Digital Signal Analysis and Processing

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FIR Filter Design

Digital Systems and Filters

FIR and IIR Systems and their choice

Windows method for FIR Filter Design

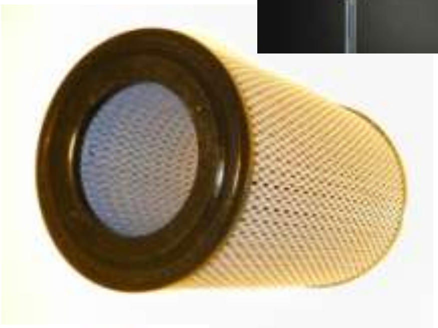
Frequency Sampling Method

Optimum Equiripple FIR Filter Design

Filters

- Any system that modifies certain frequencies relative to others is called a filter.

- Filter is a system that passes certain frequency components and totally rejects all others.



- Frequency selective filters

- Frequency shaping filters

- Applications:

Filters

Examples of filtering operations

Noise suppression

- received radio signals

- signals received by image sensors (TV, infrared imaging devices)

- electrical signals measured from human body (brain heart, neurological signals)

- signals recorded on analog media such as analog magnetic tapes

Filters

Examples of filtering operations

Enhancement of selected frequency ranges

- treble and bass control or graphic equalizers

increase sound level and high and low level frequencies to compensate for the lower sensitivity of the ear at those frequencies or for special sound effects

- enhancement of edges in images

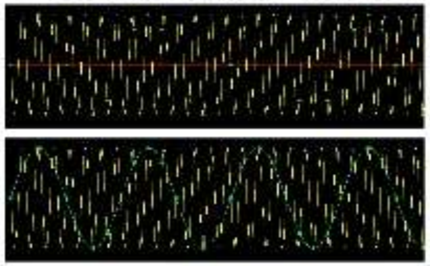
improve recognition of object (by human or computer)

edge – a sharp transition in the image brightness, sharp transitions in a signal (from Fourier theory) appear as high-frequency components which can be amplified

Filters

Examples of filtering operations

Bandwidth limiting



- means of aliasing prevention in sampling

- communication

  radio or TV signal transmitted over specific channel has to have a limited bandwidth to prevent interference with neighbouring channels

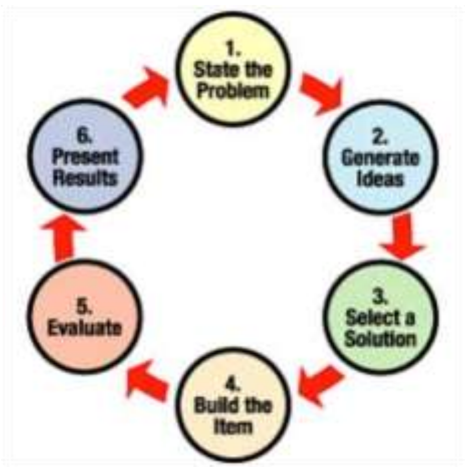
  frequency components outside the permitted band are attenuated below a specific power level

Filter Design

- Specification: desired properties of the system

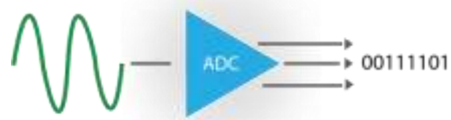
- Approximation: choosing appropriate type and determining the system function to meet specification

- Realization: realize the system structure with appropriate technology (hardware) or algorithm (software)

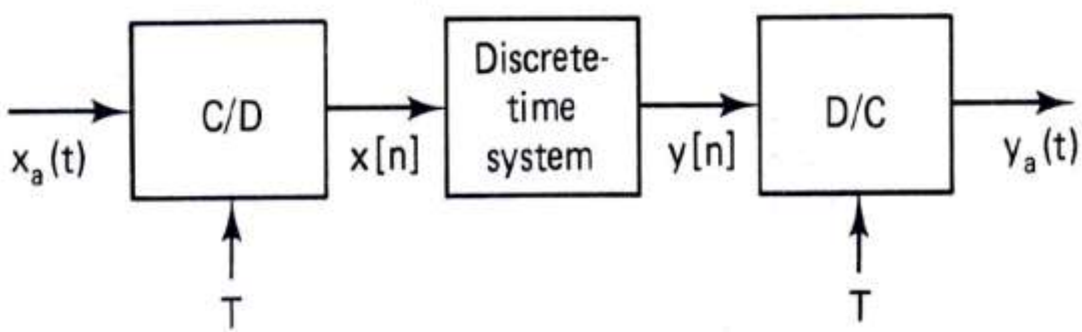


Analog Vs Digital Specification

- Usually signals to be processed by DT filters originate as analog signals.



- So, specifications are also provided in analog frequency domain as effective CT frequency response $H_{eff}(j\omega)$

