3.1. Introduction And Preliminaries.

P.P. interarrival times $X_1, X_2, \cdots \overset{iid}{\sim} \exp(\lambda)$

Define C.P. $N(t) = \max\{n, \sum_{i=1}^{n} X_i \leq t\}$.

$N(t)$ is a so-called P.P. with rate λ.

Renewal Process. interarrival times $X_1, X_2, \cdots \overset{iid}{\sim} F$ nonegative.

Define C.P. $N(t) = \sup\{n, \sum_{i=1}^{n} X_i \leq t\}$

$N(t)$ is a Renewal Process with interarrival time distribution F

X_i : time between $(i-1)$th event and ith event

event i is also called renewal.

Notice. P.P. could restart at any point in time.

R.P. could restart at any renewal

the mean time between successive events :

$$\mu = E[X_n] = \int_0^{\infty} x \, dF(x).$$

Notice : we should assume $P(X_n = 0) < 1 \Rightarrow 0 < \mu < \infty$.

arrival time of nth event : $S_n = \sum_{i=1}^{n} X_i$

Question :

whether an infinite number of renewals can occur in a finite time. (t finite, but $N(t) = \infty$).

Reply : $N(t) = \sup\{n : \sum_{i=1}^{n} X_i \leq t\}$

$$P(N(t) = \infty) = P\left(\sum_{i=1}^{\infty} X_i \leq t\right). = P\left(\lim_{n\to\infty} S_n \leq t\right)$$

However, By the strong law of large numbers,

$$\frac{S_n}{n} = \frac{X_1 + \cdots + X_n}{n} \longrightarrow \mu \quad \text{with probability } 1$$

Since $\mu > 0$. $S_n \to \infty$ as $n \to \infty$.

so $P(N(t)=\infty) = P(\lim_{n\to\infty} S_n \le t) = 0$

The Answer is false.

By this Question. we know $S_n \le t$ at most a finite n. so

$N(t) = \max\{n : S_n \le t\}$.

3.2. Distribution of $N(t)$.

P.P. with rate λ

$S_n \sim \text{Gamma}(n, \lambda)$.

R.P with interarrival time distribution F.

$S_n = \sum_{i=1}^{n} X_i \sim F_n$ (n fold convolution of F).

Since $N(t) \ge n \iff S_n \le t$

$$P(N(t)=n) = P(N(t) \ge n) - P(N(t) \ge n+1)$$
$$= P(S_n \le t) - P(S_{n+1} \le t)$$
$$= F_n(t) - F_{n+1}(t)$$

Renewal function $m(t) = E[N(t)]$

P.P. with rate λ : $m(t) = E[N(t)] = \lambda t$

R.P with interarrival time distribution F.

$$m(t) = E[N(t)] = \sum_{i=1}^{\infty} P(N(t) \ge i)$$
$$= \sum_{i=1}^{\infty} P(S_i \le t)$$
$$= \sum_{i=1}^{\infty} F_i(t)$$

Proposition 3.2.2. $m(t) < \infty$ for all $0 \le t < \infty$.

Since $P(X_i = 0) < 1 \Rightarrow \exists\ \alpha > 0$, s.t. $P(X_i \ge \alpha) > 0$

Now we define a new renewal process. $\{\bar{X}_n, n \geq 1\}$.

$$\bar{X}_n = \begin{cases} 0 & X_n < \alpha \\ \alpha & X_n \geq \alpha. \end{cases}$$

$\bar{N}(t) = \sup\{n, \bar{X}_1 + \cdots \bar{X}_n \leq t\}$.

For this R.P. renewals occur at time $t = n\alpha$. so

$N(t) \leq \bar{N}(t)$. # of renewals at each of these times

are ind geometric random variables with mean $\frac{1}{P(X_n \geq \alpha)}$

$$m(t) = EN(t) \leq E\bar{N}(t) \leq \frac{\frac{t}{\alpha} + 1}{P(X_n \geq \alpha)} < \infty.$$

3.3 Some Limit Theorems

① $N(\infty) = \lim_{t\to\infty} N(t) = \infty$ with probability 1.

