Operations Research

Lecture 5: Robust Optimization–Part I

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"To be uncertain is to be uncomfortable, but to be certain is to be ridiculous."

—— Chinese proverb

Outline

1. Data Uncertainty and Robust Linear Optimization (RLO)
2. Uncertainty Sets and Robust Counterparts
3. Drug Production under Uncertainty
4. Product Design under Uncertainty
5. Extensions
6. Conclusion

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Data Uncertainty and RLO

A typical linear optimization problem (LOP)

$$\begin{aligned} \min_{\boldsymbol{x}} \quad & \boldsymbol{c}'\boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} \geq \boldsymbol{b} \end{aligned}$$

- $\boldsymbol{x}$: decision variable
- $(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c})$: data (potentially uncertainty).

Example 1: NETLIB example

Constraint # 372 of the problem PILOT4 from NETLIB:

$$
\begin{aligned}
\boldsymbol{a}'\boldsymbol{x} \quad := \quad & -15.79081x_{826} - 8.598819x_{827} - 1.88789x_{828} - 1.362417x_{829} - 1.526049x_{830} \\
& -0.031883x_{849} - 28.725555x_{850} - 10.792065x_{851} - 0.19004x_{852} - 2.757176x_{853} \\
& -12.290832x_{854} + 717.562256x_{855} - 0.057865x_{856} - 3.785417x_{857} - 78.30661x_{858} \\
& -122.163055x_{859} - 6.46609x_{860} - 0.48371x_{861} - 0.615264x_{862} - 1.353783x_{863} \\
& -84.644257x_{864} - 122.459045x_{865} - 43.15593x_{866} - 1.712592x_{870} - 0.401597x_{871} \\
& -x_{880} - 0.946049x_{898} - 0.946049x_{916} \\
& \geq b =: 23.387405
\end{aligned}
$$

- NETLIB includes about 100 not very large LOPs, mostly of real-world origin, used as the standard benchmark for LOP solvers.
- Most coefficients are ugly real numbers
- Input Uncertainty: Unlikely that real-life parameters are known to high accuracy, which therefore can be treated as "uncertain" inputs.

Data Uncertainty and RLO

Solution $\boldsymbol{x}^*$

$$
\begin{array}{lll}
x^*_{826} = 255.6112787181108 & x^*_{827} = 6240.488912232100 & x^*_{828} = 3624.613324098961 \\
x^*_{829} = 18.20205065283259 & x^*_{849} = 174397.0389573037 & x^*_{870} = 14250.00176680900 \\
x^*_{871} = 25910.00731692178 & x^*_{880} = 104958.3199274139 &
\end{array}
$$

- What would happen if data inaccuracy is only 0.1%?

$$\frac{\left|a_i^{\text{true}} - \hat{a}_i\right|}{|\hat{a}_i|} \le 0.1\%, \ \forall\ i \tag{1}$$

- In the worst case, the constraints can be violated relative to RHS term b by 450%!!:

$$\frac{\underset{\boldsymbol{a}^{\text{true}}}{\text{Min}} \left\{ [\boldsymbol{a}^{\text{true}}]' \boldsymbol{x}^* - b : \boldsymbol{a}^{\text{true}} \text{ satisfies } (1) \right\}}{b} = \frac{-128.2}{23.387405} \approx -4.5$$

Data Uncertainty and RLO

- Similarly, assuming "random uncertainty":

$$\tilde{a}_i^{\text{true}} = (1 + \tilde{\epsilon}_i)a_i, \tilde{\epsilon}_i \sim \text{Uniform}[-0.001, 0.001], \forall\ i$$

and define a relative *Violation Ratio*:

$$\tilde{V} := \text{Max}\left[\frac{b - [\tilde{\boldsymbol{a}}^{\text{true}}]'\boldsymbol{x}^*}{b}, 0\right].$$

We then have

$\text{Prab}\{\tilde{V} > 0\}$	$\text{Prab}\{\tilde{V} > 150\%\}$	$\mathbb{E}[\tilde{V}]$
50%	18%	125%

Data Uncertainty and RLO

Danger!!

- Small perturbation in coefficients can make the constraint severely infeasible with respect to the RHS.
- Perturbation in objective can lead to large deviation.

Question:

- How to handle such **Data Uncertainty ???**.

Data Uncertainty and RLO

Chance Constrained Model (Charnce, Cooper & Symonds 1958)

$$\begin{bmatrix} \underset{\boldsymbol{x}}{\text{Min}} & \boldsymbol{c}'\boldsymbol{x} \\ \text{s.t.} & \tilde{\boldsymbol{A}}\boldsymbol{x} \geq \tilde{\boldsymbol{b}}, \end{bmatrix} \Longrightarrow \begin{bmatrix} \underset{\boldsymbol{x}}{\text{Min}} & \boldsymbol{c}'\boldsymbol{x} \\ \text{s.t.} & \mathbb{P}\left\{\tilde{\boldsymbol{A}}\boldsymbol{x} \geq \tilde{\boldsymbol{b}}\right\} \geq 1-\alpha \end{bmatrix}$$

where $\tilde{\boldsymbol{A}}, \tilde{\boldsymbol{b}}$ are the matrix and vector with random entries.

- The Chance Constrained Model (CCM) is popular in finance and engineering.

Yet some issues on Chance Constrained Model (CCM):

- From modelling point of view: We need to know the distributions of $\tilde{\boldsymbol{A}}, \tilde{\boldsymbol{b}}$, which in practice might not be available.
- From computational point of view: Computing the probability function in general is **NP-hard**.

Example (Difficulty of computing CCM)

Let us look at the following chance constraint:

$$\mathbb{P}\{\tilde{\boldsymbol{a}}'\boldsymbol{x} \geq 1\}$$

where $\boldsymbol{x} \geq 0$, $\tilde{\boldsymbol{a}} = (\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n)'$, and $\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n$ are uniform random variables which are independent and identically distributed (i.i.d.) in [-1,1].

Example (Difficulty of computing CCM, cont'd.)

Note that

$$\mathbb{P}\{\tilde{\boldsymbol{a}}'\boldsymbol{x} \geq 1\} = \mathbb{P}\{\tilde{z} \geq 1\} = \int_1^\infty f_n(z)\mathrm{d}z,$$

where $\tilde{z} = \sum_{i=1}^n \tilde{a}_i x_i$. Note also that (Bradley and Gupta 2004)

$$f_n(z) = \frac{\left[\sum_{\boldsymbol{\epsilon}\in\{-1,1\}^n} \left(z + \sum_{i=1}^n \epsilon_i x_i\right)^{n-1} \operatorname{sign}\left(z + \sum_{i=1}^n \epsilon_i x_i\right) \prod_{i=1}^n \epsilon_i\right]}{(n-1)!2^{n+1} \prod_{i=1}^n \epsilon_i}$$

where the sum is over all 2^n vectors of signs: $(\epsilon_1, \epsilon_2, \ldots, \epsilon_n) \in \{-1, 1\}^n$, and function $\operatorname{sign}(r) = 1$ if $r > 0$, -1 if $r < 0$, and 0 if $r = 0$.

