Digital Signal Analysis and Processing

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IIR Filter Design

Principles of IIR Filter Design

Transformation Used

Impulse Invariance Transformation

Bilinear Transformation

Butterworth and Chebyshev Filter Design

Spectral Transformation

Digital Filter Design

- Filter characteristics are specified in frequency domain in terms of the desired magnitude and phase response.

- Designing a specified filter is to determine the coefficients of a causal FIR or IIR filter that closely approximates the desired frequency response specifications.

- The issue of which filter to choose depends on the nature if the problem and on the specification of the desired frequency response.

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IIR or FIR?

- FIR filter is chosen when there is a requirement of linear phase characteristics within the pass band of the filter.

- IIR filters have lower side lobes in the stop band than an FIR filter with same number of parameters.

- For similar specifications, implementation of IIR filter involves fewer parameters, requires less memory and has lower computational complexity than FIR filter.

- So, when there is no requirement of linear phase or a slight phase distortion is tolerable, IIR filters are chosen in practice.

Design Principle for IIR filters

- Design methods used for FIR filters are unique to digital domain and they are related to filter design methods used for analog filters.

- Design of IIR filters however are based on analog filter design methods.

- So we first design a analog filter based on provided filter specifications and then convert it to the digital filter using some transformation.

- Analog filter design is a matured and rich field. So, it is advantageous to use its knowledge and resources.

Design Principle for IIR filters

- A analog filter is described by its transfer function:

$$H_a(s) = \frac{B(s)}{A(s)} = \frac{\sum_{k=0}^{M} b_k s^k}{\sum_{k=0}^{N} a_k s^k}$$

which is a function of complex variable s.

- Converting such filter to a digital is to find the system function $H(z)$ and hence the coefficients of corresponding digital filter that has similar frequency response.

- The transformations that perform the conversion must convert the stable analog filter to a stable digital filter.

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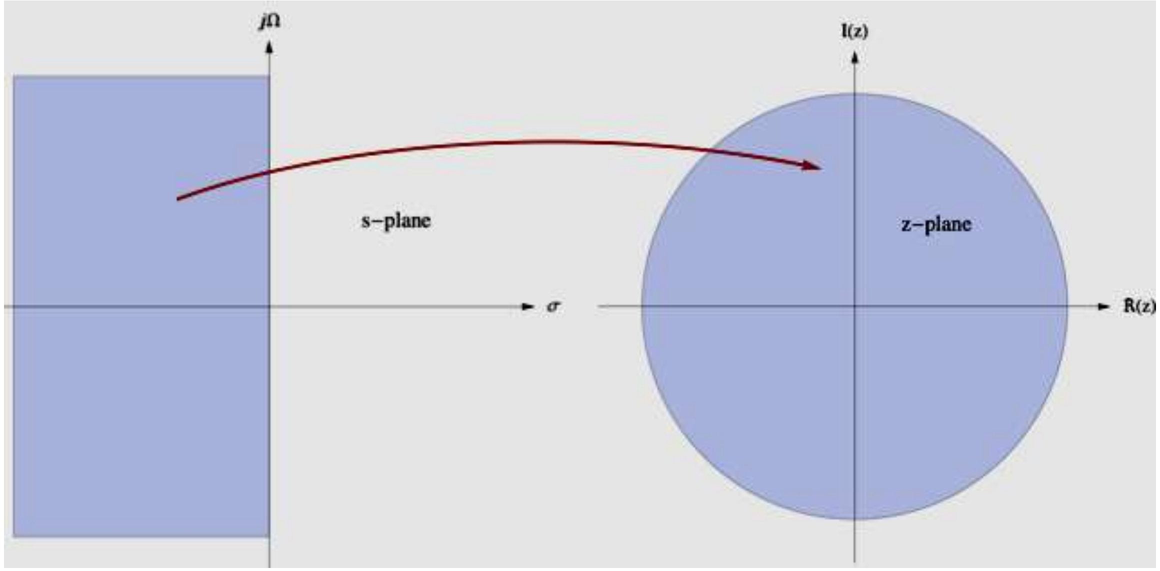
Design Principle for IIR filters

- So, left-half of s-plane must be mapped to inside of unit circle in z-plane. It is desirable that $j\omega$ axis be mapped to unit circle.

- A causal and stable IIR filter cannot have a linear phase. So, while designing a IIR filter, we consider only the magnitude response. Since magnitude and phase responses are related, we accept whatever phase response results.



IIR Filter by Impulse Invariance

- Based on approximating a CT impulse response $h(t)$ by its samples.

- So the transformation designs a digital IIR filter with impulse response $h[n]$ that is the sampled version of the impulse response of the corresponding analog filter $h(t)$.

$$h[n] \equiv h[nT] \qquad n = 0, 1, 2, \ldots..$$

where T is sampling interval.

- Suitable only for low pass filters. Cannot be used for high pass filter design.

Impulse Invariance Transformation

- Consider an analog filter with N poles at p_k and the transfer function as:

$$H_a(s) = \sum_{k=1}^{N} \frac{c_k}{s - p_k}$$

- The impulse response of the filter is :

$$h_a(t) = \sum_{k=1}^{N} c_k e^{p_k t}, \qquad t \geq 0$$

- Samples of $h_a(t)$ are:

$$h(n) = \sum_{k=1}^{N} c_k e^{p_k T n}$$

- The system function of DT system then is:

$$H(z) = \sum_{n=0}^{\infty} h(n) z^{-n}$$

Impulse Invariance Transformation

- $$H(z) = \sum_{n=0}^{\infty} \left(\sum_{k=1}^{N} c_k e^{p_k T n} \right) z^{-n}$$

  $$= \sum_{k=1}^{N} c_k \sum_{n=0}^{\infty} (e^{p_k T} z^{-1})^n$$

- Since, $$\sum_{n=0}^{\infty} (e^{p_k T} z^{-1})^n = \frac{1}{1 - e^{p_k T} z^{-1}}$$
- System function becomes:

