OM9103: Stochastic Process

Lecture 5&6:
Renewal Process

A. Introduction

Recall the following characterization of the Poisson process:

- Let X_1, X_2,… be i.i.d. exponentially distributed random variables with parameter λ.
- Define $N(t) = \max\{n : \sum_{i=1}^{n} X_i \le t\}$.

Then $\{N(t), t \ge 0\}$is a Poisson process with rate λ.

We will now study a more general class of *counting processes*. We generalize the Poisson process by allowing the interarrival times X_i to have another distribution than the exponential one. Let X_1, X_2,… be i.i.d. random variables with (nonnegative) distribution F. Define,

$$N(t) = \sup\{n : \sum_{i=1}^{n} X_i \le t\}$$

Then $\{N(t), t \ge 0\}$ is a ***renewal process with interarrival time distribution F***. Clearly, X_i denotes the time between the $(i\text{-}1)^{st}$ and i^{th} event. An event of such a process is also called a *renewal*.

- Poisson process: Restarts probabilistically at any point in time;
- Renewal process: Restarts probabilistically at each renewal.

Denote the *mean* of the distribution F by $\mu = E(X_i) = \int_0^\infty x dF(x).$ We assume that $P(X_i = 0)$ < 1, so that $0 < \mu \le \infty$. Denote the arrival time of the n^{th} event by $S_n = \sum_{i=1}^{n} X_i$.

- Could we have an infinite number of events in finite time, i.e., could we have $N(t) = \infty$ for some finite t?

Note that, by *the strong law of large numbers*:

$$\lim_{n\to\infty} \frac{S_n}{n} = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} X_i = \mu \text{ with probability 1.}$$

But since $\mu > 0$, this means that $\lim_{\{n\to\infty\}} S_n = \infty$ with probability 1. Therefore, $N(t)$ is finite with probability 1:

$$N(t) = \max\{n : \sum_{i=1}^{n} X_i \le t\}$$

B. Distribution of *N*(*t*)

B.1 Arrival Times and Convolution

Clearly, the arrival times S_n are very important in studying renewal processes. Recall that in the Poisson process with rate λ, $S_n \sim \text{gamma}(n, \lambda)$. For a general renewal process, we are interested in the distribution of the sum of n i.i.d. random variables, which is n^{th} *convolution* of the common distribution F.

First consider two independent random variables: $X \sim F$ and $Y \sim G$. The distribution of $Z = X + Y$ is then given by:

$$H(z) = P(Z \leq z) = P(X + Y \leq z) = \int P(X + Y \leq z \mid Y = y)\, dG(y)$$
$$= \int P(X \leq z - y)\, dG(y) = \int F(z - y)\, dG(y)$$

We call the distribution H the ***convolution*** of F and G. This is denoted as: $H = F*G$. Note that similarly:

$$H(z) = \int G(z - x)\, dF(x),$$

which implies that $H = F*G = G*F$.

For the arrival times of a renewal process, we have:

$$S_2 = F*F \equiv F_2, \ldots, S_n = S_{n-1} + X_n = F_{n-1}*F \equiv F_n.$$

We say that S_n is distributed as the ***n-fold convolution*** of F with itself.

B.2 Expected Number of Events

Recall that in the Poisson process with rate λ, we have $E(N(t)) = \lambda t$. How can we characterize the expected number of events up to time t for a general renewal process?

Use the fact that

$$E(N(t)) = \sum_{n=0}^{\infty} P(N(t) > n)$$

Recall from the Poisson process that $N(t) \geq n \Leftrightarrow S_n \leq t$. This result does not depend on the distribution of interarrival times! Therefore,

$$P(N(t) \geq n) = P(S_n \leq t) = F_n(t).$$

This leads to the following expression for $E(N(t))$:

$$E(N(t)) = \sum_{n=0}^{\infty} P(N(t) > n) = \sum_{n=1}^{\infty} P(N(t) \geq n) = \sum_{n=}^{\infty} F_n(t)$$

As in the (nonhomogeneous) Poisson process, we often denote *the mean value function* by m:

$$m(t) = E(N(t)),$$

which is also known as the ***renewal function***. An important part of renewal theory is concerned with studying the properties of this function.

We know that $N(t)$ is finite with probability one. However, is $E(N(t)) < \infty$ as well? Since $\Pr(X_i = 0) < 1$, there exists some $\alpha > 0$ such that $\Pr(X_i \geq \alpha) > 0$ (why?)

Now define a new renewal process with the following interarrival times:

$$\bar{X}_n = \begin{cases} 0 & \text{if } X_n < \alpha \\ \alpha & \text{if } X_n \geq \alpha \end{cases}$$

and

$$\bar{N}(t) = \max\{n : \textstyle\sum_{i=1}^{n} \bar{X}_i \leq t\}$$

Clearly, since $\bar{X}_n \leq X_n$ for all *n*, we have $\bar{N}(t) \geq N(t)$ for all *t*. Hence $E(\bar{N}(t)) \geq E(N(t))$ for all *t*, which implies that if $E(\bar{N}(t)) < \infty$, we must have $E(N(t)) < \infty$.

Notice that in the new process, interarrival times can only be 0 or α. Therefore, event times can only be integer multiples of α. Now note that the number of events at each time $t = k\alpha$ has a geometric distribution with parameter $\Pr(X \geq \alpha)$. This now implies that

$$E(\bar{N}(t)) = \sum_{i=0}^{\lfloor t/\alpha \rfloor} \frac{1}{P(X_n \geq \alpha)} - 1 = \frac{\lfloor t/\alpha \rfloor + 1}{P(X_n \geq \alpha)} - 1 < \infty$$

C. Limiting Behavior

C.1 Limiting Behavior of N(t)

We will first study the limiting behavior of the random variable $N(t)$. Consider the total number of renewals, which can be finite only if one of the interarrival times is infinite. This happens with probability zero, implying that

$$\lim_{t\to\infty} N(t) = \infty \text{ with probability 1.}$$

What is the rate at which $N(t)$ converges to infinity? Intuitively, we may expect $N(t)$ to grow *linearly* in *t*. We will make this more formal by studying

$$\lim_{t\to\infty} \frac{N(t)}{t}.$$

First, note that (i) $S_{N(t)}$ is the time of the last event up to time *t*, and (ii) $S_{N(t)+1}$ is the time of the first event after time *t*. Therefore,

$$S_{N(t)} \leq t < S_{N(t)+1}.$$

This means that

$$\frac{N(t)}{S_{N(t)+1}} < \frac{N(t)}{t} \leq \frac{N(t)}{S_{N(t)}}.$$

Since $\lim_{t\to\infty} N(t) = \infty$ with probability 1, we have, with probability 1,

$$\lim_{t\to\infty} \frac{N(t)}{S_{N(t)}} = \left(\lim_{t\to\infty} \frac{S_{N(t)}}{N(t)}\right)^{-1} = \left(\lim_{\tau\to\infty} \frac{S_\tau}{\tau}\right)^{-1} = \frac{1}{\mu}.$$

Similarly,

$$\lim_{t\to\infty} \frac{N(t)}{S_{N(t)+1}} = \lim_{t\to\infty} \frac{N(t)+1}{S_{N(t)+1}} \cdot \frac{N(t)}{N(t)+1} = \left(\lim_{t\to\infty} \frac{S_{N(t)+1}}{N(t)+1}\right)^{-1} = \left(\lim_{\tau\to\infty} \frac{S_\tau}{\tau}\right)^{-1} = \frac{1}{\mu}.$$

This implies

$$\lim_{t\to\infty} \frac{N(t)}{t} = \frac{1}{\mu} \text{ with probability 1.}$$

So the long-run *rate* at which renewals occur is $1/\mu$. And we say that $1/\mu$ is the *rate of the renewal process*.