Robust Optimization

$$\left[\begin{array}{ll} \underset{\boldsymbol{x}}{\text{Min}} & \boldsymbol{c}'\boldsymbol{x} \\ \text{s.t.} & \tilde{\boldsymbol{A}}\boldsymbol{x} \geq \tilde{\boldsymbol{b}}, \end{array}\right] \Longrightarrow \left[\begin{array}{ll} \underset{\boldsymbol{x}}{\text{Min}} & \boldsymbol{c}'\boldsymbol{x} \\ \text{s.t.} & \boldsymbol{A}\boldsymbol{x} \geq \boldsymbol{b},\ \forall\,(\boldsymbol{A},\boldsymbol{b}) \in \mathcal{U}, \end{array}\right]$$

which is called *Robust Counterpart* with *Uncertainty Set* $\mathcal{U}$ of the original uncertainty problem.

1. The robust constraint $\tilde{\boldsymbol{A}}\boldsymbol{x} \geq \tilde{\boldsymbol{b}},\ \forall\,(\boldsymbol{A},\boldsymbol{b}) \in \mathcal{U}$ is defined in a constraint-wise manner: each constraint should be satisfied for any $(\boldsymbol{A},\boldsymbol{b}) \in \mathcal{U}$.
2. The robust counterpart in current form is semi-infinite dimensional optimization problem, therefore cannot be handled by efficient linear optimization algorithms.

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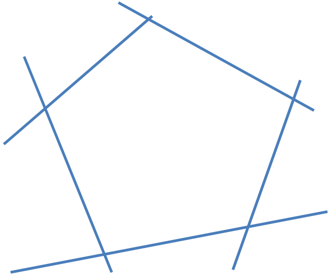
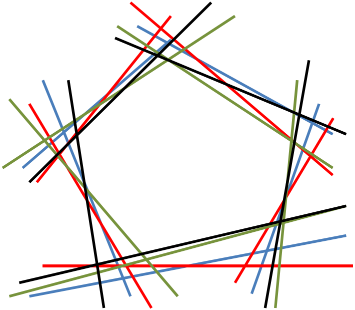
Robust Counterpart

$$
\begin{aligned}
\underset{\boldsymbol{x}}{\text{Min}} \quad & \boldsymbol{c}'\boldsymbol{x} \\
\text{s.t.} \quad & \boldsymbol{a}_i'\boldsymbol{x} \geq b, \ \forall\ \boldsymbol{a}_i \in U_i = \{\boldsymbol{a}_i^1, \boldsymbol{a}_i^2, \boldsymbol{a}_i^3, \boldsymbol{a}_i^4\}, i = 1, 2, 3, 4, 5.
\end{aligned}
$$



(a) Feasible set without uncertainty



(b) Feasible set with uncertainty

We now focus on the following single-constraint robust counterpart without loss of generality (w.l.o.g):

Robust Counterpart

$$\begin{aligned} \min_{\boldsymbol{x}} \quad & \boldsymbol{c}'\boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{a}'\boldsymbol{x} \geq b, \ \forall \ \boldsymbol{a} \in \mathcal{U}. \end{aligned}$$

The interpretation of the robust constraint $\boldsymbol{a}'\boldsymbol{x} \geq b, \ \forall \ \boldsymbol{a} \in \mathcal{U}$:

- The constraint $\boldsymbol{a}'\boldsymbol{x} \geq b$ should be satisfied for any $\boldsymbol{a} \in \mathcal{U}$;
- or, equivalently, the worst-case value of $\boldsymbol{a}'\boldsymbol{x}$ should be no less than b:

$$\min_{\boldsymbol{a} \in \mathcal{U}} \boldsymbol{a}'\boldsymbol{x} \geq b.$$

Interval Uncertainty Set $\mathcal{U}_I$

- $\mathcal{U}_I = [a_1^-, a_1^+] \times [a_2^-, a_2^+] \times \cdots \times [a_n^-, a_n^+]$
- Or equivalently, $a_i \in [a_i^-, a_i^+]$, for any $i = 1, 2, \ldots, n$.
- By representing

$$a_i = \left[\frac{a_i^- + a_i^+}{2}\right] + \delta_i \left[\frac{a_i^+ - a_i^-}{2}\right], \; \delta_i \in [-1, 1], \forall i \in [n],$$

Interval Uncertainty Set $\mathcal{U}_I$

When $a_i \in [a_i^-, a_i^+]$, for any $i = 1, 2, \ldots, n$. By representing

$$a_i = \left[\frac{a_i^- + a_i^+}{2}\right] + \delta_i \left[\frac{a_i^+ - a_i^-}{2}\right], \ \delta_i \in [-1, 1], \forall i.$$

we then have

$$\begin{aligned} & \boldsymbol{a}'\boldsymbol{x} \geq b, \ \forall \ \boldsymbol{a} \in \mathcal{U}_I \\ \Leftrightarrow \quad & \underset{a_i \in [a_i^-, a_i^+], \forall i}{\text{Min}} \left\{ \sum_{i=1}^n a_i x_i \right\} \geq b \\ \Leftrightarrow \quad & \underset{\delta_i \in [-1,1], \forall i}{\text{Min}} \left\{ \sum_{i=1}^n \left[\frac{a_i^- + a_i^+}{2}\right] x_i + \delta_i \left[\frac{a_i^+ - a_i^-}{2}\right] x_i \right\} \geq b \end{aligned}$$

Uncertainty Sets and Robust Counterparts

Interval Uncertainty Set $\mathcal{U}_I$

$$\boldsymbol{a}'\boldsymbol{x} \geq b,\ \forall\ \boldsymbol{a} \in \mathcal{U}_I \quad \Leftrightarrow \quad \sum_{i=1}^{n} \left[\frac{a_i^- + a_i^+}{2}\right] x_i - \left[\frac{a_i^+ - a_i^-}{2}\right] |x_i| \geq b \quad (*)$$

$$\Leftrightarrow \left\{ \begin{array}{l} \sum_{i=1}^{n} \left[\frac{a_i^- + a_i^+}{2}\right] x_i - \left[\frac{a_i^+ - a_i^-}{2}\right] \lambda_i \geq b \\ \lambda_i \geq -x_i, \forall i = 1, 2, \ldots, n \\ \lambda_i \geq x_i, \forall i = 1, 2, \ldots, n \end{array} \right\}$$

- This is a system of linear constraints !!!