  $$H(z) = \sum_{k=1}^{N} \frac{c_k}{1 - e^{p_k T} z^{-1}}$$

- Clearly, the system has poles at

  $$z_k = e^{p_k T}$$

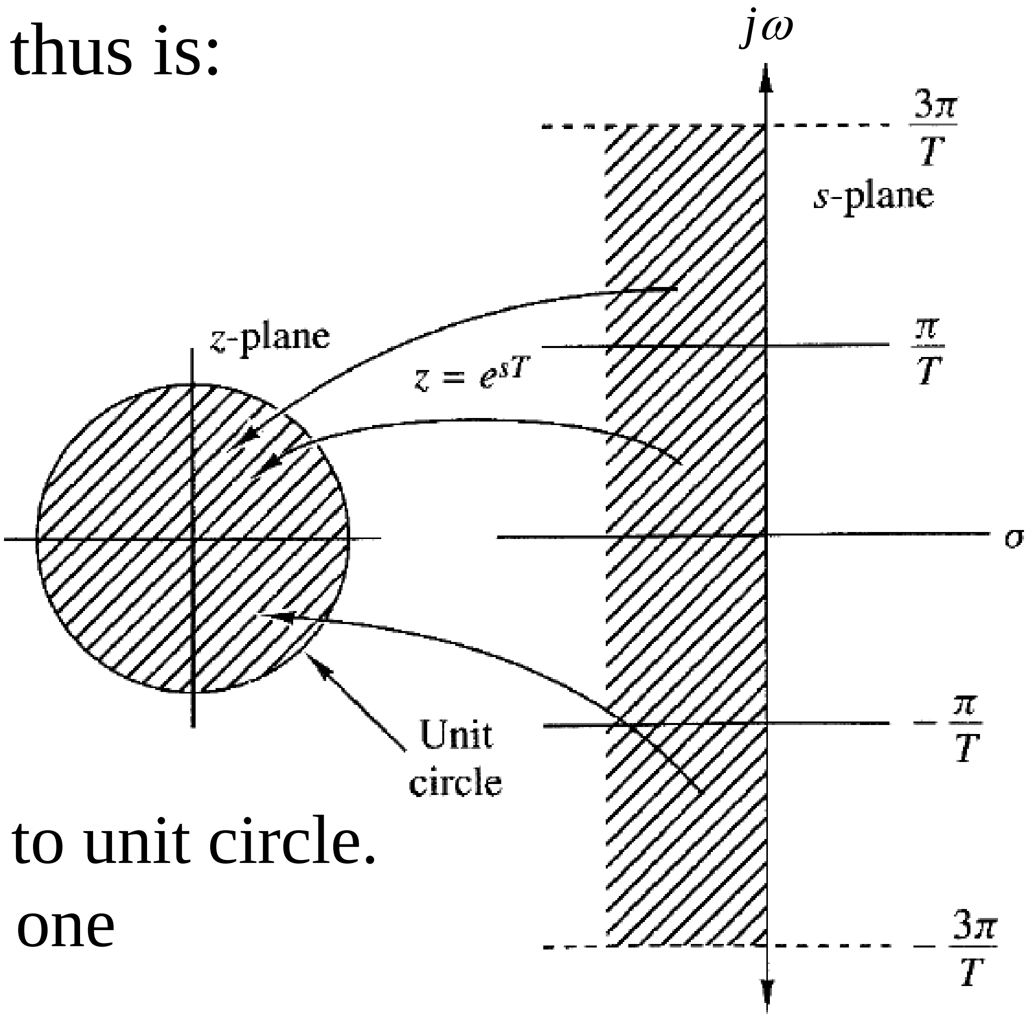
Impulse Invariance-Mapping

- The point $s=p_k$ in s-plane is mapped at $z = e^{p_k T}$
- The mapping involved thus is:

$$z = e^{sT}$$

This corresponds to

$$r = e^{\sigma T} \qquad \Omega = \omega T$$



- Thus LH of s-plane is mapped inside unit circle
- RH of s-plane is mapped outside unit circle.
- Imaginary axis is mapped to unit circle.
- But mapping is not one to one but many to one

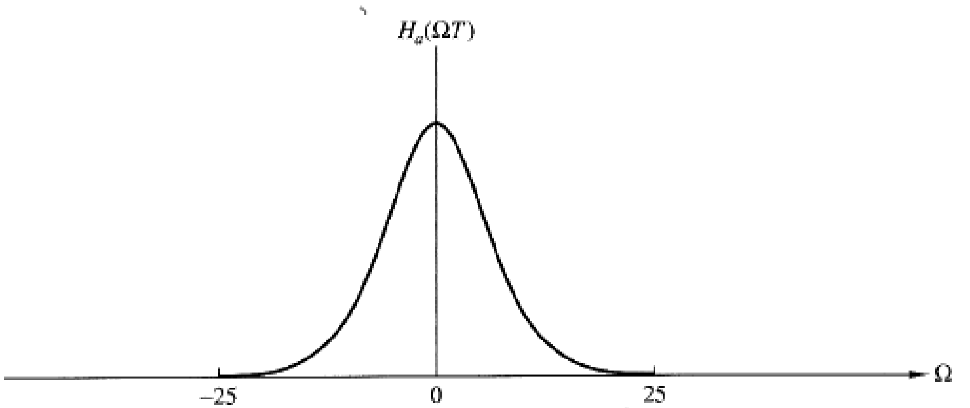
Impulse Invariance Transformation

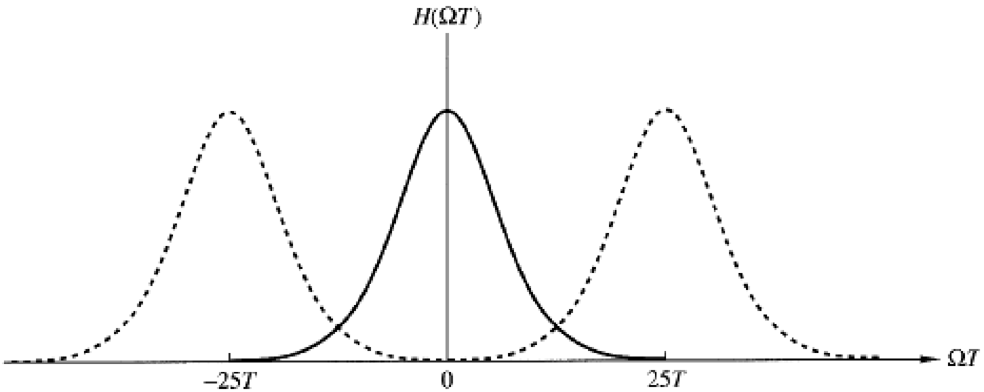
- Spectrum are related as:

$$H(f) = F_s \sum_{k=-\infty}^{\infty} H_a[(f-k)F_s]$$

$$H(\omega) = F_s \sum_{k=-\infty}^{\infty} H_a[(\omega - 2\pi k)F_s]$$

$$H(\Omega T) = \frac{1}{T} \sum_{k=-\infty}^{\infty} H_a\left(\Omega - \frac{2\pi k}{T}\right)$$





- Thus there is overlapping.

