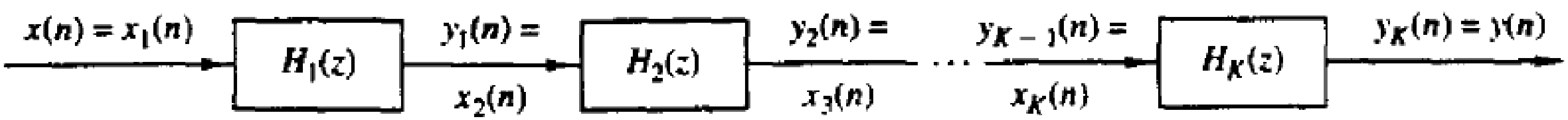
- In such case, number of multiplications may be reduced.



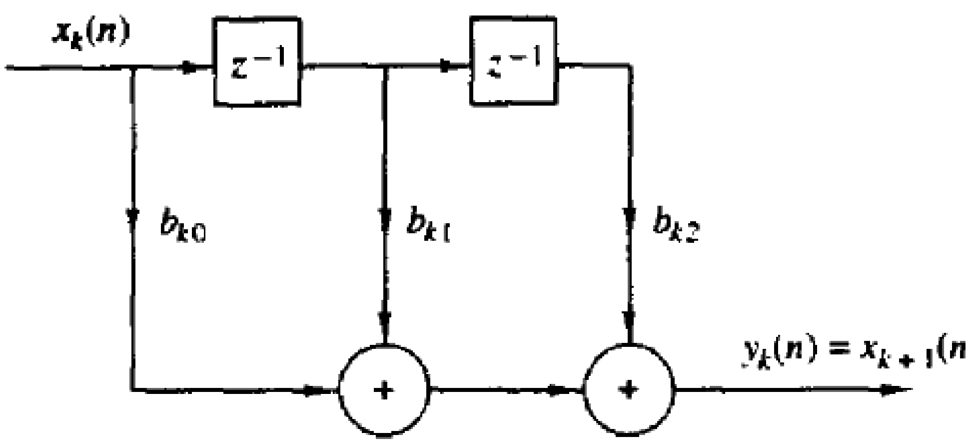
Cascade Form Structure

- Realize higher order system as cascade of smaller order sections. To avoid the complex coefficients, second order sections are chosen.

$$H(z) = \prod_{k=1}^{K} H_k(z)$$



$$H_k(z) = b_{k0} + b_{k1} z^{-1} + b_{k2} z^{-2} \qquad k = 1,\ 2\ ,........K$$



Frequency Sampled Structures

- Based on frequency domain specification of the system.

Let the frequency response desired be specified as $H(\omega)$

Since frequency response is given as $H(\omega)=\sum_{n=0}^{M-1} h[n]\, e^{-j\omega n}$,

Its samples are:

$$H[k+\alpha]=\sum_{n=0}^{M-1} h[n]\, e^{-j\frac{2\pi(k+\alpha)}{M}n} \qquad \alpha=0\, or\, \frac{1}{2}$$

The impulse response of desired system will thus be:

$$h[n]=\frac{1}{M}\sum_{k=0}^{M-1} H[k+\alpha]\, e^{j\frac{2\pi(k+\alpha)}{M}n} \qquad \alpha=0\, or\, \frac{1}{2}$$

Frequency Sampled Structures

The system function of the system will thus be:

$$H(z) = \sum_{n=0}^{M-1} h[n] z^{-n}$$

Substituting for h[n],

$$H(z) = \frac{1 - z^{-M} e^{j2\pi\alpha}}{M} \sum_{k=0}^{M-1} \frac{H[k+\alpha]}{1 - e^{j\frac{2\pi(k+\alpha)}{M}} z^{-1}}$$

$$= H_1(z) H_2(z)$$

Thus the resulting system is an all zero system of order M in cascade with a parallel bank of M single pole systems.

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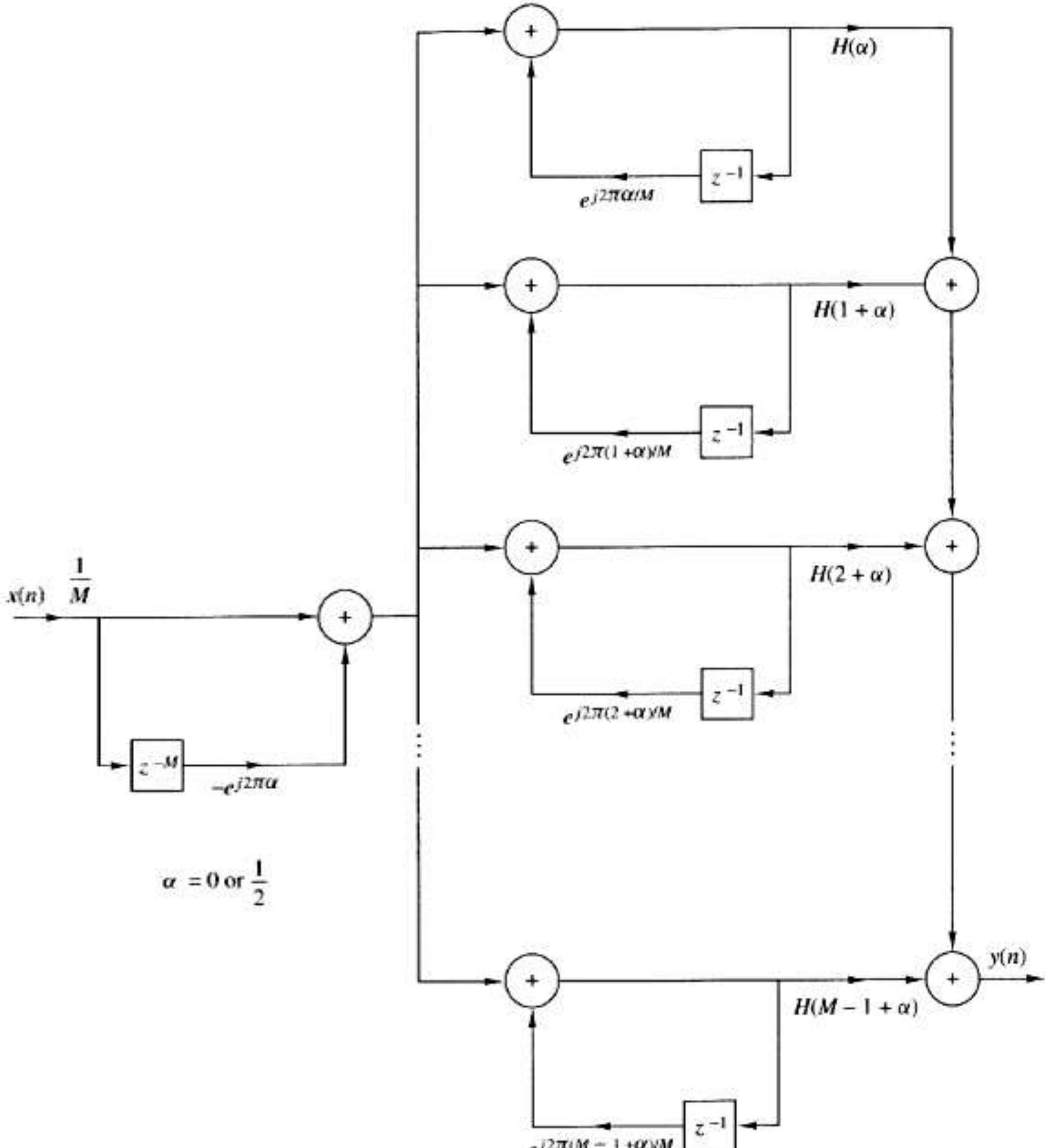
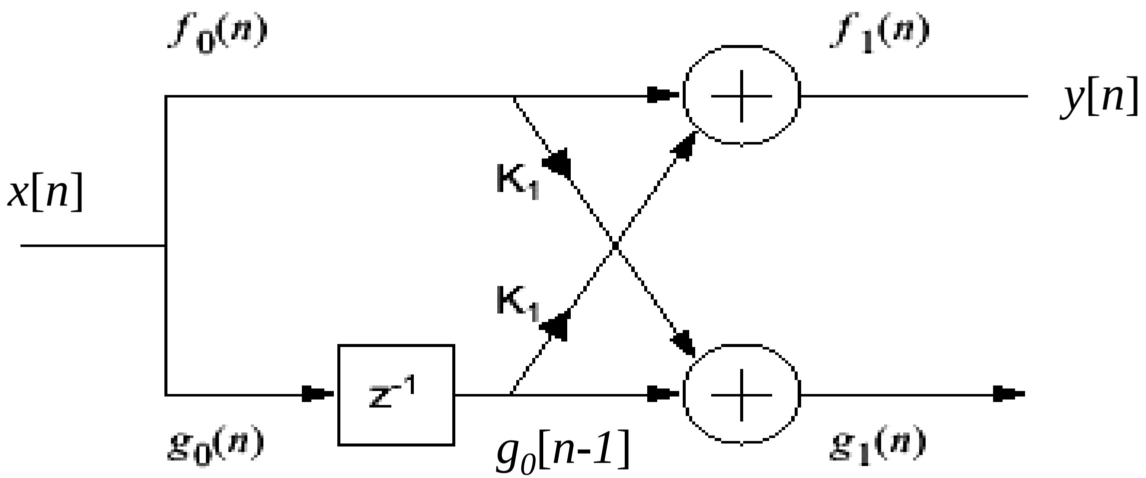


x(n)
1/M
z^{-M}
−e^{j2πα}
α = 0 or 1/2
H(α)
z^{-1}
e^{j2πα/M}
H(1 + α)
e^{j2π(1+α)/M}
H(2 + α)
e^{j2π(2+α)/M}
H(M − 1 + α)
e^{j2π(M−1+α)/M}
y(n)

Lattice structures

- Lattice structures are popular in ……….



