# Digital Signal Analysis and Processing

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# Z-Transform

Definition of z-transform

Region of Convergence

Properties of z-transform

Inverse z-transform

Convolution using z-transform

LTI System and System Function

Rational z-transform and Pole-zero

# Definition

- For a DT signal x[n], its z-transform is a power series defined as:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

- where, z is a complex variable represented as $z = re^{j\omega}$
- thus, $|z| = r \quad and \ \angle z = \omega$

- Z-transform pairs are usually shown as:

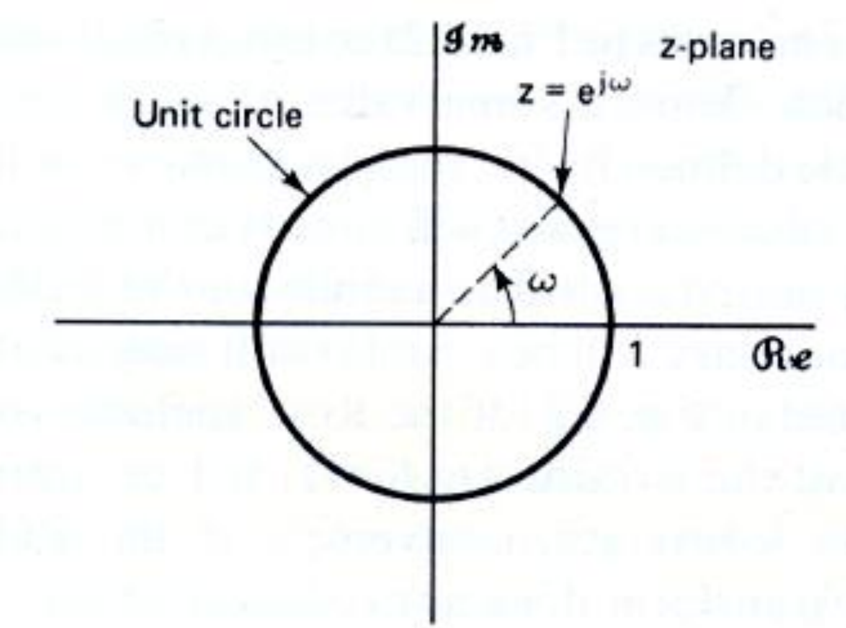
$$x[n] \xleftrightarrow{Z} X(z)$$

- $$X(z) = \sum_{n=-\infty}^{\infty} x[n](re^{j\omega})^{-n}$$

# The Z-plane

- Z-transforms a time domain signal to a complex frequency domain z-plane.

- Z-transform may be considered as generalization of Fourier transform. Z-transform evaluated at unit circle is the Fourier transform



# Region of Convergence (ROC)

- Z-transform is a power series. So, it may not always converge.

- ROC is the set of values of z for which the z-transform converges.

- Area in z-plane where the z transform has finite value.

$$|X(z)| = \left| \sum_{n=-\infty}^{\infty} x[n] z^{-n} \right| < \infty$$

- Without ROC z-transform is not unique. (More than one time domain signals may have same z-transform but will have different ROC)

# ROC IS A RING IN Z-PLANE

ROC is set of z for which, $|X(z)| < \infty$

$$|X(z)| = \left| \sum_{n=-\infty}^{\infty} x[n] z^{-n} \right|$$

$$\leq \sum_{n=-\infty}^{\infty} \left| x[n] z^{-n} \right| = \sum_{n=-\infty}^{\infty} \left| x[n] r^{-n} e^{-j\omega n} \right| = \sum_{n=-\infty}^{\infty} \left| x[n] r^{-n} \right|$$

So, $X(z)$ converges when $x[n]r^{-n}$ is absolutely summable.

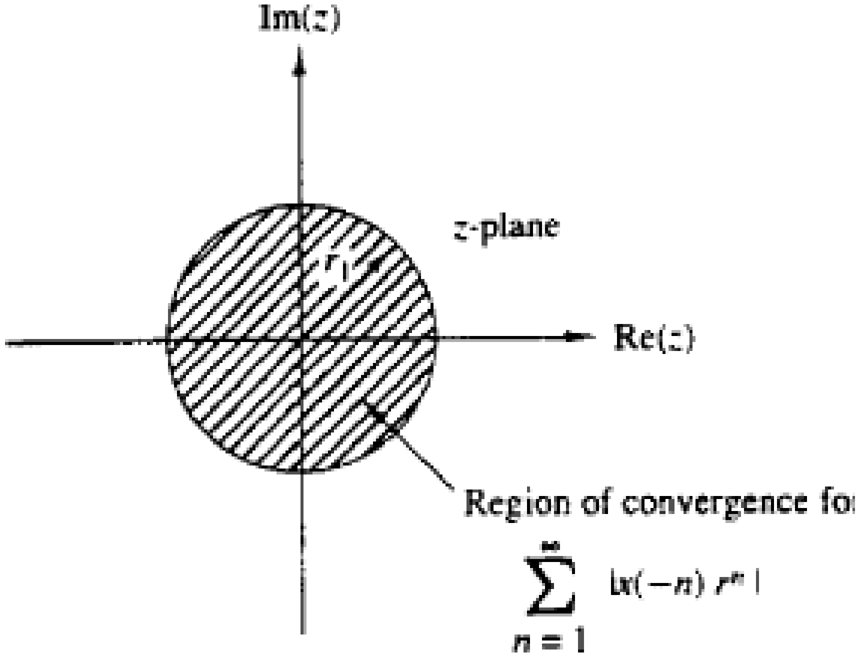
$$|X(z)| \leq \sum_{n=-\infty}^{-1} \left| x[n] r^{-n} \right| + \sum_{n=0}^{\infty} \left| x[n] r^{-n} \right|$$

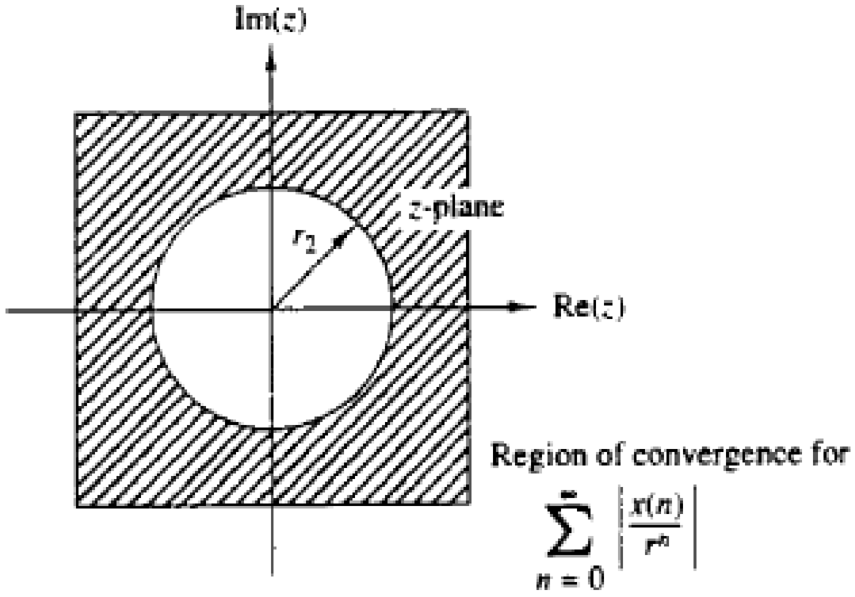
$$\leq \sum_{n=1}^{\infty} \left| x[-n] r^{n} \right| + \sum_{n=0}^{\infty} \left| \frac{x[n]}{r^{n}} \right|$$

# ROC IS A RING IN Z-PLANE

The ROC of the first sum consists of all points in a circle of radius $r_1 \quad (r_1 < \infty)$ .