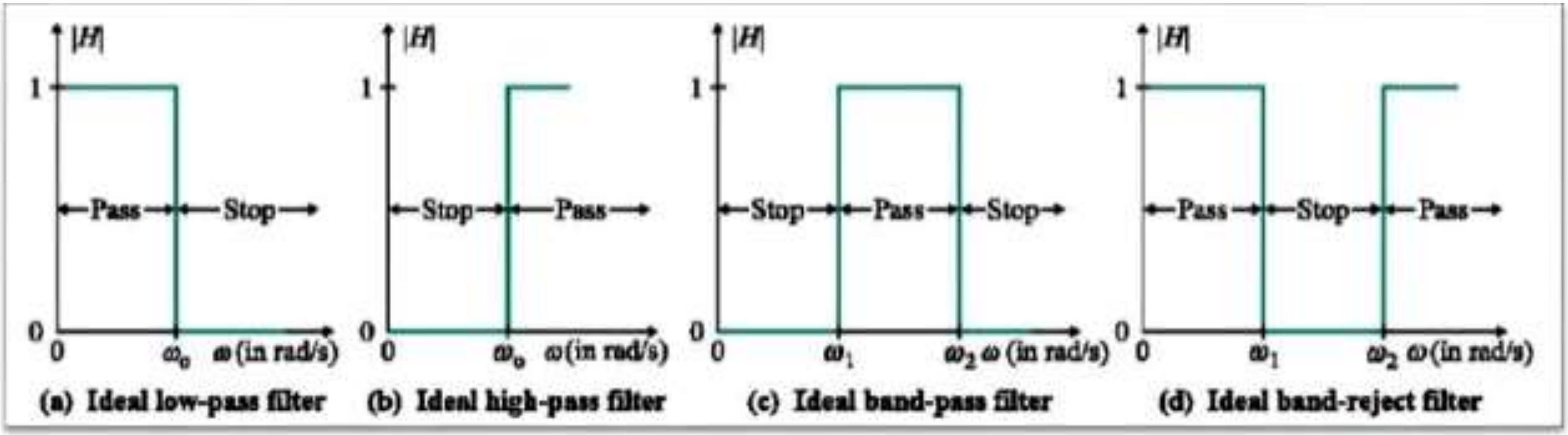


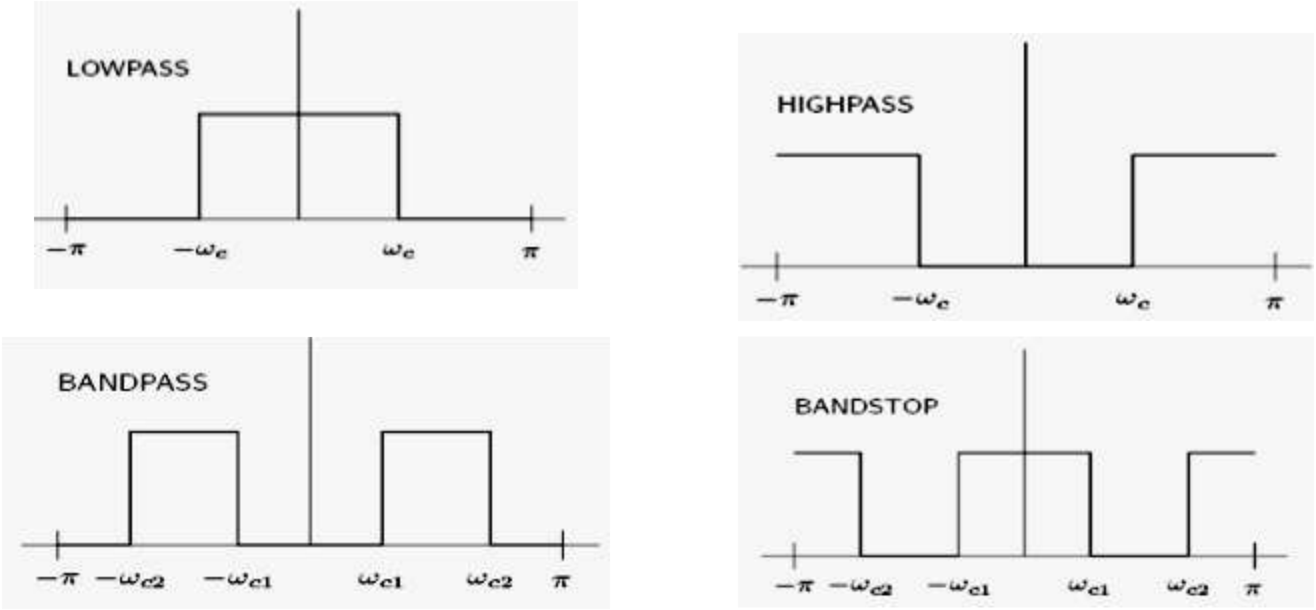
- Specification for digital frequency domain can be obtained from analog specification as:

$$H(e^{j\Omega}) = H_{eff}(j\frac{\Omega}{T})$$

where T is sampling period

Ideal and Practical Filters





Ideal and Practical Filters

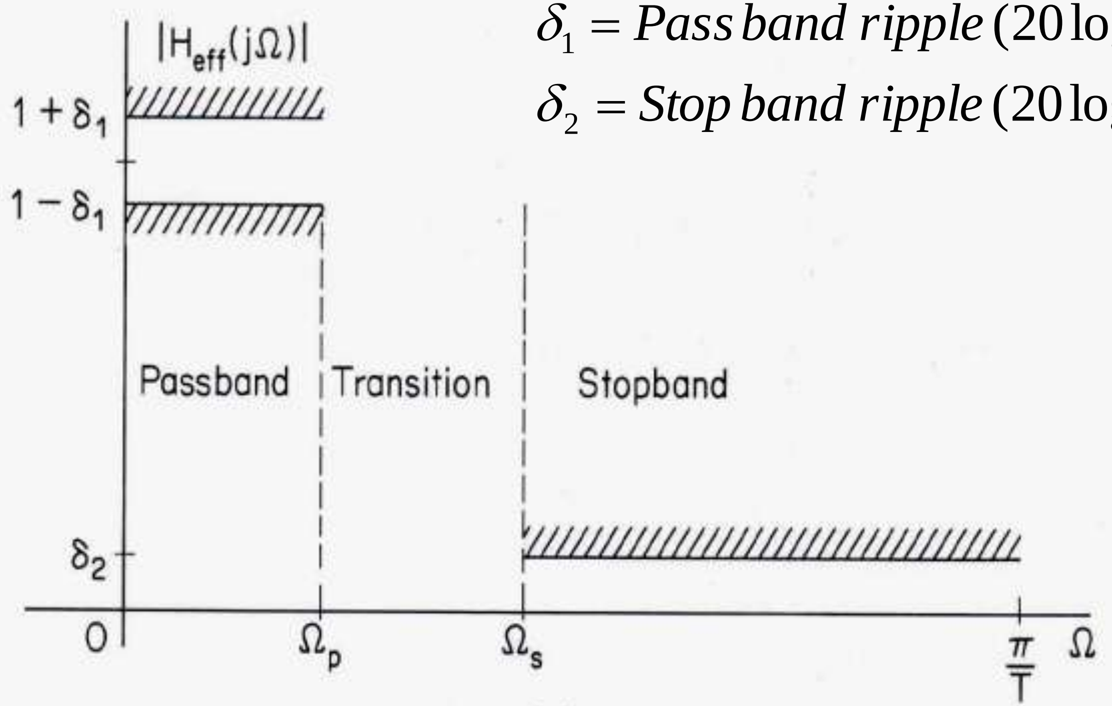
- Non zero gain in stop band
- Small variation in gain in pass band
- Transition band between pass band and stop band

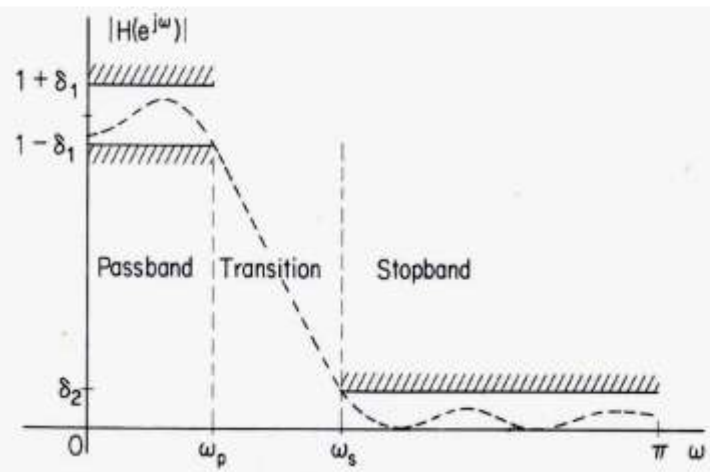
$$\Omega_p = Pass\ band\ edge\ frequency$$

$$\Omega_s = Stop\ band\ edge\ frequency$$

$$\delta_1 = Pass\ band\ ripple\ (20\log_{10}\delta_1\ in\ d\mathrm{B})$$

$$\delta_2 = Stop\ band\ ripple\ (20\log_{10}\delta_2\ in\ d\mathrm{B})$$





FIR and IIR Filters

- FIR filters with exactly linear phase can be designed.

- Several computationally efficient recursive and non recursive realization are available for FIR systems

- Non recursively realized FIR filters are inherently stable and free of limit cycle oscillations.

- Sensitivity to variations in filter coefficient is relatively low.

- For same specifications (particularly the transition bandwidth), FIR filters need higher order leading to more arithmetic computations, hardware components like multipliers, adders and delays.

Linear Phase FIR Filter

- System function of an FIR filter of length N (order N-1) is:

$$H(z) = \sum_{n=0}^{N-1} h[n] z^{-n}$$

- The corresponding frequency response hence is:

$$H(e^{j\Omega}) = \sum_{n=0}^{N-1} h[n] e^{-j\Omega n}$$

- The frequency response can be written as:

$$H(e^{j\Omega}) = H(\Omega) e^{j\phi(\Omega)}$$

where, $H(\Omega)$ is zero-phase frequency response (real valued frequency response) and $\phi(\Omega)$ is the phase term.