$$f_1[n] = f_0[n] + k_1 g_0[n-1]$$

$$g_1[n] = k_1 f_0[n] + g_0[n-1]$$

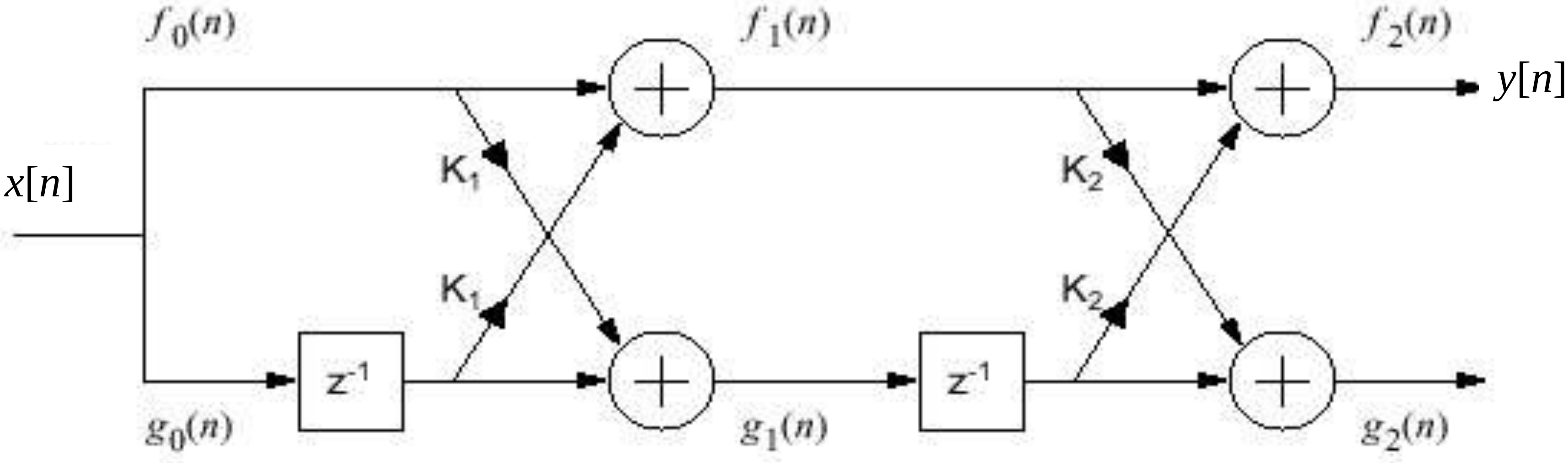
$$g_0[n] = f_0[n] = x[n]$$

$$f_1[n] = y[n]$$

$$y[n] = x[n] + k_1 x[n-1]$$

So, output is same to that of a first order FIR direct form if, $K_1=b_1$

Lattice structures

- 

$$
\begin{aligned}
f_2[n] &= f_1[n] + k_2 g_1[n-1] \\
&= f_0[n] + k_1 g_0[n-1] + k_2\left[k_1 f_0[n-1] + g_0[n-2]\right] \\
&= x[n] + k_1 x[n-1] + k_1 k_2 x[n-1] + k_2 x[n-2] \\
&= x[n] + \left[k_1 + k_1 k_2\right] x[n-1] + k_2 x[n-2]
\end{aligned}
$$

Lattice structures

- $$g_2[n] = k_2 f_1[n] + g_1[n-1]$$
$$= k_2\left[f_0[n] + k_1 g_0[n-1]\right] + k_1 f_0[n-1] + g_0[n-2]$$
$$= k_2 x[n] + k_1 k_2 x[n-1] + k_1 x[n-1] + x[n-2]$$
$$= k_2 x[n] + \left[k_1 + k_1 k_2\right] x[n-1] + x[n-2]$$

$$y[n] = f_2[n]$$

Output is same to that of a second order FIR direct form if,

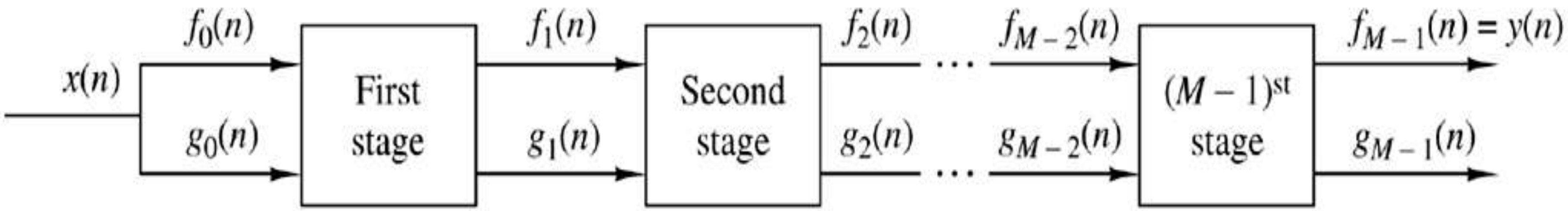
$k_2 = b_2$, $k_1 + k_1 k_2 = b_1$

i.e, $k_1 = b_1/(1+b_2)$

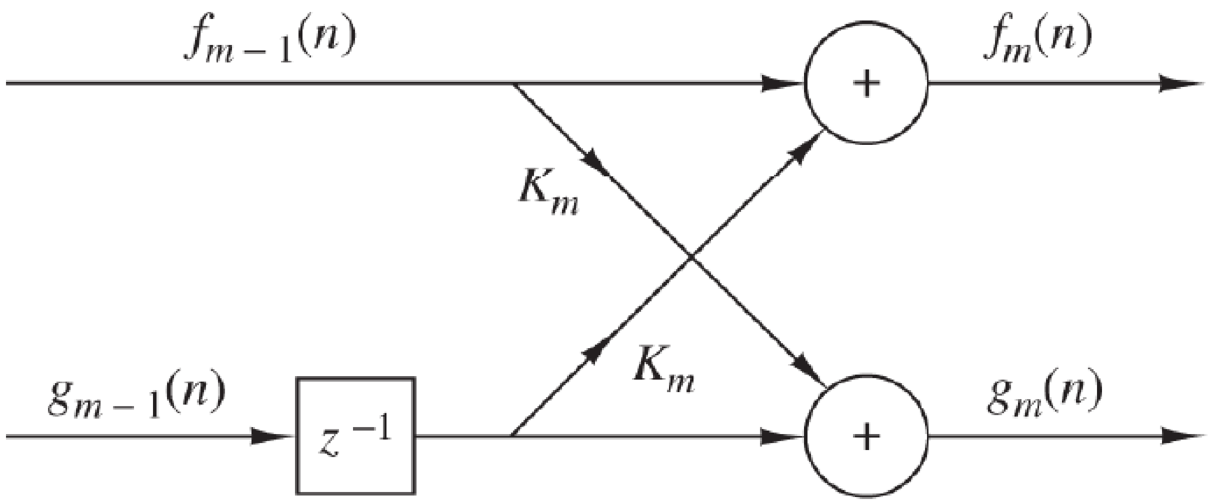
So, adding more stages higher order systems can be obtained.

Lattice structures

- A higher order (M) length FIR lattice looks like:



- Each intermediate stage is of the form:



Lattice structures

$$H_m(z) = A_m(z)$$

$$A_m(z) = \sum_{k=0}^{M} \alpha_m(k) z^{-k} \qquad \alpha_m(0) = 1 \quad \textit{and hence} \quad A_m(0) = 1$$

- For a lattice with M-1 stages,

$$f_m[n] = f_{m-1}[n] + k_m g_{m-1}[n-1]$$

$$g_m[n] = k_m f_{m-1}[n] + g_{m-1}[n-1]$$

$$g_0[n] = f_0[n] = x[n]$$

$$f_{M-1}[n] = y[n]$$

- If m^{th} stage is the output,

$$f_m[n] = \sum_{k=0}^{m} \alpha_m(k)\, x[n-k]$$

Lattice to Direct Form and vice versa

- Lattice to Direct Form:

$$A_0(z) = B_0(z) = 1$$

$$A_m(z) = A_{m-1}(z) + k_m z^{-1} B_{m-1}(z) \qquad m = 1, 2, \ldots. M-1$$

$$B_m(z) = z^{-m} A_m(z^{-1}) \qquad m = 1, 2, \ldots\ldots M-1$$

- Direct Form to Lattice:

$$A_{m-1}(z) = \frac{A_m(z) - k_m B_m(z)}{1 - k_m^{\ 2}} \qquad m = M-1, M-2, \ldots\ldots 1$$

$$B_m(z) = z^{-m} A_m(z^{-1}) \qquad m = M-1, M-2, \ldots\ldots 1$$

$$K_m = \alpha_m(m)$$

Examples:

- Determine the direct form realization of a three stage (M=4) lattice filter with coefficients K_1=1/2, K_2=1/2, K_3=1/4

We know, $A_0(z) = B_0(z) = 1$

Using

$$A_m(z) = A_{m-1}(z) + k_m z^{-1} B_{m-1}(z)$$

$$A_1(z) = A_0(z) + k_1 z^{-1} B_0(z) = 1 + \frac{1}{2} z^{-1}$$

$$B_m(z) = z^{-m} A_m(z^{-1})$$

$$B_1(z) = \frac{1}{2} + z^{-1}$$

$$A_2(z) = A_1(z) + k_2 z^{-1} B_1(z) = 1 + \frac{1}{2} z^{-1} + \frac{1}{2} z^{-1} \left\{ \frac{1}{2} + z^{-1} \right\}$$

Examples:

- $$A_2(z) = 1 + \frac{3}{4}z^{-1} + \frac{1}{2}z^{-2}$$

$$B_2(z) = \frac{1}{2} + \frac{3}{4}z^{-1} + z^{-2}$$

$$A_3(z) = A_2(z) + k_3 z^{-1} B_2(z) = 1 + \frac{3}{4}z^{-1} + \frac{1}{2}z^{-2} + \frac{1}{4}z^{-1}\left\{\frac{1}{2} + \frac{3}{4}z^{-1} + z^{-2}\right\}$$

$$= 1 + \frac{7}{8}z^{-1} + \frac{11}{16}z^{-2} + \frac{1}{4}z^{-3}$$

Examples:

- Determine the lattice representation for:

$$H(z) = A_3(z) = 1 + \frac{7}{8}z^{-1} + \frac{11}{16}z^{-2} + \frac{1}{4}z^{-3}$$

Here, $\boxed{K_3 = \alpha_3(3) = \frac{1}{4}}$ and $B_3(z) = \frac{1}{4} + \frac{11}{16}z^{-1} + \frac{7}{8}z^{-2} + z^{-3}$

We know,

$$A_{m-1}(z) = \frac{A_m(z) - k_m B_m(z)}{1 - {k_m}^2}$$

$$A_2(z) = \frac{A_3(z) - k_3 B_3(z)}{1 - {k_3}^2}$$

$$= \frac{1 + \frac{7}{8}z^{-1} + \frac{11}{16}z^{-2} + \frac{1}{4}z^{-3} - \frac{1}{4}\left\{\frac{1}{4} + \frac{11}{16}z^{-1} + \frac{7}{8}z^{-2} + z^{-3}\right\}}{1 - \left(\frac{1}{4}\right)^2} = 1 + \frac{3}{4}z^{-1} + \frac{1}{2}z^{-2}$$

