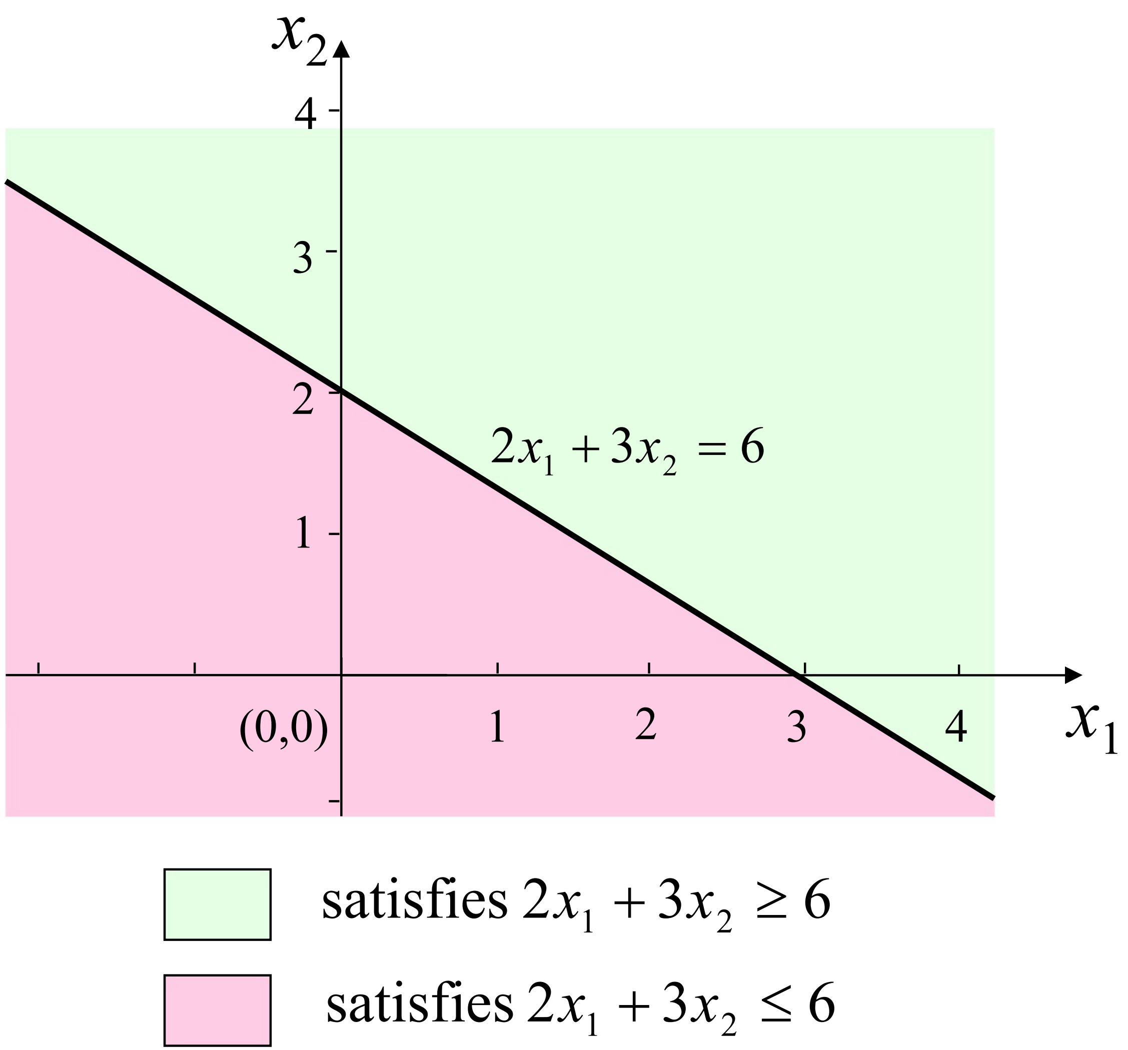
Graphical Solution of Two-Variable LP Problems

Based on

Chapters 3.2 and 3.3 of "*Operations Research*: *Application & Algorithms*," 4th edition

Graphic representation of inequality



The Giapetto's Example

$$\max \ z = 3x_1 + 2x_2 \qquad (1)$$

s.t.

