- What Is a **Linear Programming** Problem?

Reference: *Operations Research: Applications & Algorithms*, 4th Edition, by W. L. Winston, Duxbury Press, Chapter 3

- A tool for solving optimization problems
- Optimization of a linear objective function, subject to linear equality and inequality constraints

Linear Functions and Inequalities

Definition: A function $f(x_1,x_2,\ldots,x_n)$ of $x_1,x_2,\ldots,x_n$ is a linear function if and only if for some constants $c_1,c_2,\ldots,c_n,$

$$f(x_1,x_2,\ldots,x_n)= c_1 x_1+ c_2 x_2 +\ldots+ c_n x_n$$

For example: $f(x_1, x_2) = 2x_1 + x_2$ is a linear function,

but $f(x_1, x_2) = x_1x_2$ is not.

Definition: For any linear function $f(x_1,x_2,\ldots,x_n)$ and any number b, $f(x_1,x_2,\ldots,x_n) \leq b$ and $f(x_1,x_2,\ldots,x_n) \geq b$ are linear inequalities.

For example: $2x_1 + 3x_2 \geq 6$ and $2x_1 + 3x_2 \leq 6$.

Linear Programming Problem

A linear programming (LP) problem is an optimization problem for which we do the following:

1. We attempt to maximize (or minimize) a linear function of the decision variables. The function that is to be maximized or minimized is called the *objective function*.

   ---- objective function coefficient of the variable

2. The values of the decision variables must satisfy a set of *constraints*. Each constraint must be a linear equation or linear inequality.

   ---- technological coefficients

   ---- right-hand-side (rhs)

3. A *sign restriction* is associated with each variable. For any variable x_i, the sign restriction specifies either that x_i must be nonnegative ($x_i \geq 0$) or that x_i may be unrestricted in sign.

Assumptions of LP

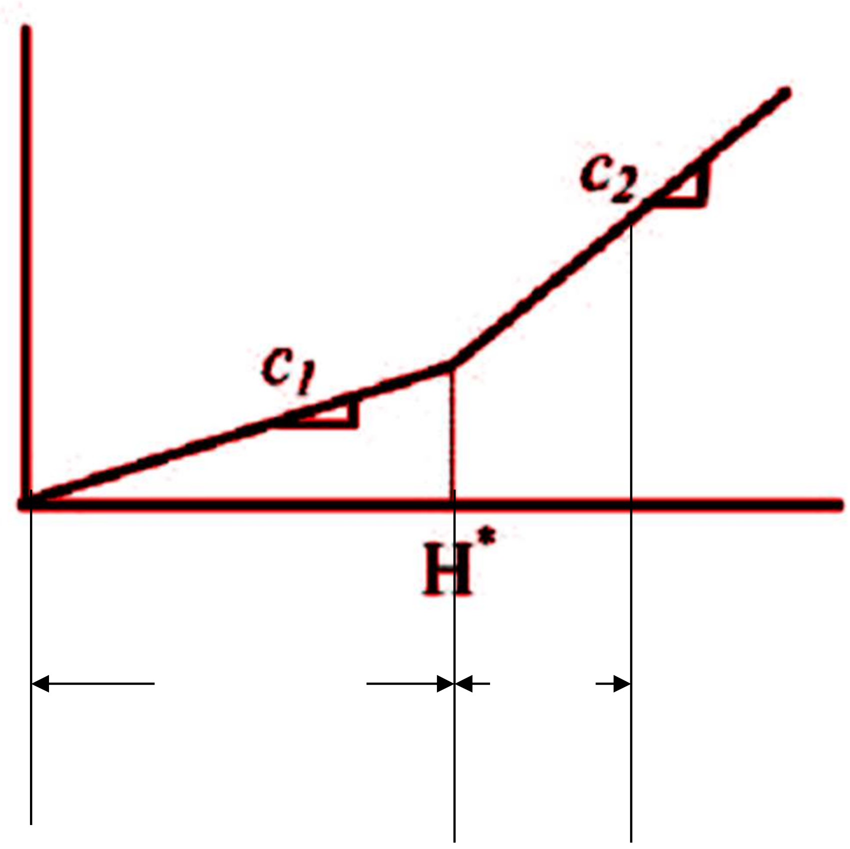
- Proportionality
  - What if the unit cost is not constant (e.g., discount)?
  - Setup cost?
- Additivity
  - D.V. are independent and the terms could be added
- Divisibility
  - D.V. are allowed to be fractional values.
- Certainty

$$
\begin{aligned}
\max\ & z = 3x_1 + 2x_2 \\
\text{s.t.}\ & 2x_1 + x_2 \le 100 \\
& x_1 + x_2 \le 80 \\
& x_1 \le 40 \\
& x_1 \ge 0 \\
& x_2 \ge 0
\end{aligned}
$$

- In order to build up an LP model, all of cost functions must be linear in objective function. Thus, we assume that all related cost functions are linear.
- Convex piecewise linear function is OK too.

$$\min \quad + \sum_{t=1}^{\bar{t}} (c_1 H_{1t} + c_2 H_{2t})$$
$$\text{s.t.} \quad H_t = H_{1t} + H_{2t}$$
$$0 \le H_{1t} \le H^{\#}$$
$$0 \le H_{2t}$$

$$c_1 < c_2$$



- See Example of Sailco Inventory.