- A filter is linear phase if phase term has the form:

$$\phi(\Omega) = \alpha\Omega + \beta$$

Types of Linear Phase FIR Filter

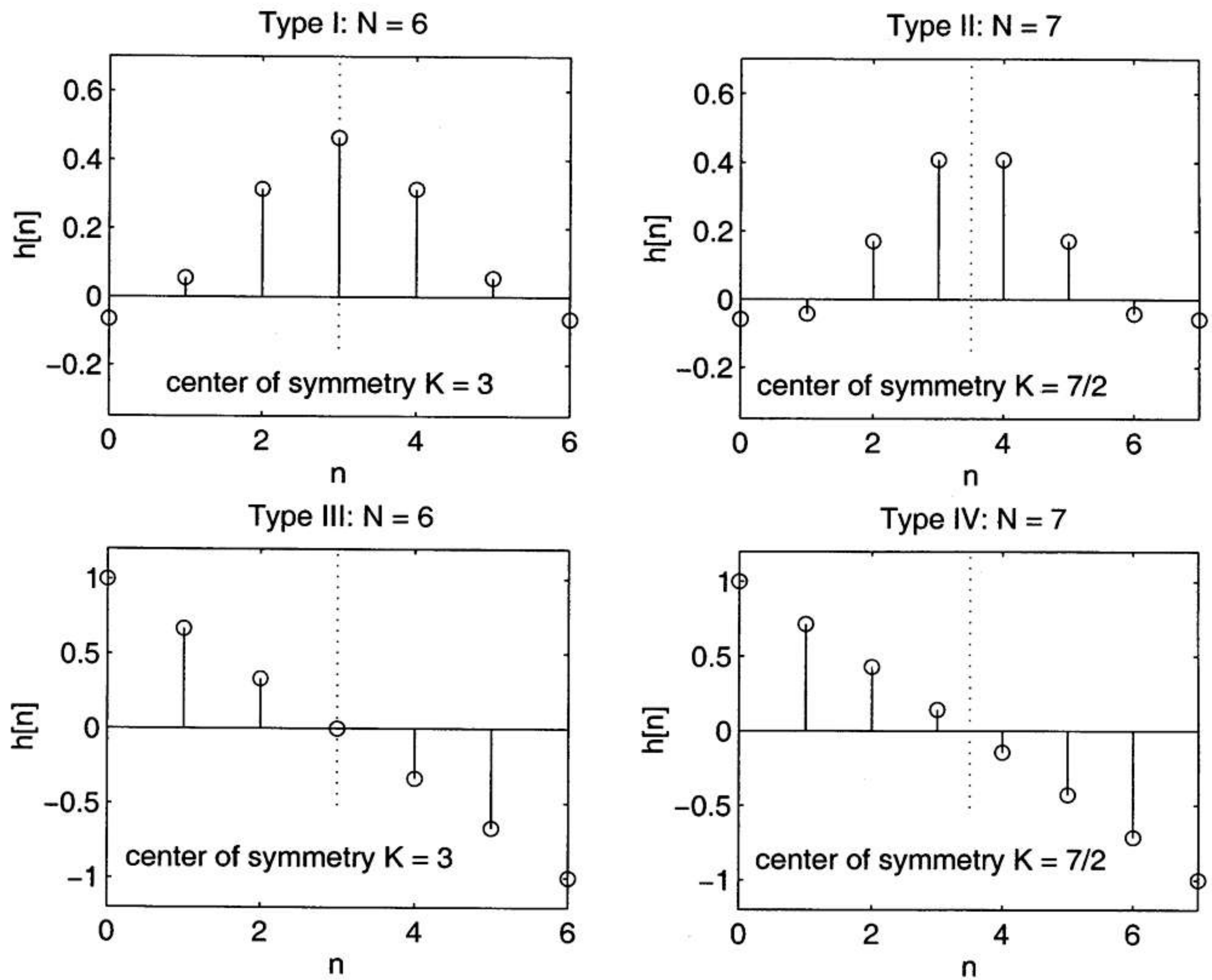
- A FIR filter will have linear phase if its impulse response is symmetric or anti symmetric.

$$h[N-1-n] = \pm h[n] \quad for \quad all \quad n$$

- So, four types of linear phase FIR filters are possible:

  - Type I: N odd and $h[n]$ symmetrical
  - Type II: N even and $h[n]$ symmetrical
  - Type III: N odd and $h[n]$ anti symmetrical
  - Type IV: N even and $h[n]$ anti symmetrical

Types of Linear Phase FIR Filter



FIR Filter Design Using Windows

FIR Filter Design Using Windows

- Let us consider the desired frequency response of the digital filter be $H_d(e^{j\Omega})$.

- The impulse response of the required filter is then $h_d[n]$ given as:

$$h_d[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d\left(e^{j\Omega}\right)e^{j\Omega n}\,d\Omega$$

The impulse response thus obtained however may not be of finite duration and hence does not represent a FIR system.

- An approximate FIR filter can be obtained by truncating the impulse.

- There are several ways for truncation.

FIR Filter Design Using Windows

- The simplest way to truncate is to abruptly truncate. Such that,

$$h[n] = \begin{cases} h_d[n], & for \ 0 \le n \le M-1 \\ 0, & otherwise \end{cases}$$

- Such truncation is equivalent to multiplying the infinitely long impulse response by a rectangular window,

$$h[n] = h_d[n]w[n] \qquad where, \ w[n] = \begin{cases} 1, & for \ 0 \le n \le M-1 \\ 0, & otherwise \end{cases}$$

- Such window functions are called rectangular windows.
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FIR Filter Design Using Windows

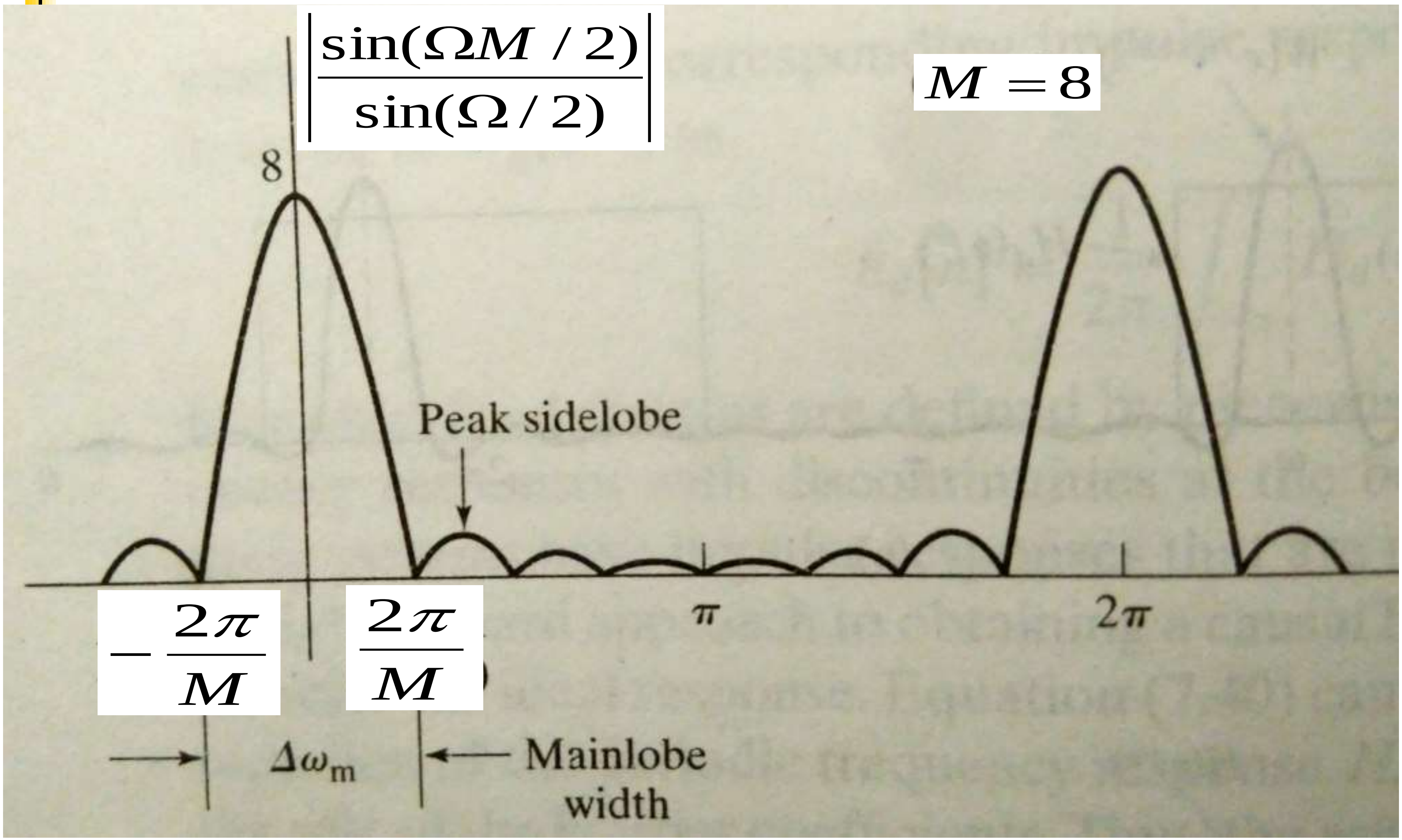
- Frequency response of rectangular windows is:

$$W(e^{j\Omega}) = \sum_{n=0}^{M-1} w[n]e^{-j\Omega n} = \sum_{n=0}^{M-1} e^{-j\Omega n} = \frac{1-e^{j\Omega M}}{1-e^{j\Omega}} = e^{-j\Omega(M-1)/2}\,\frac{\sin(\Omega M/2)}{\sin(\Omega/2)}$$

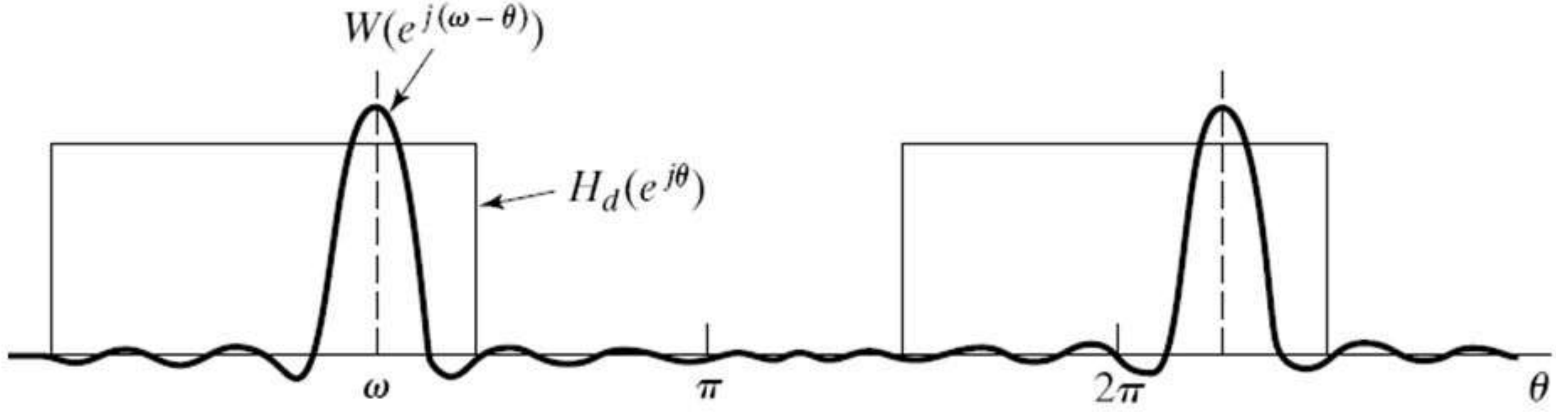
- Multiplication in time domain is equivalent to convolution in frequency domain. So, frequency response of the FIR filter designed will be same to convolution of desired frequency response and frequency response of the windows.

- $$H(e^{j\Omega}) = \frac{1}{2}\int_{-\pi}^{\pi} H_d(\theta)W(e^{j(\Omega-\theta)})d\theta$$

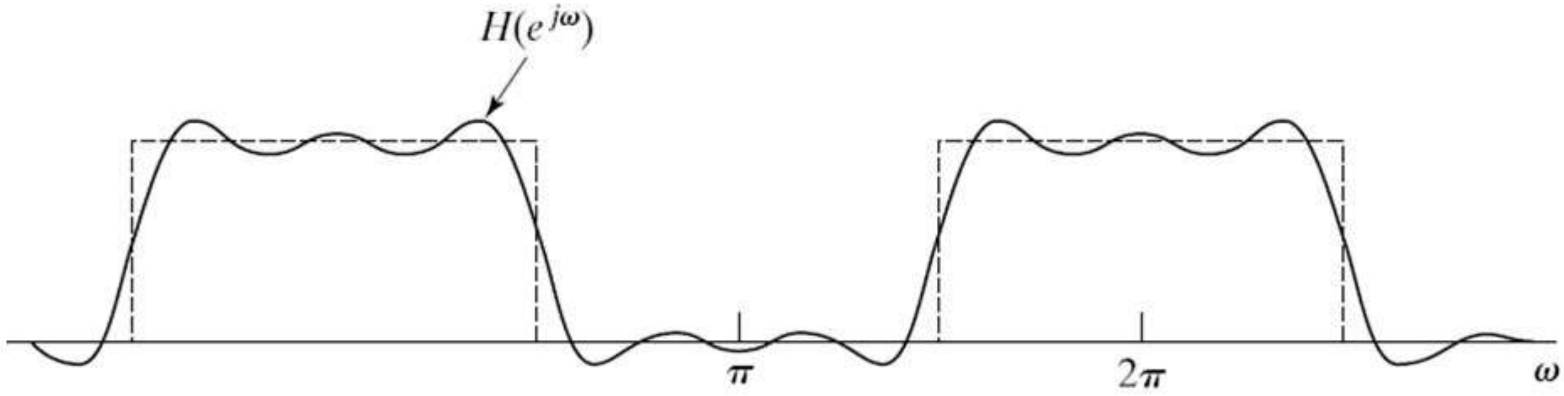
Definition of Some Popular Windows



FIR Filter Design Using Windows



Convolution process implied by truncation of the ideal impulse response (i.e., rectangular window)



Typical approximation resulting from windowing the ideal impulse response (rectangular-window case)

Problem with Rectangular Windows

- As the length of window is increased, width of main lobe in frequency response is decreased. However the sidelobes are unaffected.

- Large side lobes in $W(\Omega)$ cause undesirable ringing in the filter designed. This also causes high side lobes in filter designed.

- Other windows with less abrupt discontinuity in time domain and hence lower side lobes in frequency response reduce the problem of Gibb's phenomenon.