Examples:

SO, $\boxed{K_2 = \alpha_2(2) = \frac{1}{2}}$ and $B_2(z) = \frac{1}{2} + \frac{3}{4}z^{-1} + z^{-2}$

$$A_1(z) = \frac{A_2(z) - k_2 B_2(z)}{1 - k_2^{\,2}}$$

$$= \frac{1 + \frac{3}{4}z^{-1} + \frac{1}{2}z^{-2} - \frac{1}{2}\left\{\frac{1}{2} + \frac{3}{4}z^{-1} + z^{-2}\right\}}{1 - \left(\frac{1}{2}\right)^2} = 1 + \frac{1}{2}z^{-1}$$

$$\boxed{K_1 = \alpha_1(1) = \frac{1}{2}}$$

Transposed Structures

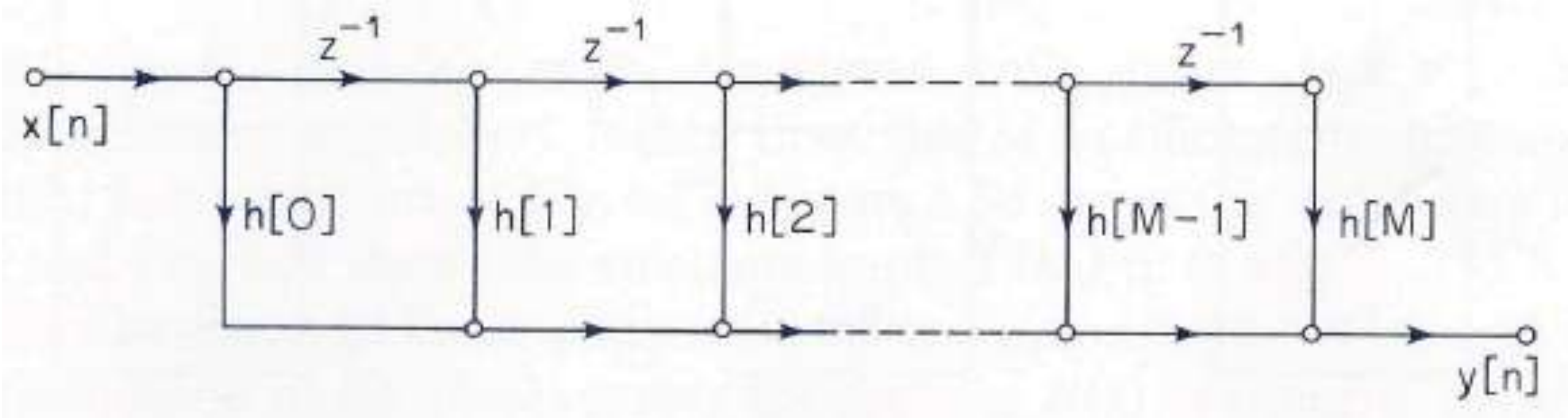
- Transposed structures are obtained by transposing the structures based on transposition theorem of signal flow graph.

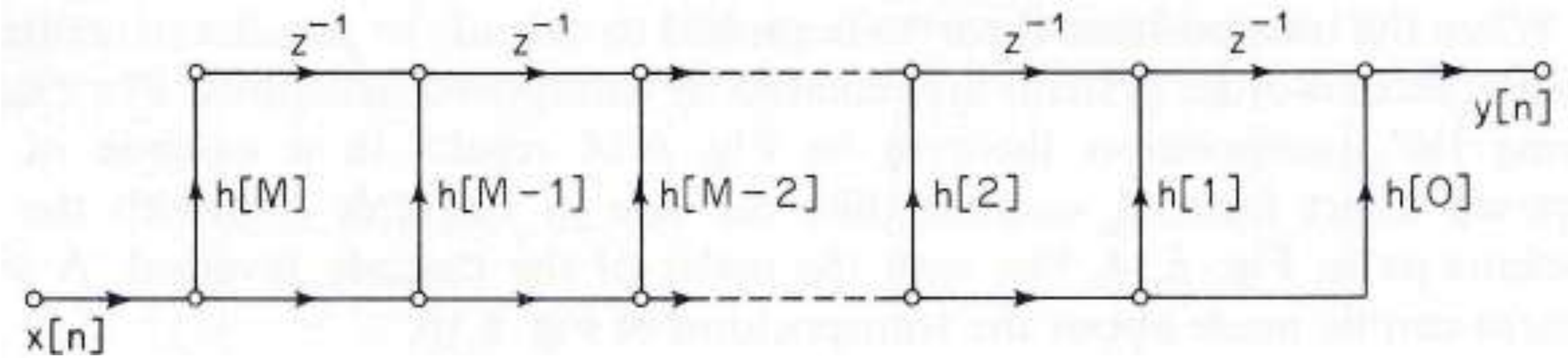
- According to the theorem, if we transpose the signal flow graph by reversing the direction of all branches in the network while keeping the branch transmittances as they were and reverse the role of the input and output, the system remains same.

- This principle can be applied to various structures.

Examples:

- For example a direct form structure can be transposed as:





Transpose form

Structures for IIR Systems

- Direct Form Structures (Form I and Form II)

  -Direct implementation of non recursive difference equation or convolution sum

- Cascade Structure

  -Larger system realized as cascade of several small order systems.

- Parallel Structure

  - Larger system realized as parallel connection of several small order systems.

- Lattice and Lattice Ladder Structure

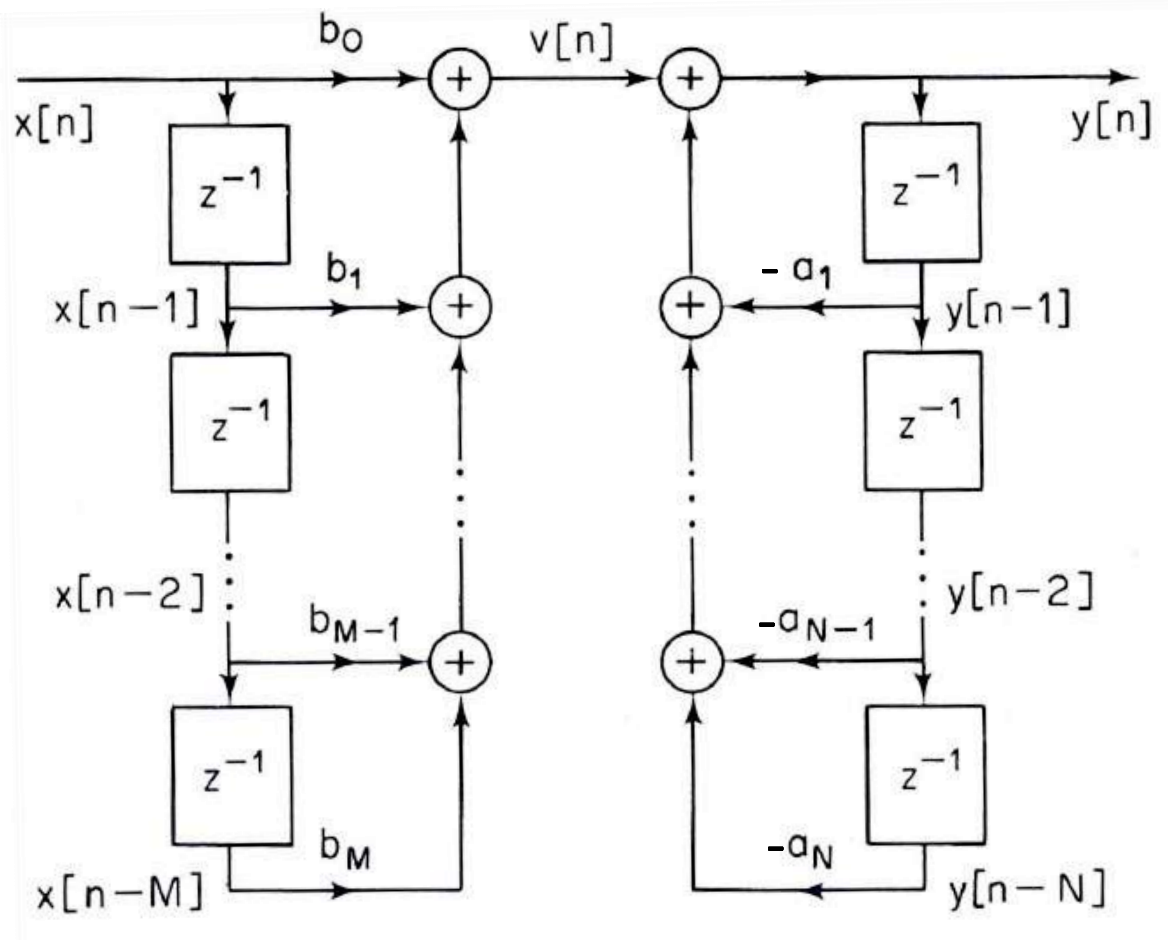
  -Realization of larger order systems by repeated regular lattice stages or lattice and ladder stages.