LP Model Formulation

Example 1: Two Toys Example

Giapetto's Woodcarving, Inc., manufactures two types of wooden toys: soldiers and trains. A soldier sells for $27 and uses $10 worth of raw materials. Each soldier that is manufactured increases Giapetto's variable labor and overhead costs by $14. A train sells for $21 and uses $9 worth of raw materials. Each train built increases Giapetto's variable labor and overhead costs by $10. The manufacture of wooden soldiers and trains requires two types of skilled labor: carpentry and finishing. A soldier requires 2 hours of finishing labor and 1 hour of carpentry labor. A train requires 1 hour of finishing labor and 1 hour of carpentry labor. Each week, Giapetto can obtain all the needed raw material but only 100 finishing hours and 80 carpentry hours. Demand for the trains is unlimited. But at most 40 soldiers are bought each week. Giapetto wishes to maximize weekly profit (revenues-costs). Formulate a mathematical model of Giapetto's situation that can be used to maximize Giapetto's weekly profit.

per each		
Selling Price ($)	27	21
Raw materials cost ($)	10	9
Variable labor and overhead cost ($)	14	10
Finishing labor (hours)	2	1
Carpentry labor (hours)	1	1

Available finishing hours: 100

Available carpentry hours: 80

Maximum demand for soldiers: 40

1. Decision variables:

x_1 = Number of soldiers produced each week

x_2 = Number of trains produced each week

2. Objective function:

maximize $z = 3x_1 + 2x_2$

weekly profit = weekly revenue - weekly costs

3. Constraints:

Total finishing hours = $2x_1 + 1x_2$

Total carpentry hours = $1x_1 + 1x_2$

Maximum demand for soldiers is 40

4. Sign restrictions:

$x_1 \geq 0$

$x_2 \geq 0$

~~x_1, x_2: integer~~

$$
\begin{array}{llcl}
\max & z = 3x_1 + 2x_2 & & \\
\text{s.t.} & 2x_1 + x_2 & \leq & 100 \\
& x_1 + x_2 & \leq & 80 \\
& x_1 & \leq & 40 \\
& x_1 & \geq & 0 \\
& x_2 & \geq & 0
\end{array}
$$

Example 2: A Diet Problem

My diet requires that all the food I eat come from one of the four “basic food groups” (chocolate cake, ice cream, soda, and cheesecake). At present, the following four foods are available for consumption: brownies, chocolate ice cream, cola, pineapple cheesecake. Each brownie costs 50¢, each scoop of chocolate ice cream cost 20¢, each bottle of cola costs 30¢, and each piece of pineapple cheesecake costs 80¢. Each day, I must ingest at least 500 calories, 6 oz of chocolate, 10 oz of sugar, and 8 oz of fat. The nutritional content per unit of each food in shown in Table 1. Formulate a LP model that can be used to satisfy my daily nutritional requirements at minimum cost.

Table 1: Nutrition content

	CALORIES	CHOCOLATE (ounces)	SUGAR (ounces)	FAT (ounces)	COST (cents)
Brownie	400	3	2	2	50
Chocolate ice cream (1 scoop)	200	2	2	4	20
Cola (1 bottle)	150	0	4	1	30
Pineapple cheese-cake (1 piece)	500	0	4	5	80

Daily nutrition requirement:

- at least 500 calories
- at least 6 oz of chocolate
- at least 10 oz of sugar
- at least 8 oz of fat

Example 2 solution

1. Decision variables:

x_1 = number of brownies eaten daily

x_2 = number of scoops of chocolate ice cream eaten daily

x_3 = bottles of cola drunk daily

x_4 = pieces of pineapple cheesecake eaten daily

2. Objective function:

Cost = cost of brownies + cost of chocolate ice cream + cost of cola

+ cost of pineapple cheesecake

$= 50x_1 + 20\,x_2 + 30\,x_3 + 80\,x_4$

minimize $z = 50x_1 + 20x_2 + 30x_3 + 80x_4$

3. Constraints:

Calorie: $400x_1 + 200x_2 + 150x_3 + 500x_4 \geq 500$

Chocolate: $3x_1 + 2x_2 + 0x_3 + 0x_4 \geq 6$

Sugar: $2x_1 + 2x_2 + 4x_3 + 4x_4 \geq 10$

Fat: $2x_1 + 4x_2 + 1x_3 + 5x_4 \geq 8$

4. Sign restrictions: $x_i \geq 0 \quad (i = 1,2,3,4)$

min $\quad z = 50x_1 + 20x_2 + 30x_3 + 80x_4$

s.t.

$$\begin{aligned}
400x_1 + 200x_2 + 150x_3 + 500x_4 &\geq 500 \\
3x_1 + 2x_2 &\geq 6 \\
2x_1 + 2x_2 + 4x_3 + 4x_4 &\geq 10 \\
2x_1 + 4x_2 + x_3 + 5x_4 &\geq 8 \\
x_i \geq 0 \quad (i = 1,2,3,4)
\end{aligned}$$

Optimal solution:

$$x_1 = x_4 = 0,\ x_2 = 3,\ x_3 = 1,\ z = 90$$

Example 3: Workforce Scheduling

A post office requires different numbers of full-time employees on different days of the week. The number of full-time employees required on each day is given in Table 2. Union rules state that each full-time employee must work five consecutive days and then receive two days off. For example, an employee who works Monday to Friday must be off on Saturday and Sunday. The post office wants to meet its daily requirements using only full-time employees. Formulate an LP that the post office can use to minimize the number of full-time employees that must be hired.

