Chapter 2 The Poisson Process

2.1 The Poisson Process

Counting Process : $\{N(t), t \geq 0\}$ a S.P. represents # of events occurred up time t, and satisfy

(1) $N(t) \geq 0$

(2) $N(t)$ is integer valued

(3) If $s < t$, then $N(s) \leq N(t)$

(4) For $s < t$, $N(t) - N(s)$ = # of events occurred in the interval $(s, t]$

Independent Increments : # of events occurred in disjoint time ind.

For example. $N(t)$ and $N(t+s) - N(t)$ ind.

Stationary Increments : distribution of # of events depends on the length of the interval.

# of events in $(t_1+s, t_2+s]$: $N(t_2+s) - N(t_1+s)$

$\stackrel{d}{=}$ # of events in $(t_1, t_2]$: $N(t_2) - N(t_1)$

Definition 2.1.1 (Poisson Process).

$\{N(t), t \geq 0\}$ C.P. with rate $\lambda > 0$, if.

(i) $N(0) = 0$

(ii). the process has independent increments

(iii). # of events in any interval of length t is Poisson distributed with mean λt. that is $\forall s, t > 0$

$$P\{N(s+t)-N(s)=n\} = e^{-\lambda t}\frac{(\lambda t)^n}{n!}$$

(iii) $\Rightarrow$ stationary increments and $E(N(t)) = \lambda t$

Definition of $o(h)$.

the function f is said to be $o(h)$ if

$$\lim_{h\to 0}\frac{f(h)}{h} = 0$$

Definition 2.1.2 (Alternative Poisson Process)

$\{N(t), t\geq 0\}$ C.P. with rate $\lambda > 0$ if.

(i) $N(0)=0$

(ii) stationary increment and independent increment

(iii) $P\{N(h)=1\} = \lambda h + o(h)$.

(iv) $P\{N(h)\geq 2\} = o(h)$.

Theorem 2.1.1.

Definition 2.1.1 $\Leftrightarrow$ Definition 2.1.2

proof. we just prove : Defintion 2.1.1 $\Leftarrow$ Definition 2.1.2

we denote $P_n(t) = P\{N(t)=n\}$.

Now, we consider $P_0(t+h)$

$$\begin{aligned} P_0(t+h) &= P(N(t+h)=0) \\ &= P(N(t)=0,\ N(t+h)-N(t)=0) \\ &= P(N(t)=0)\ P(N(t+h)-N(t)=0) \\ &= P_0(t)\cdot(1-\lambda h+o(h)) \end{aligned}$$

$[P(N(h)\geqslant 1) = P(N(h)=1) + P(N(h)\geqslant 2) = \lambda h + o(h)]$

Hence, $\dfrac{P_0(t+h)-P_0(t)}{h} = -\lambda P_0(t) + P_0(t)\cdot \dfrac{o(h)}{h}$

$\Rightarrow \quad P_0'(t) = -\lambda P_0(t) \quad (\text{as } h\to 0)$

$P_0(t) = k e^{-\lambda t}$

Since $P_0(0)=0$, then $k=1$ and $P_0(t) = e^{-\lambda t}$

Similarly, $P_n(t+h) = P(N(t+h)=n)$

$= P(N(t)=n,\ N(t+h)-N(t)=0) +$

$P(N(t)=n-1,\ N(t+h)-N(t)=1) +$

$P(N(t)\leqslant n-2,\ N(t+h)-N(t)\geqslant 2)$.

$= P_n(t)(1-\lambda h+o(h)) + P_{n-1}(t)\cdot(\lambda h+o(h)) + o(h)$.

$= P_n(t) + \lambda h(P_{n-1}(t)-P_n(t)) + o(h)$.

$\dfrac{P_n(t+h)-P_n(t)}{h} = \lambda P_{n-1}(t) - \lambda P_n(t) + \dfrac{o(h)}{h}$

$\Rightarrow \quad P_n'(t) = \lambda P_{n-1}(t) - \lambda P_n(t) \quad (h\to 0)$

$e^{\lambda t}(P_n'(t)+\lambda P_n(t)) = \lambda e^{\lambda t} P_{n-1}(t)$

$(e^{\lambda t}P_n(t))' = \lambda e^{\lambda t} P_{n-1}(t)$

Now we induct $P_n(t) = e^{-\lambda t}\dfrac{(\lambda t)^n}{n!}$

assume for $n-1$. $P_{n-1}(t) = e^{-\lambda t}\dfrac{(\lambda t)^{n-1}}{(n-1)!}$

$(e^{\lambda t}P_n(t))' = \lambda e^{\lambda t}\cdot e^{-\lambda t}\dfrac{(\lambda t)^{n-1}}{(n-1)!}$

$= \lambda^n \dfrac{t^{n-1}}{(n-1)!}$

$e^{\lambda t}P_n(t) = \lambda^n \dfrac{t^n}{n!} + C$

Since $P_n(0)=0 \Rightarrow C=0 \Rightarrow P_n(t) = e^{-\lambda t}\dfrac{(\lambda t)^n}{n!}$

So we prove (iii) in Def 2.1.1. and (i) (ii) obviously

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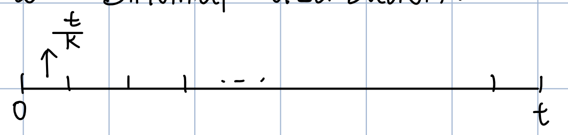
Remark. $N(t)$ has a Poisson Distribution since Poisson approximation to Binomial distribution.



$k \to \infty$.

Since $P(\#$ of events ≥ 2 happens in any one interval$) \longrightarrow 0 \quad (k \to \infty)$

$N(t) = \#$ of intervals with events occurred.