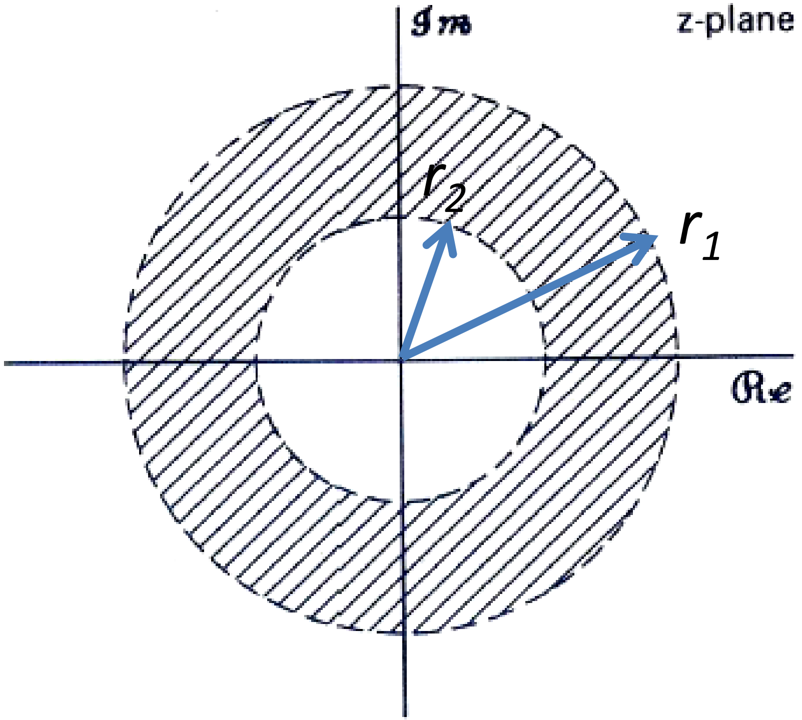
The ROC of the second sum consists of all points outside a circle radius $r > r_2$



# ROC is a ring in z-plane

Requires both terms to be finite, the ROC is thus the annular ring $r_2 < r < r_1$



# ROC for Different Signals

- Nature of ROC is different for different type of signals.
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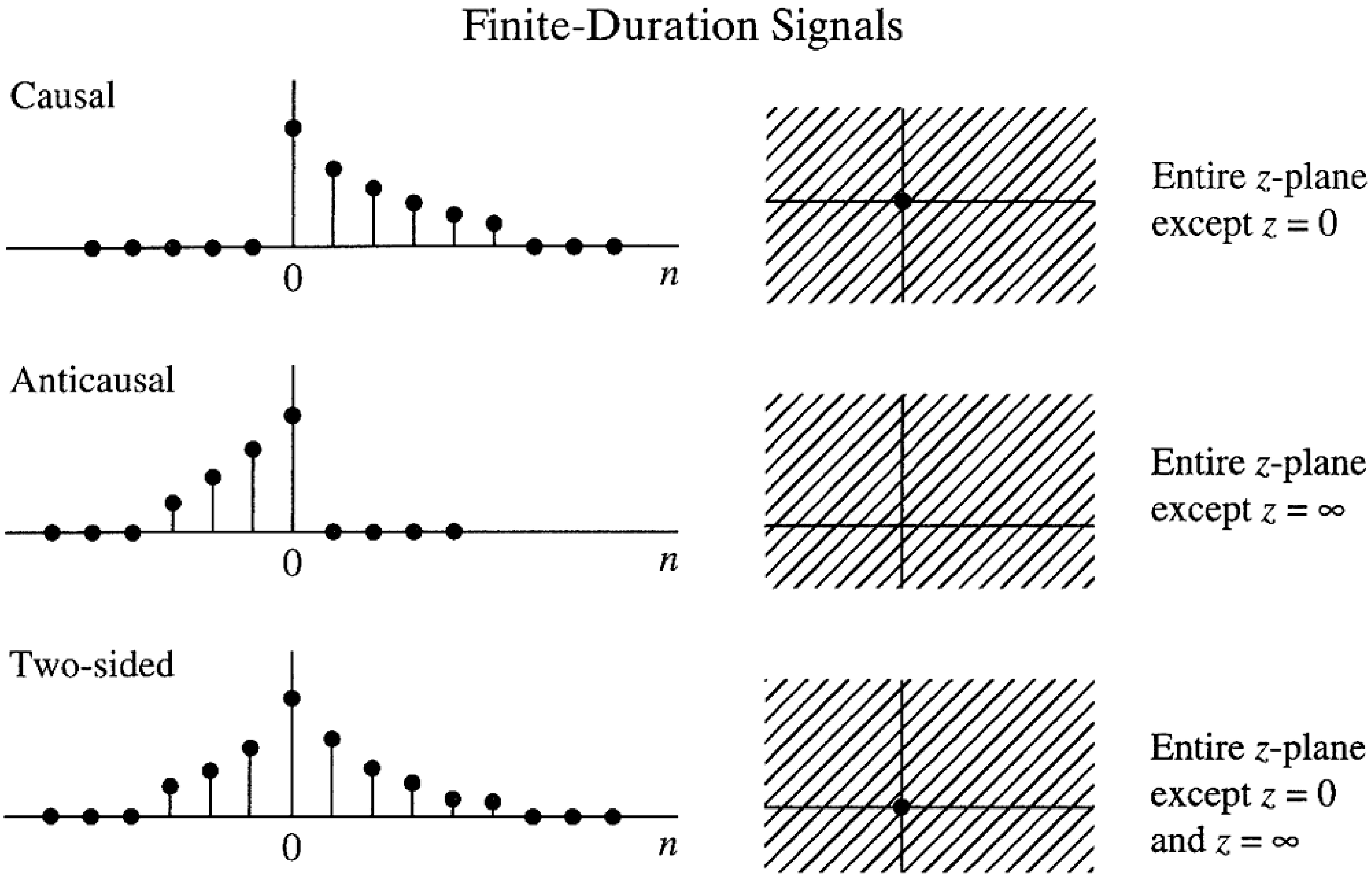