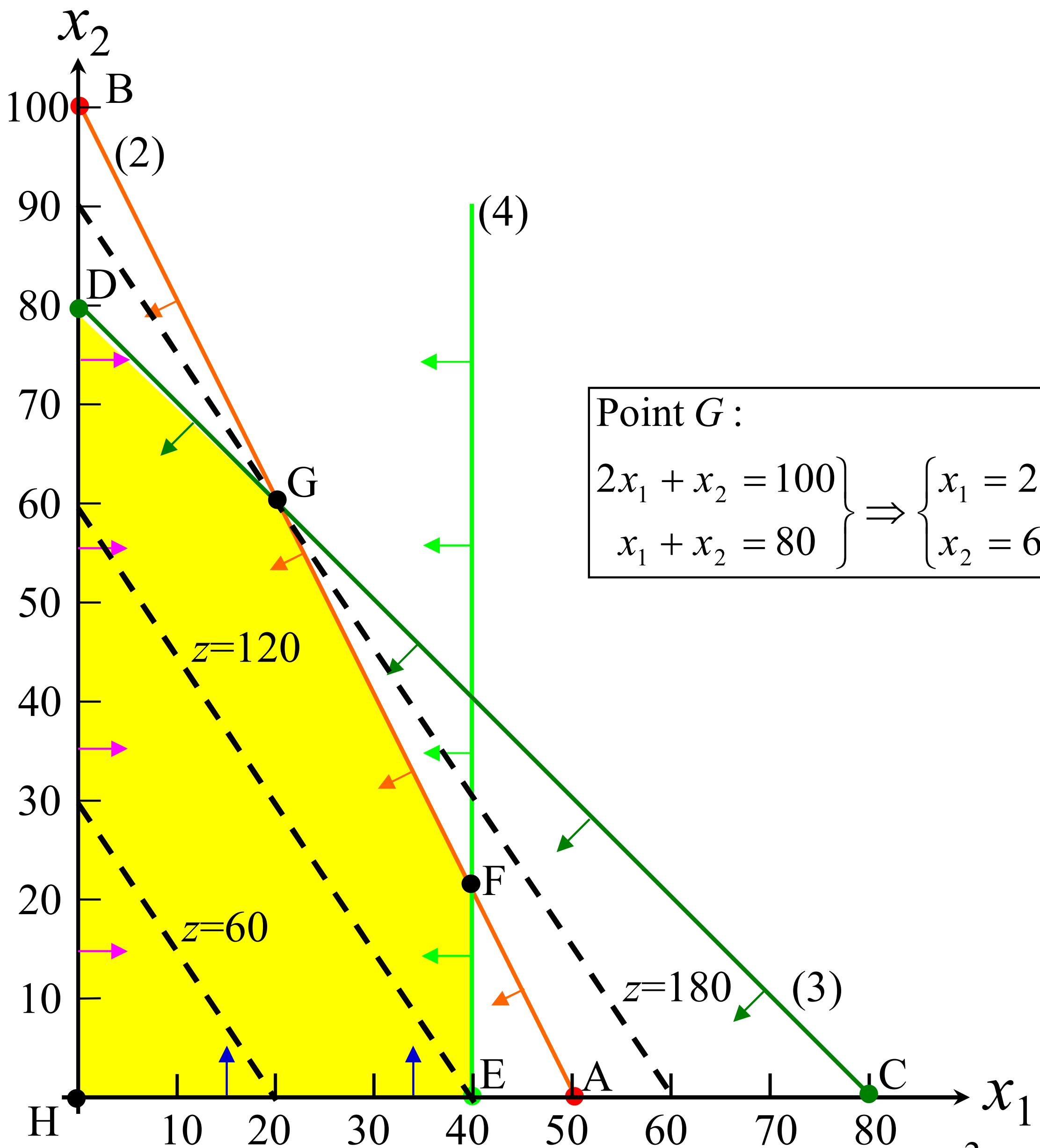
$$2x_1 + x_2 \le 100 \qquad (2)$$

$$x_1 + x_2 \le 80 \qquad (3)$$

$$x_1 \le 40 \qquad (4)$$

$$x_1 \ge 0 \qquad (5)$$

$$x_2 \ge 0 \qquad (6)$$



Point G :

$$\left.\begin{aligned} 2x_1 + x_2 &= 100 \\ x_1 + x_2 &= 80 \end{aligned}\right\} \Rightarrow \left\{\begin{aligned} x_1 &= 20 \\ x_2 &= 60 \end{aligned}\right.$$

Steps for graphical method

- Step1: determine the feasible region
  - Draw a line for each constraint and sign restriction
  - Determine the side of a line that satisfies the inequality
  - Find the intersection of all constraints and sign restrictions
- Step 2: find the optimal solution
  - Draw isoprofit line (max problem) or isocost line (min problem)
    - The line on which all points have the same z-value.
    - Choose a feasible point and calculate its objective value
    - Draw a isoprofit (or isocost) line passing through this feasible point
    - Find other isoprofit (or isocost) lines by moving parallel to this isoprofit (or isocost) line in a direction that increase z (for a max problem) or decreases z (for a min problem)
  - Find the last isoprofit (or isocost) line to intersect the feasible region (if possible)
- Step 3: find the binding constraints and solve the optimal solution (x_1, x_2) and the optimal objective value (z^*)

Feasible Region & Optimal Solution

- The feasible region for an LP is the set of all points satisfying all the LP's constraints and all the LP's sign restrictions
- For a maximization problem, an optimal solution to an LP is a point in the feasible region with the largest objective function value. Similarly, for a minimization problem, an optimal solution is a point in the feasible region with the smallest objective function value.

Binding and nonbinding constraints, Extreme points

- A constraint is **binding** if the left-hand side and the right-hand side of the constraint are equal when the optimal values of the decision variables are substituted into the constraint
- A constraint is **nonbinding** if the left-hand side and the right-hand side of the constraint are unequal when the optimal values of the decision variables are substituted into the constraint
- **Extreme points:** corner points

  Theorem 1 The feasible region for any LP has only a finite number of extreme points.

  Theorem 2 Any LP that has an optimal solution has an extreme point that is optimal.

Summary

Every LP (*not just LPs with two variables*) must fall into one of the following four cases:

Case1: The LP has a unique optimal solution: the optimal solution is one of the extreme points of the feasible region

Three special cases

Case 2: The LP has alternative or multiple optimal solutions (infinite number of optimal solutions)

Case 3: The LP is infeasible: the feasible region is empty

Case 4: The LP is unbounded: the feasible region is unbounded and there are points in the feasible region with arbitrarily large (in a max problem) z-value.

If the feasible region is unbounded, the LP is either unbounded or has optimal solution(s) (unique or infinite).