$N(t) \sim \text{Binomial}(p = \lambda h + o(h), k)$.

Since $\lim_{k \to \infty} kp = \lim_{k \to \infty} \lambda \frac{t}{k} \cdot k + o(\frac{t}{k}) \cdot k = \lambda t$.

$N(t) \sim \text{Poisson}(\lambda t) \quad (k \to \infty)$.

2.2 Interarrival and waiting time distributions

P.P. X_1 : the time of the first event.

X_n : the time between the $(n-1)$th and nth event.

Sequence of interarrival times

$\{X_n, n \geq 1\}$ defined above.

Proposition 2.2.1.

X_n, $n = 1, 2, \cdots$, are independent identically distributed $\exp(\lambda)$.

proof. We need to determine the distribution of X_n.

$P(X_1 > t) = P(N(t) = 0) = e^{-\lambda t}$

$P(X_2 > t \mid X_1 = s) = P(0 \text{ events in } (s, s+t] \mid X_1 = s)$

$= P(0 \text{ events in } (s, s+t])$

$= P(N(t) = 0) = e^{-\lambda t}$

Since X_1, X_2 independent, $X_2 \sim \exp(\lambda)$.

Similarly, we can get $x_n \sim exp(\lambda)$.

#

Remark. Since the stationary and independent increments, the process from any point is independent of all previous, which is memoryless.

waiting time (the arrival time of nth event).

$$S_n = \sum_{i=1}^{n} X_i$$

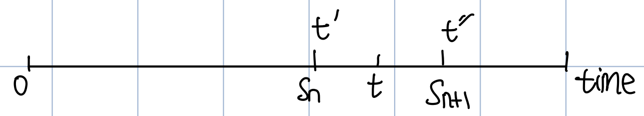
By Proposition 2.2.1 and MGF we could know

$$S_n \sim Gamma(n, \lambda)$$

The density function is

$$f(x) = \lambda e^{-\lambda t} \frac{(\lambda t)^{n-1}}{(n-1)!} \qquad t \geq 0$$

How to get density function (1).



$$S_n \leq t \Leftrightarrow N(t) \geq n$$

Hence, $P(S_n \leq t) = P(N(t) \geq n) = \sum_{i=n}^{\infty} \frac{(\lambda t)^i}{i!} e^{-\lambda t}$

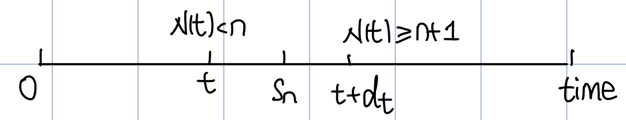
density function $f = \sum_{i=n}^{\infty} \left[\frac{(\lambda t)^{i-1}}{(i-1)!} \cdot \lambda e^{-\lambda t} - \lambda e^{-\lambda t} \frac{(\lambda t)^i}{i!} \right]$

$$= \lambda e^{-\lambda t} \left[\sum_{i=n-1}^{\infty} \frac{(\lambda t)^i}{i!} - \sum_{i=n}^{\infty} \frac{(\lambda t)^i}{i!} \right]$$

$$= \lambda e^{-\lambda t} \frac{(\lambda t)^{n-1}}{(n-1)!}$$

How to get density function (2).

we could use the independent increment assumption



$$P(t < S_n < t+dt) = P(N(t)=n-1,\ 1 \text{ event} \in (t, t+dt)) + o(dt)$$

$$= P(N(t)=n-1)\, P(1 \text{ event} \in (t, t+dt)) + o(dt)$$

$$= e^{-\lambda t} \frac{(\lambda t)^{n-1}}{(n-1)!} \cdot \lambda dt + o(dt)$$

$$\Rightarrow f(t) = \frac{P(t<S_n<t+dt)}{dt} = \lambda e^{-\lambda t} \frac{(\lambda t)^{n-1}}{(n-1)!} + \frac{o(dt)}{dt} \longrightarrow \lambda e^{-\lambda t} \frac{(\lambda t)^{n-1}}{(n-1)!} \quad (dt \to 0)$$

Alternative Definition of P.P.

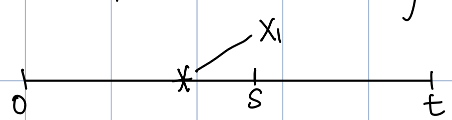
a sequence $\{X_i\} \overset{i.i.d}{\sim} \exp(\lambda)$.

Define C.P. as 1 event occurs at $X_1 = S_1$

n event occur at S_n

where $S_n = \sum_{i=1}^{n} X_i$

2.3 Conditional Distribution of The Arrival Times.



(have know event $1 \in (0, t]$)

$$P(X_1 < s \mid N(t)=1) = \frac{P(X_1<s,\ N(t)=1)}{P(N(t)=1)}$$

$$= \frac{P(1 \text{ event} \in (0,s),\ 0 \text{ event} \in [s,t])}{P(N(t)=1)}$$

$$= \frac{P(N(s)=1) \cdot P(N(t)-N(s)=0)}{P(N(t)=1)}$$

$$= \frac{e^{-\lambda s} \cdot \lambda s \cdot e^{-\lambda(t-s)}}{e^{-\lambda t} \cdot \lambda t} = \frac{s}{t}$$

which means conditional distribution is $U(0,t)$.