Example 3.3(A): (p.103). A container contains an infinite collection of coins. Each coin has a particular probability of landing heads. The *distribution* of this probability over the collection of coins is uniform in (0, 1). We flip coins sequentially. At any time, we may either flip a new coin or a previously used one. Which strategy maximizes the long-run proportion of flips that land on heads? Is there a strategy that makes the long-run proportion of heads equal to one?

Solution: Consider the following strategy:

- Choose a coin
- Flip it until it comes up tails
- Then discard it and select a new coin

Let $N(n)$ denote the number of tails in the first n flips. We are interested in $(1 - \lim_{n\to\infty} \frac{N(n)}{n})$. Notice that that the above strategy makes $\{N(n), n = 0, 1,\ldots\}$a renewal process: each time we obtain tails the process probabilistically restarts. Therefore, we know that

$$\lim_{n\to\infty} \frac{N(n)}{n} = \frac{1}{E(\text{time between events})}$$

The time between events is equal to the number of coin flips between successive tails. This time has a geometric distribution if the probability of tails, say $1-p$, is known. The expected time between events is therefore

$$\int_0^1 \frac{1}{1-p}\, dp = \infty$$

And the proportion of heads is equal to

$$1 - \lim_{n\to\infty} \frac{N(n)}{n} = 1$$

C.2 Limiting Behavior of m(t)

Recall that for a Poisson process we have $m(t) = \lambda t$, which implies that $m(t)/t = \lambda$. For a general renewal process, we may expect that:

$$\lim_{t\to\infty} \frac{N(t)}{t} = \frac{1}{\mu} \text{ with probability 1}$$

implies that

$$\lim_{t\to\infty} \frac{E\left(N(t)\right)}{t} = \lim_{t\to\infty} \frac{m(t)}{t} = \frac{1}{\mu}$$

This result is indeed true, but surprisingly hard to prove. It is called the ***Elementary Renewal Theorem***.

Let us first try the same approach as we used to study the limiting behavior of $N(t)/t$. Clearly,

$$E(S_{N(t)}) \le E(N(t)) < E(S_{N(t)+1})$$

So that

$$\frac{E(N(t))}{E(S_{N(t)+1})} < \frac{E(N(t))}{t} \le \frac{E(N(t))}{E(S_{N(t)})}$$

Now,

$$E(S_{N(t)}) = E\left(\sum_{n=1}^{N(t)} X_n\right) \stackrel{?}{=} \mu E(N(t)) = \mu m(t)$$

$$E(S_{N(t)+1}) = E\left(\sum_{n=1}^{N(t)+1} X_n\right) \stackrel{?}{=} \mu E(N(t)+1) = \mu(m(t)+1)$$

The tricky part here is that the X_i's are not independent of $N(t)$ or $N(t)+1$. So we cannot use the earlier result on the expected value of the sum of a random number of random variables! But let us for the moment assume these results are true. Then we naturally have:

$$\lim_{t\to\infty} \frac{E\left(N(t)\right)}{t} = \frac{1}{\mu}$$

Let's assume that $\{N(t), t \ge 0\}$ is a Poisson process. Then,

$$E(S_{N(t)+1}) = E(t + X_{N(t)+1}) = t + \frac{1}{\lambda} = \frac{1}{\lambda}(\lambda t + 1) = \mu E(N(t)+1)$$

However,

$$E(S_{N(t)}) = t - E(A(t)) < t = \frac{1}{\lambda}(\lambda t) = \mu E(N(t))$$

So the desired result does not even hold for the Poisson process. Therefore, the above "proof" is obviously incorrect; and we must find another way to prove the limiting result of $m(t)$.

Wald's Equation

Let X_1, X_2, … be i.i.d random variables, and N be an integral random variable. If the X_n's are independent of N, we know that

$$E\left(\sum_{n=1}^{N} X_n\right) = E(X_1)E(N)$$

But as shown before the result also holds if

- the X_n's are the interarrival times of a Poisson process (i.e., exponentially distributed), and
- $N = N(t) + 1$

Apparently, independence is sufficient, but not necessary!

Using conditioning on $N = n$ we need independence to show the result. To generalize the result, we need another proof technique. First assume that X_1, X_2,… are nonnegative i.i.d. random variables. Then define the binary random variables:

$$I_n = \begin{cases} 1 & \text{if } n \le N \\ 0 & \text{otherwise} \end{cases}$$

This means that we can write

$$\sum_{n=}^{N} X_n = \sum_{n=}^{\infty} X_n I_n$$

Taking expectations yield

$$E\left(\sum_{n=}^{N} X_n\right) = E\left(\sum_{n=}^{\infty} X_n I_n\right) = \sum_{n=1}^{\infty} E(X_n I_n).$$

Thus, if X_n and I_n are *independent*, we have

$$E\left(\sum_{n=}^{N} X_n\right) = \sum_{n=1}^{\infty} E(X_n)E(I_n) = E(X_1)\sum_{n=1}^{\infty} P(N \geq n) = E(X_1)E(N).$$

Now note that X_n and I_n are independent if the X_n's are independent of N. However, this is sufficient, but not necessary. In particular, suppose that the value of I_n depends on X_1, …, X_{n-1}, but *not* on X_n, X_{n+1}, … In other words, whether or not X_n is counted in the sum should depend only on the values X_1, …, X_{n-1}, but not X_n itself. This implies that the event $\{N = n\}$ should be independent of X_{n+1}, X_{n+2}, …. This leads to the following concept:

Definition (**Stopping Time**): An integer-valued random variable N is called a ***stopping time*** for the sequence $\{X_n: n = 1, 2\ldots\}$ if the event $\{N = n\}$ is independent of X_{n+1}, X_{n+2}, … for all $n = 1, 2, \ldots$

Wald's Equation: If N is a stopping time for the nonnegative and i.i.d. sequence X_1, X_2, …, then

$$E\left(\sum_{n=}^{N} X_n\right) = E(X_1)E(N)$$

Remark: Nonnegativity condition can be relaxed when $E(N) < \infty$ and $E(|X_1|) < \infty$.

In terms of the renewal process, the key is to decide whether (i) $N(t)$ is a stopping time for X_1, X_2,… and (ii) $N(t) + 1$ is a stopping time for X_1, X_2,…. As shown earlier, the former is clearly not true for the Poisson process. This is evident since, for $N = N(t)$, $I_n = 1$ corresponds to the event that $N(t) \geq n$, or equivalently, $\sum_{i=1}^{n} X_i \leq t$, which indicates that I_n is not independent of X_n. Hence $N(t)$ is not a stopping time for X_1, X_2, ….

