Operations Research

Lecture 6: Chance-Constrained Optimization

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What is your choice?

Q1: On study: which one between A and B is your choice:

- A: Try my best to be No.1, or to be as close to No.1 as possible.
- B: You will be satisfied once you can reach some good level, say top 1/4.

What is your choice?

Q2: On money making: which one between A and B is your choice?

- A: Making as much money as possible.
- B: You actually has a target, say 10 M, and you won't pursue more (endlessly) once it has been achieved.

What is your choice?

Q3: On marriage: which one between A and B can reflect your preference:

- A: My husband should be as rich (handsome, successful) as possible/My wife should be as pretty as possible.
- B: You will be satisfied (then married) when you meet some one who meets your requirement.

Outline

1. Chance-Constrained Program (CCP)
2. CCP vs. Value-at-Risk (V@R)
3. CCP vs. Conditional Value-at-Risk (V@R)
4. Computation of CCP
5. Conclusion

Outline

Chance-Constrained Program (CCP)

A typical linear optimization problem (LOP)

$$\begin{aligned} \min_{\boldsymbol{x}} \quad & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} \geq \boldsymbol{b} \end{aligned}$$

- $\boldsymbol{x}$: decision variable
- $(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c})$: data (potentially uncertainty).

Chance-Constrained Program (CCP)

Modelling constraints: When cost coefficients $\boldsymbol{c}$ are fixed, and we want to protect the constraint $\tilde{\boldsymbol{A}}\boldsymbol{x} \geq \tilde{\boldsymbol{b}}$:

Chance Constrained Optimization Model (Charnce, Cooper & Symonds 1958)

$$\begin{aligned} \min_{\boldsymbol{x}} \quad & \boldsymbol{c}^\top \boldsymbol{x} && (1) \\ \text{s.t.} \quad & \mathbb{P}\left\{\tilde{\boldsymbol{A}}\boldsymbol{x} \geq \tilde{\boldsymbol{b}}\right\} \geq 1-\alpha && (2) \end{aligned}$$

where $\tilde{\boldsymbol{A}}, \tilde{\boldsymbol{b}}$ are the matrix and vector with random entries, and $1-\alpha$ is a given probability level.

- We need to know the distributions of $\tilde{\boldsymbol{A}}, \tilde{\boldsymbol{b}}$.
- Computing the probability function in general is **NP-hard**.

Chance-Constrained Program (CCP)

Equivalent form: joint chance constraint

$$\min_{\boldsymbol{x}} \quad \boldsymbol{c}^\top \boldsymbol{x} \tag{3}$$

$$\text{s.t.} \quad \mathbb{P}\left\{\tilde{\boldsymbol{a}}_i^\top \boldsymbol{x} \geq \tilde{\boldsymbol{b}}_i, i = 1, 2, \cdots, m\right\} \geq 1 - \alpha. \tag{4}$$

- Assume matrix $\boldsymbol{A} \in \boldsymbol{R}^{m \times n}$ and $\boldsymbol{a}_i$ is the ith row
- (1)-(2) $\Leftrightarrow$ (3)-(4)
- Joint chance constrained problem is generally much more difficult to deal with.

Chance-Constrained Program (CCP)

Equivalent form: joint chance constraint

$$\underset{\boldsymbol{x}}{\text{Min}} \quad \boldsymbol{c}^\top \boldsymbol{x} \tag{5}$$

$$\text{s.t.} \quad \mathbb{P}\left\{ \underset{i=1}{\overset{m}{\text{Min}}} \{\tilde{\boldsymbol{a}}_i^\top \boldsymbol{x} - \tilde{\boldsymbol{b}}_i\} \geq 0 \right\} \geq 1 - \alpha. \tag{6}$$

- (3)-(4) $\Leftrightarrow$ (5)-(6) .

Chance-Constrained Program (CCP)

Example

- We consider n investable assets with random return (ratios) $\tilde{R}_1, \cdots, \tilde{R}_n$ in the next year, where

$$\tilde{R}_i \longleftarrow \hat{R}_i^k := \frac{P_i^k(t+T) - P_i^k(t)}{P_i^k(t)}, k = 1, 2, \cdots, K.(\text{WHY?})$$

  where $P_i^k(t)$ is the kth sample price of asset i at time t, and $T = 1$ Year.

- We have certain amount of capital and our aim is to maximize the expected return conditional that the chance of losing no more than a given fraction 10% of the capital is 95%.

- Let $x_1, \cdots, \boldsymbol{x}_n$ be the fractions of our capital invested in the n assets. Next year, the total investment return of our portfolio becomes

$$\sum_{i=1}^{n} \tilde{R}_i x_i.$$

Chance-Constrained Program (CCP)

Example (Cont'd.)