Order Statistics:

$Y_1, \cdots, Y_n$ r.v. $\Rightarrow$ $Y_{(1)}, \cdots, Y_{(n)}$ is order statistics if

$Y_{(1)} \leq Y_{(2)} \leq \cdots \leq Y_{(n)}$

If $Y_i \overset{i.i.d}{\sim}$ density f, then joint density of $(Y_{(1)}, \cdots, Y_{(n)})$

$$f(y_1, \cdots, y_n) = n! \prod_{i=1}^{n} f(y_i) \qquad y_1 < y_2 < \cdots < y_n$$

Since (1) $(Y_{(1)}, \cdots, Y_{(n)})$ is equal to $(y_1, \cdots, y_n)$ if $(Y_1, \cdots, Y_n)$ is equal to any of the $n!$ permutations of $(y_1, \cdots, y_n)$

(2) the probability density that

$$f(Y_1 = y_{t_1}, \cdots, Y_n = y_{t_n}) = f(Y_1 = y_{t_1}) \cdots f(Y_n = y_{t_n})$$
$$= \prod_{i=1}^{n} f(y_i)$$

$(y_{t_1}, \cdots, y_{t_n})$ is a permutation of $(y_1, \cdots, y_n)$

If $Y_1, \cdots, Y_n \overset{i.i.d}{\sim} U(0,t)$. then $Y_{(1)}, \cdots, Y_{(n)}$ density

$$f(y_1, \cdots, y_n) = n! \prod_{i=1}^{n} f(y_i) = \frac{n!}{t^n} \qquad y_1 < \cdots < y_n$$

Theorem 2.3.1

Given $N(t) = n$, the n arrival time $S_1, \cdots, S_n$ have the same distribution as the order statistics corresponding to n i.i.d r.v $\sim U(0,t)$.

proof. We need to compute the conditional density of $S_1, \cdots, S_n$ given that $N(t) = n$. let $0 < t_1 < t_2 < \cdots t_{n+1} = t$, and h_i be small enough so that $t_i + h_i < t_{i+1}$

$$P(t_i \leq S_i \leq t_i + h,\ i = 1, 2, \cdots, n \mid N(t) = n)$$

$$= \frac{P(1 \text{ event} \in [t_i, t_i + h],\ \text{no event elsewhere in } [0,t])}{P(N(t) = n)}$$

$$= \frac{\lambda h_1 e^{-\lambda h_1} \cdots \lambda h_n e^{-\lambda h_n} e^{-\lambda(t-h_1-\cdots-h_n)}}{e^{-\lambda t}\frac{(\lambda t)^n}{n!}}$$

$$= \frac{n!}{t^n} h_1 h_2 \cdots h_n$$

$$\Rightarrow \frac{P(t_i \le S_i \le t_i+h_i,\ i=1,2,\cdots,n \mid N(t)=n)}{h_1 \cdots h_n} = \frac{n!}{t^n}$$

let $h_1, \cdots, h_n \longrightarrow 0$, we have $f(t_1, \cdots, t_n) = \frac{n!}{t^n}$ $\quad 0 < t_1 < \cdots < t_n$

Example 2.3(A).

travelers arrive — P.P with rate λ.

train leaves at time t.

Compute expected sum of the waiting time of travelers arriving in $(0,t)$. $E[\sum_{i=1}^{N(t)}(t-S_i)]$.

$$E[\sum_{i=1}^{N(t)}(t-S_i) \mid N(t)=n]$$

$$= nt - E(\sum_{i=1}^{n} S_i \mid N(t)=n)$$

Now let U_i denote i.i.d r.v. $\sim U(0,t)$. then

$$E(\sum_{i=1}^{n} S_i \mid N(t)=n) = E(\sum_{i=1}^{n} U_{(i)}) = E(\sum_{i=1}^{n} U_i) = \frac{nt}{2}$$

Hence, $E[\sum_{i=1}^{N(t)}(t-S_i) \mid N(t)=n] = \frac{nt}{2}$

$$E[\sum_{i=1}^{N(t)}(t-S_i)] = E[\frac{t}{2}N(t)] = \frac{t}{2}\cdot\lambda t = \frac{\lambda}{2}t^2$$

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Event of P.P. with rate λ is classified into Type I or Type II

Event occur at s classified as Type I with probability $P(s)$.

classified as Type II with probability $1-P(s)$.

Proposition 2.3.2.

$N_i(t)$ = # of type i events occur by time t, $i=1,2$. then $N_1(t)$ and $N_2(t)$ are ind. Poisson r.v. have respective means $\lambda t p$ and $\lambda t(1-p)$ where

$$p = \frac{1}{t}\int_0^t P(s)\,ds.$$

proof.

$$P(N_1(t)=n,\ N_2(t)=m) = \sum_{k=0}^{\infty} P(N_1(t)=n,\ N_2(t)=m \mid N(t)=k)\,P(N(t)=k)$$

$$= P(N_1(t)=n,\ N_2(t)=m \mid N(t)=n+m)\,P(N(t)=n+m)$$

If an event occurs at s

$\text{Prob}(\text{Type I}) = P(s)$. $\quad \text{Prob}(\text{Type II}) = 1-P(s)$.

在 ds 发生事件的概率为 $\frac{1}{t}$，是 Type I 概率为 $P(s)$.

在 $[0,t]$ 上发生 TypeI 的概率：$p = \int_0^t \frac{1}{t}P(s)\,ds = \frac{1}{t}\int_0^t P(s)\,ds$

Type II：$1-p$.

$$P(N_1(t)=n,\ N_2(t)=m \mid N(t)=n+m)$$

$$= \binom{n+m}{n} p^n (1-p)^m$$

$$P(N_1(t)=n,\ N_2(t)=m) = \frac{(n+m)!}{n!\,m!}p^n(1-p)^m e^{-\lambda t}\frac{(\lambda t)^{n+m}}{(n+m)!}$$

$$= e^{-\lambda t p}\frac{(\lambda t p)^n}{n!}\ e^{-\lambda t(1-p)}\frac{(\lambda t(1-p))^m}{m!}$$

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Example 2.3(B). (The Infinite Server Poisson Queue)

Customer arrival ~ P.P with rate λ.