- Transposed Structures

Direct Form-I Structure

Direct implementation of the difference equation.

$$y[n] = -\sum_{k=1}^{N} a_k y[n-k] + \sum_{k=0}^{M} b_k x[n-k]$$



Direct Form-I Structure

$$y[n] = -\sum_{k=1}^{N} a_k y[n-k] + \sum_{k=0}^{M} b_k x[n-k]$$

$$\boldsymbol{v[n] = \sum_{k=0}^{M} b_k x[n-k]} \qquad H_1(z) = \frac{V(z)}{X(z)} = \sum_{k=0}^{M} b_k z^{-k}$$

$$y[n] = -\sum_{k=0}^{M} a_k y[n-k] + v[n] \qquad H_2(z) = \frac{Y(z)}{V(z)} = \frac{1}{1+\sum_{k=1}^{N} a_k z^{-k}}$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1+\sum_{k=1}^{N} a_k z^{-k}} = H_1(z)H_2(z)$$

So, an all zero system in cascade with an all pole system

Direct Form-II Structure

Above system function can be arranged to:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}}$$

$$H_1(z) = \frac{W(z)}{X(z)} = \frac{1}{1 + \sum_{k=1}^{N} a_k z^{-k}}$$

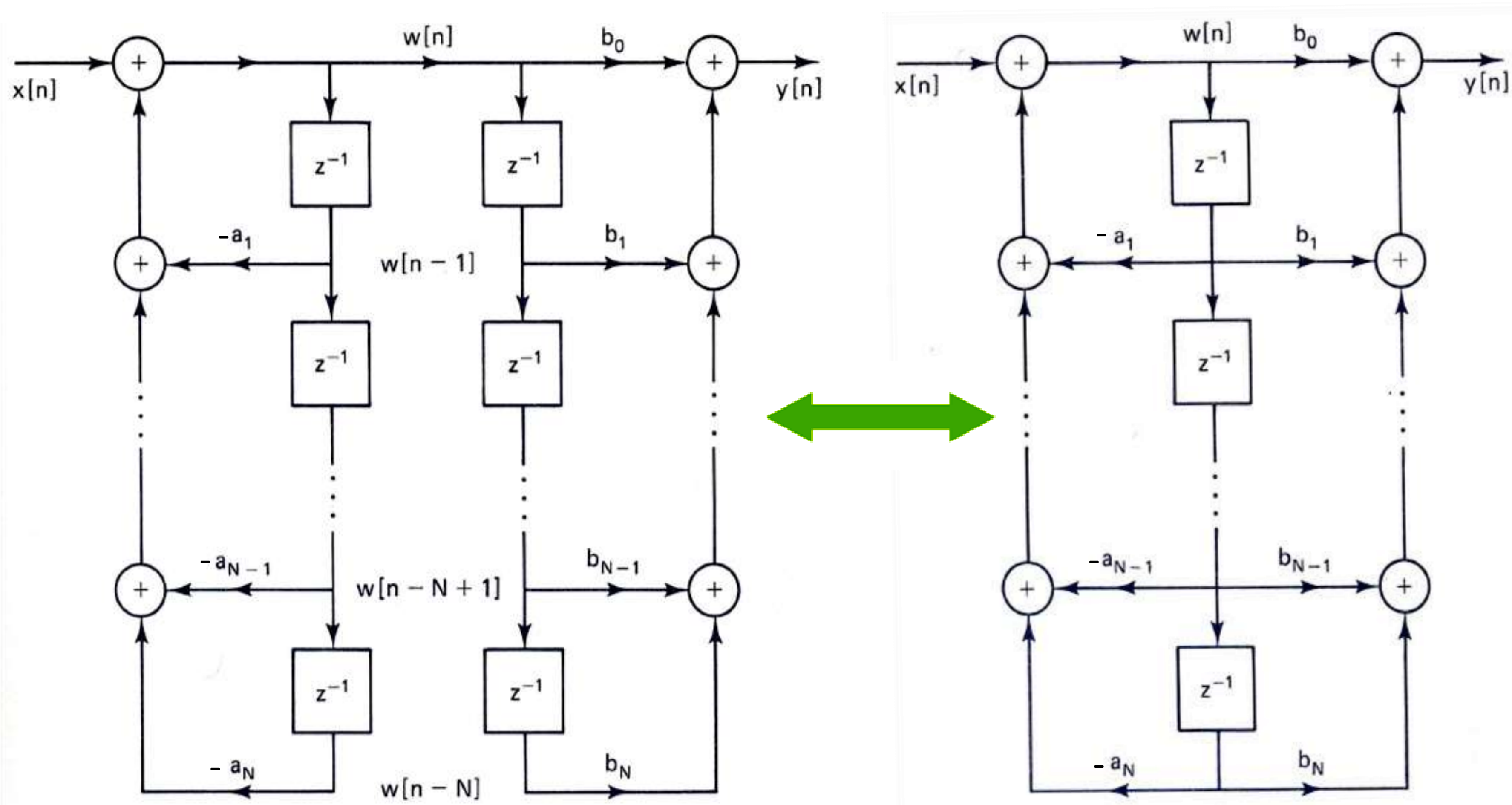
$$\boldsymbol{H_2(z) = \frac{Y(z)}{W(z)} = \sum_{k=0}^{M} b_k z^{-k}}$$

$$H(z) = H_1(z) H_2(z)$$

However, the first system here is an all pole system which is followed by all zero system.

This idea can also be explained considering that order of cascading does not matter in cascade connection of LTI systems.

Direct Form-II Structure



Saves memory elements required.

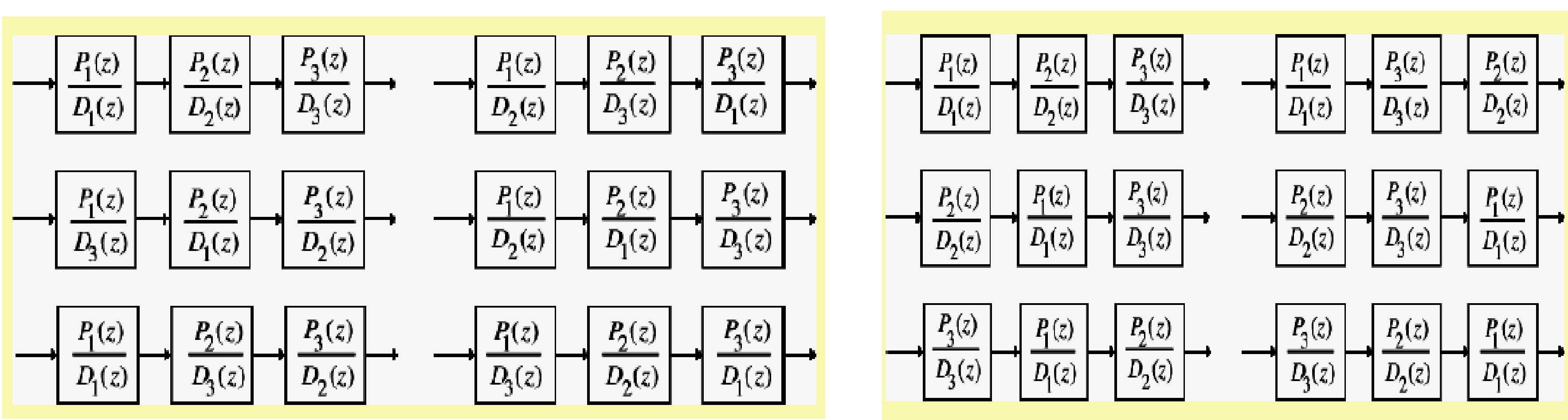
Cascaded Structure

- Realize higher order system as cascade of smaller order sections.

$$H(z) = \prod_{k=1}^{K} H_k(z)$$

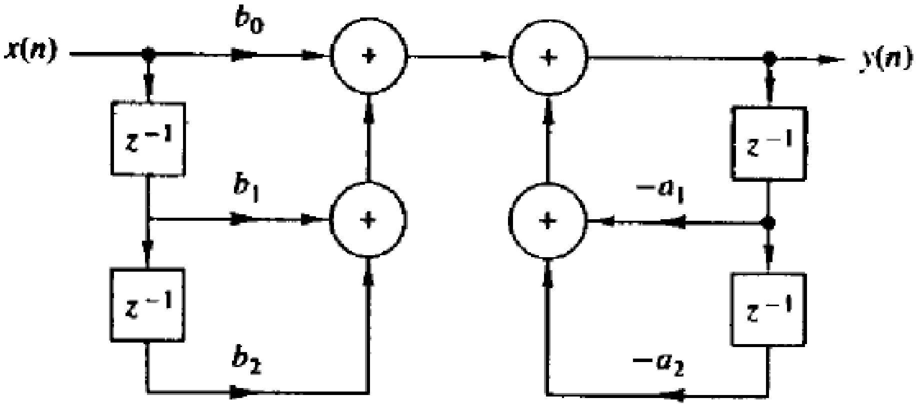
- If $P_k(z)\, and\, D_k(z)$ are first order factors of numerator and denominator polynomial, $H(z) = \dfrac{P(z)}{D(z)} = \dfrac{P_1(z)P_2(z)P_3(z)}{D_1(z)D_2(z)D_3(z)}$.

- The resulting combination for cascade connections are:

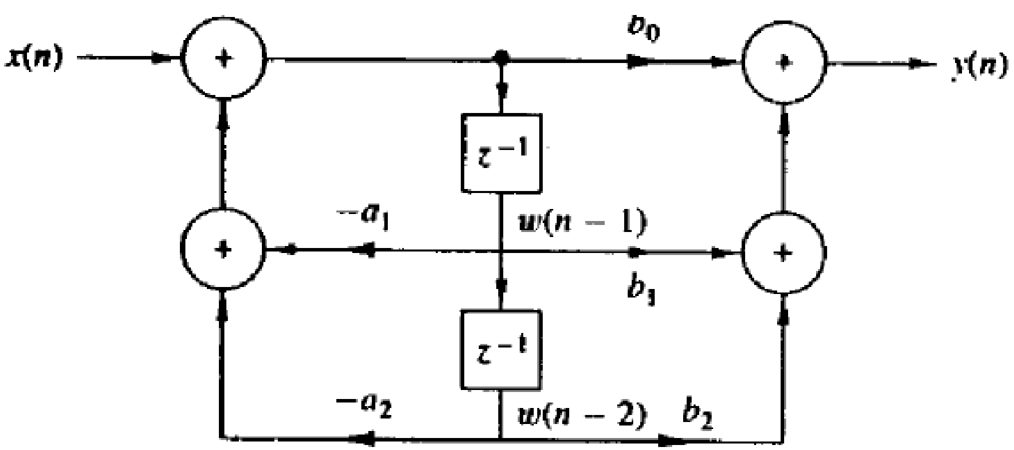


Cascaded Structure

- However, first order sections are not used. Rather second order sections are formed by combining two zeros and poles whenever possible.



$$H_k(z) = \frac{b_{k0} + b_{k1}z^{-1} + b_{k2}z^{-2}}{1 + a_{k1}z^{-1} + a_{k2}z^{-2}}$$