This portfolio problem can be formulated into the following stochastic optimization problem with a probabilistic constraint:

$$
\begin{array}{ll}
\underset{\boldsymbol{x}}{\text{Max}} & \displaystyle\sum_{i=1}^{n} \mathbb{E}[\tilde{R}_i] x_i \\
\text{s.t.} & \mathbb{P}\left\{ \displaystyle\sum_{i=1}^{n} \tilde{R}_i x_i \geq 1 - 0.1 \right\} \geq 0.95 \\
& \displaystyle\sum_{i=1}^{n} x_i = 1, \boldsymbol{x} \geq \mathbf{0}.
\end{array}
$$

Chance-Constrained Program (CCP)

Example (Normality and Single Constraint)

When the normality is assumed, e.g., $\tilde{\boldsymbol{a}} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, and consider the following single constraint:

$$\mathbb{P}\{\tilde{\boldsymbol{a}}^\top \boldsymbol{x} \geq b\} \geq \beta \Leftrightarrow \boldsymbol{\mu}^\top \boldsymbol{x} - \Phi^{-1}(\beta)\sqrt{\boldsymbol{x}^\top \boldsymbol{\Sigma} \boldsymbol{x}} \geq b,$$

where $\beta > 0.5$.

- This is a tractable case (SOCP).

Chance-Constrained Program (CCP)

CCP & RO

$$\boldsymbol{a}^\top \boldsymbol{x} \geq b,\ \forall\, \boldsymbol{a} \in \mathcal{U}_I \quad \Leftrightarrow \quad \underbrace{\boldsymbol{x}^\top \boldsymbol{\mu}}_{\text{Mean}} - \underbrace{\sum_{i=1}^n d_i |x_i|}_{\text{Penalty}} \geq b$$

$$\boldsymbol{a}^\top \boldsymbol{x} \geq b,\ \forall\, \boldsymbol{a} \in \mathcal{U}_\sharp^\gamma \quad \Leftrightarrow \quad \underbrace{\boldsymbol{x}^\top \boldsymbol{\mu}}_{\text{Mean}} - \underbrace{\gamma \left\| (\mathbf{M}^{-1})^\top \boldsymbol{x} \right\|_\sharp^*}_{\text{Penalty}} \geq b$$

$$\mathbb{P}\{\tilde{\boldsymbol{a}}^\top \boldsymbol{x} \geq b\} \geq \beta \quad \Leftrightarrow \quad \underbrace{\boldsymbol{\mu}^\top \boldsymbol{x}}_{\text{Mean}} - \underbrace{\Phi^{-1}(\beta)\sqrt{\boldsymbol{x}^\top \boldsymbol{\Sigma} \boldsymbol{x}}}_{\text{Penalty}} \geq b \quad \text{(Normality)}$$

- $\mathcal{U}_I = [a_1^-, a_1^+] \times [a_2^-, a_2^+] \times \cdots \times [a_n^-, a_n^+]$, $\mu_i := \frac{a_i^- + a_i^+}{2}$, $d_i := \frac{a_i^+ - a_i^-}{2}$
- $\mathcal{U}_\sharp^\gamma := \{\boldsymbol{a} : \|\mathbf{M}(\boldsymbol{a} - \boldsymbol{\mu})\|_\sharp \leq \gamma\}$
- Parameters γ and β controls the penalty level in RO and CCP, respectively.

When constraint coefficients $\boldsymbol{A}, \boldsymbol{b}$ are fixed, and we have random cost coefficients $\tilde{\boldsymbol{c}}$:

Probabilistic Optimization Model

$$\underset{\boldsymbol{x}}{\text{Max}} \quad \mathbb{P}\left\{\tilde{\boldsymbol{c}}^\top \boldsymbol{x} \leq \tau\right\} \tag{7}$$

$$\text{s.t.} \quad \boldsymbol{A}\boldsymbol{x} \geq \boldsymbol{b} \tag{8}$$

where τ is a given cost target level.

- Also called aspirational optimization or Target-based Optimization.
- Probabilistic Optimization Model can also be represented in the form of CCP.

Chance-Constrained Program (CCP)

- Probabilistic Optimization Model can also be represented in the form of CCP:

$$\begin{aligned} \underset{\boldsymbol{x},\lambda}{\text{Max}} \quad & \lambda \\ \text{s.t.} \quad & \mathbb{P}\left\{\tilde{\boldsymbol{c}}^\top \boldsymbol{x} \leq \tau\right\} \geq \lambda \\ & \boldsymbol{A}\boldsymbol{x} \geq \boldsymbol{b} \end{aligned}$$

  with λ being the auxiliary decision variable.

- Target τ could be adjusted.
- Can we also optimize τ? $\Longrightarrow$ V@R

Outline

CCP vs. Value-at-Risk (V@R)

Definition (Value-at-Risk (V@R))

Let $\tilde{L}$ be the loss variable, then the Value-at-Risk, or shorthanded as V@R, at confidence level of $1-\alpha$ is defined as:

$$\mathrm{V@R}_{1-\alpha}(\tilde{L}) := \inf\left\{r \mid \mathbb{P}\left\{\tilde{L} \leq r\right\} \geq 1-\alpha\right\}.$$

- $\mathrm{V@R}_{1-\alpha}(\tilde{L})$ measures the worst-case loss that can be expected with some small probability α according to the distribution of $\tilde{L}$.
- It is the $(1-\alpha)$-quantile of $\tilde{L}$.
- Risk management instrument: used to measure the Risk Capital in finance

CCP vs. Value-at-Risk (V@R)