# ROC for Different Signals

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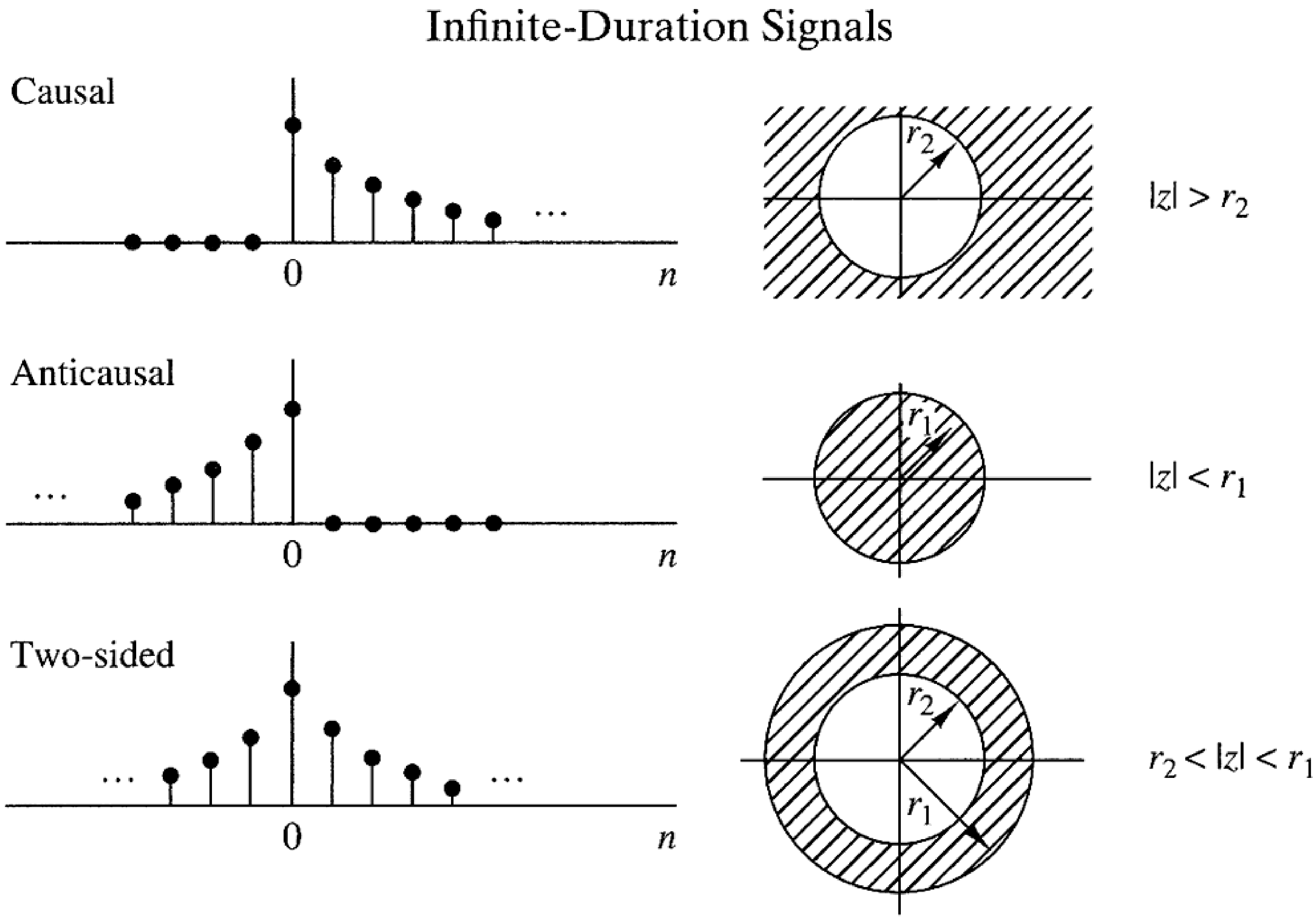


# Some examples

- Finite duration signals:

(a) $x_1(n) = \{1, 2, 5, 7, 0, 1\}$

(b) $x_2(n) = \{1, 2, \underset{\uparrow}{5}, 7, 0, 1\}$ ROC?????

$x_5(n) = \delta(n)$

$x_6(n) = \delta(n-k), k > 0$

- Infinite duration signals:

$x(n) = (\frac{1}{2})^n u(n)$ $\quad$ $x(n) = \alpha^n u(n)$

$x(n) = u(n)$

$$x(n) = -\alpha^n u(-n-1) = \begin{cases} 0, & n \geq 0 \\ -\alpha^n, & n \leq -1 \end{cases}$$

$x(n) = \alpha^n u(n) + b^n u(-n-1)$

# Some examples

(a) $X_1(z) = 1 + 2z^{-1} + 5z^{-2} + 7z^{-3} + z^{-5}$, ROC: entire $z$-plane except $z = 0$

(b) $X_2(z) = z^2 + 2z + 5 + 7z^{-1} + z^{-3}$, ROC: entire $z$-plane except $z = 0$ and $z = \infty$

$X_5(z) = 1$[i.e., $\delta(n) \overset{z}{\longleftrightarrow} 1$], ROC: entire $z$-plane

$X_6(z) = z^{-k}$[i.e., $\delta(n-k) \overset{z}{\longleftrightarrow} z^{-k}$], $k > 0$, ROC: entire $z$-plane except $z = 0$

- $x[n] = (1/2)^n u[n]$

- The z-transform of x[n] is power series $X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$

$$X(z) = \sum_{n=0}^{\infty} (1/2)^n z^{-n} = \sum_{n=0}^{\infty} \left(1/2\, z^{-1}\right)^n = \frac{1}{1 - \frac{1}{2} z^{-1}} \qquad ROC: |z| > \frac{1}{2}$$

Remember: $\boxed{\sum_{n=0}^{\infty} \gamma^n = \frac{1}{1-\gamma} \quad for\ |\gamma| < 1}$