Uncertainty Sets and Robust Counterparts

Robust Counterpart with $\mathcal{U}_I$: LP

$$\begin{aligned} \min_{\boldsymbol{x}} \quad & \boldsymbol{c}'\boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{a}'\boldsymbol{x} \geq b, \ \forall\ \boldsymbol{a} \in \mathcal{U}_I. \end{aligned}$$

$$\Updownarrow$$

$$\begin{aligned} \min_{\boldsymbol{x},\boldsymbol{\lambda}} \quad & \boldsymbol{c}'\boldsymbol{x} \\ \text{s.t.} \quad & \sum_{i=1}^{n} \left[\frac{a_i^- + a_i^+}{2}\right] x_i - \left[\frac{a_i^+ - a_i^-}{2}\right] \lambda_i \geq b \\ & \lambda_i \geq -x_i, \forall i = 1, 2, \ldots, n \\ & \lambda_i \geq x_i, \forall i = 1, 2, \ldots, n \end{aligned}$$

- The robust counterpart with $\mathcal{U}_I$ can be transformed into an LP !!!

Uncertainty Sets and Robust Counterparts

Norm-based Uncertainty Sets $\mathcal{U}_{\#}$

- $\mathcal{U}_1^\gamma := \{\boldsymbol{a} : \|\mathbf{M}(\boldsymbol{a} - \boldsymbol{\mu})\|_1 \le \gamma\}$
- $\mathcal{U}_2^\gamma := \{\boldsymbol{a} : \|\mathbf{M}(\boldsymbol{a} - \boldsymbol{\mu})\|_2 \le \gamma\}$
- $\mathcal{U}_\infty^\gamma := \{\boldsymbol{a} : \|\mathbf{M}(\boldsymbol{a} - \boldsymbol{\mu})\|_\infty \le \gamma\}$

The above norm-based uncertainty sets $\mathcal{U}_{\#}^\gamma, \# = 1, 2, \infty$ can be regarded as the variation regions defined by the deviation of $\boldsymbol{a}$ from its mean $\boldsymbol{\mu}$ transformed (twisted) by some nonsingular matrix $\mathbf{M}$ whose inverse is $\mathbf{M}^{-1}$, measured by different distance metrics $\|\cdot\|_1, \|\cdot\|_2$ and $\|\cdot\|_\infty$.

- parameter γ is regarded as the Uncertainty Budget.
- ℓ_1-norm: $\|\boldsymbol{x}\|_1 = \sum_{i=1}^n |x_i|$
- ℓ_2-norm: $\|\boldsymbol{x}\|_2 = \sqrt{\boldsymbol{x}'\boldsymbol{x}} = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$
- ℓ_∞-norm: $\|\boldsymbol{x}\|_\infty = \mathrm{Max}_{i=1}^n |x_i|$

Interval Uncertainty Set $\mathcal{U}_{\#}^{\gamma}$

$$
\begin{aligned}
\boldsymbol{a}'\boldsymbol{x} \geq b,\ \forall\ \boldsymbol{a} \in \mathcal{U}_{\#}^{\gamma} \quad &\Leftrightarrow \quad \operatorname*{Min}_{\boldsymbol{a}:\|\mathbf{M}(\boldsymbol{a}-\boldsymbol{\mu})\|_{\#}\leq\gamma} \boldsymbol{a}'\boldsymbol{x} \geq b \\
&\Leftrightarrow \quad -\operatorname*{Max}_{\boldsymbol{a}:\|\mathbf{M}(\boldsymbol{a}-\boldsymbol{\mu})/\gamma\|_{\#}\leq 1} -\boldsymbol{a}'\boldsymbol{x} \geq b \\
&\Leftrightarrow \quad -\operatorname*{Max}_{\|\boldsymbol{d}\|_{\#}\leq 1} \boldsymbol{x}'\left[-\gamma\mathbf{M}^{-1}\boldsymbol{d} - \boldsymbol{\mu}\right] \geq b \\
&\Leftrightarrow \quad \boldsymbol{x}'\boldsymbol{\mu} - \gamma\left\|(\mathbf{M}^{-1})'\boldsymbol{x}\right\|_{\#}^{*} \geq b \qquad (*)
\end{aligned}
$$

where $\boldsymbol{d} := \mathbf{M}(\boldsymbol{a}-\boldsymbol{\mu})/\gamma$.

Recall that dual norm $\|\boldsymbol{y}\|_{\#}^{*} = \sup_{\|\mathbf{z}\|_{\#}\leq 1} \mathbf{z}'\boldsymbol{y}$.

Robust Counterpart with $\mathcal{U}_{\#}^{\gamma}$

$$
\begin{aligned}
\min_{\boldsymbol{x}} \quad & \boldsymbol{c}'\boldsymbol{x} \\
\text{s.t.} \quad & \boldsymbol{a}'\boldsymbol{x} \geq b, \ \forall\ \boldsymbol{a} \in \mathcal{U}_{\#}^{\gamma}.
\end{aligned}
$$

$$\Updownarrow$$

$$
\begin{aligned}
\min_{\boldsymbol{x}} \quad & \boldsymbol{c}'\boldsymbol{x} \\
\text{s.t.} \quad & \boldsymbol{x}'\boldsymbol{\mu} - \gamma \left\| (\mathbf{M}^{-1})'\boldsymbol{x} \right\|_{\#}^{*} \geq b,
\end{aligned}
$$

where $\mathcal{U}_{\#} = \mathcal{U}_1^{\gamma}, \mathcal{U}_2^{\gamma}$ and $\mathcal{U}_{\infty}^{\gamma}$.

Uncertainty Sets and Robust Counterparts

- Recall that $\|\boldsymbol{y}\|_1^* = \|\boldsymbol{y}\|_\infty = \text{Max}_{i=1}^n |y_i|$

Robust Counterpart with $\mathcal{U}_1^\gamma$

$$\begin{aligned} \underset{\boldsymbol{x}}{\text{Min}} \quad & \boldsymbol{c}'\boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{a}'\boldsymbol{x} \geq b, \ \forall\ \boldsymbol{a} \in \mathcal{U}_1^\gamma. \end{aligned}$$

$$\Updownarrow$$

$$\begin{aligned} \underset{\boldsymbol{x},\lambda}{\text{Min}} \quad & \boldsymbol{c}'\boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{x}'\boldsymbol{\mu} - \gamma\lambda \geq b, \\ & \lambda \geq (\mathbf{M}^{-1})_i'\boldsymbol{x}, \forall i = 1, 2, \ldots, n \\ & \lambda \geq -(\mathbf{M}^{-1})_i'\boldsymbol{x}, \forall i = 1, 2, \ldots, n \end{aligned}$$

where $(\mathbf{M}^{-1})_i$ is the ith row of the matrix $\mathbf{M}^{-1}$.