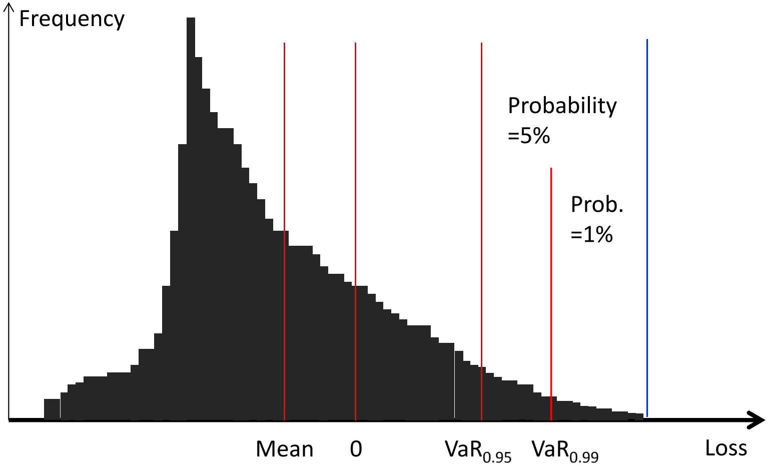


Figure 1: 95% and 99% Values-at-Risk

CCP vs. Value-at-Risk (V@R)

The insights of V@R:

- Optimization-based formulation:

$$\begin{aligned} \mathrm{V@R}_{1-\alpha}(\tilde{L}) := \underset{r}{\mathrm{Min}} \quad & r \\ \text{s.t.} \quad & \mathbb{P}\left\{\tilde{L} \leq r\right\} \geq 1-\alpha. \end{aligned}$$

- Evaluation of V@R essentially solves a CCP.

CCP vs. Value-at-Risk (V@R)

Observation (Cash Invariance)

Given any constant $r \in \Re$, *we have*

$$\text{V@R}_{1-\alpha}(\tilde{L} - r) = \text{V@R}_{1-\alpha}(\tilde{L}) - r, \ \forall \ \alpha \in [0,1].$$

CCP vs. Value-at-Risk (V@R)

Observation (Cash Invariance)

Given any constant $r \in \Re$, *we have*

$$\text{V@R}_{1-\alpha}(\tilde{L} - r) = \text{V@R}_{1-\alpha}(\tilde{L}) - r, \ \forall \ \alpha \in [0, 1].$$

Risk Measure: $\rho(\cdot) : \mathcal{R} \longmapsto \Re^{+}$, $\mathcal{R}$ is a collection of uncertain rewards.

1. **Monotonicity:** $\rho(\xi) \leq \rho(\eta)$, for any $\xi, \eta \in \mathcal{R}$ and $\xi \geq \eta$.
2. **Cash invariance:** For all $\xi \in \mathcal{R}$ and $r \in \Re$, $\rho(\xi + r) = \rho(\xi) - r$.

- V@R is a risk measure.

CCP vs. Value-at-Risk (V@R)

Example (V@R for a discrete return distribution)

We consider the following distribution of profit for asset A:

$$\mathbb{P}\{\tilde{P}_A = -\$500\} = 0.02, \mathbb{P}\{\tilde{P}_A = -\$200\} = 0.1,$$

$$\mathbb{P}\{\tilde{P}_A = \$300\} = 0.5, \mathbb{P}\{\tilde{P}_A = \$600\} = 0.38.$$

- $\text{V@R}_{99\%} = ?$
- $\text{V@R}_{95\%} = ?$
- $\text{V@R}_{90\%} = ?$

CCP vs. Value-at-Risk (V@R)

Example (V@R for a single normal return)

Now we assume the return (ratio) of asset B follows a normal distribution:

$$\tilde{R}_B \sim \mathcal{N}(\mu, \sigma) = N(0.2, 0.01).$$

- $\text{V@R}_{95\%} = ?$

1. $\tilde{L}_B \sim N(-0.2, 0.01)$ which is a continuous distribution.
2. $$\mathbb{P}\{\tilde{L}_B \leq r\} = \Phi\left(\frac{r+0.2}{0.01}\right) \geq 0.95 \Rightarrow r \geq 0.01\Phi^{-1}(0.95) - 0.2$$
3. $\text{V@R}_{95\%} = -\mu + \sigma\Phi^{-1}(0.95) = 0.01\Phi^{-1}(0.95) - 0.2$

CCP vs. Value-at-Risk (V@R)

Example (V@R for a single normal return. H.W.)

- Consider the returns $\tilde{R}_i, i = 1, 2, \cdots, n$ of n assets. Assume each $\tilde{R}_i$ follows a normal distribution, and we have $\mathbb{E}[\tilde{R}_i] = \mu_i$, and the covariance matrix of $(\tilde{R}_1, \tilde{R}_2, \cdots, \tilde{R}_n)$ is Σ.
- Construct a portfolio:

$$\sum_{i=1}^{n} x_i \tilde{R}_i.$$

- Derive the $\text{V@R}_{95\%}$ of the portfolio.