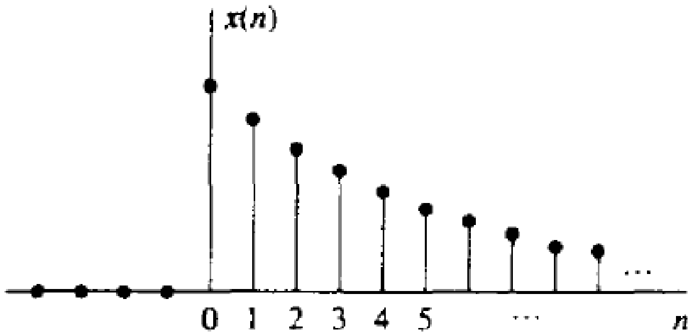
# Some examples

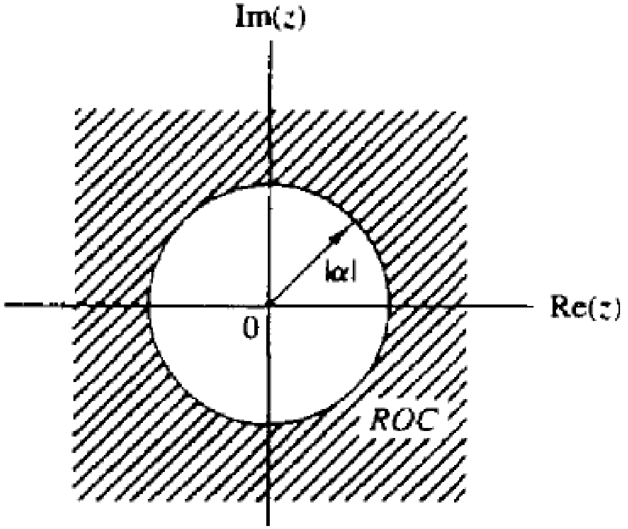
- $x[n] = \alpha^n u[n]$

$$X(z) = \frac{1}{1-\alpha z^{-1}} \qquad ROC: |z| > |\alpha|$$

- $x[n] = u[n]$

$$X(z) = \frac{1}{1-z^{-1}} \qquad ROC: |z| > 1 \qquad \because \alpha = 1$$

- 

# Some examples

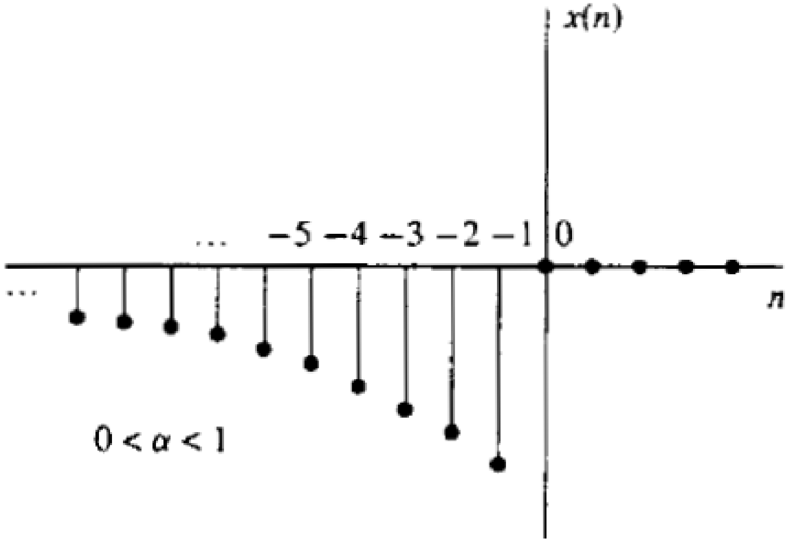
- $x[n] = -\alpha^n u[-n-1]$

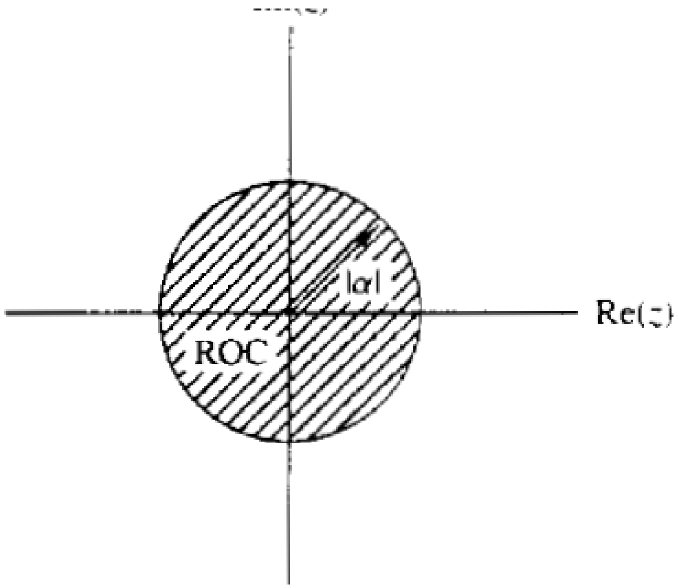
$$X(z) = \sum_{n=-\infty}^{-1} -\alpha^n z^{-n} = -\sum_{n=-\infty}^{-1} \left(\alpha z^{-1}\right)^n = -\sum_{n=1}^{\infty} \left(\alpha^{-1} z\right)^n = -\frac{\alpha^{-1} z}{1-\alpha^{-1} z}$$

$$X(z) = \frac{1}{1-\alpha z^{-1}}$$

Remember: $\boxed{\sum_{n=1}^{\infty} \gamma^n = \frac{\gamma}{1-\gamma} \quad \text{for } |\gamma| < 1}$

$$ROC: \left|\alpha^{-1} z\right| < 1 \Rightarrow |z| < |\alpha|$$



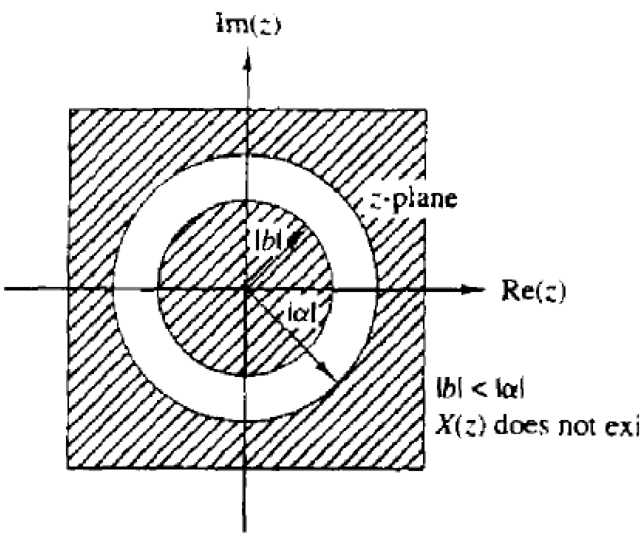
# Some examples

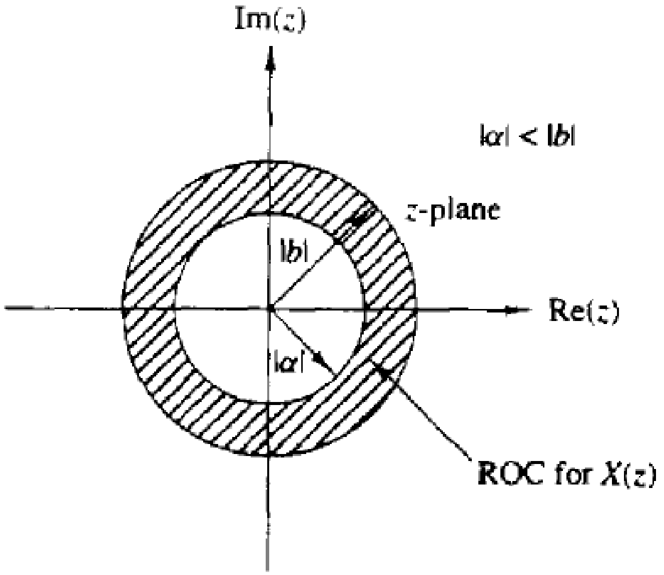
- $x[n] = \alpha^n u[n] + \beta^n u[-n-1]$

$$X(z) = \sum_{n=-\infty}^{-1} \beta^n z^{-n} + \sum_{n=0}^{\infty} \alpha^n z^{-n} = \sum_{n=-\infty}^{-1} \left(\beta z^{-1}\right)^n + \sum_{n=0}^{\infty} (\alpha z^{-1})^n$$

$$= \sum_{n=1}^{\infty} \left(\beta^{-1} z\right)^n + \sum_{n=0}^{\infty} (\alpha z^{-1})^n = \frac{\beta^{-1} z}{1-\beta^{-1} z} + \frac{1}{1-\alpha z^{-1}}$$

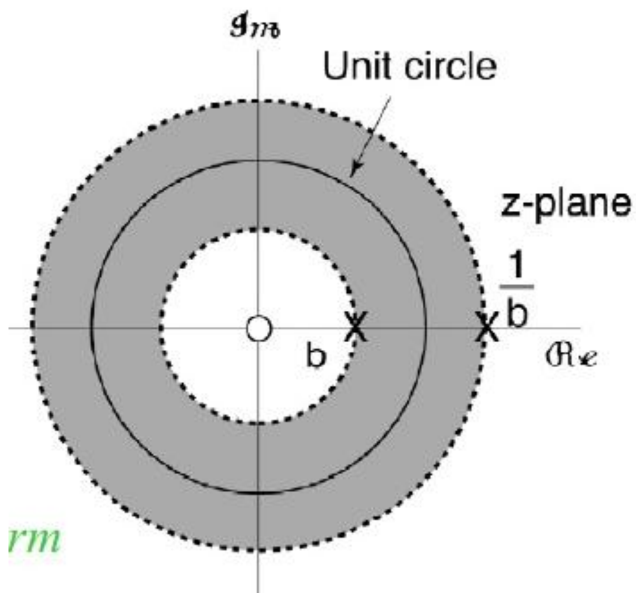
$$= \frac{1}{1-\alpha z^{-1}} - \frac{1}{1-\beta z^{-1}} \qquad ROC: |z| > |\alpha| \ \ and \ \ |z| < |\beta| \Rightarrow |\alpha| < |z| < |\beta|$$



# Some Points about ROC

- The DTFT of a signal x[n] exists (converges) if and only if the ROC of its z-transform includes the unit circle.