Now consider $N = N(t) + 1$. Note that $I_n = 1$ corresponds to the event that $N(t) + 1 \geq n$, or equivalently, $\sum_{i=1}^{n-1} X_i \leq t$. Therefore, I_n is independent of X_n and $N(t) + 1$ is a stopping time for X_1, X_2, …. Hence from Wald's equation, we have

$$E(S_{N(t)+1}) = \mu E(N(t)+1) = \mu(m(t)+1)$$

which implies that

$$\lim_{t\to\infty} \frac{m(t)}{t} \geq \frac{1}{\mu}$$

However, we still have that, even for a Poisson process, $E(S_{N(t)}) \neq \mu m(t)$.

For the time being, assume that the interarrival times are bounded from above by, say, $M < \infty$. Then,

$$\mu(m(t)+1) = E(S_{N(t)+1}) \leq t + M \Rightarrow m(t) \leq \frac{t+M}{\mu} - 1$$

This clearly implies that

$$\lim_{t\to\infty}\frac{m(t)}{t}\le\lim_{t\to\infty}\frac{\frac{t+M}{\mu}-1}{t}=\lim_{t\to\infty}\left(\frac{1}{\mu}+\frac{M/\mu-1}{t}\right)=\frac{1}{\mu}$$

Now for any renewal process, define a related renewal process $\{N^M(t), t\ge 0\}$by *truncating* the interarrival time distribution at M. Denote it's mean value function (i.e., the renewal function) by $m^M(t)$, and note that $m(t)\le m^M(t)$. From the fact that

$$\lim_{t\to\infty}\frac{m^M(t)}{t}=\frac{1}{\mu_M},$$

we can then conclude that

$$\lim_{t\to\infty}\frac{m(t)}{t}\le\lim_{t\to\infty}\frac{m^M(t)}{t}=\lim_{M\to\infty}\frac{1}{\mu_M}=\frac{1}{\mu}$$

Summarizing the above discussions, we have the following important result.

Theorem 3.3.4 (Elementary Renewal Theorem)

$$\lim_{t\to\infty}\frac{m(t)}{t}=\frac{1}{\mu}$$

D. The Key Renewal Theorem

The Elementary Renewal Theorem suggests that

$$m(t)\approx\frac{t}{\mu}\text{ for large }t.$$

So we might expect that, for large t, the function $m(t)$ can be approximated by a linear function. However, the elementary renewal theory only says that the ***deviation*** of $m(t)$ from the linear function t/μ increases more slowly than t.

To appreciate the difference between the two statements, note that for any $s>0$, we have $m(t+s)\approx (t+s)/\mu$ for large t. Now $m(t)$ can locally be approximated by a linear function if we can conclude

$$\lim_{t\to\infty}(m(t+s)-m(s))=\frac{s}{\mu}.$$

However, suppose that the interarrival times can only take on integral values. In other words, $N(t)=N\left(\lfloor t\rfloor\right)$ for all t so that

$$m(t)=m\left(\lfloor t\rfloor\right)\text{ for all }t,$$

i.e., m is a step function, and the approximation

$$m(t+s)-m(s)\approx\frac{s}{\mu}.$$

may not be very good, even for large t. For example, suppose that $m(t)=\lfloor t\rfloor$ for all t. Then, even though,

$$\lim_{t\to\infty}\frac{m(t)}{t}=\frac{\lfloor t\rfloor}{t}=1,$$

but $m(t)$ is not approximately linear locally.

For a more specific example, consider the following the renewal process with interarrival time distribution:

$$X_n = \begin{cases} 0 & \text{with probability } 1 - p \\ \alpha & \text{otherwise} \end{cases}$$

Note that $E(X_n) = \alpha p$. For this process, we have

$$m(t) = \frac{\lfloor t/\alpha \rfloor + 1}{p} - 1.$$

Now if $s < \alpha$, we have that

$$m(t + s) - m(s) = 0 \text{ or } 1/p,$$

depending on the value of t. In other words, $m(t + s) - m(s)$ does not converge at all! However, we do have

$$m(t + \alpha) - m(s) = 1/p$$

How can we characterize when the renewal function is eventually locally linear? What can we say when this is not the case?

In general, the problems occur if $m(t)$ is a step function. This happens if the interarrival times are integer multiples of some common number, called the ***period*** of the distribution. More formally, if there exists some $d > 0$ such that

$$\sum_{n=0}^{\infty} P(X_i = nd) = 1$$

we say that the distribution is ***lattice***. Note that this does not necessarily mean that we need to have $P(X_i = nd) > 0$ for all n.

D.1 Blackwell's Theorem

Theorem 3.4.1 (Blackwell's Theorem)

(a) If F is not lattice, then for all $a \geq 0$,

$$m(t + a) - m(t) \to a/\mu \text{ as } t \to \infty.$$

(b) If F is lattice with period d, then

$$m(t + d) - m(t) \to d/\mu \text{ as } t \to \infty.$$

Equivalently,

$$E(\text{number of renewals at } nd) \to d/\mu \text{ as } t \to \infty.$$

Remark:

- Note that the above result is consistent with what we already saw in the example:

$$m(t + \alpha) - m(s) = 1/p = (\alpha / \alpha p).$$

D.2 Key Renewal Theorem

An equivalent theorem for the non-lattice case is ***the Key Renewal Theorem***.

Let h be a function defined on $[0, \infty]$. For any $a > 0$, let $\underline{m}_n(a)$ be supremum and $\overline{m}_n(a)$ the infinum of $h(t)$ over the interval $t \in [(n-1)a, na]$.

Definition: We say the function h is ***directly Riemann integrable*** if

(a) both $\sum_{n=1}^{\infty} \bar{m}_n(a)$ and $\sum_{n=1}^{\infty} \underline{m}_n(a)$ are finite for all $a > 0$;

(b) $\lim_{a\to 0} a\sum_{n=1}^{\infty} \bar{m}_n(a) = \lim_{a\to 0} a\sum_{n=1}^{\infty} \underline{m}_n(a)$.

A sufficient condition for h to be directly Riemann integrable is that:

(i) $h(t) \geq 0$ for all $t \geq 0$;

(ii) $h(t)$ is nonincreasing;

(iii) $\int_0^{\infty} h(t)dt < \infty$.

Theorem 3.4.2 (**Key Renewal Theorem**) If F is not lattice, and the function $h(t)$ is directly Riemann integrable, then

$$\lim_{t\to\infty} \int_0^t h(t-x)dm(x) = \frac{1}{\mu}\int_0^{\infty} h(t)dt$$

where

$$m(t) = \sum_{n=1}^{\infty} F_n(t) \text{ and } \mu = \int_0^{\infty} \bar{F}(t)dt.$$

Remarks:

- **Intuition** behind the Key Renewal Theorem:
  - For large x, the function $m(x)$ is approximately linear with slope $1/\mu$;
  - $dm(x) \approx 1/\mu\, dx$;
  - Now note that large values of x are the most important with respect to h, since h decreases to 0 very rapidly.
- It is important to notice that we are integrating $h(t-x)$, not $h(x)$, with respect to $m(x)$.

Before we see some examples of its use, let's first look at the ***equivalence*** of the Key Renewal Theorem and Blackwell's Theorem.