Since $P(\lim_{t\to\infty} N(t) < \infty)$

$= P(X_n = \infty$ for some $n)$

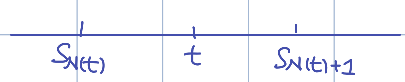
$= P\{\bigcup_{n=1}^{\infty} \{X_n = \infty\}\}$

$\leq \sum_{n=1}^{\infty} P(X_n = \infty) = 0$

② Proposition 3.3.1.

$$\frac{N(t)}{t} \longrightarrow \frac{1}{\mu} \text{ as } t \to \infty \text{ with prob 1.}$$

proof



Since $S_{N(t)} \leq t < S_{N(t)+1}$,

$$\frac{S_{N(t)}}{N(t)} < \frac{t}{N(t)} < \frac{S_{N(t)+1}}{N(t)} = \frac{S_{N(t)+1}}{N(t)+1} \cdot \frac{N(t)+1}{N(t)}.$$

By strong law of large numbers. $\frac{S_n}{n} \longrightarrow \mu$.

and $t \to \infty$, $N(t) \longrightarrow \infty$. Hence

$$\frac{S_{N(t)}}{N(t)} \to \mu, \quad \frac{S_{N(t)+1}}{N(t)} \longrightarrow \mu.$$

$$\Rightarrow \frac{N(t)}{t} \longrightarrow \frac{1}{\mu}$$

Example 3.3(A)

An infinite collection of coins.

Each coin lands on heads with prob $\sim U[0,1]$.

Now flipping either a new coin or previous one.

maximize the long-run proportion of lands on head.

Solution: choose a coin. Flip until it comes up with tails

then discard and choose another one.

the prob(head) = 1 under this strategy

proof. let $N(n)$: # of tails in the first n flips.

$$P_h = \lim_{n\to\infty} \frac{n-N(n)}{n} = 1 - \lim_{n\to\infty} \frac{N(n)}{n}$$

$\{N(n)\}$ is a R.P. under the strategy.

$$\lim_{n\to\infty} \frac{N(n)}{n} = \frac{1}{E[\text{times between two tails}]}$$

we suppose a coin lands on head with p

p from $U(0,1)$

$$E[\text{times between two tails}] = \int_0^1 \frac{1}{1-p}\,dp = \infty$$

Hence $\lim_{n\to\infty} \frac{N(n)}{n} = 1$.

3.3.1. Wald's Equation.

we now prove $\frac{m(t)}{t} \longrightarrow \frac{1}{\mu}$.

Stopping time N integer-valued r.v. is a stopping time for the sequence if the event $\{N=n\}$ ind. $X_{n+1}, X_{n+2}, \cdots$

Example 3.3(B). let X_n, $n=1,2,\cdots$, independent and

$P(X_n = 0) = P(X_n = 1) = \frac{1}{2}$

we let $N = \min\{n \quad X_1 + \cdots + X_n = 10\}$

N is a stopping time.

Example 3.3(C). let X_n, $n = 1, 2, \cdots$, independent and

$P(X_n = -1) = P(X_n = 1) = \frac{1}{2}$

we let $N = \min\{n \quad X_1 + \cdots + X_n = 1\}$.

N is a stopping time.

Theorem 3.3.2 (Wald's Equation)

If $X_1, X_2, \cdots$ i.i.d r.v. with finite expectations, N is a stopping time. for $X_1, X_2, \cdots$, $E[N] < \infty$, then

$$E\left[\sum_{i=1}^{N} X_n\right] = E[N]\, E[X]$$

proof let

$$I_n = \begin{cases} 1 & \text{if } N \geq n \\ 0 & \text{if } N < n \end{cases}$$

$$\sum_{n=1}^{N} X_n = \sum_{n=1}^{\infty} X_n I_n$$

$$E\left[\sum_{n=1}^{N} X_n\right] = E\left[\sum_{n=1}^{\infty} X_n I_n\right] = \sum_{n=1}^{\infty} E[X_n I_n]$$

$I_n = 1 \iff$ we have not stopped after observing $X_1, \cdots, X_{n-1}$.

therefore, I_n determined by $X_1, \cdots, X_{n-1}$.

$$E\left[\sum_{n=1}^{N} X_n\right] = \sum_{n=1}^{\infty} EX \cdot E[I_n] = E[X] \sum_{n=1}^{\infty} P(N \geq n) = EX\, EN.$$