- ROC cannot contain any poles.

- ROC must be a connected region.

# Properties of z-Transform

- Linearity:

If, $x_1[n] \xleftrightarrow{Z} X_1(z)$ and $x_2[n] \xleftrightarrow{Z} X_2(z)$

then, $\boxed{x[n] = A x_1[n] + B x_2[n] \xleftrightarrow{Z} X(z) = A X_1(z) + B X_2(z)}$

ROC: Overall ROC is the intersection of individual ROC.

- Proof:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} = \sum_{n=-\infty}^{\infty} \left\{ A x_1[n] + B x_2[n] \right\} z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} A x_1[n] z^{-n} + \sum_{n=-\infty}^{\infty} B x_2[n] z^{-n} = A X_1(z) + B X_2(z)$$

# Properties of z-Transform

- Time Shifting:

If $x[n] \xleftrightarrow{Z} X(z)$

then, $\boxed{x[n-n_0] \xleftrightarrow{Z} z^{-n_0} X(z)}$

ROC: same as that of $X(z)$ except $z=0$ for $n_0>0$ and $z=\infty$ for $n_0<0$.

- Proof: Let, $x[n-n_0] \xleftrightarrow{Z} X_1(z)$

$$X_1(z) = \sum_{n=-\infty}^{\infty} x[n-n_0] z^{-n}$$ Let, $p=n-n_0$

$$X_1(z) = \sum_{p=-\infty}^{\infty} x[p] z^{-(p+n_0)} = \sum_{p=-\infty}^{\infty} x[p] z^{-p} z^{-n_0}$$

$$= z^{-n_0} \sum_{p=-\infty}^{\infty} x[p] z^{-p} = z^{-n_0} X(z)$$

# Properties of z-Transform

- Scaling in z domain:

If $x[n] \overset{Z}{\longleftrightarrow} X(z)$

then, $\boxed{a^n x[n] \overset{Z}{\longleftrightarrow} X\left(\frac{z}{a}\right)}$

ROC: If ROC of $X(z)$ is $r_1 < |z| < r_2$ then new ROC is $|a|r_1 < |z| < |a|r_2$

- Proof: Let, $a^n x[n] \overset{Z}{\longleftrightarrow} X_1(z)$

$$X_1(z) = \sum_{n=-\infty}^{\infty} a^n x[n] z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x[n] \left(a^{-1} z\right)^{-n} = X\left(\frac{z}{a}\right)$$

# Properties of z-Transform

- Time Reversal:

If $x[n] \overset{Z}{\longleftrightarrow} X(z)$ ROC: $r_1 < |z| < r_2$

then, $\boxed{x[-n] \overset{Z}{\longleftrightarrow} X\left(z^{-1}\right)}$ ROC: $\frac{1}{r_2} < |z| < \frac{1}{r_1}$

Proof: Let,

$$x[-n] \overset{Z}{\longleftrightarrow} X_1(z)$$

$$X_1(z) = \sum_{n=-\infty}^{\infty} x[-n] z^{-n}$$

Let, $p=-n$

$$= \sum_{p=\infty}^{-\infty} x[p] z^{p} = \sum_{p=-\infty}^{\infty} x[p] \left(z^{-1}\right)^{-p} = X\left(z^{-1}\right)$$

# Properties of z-Transform

- Differentiation in z domain:

If $x[n] \xleftrightarrow{z} X(z)$

$$\boxed{n\,x[n] \xleftrightarrow{z} -z\frac{d\,X(z)}{d\,z}}$$

ROC: same as that of $X(z)$

Proof:

$$\frac{d\,X(z)}{d\,z} = \sum_{n=-\infty}^{\infty} x[n](-n)z^{-n-1} = -z^{-1}\sum_{n=-\infty}^{\infty} n\,x[n]z^{-n}$$

$$-z\frac{d\,X(z)}{d\,z} = \sum_{n=-\infty}^{\infty} n\,x[n]z^{-n}$$

So, $n\,x[n] \xleftrightarrow{z} -z\dfrac{d\,X(z)}{d\,z}$

# PROPERTIES OF Z-TRANSFORM

- Convolution of two sequences:

If $x_1[n] \xleftrightarrow{Z} X_1(z)$ and $x_2[n] \xleftrightarrow{Z} X_2(z)$

then, $\boxed{x[n] = x_1[n] * x_2[n] \xleftrightarrow{Z} X(z) = X_1(z)X_2(z)}$

ROC: ROC of $X(z)$ is at least the intersection of that of $X_1(z)$ and $X_2(z)$

Proof:

$$X(z) = \sum_{n=-\infty}^{\infty} \left(x_1[n] * x_2[n]\right) z^{-n} = \sum_{n=-\infty}^{\infty} \left[ \sum_{k=-\infty}^{\infty} x_1[k] x_2[n-k] \right] z^{-n}$$

$$= \sum_{k=-\infty}^{\infty} x_1[k] \left[ \sum_{n=-\infty}^{\infty} x_2[n-k] z^{-n} \right]$$

$$= X_2(z) \sum_{k=-\infty}^{\infty} x_1[k] z^{-k} = X_2(z) X_1(z)$$

# Properties of z-Transform

- Correlation of two sequences:

If $x_1[n] \xleftrightarrow{Z} X_1(z)$ and $x_2[n] \xleftrightarrow{Z} X_2(z)$

then, $$\boxed{r_{x_1x_2}[l] = \sum_{n=-\infty}^{\infty} x_1[n]x_2[n-l] \xleftrightarrow{Z} R_{x_1x_2}(z) = X_1(z)X_2\left(z^{-1}\right)}$$

ROC: ROC is at least the intersection of that of $X_1(z)$ and $X_2(z^{-1})$

# Some Examples

- $x[n] = \left[3\left(2^n\right) - 4\left(3^n\right)\right]u[n]$

Using linearity, we get: $X(z) = \dfrac{3}{1-2z^{-1}} - \dfrac{4}{1-3z^{-1}}$

ROC: $|z| > 3$

- $x[n] = \left(\cos\omega_0 n\right)u[n]$

$$(\cos\omega_0 n)u[n] = \frac{1}{2}e^{j\omega_0 n}u[n] + \frac{1}{2}e^{-j\omega_0 n}u[n]$$

Using linearity,

$$X(z) = \frac{1/2}{1-e^{j\omega_0}z^{-1}} - \frac{1/2}{1-e^{-j\omega_0}z^{-1}} = \frac{1-z^{-1}\cos\omega_0}{1-2z^{-1}\cos\omega_0 + z^{-2}}$$

ROC: $|z| > \left|e^{\pm j\omega_0}\right| \Rightarrow |z| > 1$

# Some Examples

- $x[n] = (\sin \omega_0 n) u[n]$

Similarly,

$$X(z) = \frac{z^{-1} \sin \omega_0}{1 - 2z^{-1} \cos \omega_0 + z^{-2}}$$

ROC: $|z| > \left| e^{\pm j\omega_0} \right| \Rightarrow |z| > 1$

- $x(n) = \begin{cases} 1, & 0 \le n \le N-1 \\ 0, & \text{elsewhere} \end{cases}$