CCP vs. Value-at-Risk (V@R)

Observation (V@R vs. Chance Constraint)

Given any constant $t^o \in \Re$, *we have*

$$\mathbb{P}\left\{\tilde{L} \le t^o\right\} \ge 1 - \alpha \Longleftrightarrow \mathrm{V@R}_{1-\alpha}(\tilde{L}) \le t^o, \ \forall\ \alpha \in [0, 1].$$

CCP vs. Value-at-Risk (V@R)

Observation (V@R vs. Chance Constraint)

Given any constant $t^o \in \Re$, we have

$$\mathbb{P}\left\{\tilde{L} \leq t^o\right\} \geq 1-\alpha \Longleftrightarrow \mathrm{V@R}_{1-\alpha}(\tilde{L}) \leq t^o,\ \forall\ \alpha \in [0,1].$$

Proof.

- $\Longrightarrow$: Trivial.
- $\Longleftarrow$:

$$\mathbb{P}\left\{\tilde{L} \leq \mathrm{V@R}_{1-\alpha}(\tilde{L}) + \triangle\right\} \geq 1-\alpha, \forall \triangle > 0$$

CCP vs. Value-at-Risk (V@R)

Observation (V@R vs. Chance Constraint)

Given any constant $t^o \in \Re$, we have

$$\mathbb{P}\left\{\tilde{L} \leq t^o\right\} \geq 1-\alpha \Longleftrightarrow \mathrm{V@R}_{1-\alpha}(\tilde{L}) \leq t^o,\ \forall\ \alpha \in [0,1].$$

- V@R constraint is nothing but Chance Constraint!
- V@R optimization $\Longleftrightarrow$ CCP!

Outline

CCP vs. Value-at-Risk (V@R)

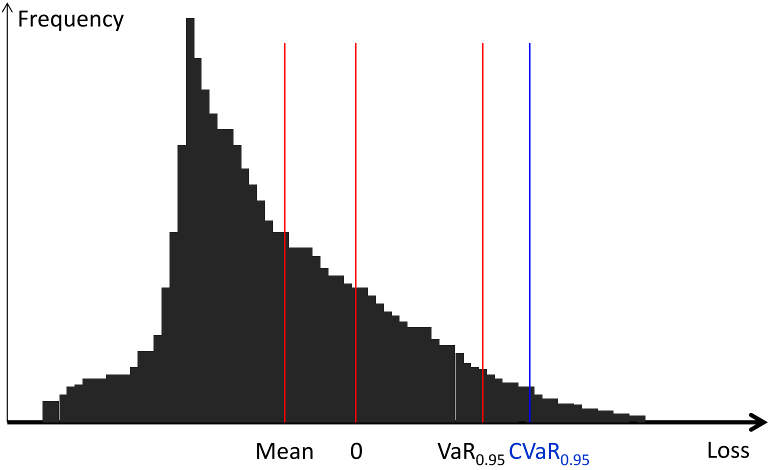
Definition (CV@R, Rockfellar & Uryasev 2000)

Let $\tilde{L}$ be the loss variable, and $\text{V@R}_{1-\alpha}$ be the Value-at-Risk at confidence level of $1-\alpha$. The Conditional Value-at-Risk at confidence level of $1-\alpha$, denoted by $\text{CV@R}_{1-\alpha}$, is the expected loss conditional that $\tilde{L} \geq \text{V@R}_{1-\alpha}$:

$$\text{CV@R}_{1-\alpha}(\tilde{L}) := \mathbb{E}\left[\tilde{L} \mid \tilde{L} \geq \text{V@R}_{1-\alpha}\right].$$

- Conditional Expected Value.
- More conservative than V@R: $\text{CV@R}_{1-\alpha}(\tilde{L}) \geq \text{V@R}_{1-\alpha}(\tilde{L})$.
- Other names: Expected Shortfall, Average Value at Risk (AV@R), etc.

CCP vs. Value-at-Risk (V@R)



CCP vs. Value-at-Risk (V@R)

Recall that

$$\mathbb{E}\left[\tilde{L} \mid \tilde{L} \in \boldsymbol{A}\right] = \frac{1}{\mathbb{P}(\boldsymbol{A})} \int_{\boldsymbol{A}} \tilde{L} \mathrm{d}\mathbb{P}$$

- Discrete $\tilde{L}$:

$$\mathbb{E}\left[\tilde{L} \mid \tilde{L} \in \boldsymbol{A}\right] = \frac{1}{\mathbb{P}(\boldsymbol{A})} \sum_{L_k \in \boldsymbol{A}} L_k \mathbb{P}\{\tilde{L} = L_k\}$$

- Continuous $\tilde{L}$:

$$\mathbb{E}\left[\tilde{L} \mid \tilde{L} \in \boldsymbol{A}\right] = \frac{1}{\mathbb{P}(\boldsymbol{A})} \int_{x \in \boldsymbol{A}} x f_{\tilde{L}}(x) \mathrm{dx}$$

CCP vs. Value-at-Risk (V@R)

Example (CV@R for a discrete return distribution)