3.3.2. Back to Renewal Theory.

for $\sum_{n=1}^{N(t)} X_n$. $N(t)$ is not a stopping time, $\{N(t) = n\}$ is depend on X_n's value for $\sum_{n=1}^{N(t)+1} X_n$. $N(t)+1$ is a stopping time, $\{N(t)+1=n\} \Longleftrightarrow$

$X_1+\cdots+X_{n-1} \le t$, $X_1+\cdots+X_n > t$. not depend on $X_{n+1} \cdots$

Corollary 3.3.3. $E[S_{N(t)+1}] = \mu(m(t)+1)$

proof $E[S_{N(t)+1}] = \sum_{n=1}^{N(t)+1} X_n = EX \cdot E[N(t)+1] = \mu \cdot (m(t)+1)$

Theorem 3.3.4 (Elementary Renewal Theorem).

$$\frac{m(t)}{t} \to \frac{1}{\mu} \quad \text{as} \quad t \to \infty$$

proof. Suppose $\mu < \infty$. If. interarrival time is bounded by M.

$S_{N(t)+1} = S_{N(t)} + X_{N(t)+1} \le t+M$. $\Rightarrow$ $E\, S_{N(t)+1} \le t+M$.

by corollary 3.3.3. $\mu(m(t)+1) = E\, S_{N(t)+1}$.

$\Rightarrow \mu(m(t)+1) \le t+M \Rightarrow m(t) \le \frac{t+M}{\mu} - 1$.

$$\lim_{t\to\infty} \frac{m(t)}{t} \le \lim_{t\to\infty} \frac{t+M}{\mu t} - \frac{1}{t} = \frac{1}{\mu}$$

If unbounded.

define $\{N^M(t), t\ge 0\}$ truncting $N(t)$ at M. which means.

$$\bar{X}_n = \begin{cases} X_n & \text{if } X_n \le M \\ M & \text{if } X_n > M. \end{cases}$$

发生频率高于 $N(t)$.

$EN(t) < EN^M(t)$.

$$\lim_{t\to\infty} \frac{m^M(t)}{t} = \frac{1}{\mu^M}, \quad \lim_{M\to\infty} \frac{1}{\mu^M} = \frac{1}{\mu}$$

$$\lim_{t\to\infty} \frac{m(t)}{t} \le \lim_{t\to\infty} \frac{m^M(t)}{t} = \frac{1}{\mu^M} \to \frac{1}{\mu} \quad \text{as } M\to\infty.$$

#

Theorem 3.3.5

let μ and σ^2 represent the mean and variance of an interarrival time. $\mu \cdot \sigma^2 < \infty$.

$$P\left(\frac{N(t)-\frac{t}{\mu}}{\sigma\sqrt{\frac{t}{\mu^3}}} < y\right) \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-\frac{x^2}{2}} dx$$

proof Let $r_t = \frac{t}{\mu} + y\sigma\sqrt{\frac{t}{\mu^3}}$, then

$$P\left(\frac{N(t) - \frac{t}{\mu}}{\sigma\sqrt{\frac{t}{\mu^3}}} < y \right) = P(N(t) < r_t)$$

$$= P(S_{r_t} > t).$$

$$= P\left(\frac{S_{r_t} - r_t\mu}{\sigma\sqrt{r_t}} > \frac{t - r_t\mu}{\sigma\sqrt{r_t}}. \right)$$

$$= P\left(\frac{S_{r_t} - r_t\mu}{\sigma\sqrt{r_t}} > -y\left(1 + \frac{y\sigma}{\sqrt{t\mu}}\right)^{-\frac{1}{2}} \right)$$

By central limit theorem.

$$\frac{S_{r_t} - r_t\mu}{\sigma\sqrt{r_t}} \longrightarrow N(0,1). \qquad -y\left(1 + \frac{y\sigma}{\sqrt{t\mu}}\right)^{-\frac{1}{2}} \longrightarrow -y.$$

$$\Rightarrow P\left(\frac{N(t) - \frac{t}{\mu}}{\sigma\sqrt{\frac{t}{\mu^3}}} < y \right) \longrightarrow \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-\frac{x^2}{2}} dx$$

3.4 The key Renewal Theorem and Applications.

lattice. : there exists $d \geq 0$ such that $\sum_{n=0}^{\infty} P(X = nd) = 1$.

X is lattice $\Leftrightarrow$ it only takes on integral multiples of d.