Table 2: Employee requirements

	Number of full-time employees required
Day 1 = Monday	17
Day 2 = Tuesday	13
Day 3 = Wednesday	15
Day 4 = Thursday	19
Day 5 = Friday	14
Day 6 = Saturday	16
Day 7 = Sunday	11

1. Decision Variables:

$x_i =$ number of employees working on Day i, $i = 1,\dots,7$.

2. Objective Function:

Total number of employees = Summation of the number of employees working on each day

$$= x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7$$

3. Constraints:

$$x_1 \geq 17, x_2 \geq 13, x_3 \geq 15, x_4 \geq 19, x_5 \geq 14, x_6 \geq 16, x_7 \geq 11$$

4. Sign Restrictions:

x_i $(i = 1,2,3,4,5,6,7)$ nonegative integers

Wrong model!!!

Example 3 solution: Workforce Scheduling

1. Decision Variables:

x_i = number of employees beginning working on Day i, $i = 1,....,7$.

Day 1	Day 2	Day 3	Day 4	Day 5	Day 6	Day 7
x_1	x_1	x_1	x_1	x_1		
	x_2	x_2	x_2	x_2	x_2	
		x_3	x_3	x_3	x_3	x_3
x_4			x_4	x_4	x_4	x_4
x_5	x_5			x_5	x_5	x_5
x_6	x_6	x_6			x_6	x_6
x_7	x_7	x_7	x_7			x_7

3. Constraints:

Day1 : $x_1 + x_4 + x_5 + x_6 + x_7 \geq 17$

$\vdots$

Day7 : $x_3 + x_4 + x_5 + x_6 + x_7 \geq 11$

4. Sign Restrictions:

x_i $(i = 1,2,3,4,5,6,7)$ nonegative integers

2. Objective Function:

Total number of employees

= Summation of the number of employees beginning working on each day

$= x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7$

Min $z = x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7$

If relax the integer restriction:

$$x_1 = \frac{4}{3}, x_2 = \frac{10}{3}, x_3 = 2, x_4 = \frac{22}{3}, x_5 = 0, x_6 = \frac{10}{3}, x_7 = 5, z = \frac{67}{3}$$

If round the fractional variables up:

$$x_1 = 2, x_2 = 4, x_3 = 2, x_4 = 8, x_5 = 0, x_6 = 4, x_7 = 5, z = 25$$

Optimal solution from integer programming:

$$x_1 = 4, x_2 = 4, x_3 = 2, x_4 = 6, x_5 = 0, x_6 = 4, x_7 = 3, z = 23$$

There is no way that the optimal linear programming solution could have been rounded to obtain the optimal all-integer solution.

Extension of Example 3

Suppose that each full-time employee works 8 hours per day. Thus, Monday's requirement of 17 workers may be reviewed as a requirement of 8(17)=136 hours. The post office may meet daily labor requirement by using both full-time and part-time employees. During each week, a full-time employee works 8 hours a day for five consecutive days, and a part-time employee works 4 hours a day for five consecutive days. A full-time employee costs \$15 per hour, whereas a part-time employee costs only \$10 per hour. Union requirements limit part-time labor to 25% of weekly labor requirements. Formulate an LP to minimize the weekly's labor costs.

Extension of Example 3: solution

- **Decision Variables:**

  x_i = Number of fulltime employees (FTE) who start work on day i;

  y_i = Number of part-time employees (PTE) who start work on day i;

  i=1,2…,7 (day 1 = Sunday).

- min $z = 15(8)(5)(x_1 + x_2 + ...x_7)+10(4)(5)(x_8 + ...x_{14})$

s.t. $8(x_1+x_4+x_5+x_6+x_7)+4(y_1+y_4+y_5+y_6+y_7)\geq 88$ (Sunday)

$8(x_1+x_2+x_5+x_6+x_7)+4(y_1+y_2+ y_5+y_6+y_7)\geq 136$ (Monday)

$8(x_1+x_2+x_3+x_6+x_7)+4(y_1+y_2+y_3+y_6+y_7)\geq 104$ (Tuesday)

$8(x_1+x_2+x_3+x_4+x_7)+4(y_1+y_2+y_3+ y_4+y_7)\geq 120$ (Wednesday)

$8(x_1+x_2+x_3+x_4+x_5)+4(y_1+y_2+y_3+ y_4+y_5)\geq 152$ (Thursday)

$8(x_2+x_3+x_4+x_5+x_6)+4(y_2 +y_3+y_4+y_5+y_6) \geq 112$ (Friday)

$8(x_3+x_4+x_5+x_6+x_7)+4(y_3+y_4+y_5+y_6+y_7)\geq 128$ (Saturday)

$20(y_1+y_2+y_3+ y_4+y_5+y_6+y_7)\leq 0.25(136+104+120+152+12+128+88)$

(this constraint ensures that part-time labor will fulfill

at most 25% of all labor requirements)

x_i, y_i ≥ 0 i=1,2…,7.