We consider the following distribution of profit for asset A:

$$\mathbb{P}\{\tilde{P}_A = -\$500\} = 0.02, \mathbb{P}\{\tilde{P}_A = -\$200\} = 0.1,$$

$$\mathbb{P}\{\tilde{P}_A = \$300\} = 0.5, \mathbb{P}\{\tilde{P}_A = \$600\} = 0.38.$$

$$\text{V@R}_{99\%} = \$500; \text{V@R}_{95\%} = \text{V@R}_{90\%} = \$200;$$

- $\text{CV@R}_{99\%} = \mathbb{E}[\tilde{L} \mid \tilde{L} \geq \$500] = \$500$
- $\text{CV@R}_{95\%} = \mathbb{E}[\tilde{L} \mid \tilde{L} \geq \$200] = \$500 \times \frac{0.02}{0.12} + \$200 \times \frac{0.1}{0.12} = \$250$
- $\text{CV@R}_{90\%} = \$250$

Example (CV@R for the normal distribution)

Assume $\tilde{L} \sim \mathcal{N}\left(\mathbb{E}[\tilde{L}], \sigma(\tilde{L})\right)$, then

$$\mathrm{CV@R}_{1-\beta}(\tilde{L}) = \mathbb{E}(\tilde{L}) + \left[\frac{\phi\left(\Phi^{-1}(\beta)\right)}{\beta}\right]\sigma(\tilde{L}).$$

Proof. By definition,

$$\begin{aligned}
\mathrm{CV@R}_{1-\beta}(\tilde{L}) &= \frac{1}{\beta}\int_{x \geq \mathrm{V@R}_{1-\beta}(\tilde{L})} x \mathrm{d}F_{\tilde{L}}(x) \\
&= \frac{1}{\beta}\int_{v=1-\beta}^{v=1} F_{\tilde{L}}^{-1}(v)\mathrm{d}v \\
&= \frac{1}{\beta}\int_{1-\beta}^{1} \mathrm{V@R}_{v}(\tilde{L})\mathrm{d}v = \frac{1}{\beta}\int_{0}^{\beta} \mathrm{V@R}_{1-v}(\tilde{L})\mathrm{d}v
\end{aligned}$$

by noting that $\mathrm{V@R}_{1-\beta}(\tilde{L}) = F_{\tilde{L}}^{-1}(1-\beta)$.

CCP vs. Value-at-Risk (V@R)

Furthermore, recall that

$$\mathrm{V@R}_{1-\beta}(\tilde{L}) = \mathbb{E}(\tilde{L}) + \sigma(\tilde{L})\Phi^{-1}(1-\beta), \forall \beta \in [0,1],$$

we have

$$\begin{aligned}
\mathrm{CV@R}_{1-\beta}(\tilde{L}) &= \mathbb{E}(\tilde{L}) + \frac{\sigma(\tilde{L})}{\beta}\int_0^{\beta} \Phi^{-1}(1-v)\mathrm{d}v \\
&= \mathbb{E}(\tilde{L}) + \frac{\sigma(\tilde{L})}{\beta}\int_{-\infty}^{\Phi^{-1}(\beta)} \Phi^{-1}(1-\Phi(y))\mathrm{d}\Phi(y) \\
&= \mathbb{E}(\tilde{L}) + \frac{\sigma(\tilde{L})}{\beta}\int_{-\infty}^{\Phi^{-1}(\beta)} -y\phi(y)\mathrm{d}y
\end{aligned}$$

CCP vs. Value-at-Risk (V@R)

Plugging the standard normal density

$$\phi(y) = \frac{1}{\sqrt{2\pi}} e^{-y^2/2},$$

we have

$$\begin{aligned}
\mathrm{CV@R}_{1-\beta}(\tilde{L}) &= \mathbb{E}(\tilde{L}) + \frac{\sigma(\tilde{L})}{\beta}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\Phi^{-1}(\beta)} -ye^{-y^2/2}\mathrm{d}y \\
&= \mathbb{E}(\tilde{L}) + \frac{\sigma(\tilde{L})}{\beta}\frac{1}{\sqrt{2\pi}} e^{\frac{-[\Phi^{-1}(\beta)]^2}{2}} \\
&= \mathbb{E}(\tilde{L}) + \left[\frac{\phi\left(\Phi^{-1}(\beta)\right)}{\beta}\right]\sigma(\tilde{L}).
\end{aligned}$$

Q.E.D.

CCP vs. Value-at-Risk (V@R)

A more useful formula of CV@R for computation and optimization.

Theorem (Rockfellar & Uryasev 2000)

$$\text{CV@R}_{1-\alpha}(\tilde{L}) = \underset{r}{\text{Min}} \left\{ r + \frac{1}{\alpha}\mathbb{E}\left[\tilde{L} - r\right]_+ \right\}$$

where $[a]_+ = \text{Max}\{a, 0\}$.

- How to understand this formula?
- Why it is useful?
- CV@R is also a risk measure, even better!