Theorem 3.4.1 (Blackwell's theorem)

(i) If F is not lattice, then

$$m(t+a) - m(t) \longrightarrow \frac{a}{\mu} \qquad t \to \infty.$$

for all $a > 0$

(ii) If F is lattice with period d.

$$E[\text{number of renewals at } nd] \longrightarrow \frac{d}{\mu} \qquad n \to \infty$$

If F is not lattice. then the expected number of renewals in an interval of length a, is approximately $\frac{a}{\mu}$

h be a function defined on $[0, \infty)$ $\qquad \forall a \geq 0$

$\underline{m}_n(a)$: the supremum of $h(t)$ in $(n-1)a < t < na$.

$\overline{m}_n(a)$: the infinum of $h(t)$ in $(n-1)a < t < na$.

h is Directly Riemann Integrable if

a) both $\sum_{n=1}^{\infty} \underline{m}_n(a)$ and $\sum_{n=1}^{\infty} \overline{m}_n(a)$ are finite

b). $\lim_{a\to 0} \sum_{n=1}^{\infty} \underline{m}_n(a)\, a = \lim_{a\to 0} \sum_{n=1}^{\infty} \overline{m}_n(a)\, a$

sufficient condition for h to be directly Riemann Integrable.

(i) $h(t) \geq 0$ for all $t \geq 0$

(ii) $h(t)$ is nonincreasing

(iii) $\int_0^\infty h(t)\,dt < \infty$ (Lesbegue Integrable)

Theorem 3.4.2 (The key Renewal theorem)

If F is not lattice, and if $h(t)$ is directly Riemann Integrable, then

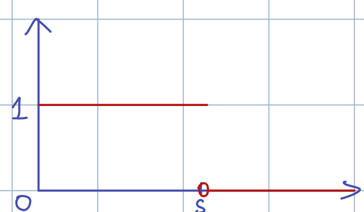
$$\lim_{t\to\infty} \int_0^t h(t-x)\,dm(x) = \frac{1}{\mu} \int_0^\infty h(t)\,dt.$$

where

$$m(x) = \sum_{n=1}^{\infty} F_n(x), \qquad \mu = \int_0^{+\infty} \bar{F}(t)\,dt$$

$KRT \Rightarrow BT$.

let $h(t) = \begin{cases} 1 & 0 \leq t \leq s \\ 0 & t > s \end{cases}$



$\int_0^\infty h(t)\,dt = \int_0^s dt = s < \infty$, $h(t)$ directly Riemann Integrable

$RHS\ (KRT) = \frac{1}{\mu} \int_0^\infty h(t)\,dt = \frac{s}{\mu}$

$LHS\ (KRT) = \lim_{t\to\infty} \int_0^t h(t-x)\,d\,m(x)$

$$= \lim_{t\to\infty} \int_{t-s}^{t} h(t-x)\,dm(x).$$

$$= \lim_{t\to\infty} \int_{t-s}^{t} dm(x) = \lim_{t\to\infty} (m(t)-m(t-s))$$

故有: $\frac{s}{\mu} = \lim_{t\to\infty} (m(t)-m(t-s))$, BT 成立.

BT $\Rightarrow$ KRT.

(积分拆成离散形式).

$$\lim_{t\to\infty} \int_0^t h(t-x)\,dm(x) = \lim_{t\to\infty} \lim_{n\to\infty} \sum_{i=1}^{n} \int_{\frac{i-1}{n}t}^{\frac{i}{n}t} h(t-\tfrac{i}{n}t)\,dm(x).$$

$$= \lim_{t\to\infty} \lim_{n\to\infty} \sum_{i=1}^{n} h(t-\tfrac{i}{n}t)\left[m(\tfrac{i}{n}t)-m(\tfrac{i-1}{n}t)\right]$$

$$= \lim_{t\to\infty} \lim_{n\to\infty} \sum_{i=1}^{n} \frac{t}{n\mu} h(t-\tfrac{i}{n}t)$$

$$= \lim_{t\to\infty} \lim_{n\to\infty} \frac{1}{\mu} \times \frac{t}{n} \sum_{i=1}^{n} h(t-\tfrac{i}{n}t)$$

$$= \frac{1}{\mu} \lim_{t\to\infty} \int_0^t h(t-x)\,dx = \frac{1}{\mu} \int_0^\infty h(x)\,dx.$$