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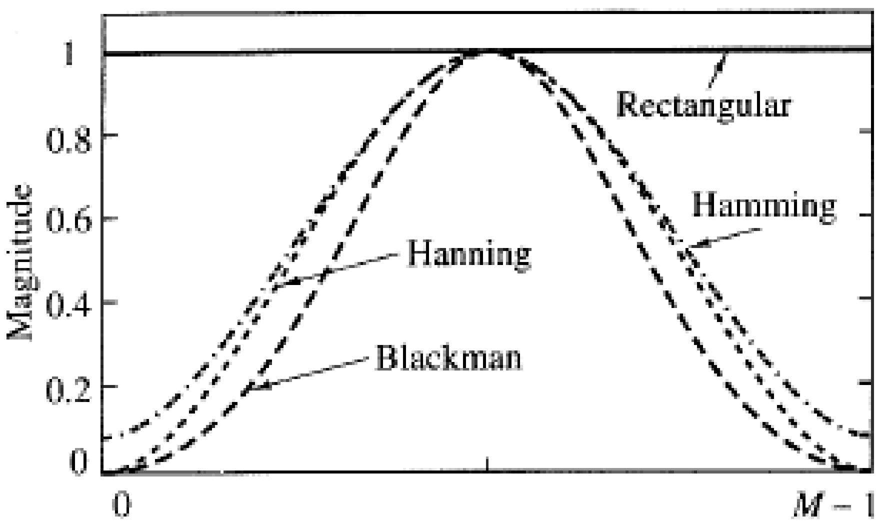
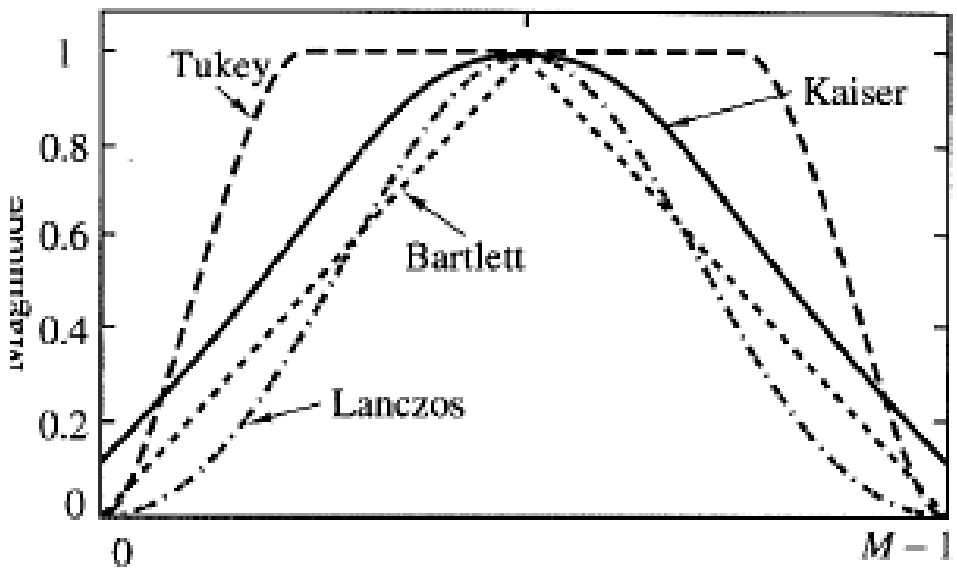
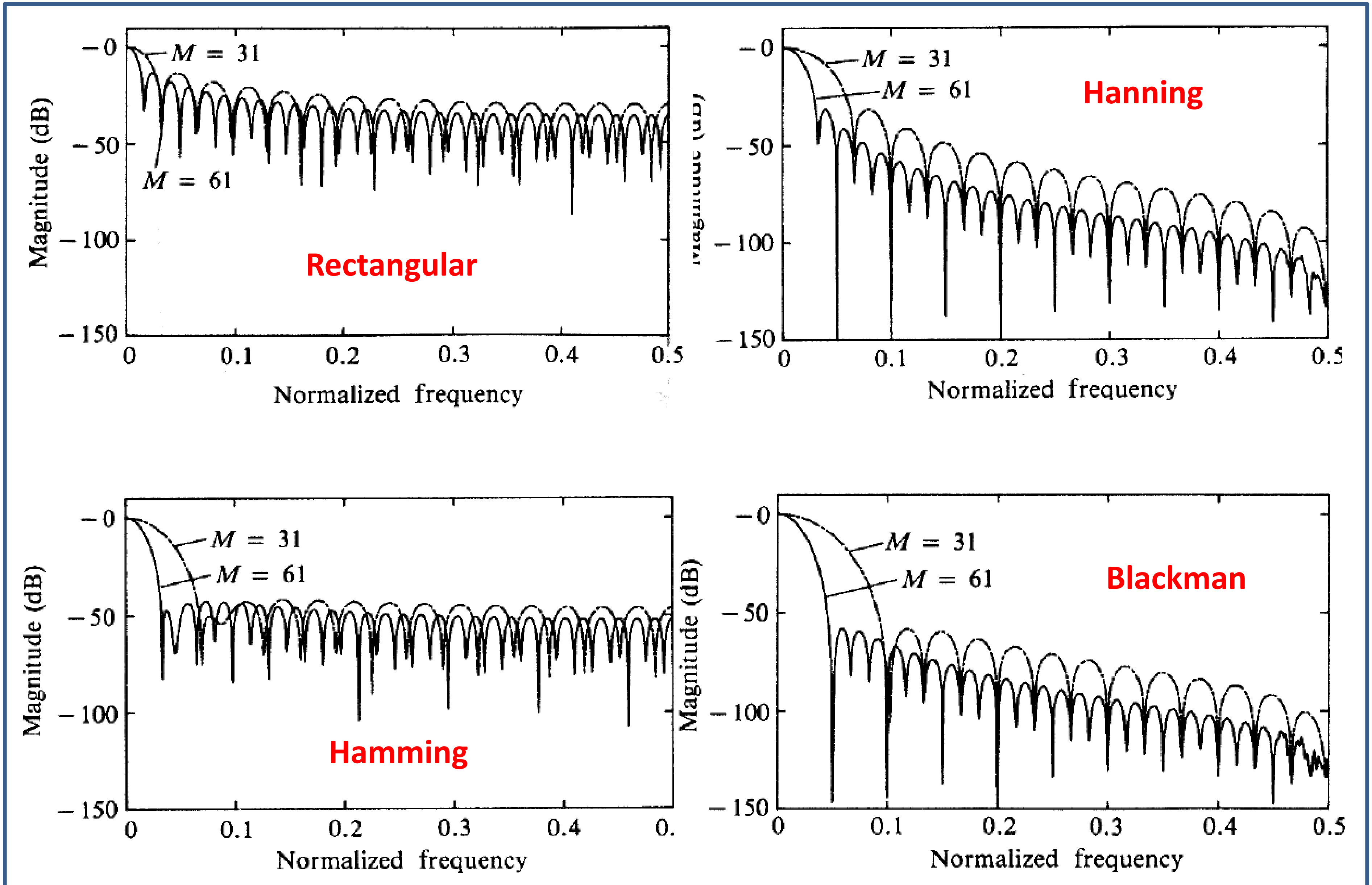
Some Popular Windows



Bartlett (triangular)	$1-\dfrac{2\left\lvert n-\dfrac{M-1}{2}\right\rvert}{M-1}$
Blackman	$0.42-0.5\cos\dfrac{2\pi n}{M-1}+0.08\cos\dfrac{4\pi n}{M-1}$
Hamming	$0.54-0.46\cos\dfrac{2\pi n}{M-1}$
Hanning	$\dfrac{1}{2}\left(1-\cos\dfrac{2\pi n}{M-1}\right)$
Kaiser	$\dfrac{I_0\left[\alpha\sqrt{\left(\dfrac{M-1}{2}\right)^2-\left(n-\dfrac{M-1}{2}\right)^2}\right]}{I_0\left[\alpha\left(\dfrac{M-1}{2}\right)\right]}$
Lanczos	$\left\{\dfrac{\sin\left[2\pi\left(n-\dfrac{M-1}{2}\right)\Big/(M-1)\right]}{2\pi\left(n-\dfrac{M-1}{2}\right)\Big/\left(\dfrac{M-1}{2}\right)}\right\}^L, \quad L>0$