Remember: $\boxed{\sum_{n=0}^{N} \gamma^n = \frac{1-\gamma^{N+1}}{1-\gamma} \quad \mathit{for}\ |\gamma| < 1}$

$$x[n] = u[n] - u[n-N]$$

Using linearity and time shifting,

$$X(z) = \frac{1}{1-z^{-1}} - \frac{z^{-N}}{1-z^{-1}}$$

ROC: $|z| > 1$

# Some Examples

- $x[n]=a^n(\cos\omega_0 n)u[n]$

Using Scaling in z domain property,

$$X(z)=\frac{1-(z/a)^{-1}\cos\omega_0}{1-2(z/a)^{-1}\cos\omega_0+(z/a)^{-2}}$$

$$=\frac{1-az^{-1}\cos\omega_0}{1-2az^{-1}\cos\omega_0+a^2z^{-2}}$$

ROC: $|z|>|a|$

- $x[n]=u[-n]$

By time inversion property,

$$X(z)=\frac{1}{1-(z^{-1})^{-1}}=\frac{1}{1-z}$$

ROC: $|z|<1$

# Some Examples

- $x[n] = n\,a^n u[n]$

Let $x_1[n] = a^n u[n] \Rightarrow X_1(z) = \dfrac{1}{1 - az^{-1}}$

Using differentiation in z domain property,

$$X(z) = -z\,\frac{d\,\dfrac{1}{1-az^{-1}}}{dz} = \frac{a\,z^{-1}}{\left(1 - a\,z^{-1}\right)^2}$$

ROC: $|z| > |a|$

# Convolution by z-transform

- z-transform can be utilized to simplify the convolution.
- Steps to compute convolution using z-transform are:

**1.** Compute the $z$-transforms of the signals to be convolved.

$$X_1(z) = Z\{x_1(n)\}$$

$$\text{(time domain} \longrightarrow z\text{-domain)}$$

$$X_2(z) = Z\{x_2(n)\}$$

**2.** Multiply the two $z$-transforms.

$$X(z) = X_1(z)X_2(z), \qquad (z\text{-domain})$$

**3.** Find the inverse $z$-transform of $X(z)$.

$$x(n) = Z^{-1}\{X(z)\}, \qquad (z\text{-domain} \longrightarrow \text{time domain})$$

# Convolution -Example

- Compute the convolution $x(n)$ of the signals

$$x_1(n) = \{1, -2, 1\}$$

$$x_2(n) = \begin{cases} 1, & 0 \le n \le 5 \\ 0, & \text{elsewhere} \end{cases}$$

$$X_1(z) = 1 - 2z^{-1} + z^{-2}$$

$$X_2(z) = 1 + z^{-1} + z^{-2} + z^{-3} + z^{-4} + z^{-5}$$

$$X(z) = X_1(z)X_2(z) = 1 - z^{-1} - z^{-6} + z^{-7}$$

$$x(n) = \{\underset{\uparrow}{1}, -1, 0, 0, 0, 0, -1, 1\}$$

# Inverse z-transform

- The procedure for transforming from the z-domain to the time domain is called the *inverse z-transform*.

- There are several methods for inversion.
  - Direct inversion based on contour integration.
  - Inversion based on partial fraction expansion
  - Inversion based on power series expansion by long division

- Direct Inversion:
  - It is based on Cauchy integral theorem and uses contour integration. Rarely used in practice for inversion.

$$\boxed{x(n) = \frac{1}{2\pi j} \oint_C X(z) z^{n-1}\, dz}$$

Where, *C* is any contour within the ROC of *X*(*z*) enclosing the origin.

# Inverse Z-transform

- Power series expansion:

  - Expand $X(z)$ into a power series of the form:

$$X(z) = \sum_{n=-\infty}^{\infty} c_n z^{-n}$$

  - then, $x[n]=c_n$ for all $n$.
  - when $X(z)$ is rational, the expansion is obtained by long division.
  - when the ROC of X(z) is such that x[n] is to be a causal signal, we arrange the terms in numerator and denominator such that we get expansion in negative powers of z.
  - Similarly for anti causal X(z), we expect expansion in positive powers of z.

# Inverse Z-transform

- $X(z)=\dfrac{1}{1-1.5z^{-1}+0.5z^{-2}}$ ROC: $|z|>1$

Looking at ROC, the signal must be a causal signal. So,

$$X(z)=\frac{1}{1-\frac{3}{2}z^{-1}+\frac{1}{2}z^{-2}}=1+\frac{3}{2}z^{-1}+\frac{7}{4}z^{-2}+\frac{15}{8}z^{-3}+\frac{31}{16}z^{-4}+\cdots$$

By comparing

$$x(n)=\{\underset{\uparrow}{1},\frac{3}{2},\frac{7}{4},\frac{15}{8},\frac{31}{16},\ldots\}$$

- $X(z)=\dfrac{1}{1-1.5z^{-1}+0.5z^{-2}}$ ROC: $|z|<0.5$

Looking at ROC, the signal must be an anticausal signal. So,

# Inverse z-transform

$$X(z) = \frac{1}{1 - 1.5z^{-1} + 0.5z^{-2}} \qquad \text{ROC: } |z| < 0.5$$

Looking at ROC, the signal must be an anticausal signal. So,

$$
\begin{array}{r|l}
 & 2z^2 + 6z^3 + 14z^4 + 30z^5 + 62z^6 + \cdots \\
\hline
\frac{1}{2}z^{-2} - \frac{3}{2}z^{-1} + 1 & 1 \\
 & \underline{1 - 3z + 2z^2} \\
 & \quad 3z - 2z^2 \\
 & \quad \underline{3z - 9z^2 + 6z^3} \\
 & \qquad 7z^2 - 6z^3 \\
 & \qquad \underline{7z^2 - 21z^3 + 14z^4} \\
 & \qquad\quad 15z^3 - 14z^4 \\
 & \qquad\quad \underline{15z^3 - 45z^4 + 30z^5} \\
 & \qquad\qquad 31z^4 - 30z^5
\end{array}
$$

$$x(n) = \{\cdots 62, 30, 14, 6, 2, 0, \underset{\uparrow}{0}\}$$

# Inverse z-transform

- Partial Fraction Expansion and Lookup Table:

  - Express the function $X(z)$ as a linear combination

$$X(z) = \alpha_1 X_1(z) + \alpha_2 X_2(z) + \cdots + \alpha_K X_K(z)$$

  - where $X_1(z)$, $X_2(z)$……..are standard functions whose inverse are known. Then by linearity property,

$$x(n) = \alpha_1 x_1(n) + \alpha_2 x_2(n) + \cdots + \alpha_K x_K(n)$$

  - Partial fraction expansion is commonly used when X(z) is a rational fraction of the form

$$X(z) = \frac{B(z)}{A(z)} = \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{1 + a_1 z^{-1} + \cdots + a_N z^{-N}}$$

# Inverse z-transform

Partial Fraction Expansion:

$$X(z) = \frac{B(z)}{A(z)} = \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{1 + a_1 z^{-1} + \cdots + a_N z^{-N}}$$

- A rational fraction shown above is called proper if, *M*<*N*.

- If it is improper, it should be converted to proper by dividing numerator by denominator resulting in a polynomial and a proper fraction

$$X(z) = \frac{B(z)}{A(z)} = c_0 + c_1 z^{-1} + \cdots + c_{M-N} z^{-(M-N)} + \frac{B_1(z)}{A(z)}$$

- If *X*(z) is a proper fraction, it can be written in the form:

$$X(z) = \frac{b_0 z^N + b_1 z^{N-1} + \cdots + b_M z^{N-M}}{z^N + a_1 z^{N-1} + \cdots + a_N}$$

with only positive powers of z however is not proper

# Inverse Z-transform

- However,

$$\frac{X(z)}{z} = \frac{b_0 z^{N-1} + b_1 z^{N-2} + \cdots + b_M z^{N-M-1}}{z^N + a_1 z^{N-1} + \cdots + a_N}$$

will always be proper.