Key Renewal Theorem $\Rightarrow$ Blackwell's Theorem

We can show that Blackwell's Theorem follows from the Key Renewal Theorem by choosing (for some $s \geq 0$):

$$h(t) = \begin{cases} 1 & 0 \leq t \leq s \\ 0 & t > s \end{cases}$$

Then $h(t)$ is directly Riemann integrable, and

$$\int_0^{\infty} h(t)ds = s$$

Now note that

$$\frac{s}{\mu} = \frac{1}{\mu}\int_0^{\infty} h(t)dt = \lim_{t\to\infty}\int_0^t h(t-x)dm(x) = \lim_{t\to\infty}\int_{t-s}^t dm(x) = \lim_{t\to\infty}(m(t) - m(t-s))$$

Blackwell's Theorem $\Rightarrow$ Key Renewal Theorem

We can show that the Key Renewal Theorem follows from Blackwell's Theorem by approximating h by a step function.

$$\lim_{t\to\infty}\int_0^t h(t-x)dm(x) = \lim_{t\to\infty}\lim_{n\to\infty}\sum_{i=1}^{n}\int_{\frac{(i-1)t}{n}}^{\frac{it}{n}} h\left(\frac{it}{n}\right)dm(x)$$

$$= \lim_{t\to\infty}\lim_{n\to\infty}\sum_{i=1}^{n} h\left(\frac{it}{n}\right)\left(m\left(\frac{it}{n}\right)-m\left(\frac{(i-1)t}{n}\right)\right)$$

$$= \lim_{n\to\infty}\lim_{t\to\infty}\sum_{i=1}^{n} h\left(\frac{it}{n}\right)\left(m\left(\frac{it}{n}\right)-m\left(\frac{(i-1)t}{n}\right)\right)$$

$$= \lim_{n\to\infty}\lim_{t\to\infty}\sum_{i=1}^{n} h\left(\frac{it}{n}\right)\frac{t}{n\mu} = \frac{1}{\mu}\lim_{t\to\infty}\int_0^t h(x)dx = \frac{1}{\mu}\int_0^\infty h(x)dx$$

The Key Renewal Theorem can be used to compute the long-run behavior of random variables $W(t)$ that depend on:

- the renewal process, but
- *only* on a particular renewal cycle, and
- *not* on the time at which this cycle takes place.

For example, $W(t) = Y(t)$ = the remaining duration of the cycle at time t. We can then compute this expectation by conditioning on the time of the previous renewal, $S_{N(t)}$.

Lemma 3.4.3: $P(S_{N(t)} \le s) = \bar{F}(t) + \int_0^s \bar{F}(t-y)dm(y)$ for $t \ge s \ge 0$.

Proof.

$$P(S_{N(t)} \le s) = \sum_{n=0}^{\infty} P(S_n \le s, S_{n+1} > t) = \bar{F}(t) + \sum_{n=1}^{\infty} P(S_n \le s, S_{n+1} > t)$$

$$= \bar{F}(t) + \sum_{n=1}^{\infty}\int_0^\infty P(S_n \le s, S_{n+1} > t \mid S_n = y)dF_n(y)$$

$$= \bar{F}(t) + \sum_{n=1}^{\infty}\int_0^s P(S_{n+1} > t \mid S_n = y)dF_n(y) = \bar{F}(t) + \sum_{n=1}^{\infty}\int_0^s P(X_{n+1} > t-y)dF_n(y)$$

$$= \bar{F}(t) + \sum_{n=1}^{\infty}\int_0^s \bar{F}(t-y)dF_n(y) = \bar{F}(t) + \int_0^s \bar{F}(t-y)d\left(\sum_{n=1}^{\infty} F_n(y)\right)$$

$$= \bar{F}(t) + \int_0^s \bar{F}(t-y)dm(y)$$

Now let $W(t)$ be a random variable depending on the renewal process. Conditioning on $S_{N(t)}$:

$$E(W(t)) = E(W(t) \mid S_{N(t)} = 0)\cdot\bar{F}(t) + \int_0^t E(W(t) \mid S_{N(t)} = y)\bar{F}(t-y)dm(y)$$

If $E(W(t)|S_{N(t)} = y)$ only depends on the amount of time since the last renewal, i.e., we can define a function h as follows:

$$h(t-y) = E(W(t) \mid S_{N(t)} = y)\bar{F}(t-y) \text{ and } h(t) = E(W(t) \mid S_{N(t)} = 0)\bar{F}(t).$$

Then we obtain:

$$E(W(t)) = E(W(t) \mid S_{N(t)} = 0)\cdot \bar{F}(t) + \int_0^t E(W(t) \mid S_{N(t)} = y)\bar{F}(t-y)dm(y)$$

$$= h(t) + \int_0^t h(t-y)dm(y) \to \frac{1}{\mu}\int_0^\infty h(y)dy$$

assuming that *h* satisfies the conditions in the Key Renewal Theorem.

Concluding, we have that if $h(t-y) = E(W(t) \mid S_{N(t)} = y)\bar{F}(t-y)$ is a function that depends only on *t* – *y*, and *h* is directly Riemann integrable, then

$$\lim_{t\to\infty} E(W(t)) = \frac{1}{\mu}\int_0^\infty h(y)dy.$$

D.3 Applications

D.3.1 Another Limiting Property of m(t)

Can we say something about $\lim_{t\to\infty}\left(m(t) - t/\mu\right)$? Write the time of the first renewal after *t* as

$$S_{N(t)+1} = t + Y(t)$$

where *Y*(*t*) is the remaining lifetime of the “unit” in use at time *t*. Hence we have

$$E\left(S_{N(t)+1}\right) = E\left(t + Y(t)\right) \Rightarrow \mu(m(t)+1) = t + E(Y(t)) \Rightarrow m(t) - \frac{t}{\mu} = \frac{E(Y(t))}{\mu} - 1$$

So we wish to say something about the limiting behavior of *E*(*Y*(*t*)). Conditioning on $S_{N(t)}$, we have the following:

$$E(Y(t)) = E(Y(t) \mid S_{N(t)} = 0)\cdot \bar{F}(t) + \int_0^t E(Y(t) \mid S_{N(t)} = y)\bar{F}(t-y)dm(y)$$

Now note that

$$E(Y(t) \mid S_{N(t)} = y) = E(X - (t - y) \mid X < t - y).$$

Then,

$$E(Y(t)) = E(X - t \mid X > t)\cdot \bar{F}(t) + \int_0^t E(X - (t-y) \mid X > t-y)\bar{F}(t-y)dm(y)$$

Define

$$h(t) = E(X - t \mid X > t)\cdot \bar{F}(t) = \int_t^\infty (x-t)dF(x)$$

Since $h'(t) = -\bar{F}(t) \le 0$, it follows that, besides being nonnegative, *h* is also nonincreasing. Moreover,

$$\int_0^\infty h(t)dt = \int_0^\infty\int_t^\infty (x-t)dF(x)dt = \int_0^\infty\int_t^\infty (x-t)dtdF(x) = \int_0^\infty \frac{x^2}{2}dF(x) = E\left(\tfrac{1}{2}X^2\right)$$

If $E(X^2) < \infty$, then *h* is directly Riemann integrable, and we have (by the Key Renewal Theorem)

$$\lim_{t\to\infty} E(Y(t)) = \frac{1}{\mu}\int_0^\infty h(t)dt = \frac{E(X^2)}{2\mu} \Rightarrow \lim_{t\to\infty}\left(m(t) - \frac{t}{\mu}\right) = \frac{E(X^2)}{2\mu^2} - 1$$

D.3.2 Age and Remaining Lifetime Distributions

Consider the renewal process $\{N(t), t \geq 0\}$. Denote the time since the last renewal by $A(t)$. What is the limiting distribution of $A(t)$?