Example 4: Multiperiod Workforce Scheduling

CSL is a chain of computer service stores. The number of hours of skilled repair time that CSL requires during the next five months is as follows:

month	1	2	3	4	5
hours d_t	6,000	7,000	8,000	9,500	11,000

At the beginning of Month 1, 50 skilled technicians work for CSL.

Each skilled technician can work up to 160 hours per month. To meet further demands, new technicians must be trained. It takes one month to train a new technician. During the month of training, a trainee must be supervised for 50 hours by an experienced technician.

At the end of each month, 5% of CSL's experienced technicians quit to join Plum computers.

Each experienced technician is paid $2,000 a month (even if he or she does not work the full 160 hours). During the month of training, a trainee is paid $1,000 a month.

Formulate an LP whose solution will enable CSL to minimize the labor cost incurred in meeting the service requirements for the next five months.

Example 4 solution: Multiperiod Workforce Scheduling

1. Decision Variables:

x_t = number of technicians trained during month t, $t = 1,2,3,4,5.$

y_t = number of experienced technicians at the beginning of month t, $t = 1,2,3,4,5.$

2. Objective Function:

Total labor cost over the five months

= cost of paying trainees + cost of paying experienced technicians

$= 1{,}000(x_1 + x_2 + x_3 + x_4 + x_5) + 2{,}000(y_1 + y_2 + y_3 + y_4 + y_5)$

$$\text{Min } z = 1{,}000(x_1 + x_2 + x_3 + x_4 + x_5) + 2{,}000(y_1 + y_2 + y_3 + y_4 + y_5)$$

3. Constraints:

(a) Number of available technician hours during month t

$\geq$ number of technician hours required during month t

Number of available technician hours during month t is $160y_t$ - $50x_t$

$$160y_t - 50x_t \geq d_t \quad (t = 1,2,3,4,5)$$

(b) Experienced technicians available at beginning of month t

= experienced technicians available at beginning of month (t-1) + technicians trained during month (t-1) – experienced technicians who quit during month (t-1)

$$y_t = y_{t-1} + x_{t\text{-}1} - 0.05y_{t-1} \quad \text{or} \quad y_t = 0.95y_{t-1} + x_{t\text{-}1} \quad (t = 2,3,4,5)$$

$$y_1 = 50 \ \text{(initial condition)}$$

4. Sign Restrictions:

$x_t, y_t \ (t = 1, 2, 3, 4, 5)$ nonegative integers

If we ignore the integer constraints on the variables x_t and y_t, then we obtain the LP model.

$$
\begin{array}{ll}
\min & z = 1{,}000(x_1 + x_2 + x_3 + x_4 + x_5) + 2{,}000(y_1 + y_2 + y_3 + y_4 + y_5) \\
\text{s.t.} & \\
& 160y_1 - 50x_1 \ge 6{,}000 \qquad y_1 = 50 \\
& 160y_2 - 50x_2 \ge 7{,}000 \qquad 0.95y_1 + x_1 = y_2 \\
& 160y_3 - 50x_3 \ge 8{,}000 \qquad 0.95y_2 + x_2 = y_3 \\
& 160y_4 - 50x_4 \ge 9{,}500 \qquad 0.95y_3 + x_3 = y_4 \\
& 160y_5 - 50x_5 \ge 11{,}000 \qquad 0.95y_4 + x_4 = y_5 \\
& x_t, y_t \ge 0 \quad (t = 1,2,3,4,5)
\end{array}
$$

Example 5: Multiperiod Production and Inventory Planning

- Sailco Corporation must determine how many sailboats should be produced during each of the next four quarters (one quarter = three months). The demand during each of the next four quarters is as follows: first quarter, 40 sailboats; second quarter, 60 sailboats; third quarter, 75 sailboats; fourth quarter, 25 sailboats. <u>Sailco must meet demands on time.</u>
- At the beginning of the 1st quarter, Sailco has an inventory of 10 sailboats. At the beginning of each quarter, Sailco must decide how many sailboats should be produced during that quarter. For simplicity, we assume that sailboats manufactured during a quarter can be used to meet demand for that quarter.
- During each quarter, Sailco can produce up to 40 sailboats with regular-time labor at a total cost of 400 per sailboat. By having employees work overtime during a quarter, Sailco can produce additional sailboats with overtime labor at a total cost of $450 per sailboat.
- At the end of each quarter (after production has occurred and the current quarter's demand has been satisfied), a carrying or holding cost of $20 per sailboat is incurred. Use LP to determine a production schedule to minimize the sum of production and inventory costs during the next four quarters.

Example 5 solution: Multiperiod Production and Inventory planning

- Decision variables:

x_t = # of sailboats produced by regular-time labor during quarter *t* (*t*=1,2,3,4)

y_t = # of sailboats produced by over-time labor during quarter *t* (*t*=1,2,3,4)

s_t = # of sailboats sold at quarter *t* (*t*=1,2,3,4)

I_t = # of sailboats on hand at end of quarter *t* (*t*=1,2,3,4)

- Total cost over 4 quarters:

$$\min z = \sum_{t=1}^{4} (400x_t + 450y_t + 20I_t)$$

- Subject to

$$\begin{aligned}
&I_{t-1} + x_t + y_t - s_t = I_t && t = 1,2,3,4 \\
&s_t = d_t && t = 1,2,3,4 \\
&x_t \le 40 && t = 1,2,3,4 \\
&I_0 = 10 && \\
&x_t, y_t, s_t, I_t \ge 0 && t = 1,2,3,4
\end{aligned}$$

Extension: Applications of LP in Production Planning and Inventory Management (Aggregate Production Planning)

Example 6-1: Production Process

Rylon Corporation manufactures B ("Brute") and C ("Chanelle") perfumes. The raw material needed to manufacture each type of perfume can be purchased for $3 a pound.