Service time ~ G

At time t, # of completed service — Type I

# of in service — Type II

If a customer enter at $s \le t$.

he is type I if his service time < $t-s$.

In other words. Prob $= G(t-s)$. for Type I $\triangleq P(s)$.

Thus $N_1(t) \sim$ Poisson $(\lambda t p)$.

$$E N_1(t) = \lambda t p = \lambda t \cdot \frac{1}{t} \int_0^t G(t-s)\, ds.$$
$$= \lambda \int_0^t G(t-s)\, ds.$$

$N_2(t) \sim$ Poisson $(\lambda t(1-p))$

$$E N_2(t) = \lambda t (1-p) = \lambda \int_0^t \bar{G}(t-s)\, ds.$$

$N_1(t)$. $N_2(t)$ ind.

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Example 2.3(C)

shocks arrive according to P.P. with rate λ

ith shock results a damage D_i

D_i i.i.d and independent to $\{N(t), t \geq 0\}$

$N(t)$: # of shocks in $[0,t]$

Damage D occur at time 0. decrease to De^{-at} at time t.

The total damage at time t

$$D(t) = \sum_{i=1}^{N(t)} D_i e^{-a(t-S_i)}$$

S_i: the arrival time of ith shock

$$E[D(t) \mid N(t)=n] = E\Big[\sum_{i=1}^{N(t)} D_i e^{-a(t-S_i)} \Big| N(t)=n\Big]$$
$$= E(D) e^{-at} E\Big[\sum_{i=1}^{n} e^{aS_i} \Big| N(t)=n\Big]$$

let $U_1, \cdots, U_n \overset{i.i.d}{\sim} U(0,t)$

$$= E[D] e^{-at} E\Big[\sum_{i=1}^{n} e^{aU_{(i)}}\Big]$$
$$= E[D] e^{-at} E\Big[\sum_{i=1}^{n} e^{aU_i}\Big]$$
$$= E[D] e^{-at}\, n \int_0^t \frac{1}{t} e^{ax}\, dx$$

$$= \frac{n}{t} E[D] e^{-\alpha t} \cdot \frac{1}{\alpha} e^{\alpha x}\Big|_0^t$$

$$= \frac{n}{\alpha t} E[D] e^{-\alpha t} (e^{\alpha t} - 1)$$

$$= \frac{n}{\alpha t} E[D] (1 - e^{-\alpha t})$$

$$E[D(t)] = E[E[D(t) \mid N(t) = n]]$$

$$= E\left[\frac{N(t)}{\alpha t} E[D] (1 - e^{-\alpha t})\right]$$

$$= \frac{\lambda}{\alpha} E[D] (1 - e^{-\alpha t})$$

Another Approach :

Break up $(0, t)$ into nonoverlapping intervals of length h.

X_i : the sum of the damages at time t of shocks in ith interval , $i = 0, 1, \cdots [\frac{t}{h}]$

$$D(t) = \sum_{i=0}^{[\frac{t}{h}]} X_i$$

$$E[D(t)] = E\left[\sum_{i=0}^{[\frac{t}{h}]} X_i\right]$$

$$= \sum_{i=0}^{[\frac{t}{h}]} E[X_i].$$

$$E[X_i] = P(N(h) = 0) \cdot 0 + P(N(h) = 1) \cdot E[D] \cdot e^{-\alpha(t - S_i)} + o(h))$$

$$= o(h) + E[D] \cdot e^{-\alpha(t - t_i)} \cdot \lambda h \qquad (t_i : \text{occured time} \in (ih, (i+1)h))$$

$$E[D(t)] = \sum_{i=0}^{[\frac{t}{h}]} E[D] \cdot \lambda h \cdot e^{-\alpha(t - t_i)}$$

$$= \int_0^t E[D] \lambda e^{-\alpha(t - x)} dx$$

$$= \lambda E[D] \cdot e^{-\alpha t} \int_0^t e^{\alpha x} dx$$

$$= \frac{\lambda E[D]}{\alpha} e^{-\alpha t} (e^{\alpha t} - 1)$$

$$= \frac{\lambda}{\alpha} E[D] (1 - e^{-\alpha t})$$

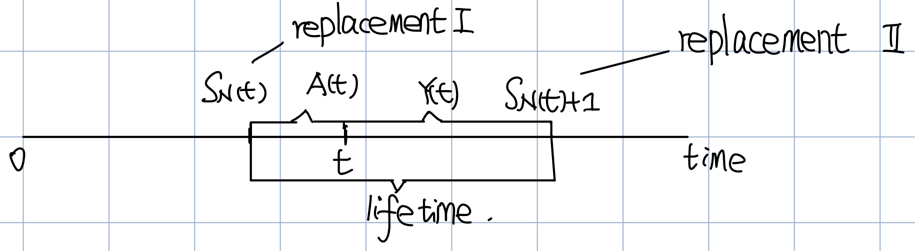
Example (Inspection Paradox & Distribution of Age).

light bulb $\sim \exp(\lambda)$

# of replacement $\{N(t), t \geq 0\}$ P.P with rate λ

Now fix time t, consider the bulb in use.

* the expected lifetime
* the distribution of lifetime.



The lifetime $X_{N(t)+1} = A(t) + Y(t)$.

$Y(t) \sim \exp(\lambda)$ since memoryless. $E[Y(t)] = \frac{1}{\lambda}$

$E[X_{N(t)+1}] = E[A(t)] + E[Y(t)] > \frac{1}{\lambda}$

the inspection paradox:

the expected time of item in use > that of typical one.