For fixed $x \geq 0$, let $W(t) = 1_{\{A(t+x)>x\}}$, that is, $W(t) = 1$ if and only if the term that is in use at time $t + x$ was already in use at time t. Clearly,

$$E(W(t)) = P(A(t + x) > x)$$

And note that

$$\lim_{t\to\infty} E(W(t)) = \lim_{t\to\infty} P(A(t+x) > x) = \lim_{t\to\infty} A(t) > x)$$

yields the limiting distribution of $A(t)$.

Now let

$$h(t-y) = E(W(t) \mid S_{N(t)} = y)\bar{F}(t-y) = P(A(t+x) > x \mid S_{N(t)} = y)\bar{F}(t-y)$$

$$= P(X > t+x-y \mid X > t-y)\bar{F}(t-y) = \frac{\bar{F}(t+x-y)}{\bar{F}(t-y)}\bar{F}(t-y) = \bar{F}(t+x-y)$$

$$\Rightarrow h(t) = \bar{F}(t+x).$$

Clearly,

- h is nonnegative;
- h is nonincreasing;
- $\int_0^\infty h(t)dt = \int_0^\infty F(t+x)dt = \int_x^\infty \bar{F}(t)dt \leq \int_0^\infty \bar{F}(t)dt = \mu$

So h is directly Riemann integrable if $\mu < \infty$. So, by the Key Renewal Theorem,

$$\lim_{t\to\infty} P(A(t) > x) = \frac{1}{\mu}\int_0^\infty h(t)dt = \frac{1}{\mu}\int_x^\infty \bar{F}(t)dt = \frac{\int_x^\infty \bar{F}(t)dt}{\int_0^\infty \bar{F}(t)dt}$$

Now note that

$$\lim_{t\to\infty} P(Y(t) > x) = \lim_{t\to\infty} P(A(t+x) > x) = \lim_{t\to\infty} P(A(t) > x)$$

so that, asymptotically, the forward process is stochastically equal to the backward process. Recall that

$$\lim_{t\to\infty} P(A(t) \leq x) = \frac{1}{\mu}\int_0^x \bar{F}(t)dt$$

Now note

$$\int_0^x \bar{F}(t)dt = \int_0^x\int_t^\infty dF(y) = \int_0^\infty \int_0^{\min(y,x)} dtdF(y) = \int_0^x \min(y,x)dF(y)$$

where $X \sim F$.

Combining these two results, we obtain

$$\lim_{t\to\infty} P(A(t) \leq x) = \frac{1}{\mu}\int_0^x \bar{F}(t)dt = \frac{E(\min(X,x))}{E(X)}$$

which says that the limiting distribution that the age is at most x is the ratio of

- the expected amount of time in a renewal cycle that the life for an item is at most x;
- the expected length of a renewal cycle.

More formally, let us consider the following alternative analysis of the limiting distribution of $A(t)$. Define a stochastic process $\{B(t): t \geq 0\}$ which

- has the value 1 if the item in use is no older than x,
- has the value 0 otherwise.

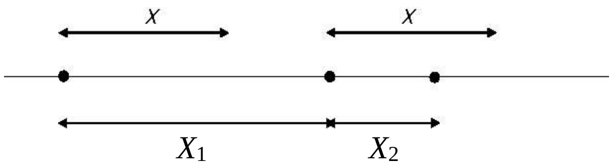
Then

$$\lim_{t\to\infty} P(A(t) \leq x) = \lim_{t\to\infty} P(B(t) = 1)$$

Clearly, we have

$$\lim_{t\to\infty} P(B(t) = 1) = \frac{E(\min(X, x))}{E(X)}$$

Now note that $E(\min(X, x))$ is precisely equal to the expected amount of time that the process $\{B(t): t \geq 0\}$ has the value 1 during a renew cycle:



D.4 Alternating Renewal Process

D.4.1 Basic Limiting Result of Alternating Renewal Process

The above result on the limiting distribution of age can be generalized. Consider a renewal process $\{N(t), t \geq 0\}$. Let $\{B(t), t \geq 0\}$ be a binary process that, during the n^{th} renewal cycle of length X_n such that it takes on the value 1 for Z_n units of time (i.e., the system is "on") and takes on the value 0 for the remaining $Y_n = X_n - Z_n$ units of time (i.e., the system is "off".) .

Put differently, the length X_n of the n^{th} renewal cycle is given by $X_n = Z_n + Y_n$, where the vectors (Z_n, Y_n) are i.i.d., which implies that both the sequence of $\{Z_n\}$ and the sequence of $\{Y_n\}$ are also i.i.d. But we may allow Y_n and Z_n to be dependent.

We call the process $\{B(t), t \geq 0\}$ ***an alternating renewal process***. We say that the system modeled by this process is "on" if $B(t) = 1$; "off" if $B(t) = 0$.

Let $P(t)$ denote the probability that the system is *on* at time t. We can use the Key Renewal Theorem to prove the following result.

Theorem 3.4.4: If $E(X_n) = E(Z_n + Y_n) = \mu < \infty$, then

$$\lim_{t\to\infty} P(t) = \lim_{t\to\infty} P(B(t) = 1) = \frac{E(Z_1)}{E(Y_1) + E(Z_1)}$$

Remark: In the case where we studied the limiting distribution of age, we have

- $Z_n = \min(X_n, x)$ and $Y_n = X_n - Z_n = \max(0, X_n - x)$

Proof. Denote the distribution of Z_n by H. Clearly, since $\mu < \infty$, we have $E(Z_n) < \infty$. Note that $P(B(t) = 1) = E(B(t))$. We can prove the result using the Key Renewal Theorem, by considering the conditional expectation of $B(t)$ given the time of the last arrival:

$$P(B(t)=1 \mid S_{N(t)}=y) = P(Z > t-y \mid Z+Y > t-y)$$
$$= \frac{P(Z > t-y, Z+Y > t-y)}{P(Z+Y > t-y)} = \frac{P(Z > t-y)}{P(X > t-y)} = \frac{\bar{H}(t-y)}{\bar{F}(t-y)}$$

Hence,

$$P(t) = P(B(t)=1 \mid S_{N(t)}=0)\cdot \bar{F}(t) + \int_0^t P(B(t)=1 \mid S_{N(t)}=y)\cdot F(t-y)dm(y)$$
$$= \frac{\bar{H}(t)}{\bar{F}(t)}\cdot \bar{F}(t) + \int_0^t \frac{\bar{H}(t-y)}{\bar{F}(t-y)}\bar{F}(t-y)dm(y) = \bar{H}(t) + \int_0^t \bar{H}(t-y)dm(y)$$

Note that

- $\bar{H}(t)$ is nonnegative and nonincreasing;
- $\int_0^\infty \bar{H}(t)dt = E(Z_1) < \infty$;
- $\lim_{t\to\infty} \bar{H}(t) = 0$.