Lemma 3.4.3 (the distribution of $S_{N(t)}$. — the time of the last renewal piror to time t).

$$P(S_{N(t)} \le s) = \bar{F}(t) + \int_0^t \bar{F}(t-y)\,dm(y). \qquad t \ge s \ge 0.$$

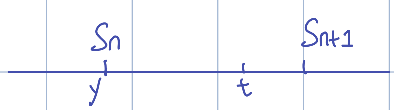
proof.

$$P(S_{N(t)} \le s) = \sum_{n=0}^{\infty} P(S_n \le s,\ S_{n+1} > t).$$

$$= P(S_1 > t) + \sum_{n=1}^{\infty} P(S_n \le s,\ S_{n+1} > t)$$

$$= \bar{F}(t) + \sum_{n=1}^{\infty} \int_0^\infty P(S_n \le s,\ S_{n+1} > t \mid S_n = y\}\,dF_n(y)$$

$$= \bar{F}(t) + \sum_{n=1}^{\infty} \int_0^s P(S_{n+1} > t \mid S_n = y)\,dF_n(y).$$



$$= \bar{F}(t) + \sum_{n=1}^{\infty} \int_0^s P(X_{n+1} > t-y)\,dF_n(y)$$

$$= \bar{F}(t) + \int_0^s \bar{F}(t-y)\,d\sum_{n=1}^{\infty} F_n(y)$$

$$= \bar{F}(t) + \int_0^s \bar{F}(t-y)\,dm(y)$$

Remarks.

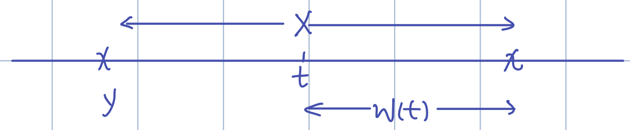
$$P(S_{N(t)} = s) = \left(\bar{F}(t) + \int_0^s \bar{F}(t-y)\,dm(y)\right)' = \bar{F}(t-s)\,dm(s)$$

$$P(S_{N(t)} = 0) = \bar{F}(t)$$

3.4.2. Limiting Mean Excess and the Expansion of $m(t)$.

$W(t)$: excess of a nonlattice renewal process.

$$E[W(t)] = E[W(t) \mid S_{N(t)}=0]\,\bar F(t) + \int_0^t E[W(t) \mid S_{N(t)}=y]\,\bar F(t-y)\,dm(y).$$



$$E[W(t) \mid S_{N(t)}=0] = E[X-t \mid X>t]$$

$$E[W(t) \mid S_{N(t)}=y] = E[X-(t-y) \mid X>(t-y)]$$

we let $h(t) = E[W(t) \mid S_{N(t)}=0]\,\bar F(t)$

then $h(t-y) = E[W(t) \mid S_{N(t)}=y]\,\bar F(t-y)$.

$$E[W(t)] = h(t) + \int_0^t h(t-y)\,dm(y) \xrightarrow{KRT} \frac{1}{\mu}\int_0^\infty h(x)\,dx.$$

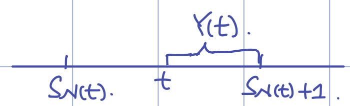
Hence, If h is directly Riemann Integrable, then.

$$E[W(t)] \longrightarrow \frac{1}{\mu}\int_0^\infty h(x)dx \qquad (t\to\infty).$$

PROPOSITION 3.4.6.

$$\lim_{t\to\infty}\left(m(t)-\frac{t}{\mu}\right) = \frac{EX^2}{2\mu^2} - 1.$$

proof. $S_{N(t)+1} = t + Y(t)$



$Y(t)$: remaining life time of the unit in use at time t.

$$E[S_{N(t)+1}] = E[t+Y(t)] \Rightarrow t + EY(t) = \mu(m(t)+1)$$

$$\Rightarrow m(t) - \frac{t}{\mu} = \frac{E[Y(t)]}{\mu} - 1$$

$$E[Y(t)] = E[Y(t) \mid S_{N(t)}=0]\,\bar F(t) + \int_0^t E[Y(t) \mid S_{N(t)}=y]\,\bar F(t-y)\,dm(y).$$

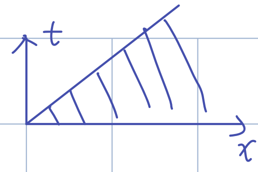
$$E[Y(t) \mid S_{N(t)}=y] = E[X-(t-y) \mid X>t-y]$$