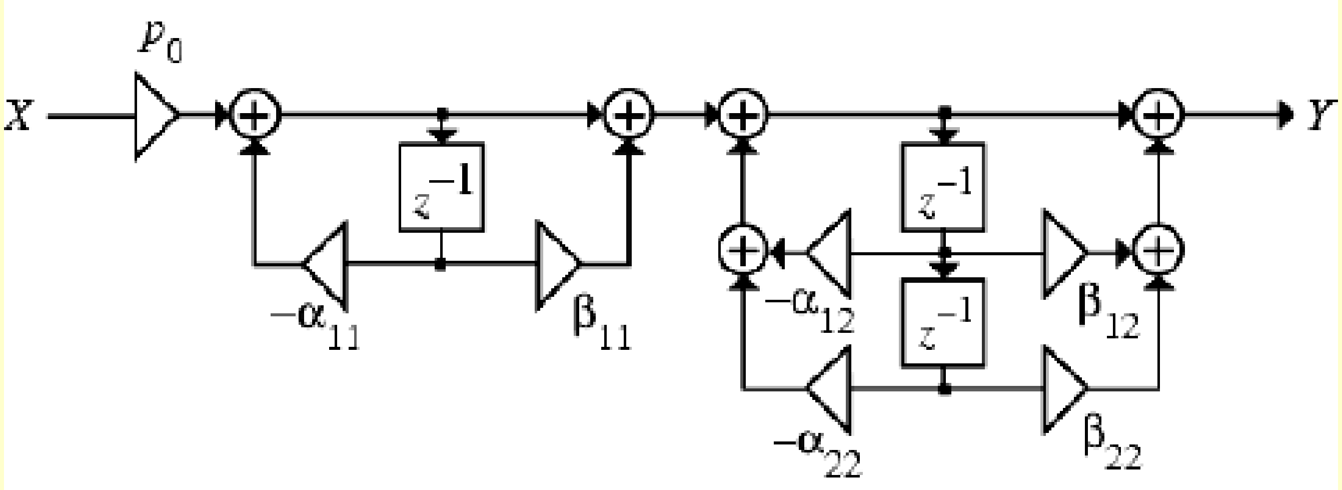


Cascaded Structure

- Consider the 3rd-order transfer function

$$H(z) = p_0\left(\frac{1+\beta_{11}z^{-1}}{1+\alpha_{11}z^{-1}}\right)\left(\frac{1+\beta_{12}z^{-1}+\beta_{22}z^{-2}}{1+\alpha_{12}z^{-1}+\alpha_{22}z^{-2}}\right)$$

One possible realization is shown below

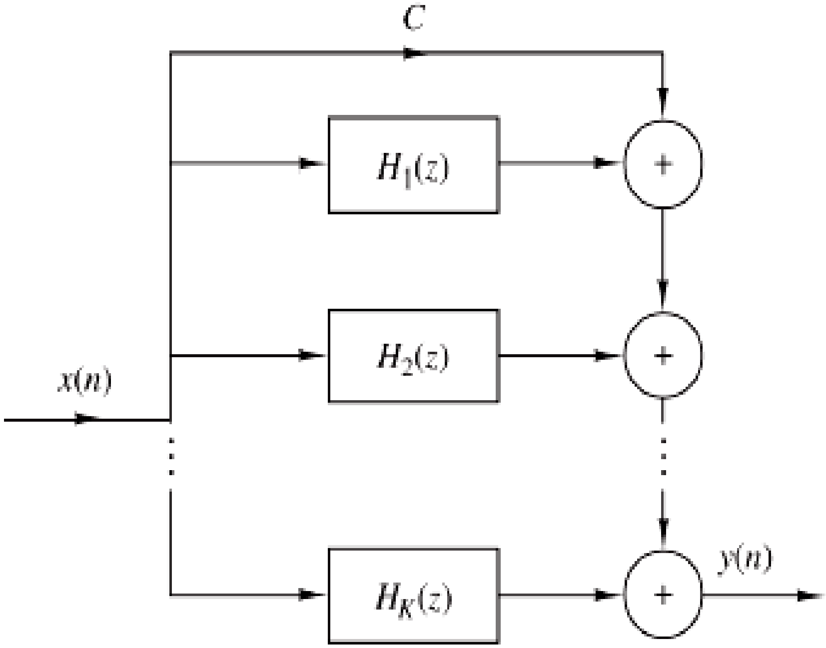


Parallel Form Structure

- Larger order systems are realized as parallel combination of smaller order sections.
- These sections are obtained by partial fraction expansion of H(z).

$$H(z) = C + \sum_{k=1}^{N} \frac{A_k}{1 - P_k z^{-1}}$$

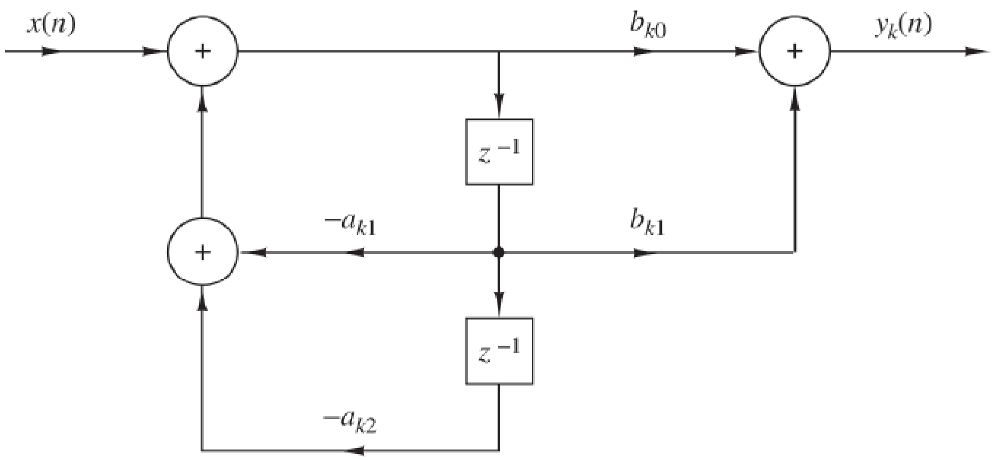
- However, to avoid complex poles, two poles are combined to result second order sections.



Parallel Form Structure

- A standard second order section for such case is

$$H_k(z) = \frac{b_{k0} + b_{k1} z^{-1}}{1 + a_{k1} z^{-1} + a_{k2} z^{-2}}$$



An Example

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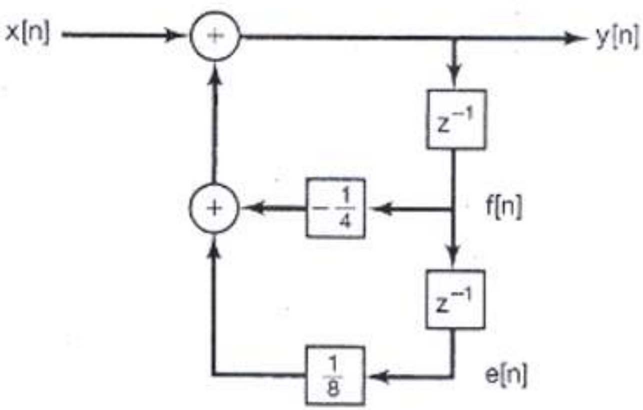
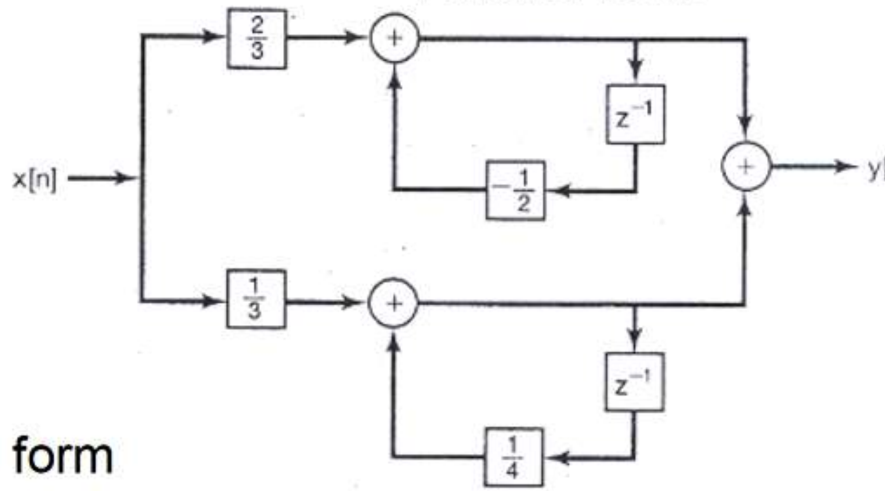
$$H(s)=\frac{1}{1+\frac{1}{4}z^{-1}-\frac{1}{8}z^{-2}}=\left(\frac{1}{1+\frac{1}{2}z^{-1}}\right)\left(\frac{1}{1-\frac{1}{4}z^{-1}}\right)=\frac{\frac{2}{3}}{1+\frac{1}{2}z^{-1}}+\frac{\frac{1}{3}}{1-\frac{1}{4}z^{-1}}$$