Then we must have

$$\lim_{t\to\infty} P(t) = \frac{1}{\mu}E(Z_1) = \frac{E(Z_1)}{E(Y_1)+E(Z_1)}$$

D.4.2 Average Behavior

We have analyzed the long-run probability that the system is “on”. Another interesting quantity is the long-run *average* amount of time that the system is “on”, that is, we are interested in

$$\lim_{x\to\infty} \frac{1}{t}\int_0^t B(s)ds$$

We can determine this as follows:

$$\lim_{x\to\infty} \frac{1}{t}\int_0^t B(s)ds = \lim_{t\to\infty} \frac{1}{t}\sum_{n=1}^{N(t)} Z_n = \lim_{x\to\infty}\left(\frac{1}{N(t)}\sum_{n=1}^{N(t)} Z_n\right)\cdot\left(\frac{N(t)}{t}\right)$$
$$= E(Z_1)\frac{1}{\mu} = \frac{E(Z_1)}{E(Z_1)+E(Y_1)} = \lim_{t\to\infty} P(B(t)=1)$$

In other words,

- the long-run probability that the system is “on” is equal to the long-run average amount of time the system is “on”.

However, we should bear in mind that the former only exists if F is nonlattice!

D.4.3 Reward Structure & Renewal Reward Process

Consider a renewal process $\{N(t): t \geq 0\}$. Suppose that you earn a ***reward*** of

- 1 per time unit during the first Z_n periods in a renewal cycle;
- 0 per time unit during the following Y_n periods in a renewal cycle.

where $X_n = Z_n + Y_n$. We can say that the system is "on" while you are earning a reward, and "off" otherwise. With this interpretation,

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t B(s)ds$$

is equal to the long-run average reward earned per unit time.

This result can be generalized. Consider a renewal process $\{N(t): t \geq 0\}$. Let $\{R(t): t \geq 0\}$ be a process that

- keeps track of the total *reward* of some kind gathered so far;
- where the (incremental) rewards gathered depend only on the time since the last reward.

We call the process $\{R(t): t \geq 0\}$ a ***renewal reward process***.

Let R_n denote the reward earned during the nth renewal cycle. Clearly, the random variables R_1, R_2, … are i.i.d., and their distribution depends on the interarrival distribution F. But R_n and X_n may be dependent.

Consider the long-run *average* reward per unit time, i.e., we are interested in

$$\lim_{t\to\infty}\frac{R(t)}{t} \quad \text{or, } \textit{similarly} \quad \lim_{t\to\infty}\frac{1}{t}\sum_{n=1}^{N(t)} R_n$$

Assume that $E(X_1) < \infty$ and $E(R_1) < \infty$. Using the same approach as before,

$$\lim_{t\to\infty}\frac{1}{t}\sum_{n=1}^{N(t)} R_n = \lim_{t\to\infty}\left(\frac{1}{N(t)}\sum_{n=1}^{N(t)} R_n\right)\left(\frac{N(t)}{t}\right) = E(R_1)\frac{1}{\mu} = \frac{E(R_1)}{E(X_1)}$$

Now note that

$$\lim_{t\to\infty}\frac{1}{t}\sum_{n=1}^{N(t)} R_n \leq \lim_{t\to\infty}\frac{R(t)}{t} \leq \lim_{t\to\infty}\frac{1}{t}\sum_{n=1}^{N(t)+1} R_n$$

and

$$\lim_{t\to\infty}\frac{1}{t}\sum_{n=1}^{N(t)+1} R_n = \lim_{t\to\infty}\frac{1}{t}\sum_{n=1}^{N(t)} R_n = \frac{E(R_1)}{E(X_1)}$$

Remark:

- Note that any reward can be either accumulated gradually during each renewal cycle or earned at the end of each renewal cycle. Depending on the problem, either of these possibilities may be the preferred way of modeling the reward structure.

Using the Key Renewal Theorem, we can show the following limiting property of the mean reward function:

$$\lim_{t\to\infty}\frac{E(R(t))}{t} = \frac{E(R_1)}{E(X_1)}$$

assuming that the interarrival distribution F is nonlattice.

D.5 Examples

Example 1 (Reliability): A computer system consists of one operating unit and has a single spare part as protection against failures. If the operating unit fails, it is replaced immediately by the spare part if one is available. And the failed unit is sent to a repair facility, which can only repair one unit at a time. The system is said to be down when both units are broken down. The performance measure that we are interested in is the long-run fraction of time that the system is up.

The time of failure of the operating unit is distributed according to G. The repair time is distributed according to H. The operating and repair times are independent. Let us now define a renewal process. Let a new cycle begin each time when a new operating unit is installed and the other unit enters repair. Let L denote the lifetime of an operating unit and R the repair time of a failed unit.

The length of a cycle is equal to L is $L \geq R$ and equal to R otherwise. This implies that the *interarrival time* X is given by $\max(L, R)$. The *uptime* during a cycle is equal to L and

$$E(L) = \int_0^\infty (1 - G(x))dx$$

The distribution of the cycle length is

$$P(X \leq x) = P(\max(L,R) \leq x) = P(L \leq x, R \leq x) = G(x) \cdot H(x)$$

The expected cycle length is equal to

$$E(\max(L,R)) = \int_0^\infty \left[1 - G(x) \cdot H(x)\right] dx$$

Hence, the long-run fraction of time the system is up is

$$\frac{\int_0^\infty (1 - G(x))dx}{\int_0^\infty \left[1 - G(x) \cdot H(x)\right] dx};$$

and the long-run fraction of time the system is down is

$$1 - \frac{\int_0^\infty (1 - G(x))dx}{\int_0^\infty \left[1 - G(x) \cdot H(x)\right] dx} = \frac{\int_0^\infty G(x) \cdot (1 - H(x))dx}{\int_0^\infty \left[1 - G(x) \cdot H(x)\right] dx}.$$

Example 2 (Repair Problem): Consider an item that deteriorates by incurring an amount of damage each period. The damages incurred in the successive periods are independent exponentially distributed random variables with parameter λ. The successive damages accumulate. At the end of each period, the item is inspected. The item has to be repaired at a high cost of c_0 if the inspection reveals an accumulated damage of more than z_0. A preventative repair at a lower cost of c_1 is possible when the accumulative damage is at most z_0. Assume that a repair takes negligible time, and the returns the item as good as new.

Consider the following *control rule*:

- Perform a repair if the cumulative damage is more than $z \le z_0$.
- Note that the repair is preventative is the cumulative damage exceeds z_0.

How would we determine the value of z that minimizes average cost?

We can model this problem as follows:

- First, consider a renewal process $\{N(t)\text{: } t \ge 0\}$, where a renewal occurs when the item is repaired and returned good as new.
    - Note that the interarrival distribution is lattice!
- Now study the interarrival times using another stochastic process:
    - Let the "time parameter" be the cumulative damage level;
    - An event corresponds to the passage of an inspection period.
    - That is, let $\{D(t)\text{: } t \ge 0\}$ denote the number of periods until cumulative damage of at least t is measured.
- The process renews when the item is repaired and returned good as new.