- The robust counterpart with $\mathcal{U}_1^\gamma$ can be transformed into an LP !

Uncertainty Sets and Robust Counterparts

- Recall that $\|\boldsymbol{y}\|_\infty^* = \|\boldsymbol{y}\|_1 = \sum_{i=1}^n |y_i|$

Robust Counterpart with $\mathcal{U}_\infty^\gamma$

$$\begin{aligned} \min_{\boldsymbol{x}} \quad & \boldsymbol{c}'\boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{a}'\boldsymbol{x} \geq b, \ \forall\ \boldsymbol{a} \in \mathcal{U}_\infty^\gamma. \end{aligned}$$

$$\Updownarrow$$

$$\begin{aligned} \min_{\boldsymbol{x},\boldsymbol{\lambda}} \quad & \boldsymbol{c}'\boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{x}'\boldsymbol{\mu} - \gamma \sum_{i=1}^n \lambda_i \geq b, \\ & \lambda_i \geq (\mathbf{M}^{-1})_i'\boldsymbol{x}, \forall i = 1, 2, \ldots, n \\ & \lambda_i \geq -(\mathbf{M}^{-1})_i'\boldsymbol{x}, \forall i = 1, 2, \ldots, n \end{aligned}$$

where $(\mathbf{M}^{-1})_i$ is the ith row of the matrix $\mathbf{M}^{-1}$.

- The robust counterpart with $\mathcal{U}_\infty^\gamma$ can be transformed into an LP !

- Recall that $\|\boldsymbol{y}\|_2^* = \|\boldsymbol{y}\|_2$

Robust Counterpart with $\mathcal{U}_2^\gamma$

$$\begin{array}{ll} \underset{\boldsymbol{x}}{\text{Min}} & \boldsymbol{c}'\boldsymbol{x} \\ \text{s.t.} & \boldsymbol{a}'\boldsymbol{x} \geq b, \ \forall\ \boldsymbol{a} \in \mathcal{U}_2^\gamma. \end{array}$$

$$\Updownarrow$$

$$\begin{array}{ll} \underset{\boldsymbol{x}}{\text{Min}} & \boldsymbol{c}'\boldsymbol{x} \\ \text{s.t.} & \left\|(\mathbf{M}^{-1})'\boldsymbol{x}\right\|_2 \leq (\boldsymbol{x}'\boldsymbol{\mu} - b)/\gamma. \end{array}$$

- The robust counterpart with $\mathcal{U}_2^\gamma$ can be transformed into an SOCP !

Uncertainty Sets and Robust Counterparts

Insight of Robust Constraints

$$\boldsymbol{a}'\boldsymbol{x} \geq b,\ \forall\ \boldsymbol{a} \in \mathcal{U}_I \Leftrightarrow \underbrace{\sum_{i=1}^{n} \left[\frac{a_i^- + a_i^+}{2}\right] x_i}_{\text{Mean}} \underbrace{- \sum_{i=1}^{n} \left[\frac{a_i^+ - a_i^-}{2}\right] |x_i|}_{\text{Penalty}} \geq b \quad (*)$$

$$\boldsymbol{a}'\boldsymbol{x} \geq b,\ \forall\ \boldsymbol{a} \in \mathcal{U}_{\#}^{\gamma} \Leftrightarrow \underbrace{\boldsymbol{x}'\boldsymbol{\mu}}_{\text{Mean}} \underbrace{-\gamma \left\|(\mathbf{M}^{-1})'\boldsymbol{x}\right\|_{\#}^{*}}_{\text{Penalty}} \geq b \qquad (*)$$

- Pattern of Uncertainty Set: "Mean" + "Deviation"
- Pattern of Robust Counterpart: "Mean" + "Penalty"
- Parameter γ controls the penalty level

Polyhedron-based Uncertainty Sets $\mathcal{U}_{\mathrm{P}}$

- $\mathcal{U}_{\mathrm{P}} := \{\boldsymbol{a} : \boldsymbol{B}\boldsymbol{a} \geq \boldsymbol{r}\}$

- Recalling that

$$\left[\begin{array}{ll} \underset{\boldsymbol{x}}{\text{Min}} & \boldsymbol{c}'\boldsymbol{x} \\ \text{s.t.} & f(\boldsymbol{x},\boldsymbol{y}) \geq \mathbf{b} \\ & \boldsymbol{y} \in D(\boldsymbol{x}). \end{array}\right] = \left[\begin{array}{ll} \underset{\boldsymbol{x}}{\text{Min}} & \boldsymbol{c}'\boldsymbol{x} \\ \text{s.t.} & \underset{\boldsymbol{y}\in D(\boldsymbol{x})}{\text{Max}} f(\boldsymbol{x},\boldsymbol{y}) \geq \mathbf{b} \end{array}\right]$$

Uncertainty Sets and Robust Counterparts

Polyhedron-based Uncertainty Sets $\mathcal{U}_{\mathrm{P}}$

- $\mathcal{U}_{\mathrm{P}} := \{\boldsymbol{a} : \boldsymbol{B}\boldsymbol{a} \geq \boldsymbol{r}\}$

$$
\begin{aligned}
\boldsymbol{a}'\boldsymbol{x} \geq b, \ \forall\, \boldsymbol{a} \in \mathcal{U}_{\mathrm{P}} \quad &\Leftrightarrow \quad \min_{\boldsymbol{a}:\boldsymbol{B}\boldsymbol{a}\geq\boldsymbol{r}} \ \boldsymbol{a}'\boldsymbol{x} \geq b \\
&\Leftrightarrow \quad \max_{\boldsymbol{p}:\boldsymbol{p}'\boldsymbol{B}=\boldsymbol{x}',\boldsymbol{p}\geq\boldsymbol{0}} \ \boldsymbol{p}'\boldsymbol{r} \geq b \\
&\Leftrightarrow \quad \left\{ \boldsymbol{x} \ \middle| \ \begin{array}{l} \boldsymbol{p}'\boldsymbol{r} \geq b \\ \boldsymbol{p}'\boldsymbol{B} = \boldsymbol{x}' \\ \boldsymbol{p} \geq \boldsymbol{0} \end{array} \right\}
\end{aligned}
$$

- Using strong duality of LP.