令 $h(t) = E[Y(t) \mid S_{N(t)}=0]\,\bar F(t)$

$$= E[X-t \mid X>t]\,\bar F(t) = \int_t^\infty (x-t)\,dF(x) \geq 0$$

$$h'(t) = -\bar F(t) \leq 0$$

$$\int_0^\infty h(t)\,dt = \int_0^\infty \int_t^\infty (x-t)\,dF(x)\,dt$$
$$= \int_0^\infty \int_0^x (x-t)\,dt\,dF(x)$$
$$= \int_0^\infty \frac{1}{2}x^2\,dF(x) = \frac{1}{2}EX^2$$



If $EX^2 < \infty$, then h is Riemann Integrable.

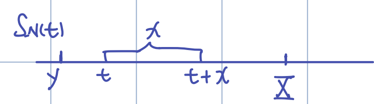
KRT $\Rightarrow$ $E[Y(t)] \Rightarrow \frac{1}{\mu}\int_0^\infty h(t)\,dt = \frac{1}{2\mu}EX^2$

$$\Rightarrow \lim_{t\to\infty} m(t) - \frac{t}{\mu} = \frac{1}{2\mu^2}EX^2 - 1$$

Age: $A(t)$. For $x \geq 0$. we denote $w(t) = 1(A(t+x) > x)$.

$w(t) = 1 \Leftrightarrow$ The item in use at $t+x$ was in use at t.

$E\,w(t) = P(A(t+x) > x). = P(Y(t) > x)$



$$\lim_{t\to\infty} E[w(t)] = \lim_{t\to\infty} P(A(t+x) > x) = \lim_{t\to\infty} P(A(t) > x).$$

令 $h(t-y) = E[w(t) \mid S_{N(t)} = y]\,\bar{F}(t-y) = P(A(t+x) > x \mid S_{N(t)} = y)\,\bar{F}(t-y)$

$$= P(X > t+x-y \mid X > t-y)\,\bar{F}(t-y)$$
$$= \frac{\bar{F}(t+x-y)}{\bar{F}(t-y)} \cdot \bar{F}(t-y). = \bar{F}(x+t-y)$$

obviously. $h \geq 0$, $h \downarrow$

$$\int_0^\infty h(t)\,dt = \int_0^\infty \bar{F}(x+t)\,dt = \int_x^\infty \bar{F}(t)\,dt \leq \mu.$$

Hence. h is directly Riemann Integrable.

By K.R.T

$$\lim_{t\to\infty} P(A(t) > x) = \frac{1}{\mu}\int_0^\infty h(t)\,dt = \frac{\overset{EX - E\min(X,x).}{\overset{\parallel}{\int_x^\infty \bar{F}(t)\,dt}}}{\int_0^\infty \bar{F}(t)\,dt. = \mu}$$

$$\lim_{t\to\infty} P(Y(t) > x) = \lim_{t\to\infty} P(A(t+x) \geq x) = \lim_{t\to\infty} P(A(t) \geq x).$$

$\Rightarrow$ Asymptotically $Y(t) = A(t)$ in distribution.

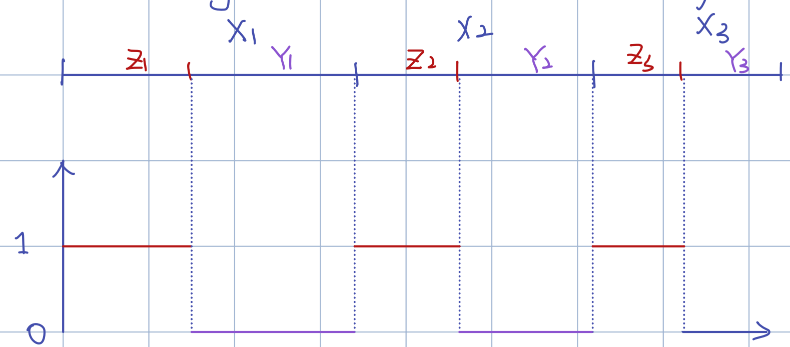
Denote $\{B(t): t \geq 0\}$. $B(t) = \begin{cases} 1 & \text{if the item in use is not older than } x \\ 0 & \text{otherwise.} \end{cases}$

$$\lim_{t\to\infty} P(A(t)\le x) = \lim_{t\to\infty} P(B(t)=1) = \frac{E\min(X,x)}{EX}.$$

Alternating Renewal Process

$\{N(t), t\ge 0\}$ is R.P. $\{B(t), t\ge 0\}$ 取值 0, 1 二元变量.