A repair takes place the first time the measured damages exceeds z, i.e., the number of periods until a repair takes place is $D(z) + 1$. Then the expected length of a cycle is

$$E(D(t) + 1) = m(z) + 1 = \lambda z + 1.$$

The total cost incurred in a cycle is c_0 if the damage at inspection exceeds z_0 and is c_1 otherwise. To compute the total expected costs per cycle, we need to know the amount of damage incurred from the time the limit z is reached until the next inspection. This amount is clearly exponentially distributed, so the expected costs per cycle are

$$\begin{aligned} & c_0 P(Z > z_0 - z) + c_1 P(Z \le z_0 - z) \\ & = c_0 e^{-\lambda(z_0 - z)} + c_1\left(1 - e^{-\lambda(z_0 - z)}\right) = c_1 + (c_0 - c_1)e^{-\lambda(z_0 - z)} \end{aligned}$$

Then the long-run average cost per unit time is equal to

$$\frac{c_1 + (c_0 - c_1)e^{-\lambda(z_0 - z)}}{1 + \lambda z}$$

From this, we can find the value of z that minimizes the above function.

<u>Question</u>: Can you generalize the above procedure to non-exponential damage distributions?

E. Delayed Renewal Process

E.1 Delayed Renewal Process and the Basic Limiting Results

Note that in the above reliability example, we may not necessarily start observing the system at a renewal. In particular, we may start observing the system with a new unit and a new spare part. The time until the *first* renewal is then different from the time between the next renewals.

In general, let the time until the first renewal be distributed according to G, and the time between all subsequent renewals as F. A process in which the time until the first renewal is distributed differently than the time until the next renewal is called a ***delayed renewal process***.

Analogous results as for the ordinary renewal processes can be derived for delayed renewal processes. Denote the delayed renewal process by $\{N_D(t): t \geq 0\}$. Conditioning on the time until the first renewal gives:

$$P(S_n \leq t) = \int_0^\infty P(S_n \leq t \mid S_1 = y)dG(y) = \int_0^\infty F_{n-1}(t-y)dG(y) = G * F_{n-1}(t)$$

Thus,

$$P(N_D(t) \geq n) = P(S_n \leq t) = G * F_{n-1}(t).$$

Therefore,

$$m_D(t) = E(N_D(t)) = \sum_{n=1}^\infty P(N_D(t) \geq n) = \sum_{n=1}^\infty G * F_{n-1}(t)$$

As for the ordinary renewal process,

$$\lim_{t\to\infty} \frac{N_D(t)}{t} = \frac{1}{\mu} \text{ with probability 1.}$$

And the elementary renewal theory generalizes to

$$\lim_{t\to\infty} \frac{m_D(t)}{t} = \frac{1}{\mu}.$$

Blackwell's Theorem and the Key Renewal Theorem generalizes as follows:

- If F is not lattice, then

$$\lim_{t\to\infty}(m_D(t+s) - m_D(t)) = \frac{s}{\mu}$$

- If, in addition, h is directly Riemann integrable, then

$$\lim_{t\to\infty} \int_0^t h(t-x)dm_D(x) = \frac{1}{\mu}\int_0^\infty h(t)dt$$

E.2 Equilibrium Renewal Process

For an ordinary renewal process with F nonlattice, recall that the limiting distribution of remaining lifetime is

$$\lim_{t\to\infty} P(Y(t) \leq x) = \frac{1}{\mu}\int_0^x \bar{F}(t)dt \equiv F_e(x)$$

This distribution is called the ***equilibrium distribution***.

If we start observing the process at some time t very long after the process started, the distribution of the time until the first renewal is approximately F_e (if we have no information about the time of the last renewal). The delayed renewal process with $G = F_e$ is called the ***equilibrium renewal process***.

This process has some interesting properties. First, consider its mean value function, $m_D(t)$. Its Laplace transform is

$$\tilde{m}_D(s) = \int_0^\infty e^{-sx} dm_D(x) = \int_0^\infty e^{-sx} d\left(\sum_{n=1}^\infty G * F(x)\right)$$

Now use the fact the Laplace transform of the sum of random variables is the product of the corresponding Laplace transforms:

$$\tilde{m}_D(s) = \int_0^\infty e^{-sx} d\left(\sum_{n=1}^\infty G * F(x)\right) = \sum_{n=1}^\infty \tilde{G}(s) \cdot \tilde{F}^n(s) = \frac{\tilde{G}(s)}{1-\tilde{F}(s)}$$

Now determine the Laplace transform of $G = F_e$:

$$\tilde{F}_e(s) = \int_0^\infty e^{-sx} dF_e(x) = \int_0^\infty e^{-sx} \frac{\bar{F}(x)}{\mu} dx = \frac{1}{\mu}\int_0^\infty \int_x^\infty e^{-sx} dF(y)dx = \frac{1}{\mu}\int_0^\infty \int_0^y e^{-sx} dx dF(y)$$

$$= \frac{1}{\mu s}\int_0^\infty \left(1-e^{-sy}\right) dF(y) = \frac{1-\tilde{F}(s)}{\mu s}$$

Thus,

$$\tilde{m}_D(s) = \frac{\tilde{F}_e(s)}{1-\tilde{F}_e(s)} = \frac{1}{\mu s}.$$

Now note that

$$\tilde{m}_D(s) = \frac{1}{\mu s} = \int_0^\infty e^{-st} d\left(\frac{t}{\mu}\right)$$

Since a function is uniquely determined by its Laplace transform, we must have,

$$m_D(t) = \frac{t}{\mu}$$

Similarly to the ordinary renewal process, we can express the distribution of the time of the $N_D(t)$-th renewal as

$$P(S_{N_D(t)} \le s) = \bar{G}(t) + \int_0^s \bar{F}(t-y) dm_D(y)$$

We can use this to determine the distribution of the remaining lifetime at time t:

$$P(Y_D(t) > x) = P(Y_D(t) > x \mid S_{N_D(t)} = 0) \cdot \bar{G}(t) + \int_0^t P(Y_D(t) > x \mid S_{N_D(t)} = y) \cdot \bar{F}(t-y) dm_D(y)$$

Furthermore,

$$P(Y_D(t) > x \mid S_{N_D(t)} = 0) = P(X_1 > t + x \mid X_1 > t) = \frac{\bar{G}(t+x)}{\bar{G}(t)}$$

and for $y > 0$,

$$P(Y_D(t) > x \mid S_{N_D(t)} = y) = P(X_2 > t + x - y \mid X_2 > t - y) = \frac{\bar{F}(t+x-y)}{\bar{F}(t-y)}$$

Thus, we get

$$P(Y_D(t) > x) = \bar{F}_e(t+x) + \int_0^t \bar{F}(t+x-s) dm_D(s)$$

$$= \frac{1}{\mu}\int_{t+x}^\infty F(y)dy + \frac{1}{\mu}\int_0^t \bar{F}(t+x-s)ds = \frac{1}{\mu}\left(\int_{t+x}^\infty F(y)dy + \int_x^{t+x} \bar{F}(y)dy\right) = \bar{F}_e(x)$$

Finally, consider the distribution of increments $N_D(t + s) - N_D(s)$. This is the number of renewals in the first t-time units or a delayed renewal process with initial arrival time distribution equal to $Y_D(s)$. Since $Y_D(s) \sim F_e$, it follows that $\{N_D(t), t \geq 0\}$ has ***stationary increments***.

Summarizing the above discussions, we conclude that for an *equilibrium renewal process* $\{N_D(t), t \geq 0\}$, we have

- For all t, $m_D(t) = \dfrac{t}{\mu}$;
- For all t, $P(Y_D(t) \leq x) = F_e(t)$;
- $\{N_D(t), t \geq 0\}$ has stationary increments.