$P[A(t) > x] = P(0 \text{ event in } (t-x, t))$

$$= \begin{cases} e^{-\lambda x} & x \leq t \\ 0 & \text{otherwise.} \end{cases}$$

$$E[A(t)] = \int_0^t e^{-\lambda x} \cdot dx = \frac{1}{\lambda}(1 - e^{-\lambda t}).$$

$$E[X_{N(t)+1}] = \frac{2}{\lambda} - \frac{1}{\lambda} e^{-\lambda t}$$

$t \to \infty$. $E[X_{N(t)+1}] \longrightarrow \frac{2}{\lambda}$

Example Filtered Process

$\{N(t) : t \geq 0\}$ P.P. with rate λ

type I with prob p

type II with prob $1-p$

$\{N_k(t): t\geq 0\}$: # of type k events (filtering process).

$N_k(0)=0$, $k=1,2$.

$$P(N_1(t)=m)=\sum_{n=m}^{\infty} P(N_1(t)=m \mid N(t)=n)\cdot P(N(t)=n)$$
$$=\sum_{n=m}^{\infty}\binom{n}{m}p^m(1-p)^{n-m}\cdot e^{-\lambda t}\frac{(\lambda t)^n}{n!}$$
$$=\sum_{n=m}^{\infty}\frac{n!}{m!(n-m)!}p^m(1-p)^{n-m}e^{-\lambda t}\frac{(\lambda t)^n}{n!}$$
$$=\frac{(\lambda t p)^m}{m!}e^{-\lambda t}\sum_{n=m}^{\infty}\frac{[(1-p)\lambda t]^{n-m}}{(n-m)!}$$
$$=\frac{(\lambda t p)^m}{m!}e^{-\lambda t}\cdot e^{(1-p)\lambda t}$$
$$=e^{-\lambda t p}\frac{(\lambda t p)^m}{m!}$$

故 $\{N_1(t): t\geq 0\}$ is P.P. with rate λp.

Proposition : $\{N(t): t\geq 0\}$ P.P. with rate λ

type I with prob p

type II with prob $1-p$

Then the derived filtered process

$\{N_k(t): t\geq 0\}$ ind. P.P. with rate $\lambda p^{2-k}(1-p)^{k-1}$, $k=1,2$

proof. We just need to prove independence.

$$P(N_1(t)=n,\ N_2(t)=m)$$
$$=P(N_1(t)=n,\ N_2(t)=m \mid N(t)=m+n)\cdot P(N(t)=m+n).$$
$$=\binom{n+m}{n}p^n(1-p)^m e^{-\lambda t}\frac{(\lambda t)^{m+n}}{(m+n)!}$$
$$=\frac{(\lambda t p)^n}{n!}e^{-\lambda t p}\cdot\frac{[\lambda t(1-p)]^m}{m!}\cdot e^{-\lambda t(1-p)}$$
$$=P(N_1(t)=n)\cdot P(N_2(t)=m)$$

#

Generalization to m types :

$\{N(t), t\geq 0\}$ P.P. with rate λ.

m types of events with prob. p_m

$p_1 + \cdots + p_m = 1$.

Then $\{N_k(t), t \geq 0\}$ p.p. with rate $p_k \lambda$. ind.

Aggregate P.P:

If $\{N_j(t) : t \geq 0\}$ ind. p.p. with rate λ_j, then

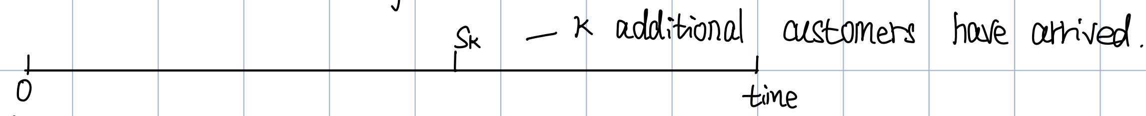
$\{N(t) = \sum_{j=1}^{M} N_j(t) : t \geq 0\}$ P.P. with rate $\lambda = \sum_{j=1}^{M} \lambda_j$.

2.3.1. M/G/1 Busy Period.

M: arrival process, we suppose P.P. with rate λ.

G: service time: general distribution.

1: The number of servers



↑ busy period starts

busy period starts: when a customer arrives and sees the server is free.

$Y_1, Y_2, \cdots, Y_n$: sequence of service time. $\sim G$

we say busy period last t, consist of n service

$\Leftrightarrow$ 1). $S_k \leq Y_1 + \cdots + Y_k$ $\quad k = 1, \cdots, n-1$ (第k个人来时，前面服务未结束).

2) $Y_1 + \cdots + Y_n = t$ (总服务时间 t).

3) There are $n-1$ arrivals in $(0, t)$.

P(busy period of length t and consists of n services):

$= P(Y_1+\cdots+Y_n=t,\ n-1 \text{ arrivals in } (0,t),\ S_k \le Y_1+\cdots+Y_k,\ k=1,\cdots,n-1)$

$= P(S_k \le Y_1+\cdots+Y_k,\ k=1,\cdots,n-1 \mid n-1 \text{ arrivals in } (0,t),\ Y_1+\cdots+Y_n=t) \times$

$P(n-1 \text{ arrivals in } (0,t),\ Y_1+\cdots+Y_n=t)$

而 $P(n-1 \text{ arrivals in } (0,t),\ Y_1+\cdots+Y_n=t)$.

$= e^{-\lambda t}\dfrac{(\lambda t)^{n-1}}{(n-1)!}\, dG_n(t)$.