Alternative or Multiple Optimal

$$\max \ z = 3x_1 + 2x_2 \qquad (1)$$

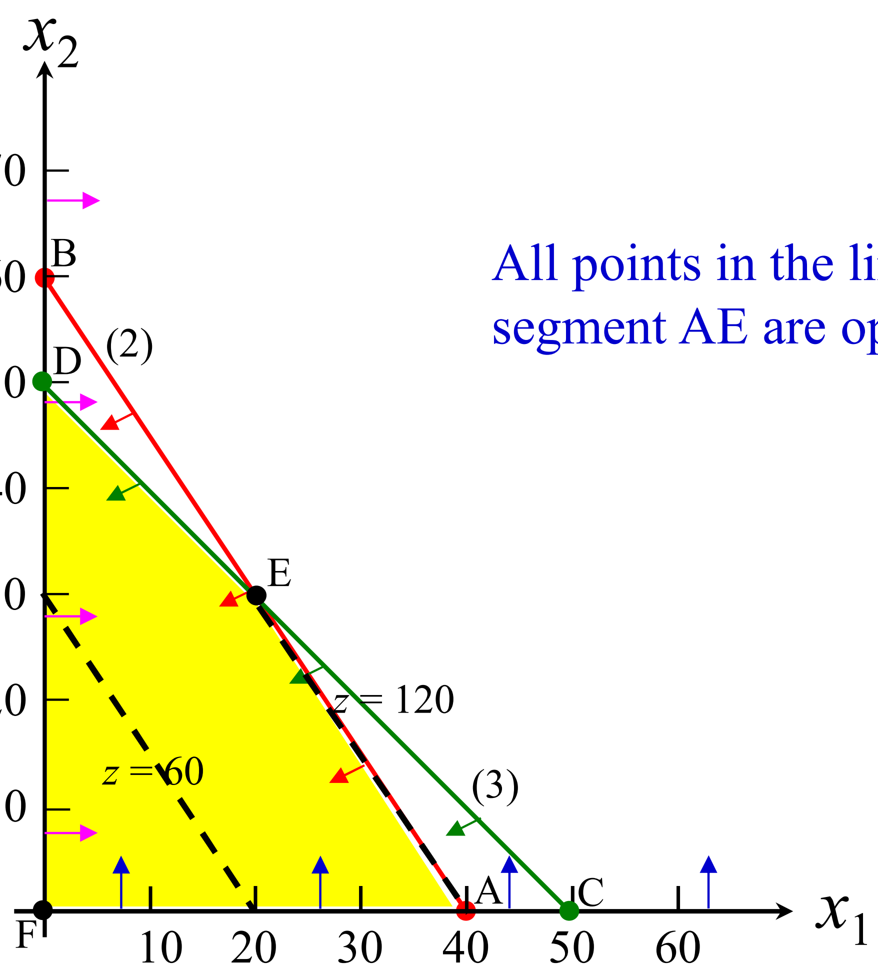
s.t.

$$\frac{1}{40}x_1 + \frac{1}{60}x_2 \le 1 \qquad (2)$$

$$\frac{1}{50}x_1 + \frac{1}{50}x_2 \le 1 \qquad (3)$$

$$x_1 \ge 0 \qquad (4)$$

$$x_2 \ge 0 \qquad (5)$$



All points in the line segment AE are optimal

Infeasible

$$\max \quad z = 3x_1 + 2x_2 \qquad (1)$$

s.t.

$$\frac{1}{40}x_1 + \frac{1}{60}x_2 \le 1 \qquad (2)$$

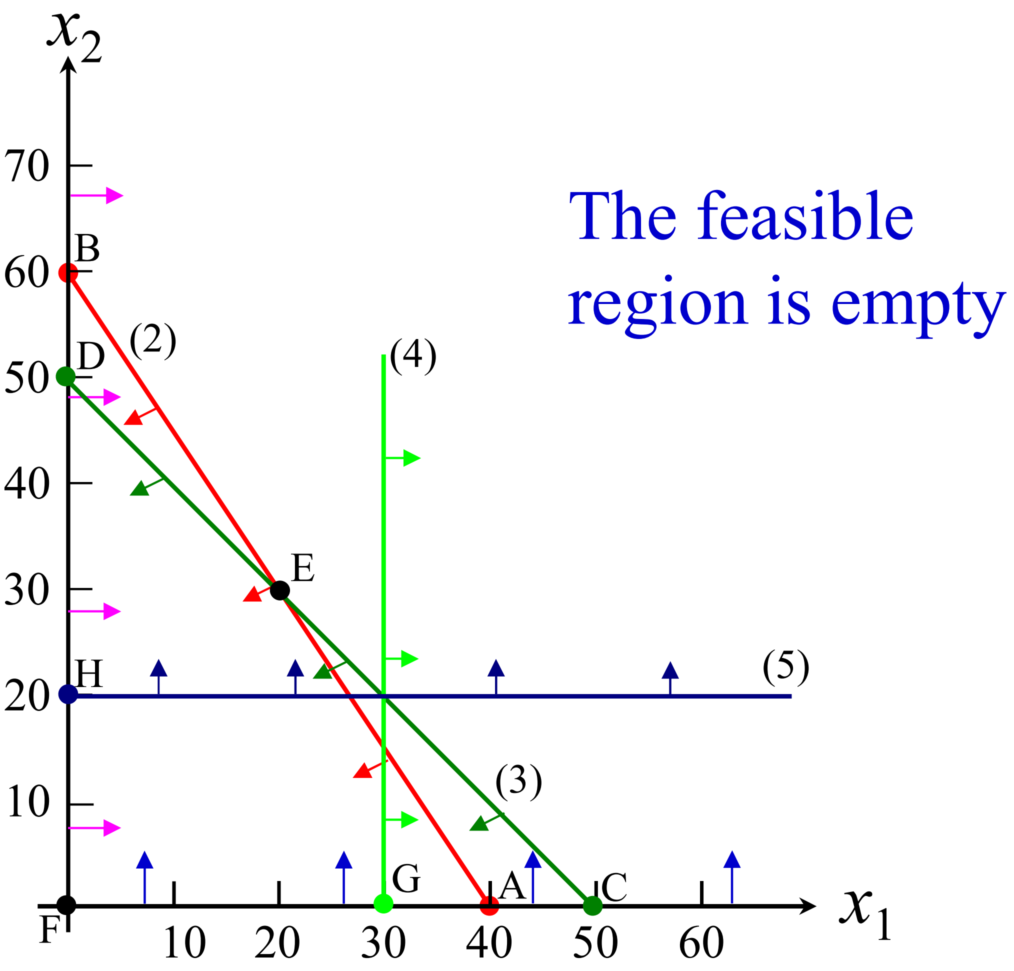
$$\frac{1}{50}x_1 + \frac{1}{50}x_2 \le 1 \qquad (3)$$