Uncertainty Sets and Robust Counterparts

Robust Counterpart with $\mathcal{U}_{\mathrm{P}}$

$$
\begin{aligned}
\min_{\boldsymbol{x}} \quad & \boldsymbol{c}'\boldsymbol{x} \\
\text{s.t.} \quad & \boldsymbol{a}'\boldsymbol{x} \geq b, \ \forall \ \boldsymbol{a} \in \mathcal{U}_{\mathrm{P}}.
\end{aligned}
$$

$$\Updownarrow$$

$$
\begin{aligned}
\min_{\boldsymbol{x},\boldsymbol{p}} \quad & \boldsymbol{c}'\boldsymbol{x} \\
\text{s.t.} \quad & \boldsymbol{p}'\boldsymbol{r} \geq b \\
& \boldsymbol{p}'\boldsymbol{B} = \boldsymbol{x}' \\
& \boldsymbol{p} \geq \boldsymbol{0}
\end{aligned}
$$

- The robust counterpart with $\mathcal{U}_P$ can be transformed into an LP !

Example (Another Budgeted Uncertainty Set)

Define

$$\mathcal{U}^{\Gamma} = \left\{ \boldsymbol{a} \in \Re^n : |a_j - \mu_j| \leq \Delta_j, j = 1, 2, \ldots, n, \sum_{j=1}^{n} \frac{|a_j - \mu_j|}{\Delta_j} \leq \Gamma \right\},$$

Where each $\mu_j \in \Re$ and $\Delta_j > 0$ are given inputs. Then,

$$\begin{array}{ll} \underset{\boldsymbol{x}}{\text{Min}} & \boldsymbol{c}'\boldsymbol{x} \\ \text{s.t.} & \boldsymbol{a}'\boldsymbol{x} \geq b, \ \forall \ \boldsymbol{a} \in \mathcal{U}^{\Gamma}. \end{array}$$

$$\Updownarrow$$

H.W.

- The robust counterpart with $\mathcal{U}^{\Gamma}$ can be transformed into an LP !

- Recalling what we have learned in modeling with LPs.

Example

$$\begin{array}{rl} \underset{\boldsymbol{x}}{\text{Min}} & \sum_{i=1}^{N} c_i|x_i| \\ \text{s.t.} & \boldsymbol{A}\boldsymbol{x} \geq \boldsymbol{b} \end{array}$$

where $c_i > 0$.

Example

$$\begin{array}{rl} \underset{\boldsymbol{x}}{\text{Min}} & \boldsymbol{c}'\boldsymbol{x} \\ \text{s.t.} & \sum_{i=1}^{N} |x_i| \leq b \end{array}$$

- $$|x_i| = \text{Max}\{x_i, -x_i\} = \text{Min}\{\lambda_i : \lambda_i \geq x_i; \lambda_i \geq -x_i\}.$$

Uncertainty Sets and Robust Counterparts

Optimality vs. Robustness

$$\begin{aligned} \mathcal{Z}(\gamma) := \underset{\boldsymbol{x}}{\text{Min}} \quad & \boldsymbol{c}'\boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{a}'\boldsymbol{x} \geq b, \ \forall\, \boldsymbol{a} \in \mathcal{U}(\gamma). \end{aligned}$$

where $\mathcal{U}(\gamma_1) \supseteq \mathcal{U}(\gamma_2)$ if $\gamma_1 > \gamma_2$.

- The (nominal) optimal robust objective value $\mathcal{Z}(\gamma)$ in nondecreasing in γ, i.e.

$$\gamma_1 > \gamma_2 \Rightarrow \mathcal{U}(\gamma_1) \supseteq \mathcal{U}(\gamma_2) \Rightarrow \mathcal{Z}(\gamma_1) \geq \mathcal{Z}(\gamma_2).$$

Example (Two-Person Zero-Sum Game)

Randomized Strategy:

- Suppose rowboy picks i with probability y_i.
- Suppose colgirl picks j with probability x_j.
- Now, $\boldsymbol{y} = [y_1, y_2, \dots, y_m]'$ and $\boldsymbol{x} = [x_1, x_2, \dots, x_n]'$ will be rowboy's and colgirl's decisions, respectively.
  $\sum_{i=1}^m y_i = 1, y_i \geq 0; \quad \sum_{j=1}^n x_j = 1, x_j \geq 0.$

If rowboy uses random strategy $\boldsymbol{y}$ and colgirl uses $\boldsymbol{x}$, then expected payoff from rowboy to colgirl is

$$\sum_{i=1}^m \sum_{j=1}^n y_i a_{ij} x_j = \boldsymbol{y}' \boldsymbol{A} \boldsymbol{x}.$$

Example (Two-Person Zero-Sum Game)

- Rowboy's problem (Worst-case loss minimization):

$$\operatorname*{Min}_{\boldsymbol{y}\in\mathcal{Y}} \left[\operatorname*{Max}_{\boldsymbol{x}\in\mathcal{X}} \boldsymbol{y}'\boldsymbol{A}\boldsymbol{x}\right] \Longleftrightarrow$$

- Colgirl's problem (Worst-case return maximization):

$$\operatorname*{Max}_{\boldsymbol{x}\in\mathcal{X}} \left[\operatorname*{Min}_{\boldsymbol{y}\in\mathcal{Y}} \boldsymbol{y}'\boldsymbol{A}\boldsymbol{x}\right] \Longleftrightarrow$$

where

$$\mathcal{Y} := \left\{\boldsymbol{y} : \sum_{i=1}^{m} y_i = 1, y_i \geq 0\right\}; \quad \mathcal{X} := \left\{\boldsymbol{x} : \sum_{j=1}^{n} x_j = 1, x_j \geq 0\right\}.$$

Proposition (Decomposition of support function)

let $S_1, S_2, \ldots, S_n$ be closed convex sets, such that $\cap_i \mathrm{ri}(S_i) \neq \emptyset$, and $S = \cap_i S_i$. Then we have

$$\delta^*(\boldsymbol{x} \mid S) = \mathrm{Min}\left\{ \sum_{i=1}^{n} \delta^*(\boldsymbol{y}_i \mid S_i) \mid \sum_{i=1}^{n} \boldsymbol{y}_i = \boldsymbol{x} \right\}.$$

where $\delta^*(\boldsymbol{x} \mid S)$ is the support function of set S.