- So to avoid or minimize, aliasing, sampling time T must be sufficiently small.

Impulse Invariance-Example

- Convert the analog filter system function to digital using impulse invariance transformation:

$$H_a(s) = \frac{s + 0.1}{(s + 0.1)^2 + 9}$$

- The analog filter has poles at: $p_k = -0.1 \pm j3$

The partial fraction expansion is thus:

$$H(s) = \frac{\frac{1}{2}}{s + 0.1 - j3} + \frac{\frac{1}{2}}{s + 0.1 + j3}$$

The system function of the corresponding IIR filter based on impulse invariance transformation is:

$$H(z) = \frac{\frac{1}{2}}{1 - e^{-0.1T} e^{j3T} z^{-1}} + \frac{\frac{1}{2}}{1 - e^{-0.1T} e^{-j3T} z^{-1}}$$

$$H(z) = \frac{1 - (e^{-0.1T} \cos 3T) z^{-1}}{1 - (2e^{-0.1T} \cos 3T) z^{-1} + e^{-0.2T} z^{-1}}$$

Bilinear Transformation

- Conformal mapping that maps imaginary axis in s-plane to unit circle in z-plane only once. So, no aliasing.

- Based on approximation of integration by trapezoidal formula.

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Bilinear Transformation

- Consider a analog filter with transfer function

$$H(s) = \frac{b}{s+a}$$

The system is also described by differential equation as:

$$\frac{dy(t)}{dt} + ay(t) = bx(t)$$

Evaluating differential equation at $t=nT$,

$$y'[nT] = -ay[nT] + bx[nT]$$

Let us consider an integral

$$y(t) = \int_{t_0}^{t} y'(\tau)d\tau + y(t_0)$$

Approximating with trapezoidal formula at $t=nT$,

$$y[nT] = \frac{T}{2}\left(y'[nT] + y'[nT-T]\right) + y[nT-T]$$

Bilinear Transformation

- Substituting for $y'[nT]$

$$y[nT]=\frac{T}{2}\left(-ay[nT]+bx[nT]-ay[nT-T]+bx[nT-T]\right)+y[nT-T]$$

- Using $y[n]$ for $y[nT]$

$$y[n]=\frac{T}{2}\left(-ay[n]+bx[n]-ay[n-1]+bx[n-1]\right)+y[n-1]$$

$$\left(1+a\frac{T}{2}\right)y[n]=\left(1-a\frac{T}{2}\right)y[n-1]+b\frac{T}{2}x[n]+b\frac{T}{2}x[n-1]$$

Using Z-transform,

$$\left(1+a\frac{T}{2}\right)Y(z)=\left(1-a\frac{T}{2}\right)z^{-1}Y(z)+b\frac{T}{2}X(z)+b\frac{T}{2}z^{-1}X(z)$$

$$\left(1+a\frac{T}{2}-\left(1-a\frac{T}{2}\right)z^{-1}\right)Y(z)=b\frac{T}{2}\left(1+z^{-1}\right)X(z)$$

Bilinear Transformation

- The system function thus is:
- $$H(z) = \frac{b\frac{T}{2}\left(1+z^{-1}\right)}{\left(1+a\frac{T}{2}-z^{-1}+a\frac{T}{2}z^{-1}\right)} = \frac{b\frac{T}{2}\left(1+z^{-1}\right)}{\left(1-z^{-1}+a\frac{T}{2}\left(1+z^{-1}\right)\right)}$$

$$H(z) = \frac{b}{\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)+a}$$

- Comparing H(z) with H(s), we can conclude that

$$H(z) = H(s)\Big|_{s=\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$

- The mapping involved with bilinear transformation thus is

$$s = \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}$$

Bilinear Transformation

- If we express s and z as, $z = re^{j\Omega}$ and $s = \sigma + j\omega$
- The mapping equation results:

$$\sigma = \frac{2}{T}\frac{r^2 - 1}{1 + r^2 + 2r\cos\Omega} \qquad \omega = \frac{2}{T}\frac{2r\sin\Omega}{1 + r^2 + 2r\cos\Omega}$$

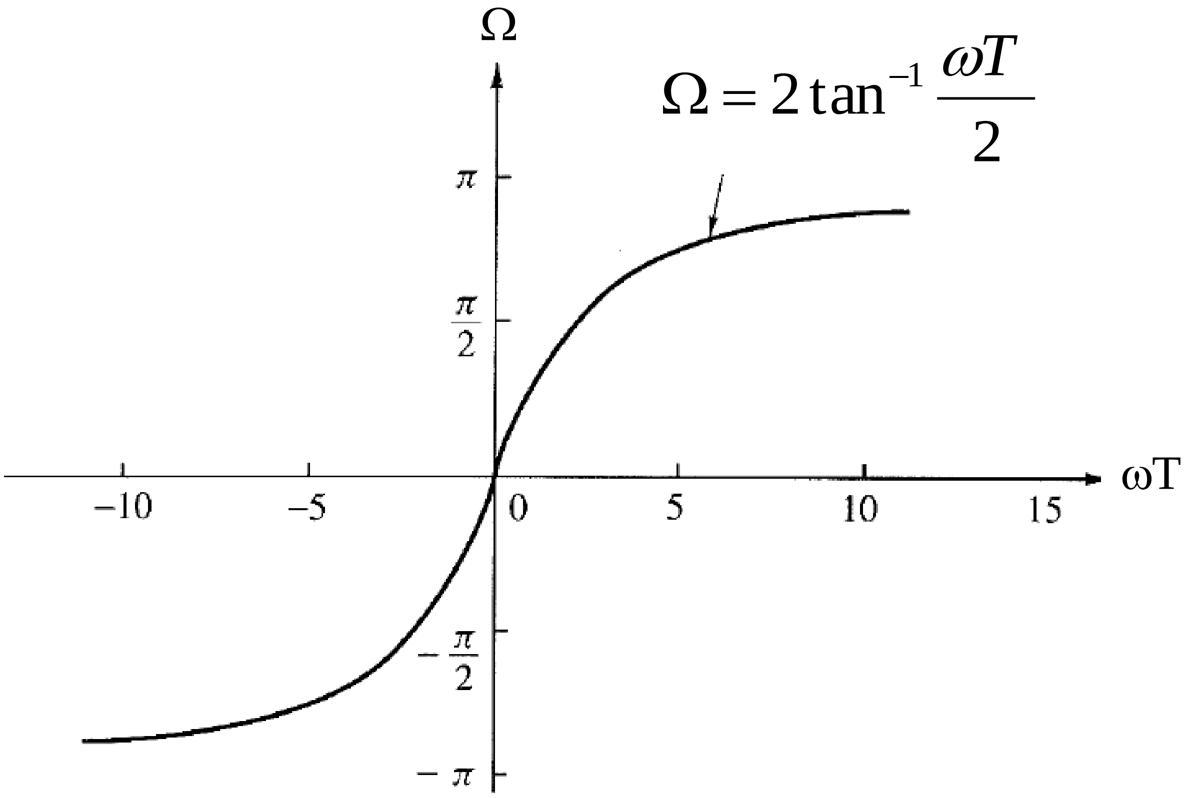
- LH of s-plane maps to inside of unit circle and RH of s-plane maps to outside of unit circle.
- In addition, imaginary axis of s-plane maps to the unit circle.
- Mapping to the unit circle is one to one. Hence the is a compression.
- This frequency compression is called frequency warping.