G_n — the n-fold convolution of G

We denote τ_k ordered statistics $\overset{iid}{\sim} U(0,t)$

$P($busy period of length t and consists of n services$)$

$= P(\tau_k \le Y_1+\cdots+Y_k,\ k=1,\cdots,n-1 \mid Y_1+\cdots+Y_n=t) \times e^{-\lambda t}\dfrac{(\lambda t)^{n-1}}{(n-1)!}\, dG_n(t)$.

Lemma 2.3.3.

Let $Y_1,\cdots,Y_n$ be independent and identically distributed nonnegative random variables Then

$$E[Y_1+\cdots+Y_k \mid Y_1+\cdots+Y_n=y] = \frac{k}{n}y. \quad k=1,\cdots,n$$

proof $\quad nE[Y_i \mid Y_1+\cdots+Y_n=y] = E[Y_1+\cdots+Y_n \mid Y_1+\cdots+Y_n=y]$

$= y\cdot 1 = y.$

$\Rightarrow E[Y_i \mid Y_1+\cdots+Y_n=y] = \dfrac{y}{n}.$

$E[Y_1+\cdots+Y_k \mid Y_1+\cdots+Y_n=y] = kE[Y_i \mid Y_1+\cdots+Y_n=y] = \dfrac{k}{n}y.$

#

Lemma 2.3.4

$\tau_1,\cdots,\tau_n$ ordered statistics $\overset{iid}{\sim} U(0,t)$

$Y_1,\cdots,Y_n$ iid nonnegative r.v. ind of $\{\tau_1,\cdots,\tau_n\}$

then $\quad P(Y_1+\cdots+Y_k < \tau_k,\ k=1,\cdots,n \mid Y_1+\cdots+Y_n=y)$

$$= \begin{cases} 1-\frac{y}{t} & 0<y<t \\ 0 & \text{otherwise} \end{cases}$$

proof. the proof is by induction on n

$n=1$, $P(Y_1 < T_1 \mid Y_1 = y) = P(T_1 > y) = 1-\frac{y}{t} \quad 0<y<t$

Assume $n-1$ ✓, consider n case, suppose $0<y<t$

$P(Y_1+\cdots+Y_k < T_k,\ k=1,\cdots,n \mid Y_1+\cdots+Y_{n-1}=s,\ T_n=u.\ Y_1+\cdots+Y_n=y)$

$= P(Y_1+\cdots+Y_k < T_k^*,\ k=1,\cdots,n-1 \mid Y_1+\cdots+Y_{n-1}=s) \qquad y<u.$

T_k^* : the ordered statistics iid $U(0,u)$

$$= \begin{cases} 1-\frac{s}{u} & y<u \\ 0 & \text{otherwise} \end{cases}$$

Hence,

$P(Y_1+\cdots+Y_k < T_k,\ k=1,\cdots,n \mid Y_1+\cdots+Y_{n-1},\ T_n=u.\ Y_1+\cdots+Y_n=y)$

$= 1-\frac{Y_1+\cdots+Y_{n-1}}{T_n}$

$P\{Y_1+\cdots+Y_k<T_k,\ k=1,\cdots,n \mid T_n=u,\ Y_1+\cdots+Y_n=y\}$

$= \sum_{Y_1+\cdots+Y_{n-1}} P[[Y_1+\cdots+Y_k<T_k,\ k=1\cdots n \mid T_n=u,\ Y_1+\cdots+Y_n=y] \mid Y_1+\cdots+Y_{n-1})\ P(Y_1+\cdots+Y_{n-1})$

$= E[Y_1+\cdots+Y_k<T_k,\ k=1,\cdots n \mid T_n=u.\ Y_1+\cdots+Y_n=y.\ Y_1+\cdots+Y_{n-1}]$

$= E[1-\frac{Y_1+\cdots+Y_{n-1}}{T_n} \mid T_n=u,\ Y_1+\cdots+Y_n=y]$

$= 1-\frac{1}{u}E[Y_1+\cdots+Y_{n-1} \mid Y_1+\cdots+Y_n=y]$

$= 1-\frac{1}{u}\cdot\frac{n-1}{n}y$

$P(Y_1+\cdots+Y_k<T_k,\ k=1,\cdots,n \mid Y_1+\cdots+Y_n=y)$

$= E[1-\frac{y}{T_n}\cdot\frac{n-1}{n} \mid y<T_n]\ P(y<T_n) \quad (P(y>T_n)$ 一项为 $0)$.

$= P(y<T_n) - \frac{n-1}{n}y\,E[\frac{1}{T_n} \mid y<T_n]\ P(y<T_n)$

而 $P(T_n<x) = P[\max_{1\le t\le n} U_t < x]$

$= P(U_i<x,\ i=1,\cdots,n) = (\frac{x}{t})^n$

故 $f_{T_n}(x) = \frac{n}{t}(\frac{x}{t})^{n-1} \qquad 0<x<t$

故 $E[\frac{1}{T_n} \mid y<T_n]P(y<T_n) = \int \frac{1}{x} P(x \mid y<x) \cdot P(y<x)\, dx$