Frequency response of Windows

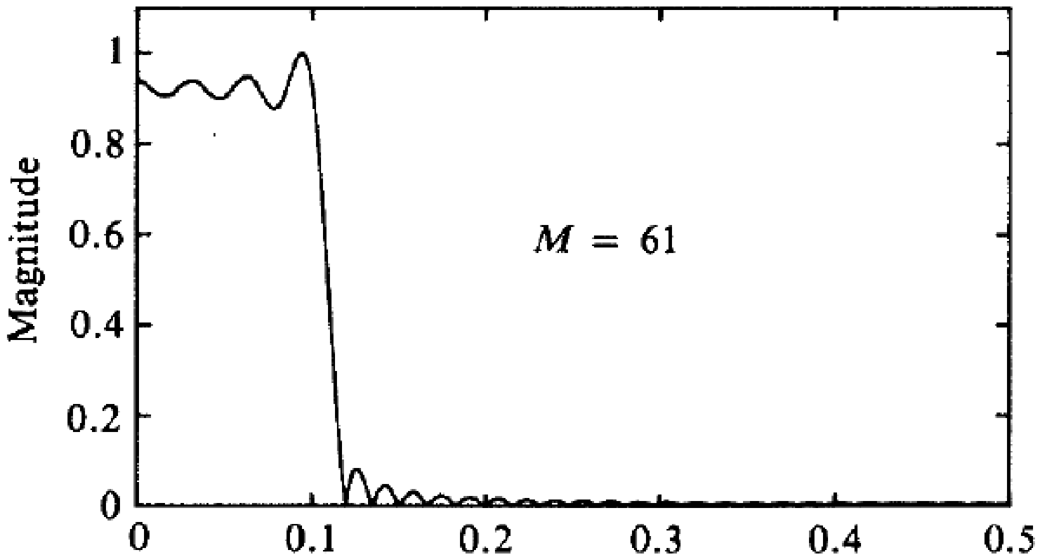


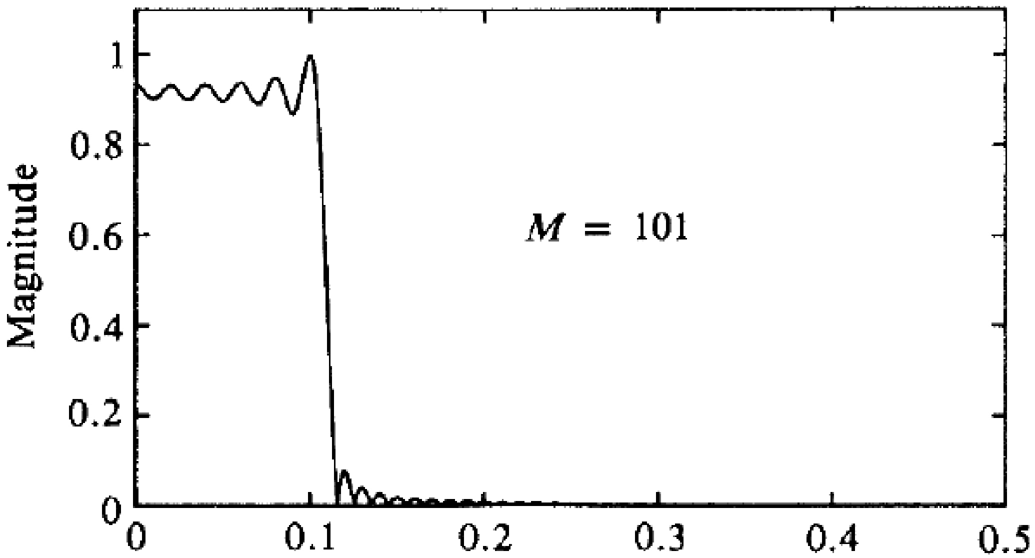
Properties of Some Windows

Name	Peak side Lobe (dB)	Width of main lobe (radian frequency)	Stopband Attenuation (dB) $20log_{10}\left(\frac{1}{\delta_s}\right)$	Passband Ripple(dB) $20log_{10}\left(1+\delta_p\right)$
Rectangular	-13	$4\pi/M$	21	0.7416
Bartlett	-25	$8\pi/M$	25	0.4752
Hanning Hamming	-31 -41	$8\pi/M$ $12\pi/M$	44 53	0.0546 0.0194
Blackmann	-57	$4\pi/M$	74	0.0017

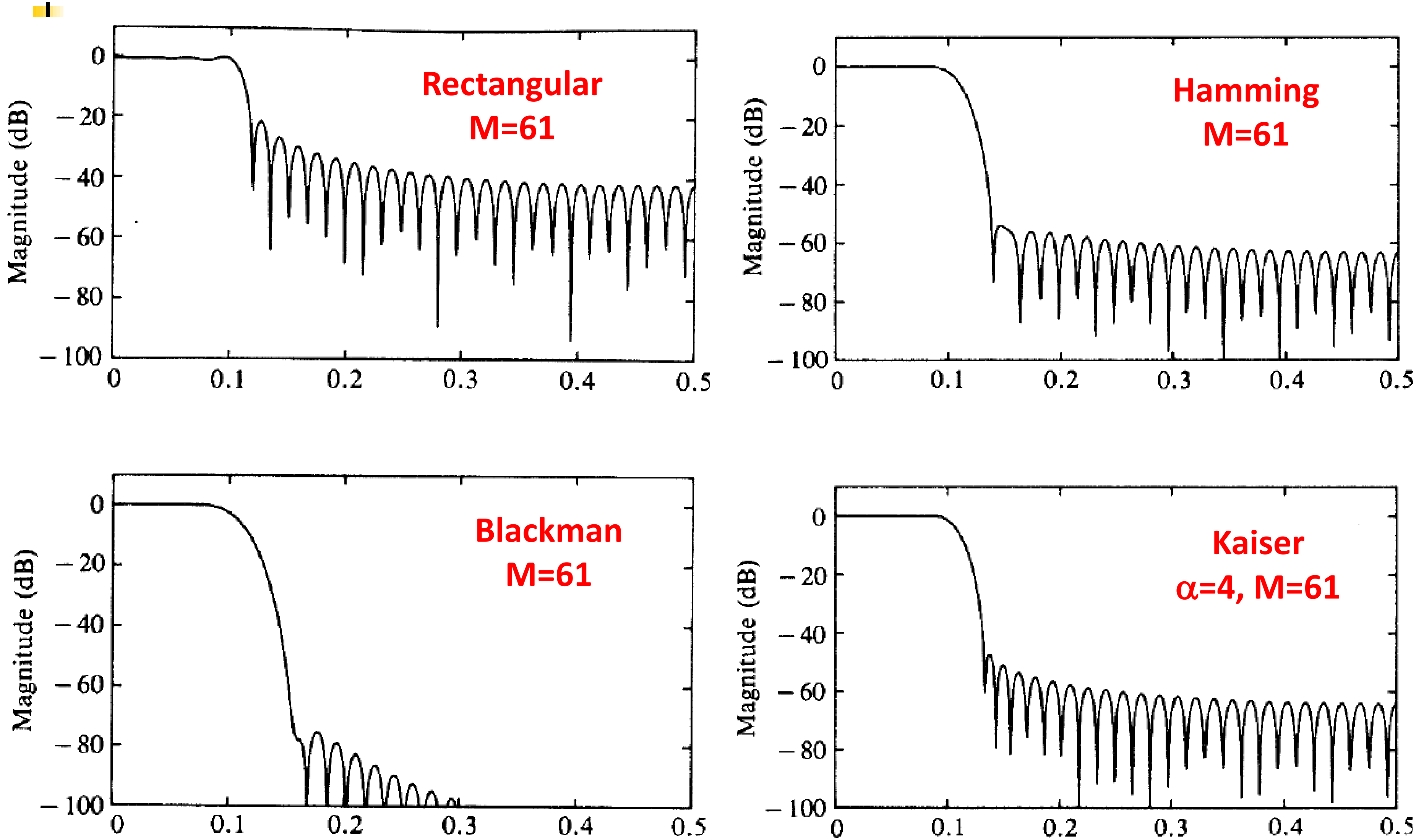
Name	Normalized Tx Bandwidth (Hz)	Peak Approximation Error $20log_{10}\left(\delta_s\right)$	Equivalent Kaiser Window β
Rectangular	0.9/M	-21	0
Bartlett		-25	1.33
Hanning Hamming	3.1/M 3.3/M	-44 -53	3.86 4.86
Blackmann	5.5/M	-74	7.04