Bilinear Transformation

- For imaginary axis, σ=0 and hence, r=1. The mapping equation reduces to:

$$\omega = \frac{2}{T}\frac{\sin\Omega}{1+\cos\Omega} = \frac{2}{T}\tan\left(\frac{\Omega}{2}\right)$$

$$\Omega = 2\tan^{-1}\frac{\omega T}{2}$$



Bilinear Transformation-Example

- Convert the following analog transfer function to digital IIR filter using bilinear transformation.

$$H_a(s) = \frac{s+0.1}{(s+0.1)^2+16}$$

The resonant frequency of the digital filter is to be $\Omega_r = \frac{\pi}{2}$

- Here, $\omega_r = 4$. So using the mapping equation, $\Omega = 2\tan^{-1}\frac{\omega T}{2}$ we get, $T = \frac{1}{2}$. Thus,

$$s = 4\frac{1-z^{-1}}{1+z^{-1}}$$

Finally, system function is:

$$H(z) = \frac{0.128+0.006z^{-1}-0.122z^{-1}}{1+0.0006z^{-1}+0.975z^{-2}} \qquad H(z) = \frac{0.128+0.006z^{-1}-0.122z^{-2}}{1+0.975z^{-2}}$$

Bilinear Transformation-Example

- Design a single pole lowpass digital filter with 3-dB bandwidth of 0.2π, using bilinear transformation for following analog filter.
$$H_a(s) = \frac{\omega_c}{s+\omega_c}$$

- Using the mapping equation, $\omega_c = \frac{2}{T}\tan\left(\frac{0.2\pi}{2}\right) = \frac{0.65}{T}$

Using this the analog transfer function becomes:

$$H_a(s) = \frac{0.65/T}{s+0.65/T}$$

Finally the system function of digital filter is:

$$H(z) = \frac{0.65/T}{\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}+0.65/T} = \frac{0.245\left(1+z^{-1}\right)}{1-0.509z^{-1}}$$

Butterworth Filters

- A low pass Butterworth analog filter is an all pole filter with magnitude squared frequency response given as:

$$|H(\omega)|^2 = \frac{1}{1+(\omega/\omega_c)^{2N}}$$

where, ω_c is 3 dB cutoff frequency and N is the order.

- Simple maximally flat response.

- Monotonic decay in both pass band and stop band.

- All poles lie on the circle of radius ω_c and different angles.

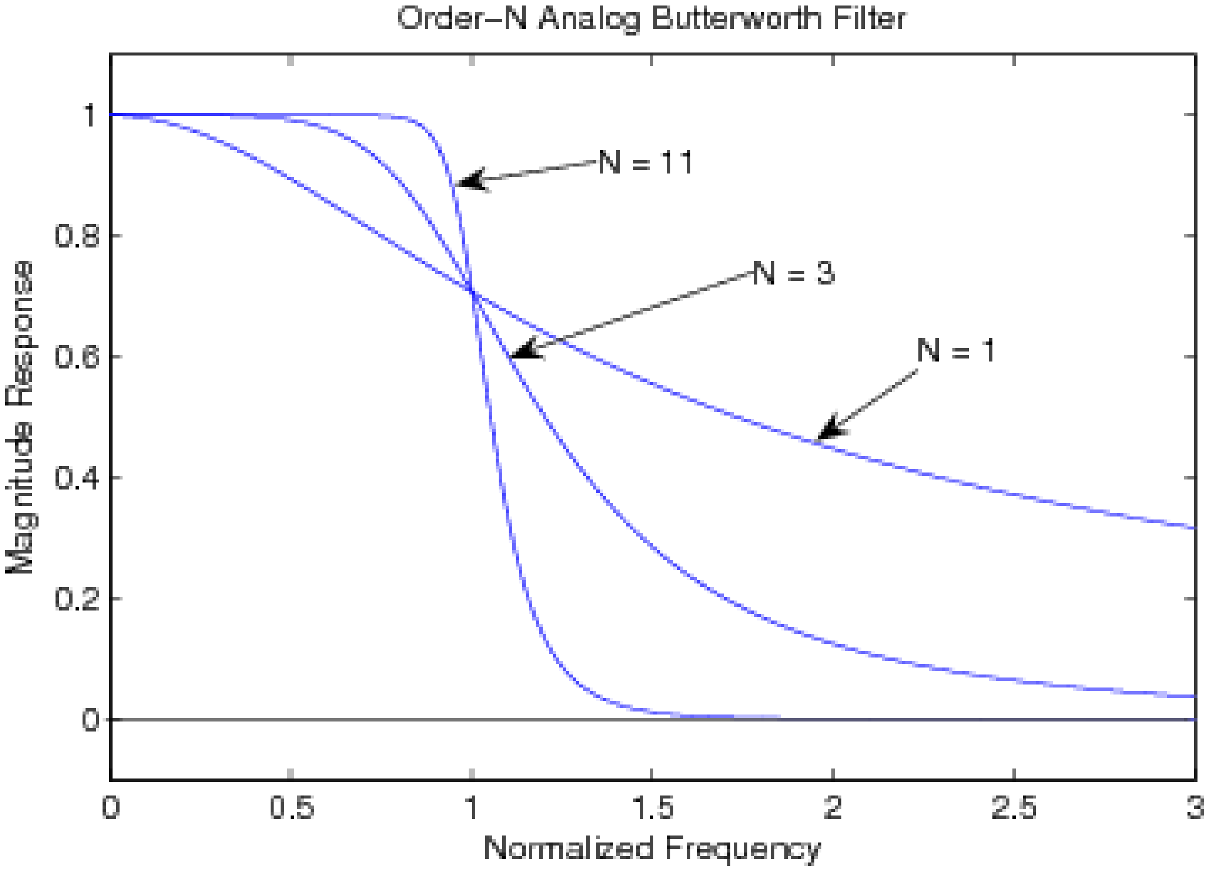
- Angles given by: $\theta_k = \frac{\pi}{2} + (2k+1)\frac{\pi}{2N} \qquad k = 0,1,2,......N-1$

Butterworth Filters

- As the order increases, transition width decreases.

$$\left|H(\omega)\right|_{\omega=0} = 1 \qquad \textit{for all } N$$

$$\left|H(\omega)\right|_{\omega=\omega_c} = \frac{1}{\sqrt{2}} \qquad \textit{for all } N$$



Butterworth Filters-Order

- Let $\omega_p, \omega_s, \alpha_p, \alpha_s$ be pass band edge frequency, stop band edge frequency, maximum pass band attenuation and minimum stop band attenuation.

