# NLP Exercises

**3** A company uses a raw material to produce two types of products. When processed, each unit of raw material yields 2 units of product 1 and 1 unit of product 2. If $x_1$ units of product 1 are produced, then each unit can be sold for $\$49 - x_1$, if $x_2$ units of product 2 are produced, then each unit can be sold for $\$30 - 2x_2$. It costs $\$5$ to purchase and process each unit of raw material. Use the Kuhn–Tucker conditions to determine how the company can maximize profits.

**Solution:**

Let R $=$ units of raw material purchased.

We wish to solve

$$\begin{array}{lll} max & z = (49 - x_1)x_1 + (30 - 2x_2)x_2 - 5R & \\ s.t. & x_1 \le 2R & (1) \\ & x_2 \le R & (2) \\ & x_1, x_2 \ge 0 & \end{array}$$

Since the objective function is concave and the constraints are linear, the K-T conditions will yield an optimal solution. The K- T conditions are

$$\begin{array}{ll} 49 - 2x_1 - \lambda_1 = 0 & (3) \\ 30 - 4x_2 - \lambda_2 = 0 & (4) \\ -5 + 2\lambda_1 - \lambda_2 = 0 & (5) \\ \lambda_1(x_1 - 2R) = 0 & (6) \\ \lambda_2(x_2 - R) = 0 & (7) \\ x_1, x_2, \lambda_1, \lambda_2 \ge 0 & \end{array}$$

Try $\lambda_1 > 0$ and $\lambda_2 = 0$. Then (6) yields $x_1 = 2R$. From (5), $\lambda_1 = 2.5$. Then (3) yields $x_1 = 23.25$ and (4) yields $x_2 = 7.5$.

Since $x_1 = 2R$, we find that $R = 11.625$. All K-T conditions and original constraints are satisfied so we have found an optimal solution. Optimal solution: $R = 11.625, x_1 = 23.25, x_2 = 7.5$.

**5** Use Golden Section Search to locate, within 0.5, the optimal solution to

$$\begin{array}{ll} max & 3x - x^2 \\ s.t. & 0 \le x \le 2R \end{array}$$

**Solution:**

$x_1 = 5 - .618(5) = 1.91,\ x_2 = 0 + .618(5) = 3.09$

$f(x_1) = 2.08 > f(x_2) = -.28$, so new interval of uncertainty is $[0, 3.09]$.

$x_3 = 3.09 - .619(3.09) = 1.18,\ x_4 = 1.91.$

$f(x_3) = 2.15 > f(x_4) = 2.08$, so new interval of uncertainty is $[0, 1.91)$.

$x_6 = 1.18,\ x_5 = 1.91 - .618(1.91) = .73$

$f(x_6) = 2.15 > f(x_5) = 1.66$, so new interval of uncertainty is $(.73, 1.91]$.

$x_7 = 1.18,\ x_8 = .73 + .618(1.18) = 1.46$

$f(x_8) = 2.25 > f(x_7) = 2.15$, so new interval of uncertainty is $(1.18, 1.91]$.

Now $x_9 = 1.46$, $x_{10} = 1.18 + .618(.73) = 1.63$, and $f(x_9) = 2.25 > f(x_{10}) = 2.23$, so new interval of uncertainty is $(1.18, 1.63]$. This interval has width less than .50, so we are finished. (Actual maximum occurs for $x = 1.5$.)

**6** Perform two iterations of the method of steepest ascent in an attempt to maximize

$$f(x_1, x_2) = (x_1 + x_2)e^{-(x_1+x_2)} - x_1$$

Begin at the point $(0,1)$.

**Solution:**

$$\nabla f(x_1, x_2) = [(1 - x_1 - x_2)e^{-(x_1+x_2)} - 1, (1 - x_1 - x_2)e^{-(x_1+x_2)}]$$

Iteration 1:

$\nabla f(0,1) = [-1,0]$. Thus, new point is $(-t, 1)$, where $t \geq 0$.