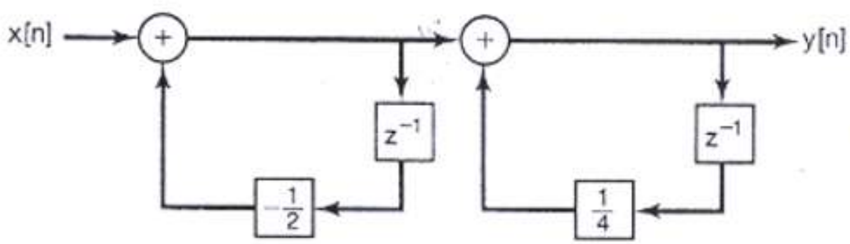
Direct form I and II



Parallel form



Cascade form



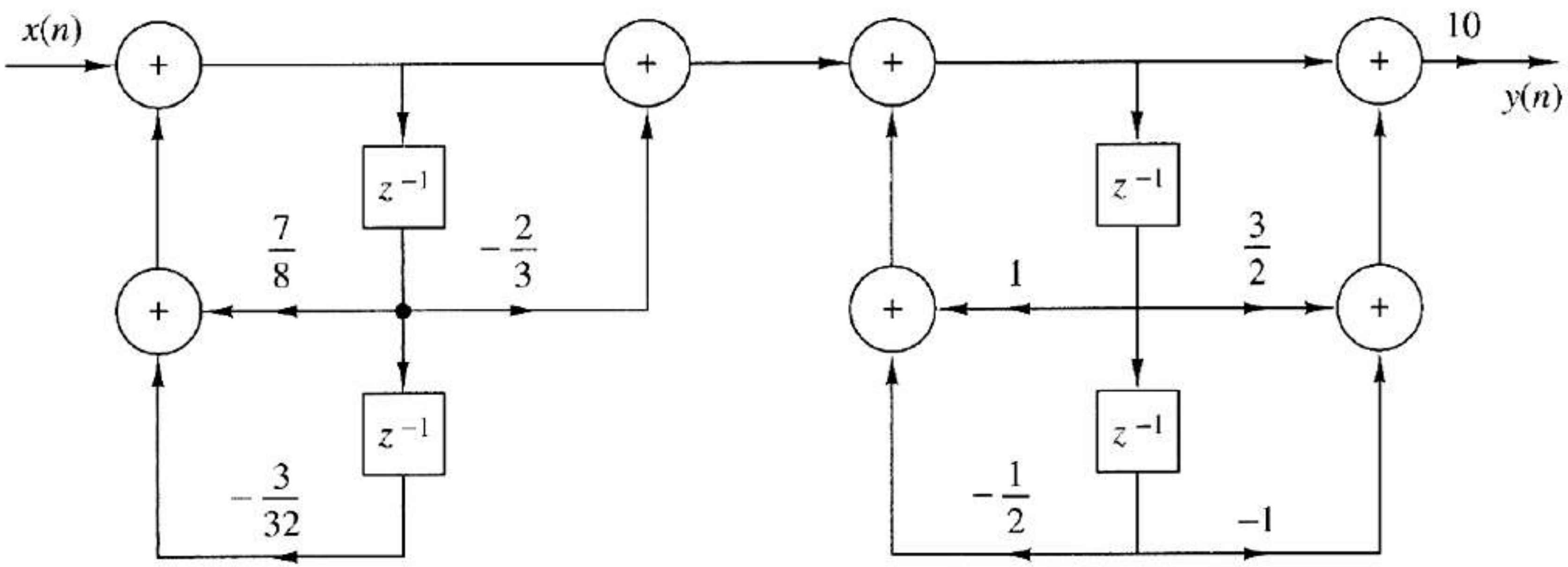
Another example

$$H(z) = \frac{10(1 - \frac{1}{2}z^{-1})(1 - \frac{2}{3}z^{-1})(1 + 2z^{-1})}{(1 - \frac{3}{4}z^{-1})(1 - \frac{1}{8}z^{-1})[1 - (\frac{1}{2} + j\frac{1}{2})z^{-1}][1 - (\frac{1}{2} - j\frac{1}{2})z^{-1}]}$$

- For cascade realization:

$$H_1(z) = \frac{1 - \frac{2}{3}z^{-1}}{1 - \frac{7}{8}z^{-1} + \frac{3}{32}z^{-2}} \qquad H_2(z) = \frac{1 + \frac{3}{2}z^{-1} - z^{-2}}{1 - z^{-1} + \frac{1}{2}z^{-2}}$$

$$H(z) = 10H_1(z)H_2(z)$$

- 

Another Example

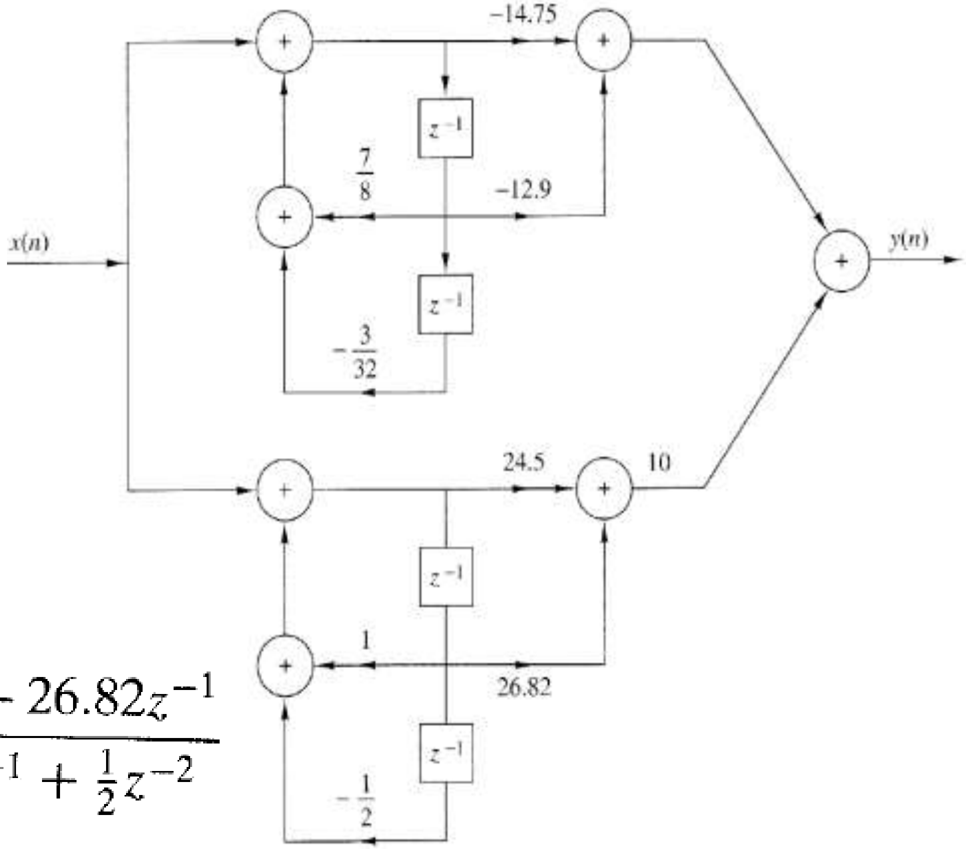
- For parallel realization:

$$H(z) = \frac{A_1}{1-\frac{3}{4}z^{-1}} + \frac{A_2}{1-\frac{1}{8}z^{-1}} + \frac{A_3}{1-(\frac{1}{2}+j\frac{1}{2})z^{-1}} + \frac{A_3^*}{1-(\frac{1}{2}-j\frac{1}{2})z^{-1}}$$

$$A_1 = 2.93, \quad A_2 = -17.68,$$

$$A_3 = 12.25 - j14.57,$$

$$A_3^* = 12.25 + j14.57$$



$$H(z) = \frac{-14.75 - 12.90z^{-1}}{1-\frac{7}{8}z^{-1}+\frac{3}{32}z^{-2}} + \frac{24.50 + 26.82z^{-1}}{1-z^{-1}+\frac{1}{2}z^{-2}}$$

Lattice Structure For All Pole IIR Systems

- System function of all pole IIR system is:

$$H(z)=\frac{1}{A_N(z)}=\frac{1}{1+\sum_{k=1}^{N}a_N(k)z^{-k}}$$

Corresponding difference equation is:

$$y[n]=x[n]-\sum_{k=1}^{N}a_N(k)y[n-k]$$

If the roles of input and output are exchanged,

$$y[n]=x[n]+\sum_{k=1}^{N}a_N(k)x[n-k]$$

Corresponding system function is:

$$H(z)=A_N(z)=1+\sum_{k=1}^{N}a_N(k)z^{-k}$$

Lattice Structure For All Pole IIR Systems

- Thus, structure of all pole IIR system can be obtained by interchanging roles of input and output of an all zero FIR system.
- To obtain an lattice of an all pole IIR filter, we assign following in an lattice for FIR filter

$$x[n] = f_N[n]$$

$$y[n] = f_0[n]$$

- The quantities $f_m[n]$ are then computed in reverse order.

$$f_{m-1}(n) = f_m(n) - K_m g_{m-1}(n-1), \qquad m = N, N-1, \ldots, 1$$

- Expression for $g_m[n]$ remain same.

Lattice Structure For All Pole IIR Systems

- The set of governing equations thus are:

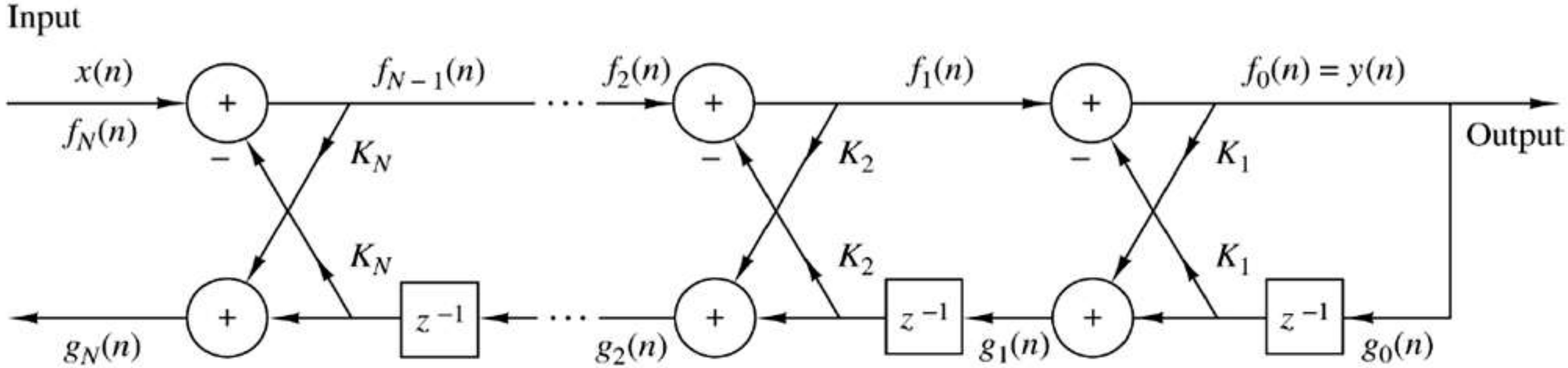
$$f_N(n) = x(n)$$

$$f_{m-1}(n) = f_m(n) - K_m g_{m-1}(n-1), \qquad m = N, N-1, \ldots, 1$$

$$g_m(n) = K_m f_{m-1}(n) + g_{m-1}(n-1), \qquad m = N, N-1, \ldots, 1$$

$$y(n) = f_0(n) = g_0(n)$$

- The resulting structure is



Lattice Structure For all Pole IIR Systems

- All-zero and all-pole lattice structures are characterized by the same set of lattice parameters (K_1, K_2, K_3……).

- The two lattice structures differ only in the interconnections of their signal flow graphs.

- The algorithms for converting between the direct form coefficients and the all-zero lattice coefficients apply exactly for all-pole lattice coefficients also.

- All-pole lattice structure is stable if and only if, $|K_m| < 1$ for all m.