Processing 1 lb of raw material requires 1 hour of laboratory time. Each pound of processed raw material yields 3 oz of Regular B Perfume and 4 oz of Regular C Perfume.

Regular B can be sold for $7/oz and regular C for $6/oz. Rylon also has the option of further processing Regular B and Regular C to produce Luxury B, sold at $18/oz, and Luxury C, sold at $14/oz.

Each ounce of Regular B processed further requires an additional 3 hours of laboratory time and $4 processing cost and yield 1 oz of Luxury B.

Each ounce of Regular C processed further required an additional 2 hours of laboratory time and $4 processing cost and yields 1 oz of Luxury C.

Each year, Rylon has 6000 hours of laboratory time available and can purchase up to 4000 lb of raw material.

Formulate an LP that can be used to determine how Rylon can maximize profits. *Assume that the cost of the laboratory hours is a fixed cost*.

1 lb raw material
(cost \$3/lb)

(1 hour lab.)

3 oz Regular B
(selling price \$7/oz)

and

4 oz Regular C
(selling price \$6/oz)

1 oz Regular B
(selling price \$7/oz)

(3 hour lab.)
(\$4 processing cost)

1 oz Luxury B
(selling price \$18/oz)

1 oz Regular C
(selling price \$6/oz)

(2 hour lab.)
(\$4 processing cost)

1 oz Luxury C
(selling price \$14/oz)

Lab. Time: 6000 hours/year

Raw Materials: 4000 lb/year

Example 6-1 solution: Production Process

1. Decision variables

x_1 = number of ounces of Regular B sold annually

x_2 = number of ounces of Luxury B sold annually

x_3 = number of ounces of Regular C sold annually

x_4 = number of ounces of Luxury C sold annually

x_5 = number of pounds of raw materials purchased annually

2. Objective function

$$\begin{aligned}\text{Contribution to profit} &= \text{revenues from perfume sales} \\ &\quad - \text{processing costs} - \text{costs of new materials} \\ &= 7x_1 + 18x_2 + 6x_3 + 14x_4 - 4x_2 - 4x_4 - 3x_5 \\ &= 7x_1 + 14x_2 + 6x_3 + 10x_4 - 3x_5\end{aligned}$$

3. Constraints

Total lab hours used annually = time for processing raw materials

+ time for processing Luxury B + time for processing Luxury C

$$= 1x_5 + 3x_2 + 2x_4$$

$$3x_2 + 2x_4 + x_5 \le 6000$$

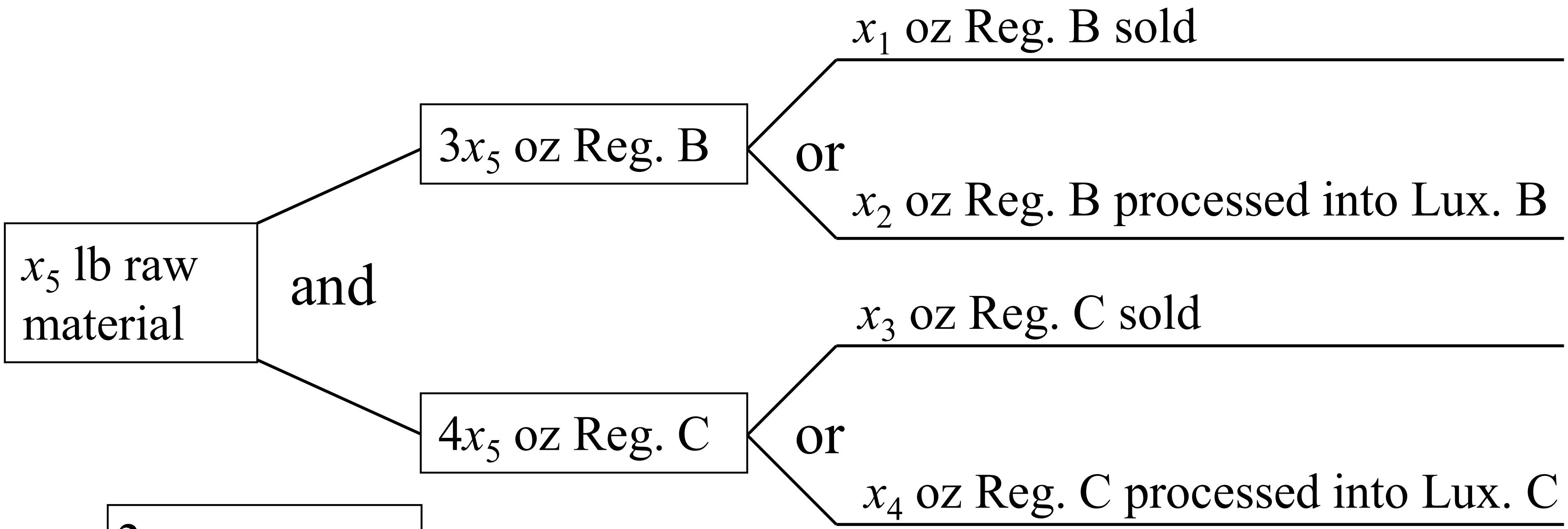
Total raw materials purchased annually = $x_5 \le 4000$