In the nth renewal cycle, length X_n, the binary process takes the value of 1, for Z_n units of time system is "on". Takes the value of 0 for the remaining $Y_n = X_n - Z_n$ units of time system is off.



let $P(t)$ be the probability that system "on" at time t.

Theorem 3.3.4

If $EX_n = E[Z_n + Y_n] = \mu < \infty$. then

$$\lim_{t\to\infty} P(t) = \lim_{t\to\infty} P(B(t)=1) = \frac{EZ_1}{EZ_1+EY_1}$$

we denote $A(t)$ $Z_n = \min(X_n, x)$.

$$Y_n = \max(0, X_n - x).$$

then $$\lim_{t\to\infty} P(A(t)\le x) = \frac{E\min(X_n, x)}{EX} = \frac{EZ_n}{EZ_n+EY_n}$$

proof. Let H be the distribution of Z.

$$P(t) = P(B(t)=1 \mid S_{N(t)}=0)\cdot \bar{F}(t) + \int_0^t P(B(t)=1 \mid S_{N(t)}=y)\cdot \bar{F}(t-y)\, dm(y).$$

$$= \frac{\bar{H}(t)}{\bar{F}(t)} \times \bar{F}(t) + \int_0^t \frac{\bar{H}(t-y)}{\bar{F}(t-y)} \bar{F}(t-y)\, dm(y)$$

$$= H(t) + \int_0^t H(t-y)\, dm(y)$$

Since $\bar{H}(t)$ nonegative. nonincreasing.

$\bar{H}(t) = P(Z_1 > t)$.

$\int_0^\infty \bar{H}(t)\, dt = EZ_1 < \infty$. $\lim_{t\to\infty} \bar{H}(t) = 0$.

故由 KRT 定理：$\lim_{t\to\infty} P(t) = \frac{1}{\mu} \int_0^\infty H(t)dt = \frac{EZ}{EZ+EY}$

Averaging Behavior.

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t B(s)\, ds = \frac{EZ}{EZ+EY}.$$

proof. $\lim_{t\to\infty} \frac{1}{t} \sum_{n=1}^{N(t)} Z_n \leq \lim_{t\to\infty} \frac{1}{t} \int_0^t B(s)\, ds \leq \lim_{t\to\infty} \frac{1}{t} \sum_{n=1}^{N(t)+1} Z_n$

而 $\lim_{t\to\infty} \frac{1}{t} \sum_{n=1}^{N(t)} Z_n = \lim_{t\to\infty} \frac{1}{N(t)} \sum_{n=1}^{N(t)} Z_n \times \frac{N(t)}{t}$

$$\xrightarrow{\text{大数定律}} EZ_1 \times \frac{1}{\mu}$$

$\lim_{t\to\infty} \frac{1}{t} \sum_{n=1}^{N(t)+1} Z_n = \lim_{t\to\infty} \frac{1}{N(t)+1} \sum_{n=1}^{N(t)+1} Z_n \times \frac{N(t)+1}{t}$

$$\xrightarrow{\text{大数定律}} EZ_1 \times \frac{1}{\mu}$$

故 $\lim_{t\to\infty} \frac{1}{t} \int_0^t B(s)\, ds = \frac{EZ_1}{\mu} = \frac{EZ_1}{EZ_1+EY_1}$

Reward Structure and Renewal Reward Process

Renewal Process $\{N(t), t\geq 0\}$

we suppose that earn reward 1 per time unit during the first Z_n periods in a renewal cycle.

earn reward 0 per time unit during the remaining Y_n

periods in a renewal cycle.

we know $P(t) = P(B(t)=1) = \frac{EZ_1}{EZ_1+EY_1}$

the long-run average reward per unit time.

$\lim_{t\to\infty} \frac{1}{t}\int_0^t B(s)\,ds = \frac{EZ}{EZ+EY}$.

Now. we denote R.P. as $\{N(t), t\geq 0\}$.

R.R.P as $\{R(t), t\geq 0\}$.