Then,
$$|H(\omega)| = \left(1 + (\omega/\omega_c)^{2N}\right)^{-\frac{1}{2}}$$

In dB, $A = 20\log_{10}|H(\omega)| = -10\log_{10}\left(1 + (\omega/\omega_c)^{2N}\right)$

Attenuation in dB $\alpha = -A = 10\log_{10}\left(1 + (\omega/\omega_c)^{2N}\right)$

At pass band edge,
$$\alpha_p = 10\log_{10}\left(1 + (\omega_p/\omega_c)^{2N}\right) \Rightarrow \left(\frac{\omega_p}{\omega_c}\right)^{2N} = 10^{\frac{\alpha_p}{10}} - 1$$

At stop band edge,
$$\alpha_s = 10\log_{10}\left(1 + (\omega_s/\omega_c)^{2N}\right) \Rightarrow \left(\frac{\omega_s}{\omega_c}\right)^{2N} = 10^{\frac{\alpha_s}{10}} - 1$$

Butterworth Filters-Order

- Dividing,

$$\left(\frac{\omega_s}{\omega_p}\right)^{2N} = \frac{10^{\alpha_s/10} - 1}{10^{\alpha_p/10} - 1}$$

Taking logarithm and solving for N,

$$N = \frac{\log_{10}\left(\dfrac{10^{\alpha_s/10} - 1}{10^{\alpha_p/10} - 1}\right)}{2\log_{10}\left(\dfrac{\omega_s}{\omega_p}\right)}$$

3-dB cutoff frequency is:

$$\omega_c = \frac{\omega_s}{\left[10^{\alpha_s/10} - 1\right]^{1/2N}}$$

Butterworth Filters-Order

- Use impulse invariance method to design a Butterworth LPF with, $\Omega_p = 0.3\pi, \Omega_s = 0.5\pi, \alpha_p = 1.41\,dB, \alpha_s = 12\,dB,\ T = 1$

➢ $\omega_p = \dfrac{\Omega_p}{T} = 0.3\pi, \omega_s = \dfrac{\Omega_s}{T} = 0.5\pi$

Computing order:

$$N = \frac{\log_{10}\left(\dfrac{10^{12/10} - 1}{10^{1.41/10} - 1}\right)}{2\log_{10}\left(\dfrac{0.5\pi}{0.3\pi}\right)} = 3.58 \approx 4$$

3-dB cutoff frequency is: $\omega_c = \dfrac{0.3\pi}{\left[10^{1.41/10} - 1\right]^{1/8}} = 0.34\pi$

Poles are at: $s_k = \omega_c e^{j\theta_k} = 0.34\pi\, e^{j\theta_k} = \ldots\ldots\ldots$

Now the BW transfer function is: $H_a(s) = \dfrac{\omega_c{}^N}{(s - s_1)(s - s_2)\ldots\ldots(s - s_N)}$

Chebyshev Filters

- Chebyshev filters are characterized by ripples either in pass band or stop band.

- Two types of Chebyshev filters:

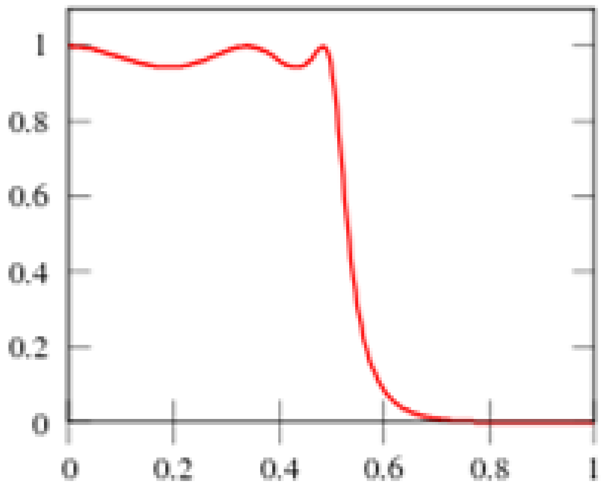
  - Type I: Equiripple behavior in pass band and monotonic in stop band.

  - Type II: Equiripple behavior in stop band and monotonic in pass band.

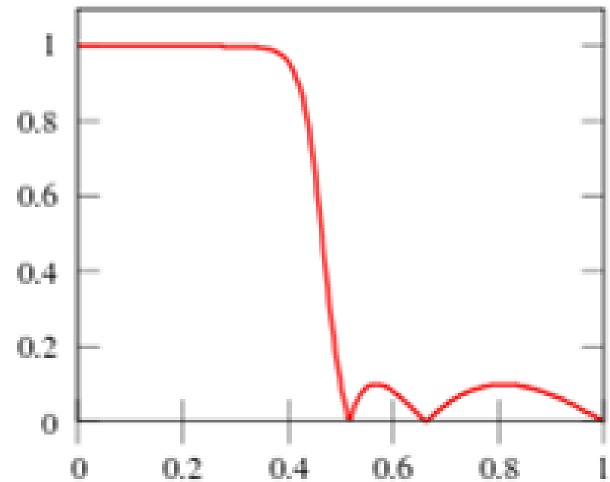
- Type I filter is an all pole system while Type II filter is a pole-zero system