Appendix: Renewal Equations

The textbook has not fully explored the potential applications of the renewal function $m(t)$. One such an area is the powerful "renewal equation". The following results are drawn from the following reference:

Samul Karlin & Howard M Taylor, *A First Course in Stochastic Processes, 2nd Edition*. Academic Press, 1975. (Chapter 5 "Renewal Processes".)

A1. Renewal Function & Renewal Equation

Recall the renewal function is defined as $m(t) = E[N(t)]$.

Lemma: The renewal function satisfies the following equation:

$$m(t) = F(t) + \int_0^t m(t-y)dF(y) \quad t \ge 0.$$

Proof: This identity will be validated by invoking the ***renewal argument***, which proceeds by conditioning on the time X_1 of the first renewal and counting the expected number of renewals thereafter. Note that the probabilistic structure of events begins anew after the moment X_1, and consequently,

$$E[N(t) \mid X_1 = x] = \begin{cases} 0, & \text{if } x > t \\ 1 + m(t-x), & \text{if } x \le t \end{cases}$$

In words, there are no renewals in (0, t] if the first lifetime X_1 exceeds t. On the other hand, where $X_1 = x < t$ there is the renewal engendered at time x plus, on the average, $m(t - x)$ further renewals occurring during the time period extending from x to t.

Therefore, we have the following

$$m(t) = E(N(t)) = \int_0^\infty E(N(t) \mid X_1 = x)dF(x)$$

$$= \int_0^t (1 + m(t-x))dF(x) = F(t) + \int_0^t m(t-x)dF(x),$$

which proves the lemma.

In general, we have the following definition:

Definition (**Renewal Equation**) An integral equation of the form:

$$A(t) = a(t) + \int_0^t A(t-x)dF(x) \quad t \ge 0,$$

is called ***a renewal equation***. The prescribed (or known) functions are $a(t)$ and the distribution function $F(t)$, while the undetermined (or unknown) quantity is $A(t)$.

Remark: For the renewal function, $a(t) = F(t)$.

A.2 Solution of the Renewal Equation

The following theorem affirms that the solution of an arbitrary renewal equation can represented in terms of the renewal function.

<u>Theorem</u>: Suppose $a(t)$ is bounded function. There exists only and only one function $A(t)$ bounded on finite intervals that satisfies

$$A(t) = a(t) + \int_0^t A(t-x)dF(x) \quad t \geq 0. \qquad \text{(A.1)}$$

This function is given by

$$A(t) = a(t) + \int_0^t a(t-x)dm(x). \qquad \text{(A.2)}$$

<u>Proof</u>: We first show that A specified by (A.2) fulfills the requisite boundedness property and indeed solves (A.1).

Since a is a bounded function and m is nondecreasing and finite, for every T, it follows that

$$\sup_{0\leq t\leq T} |A(t)| \leq \sup_{0\leq t\leq T} |a(t)| + \int_0^T \left(\sup_{0\leq t\leq T} |a(t)| \right) dm(t) = \sup_{0\leq t\leq T} |a(t)| \left(1 + m(T)\right) < \infty$$

establishing that the expression (A.2) is bounded on finite intervals. To check $A(t)$ of (A.2) satisfies (A.1), we have

$$A(t) = a(t) + \left(\sum_{k=1}^{\infty} F_k \right) * a(t) = a(t) + F * a(t) + \left(\sum_{k=2}^{\infty} F_k \right) * a(t)$$

$$= a(t) + F * \left(a(t) + \left(\sum_{k=1}^{\infty} F_k \right) * a(t) \right) = a(t) + F * A(t),$$

because of the fact that $F_k = F*F_{k-1}$ for all $k \geq 2$.

To complete the proof, we need show the uniqueness of A. This is done by showing that any solution of the renewal equation (A.1), bounded in finite intervals, is represented by (A.2). Note that (A.1) can be expressed in a brief form as follows:

$$A = a + F * A$$

and substitute for A on the right hand side to get

$$A = a + F * (a + F * A)) = a + F_1 * a + F_2 * A.$$

We iterate this process, leading to

$$A = a + \left(\sum_{k=1}^{n-1} F_k \right) * a + F_n * A$$

Next, note that

$$|F_n * A(t)| = \left| \int_0^t A(t-y)dF_n(y) \right| \leq \left(\sup_{0\leq y\leq t} |A(t-y)| \right) \times F_n(t).$$

Since A is assumed bounded in finite intervals, and $\lim_{t\to\infty} F_n(t) = 0$, it follows that

$$\lim_{t\to\infty} |F_n * A(t)| = 0 \text{ for every fixed } t.$$

Similarly, since a is bounded, we obtain

$$\lim_{n\to\infty}\left(\sum_{k=1}^{n-1} F_k\right) * a(t) = \left(\sum_{k=1}^{\infty} F_k\right) * a(t) = m * a(t).$$

Thus,

$$A(t) = a(t) + \lim_{n\to\infty}\left\{\sum_{k=1}^{n-1} F_k * a(t) + F_n * A(t)\right\} = a(t) + M * a(t),$$

implying that the general solution A of (A.1) acquires the representation (A.2). This proves the uniqueness. □

<u>Example 1</u>: $A(t) = E(S_{N(t)+1})$.

Note that

$$E[S_{N(t)+1} \mid X_1 = x] = \begin{cases} x, & \text{if } x > t \\ x + A(t-x), & \text{if } x \le t \end{cases}$$

This leads to the following renewal equation:

$$A(t) = E\left[S_{N(t)+1}\right] = \int_0^\infty E\left[S_{N(t)+1} \mid X_1 = x\right] dF(x) = E(X_1) + \int_0^t A(t-x)dF(x)$$

with $a(t) = E(X_1)$. Therefore, we obtain

$$A(t) = a(t) + \int_0^t a(t-x)dm(x) = E(X_1) + \int_0^t E(X_1)dm(x) = E(X_1)(1 + m(t)),$$

which has been previously shown by using Wald's equation.

<u>Example 2</u>: $A(t) = E(Y(t))$, where $Y(t) = S_{N(t)+1} - t$ is the remaining duration of the renewal cycle at time t.

Now, based on the discussion in the last example, we have

$$E[Y(t) \mid X_1 = x] = E[S_{N(t)+1} \mid X_1 = x] - t = \begin{cases} x - t, & \text{if } x > t \\ A(t-x), & \text{if } x \le t \end{cases}$$

Then it follows that

$$A(t) = E(Y(t)) = \int_0^\infty E(Y(t) \mid X_1 = x)dF(x)$$

$$= \int_t^\infty (x-t)dF(x) + \int_0^t A(t-x)dF(x) = a(t) + \int_0^t A(t-x)dF(x)$$

where

$$a(t) = \int_t^\infty (x-t)dF(x)$$

Hence,

$$A(t) = E(Y(t)) = a(t) + \int_0^t a(t-y)dm(y)$$

$$= \int_t^\infty (x-t)dF(x) + \int_0^t \left(\int_{t-y}^\infty (x-(t-y))dF(x)\right)dm(y)$$

We can use the Key Renewal Theorem to examine the limiting property of $A(t)$.