- To find the partial fraction expansion of above, first we find the poles (roots of denominator polynomial) $P_1, P_2, P_3 \ldots P_N$.

- If the poles are distinct and real, expansion will be of the form:

$$\frac{X(z)}{z} = \frac{A_1}{z - p_1} + \frac{A_2}{z - p_2} + \cdots + \frac{A_N}{z - p_N}$$

# Inverse z-transform

- However, $$\frac{X(z)}{z} = \frac{b_0 z^{N-1} + b_1 z^{N-2} + \cdots + b_M z^{N-M-1}}{z^N + a_1 z^{N-1} + \cdots + a_N}$$

  will always be proper.

- To find the partial fraction expansion of above, first we find the poles (roots of denominator polynomial) $P_1, P_2, P_3 \ldots P_N$.

- If the poles are **distinct and real**, expansion will be of the form:

$$\frac{X(z)}{z} = \frac{A_1}{z - p_1} + \frac{A_2}{z - p_2} + \cdots + \frac{A_N}{z - p_N}$$

- The constants are determined as:

$$A_k = \left. \frac{(z - p_k)X(z)}{z} \right|_{z=p_k}, \qquad k = 1, 2, \ldots, N$$

# Inverse z-transform

- If the poles are **distinct and complex**, expansion will be of the same form except that complex poles exist in conjugate pairs and the corresponding constants are also conjugate of each other.
- Determination of constants can be done in a similar way.
- If the poles are **same and have multiplicity of *m***, the partial fraction expansion will be of the form:

$$\frac{A_{1k}}{z-p_k}+\frac{A_{2k}}{(z-p_k)^2}+\cdots+\frac{A_{mk}}{(z-p_k)^m}$$

- where,

$$A_{mk}=\frac{X(z)}{z}(z-p_k)^m\Big|_{z=p_k} \qquad A_{m-1k}=\frac{d}{dz}\left[\frac{X(z)}{z}(z-p_k)^m\right]\Bigg|_{z=p_k}$$

and so on…….

# Examples

- Determine inverse z-transform of $X(z)=\dfrac{1}{1-1.5z^{-1}+0.5z^{-2}}$

if, **(a)** ROC: $|z|>1$ **(b)** ROC: $|z|<0.5$ **(c)** ROC: $0.5<|z|<1$

- Eliminating -ve powers of z: $X(z)=\dfrac{z^2}{z^2-1.5z+0.5}$

- Dividing by z and obtaining poles:

$$\frac{X(z)}{z}=\frac{z}{(z-1)(z-0.5)}$$

- Obtaining partial fraction:

$$\frac{X(z)}{z}=\frac{A_1}{z-1}+\frac{A_2}{z-0.5}$$

- Determining constants: $A_1=\dfrac{X(z)}{z}(z-1)\Big|_{z=1}=\dfrac{z}{(z-0.5)}\Big|_{z=1}=2$

# Examples

$$A_2 = \frac{X(z)}{z}(z-0.5)\bigg|_{z=0.5} = \frac{z}{(z-1)}\bigg|_{z=0.5} = -1$$

- The partial fraction expansion thus is $\dfrac{X(z)}{z} = \dfrac{2}{z-1} - \dfrac{1}{z-0.5}$

and $$X(z) = \frac{2}{1-z^{-1}} - \frac{1}{1-0.5z^{-1}}$$

- Finally, for ($a$), $x[n]$ must be causal. So,

$$x[n] = 2u[n] - 0.5^n u[n]$$

- for ($b$), $x[n]$ must be anticausal. So,

$$x[n] = -2u[-n-1] + 0.5^n u[-n-1]$$

- for (c), $x[n]$ must be two sided. So,

$$x[n] = -2u[-n-1] - 0.5^n u[n]$$

# Examples

Determine the causal signal $x(n)$ having the $z$-transform

$$X(z) = \frac{1}{(1+z^{-1})(1-z^{-1})^2}$$

$$\frac{X(z)}{z} = \frac{z^2}{(z+1)(z-1)^2}$$

$$\frac{X(z)}{z} = \frac{z^2}{(z+1)(z-1)^2} = \frac{A_1}{z+1} + \frac{A_2}{z-1} + \frac{A_3}{(z-1)^2}$$

$$A_1 = \left.\frac{(z+1)X(z)}{z}\right|_{z=-1} = \frac{1}{4}$$

$$A_3 = \left.\frac{(z-1)2X(z)}{z}\right|_{z=1} = \frac{1}{2}$$

$$A_2 = \frac{d}{dz}\left[\frac{(z-1)^2 X(z)}{z}\right]_{z=1} = \frac{3}{4}$$

# Examples

$$X(z) = \frac{1}{4}\frac{1}{1+z^{-1}} + \frac{3}{4}\frac{1}{1-z^{-1}} + \frac{1}{2}\frac{z^{-1}}{(1-z^{-1})2}$$

$$x(n) = \frac{1}{4}(-1)^n u(n) + \frac{3}{4}u(n) + \frac{1}{2}nu(n) = \left[\frac{1}{4}(-1)^n + \frac{3}{4} + \frac{n}{2}\right]u(n)$$

**Remember:**

$$Z^{-1}\left\{\frac{1}{1-p_k z^{-1}}\right\} = \begin{cases} (p_k)^n u(n), & \text{if ROC: } |z| > |p_k| \\ & \text{(causal signals)} \\ -(p_k)^n u(-n-1), & \text{if ROC: } |z| < |p_k| \\ & \text{(anticausal signals)} \end{cases}$$

$$Z^{-1}\left(\frac{A_k}{1-p_k z^{-1}} + \frac{A_k^*}{1-p_k^* z^{-1}}\right) = 2|A_k| r_k^n \cos(\beta_k n + \alpha_k) u(n)$$

$$Z^{-1}\left\{\frac{pz^{-1}}{(1-pz^{-1})^2}\right\} = np^n u(n)$$

# Rational z-transforms

- z-transforms for which $X(z)$ is a rational function (ratio of two polynomials in z or $z^{-1}$).

$$X(z) = \frac{B(z)}{A(z)} = \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{a_0 + a_1 z^{-1} + \cdots + a_N z^{-N}} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

- Also,

$$X(z) = \frac{B(z)}{A(z)} = \frac{b_0 z^{-M}}{a_0 z^{-N}} \frac{z^M + (b_1/b_0) z^{M-1} + \cdots + b_M/b_0}{z^N + (a_1/a_0) z^{N-1} + \cdots + a_N/a_0}$$

- Alternatively,

$$X(z) = \frac{B(z)}{A(z)} = \frac{b_0}{a_0} z^{-M+N} \frac{(z - z_1)(z - z_2) \cdots (z - z_M)}{(z - p_1)(z - p_2) \cdots (z - p_N)}$$

- Zeros: The values of z for which $X(z)=0$ *e.g.* $z_1$, $z_2$, ….
- Poles: The values of z for which $X(z)=\infty$ *e.g.*, $p_1$, $p_2$, ….