H.W.

$$\begin{aligned} \min_{\boldsymbol{x}} \quad & \boldsymbol{c}'\boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{a}'\boldsymbol{x} \leq b, \ \forall\, \boldsymbol{a} \in \mathcal{U}_A \cap \mathcal{U}_B. \end{aligned}$$

where

$$\mathcal{U}_A := \{\boldsymbol{a} \mid \|\boldsymbol{a} - \mu\|_1 \leq \gamma_A\}, \mathcal{U}_B := \{\boldsymbol{a} \mid \|\boldsymbol{a} - \mu\|_2 \leq \gamma_B\}.$$

Outline

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Drug Production under Uncertainty

Drug Production (Ben-Tal 2016)

- A company produces two kinds of drugs, DrugI and DrugII, containing a specific active agent A, which is extracted from raw materials purchased on the market.
- There are two kinds of raw materials, RawI and RawII, which can be used as sources of the active agent.
- The related production, cost and resource data are given in the following table.
- The goal is to find the production plan which maximizes the profit of the company.
- Decision variables: quantities of materials I and II to buy: $\text{Raw}_I, \text{Raw}_{II}$; quantities of two drugs to produce: $\text{Drug}_I, \text{Drug}_{II}$.

Drug Production under Uncertainty

Parameter	DrugI	DrugII
Selling price, \$ per 1000 packs	6,200	6,900
Content of agent A, g per 1000 packs	0.500	0.600
Manpower required, hours per 1000 packs	90.0	100.0
Equipment required, hours per 1000 packs	40.0	50.0
Operational costs, \$ per 1000 packs	700	800

(a) Drug production data

Raw material	Purchasing price, \$ per kg	Content of agent A, g per kg
RawI	100.00	0.01
RawII	199.90	0.02

(b) Contents of raw materials

Budget, \$	Manpower, hours	Equipment hours	Capacity of raw materials storage, kg
100,000	2,000	800	1,000

(c) Resources

Drug Production under Uncertainty

Modelling the constraints:

- Balance of active agent:

$$0.01 \cdot \text{Raw}_I + 0.02 \cdot \text{Raw}_{II} \geq 0.5 \cdot \text{Drug}_I + 0.6 \cdot \text{Drug}_{II}$$

- Storage restriction:

$$\text{Raw}_I + \text{Raw}_{II} \leq 1000$$

- Manpower restriction:

$$90.0 \cdot \text{Drug}_I + 100.0 \cdot \text{Drug}_{II} \leq 2000$$

- Equipment restriction:

$$40.0 \cdot \text{Drug}_I + 50.0 \cdot \text{Drug}_{II} \leq 800$$

- Budget restriction:

$$100.0 \cdot \text{Raw}_I + 199.90 \cdot \text{Raw}_{II} + 700 \cdot \text{Drug}_I + 800 \cdot \text{Drug}_{II} \leq 100000$$

Drug Production under Uncertainty

Nominal Problem: LP Formulation

$$
\begin{aligned}
\text{Max} \quad & (6200\text{Drug}_I + 6900\text{Drug}_{II}) \\
& -(100\text{Raw}_I + 199.90\text{Raw}_{II} + 700\text{Drug}_I + 800\text{Drug}_{II}) \\
\text{s.t.} \quad & 0.01 \cdot \text{Raw}_I + 0.02 \cdot \text{Raw}_{II} \geq 0.5 \cdot \text{Drug}_I + 0.6 \cdot \text{Drug}_{II} \\
& \text{Raw}_I + \text{Raw}_{II} \leq 1000 \\
& 90.0 \cdot \text{Drug}_I + 100.0 \cdot \text{Drug}_{II} \leq 2000 \\
& 40.0 \cdot \text{Drug}_I + 50.0 \cdot \text{Drug}_{II} \leq 800 \\
& 100.0 \cdot \text{Raw}_I + 199.90 \cdot \text{Raw}_{II} + 700 \cdot \text{Drug}_I + 800 \cdot \text{Drug}_{II} \\
& \leq 100000 \\
& \text{Raw}_I, \text{Raw}_{II}, \text{Drug}_I, \text{Drug}_{II} \geq 0
\end{aligned}
$$

- Nominal Solution: RawI = 0, RawII = 438.789, DrugI = 17.552, DrugII = 0
- Nominal Optimal Value: \$ 8819.658

Drug Production under Uncertainty

Uncertainty:

- I reality, the content of Active agent A in Raw Mat. I and Raw Mat. II drift in a 0.5% margin around their nominal values 0.01 and 0.02:

$$a_I \in [0.00995, 0.01005], \quad a_{II} \in [0.0196, 0.0204].$$

- Naturally, we assume

$$\mathbb{P}\{0.00995 \leq a_I < 0.01\} = \frac{1}{2} = \mathbb{P}\{0.0196 \leq a_{II} < 0.02\}$$

- The nominal optimal solution is infeasible with prob. 1/2 (not enough active agents to produce the required amount of DrugI)!!

$$a_I \cdot \underbrace{\mathrm{Raw}_I^*}_{0} + a_{II} \cdot \underbrace{\mathrm{Raw}_{II}^*}_{438.789} \geq 0.5 \cdot \underbrace{\mathrm{Drug}_I^*}_{17.552} + 0.6 \cdot \underbrace{\mathrm{Drug}_{II}^*}_{0}$$

Drug Production under Uncertainty

$$a_I \cdot \underbrace{\text{Raw}_I^*}_{0} + 0.0196 \cdot \underbrace{\text{Raw}_{II}^*}_{438.789} \geq 0.5 \cdot \text{Drug}_I^* + 0.6 \cdot \underbrace{\text{Drug}_{II}^*}_{0}$$

- Assume optimistically that we know when the a_I and a_{II} would fall short, but do not know the exact value.
- Simple Correction Policy: reduce production of DrugI accordingly: New DrugI$=(0.0196 \times 438.789)/0.5 = 17.2$. This gives a new (reduced) return around \$6886.8:
- Under this policy, the new expected return:

$$0.5 \times 8819.658 + 0.5 \times 6886.8 = 7853.229.$$

There is a reduction of 11% in return (compared with the nominal optimal return 8819.658).

Drug Production under Uncertainty

Conclusion 1:

- Even small (and unavoidable) fluctuations of the data my make a nominal solution infeasible.
- Moreover, a trivial adjustment to correct the infeasibility may affect severely the solutions quality.

Drug Production under Uncertainty

- $\mathcal{U}_I := \{(a_I, a_{II}) : a_I \in [0.00995, 0.01005], a_{II} \in [0.0196, 0.0204]\}$.