$$x_1 \ge 30 \qquad (4)$$

$$x_2 \ge 20 \qquad (5)$$

$$x_1 \ge 0 \qquad (6)$$

$$x_2 \ge 0 \qquad (7)$$



Unbounded

$$\max z = x_1 + x_2 \qquad (1)$$

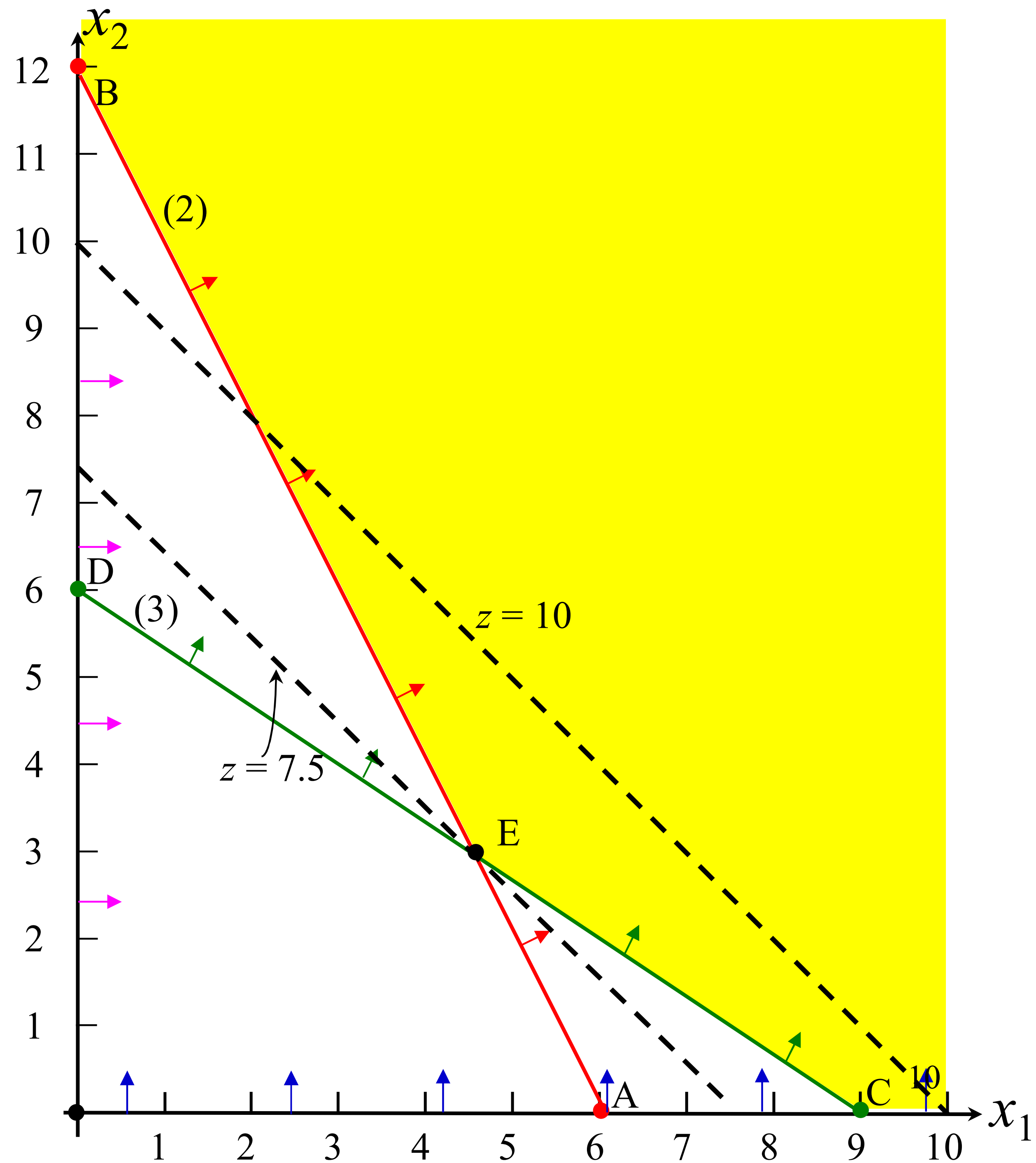
s.t.

$$2x_1 + 1x_2 \geq 12 \qquad (2)$$

$$2x_1 + 3x_2 \geq 18 \qquad (3)$$

$$x_1 \geq 0 \qquad (4)$$

$$x_2 \geq 0 \qquad (5)$$



Revised Simplex Algorithm

Based on

Chapter 4.1 - 4.8 and 4.11-4.12 of "*Operations Research*: *Application & Algorithms*," 4th edition

Converting an LP to Standard Form

- Before the simplex algorithm can be used to solve an LP, the LP must be converted into a standard form, in which
  - all the constraints are equalities and
  - all variables are nonnegative.
- Procedure:

✓ if the ith constraint is a "$\leq$" constraint, then we convert it to an "=" constraint by adding a ***slack variable*** s_i and the sign restriction $s_i \geq 0$.

✓ if the ith constraint is a "$\geq$" constraint, then we convert it to an "=" constraint by subtracting an ***excess variable*** e_i and adding the sign restriction $e_i \geq 0$.

✓ if the variable x_i is unrestricted in sign (urs or free), replace x_i in both the objective function and constraints by $x_i' - x_i''$, where $x_i' \geq 0$, $x_i'' \geq 0$.