# Rational z-transforms

- In compact form,

$$X(z) = Gz^{N-M} \frac{\prod_{k=1}^{M}(z - z_k)}{\prod_{k=1}^{N}(z - p_k)}$$

- Clearly, $X(z)$ has $M$ finite zeros at $z_1$, $z_2$, …..$z_M$ and $N$ finite poles at $p_1$, $p_2$, …$p_N$

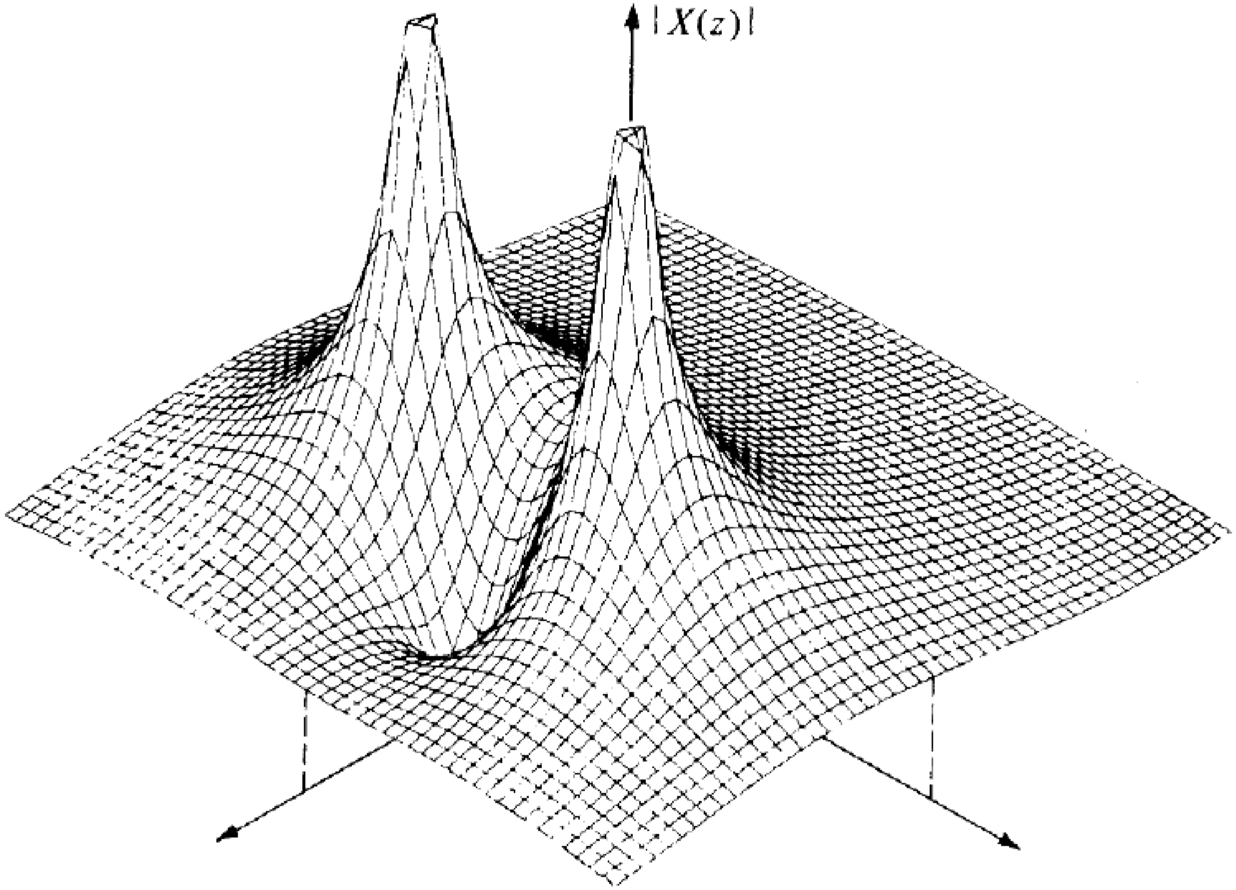
- However, total number of poles and zeros are always same.

- If $N>M$, then $N$-$M$ zeros occur at $z=0$ (origin) and if $M>N$, then $M$-$N$ poles occur at $z=0$ (origin)

- There may be zeros and poles at infinity also.
- There can be no poles in ROC.

# Pole –Zero Plot

$$X(z) = \frac{z^{-1} - z^{-2}}{1 - 1.2732z^{-1} + 0.81z^{-2}}$$

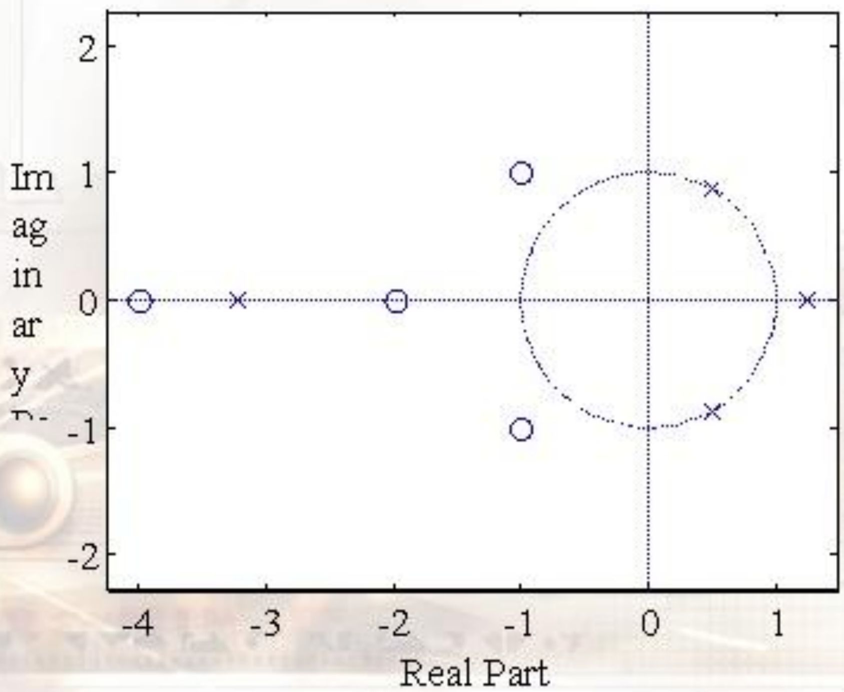


# Pole –Zero Plot

- Pole and zeros of a z-transform is usually represented by a pole-zero plot in z plane.

$$G(z) = \frac{2z^4 + 16z^3 + 44z^2 + 56z + 32}{3z^4 + 3z^3 - 15z^2 + 18z - 12}$$

obtained using MATLAB is shown below




×-pole
O-zero

# Z-transform and LTI Systems

- For a LTI system, the input output relationship in z domain looks like:

$$Y(z) = H(z)\,X(z)$$

where, $Y(z)$, $X(z)$ and $H(z)$ are the z-transforms of the output sequence, the input sequence and the impulse response of the system.

- The system function is thus,

$$H(z) = \frac{Y(z)}{X(z)}$$

-

# Z-transform and LTI Systems

- When the system is described by difference equation,

$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k)$$

$$Y(z) = -\sum_{k=1}^{N} a_k Y(z) z^{-k} + \sum_{k=0}^{M} b_k X(z) z^{-k}$$

- the system function is then,

$$\frac{Y(z)}{X(z)} = H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}}$$

# Some more points

- A LTI system is causal if its impulse response is causal. So, it can be said that it is causal if and only if, ROC of the system function is exterior of a circle including the $\infty$.

- LTI system is BIBO stable if and only if the ROC of the system function includes unit circle.

- A LTI system is causal and stable if and only if, all the poles of system function are inside the unit circle.