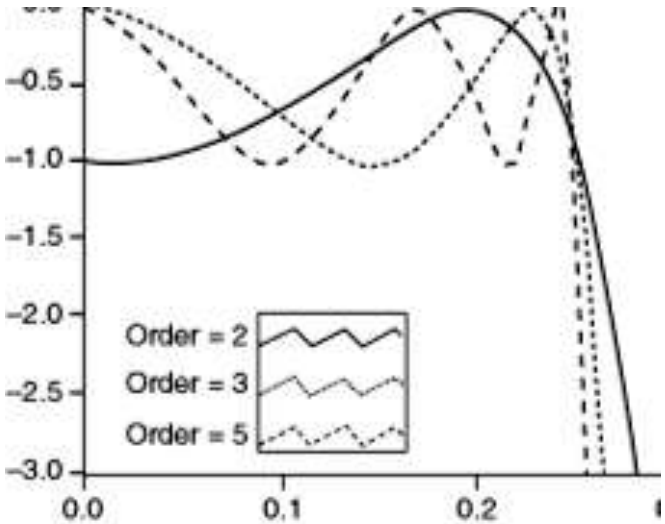
Chebyshev Filters

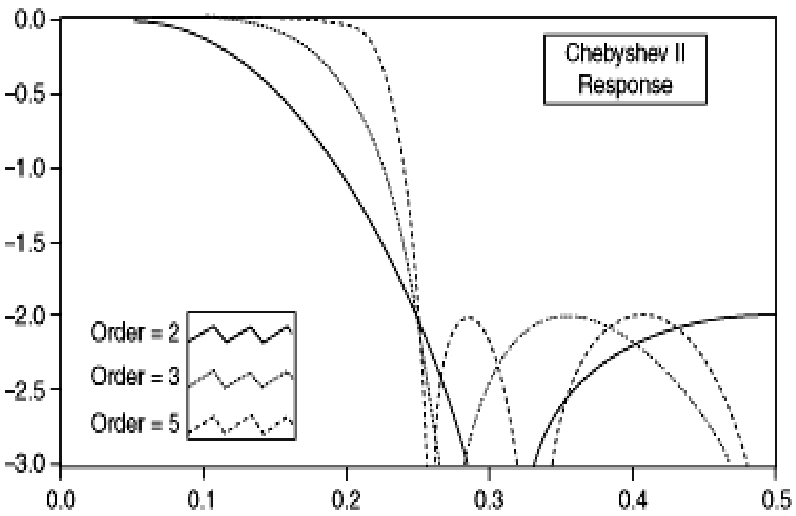
- Chebyshev Type I response



Chebyshev Type II response







Chebyshev - Type I

- A low pass Chebyshev Type I analog filter is an all pole filter with magnitude squared frequency response given as:

$$|H(\omega)|^2 = \frac{1}{1+\varepsilon^2 T_N^{\,2}\left(\omega/\omega_p\right)}$$

where, $T_N(x)$ is N^{th} order Chebyshev polynomial defined as:

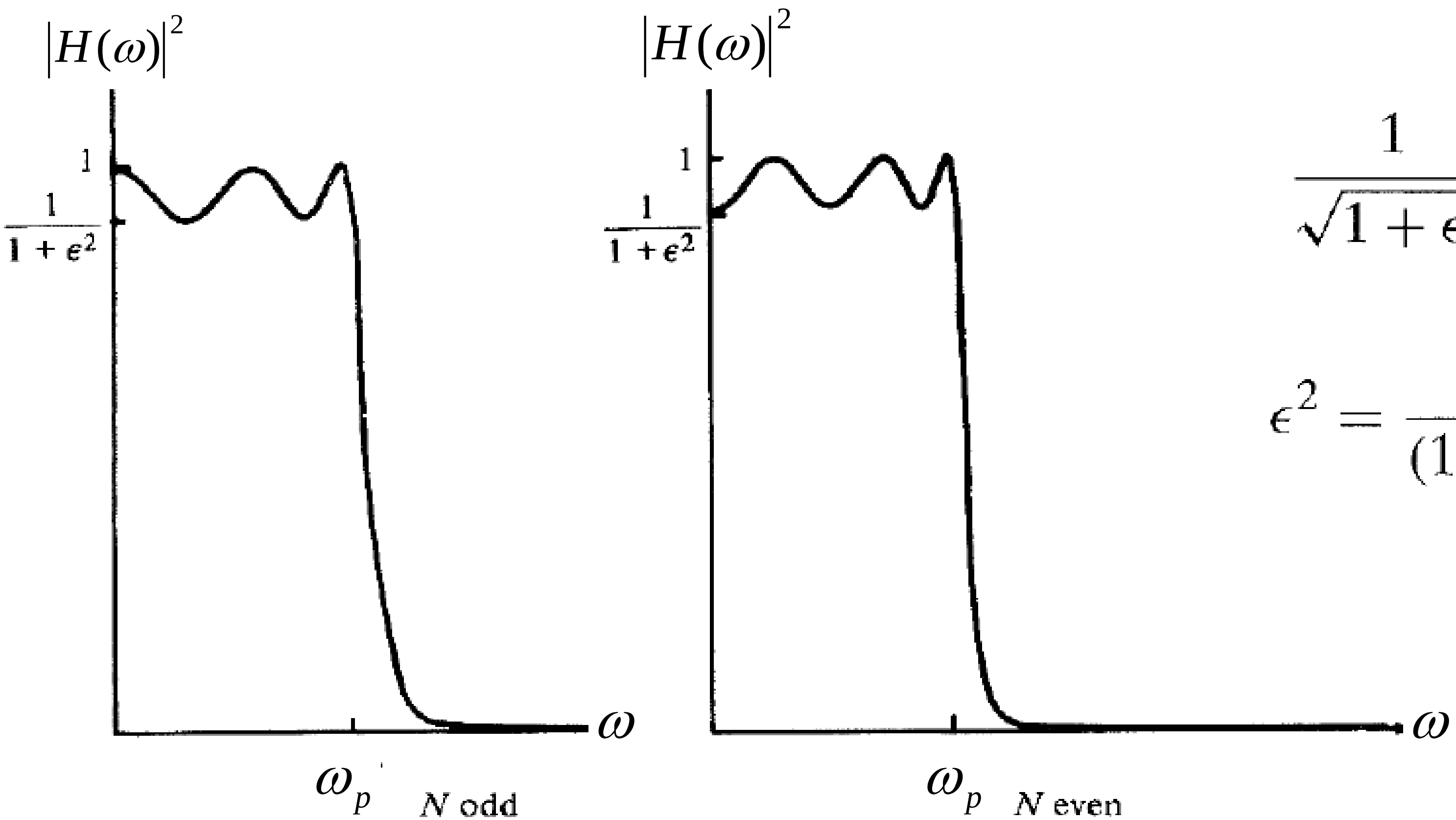
$$T_N(x) = \begin{cases} \cos(N\cos^{-1}x) &, \quad |x|\le 1 \\ \cosh(N\cosh^{-1}x), & |x|>1 \end{cases}$$

and ε is scaling factor controlling the ripple in pass band.

$$|T_N(x)|\le 1 \quad \textit{for all } N \textit{ and } |x|\le 1$$

$$|T_N(1)| = 1 \quad \textit{for all } N$$

Chebyshev - Type I



$$\frac{1}{\sqrt{1+\epsilon^2}} = 1 - \delta_1$$

$$\epsilon^2 = \frac{1}{(1-\delta_1)^2} - 1$$

δ_1 is pass band ripple.