CCP vs. Value-at-Risk (V@R)

Observation (CV@R vs. Chance Constraint)

Given any constant $t \in \Re$, we have

$$\text{CV@R}_{1-\alpha}(\tilde{L}) \le t \Longrightarrow \mathbb{P}\left\{\tilde{L} \le t\right\} \ge 1-\alpha,\ \forall\ \alpha \in [0,1].$$

- Note that

$$\mathbb{P}\left\{\tilde{L} \le t\right\} \ge 1-\alpha \Leftrightarrow \text{V@R}_{1-\alpha}(\tilde{L}) \le t,\ \forall\ \alpha \in [0,1].$$

  and

$$\text{CV@R}_{1-\alpha}(\tilde{L}) \ge \text{V@R}_{1-\alpha}(\tilde{L}).$$

- CV@R constraint provides a safe approximation for Chance Constraint!

CCP vs. Value-at-Risk (V@R)

Observation (CV@R vs. Chance Constraint)

Given any constant $t \in \Re$, we have

$$\text{CV@R}_{1-\alpha}(\tilde{L}) \leq t \Longrightarrow \mathbb{P}\left\{\tilde{L} \leq t\right\} \geq 1-\alpha,\ \forall\ \alpha \in [0,1].$$

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  and

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- CV@R constraint provides a safe approximation for Chance Constraint!

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Example (Normality and single constraint)

Under the assumption of normality, the single constraint:

$$\mathbb{P}\{\tilde{\boldsymbol{a}}^\top \boldsymbol{x} \geq b\} \geq 1-\alpha \Leftrightarrow \boldsymbol{\mu}^\top \boldsymbol{x} + \Phi^{-1}(\alpha)\sqrt{\boldsymbol{x}^\top \boldsymbol{\Sigma} \boldsymbol{x}} \geq b,$$

where $\tilde{\boldsymbol{a}} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, $1-\alpha > 0.5$.

Example (Joint Chance Constraint & log-concavity & Independence)

Consider the following joint chance constraint:

$$\mathbb{P}\Big\{x_i \leq \tilde{a}_i \leq y_i, i \in [I]\Big\} \geq \beta \Longleftrightarrow \sum_{i \in [I]} \ln \mathbb{P}\Big\{x_i \leq \tilde{a}_i \leq y_i\Big\} \geq \ln \beta$$

where $\tilde{a}_i, i \in [I]$ are independent and have log-concave densities.

- Tractability result (Prśekopa 1980): If continuous r.v. $\tilde{a}$ has a log-concave density function, then

$$\mathcal{G}(x, y) := \ln \mathbb{P}\Big\{x \leq \tilde{a} \leq y\Big\}$$

is concave in (x, y).
- $\nabla \mathcal{G}(x, y)$ can be computed.

Computation of CCP

- The key difficulty of the CCP lies in the following chance constraint:
$$\mathbb{P}\left\{\tilde{\boldsymbol{a}}^\top \boldsymbol{x} \geq \tilde{b}\right\} \geq 1-\alpha$$
- It is difficult for the case of $\tilde{\boldsymbol{a}}$ with a general distribution, due to multiple integration !
- A practical setting: a set of data
$$\left(\boldsymbol{a}^k, b^k\right), k \in [K]$$
is given for each $(\tilde{\boldsymbol{a}}^\top, \tilde{b})$.

Computation of CCP

Central Problem

Assume we have data $(\boldsymbol{a}^k, b^k), k \in [K]$ for random parameters $(\tilde{\boldsymbol{a}}, \tilde{b})$, then how to solve the following CCP:

$$\begin{aligned}
\underset{\boldsymbol{x}}{\text{Min}} \quad & \boldsymbol{c}^\top \boldsymbol{x} \\
\text{s.t.} \quad & \mathbb{P}\Big\{ \tilde{\boldsymbol{a}}^\top \boldsymbol{x} \geq \tilde{b} \Big\} \geq 1 - \alpha \\
& \boldsymbol{B}\boldsymbol{x} \geq \boldsymbol{d},
\end{aligned}$$

- Note that

$$\mathbb{P}\{\tilde{\boldsymbol{a}}^\top \boldsymbol{x} \geq \tilde{b}\} = \mathbb{E}\left[\mathbf{1}_{\{\tilde{\boldsymbol{a}}^\top \boldsymbol{x} \geq \tilde{b}\}}\right] = \frac{1}{K}\sum_{k=1}^{K} z_k,$$

where z_k models the indicator function $\mathbf{1}_{\{\cdot\}}$:

$$z_k = \begin{cases} 1, & b^k - \boldsymbol{x}^\top \boldsymbol{a}^k \leq 0 \\ 0, & otherwise \end{cases}$$

Note that

$$\left\{ \boldsymbol{x} \;\middle|\; \begin{array}{ll} \exists z_k \in \{0,1\}, & \forall\ k \in [K] \\ \frac{1}{K}\sum_{k=1}^{K} z_k \geq 1-\alpha & \\ z_k = \begin{cases} 1, & b^k - \boldsymbol{x}^\top \boldsymbol{a}^k \leq 0 \\ 0, & otherwise \end{cases}, & \forall k \in [K] \end{array} \right\}$$

$\Longleftrightarrow$

$$\left\{ \boldsymbol{x} \;\middle|\; \begin{array}{ll} \exists z_k \in \{0,1\}, & \forall\ k \in [K] \\ \frac{1}{K}\sum_{k=1}^{K} z_k \geq 1-\alpha & \\ b^k - \boldsymbol{x}^\top \boldsymbol{a}^k \leq M(1-z_k), & \forall\ k \in [K] \end{array} \right\}$$