4. Sign restrictions

$x_i \ge 0, (i = 1, \ldots, 5)$

$$\begin{aligned} \text{maximize } z = 7x_1 + 14x_2 + 6x_3 + 10x_4 - 3x_5 & \\ \text{s.t.} \quad 3x_2 + 2x_4 + x_5 &\le 6000 \\ x_5 &\le 4000 \\ x_i &\ge 0, (i = 1, \ldots, 5) \end{aligned}$$

Incorrect!!



$$\boxed{\begin{aligned} 3x_5 &= x_1 + x_2 \\ 4x_5 &= x_3 + x_4 \end{aligned}}$$

$$\begin{array}{ll} \max & z = 7x_1 + 14x_2 + 6x_3 + 10x_4 - 3x_5 \\ \text{s.t.} & \\ & 3x_2 + 2x_4 + x_5 \le 6000 \\ & x_5 \le 4000 \\ & x_1 + x_2 - 3x_5 = 0 \\ & x_3 + x_4 - 4x_5 = 0 \\ & x_i \ge 0 \; (i = 1, \cdots, 5) \end{array}$$

Question: How to modify the formulation if processing 1 oz Reg. Perfume yields 0.5 oz Lux. Perfume?

$$3x_5 = x_1 + 2x_2$$

$$4x_5 = x_3 + 2x_4$$

Example 6-2: Production and Assembly

- A division of a plastics company manufactures three basic products: sporks, packets, and school packs. A spork is a plastic utensil which purports to be a combination spoon, fork, and knife. The packets consist of a spork, a napkin, and a straw wrapped in cellophane. The school packs are boxes of 100 packets with an additional 10 loose sporks included.
- Production of 1000 sporks requires 0.8 standard hours of molding machine capacity, 0.2 standard hours of supervisory time, and $2.50 in direct costs.
- Production of 1000 packets requires 1.5 standard hours of the packaging-area capacity, 0.5 standard hours of supervisory time, and $4.00 in direct costs. *There is an unlimited supply of napkins and straws.*
- Production of 1000 school packs requires 2.5 standard hours of packaging-area capacity, 0.5 standard hours of supervisory time, and $8.00 in direct costs.
- *Any of the three products may be sold in unlimited quantities at prices of $5.00, $15.00, and $300.00 per thousand, respectively*.
- If there are 200 hours of production time in the coming month, what products, and how much of each, should be manufactured to yield the most profit?

Example 6-2 solution: Production and Assembly solution

$x_1 =$ Total number of sporks produced in thousands,
$x_2 =$ Total number of packets produced in thousands,
$x_3 =$ Total number of school packs produced in thousands,
$y_1 =$ Total number of sporks sold as sporks in thousands,
$y_2 =$ Total number of packets sold as packets in thousands,
$y_3 =$ Total number of school packs sold as school packs in thousands.