Example: Convert the following LP to standard form

Original LP

LP in standard form

$$\begin{aligned} \text{max} \quad & z = 30x_1 - 4x_2 \\ \text{s.t.,} \quad & x_1 \leq 5 \\ & 5x_1 - x_2 \leq 30 \\ & 20x_1 + 15x_2 \geq 2000 \\ & x_1 \geq 0, x_2 \text{ urs} \end{aligned}$$

$$x_2 = x_2' - x_2''$$

Preview (Theory) of LP Solutions (Revised Simplex Algorithm)

basic feasible solution (BFS)
adjacent BFSs

Definition of basic solution & bfs

- Consider a system ***A*x=*b*** of m linear equations in n variables ($n \geq m$).
- A ***basic solution*** to ***A*x=*b*** is obtained by

  (1) choosing and setting n-m variables (the *nonbasic variables*, or *NBV*) equal to 0, and

  (2) solving the values of the remaining m variables (the *basic variables*, or *BV*) that satisfy ***A*x=*b*** (m equations).
- Any basic solution in which all variables are nonnegative is called a **basic feasible solution** (or **bfs**).
- Different choices of nonbasic variables lead to different basic solutions. If setting the nonbasic variables equal to 0 yields unique values for the remaining m variables, a basic solution is generated. It is possible that a choice of n-m nonbasic variables does not generate a basic solution.

Example: no solution case

Some sets of m variables do not yield a basic solution. Consider the following system

$$\boxed{\begin{aligned} x_1 + 2x_2 + x_3 &= 1 \\ 2x_1 + 4x_2 + x_3 &= 3 \end{aligned}}$$

In this example, n=3, m=2, and the number of nonbasic variables = 3-2=1.

If $NBV=\{x_3\}$ and $BV=\{x_1,x_2\}$, then ($x_3=0$)

$$\boxed{\begin{aligned} x_1 + 2x_2 &= 1 \\ 2x_1 + 4x_2 &= 3 \end{aligned}}$$

Because this system has no solution, there is no basic solution corresponding to $BV=\{x_1,x_2\}$.

bfs and extreme point

The following theorem explains why the concept of a basic feasible solution (bfs) is of great importance in LP.

Theorem 3 A point in the feasible region of an LP is an extreme point if and only if it is a bfs to the LP.

That is, for any LP, there is a unique extreme point of the LP's feasible region corresponding to each bfs.

Theorem 1 The feasible region for any LP has only a finite number of extreme points (corner points).

Corollary 1 The LP has a finite number of bfs.

Example: bfs and extreme point

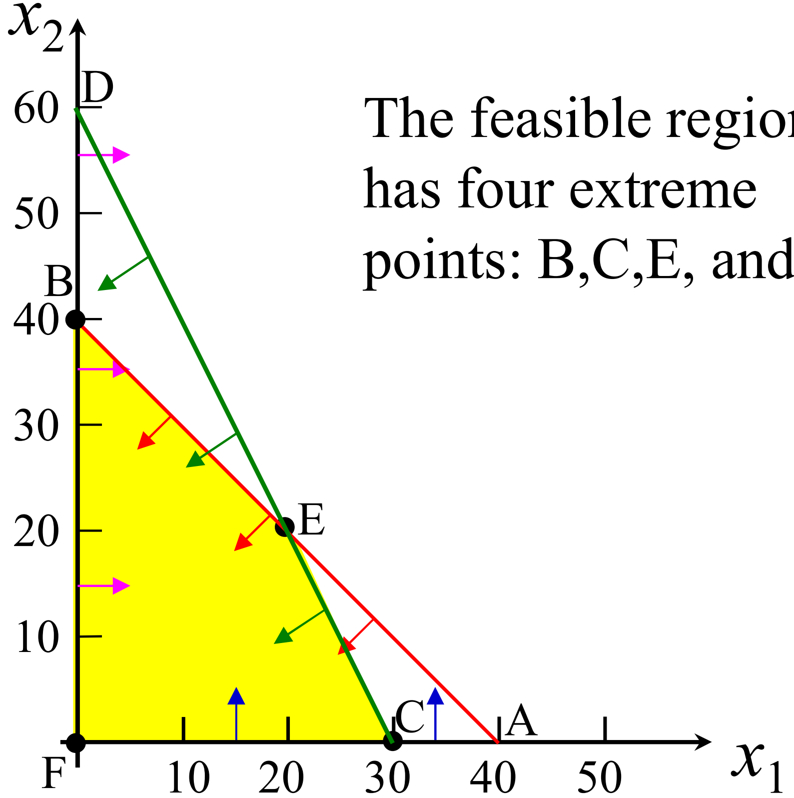
Consider the relationship between extreme points and bfs outlined in Theorem 4.1.

Original form:

$$\begin{aligned} \max\ z &= 4x_1 + 3x_2 \\ \text{s.t.}\quad x_1 + x_2 &\le 40 \\ 2x_1 + x_2 &\le 60 \\ x_1, x_2 &\ge 0 \end{aligned}$$

Standard form:

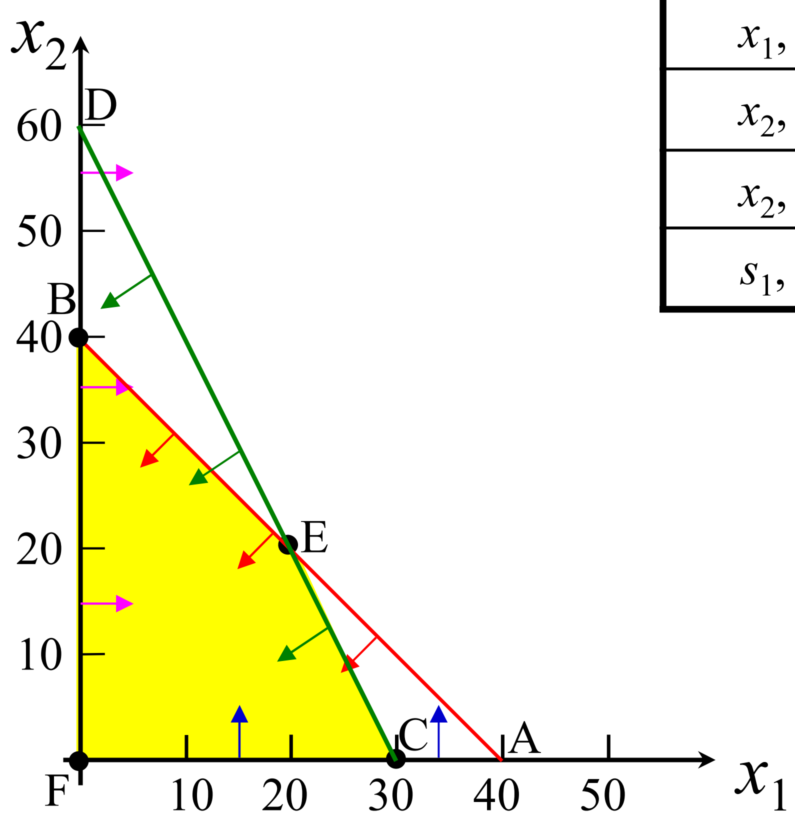
$$\begin{aligned} \max\ z &= 4x_1 + 3x_2 \\ \text{s.t.}\quad x_1 + x_2 + s_1 \quad\quad &= 40 \\ 2x_1 + x_2 \quad\quad + s_2 &= 60 \\ x_1, x_2, s_1, s_2 &\ge 0 \end{aligned}$$



The feasible region has four extreme points: B,C,E, and F

Example: bfs and extreme point

$$\begin{aligned} x_1 + x_2 + s_1 \quad\quad &= 40 \\ 2x_1 + x_2 \quad\quad + s_2 &= 60 \end{aligned}$$



Basic Variables	Nonbasic Variables	bfs to standard form	Corresponds to Extreme Point
x_1, x_2	s_1, s_2	$s_1=s_2=0,\ x_1=x_2=20$	E
x_1, s_1	x_2, s_2	$x_2=s_2=0,\ x_1=30, s_1=10$	C
x_1, s_2	x_2, s_1	$x_2=s_1=0,\ x_1=40, s_2= -20$	Not a bfs (A)
x_2, s_1	x_1, s_2	$x_1=s_2=0,\ s_1= -20,\ x_2=60$	Not a bfs (D)
x_2, s_2	x_1, s_1	$x_1=s_1=0,\ x_2=40, s_2=20$	B
s_1, s_2	x_1, x_2	$x_1=x_2=0,\ s_1=40, s_2=60$	F

The bfs to the standard form of an LP correspond to the LP's extreme points.

bfs and optimal solution

Theorem 4 If an LP has an optimal solution, then it has a bfs that is optimal.

Theorem 2 If an LP has an optimal solution, then it has an extreme point that is optimal.

This is important because any LP has only a finite number of bfs's. Searching for an optimal solution to an LP, we need only find the best bfs to $\boldsymbol{A}\mathbf{x}=\boldsymbol{b}$.

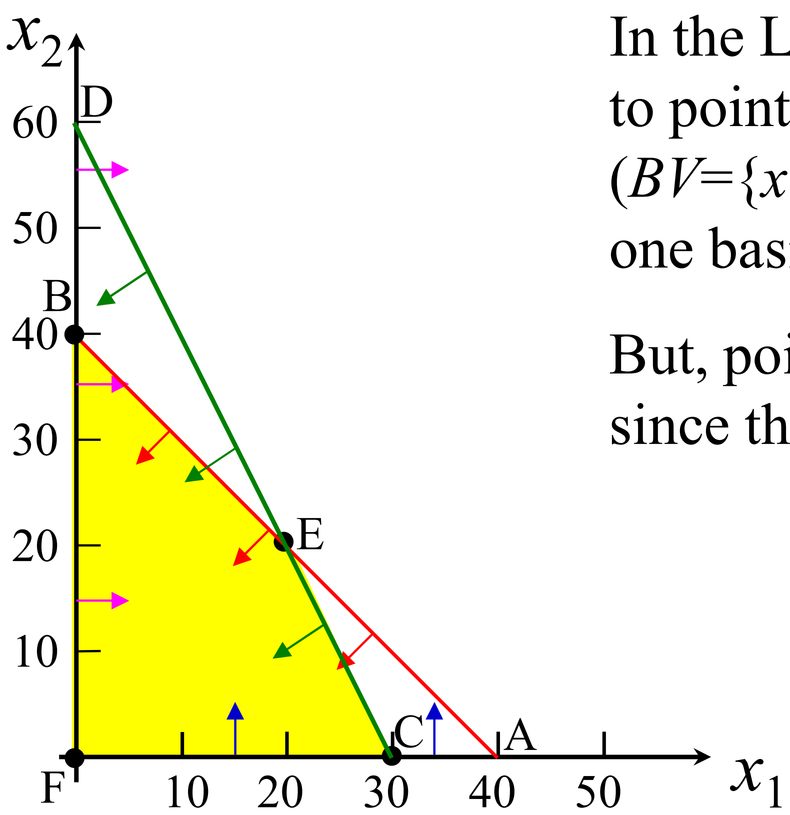
If an LP in standard form has m constraints and n variables, there may be a bfs for each choice of nonbasic variables. Therefore, an LP has the number of bfs's at most

$$\binom{n}{m} = \frac{n!}{(n-m)!\,m!}.$$

For example: $\binom{20}{10} = 184{,}756$; $\binom{200}{100} = ?$

Adjacent bfs

For any LP with m constraints, two bfs's are said to be **adjacent** if their sets of m basic variables have m-1 basic variables in common.



In the Leather Limited LP, the bfs corresponding to point E ($BV=\{x_1,x_2\}$) is adjacent to point C ($BV=\{x_1,s_1\}$), because they share (m-1=2-1=1) one basic variable x_1.