Effect of Window Length





Effect of Window Type



FIR-Example

Design a linear phase FIR LPF using rectangular window with desired frequency response given as:

$$H_d(e^{j\Omega}) = \begin{cases} e^{-j\Omega\tau}, & |\Omega| \le \Omega_c \\ 0, & otherwise \end{cases}$$

Consider the filter length to be 7 and $\Omega_c = 0.5\pi \ rad/\sec$

- The desired impulse response is:

$$h_d[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d\left(e^{j\Omega}\right)e^{j\Omega n}d\Omega = \frac{1}{2\pi}\int_{-\Omega_c}^{\Omega_c} e^{-j\Omega\tau}\, e^{j\Omega n}d\Omega$$

$$= \frac{1}{2\pi}\left[\frac{e^{j\Omega(n-\tau)}}{j(n-\tau)}\right]_{-\Omega_c}^{\Omega_c} = \frac{1}{2\pi}\left[\frac{e^{j\Omega_c(n-\tau)} - e^{-j\Omega_c(n-\tau)}}{j(n-\tau)}\right]$$

$$= \frac{1}{2\pi}\left[\frac{e^{j\frac{\pi}{2}(n-\tau)} - e^{-j\frac{\pi}{2}(n-\tau)}}{j(n-\tau)}\right] = \frac{\sin\frac{\pi}{2}(n-\tau)}{\pi(n-\tau)} = \frac{1}{2}sinc\frac{(n-\tau)}{2}$$

FIR-Example

- Since the length is M=7, $\frac{M-1}{2}=\tau=3$

Thus,

$$h_d[n]=\frac{1}{2}sinc\frac{(n-3)}{2}=\frac{1}{\pi}\frac{\sin\left(\frac{n-3}{2}\pi\right)}{n-3}$$

$$h_d[0]=\frac{1}{\pi}\frac{\sin\left(-\frac{3}{2}\pi\right)}{-3}=-\frac{1}{3\pi}=h_d[6]$$

$$h_d[1]=\frac{1}{\pi}\frac{\sin(-\pi)}{-2}=0=h_d[5]$$

Similarly, $h_d[n]=\left\{-\frac{1}{3\pi},0,\frac{1}{\pi},\frac{1}{2},\frac{1}{\pi},0,-\frac{1}{3\pi}\right\}$

Using rectangular window,

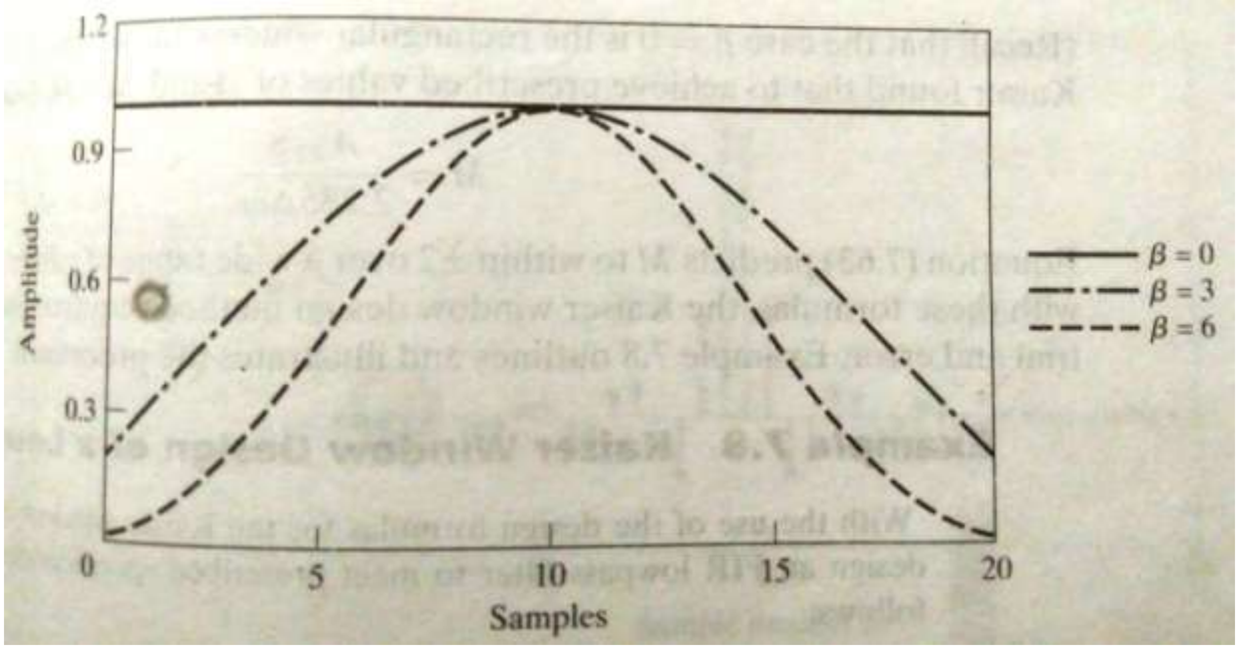
$$h[n]=h_d[n] \qquad for \;\; 0\le n\le 6$$

Kaiser Window

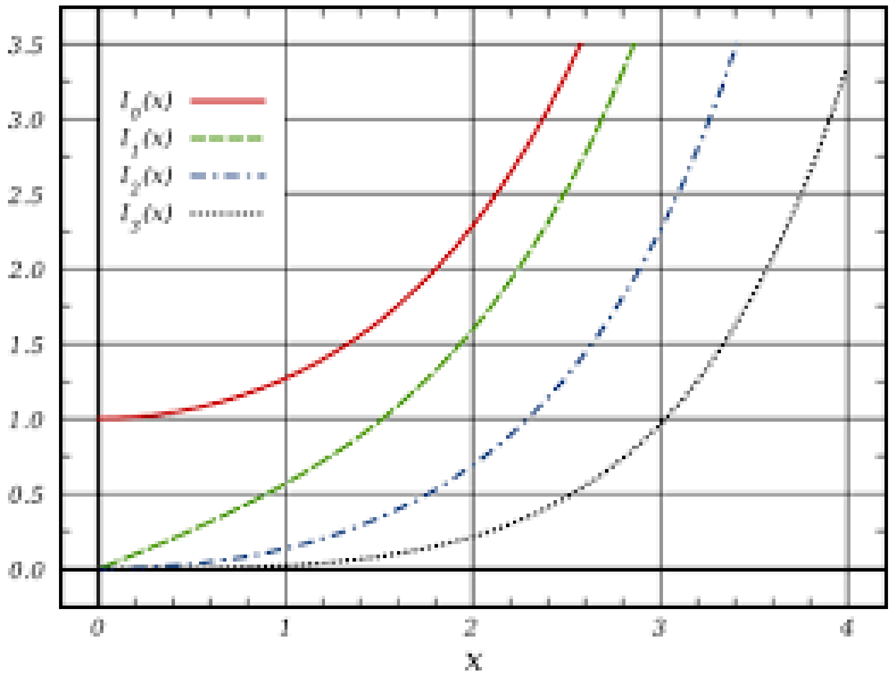
- Windows discussed before has only one control parameter: length
- We cannot independently control the transition width and the attenuation in stop band.
- So we select a particular window based on the attenuation and then find the length to satisfy the transition width.
- In contrast, Kaiser window has two parameters: length M and shape parameter β.
- So, both side lobe amplitude and transition band width can be controlled with same window.
- Defined using the zeroth-order modified Bessel function of first kind.