Max Total profit $= 5y_1 + 15y_2 + 300y_3 - 2.5x_1 - 4x_2 - 8x_3$

s.t., $y_1 = x_1 - x_2 - 10x_3$

$y_2 = x_2 - 100x_3$

$y_3 = x_3$

$0.8x_1 \leq 200$

$1.5x_2 + 2.5x_3 \leq 200$

$0.2x_1 + 0.5x_2 + 0.5x_3 \leq 200$

$x_i \geq 0, y_i \geq 0 \; for \; i = 1,2,3.$

$y_1{}^* = 116.7$

$y_2 = 133.3$

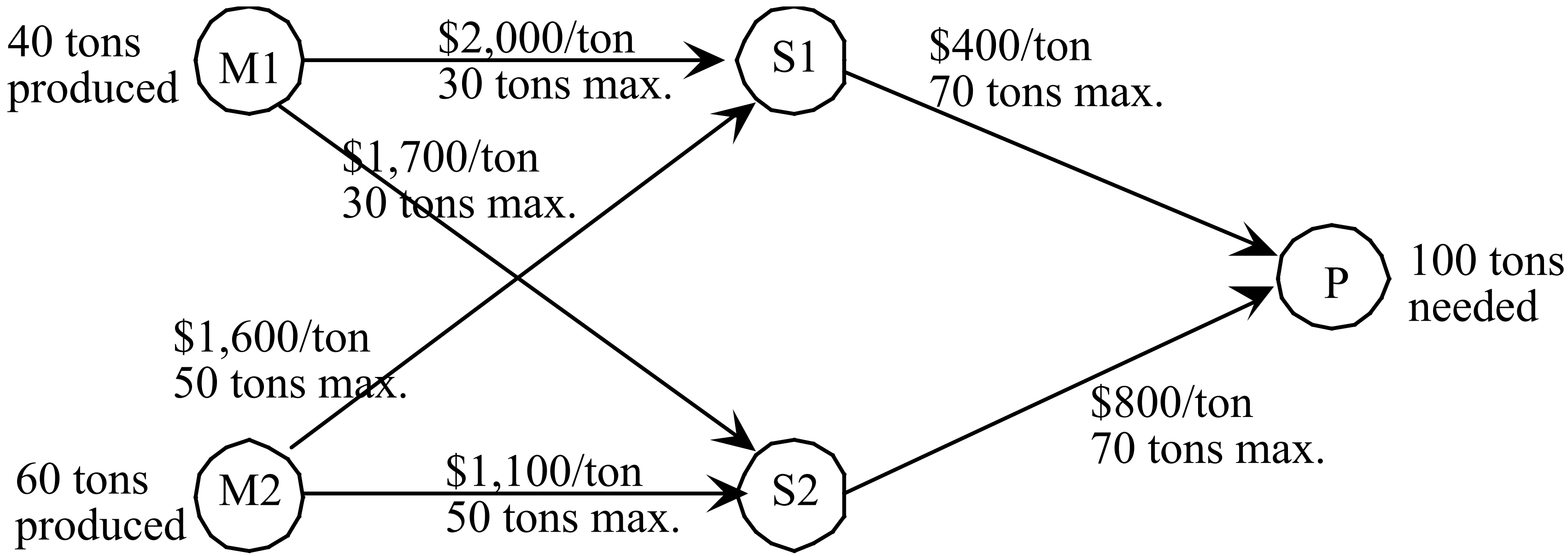
$y_3 = 0$

$\text{Maximal Total profit} = 1425$

Since the computation time required to solve a LP problem increases roughly with the cube of the number of rows of the problem, in this example the y's constitute better decision variables than the x's. The values of the x's are easily determined from the constraints.

Example 7: Supply Chain Management

- The Fagersta Steelworks currently is working two mines to obtain its iron ore. This iron ore is shipped to either of two storage facilities. When needed, it then is shipped on to the company's steel plant. The diagram below depicts this distribution network, where M1 and M2 are the two mines, S1 and S2 are the two storage facilities, and P is the steel plant. The diagram also shows the monthly amounts produced at the mines and needed at the plant, as well as the shipping cost and the maximum amount that can be shipped per month through each shipping lane.
- Management now wants to determine the most economic plan for shipping the iron ore from the mines through the distribution network to the steel plant. **Formulate a linear programming model for this problem.**


40 tons
produced
M1
$2,000/ton
30 tons max.
S1
$400/ton
70 tons max.
$1,700/ton
30 tons max.
P
100 tons
needed
$1,600/ton
50 tons max.
$800/ton
70 tons max.
60 tons
produced
M2
$1,100/ton
50 tons max.
S2

Example 7 solution: Supply Chain Management

The decision variables are defined as follows:

- $x_{m1\text{-}s1}$: number of units (tons) shipped from Mine 1 to Storage Facility 1,
- $x_{m1\text{-}s2}$: number of units (tons) shipped from Mine 1 to Storage Facility 2,
- $x_{m2\text{-}s1}$: number of units (tons) shipped from Mine 2 to Storage Facility 1,
- $x_{m2\text{-}s2}$: number of units (tons) shipped from Mine 1 to Storage Facility 2,
- $x_{s1\text{-}p}$: number of units (tons) shipped from Storage Facility 1 to the Plant,
- $x_{s2\text{-}p}$: number of units (tons) shipped from Storage Facility 2 to the Plant.

The total shipping cost is:

$$Z=2000\, x_{m1\text{-}s1} + 1700\, x_{m1\text{-}s2} + 1600\, x_{m2\text{-}s1} + 1100\, x_{m2\text{-}s2} + 400\, x_{s1\text{-}p} + 800\, x_{s2\text{-}p}$$

The constraints:

- Supply constraint on M1 and M2:
- $x_{m1\text{-}s1} + x_{m1\text{-}s2} = 40$
- $x_{m2\text{-}s1} + x_{m2\text{-}s2} = 60$
- Conservation-of-flow constraint on S1 and S2:
- $x_{m1\text{-}s1} + x_{m2\text{-}s1} - x_{s1\text{-}p} = 0$
- $x_{m1\text{-}s2} + x_{m2\text{-}s2} - x_{s2\text{-}p} = 0$
- Demand constraint on P:
- $x_{s1\text{-}p} + x_{s2\text{-}p} = 100$
- Capacity constraints:

$$x_{m1\text{-}s1} \le 30,\ x_{m1\text{-}s2} \le 30,\ x_{m2\text{-}s1} \le 50,\ x_{m2\text{-}s2} \le 50,\ x_{s1\text{-}p} \le 70,\ x_{s2\text{-}p} \le 70$$

- Nonnegativity constraints:

$$x_{m1\text{-}s1} \ge 0,\ x_{m1\text{-}s2} \ge 0,\ x_{m2\text{-}s1} \ge 0,\ x_{m2\text{-}s2} \ge 0,\ x_{s1\text{-}p} \ge 0,\ x_{s2\text{-}p} \ge 0$$

Example 8: Short-Term Financial Planning

Semicond is a small electronics company that manufactures tape recorders and radios. The per-unit labor costs, raw material costs, and selling price of each product are given. On January 1, Semicond has available raw material that is sufficient to manufacture 100 tape recorders and 100 radios. On the same date, the company's balance sheet is as shown in the table, and Semicond's asset-liability ratio (called the current ratio) is 20,000/10,000 = 2.