But, point E and F ($BV=\{s_1,s_2\}$) are not adjacent since they share no basic variables.

Note that two bfs's are adjacent if they both lie on the same edge of the boundary of the feasible region.

Formulas in terms of matrices

We use matrices to show how simplex method can be expressed in terms of the LP's parameters. The formulas developed in this section are used in our study of sensitivity analysis, duality, and advanced LP topics.

- Suppose we have converted an LP with m constraints into standard form. Assuming the standard form contains n variables (labeled for convenience $x_1, x_2, \ldots x_n$), including slack and excess variables.

$$
\begin{array}{ll}
\text{max (or min)} & c_1x_1 + c_2x_2 + \cdots + c_nx_n \\
\text{s.t.} & a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\
& a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\
& \quad \vdots \qquad\quad \vdots \qquad\qquad\quad \vdots \qquad\quad \vdots \\
& a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \\
& x_i \geq 0, i = 1, \ldots, n
\end{array}
$$

LP problem in matrix form:

$$
\begin{array}{lr}
\min(\max) & z = \mathbf{cx} \\
s.t., & \mathbf{Ax} = \mathbf{b} \\
& \mathbf{x} \geq \mathbf{0}
\end{array}
\qquad (1)
$$

LP problem in matrix form:

$$\begin{aligned} \min(\max) \quad & z = \mathbf{cx} \\ s.t., \quad & \mathbf{Ax} = \mathbf{b} \qquad\qquad (1) \\ & \mathbf{x} \geq \mathbf{0} \end{aligned}$$

where

$$\mathbf{A} = [\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$$\mathbf{c} = [c_1, \quad c_2, \quad \ldots, \quad c_n]$$

$\mathbf{a}_j$ is the column (in the constraints) for the variable x_j in (1).

$\mathbf{b}$ is an $m \times 1$ column vector of the rhs of the constraints in (1).

Dakota Example

$$\begin{aligned} z &= 60x_1 + 30x_2 + 20x_3 + 0s_1 + 0s_2 + 0s_3 \\ 8x_1 &+ 6x_2 + x_3 + s_1 = 48 \\ 4x_1 &+ 2x_2 + 1.5x_3 + s_2 = 20 \\ 2x_1 &+ 1.5x_2 + 0.5x_3 + s_3 = 8 \end{aligned}$$

$\mathbf{a}_2$ $\qquad\qquad$ $\mathbf{b}$

Suppose we have found a bfs.

$BV = \{BV_1, BV_2, \cdots, BV_m\}$: the set of basic variables
NBV = the set of nonbasic varaibles (in any desired order)

$\mathbf{x}_{BV} = \begin{bmatrix} x_{BV_1} \\ x_{BV_2} \\ \vdots \\ x_{BV_m} \end{bmatrix}$ (note the order: BV_i: the basic variable for row i)

$\mathbf{x}_{NBV}$
$= (n-m) \times 1$ vetor listing the nonbasic variables (in the NBV order)

$\mathbf{c}_{BV} = 1 \times m$ row vector of the initial objective coefficients for the basic variables, i.e., $\mathbf{c}_{BV} = [c_{BV_1}, c_{BV_2}, \cdots, c_{BV_m}]$.
$\mathbf{c}_{NBV} = 1 \times (n-m)$ row vetor of the initial objective coefficients for the nonbasic varaibeles (in the NBV order).

$\mathbf{B}$ is an $m \times m$ matrix whose j^{th} column is the column for BV_j in (1).
$\mathbf{N}$ is an $m \times (n-m)$ matrix whose columns are the columns for the nonbasic varaibles (in the NBV order) in (1).

With respect to a set of basic variables BV, the LP (1) may be written as follows

$$\begin{array}{ll} \text{Max } z = \mathbf{c}_{BV}\mathbf{x}_{BV} + \mathbf{c}_{NBV}\mathbf{x}_{NBV} & \\ \text{s.t.} \quad \mathbf{B}\mathbf{x}_{BV} + \mathbf{N}\mathbf{x}_{NBV} = \mathbf{b} & \\ \quad \mathbf{x}_{BV}, \mathbf{x}_{NBV} \geq 0 & \end{array} \qquad (2)$$

$$\begin{aligned} z &= 60x_1 + 30x_2 + 20x_3 + 0s_1 + 0s_2 + 0s_3 \\ & 8x_1 + 6x_2 + x_3 + s_1 = 48 \\ & 4x_1 + 2x_2 + 1.5x_3 + s_2 = 20 \\ & 2x_1 + 1.5x_2 + 0.5x_3 + s_3 = 8 \end{aligned}$$

Given the basic variables s_1, x_3, and x_1:

$$\text{Max } z = [0 \;\; 20 \;\; 60] \cdot \begin{bmatrix} s_1 \\ x_3 \\ x_1 \end{bmatrix} + [30 \;\; 0 \;\; 0] \cdot \begin{bmatrix} x_2 \\ s_2 \\ s_3 \end{bmatrix}$$

$$\text{s.t. } \begin{bmatrix} 1 & 1 & 8 \\ 0 & 1.5 & 4 \\ 0 & 0.5 & 2 \end{bmatrix} \cdot \begin{bmatrix} s_1 \\ x_3 \\ x_1 \end{bmatrix} + \begin{bmatrix} 6 & 0 & 0 \\ 2 & 1 & 0 \\ 1.5 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_2 \\ s_2 \\ s_3 \end{bmatrix} = \begin{bmatrix} 48 \\ 20 \\ 8 \end{bmatrix}$$

$$\begin{bmatrix} s_1 \\ x_3 \\ x_1 \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \qquad \begin{bmatrix} x_2 \\ s_2 \\ s_3 \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Reformat the LP in terms of B^{-1} and x_{NBV}