Robust Design Model

$$
\begin{aligned}
\text{Min} \quad & (100\text{Raw}_I + 199.90\text{Raw}_{II} + 700\text{Drug}_I + 800\text{Drug}_{II}) \\
& -(6200\text{Drug}_I + 6900\text{Drug}_{II}) \\
\text{s.t.} \quad & a_I\text{Raw}_I + a_{II}\text{Raw}_{II} \geq 0.5\text{Drug}_I + 0.6\text{Drug}_{II}, \forall (a_I, a_{II}) \in \mathcal{U}_I \\
& \text{Raw}_I + \text{Raw}_{II} \leq 1000 \\
& 90.0 \cdot \text{Drug}_I + 100.0 \cdot \text{Drug}_{II} \leq 2000 \\
& 40.0 \cdot \text{Drug}_I + 50.0 \cdot \text{Drug}_{II} \leq 800 \\
& 100.0 \cdot \text{Raw}_I + 199.90 \cdot \text{Raw}_{II} + 700 \cdot \text{Drug}_I + 800 \cdot \text{Drug}_{II} \\
& \leq 100000 \\
& \text{Raw}_I, \text{Raw}_{II}, \text{Drug}_I, \text{Drug}_{II} \geq 0
\end{aligned}
$$

Drug Production under Uncertainty

Robust Solution:

- RawI = 877.732, RawII = 0, DrugI = 17.467, DrugII = 0.
- Robust Optimal Value: \$ 8294.567.

Nominal Solution:

- RawI = 0, RawII = 438.789, DrugI = 17.552, DrugII = 0
- Nominal Optimal Value: \$ 8819.658

Drug Production under Uncertainty

Robust Solution:

- RawI = 877.732, RawII = 0, DrugI = 17.467, DrugII = 0.
- Robust Optimal Value: \$ 8294.567.

Nominal Solution:

- RawI = 0, RawII = 438.789, DrugI = 17.552, DrugII = 0
- Nominal Optimal Value: \$ 8819.658

Conclusion 2:

- Robust opt is 5.9% less than nominal opt and is ALWAYS feasible.
- Recall: correcting the nominal sol. to be feasible results in 11% reduction in profit.
- Robust Solution is Nontrivial: Note also the significant change of the production plan.

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Product Design under Uncertainty

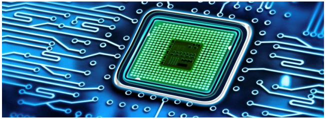
In the design of a new electronic component, the values of the design variables $\boldsymbol{x} = (x_1, x_2, x_3)' \in \Re^3_+$ need to be determined. When implemented, the component needs to meet the following minimum power output level constraint with $\mathbf{g} = (2, 1, 2)'$ as the power generation coefficients:

$$2x_1 + x_2 + 2x_3 \geq 8;$$

and the following total design budget constraint with (1,2,1)' as the cost coefficients:

$$x_1 + 2x_2 + x_3 \leq 20.$$

A feasible design $\boldsymbol{x} \in \Re^3_+$ is one which meets the above constraints when implemented.



Product Design under Uncertainty

Example (Power Generation Uncertainty Problem)

More often than not, the power generation coefficients $\mathbf{g} = (g_1, g_2, g_3)'$ in practice are not deterministic, and we known they could perturb in the following region around the nominal values $(2, 1, 2)'$:

$$\mathbf{g} \in \mathcal{U}_2^{\gamma} := \{\mathbf{g} : \|\mathbf{g} - (2, 1, 2)'\|_2 \leq \gamma\}.$$

Then the question is:

- If we want to hedge against the uncertainty in power generation coefficients, while minimize the design cost, how should we model the problem?

Robust Design Model for Power Generation Uncertainty

$$
\begin{aligned}
\min_{\boldsymbol{x}\in\Re^3_+} \quad & x_1 + 2x_2 + x_3 \\
\text{s.t.} \quad & g_1x_1 + g_2x_2 + g_3x_3 \geq 8,\ \forall\ \mathbf{g} \in \mathcal{U}_2^\gamma
\end{aligned}
$$

where

$$
\mathcal{U}_2^\gamma := \{\mathbf{g} : \|\mathbf{g} - (2,1,2)'\|_2 \leq \gamma\}.
$$

Product Design under Uncertainty

Example (Implementation Error Problem)

A design $\boldsymbol{x}$ when implemented, will inevitably contain implementation errors, so that the actual values $\tilde{\boldsymbol{x}}$ of the design variables are as follows:

$$\tilde{\boldsymbol{x}} = \boldsymbol{x} + \boldsymbol{u}$$

where $\boldsymbol{u} = (u_1, u_2, u_3)'$ is the vector of implementation errors, assumed to arise from the following set

$$\mathcal{U}_2^{\gamma} := \{\boldsymbol{u} : \|\boldsymbol{u}\|_2 \leq \gamma\}.$$

Product Design under Uncertainty

Example (Implementation Error Problem, cont'd)

Question:

- Write down the formulation of the problem of solving for a feasible design $\boldsymbol{x}^*$ that maximizes the value γ.
- Implement the model in Excel and solve it. What is the optimal design x and γ.
- What is the real-life interpretation of maximizing the value of γ

Robust Design Model for Implementation Error

$$
\begin{aligned}
\max_{\boldsymbol{x},\gamma} \quad & \gamma \\
\text{s.t.} \quad & 2(x_1+u_1)+(x_2+u_2)+2(x_3+u_3) \geq 8, \ \forall \ \boldsymbol{u} \in \mathcal{U}_2^\gamma \\
& (x_1+u_1)+2(x_2+u_2)+(x_3+u_3) \leq 20, \ \ \forall \ \boldsymbol{u} \in \mathcal{U}_2^\gamma \\
& x_i+u_i \geq 0, \forall \ \boldsymbol{u} \in \mathcal{U}_2^\gamma, i=1,2,3
\end{aligned}
$$

where

$$\mathcal{U}_2^\gamma = \{\boldsymbol{u} : \|\boldsymbol{u}\|_2 \leq \gamma\}$$

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Robust Counterpart for Nonlinear Constraints

$$
\begin{array}{ll}
\underset{\boldsymbol{x}}{\text{Min}} & \boldsymbol{c}'\boldsymbol{x} \\
\text{s.t.} & f(\boldsymbol{a}, \boldsymbol{x}) \leq b, \ \forall \ \boldsymbol{a} \in \mathcal{U}.
\end{array}
$$

- $f(\boldsymbol{a}, \boldsymbol{x})$ is concave w.r.t. $\boldsymbol{a} \in \Re^m$ for all $\boldsymbol{x} \in \Re^n$
- The uncertainty set

$$
\mathcal{U} := \left\{ \boldsymbol{a} = \bar{\boldsymbol{a}} + \mathbf{A}_{m \times r} \boldsymbol{\xi} \mid \boldsymbol{\xi} \in \mathcal{W} \right\}
$$

where $\mathcal{W} \subseteq \Re^r$ is a nonempty, convex and compact set.

Definition (More Useful Tools)

Let $f : \Re^n \longrightarrow \Re$ be a function with $\text{dom}(f) := \{\boldsymbol{x} \mid f(\boldsymbol{x}) < \infty\}$, and $g : \Re^n \longrightarrow \Re$ be a function with $\text{dom}(g) := \{\boldsymbol{x} \mid g(\boldsymbol{x}) > -\infty\}$.