$= \int \frac{1}{x} P(x,\ x>y)\, dx$

$= \int_y^t \frac{1}{x} \cdot \frac{n}{t} (\frac{x}{t})^{n-1}\, dx$

$= \frac{n}{t^n} \int_y^t x^{n-2}\, dx$

$= \frac{n}{(n-1)t^n} (t^{n-1} - y^{n-1})$

从而有. $P\{Y_1+\cdots+Y_k<T_k,\ k=1,\cdots,n \mid T_n=u,\ Y_1+\cdots+Y_n=y\}$

$= 1-(\frac{y}{t})^n - \frac{y}{t^n}(t^{n-1}-y^{n-1})$

$= 1-\frac{y}{t}$

#

Lemma 2.3.5

$P(Y_1+\cdots+Y_k<T_k,\ k=1,\cdots,n-1 \mid Y_1+\cdots+Y_n=t) = \frac{1}{n}$

* proof. $P(Y_1+\cdots+Y_k<T_k.\ k=1,\cdots,n-1 \mid Y_1+\cdots+Y_{n-1}=y,\ Y_1+\cdots+Y_n=t)$

$$= \begin{cases} 1-\frac{y}{t} & 0<y<t \\ 0 & \text{otherwise} \end{cases}$$

由双期望公式:

$P(Y_1+\cdots+Y_k<T_k,\ k=1,\cdots,n-1 \mid Y_1+\cdots+Y_n=t)$

$= E[\ 1-\frac{Y_1+\cdots+Y_{n-1}}{t} \mid Y_1+\cdots+Y_n=t\ ]$

$= 1-\frac{1}{t} \times \frac{n-1}{n} \cdot t = \frac{1}{n}$

#

Since $U \sim U(0,t) \iff t-U \sim (0,t)$.

so $T_1,\cdots,T_{n-1} \overset{d}{=} t-T_n,\cdots,t-T_1$

Hence. $P(T_k \leq Y_1+\cdots+Y_k.\ k=1,\cdots,n-1 \mid Y_1+\cdots+Y_n=t)$

$= P(t-T_{n-k} \leq t-(Y_{k+1}+\cdots+Y_n),\ k=1,\cdots,n-1 \mid Y_1+\cdots+Y_n=t)$

$= P(T_{n-k} \geq Y_{k+1}+\cdots+Y_n,\ k=1,\cdots,n-1 \mid Y_1+\cdots Y_n=t)$

Since $Y_1, \cdots, Y_n$ have the same distribution as $y_n, \cdots, y_1$, so we can replace Y_n as $Y_1 \cdots$

Hence $P(T_k \le Y_1 + \cdots + Y_k,\ k=1,\cdots,n-1 \mid Y_1 + \cdots + Y_n = t)$

$$= P(T_{n-k} \geq Y_1 + \cdots + Y_{n-k},\ k=1,\cdots,n-1 \mid Y_1 + \cdots + Y_n = t)$$

$$= \frac{1}{n}$$

Hence, P{busy period of length t and consists of n services} :

$$= e^{-\lambda t} \frac{(\lambda t)^{n-1}}{n!} \, d\, G_n(t).$$

$$P(\text{busy period of length} \le t) = \sum_{n=0}^{\infty} \int_0^t e^{-\lambda s} \frac{(\lambda s)^{n-1}}{n!} \, d\, G_n(s)$$

#

2.4 Nonhomogeneous Poisson Process.

Definition 2.4.1 Counting Process $\{N(t), t \geq 0\}$ is said to be a nonhomogeneous Poisson Process with intensity function $\lambda(t)$, $t \geq 0$ if

(1) $N(0) = 0$

(2) independent increments

(3) $N(t+s) - N(t) \sim$ Poisson with mean $\int_t^{t+s} \lambda(u)\, du$.

Def 2.4.1 (2) Counting Process $\{N(t), t \geq 0\}$ is said to be a nonhomogeneous Poisson Process with intensity function $\lambda(t)$, $t \geq 0$ if

(1) $N(0) = 0$

(2) indpendent increments

(3) $P(N(t+h) - N(t) = 1) = \lambda(t) h + o(h)$

(4) $P(N(t+h) - N(t) \geq 2) = o(h)$.

Mean value function :

$m(t) = E[N(t)] = \int_0^t \lambda(s)\, ds.$

$N(t+s) - N(t) \sim$ Poisson with parameter $m(t+s) - m(t)$.

$$P(N(t+s) - N(t) = n) = e^{-(m(t+s)-m(t))} \frac{[m(t+s)-m(t)]^n}{n!} \qquad n \geq 0$$

*proof. $P_0(s+h) = P[N(t+s+h) - N(t) = 0]$

$= P(0 \text{ event in } (t, t+s),\ 0 \text{ event in } (t+s, t+s+h)]$

$= P_0(s) \cdot (1 - \lambda(t+s)h + o(h))$

$\Rightarrow \frac{P_0(s+h) - P_0(s)}{h} = -\lambda(t+s) \cdot P_0(s) + \frac{o(h)}{h}$

$\Rightarrow P_0'(s) = -\lambda(t+s) \cdot P_0(s)$

$\ln P_0(s) = -\int_0^s \lambda(t+u)\, du$

$\Rightarrow P_0(s) = e^{-\int_0^s \lambda(t+u) du} = e^{-\int_t^{t+s} \lambda(k) dk}$

$= e^{-[m(t+s) - m(t)]}$

And then we can use similarly process and induction to proof.

#

Homogeneous P.P $\Rightarrow$ Nonhomogeneous P.P.

when the intensity function $\lambda(t)$ is bounded. we suppose that

$\lambda(t) \leq \lambda$

Then we let $\{N'(t), t \geq 0\}$ homogeneous P.P. with rate λ. Now we suppose an event of P.P. occures at time t is counted with probability $\frac{\lambda(t)}{\lambda}$, then the C.P. is nonhomogeneous P.P.

$P($ one counted event in $(t, t+h))$

$= P($ one event in $(t, t+h)) \cdot \frac{\lambda(t)}{\lambda} + o(h)$

$= \lambda h \cdot \frac{\lambda(t)}{\lambda} + o(h) = \lambda(t)h + o(h).$

Example 2.4(B). The output Process of an Infinite Server Poission

Queue $(M/G/\infty)$.