Kaiser Window

$$w[n]=\begin{cases}\dfrac{I_0\left[\beta\left(1-\left[\dfrac{n-\alpha}{\alpha}\right]^2\right)^{\frac{1}{2}}\right]}{I_0(\beta)}, & 0\le n\le M-1\\[2ex] 0, & otherwise\end{cases}$$



Kaiser Window



$$\Delta\Omega = \Omega_s - \Omega_p \qquad A = -20\log_{10}\delta$$

$$\beta = \begin{cases} 0.112(A-8.7), & A > 50 \\ 0.5842(A-21)^{0.4} + 0.07886(A-21), & 21 \le A \le 50 \\ 0.0\,, & A < 21 \end{cases}$$

$$M = \frac{A-8}{2.285\Delta\Omega}$$

Example-Kaiser Window

- Compute Kaiser window parameters for:

$$0.99 \le |H(\Omega)| \le 1.01 \qquad for \;\; 0 \le |\Omega| \le 0.19\pi$$

$$|H(\Omega)| \le 0.01 \qquad for \;\; 0.21\pi \le |\Omega| \le \pi$$

Consider

$$H_d(\Omega) = \begin{cases} e^{-j\Omega\frac{M-1}{2}}, & -\Omega_c \le \Omega \le \Omega_c \\ 0, & otherwise \end{cases}$$

- Here,

Example – Windows Method

- Design a linear phase FIR LPF having phase delay of 4 with at least -40 dB attenuation in stop band. Use $\Omega_c = 0.5\pi$

- Here, M=9 and we need to use Hamming window. Rest is same.

Frequency Sampling Method

- The desired frequency response is specified as samples at a set of equally spaced frequencies.

Let $H(\Omega)$ be the desired frequency response and its samples be taken at

$$\Omega_k = \frac{2\pi}{M}(k+\alpha), \qquad k = 0, 1, \ldots, \frac{M-1}{2}, \quad M \text{ odd}$$

$$k = 0, 1, \ldots, \frac{M}{2} - 1, \quad M \text{ even}$$

$$\alpha = 0 \quad \text{or} \quad \tfrac{1}{2}$$

If the impulse response associated with the desired frequency response is $h[n]$, it is related as:

$$H(\Omega) = \sum_{n=0}^{M-1} h[n] e^{-j\Omega n}$$

Samples of frequency response will then be:

Frequency Sampling Method

$$H(k+\alpha) \equiv H\left(\frac{2\pi}{M}(k+\alpha)\right)$$

$$H(k+\alpha) \equiv \sum_{n=0}^{M-1} h(n)e^{-j2\pi(k+\alpha)n/M}, \qquad k = 0, 1, \ldots, M-1$$

- And hence the impulse response is given by

$$h(n) = \frac{1}{M}\sum_{k=0}^{M-1} H(k+\alpha)e^{j2\pi(k+\alpha)n/M}, \qquad n = 0, 1, \ldots, M-1$$

- This method is particularly useful when the frequency response of the desired system is zero in a wide range of frequencies.

Optimum Equiripple FIR Filter Design

- Windows method and frequency sampling method are relatively simple methods.

- However, they lack in precise control of the critical frequencies (Ω_s , Ω_p and Ω_c).

- In addition, windows method does not permit individual control over the approximation errors in different bands.

- The error in each band is not uniformly distributed. Usually, error is greatest on either side of a discontinuity of the ideal frequency response.

- Such filters approximately have equal errors in pass band and stop band.

Optimum Equiripple FIR Filter Design

- Optimum equiripple linear phase FIR filter design deals with design of linear phase FIR filters that have equiripple criteria in both pass band and ripple band.

- This ensures that the maximum error in both pass band and stop band is minimized resulting in optimum design.

- For all four possible type of linear phase FIR filters, the real valued frequency response can be written in the form:

$$H_r(\Omega) = Q(\Omega)P(\Omega)$$

where,

$$P(\Omega) = \sum_{k=0}^{L} \alpha(k)\cos \Omega k$$

and

$$Q(\Omega) = \begin{cases} 1 & \text{Case 1} \\ \cos\frac{\Omega}{2} & \text{Case 2} \\ \sin\Omega & \text{Case 3} \\ \sin\frac{\Omega}{2} & \text{Case 4} \end{cases}$$

α_k are filter parameters related to impulse response.

Optimum Equiripple FIR Filter Design

- Let the desired real valued frequency response be $H_{dr}(\Omega)$.

- The error between desired and designed frequency response is thus the difference. To be able to control the relative size of error in different frequency band, we define weighted error function as:

$$E(\Omega) = W(\Omega)[H_{dr}(\Omega) - H_r(\Omega)]$$

- The weight function conveniently be normalized as:

$$W(\Omega) = \begin{cases} \delta_2/\delta_1, & \Omega \text{ in the passband} \\ 1, & \Omega \text{ in the stopband} \end{cases}$$

- Now,

$$\begin{aligned} E(\Omega) &= W(\Omega)[H_{dr}(\Omega) - Q(\Omega)P(\Omega)] \\ &= W(\Omega)Q(\Omega)\left[\frac{H_{dr}(\Omega)}{Q(\Omega)} - P(\Omega)\right] \end{aligned}$$

Optimum Equiripple FIR Filter Design

- Let,

$$\hat{W}(\Omega) = W(\Omega)Q(\Omega)$$

$$\hat{H}_{dr}(\Omega) = \frac{H_{dr}(\Omega)}{Q(\Omega)}$$

- So, $E(\Omega) = W(\Omega)[H_{dr}(\Omega) - P(\Omega)]$

- The optimization problem is now to find the filter parameters α_k such that the maximum value of the absolute error is minimized over the frequency bands.

$$\min_{\text{over } \{\alpha(k)\}} \left[\max_{\omega \in S} |E(\Omega)| \right] = \min_{\text{over } \{a(k)\}} \left[\max_{\omega \in S} |\hat{W}(\Omega)[\hat{H}_{dr}(\Omega) - \sum_{k=0}^{L} \alpha(k) \cos \Omega k]| \right]$$

Optimum Equiripple FIR Filter Design

- the solution of the above problem is provided by the alternation theorem, which says that the solution exists when the error function exhibits at least L+2 extremal frequencies in the desired frequency range.

- That is, there must exist at least L+2 frequencies $\Omega_1 < \Omega_2 < \Omega_3 ..... < \Omega_{L+2}$ such that, $|E(\Omega_i)| = -|E(\Omega_{i+1})|$ and,

$$|E(\Omega_i)| = \max_{over\ \Omega} |E(\Omega)|$$

- The set of filter parameters as required by alternation theorem can be determined by a iterative algorithm known as Remez exchange algorithm.

Remez Exchange Algorithm

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