- Convex conjugate of f:

$$f^*(\boldsymbol{y}) := \sup_{\boldsymbol{x} \in \text{dom}(f)} \{\boldsymbol{y}'\boldsymbol{x} - f(\boldsymbol{x})\}.$$

- Concave conjugate of g:

$$g_*(\boldsymbol{y}) := \inf_{\boldsymbol{x} \in \text{dom}(g)} \{\boldsymbol{y}'\boldsymbol{x} - g(\boldsymbol{x})\}.$$

- $f^*(\boldsymbol{y})$ is always a convex function, while $g_*(\boldsymbol{y})$ is always a concave function.
- $f^{**} = f, g_{**} = g$.

Definition (More Useful Tools)

- Indicator function:

$$\delta(\boldsymbol{x}|S) = \begin{cases} 0, & \boldsymbol{x} \in S \\ \infty, & \text{otherwise} \end{cases}$$

- support function:

$$\delta^*(\boldsymbol{y}|S) := \sup_{\boldsymbol{x} \in S} \boldsymbol{y}'\boldsymbol{x}$$

Let $f : \Re^n \longrightarrow \Re$ be a closed convex function with $\mathrm{dom}(f) := \{\boldsymbol{x} \mid f(\boldsymbol{x}) < \infty\}$, and $g : \Re^n \longrightarrow \Re$ be a closed concave function with $\mathrm{dom}(g) := \{\boldsymbol{x} \mid g(\boldsymbol{x}) > -\infty\}$.

- Primal Problem:

$$\inf\{f(\boldsymbol{x}) - g(\boldsymbol{x}) \mid \boldsymbol{x} \in \mathrm{dom}(f) \cap \mathrm{dom}(g)\} \quad (P)$$

- The Fenchel Dual of (P) is:

$$\sup\{g_*(\boldsymbol{y}) - f^*(\boldsymbol{y}) \mid \boldsymbol{y} \in \mathrm{dom}(g_*) \cap \mathrm{dom}(f^*)\} \quad (D)$$

Theorem (Fenchel Duality)

- *If* $\mathrm{ri}(\mathrm{dom}(f)) \cap \mathrm{ri}(\mathrm{dom}(g)) \neq \emptyset$, *then* Opt.Val. *of (P)*=Opt.Val. *of (D), and the latter is attainable.*
- *If* $\mathrm{ri}(\mathrm{dom}(g_*)) \cap \mathrm{ri}(\mathrm{dom}(f^*)) \neq \emptyset$, *then* Opt.Val. *of (P)*=Opt.Val. *of (D), and the former is attainable.*

Theorem (Fenchel Robust Counterpart, Ben-Tal, et al. 2015)

If $\bar{\boldsymbol{a}} \in \mathrm{ri}(\mathrm{dom} f(\cdot, \boldsymbol{x})), \forall \boldsymbol{x}$, *then the robust constraints*

$$\{\boldsymbol{x} \in \Re^n \mid f(\boldsymbol{a}, \boldsymbol{x}) \leq b, \ \forall \ \boldsymbol{a} \in \mathcal{U}\}$$

is equivalent to

$$\left\{\boldsymbol{x} \in \Re^n \mid \exists \boldsymbol{v} \in \Re^m, \boldsymbol{v}'\bar{\boldsymbol{a}} + \delta^*(\mathbf{A}'\boldsymbol{v} \mid \mathcal{W}) + f_*(\boldsymbol{v}, \boldsymbol{x}) \leq b\right\}$$

- Remark:

$$\begin{aligned} \operatorname*{Max}_{\boldsymbol{a} \in \mathcal{U}} f(\boldsymbol{a}, \boldsymbol{x}) &= \operatorname*{Max}_{\boldsymbol{a} \in \Re^m} \{f(\boldsymbol{a}, \boldsymbol{x}) - \delta(\boldsymbol{a} \mid \mathcal{U})\} \\ &= \operatorname*{Min}_{\boldsymbol{v} \in \Re^m} \{\delta^*(\boldsymbol{v} \mid \mathcal{U}) - f_*(\boldsymbol{v}, \boldsymbol{x})\} \end{aligned}$$

Example (Fenchel Robust Counterpart, Ben-Tal, et al. 2015)

When $f(\boldsymbol{a},\boldsymbol{x}) = \sum_{i\in[n]} f_i(\boldsymbol{a})x_i, x_i > 0$, with $f_i(\boldsymbol{a}) = -(1/2)\boldsymbol{a}'\mathbf{Q}_i\boldsymbol{a}, \mathbf{Q}_i$ is PSD and nonsingular. Then

$$\{\boldsymbol{x} \in \Re^n \mid f(\boldsymbol{a},\boldsymbol{x}) \le b,\ \forall\ \boldsymbol{a} \in \mathcal{U}\}$$

is equivalent to

$$\begin{cases} \boldsymbol{x} \in \Re^n \mid \exists \boldsymbol{v} \in \Re^m, \boldsymbol{v}'\bar{\boldsymbol{a}} + \delta^*(\mathbf{A}'\boldsymbol{v} \mid \mathcal{W}) + \frac{1}{2}\sum_{i\in[n]} \frac{(\boldsymbol{s}^i)'\mathbf{Q}_i^{-1}\boldsymbol{s}^i}{x_i} \le b \\ \sum_{i=1}^n \boldsymbol{s}^i = \boldsymbol{v} \end{cases}$$

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Conclusion

1. Is robust optimization problem convex optimization? YES!
2. Robust optimization is semi-infinite dimensional optimization
3. Robust counterpart with interval uncertainty $\longrightarrow$ LP!
4. Robust counterpart with $\|\cdot\|_1$ and $\|\cdot\|_\infty$ uncertainty $\longrightarrow$ LP!
5. Robust counterpart with polyhedral uncertainty $\longrightarrow$ LP!
6. Robust counterpart with $\|\cdot\|_2$ uncertainty $\longrightarrow$ SOCP!
7. Applications: Robust Solution may not be trivial solution.
8. Duality is the KEY!!!

Reference and Further Reading

1. Arkadi Nemirovski, Lectures on Robust Convex Optimization, Georgia Institute of Technology, 2012 (Available online).
2. David M. Bradley, Ramesh C. Gupta, On the Distribution of the Sum of n Non-Identically Distributed Uniform Random Variables, arXiv:math/0411298, 2004.
3. A. Ben-Tal, D. Hertog, J.-P. Vial, Deriving robust counterparts of nonlinear uncertain inequalities, Optimization-Online, 2015.