Maximize $f(t) = (1-t)e^{t-1} + t$. $f'(t) = (1-t)e^{t-1} - e^{t-1} + 1 = 0$ for $te^{t-1} = 1$ or $t = 1$.

Thus, new point is (-1,1).

Iteration 2:

$\nabla f(\text{-}1, 1) = [0, 1]$, so new point is $[\text{-}1, 1 + t]$.

We choose $t \geq 0$ to maximize $h(t) = te^{-t} + 1$, $h'(t) = -te^{-t} + e^{-t} = 0$ for $t = 1$.

Thus, new point is (-1, 2).

**8** Solve the following NLP:

$$\begin{aligned} max \quad & xyz \\ s.t. \quad & 2x + 3y + 4w = 36 \end{aligned}$$

**Solution:**

If we choose to maximize $\ln x + \ln y + \ln z$, then the Lagrangian is

$$\begin{aligned} L &= \ln x + \ln y + \ln w + \lambda(36 - 2x - 3y - 4w) \\ \frac{\partial L}{\partial x} &= \frac{1}{x} - 2\lambda = 0 \\ \frac{\partial L}{\partial y} &= \frac{1}{y} - 3\lambda = 0 \\ \frac{\partial L}{\partial w} &= \frac{1}{w} - 4\lambda = 0 \\ \frac{\partial L}{\partial \lambda} &= 36 - 2x - 3y - 4w = 0 \qquad (1) \end{aligned}$$

Thus, $x = \frac{1}{2}\lambda$, $y = \frac{1}{3}\lambda$, $w = \frac{1}{4}\lambda$.

Then (1) yields $\frac{3}{\lambda} = 36$ or $\lambda = \frac{1}{12}$. Then we obtain $x = 6, y = 4, w = 3$, the optimal objective function value is 72.

**9** Solve the following NLP:

$$\begin{aligned} max \quad & z = \frac{50}{x} + \frac{20}{y} + xy \\ s.t. \quad & x \geq 1, y \geq 1 \end{aligned}$$

**Solution:**

KT-Conditions yield

$$-\frac{50}{x^2}+y-\lambda_1=0 \quad (1)$$
$$-\frac{20}{y^2}+x-\lambda_2=0 \quad (2)$$
$$\lambda_1(x-1)=0 \quad (3)$$
$$\lambda_2(y-1)=0 \quad (4)$$
$$x \geq 1, y \geq 1, \lambda_1 \geq 0, \lambda_2 \geq 0$$

Case I: Trying $\lambda_1$ and $\lambda_2 > 0$ does not work.
Case II: Trying $\lambda_1 > 0$ and $\lambda_2 = 0$ violates $(1)$.
Case III: Trying $\lambda_1 = 0$ and $\lambda_2 > 0$ violates $(2)$.
Case IV: $\lambda_1 = \lambda_2 = 0$. Then $(1)$ and $(2)$ yield $x^2y = 50$ and $y^2x = 20$. Solving yields $y = 2$ and $x = 5$, which satisfies the KT- conditions and has $z = 30$. Since this is the only point satisfying KT-conditions, it must be the optimal solution.

**10** If a company charges a price p for a product and spends $\$a$ on advertising, it can sell $10{,}000 + 5\sqrt{a} - 100p$ units of the product. If the product costs $\$10$ per unit to produce, then how can the company maximize profits?

**Solution:**

We want to maximize $\pi = \left(10{,}000 + 5\sqrt{a} - 100p\right)(p-10) - a$

$$\frac{\partial \pi}{\partial p} = -100(p-10) + 10000 + s\sqrt{a} - 100p = 0$$
$$\frac{\partial \pi}{\partial a} = \frac{5(p-10)}{2\sqrt{a}} - 1 = 0$$

Solving these equations yields $p = 58$ and $a = 14{,}400$.