MIP Formulation for CCP

$$
\begin{aligned}
\min_{\boldsymbol{x},\mathbf{z}} \quad & \boldsymbol{c}^\top \boldsymbol{x} \\
\text{s.t.} \quad & \frac{1}{K}\sum_{k=1}^{K} z_k \geq 1-\alpha \\
& b^k - \boldsymbol{x}^\top \boldsymbol{a}^k \leq M(1-z_k),\ \forall\ k \in [K] \\
& z_k \in \{0,1\},\ \forall k \in [K] \\
& \boldsymbol{x} \in \mathcal{X},
\end{aligned}
$$

where M is a sufficiently large number.

- The size of the MIP grows up quickly when the sample size increases.

Computation of CCP

Central Problem (Joint Constraint)

Assume we have data $(\boldsymbol{a}_i^k, b_i^k), k = 1, 2, \cdots, K$ for random parameters $(\tilde{\boldsymbol{a}}_i^\top, \tilde{b}_i), i = 1, 2, \cdots, m$, then how to solve the following CCO:

$$\begin{aligned} \min_{\boldsymbol{x}} \quad & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{s.t.} \quad & \mathbb{P}\{\tilde{\boldsymbol{a}}_i^\top \boldsymbol{x} \geq \tilde{b}_i, i = 1, 2, \cdots, m\} \geq 1 - \alpha \\ & \boldsymbol{B}\boldsymbol{x} \geq \boldsymbol{d}, \end{aligned}$$

MIP Formulation for CCP

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\begin{aligned}
\min_{\boldsymbol{x},\boldsymbol{z}} \quad & \boldsymbol{c}^\top \boldsymbol{x} \\
\text{s.t.} \quad & \frac{1}{K}\sum_{k=1}^{K} z_k \geq 1-\alpha \\
& b_i^k - \boldsymbol{x}^\top \boldsymbol{a}_i^k \leq M(1-z_k), \ \forall\, k \in [K], i \in [m] \\
& z_k \in \{0,1\}, \ \forall\, k \in [1;K] \\
& \boldsymbol{B}\boldsymbol{x} \geq \boldsymbol{d},
\end{aligned}
$$

where M is a sufficiently large number.

- H.W. How to linearize the constraint $|x| \geq b$?

Convex Approximation: CVaR Approximation

Computation of CCP

Rewrite the problem into the following form:

Assume we have data $(\boldsymbol{a}_i^k, b_i^k), k = 1, 2, \cdots, K$ for random parameters $(\tilde{\boldsymbol{a}}_i^\top, \tilde{b}_i), i = 1, 2, \cdots, m$, then how to solve the following CCP:

$$\begin{aligned} \underset{\boldsymbol{x}}{\text{Min}} \quad & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{s.t.} \quad & \mathbb{P}\left\{ \underset{i=1}{\overset{m}{\text{Max}}} \{\tilde{b}_i - \boldsymbol{x}^\top \tilde{\boldsymbol{a}}_i\} \leq 0 \right\} \geq 1 - \alpha \\ & \boldsymbol{B}\boldsymbol{x} \geq \boldsymbol{d}, \end{aligned}$$

Recall that

$$\mathrm{CV@R}_{1-\alpha}(\tilde{L}) \leq 0 \Rightarrow \mathbb{P}\left\{ \tilde{L} \leq 0 \right\} \geq 1 - \alpha, \ \forall \ \alpha \in [0, 1],$$

we consider the CVaR constraint

$$\mathrm{CV@R}_{1-\alpha}\left(\underset{i=1}{\overset{m}{\text{Max}}} \{\tilde{b}_i - \boldsymbol{x}^\top \tilde{\boldsymbol{a}}_i\} \right) \leq 0.$$

Computation of CCP

CVaR Approximation Model

$$\begin{aligned}
\underset{\boldsymbol{x}}{\text{Min}} \quad & \boldsymbol{c}^\top \boldsymbol{x} \\
\text{s.t.} \quad & \text{CV@R}_{1-\alpha}\left(\underset{i=1}{\overset{m}{\text{Max}}}\{\tilde{b}_i - \boldsymbol{x}^\top \tilde{\boldsymbol{a}}_i\}\right) \leq 0 \\
& \boldsymbol{B}\boldsymbol{x} \geq \boldsymbol{d}.
\end{aligned}$$

Recall that

$$\text{CV@R}_{1-\alpha}(\tilde{L}) = \underset{r}{\text{Min}} \left\{ r + \frac{1}{\alpha}\mathbb{E}[\tilde{L} - r]_+ \right\}.$$