Lattice Ladder Structure For Pole Zero IIR Systems

- For pole-zero IIR systems a modification on all-pole lattice is needed to incorporate the role of zeros.

$$H(z) = \frac{C_M(z)}{A_N(z)} = \frac{\sum_{k=0}^{M} c_M(k) z^{-k}}{1 + \sum_{k=1}^{N} a_N(k) z^{-k}}$$

- If we think this system as an all-pole system (input $x[n]$ output $w[n]$) followed by all-zero system (input $w[n]$ output $y[n]$)

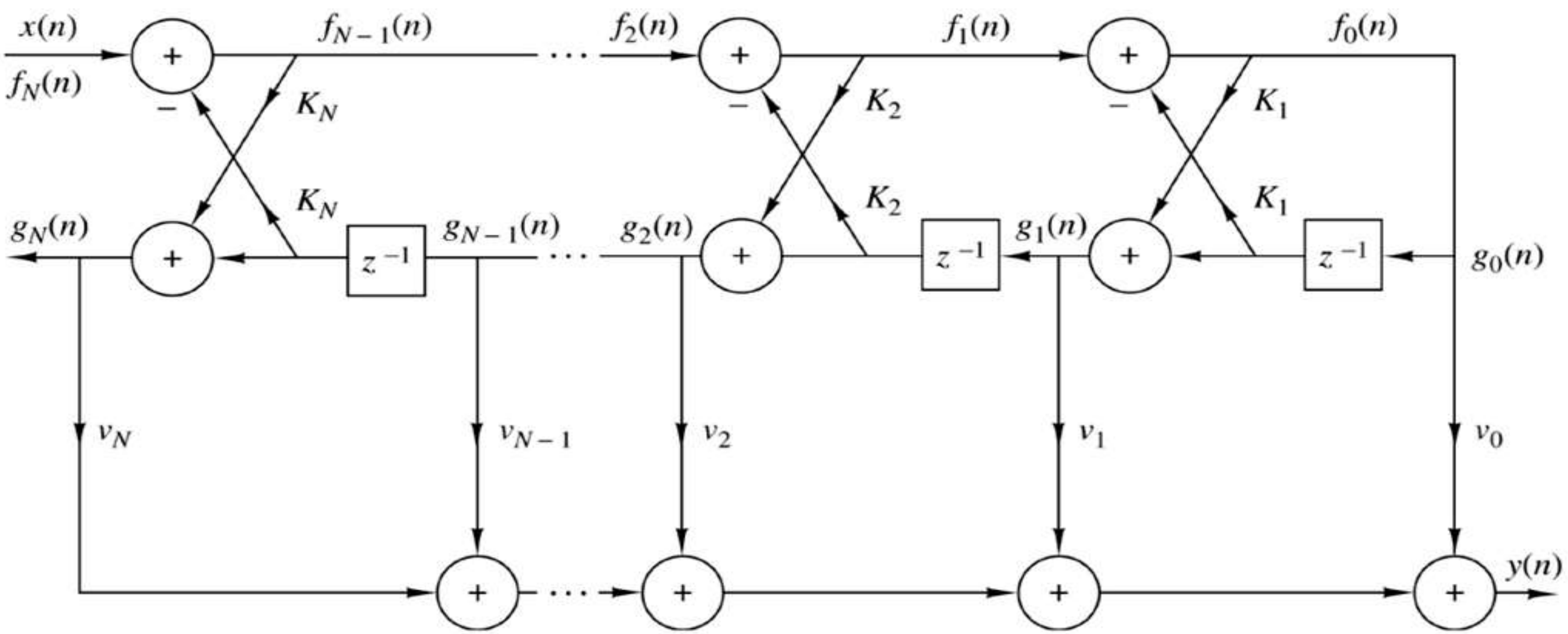
$$w[n] = x[n] - \sum_{k=1}^{N} a_N(k) w[n-k]$$

$$y[n] = \sum_{k=0}^{M} c_M(k) w[n-k]$$

- Clearly, the all-zero system provides the linear combination of all delayed outputs of the all-pole system.

Lattice Ladder Structure For Pole Zero IIR Systems

- The all-pole lattice has a backward path with $g_m[n]$ providing the linear combination of present and past outputs.
- So, addition of a ladder part to all-pole lattice provides the solution.



Lattice Ladder Structure For Pole Zero IIR Systems

- The lattice coefficients are computed using $A_N(z)$ as discussed already.
- The ladder coefficients are computed using $C_M(z)$.
- From the structure before, $y[n]=\sum_{m=0}^{M} v_m g_m[n]$

  In, z domain, $Y(z)=\sum_{m=0}^{M} v_m G_m(z)$

  The system function is thus,

$$H(z)=\frac{Y(z)}{X(z)}=\sum_{m=0}^{M} v_m \frac{G_m(z)}{X(z)}$$

$$\left[\because \quad X(z)=F_N(z) \quad \textit{and} \quad F_0(z)=G_0(z)\right]$$

$$H(z)=\sum_{m=0}^{M} v_m \frac{G_m(z)}{G_0(z)} \frac{F_0(z)}{F_N(z)}$$

Lattice Ladder Structure For Pole Zero IIR Systems

$$H(z) = \sum_{m=0}^{M} v_m \frac{B_m(z)}{A_N(z)} = \frac{\sum_{m=0}^{M} v_m B_m(z)}{A_N(z)}$$

$$\boxed{C_M(z) = \sum_{m=0}^{M} v_m B_m(z)}$$

For any *m*,

$$C_m(z) = \sum_{k=0}^{m-1} v_k B_k(z) + v_m B_m(z)$$

$$C_m(z) = C_{m-1}(z) + v_m B_m(z)$$

The ladder coefficients are then, $v_m = c_m(m) \qquad m = 0, 1, \ldots M$

Also,

$$C_{m-1}(z) = C_m(z) - v_m B_m(z)$$

An Example

- Synthesize the transfer function using lattice-ladder structure

$$H(z) = \frac{1 - z^{-1} + 0.5z^{-2}}{1 + 0.2z^{-1} - 0.15z^{-2}} = \frac{z^2 - z + 0.5}{z^2 + 0.2z - 0.15}$$

- For the lattice coefficients:

$$A_2(z) = 1 + 0.2z^{-1} - 0.15z^{-2} \rightarrow \boxed{K_2 = -0.15}$$

$$B_2(z) = -0.15 + 0.2z^{-1} + z^{-2}$$

$$A_{m-1}(z) = \frac{A_m(z) - K_m B_m(z)}{1 - K_m^2}$$

$$A_1(z) = \frac{1 + 0.2z^{-1} - 0.15z^{-2} - (-0.15)\left(-0.15 + 0.2z^{-1} + z^{-2}\right)}{1 - (-0.15)^2}$$

$$= \frac{0.9775 + 0.23z^{-1}}{0.9775} = 1 + 0.23529z^{-1} \rightarrow \boxed{K_1 = 0.23529}$$

$$B_1(z) = 0.23529 + z^{-1}$$

An Example

- For the ladder coefficients:

$$C_2(z) = 1 - z^{-1} + 0.5z^{-2} \rightarrow \boxed{v_2 = c_2(2) = 0.5}$$

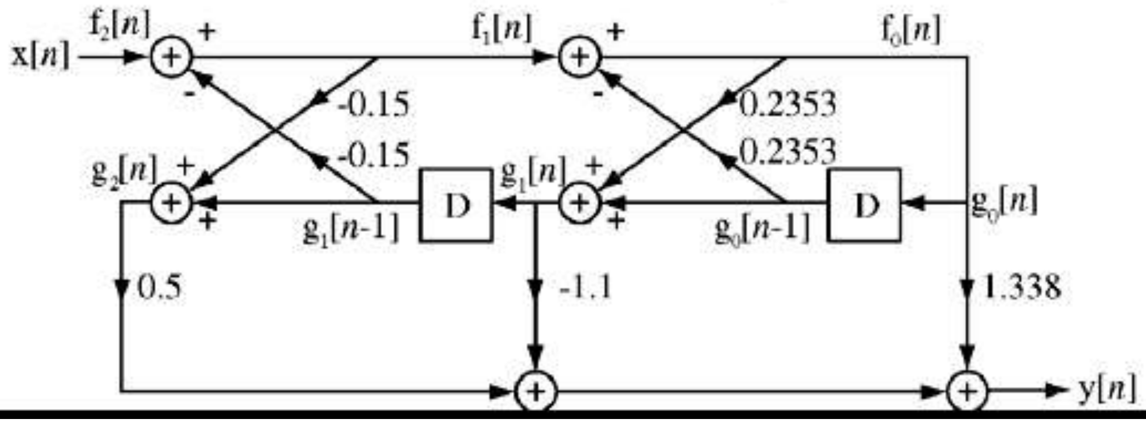
$$C_{m-1}(z) = C_m(z) - v_m B_m(z)$$

$$C_1(z) = C_2(z) - v_2 B_2(z) = 1 - z^{-1} + 0.5z^{-2} - 0.5\{-0.15 + 0.2z^{-1} + z^{-2}\}$$

$$= 1.075 - 1.1z^{-1} \rightarrow \boxed{v_1 = c_1(1) = -1.1}$$

$$C_0(z) = C_1(z) - v_1 B_1(z) = 1.075 - 1.1z^{-1} + 1.1\{0.23529 + z^{-1}\}$$

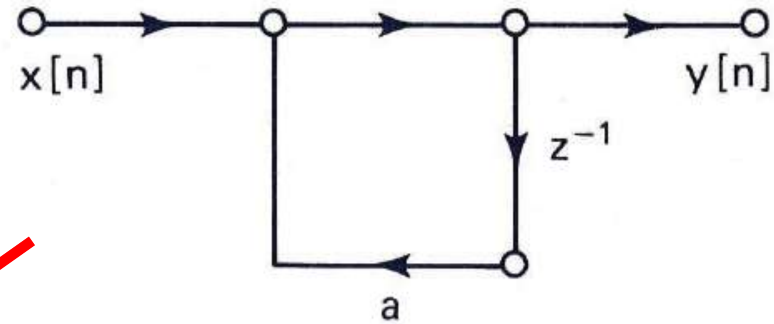
$$= 1.333819 \rightarrow \boxed{v_0 = c_0(0) = 1.333819}$$



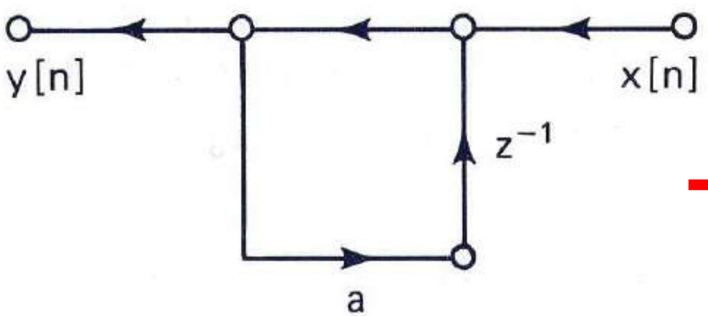
Transposed Structures

- Transposed structures are obtained by transposing the structures based on transposition theorem of signal flow graph.
-