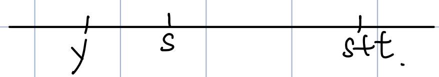
We suppose the output process of $M/G/\infty$ is a nonhomogeneous. P.P. with intensity function $\lambda(t) = \lambda G(t)$.

Two arguements before proof.

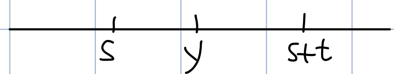
(1). # of departures in $(s, s+t) \sim \text{Poisson}(\lambda \int_s^{s+t} G(y)\,dy)$.

(2) # of departures in disjoint time ind.

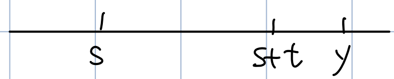
proof (1). call an arrival type I if it departs in $(s, s+t)$. Then an arrival at time y is type I:



$s-y \le g \le s+t-y$



$g \le s+t-y$



impossible.

$$P = \begin{cases} G(s+t-y) - G(s-y) & y < s \\ G(s+t-y) & s < y < s+t \\ 0 & y > s+t \end{cases}$$

Hence, # of such departures will be Poisson distributed with

mean $\quad \lambda t \frac{1}{t} \int_0^\infty P(y)\,dy = \lambda \int_0^s G(s+t-y) - G(s-y)\,dy$

$$+ \lambda \int_s^{s+t} G(s+t-y)\,dy$$

$$= \lambda \int_0^{s+t} G(s+t-y)\,dy - \lambda \int_0^s G(s-y)\,dy$$

$$= \lambda \int_s^{s+t} G(y)\,dy$$

(2) I_1, I_2 denote disjoint time intervals.

call an arrival type I if it departs in I_1

call an arrival type II if it departs in I_2

call an arrival type III, otherwise.

According to Prop 2.3.2.

# of type I and # of type II ind. Poisson r.v.

Nonhomogeneous P.P. $\Rightarrow$ homogeneous P.P.

Rescaling time.

Suppose $\{X(t), t \geq 0\}$ nonhomogeneous. P.P. indensity function $\lambda(t)$.

let. $\Lambda(t) = \int_0^t \lambda(\mu) d\mu$.

then we make a determinstic change in time scale.

$\{N(s), s \geq 0\}$ $\quad N(s) = X(t)$. $\quad s = \Lambda(t)$.

$P(N(s+\Delta s) - N(s) = 1)$

$= P(X(t+\Delta t) - X(t) = 1) = \lambda(t)\Delta t + o(\Delta t)$

$= \Delta s + o\Delta(s)$

$\Rightarrow \{N(s), s \geq 0\}$ is P.P. with rate 1.

Interval time of nonhomogeneous P.P.

$\bar{X}_1$ is similar to H.P.P.

$P(\bar{X}_1 > t) = P(N(t) = 0) = e^{-m(t)}$, $\quad P(\bar{X}_1 = t) = (1 - e^{-m(t)})' = \lambda(t) \cdot e^{-m(t)}$.

If $\lim_{t \to \infty} m(t) < \infty$, $P(\bar{X}_1 = \infty) > 0$

which means the first event maybe never occur in N.H.P.P.

$$P(\bar{X}_2 > t) = \int_0^\infty P(\bar{X}_2 > t \mid \bar{X}_1 = s)\, e^{-m(s)} \cdot \lambda(s) \, ds.$$

$$= \int_0^\infty e^{-(m(t+s) - m(s))} \cdot e^{-m(s)} \lambda(s) \, ds$$

$$= \int_0^\infty e^{-m(t+s)} \lambda(s) \, ds$$

$\bar{X}_1$, $\bar{X}_2$ not ind. & identical.

2.5 Compound Poisson Random Variables and Process.

Definition of Compound Poisson r.v.

$X_1, X_2, \dots \overset{i.i.d}{\sim}$ distribution F. ind of N.

$N \sim$ Poisson (λ). then the r.v.

$$W = \sum_{i=1}^{N} X_i$$

is said to be a compound Poisson r.v.

Moment Generating Function.

$$\phi_W(t) = E[e^{tW}] = \sum_{n=0}^{\infty} E[e^{tW} | N=n]\ P(N=n)$$

$$= \sum_{n=0}^{\infty} E[e^{tX_1+\dots+tX_n} | N=n] \cdot P(N=n)$$

$$= \sum_{n=0}^{\infty} E[e^{tX_i}]^n \cdot e^{-\lambda} \cdot \frac{(\lambda)^n}{n!}$$

$$= \sum_{n=0}^{\infty} e^{-\lambda} \frac{(\lambda E[e^{tX_i}])^n}{n!}$$

$$= e^{-\lambda} e^{\lambda \phi_{X_1}(t)}$$

$$= e^{\lambda(\phi_{X_1}(t)-1)}$$

$$EW = \phi'_W(0) = \left[e^{\lambda(\phi_{X_1}(t)-1)} \cdot \lambda \phi'_{X_1}(t)\right]\Big|_{t=0}$$

$$= \lambda EX.$$

$$EW^2 = \phi''_W(0) = \left[e^{\lambda(\phi_X(t)-1)} \cdot (\lambda \phi'_{X_1}(t))^2 + e^{\lambda(\phi_{X_1}(t)-1)} \cdot \lambda \phi''_{X_1}(t)\right]\Big|_0$$

$$= (\lambda EX)^2 + \lambda EX^2$$

$$\text{Var } W = EW^2 - (EW)^2 = (\lambda EX)^2 + \lambda EX^2 - (\lambda EX)^2$$

$$= \lambda EX^2$$