The order is given as: $$N = \frac{\cosh^{-1}\left(\frac{10^{\alpha_s/10}-1}{10^{\alpha_p/10}-1}\right)^{\frac{1}{2}}}{\cosh^{-1}\left(\frac{\omega_s}{\omega_p}\right)}$$

Chebyshev - Type I

- Determine the prder of a type I LPF Chebyshev filter that has 1-dB ripple in the passband, a cutoff frequency 1000 π, a stop band frequency of 2000 π, and an attenuation of 40 dB or more in stop band.

- Here, $\omega_p = 1000\pi,\ \omega_s = 2000\pi,\ \alpha_s = 40,\ \alpha_p = 1 dB$

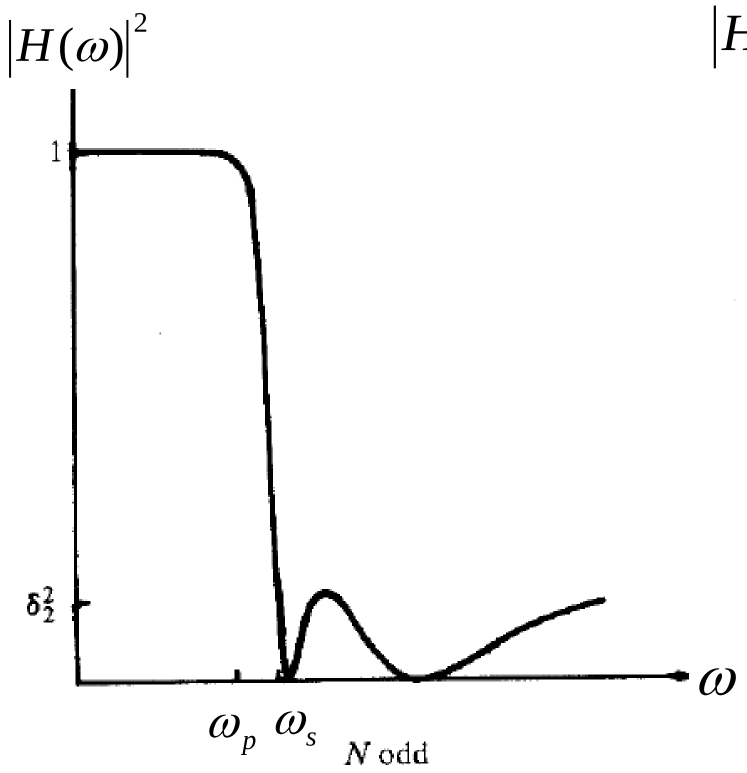
The order is given as:

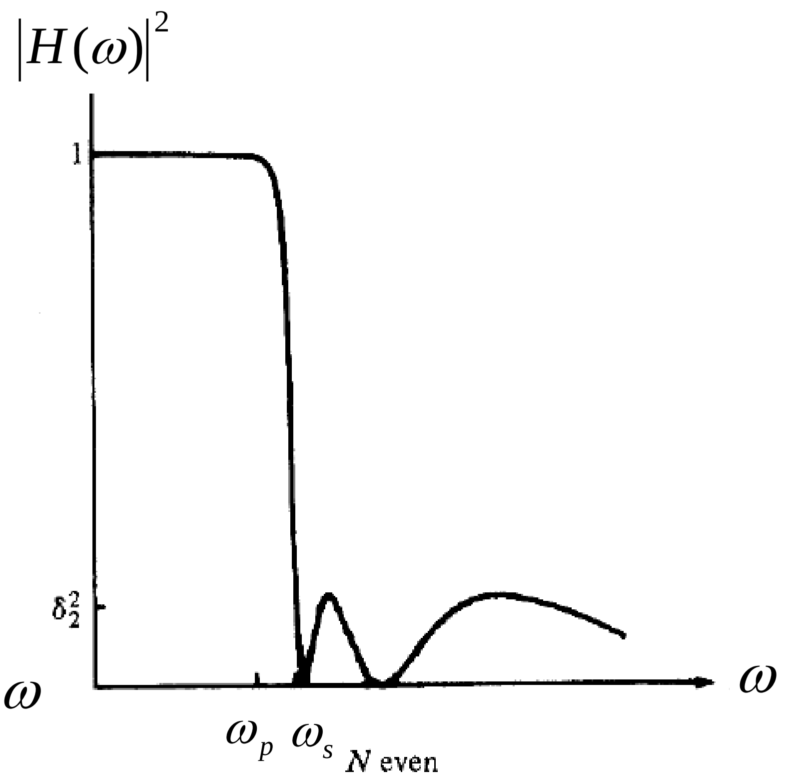
$$N = \frac{\cosh^{-1}\left(\dfrac{10^{40/10}-1}{10^{1/10}-1}\right)^{\frac{1}{2}}}{\cosh^{-1}(2)} = 3.9 \approx 4$$

Also,

$$\frac{1}{\sqrt{1+\epsilon^2}} = 1 - \delta_1 \qquad 10\log_{10}(1+\epsilon^2) = 1 \qquad \epsilon = 0.5088$$