Semicond must determine how many tape recorders and radios should be produced during January. Demand is large enough to ensure that all goods produced will be sold. All sales are on credit, however, and payment for goods produced in January will not be received until March 1. During January, Semicond will collect $2,000 in accounts receivable, and Semicond must pay off $1,000 of the outstanding loan and a monthly rent of $1,000. On February 1, Semicond will receive a shipment of raw material worth $2,000, which will be paid for on March 1. Semicond's management has decided that the cash balance on February 1, must be at least $4,000. Semicond's bank requires that the current ratio at the beginning of February be at least 2. To maximize the contribution to profit from January production, (revenues to be received) (variable production costs), what should Semicond produce during January?

	Tape Recorder	Radio
Selling price	$100	$90
Labor cost	$50	$35
Raw material cost	$30	$40

	Assets	Liabilities
cash	$10,000	
Accounts receivable	$3,000	
Inventory outstanding	$7,000	
Bank loan		$10,000

Example 8 solution: Short-Term Financial Planning

February 1 liabilities
= January 1 liabilities – January loan payment + amount due on February 1 inventory shipment
= 10,000 – 1,000 + 2,000 = \$11,000

Thus, the current ratio is

$$\frac{20{,}000 + 20x_1 + 15x_2}{11{,}000} \geq 2$$

which is equivalent to

$$20x_1 + 15x_2 \geq 2000$$

The optimal solution is z=2,500, x_1=50, x_2=100.

Thus, Semicond can maximize the contribution of January's production to profits by manufacturing 50 tape recorders and 100 radios. This will contribute 2,500 to profits.

$$\begin{array}{llrl}
\max & z = & 20x_1 + 15x_2 & \\
s.t., & & x_1 \leq 100 & \text{(tape recorder constraint)} \\
& & x_2 \leq 100 & \text{(radio constraint)} \\
& & 50x_1 + 35x_2 \leq 6{,}000 & \text{(cash position constraint)} \\
& & 20x_1 + 15x_2 \geq 2{,}000 & \text{(current ratio constraint)} \\
& & x_1, x_2 \geq 0 & \text{(sign restrictions)}
\end{array}$$

Example 9: Multiperiod Financial Models

Finco Investment Corporation must determine investment strategy for the firm during the next three years. Currently (time 0), $100,000 is available for investment. Investments A, B, C, D, and E are available. The cash flow associated with investing $1 in each investment is given. For example, $1 invested in investment B requires a $1 cash outflow at time 1 and returns 50¢ at time 2 and $1 at time 3. To ensure that the company's portfolio is diversified, Finco requires that at most $75,000 be placed in any single investment. In addition to investments A–E, Finco can earn interest at 8% per year by keeping uninvested cash in money market funds. Returns from investments may be immediately reinvested. For example, the positive cash flow received from investment C at time 1 may immediately be reinvested in investment B. Finco cannot borrow funds, so the cash available for investment at any time is limited to cash on hand. Formulate an LP that will maximize cash on hand at time 3.

	Cash Flow ($) at Time*			
	0	1	2	3
A	−1	+0.50	+1	0
B	0	−1	+0.50	+1
C	−1	+1.2	0	0
D	−1	0	0	+1.9
E	0	0	−1	+1.5

**Note:* Time 0 = present; time 1 = 1 year from now; time 2 = 2 years from now; time 3 = 3 years from now.

$$0.5A + 1.2C + 1.08S_0 = B + S_1 \tag{65}$$

At time 2, $A + 0.5B + 1.08S_1$ is available for investment, and investments E and S_2 are available. Thus, for $t = 2$, (63) reduces to

$$A + 0.5B + 1.08S_1 = E + S_2 \tag{66}$$

Let's not forget that at most $75,000 can be placed in any of investments A–E. To take care of this, we add the constraints

$$A \le 75{,}000 \tag{67}$$
$$B \le 75{,}000 \tag{68}$$
$$C \le 75{,}000 \tag{69}$$
$$D \le 75{,}000 \tag{70}$$
$$E \le 75{,}000 \tag{71}$$

Combining (62) and (64)–(71) with the sign restrictions (all variables ≥ 0) yields the following LP:

$$\begin{aligned}
\max z &= B + 1.9D + 1.5E + 1.08S_2 \\
\text{s.t.} \quad & A + C + D + S_0 = 100{,}000 \\
& 0.5A + 1.2C + 1.08S_0 = B + S_1 \\
& A + 0.5B + 1.08S_1 = E + S_2 \\
& A \le 75{,}000 \\
& B \le 75{,}000 \\
& C \le 75{,}000 \\
& D \le 75{,}000 \\
& E \le 75{,}000 \\
& A, B, C, D, E, S_0, S_1, S_2 \ge 0
\end{aligned}$$

We find the optimal solution to be $z = 218{,}500$, $A = 60{,}000$, $B = 30{,}000$, $D = 40{,}000$, $E = 75{,}000$, $C = S_0 = S_1 = S_2 = 0$. Thus, Finco should not invest in money market funds. At time 0, Finco should invest $60,000 in A and $40,000 in D. Then, at time 1, the $30,000 cash inflow from A should be invested in B. Finally, at time 2, the $60,000 cash inflow from A and the $15,000 cash inflow from B should be invested in E. At time 3, Finco's $100,000 will have grown to $218,500.

Example 9 solution: Multiperiod Financial Models