R.R.P. R_n: the reward earned at the time of the nth arrival

$R_1, R_2, \cdots$ i.i.d. distribution depends on the distribution of X.

$R(t) = \sum_{n=1}^{N(t)} R_n$. – the total reward earned by time t.

* keep track of the total reward so far.
* the incremental reward gathered depend only on the time since the last renewal.

let $E[R] = ER_n$. $EX = EX_n$. then.

long-run average reward

$$\lim_{t\to\infty} \frac{R(t)}{t} = \frac{ER}{EX}$$

$$\lim_{t\to\infty} \frac{ER(t)}{t} = \frac{ER}{EX}$$

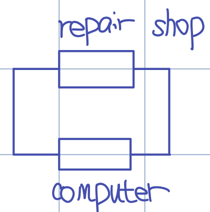
proof. $\lim_{t\to\infty} \frac{R(t)}{t} = \lim_{t\to\infty} \frac{1}{t}\sum_{n=1}^{N(t)} R_n$

$$= \lim_{t\to\infty} \frac{1}{N(t)}\sum_{n=1}^{N(t)} R_n \times \frac{N(t)}{t} = \frac{ER}{\mu} = \frac{ER}{EX}.$$

Example 1. one operating unit, one spare part

If operating unit fails then immediately replaced by the spare part, and failed unit is sent to repair.

System is down if both units are broken.



— at most repair one unit.

we suppose Time of failure $\sim G$

Repair time $\sim H$

operating time and repair time independent.

Now we want to know long-run average up time.

Solve: the Renewal Point: when one breaks down and the other immediately replaces.

the length of the Renewal Cycle:

let L be the life time of the operating time.

R be the repaired time

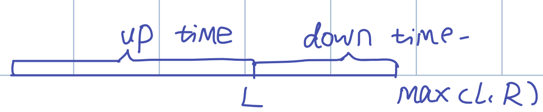
① $L \geq R$. cycle length L

② $L < R$. cycle length R.

Hence. cycle length: $\max(L, R)$.

$P(\max(L,R) \leq x) = P(L \leq x)\, P(R \leq x) = H(x)\, G(x)$.

$E \max(L,R) = \int_0^\infty 1 - H(x)\, G(x)\, dx$.



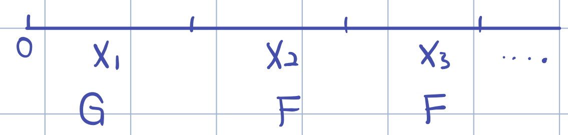
long-run average up time

$$\frac{EL}{EX} = \frac{\int_0^\infty 1-G(x)\,dx}{\int_0^\infty 1-H(x)G(x)\,dx}.$$

and long-run average down time.

$$1-\frac{EL}{EX} = 1-\frac{EL}{E[\max(L,R)]}.$$

Delayed Renewal Process.



, F 与 G 独立.

即 $X_1 \sim G$. 其余的 $X_i \sim F$. 则这样的 Renewal Process.

$\{N_D(t), t \geq 0\}$ 称为 Delayed Renewal Process.

$$P(S_n \leq t) = \int_0^\infty P(S_n \leq t \mid S_1 = y)\,dy$$

$$= \int_0^\infty F_{n-1}(t-y)\,dG(y). = G * F_{n-1}(t).$$

$$m_D(t) = E[N_D(t)] = \sum_{n=1}^{\infty} P(N_D(t) \geq n)$$

$$= \sum_{n=1}^{\infty} P(S_n \leq t)$$

$$= \sum_{n=1}^{\infty} G * F_{n-1}(t).$$

$\lim_{t\to\infty} \frac{N_D(t)}{t} = \frac{1}{\mu}$ with probability 1.

Elementary Renewal Theorem. $\lim_{t\to\infty} \frac{m_D(t)}{t} = \frac{1}{\mu}$

Blackwell's Theorem and key Renewal Theorem.

If F is nonlattice. then $\lim_{t\to\infty} m_D(t+a) - m_D(t) = \frac{a}{\mu}$

If h is directly Riemann Integrable. then

$$\lim_{t\to\infty} \int_0^t h(t-x)\, d m_D(x) = \frac{1}{\mu} \int_0^\infty h(x)\, dx.$$

Delayed Renewal Process 的结论与一般的 Renewal Process 没有太大区别. 推导过程也类似.