Computation of CCP

$$
\begin{aligned}
& \mathrm{CV@R}_{1-\alpha}\left(\operatorname*{Max}_{i=1}^{m}\{\tilde{b}_i - \boldsymbol{x}^\top \tilde{\boldsymbol{a}}_i\}\right) \le 0 \\
\Leftrightarrow\quad & \operatorname*{Min}_{r}\left\{r + \frac{1}{\alpha}\mathbb{E}\left[\operatorname*{Max}_{i=1}^{m}\{\tilde{b}_i - \boldsymbol{x}^\top \tilde{\boldsymbol{a}}_i\} - r\right]_+\right\} \le 0 \\
\Leftrightarrow\quad & r + \frac{1}{\alpha}\mathbb{E}\left[\operatorname*{Max}_{i=1}^{m}\{\tilde{b}_i - \boldsymbol{x}^\top \tilde{\boldsymbol{a}}_i\} - r\right]_+ \le 0; r \in \Re \\
\Leftrightarrow\quad & r + \frac{1}{\alpha}\cdot\frac{1}{K}\sum_{k=1}^{K}\left[\operatorname*{Max}_{i=1}^{m}\{b_i^k - \boldsymbol{x}^\top \boldsymbol{a}_i^k\} - r\right]_+ \le 0; r \in \Re \\
\Leftrightarrow\quad & r + \frac{1}{\alpha}\cdot\frac{1}{K}\sum_{k=1}^{K}\operatorname{Max}\left\{\operatorname*{Max}_{i=1}^{m}\{b_i^k - \boldsymbol{x}^\top \boldsymbol{a}_i^k\} - r, 0\right\} \le 0; r \in \Re \\
\Leftrightarrow\quad & r + \frac{1}{\alpha}\cdot\frac{1}{K}\sum_{k=1}^{K}\gamma_k \le 0; \gamma_k \ge 0, \gamma_k \ge \operatorname*{Max}_{i=1}^{m}\{b_i^k - \boldsymbol{x}^\top \boldsymbol{a}_i^k\} - r, \forall k, r \in \Re
\end{aligned}
$$

Computation of CCP

Therefore, the CVaR Approximation Model can be transformed into the following LP!

CVaR Approximation Model

$$
\begin{aligned}
\underset{\boldsymbol{x},\boldsymbol{\gamma},r}{\text{Min}} \quad & \boldsymbol{c}^\top \boldsymbol{x} \\
\text{s.t.} \quad & r + \frac{1}{\alpha} \cdot \frac{1}{K} \sum_{k=1}^{K} \gamma_k \leq 0,\ \forall\ k \in [1;K] \\
& \gamma_k \geq 0,\ \forall\ k \in [1;K] \\
& \gamma_k \geq b_i^k - \boldsymbol{x}^\top \boldsymbol{a}_i^k,\ \forall\ k \in [1;K],\ i \in [1;m] \\
& r \in \Re \\
& \boldsymbol{B}\boldsymbol{x} \geq \boldsymbol{d}.
\end{aligned}
$$

Computation of CCP

Example (Portfolio problem revisited, H.W.)

- We consider n investable assets with random return (ratios) $\tilde{R}_1, \cdots, \tilde{R}_n$ in the next year, where

$$\tilde{R}_i \longleftarrow \hat{R}_i^k := \frac{P_i^k(t+T) - P_i^k(t)}{P_i^k(t)}, k = 1, 2, \cdots, K.$$

  where $P_i^k(t)$ is the kth sample price of asset i at time t, and $T = 1$ Year.

- We have certain amount of capital and our aim is to maximize the expected return conditional that the chance of losing no more than a given fraction 10% of the capital is 95%.
- Let $x_1, \cdots, \boldsymbol{x}_n$ be the fractions of our capital invested in the n assets. Next year, the total investment return of our portfolio becomes

$$\sum_{i=1}^{n} \tilde{R}_i x_i.$$

Computation of CCP

Example (Cont'd.)

Consider the following VaR (risk) minimization portfolio problem:

$$\begin{aligned} \min_{\boldsymbol{x}} \quad & \mathrm{V@R}_{0.95}\left(-\sum_{i=1}^{n} \tilde{R}_i x_i\right) \\ \text{s.t.} \quad & \sum_{i=1}^{n} \mathbb{E}[\tilde{R}_i] x_i \geq \tau^o \\ & \sum_{i=1}^{n} x_i = 1, \boldsymbol{x} \geq \mathbf{0}. \end{aligned}$$

1. Formulation the above VaR minimization problem into its MIP formulation.
2. Write down its CVaR approximation formulation (LP).

Outline

Conclusion

1. CCP model can be used to model the both types of problems with "Constraint protection" and "Target-based Optimization".
2. A new risk measure: V@R; V@R constraint problem or V@R minimization problem can be modeled into CCP.
3. A new risk measure: CV@R; CV@R constraint provides a safe approximation for V@R constraint and hence chance constraint.
4. The computational formula for CV@R $\star\star\star\star\star$.
5. Computations of CCP with samples: (i) MIP exact formulation. (ii) CV@R approximation formulation $\rightarrow$ LP.

Reference and Further Reading

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3. Attilio Meucci, Risk and Asset Allocation, Springer-Verlag, Berlin Heidelberg, 2005.
4. P. Jaillet, S.D. Jena, T.S. Ng, M. Sim, Satisficing awakens: models to mitigate uncertainty, Optimization-online, 2017.

"If you want to live a happy life, tie it to a goal, not to people or things."

—— Albert Einstein