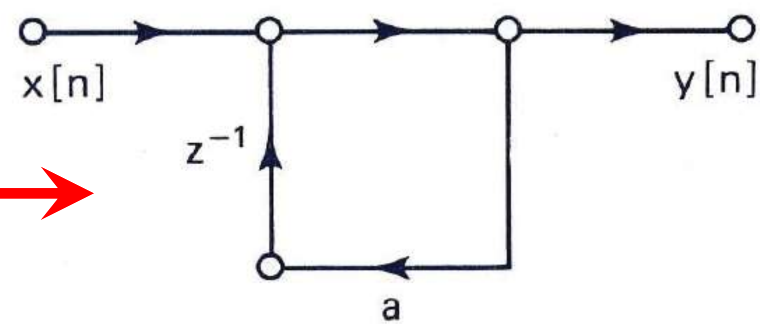
$$H(z) = \frac{1}{1 - az^{-1}}$$



Direct Form



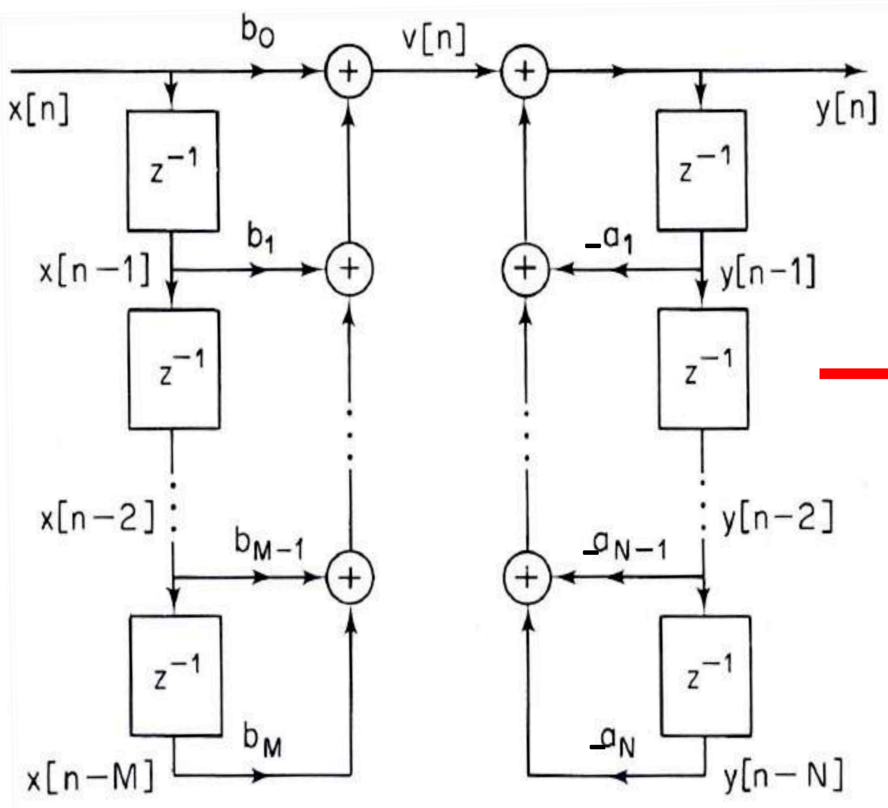
Performing the Transposition



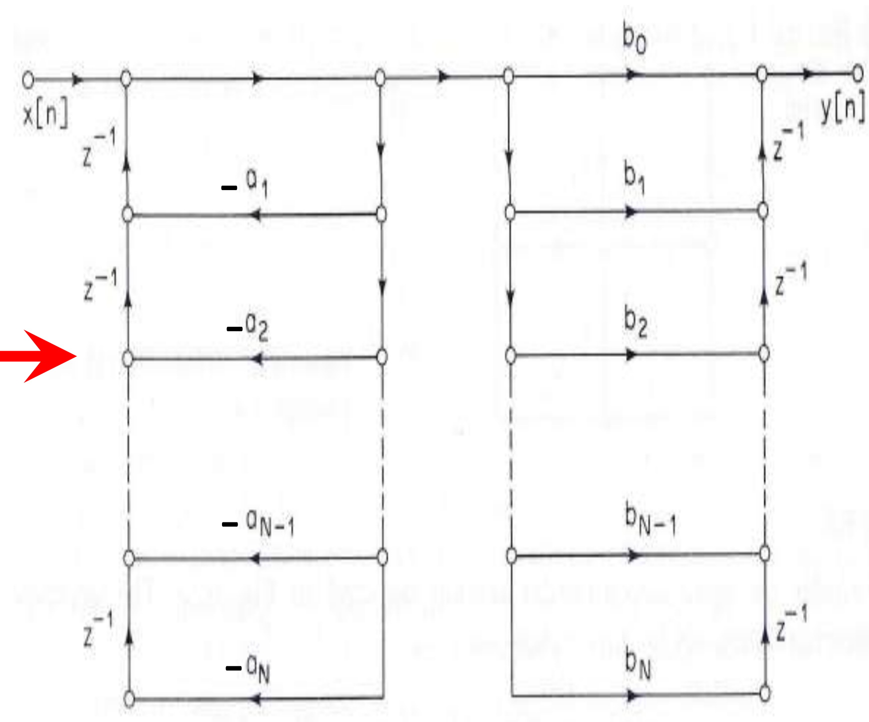
Transposed Direct Form

Transposed Structures

- Similarly for a general direct form I structure,



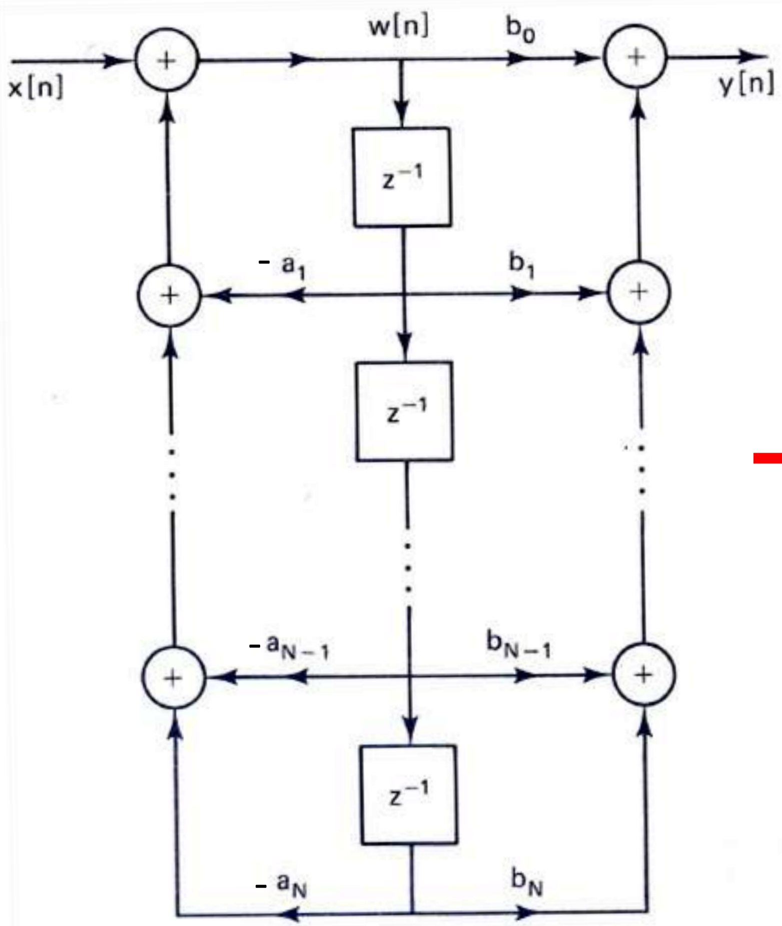
Direct Form I



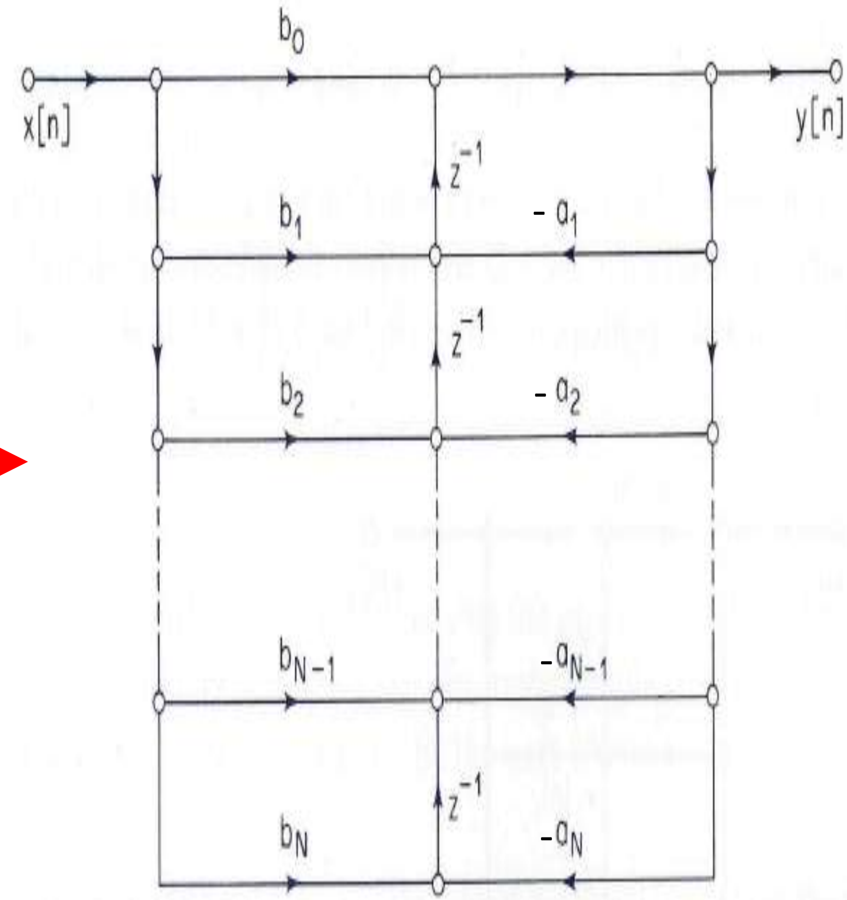
Transposed Direct Form I

Transposed Structures

- Similarly for a direct form II structure,



Direct Form II



Transposed Direct Form II