2 Establishing a Theoretical Basis

The trend to approach merge tree construction from a decentralized point of view, focusing on critical points and monotone paths connecting them, can be observed in most of the recent work in this field. [1] [2] [6] [4] In this section we will be looking into the definitions regarding merge tree construction to understand this decentralized approach. With this perspective in mind the approach in [2] comes as a natural parallelization of the original ordered union-find algorithm in [7]. This then leads us to the question of why local ordering is present in [2] and how the concept could be adapted to no longer depend on it. The answers to this question will then lead to the foundation of our novel merge tree construction algorithm.

2.1 Merge Tree Definition

Consider a smooth, real-valued function f on a manifold M. A merge tree is either a join or a split tree, the following definition covers the join tree, the definition of a split tree follows similarily with super-level sets instead of sub-level sets. A join tree represents the development of the sub-level set $M^a = f^{-1}(-\infty, a]$ with growing a, with respect to the number of connected components. There are two types of events that can change the number of connected components in the developing sub-level set at a value a: Creation of a new component and the merging of two such components. These events are represented by the two types of nodes in a join tree: leafs representing minima (extrema in general) and inner nodes representing join nodes (saddles in general). Let in the following $M'^a = f^{-1}(-\infty, a[$ be the sub-level set just "before" a. While in general $f^{-1}(a)$ may contain more than one point, when defining the two event types we will refer to a point p^a corresponding to $f^{-1}(a)$. This is meaningful because, if $f^{-1}(a)$ contains more than one point, at most one of those points will fit either of the following criteria, because f is assumed to be a Morse function.

Minima/Extrema If p^a has at least one neighborhood in M without any points whos image under f are part of the sub-level set M'^a, then p^a corresponds to a new connected component of M^a. In a join tree every value a with this property will be represented by a leaf and called minimum.

Join Nodes/Saddles If all neighborhoods in M of p^a include points whos images under f are part of at least two different connected components of M'^a, then p^a creates a connection between those connected components, joining them into one connected component of M^a. In a join tree every value a with this property will be represented by an inner node, sharing an edge with the minima or inner nodes representing the newly joined components. These nodes are called join nodes and the edges are called arcs.

For the sake of illustration one can let M be the R^2 and visualize the values of f as height of a smooth landscape embedded in R^3. Cutting through this landscape with a plane with height a and discarding any portions "above" the cut leaves us with the