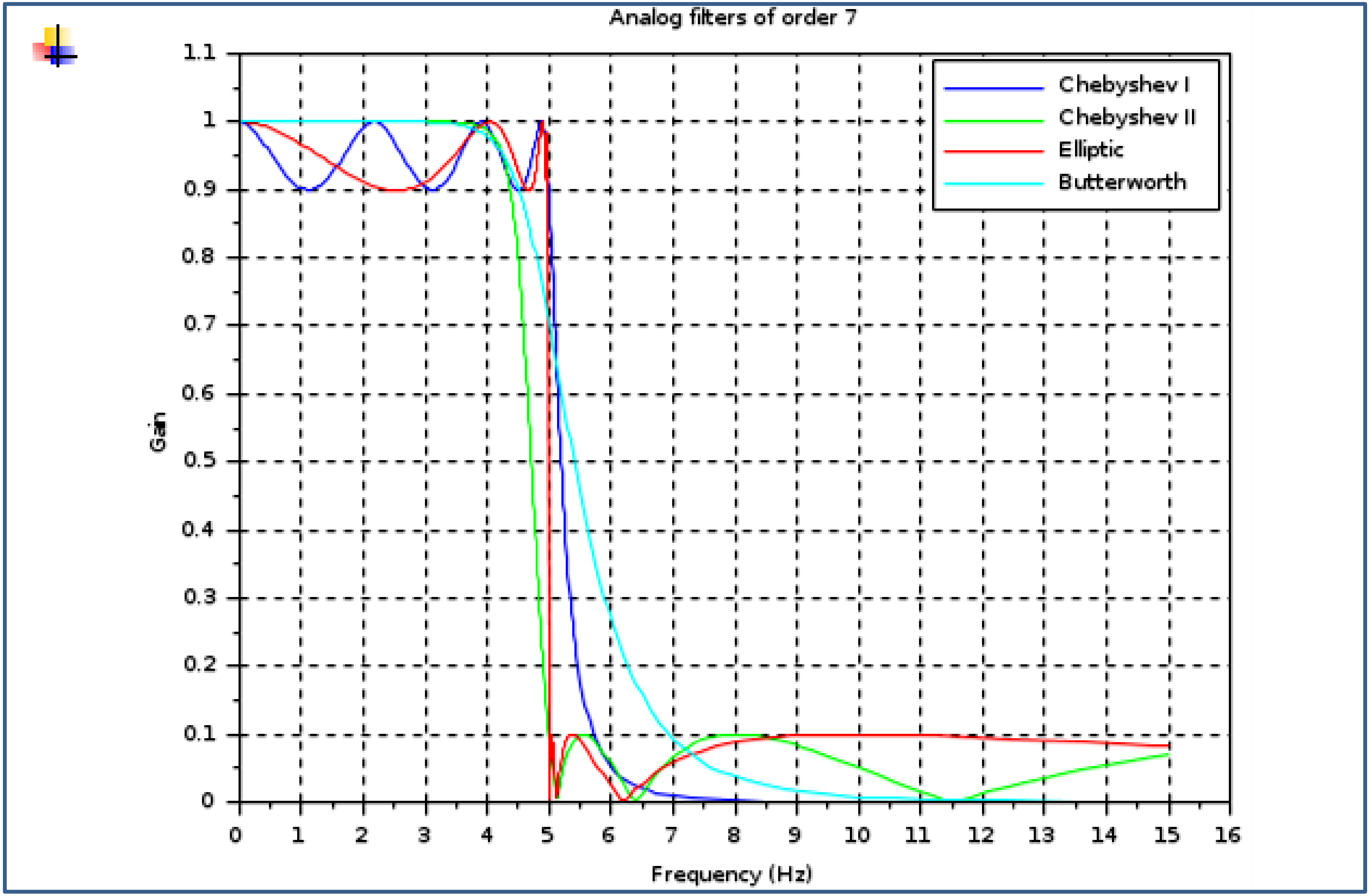
Chebyshev - Type II





δ_2 is pass band ripple.

Comparison of Frequency response



Spectral Transformation

- In course of signal processing we may need other type of filters than low pass also. That includes high pass and band pass filters.

- When filters other than low pass are needed, we design a low pass filter and then convert it to the one required. It is called spectral transformation.

- This is done to simplify the design process.

- While designing digital IIR filters, we have two options, either to perform the spectral transformation in analog domain or to perform it in digital domain.

Transformation In Analog Domain

- Low pass to low pass:

  To convert a low pass filter with pass band edge frequency ω_p to a low pass filter with pass band edge frequency ω_p', the transformation is:

$$s \rightarrow \frac{\omega_p}{\omega_p'} s$$

- Low pass to high pass:

  To convert a low pass filter with pass band edge frequency ω_p to a high pass filter with pass band edge frequency ω_p', the transformation is:

$$s \rightarrow \frac{\omega_p \omega_p'}{s}$$

Transformation In Analog Domain

- Low pass to band pass:

  To convert a low pass filter with pass band edge frequency ω_p to a low pass filter with pass band edge frequencies ω_{p1}, and ω_{p2} the transformation is:

$$s \rightarrow \omega_p \frac{s^2 + \omega_{p1}\omega_{p2}}{s(\omega_{p2} - \omega_{p1})}$$

- Low pass to band stop:

$$s \rightarrow \omega_p \frac{s(\omega_{p2} - \omega_{p1})}{s^2 + \omega_{p1}\omega_{p2}}$$

Transformation In Digital Domain

Type of transformation	Transformation	Parameters
Lowpass	$z^{-1} \longrightarrow \dfrac{z^{-1}-a}{1-az^{-1}}$	$\omega'_p =$ band edge frequency new filter $a = \dfrac{\sin[(\omega_p - \omega'_p)/2]}{\sin[(\omega_p + \omega'_p)/2]}$
Highpass	$z^{-1} \longrightarrow -\dfrac{z^{-1}+a}{1+az^{-1}}$	$\omega'_p =$ band edge frequency new filter $a = -\dfrac{\cos[(\omega_p + \omega'_p)/2]}{\cos[(\omega_p - \omega'_p)/2]}$

Transformation In Digital Domain

Bandpass

$$z^{-1} \longrightarrow -\frac{z^{-2} - a_1 z^{-1} + a_2}{a_2 z^{-2} - a_1 z^{-1} + 1}$$

$$\omega_l = \text{lower band edge frequency}$$
$$\omega_u = \text{upper band edge frequency}$$
$$a_1 = 2\alpha K/(K+1)$$
$$a_2 = (K-1)/(K+1)$$
$$\alpha = \frac{\cos[(\omega_u + \omega_l)/2]}{\cos[(\omega_u - \omega_l)/2]}$$
$$K = \cot\frac{\omega_u - \omega_l}{2}\tan\frac{\omega_p}{2}$$

Bandstop

$$z^{-1} \longrightarrow \frac{z^{-2} - a_1 z^{-1} + a_2}{a_2 z^{-1} - a_1 z^{-1} + 1}$$

$$\omega_l = \text{lower band edge frequency}$$
$$\omega_u = \text{upper band edge frequency}$$
$$a_1 = 2\alpha/(K+1)$$
$$a_2 = (1-K)/(1+K)$$
$$\alpha = \frac{\cos[(\omega_u + \omega_l)/2]}{\cos[(\omega_u - \omega_l)/2]}$$
$$K = \tan\frac{\omega_u - \omega_l}{2}\tan\frac{\omega_p}{2}$$