$$z = \mathbf{c}_{BV}\mathbf{B}^{-1}\mathbf{b} + (\mathbf{c}_{NBV} - \mathbf{c}_{BV}\mathbf{B}^{-1}\mathbf{N})\mathbf{x}_{NBV} \qquad (4)$$
$$\mathbf{x}_{BV} + \mathbf{B}^{-1}\mathbf{N}\mathbf{x}_{NBV} = \mathbf{B}^{-1}\mathbf{b}$$
$$\mathbf{x}_{NBV} \geq 0$$

$$\mathbf{B}^{-1}\mathbf{B}\mathbf{x}_{BV} + \mathbf{B}^{-1}\mathbf{N}\mathbf{x}_{NBV} = \mathbf{B}^{-1}\mathbf{b} \Rightarrow \mathbf{x}_{BV} + \mathbf{B}^{-1}\mathbf{N}\mathbf{x}_{NBV} = \mathbf{B}^{-1}\mathbf{b}$$
$$z = \mathbf{c}_{BV}(\mathbf{B}^{-1}\mathbf{b} - \mathbf{B}^{-1}\mathbf{N}\mathbf{x}_{NBV}) + \mathbf{c}_{NBV}\mathbf{x}_{NBV}$$
$$= \mathbf{c}_{BV}\mathbf{B}^{-1}\mathbf{b} + (\mathbf{c}_{NBV} - \mathbf{c}_{BV}\mathbf{B}^{-1}\mathbf{N})\mathbf{x}_{NBV}$$

(Revised) constraints: Column of a nonbasic variable $x_j = \mathbf{B}^{-1}\mathbf{a}_j$; rhs of constraints = $\mathbf{B}^{-1}\mathbf{b}$

Revised objective function (rof) in (4): coefficient of nonbasic variable x_j is (coefficient of x_j in $\mathbf{c}$) $- \mathbf{c}_{BV}\mathbf{B}^{-1}$(column of $\mathbf{A}$ for x_j) $= c_j - \mathbf{c}_{BV}\mathbf{B}^{-1}\mathbf{a}_j$; the objective value for the bfs $= \mathbf{c}_{BV}\mathbf{B}^{-1}\mathbf{b}$.

In (4), the basic variables are eliminated. The coefficients for basic variables in rof are always 0 because their effect on z is counted in the constant.

Dakota Example

$$
\begin{aligned}
z &= 60x_1 + 30x_2 + 20x_3 + 0s_1 + 0s_2 + 0s_3 \\
& \quad 8x_1 + 6x_2 + x_3 + s_1 = 48 \\
& \quad 4x_1 + 2x_2 + 1.5x_3 + s_2 = 20 \\
& \quad 2x_1 + 1.5x_2 + 0.5x_3 + s_3 = 8
\end{aligned}
$$

$\mathbf{a}_2$ $\qquad$ $\mathbf{b}$

$$
\begin{aligned}
z &= 280 - 5x_2 - 10s_2 - 10s_3 \\
& \quad -2x_2 + s_1 + 2s_2 - 8s_3 = 24 \\
& \quad -2x_2 + x_3 + 2s_2 - 4s_3 = 8 \\
& \quad x_1 + 1.25x_2 - 0.5s_2 + 1.5s_3 = 2
\end{aligned}
$$

$\mathbf{B}^{-1}\mathbf{a}_2$ $\qquad$ $\mathbf{B}^{-1}\mathbf{b}$

The basic varaibles are s_1, x_3, x_1.

Thus, $\mathbf{B}^{-1} = \begin{bmatrix} 1 & 2 & -8 \\ 0 & 2 & -4 \\ 0 & -0.5 & 1.5 \end{bmatrix}$. Thus, $\underbrace{\begin{bmatrix} s_1 \\ x_3 \\ x_1 \end{bmatrix}}_{\mathbf{x}_{BV}} + \underbrace{\begin{bmatrix} -2 & 2 & -8 \\ -2 & 2 & -4 \\ 1.25 & -0.5 & 1.5 \end{bmatrix} \begin{bmatrix} x_2 \\ s_2 \\ s_3 \end{bmatrix}}_{\mathbf{B}^{-1}\mathbf{N}\mathbf{x}_{NBV}} = \underbrace{\begin{bmatrix} 24 \\ 8 \\ 2 \end{bmatrix}}_{=\mathbf{B}^{-1}\mathbf{b}}$

e.g. column $x_2 = \mathbf{B}^{-1}\mathbf{a}_2$

$$= \begin{bmatrix} 1 & 2 & -8 \\ 0 & 2 & -4 \\ 0 & -0.5 & 1.5 \end{bmatrix} \cdot \begin{bmatrix} 6 \\ 2 \\ 1.5 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \\ 1.25 \end{bmatrix}$$

rhs of the constraints = $\mathbf{B}^{-1}\mathbf{b}$

$$= \begin{bmatrix} 1 & 2 & -8 \\ 0 & 2 & -4 \\ 0 & -0.5 & 1.5 \end{bmatrix} \cdot \begin{bmatrix} 48 \\ 20 \\ 8 \end{bmatrix} = \begin{bmatrix} 24 \\ 8 \\ 2 \end{bmatrix}$$

Because $\mathbf{c}_{\mathrm{BV}} = [0,20,60]$, then $\mathbf{c}_{\mathrm{BV}}\mathbf{B}^{-1} = [0 \quad 10 \quad 10]$.

coefficient of x_2 in rof $= c_2 - \mathbf{c}_{\mathrm{BV}}\mathbf{B}^{-1}\mathbf{a}_2 = 30 - [0,10,10]\begin{bmatrix} 6 \\ 2 \\ 1.5 \end{bmatrix} = -5$;

coefficient of s_2 in rof $= 0 - \mathbf{c}_{\mathrm{BV}}\mathbf{B}^{-1}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = -10$;

similarly, coefficient of s_3 is -10;

the constant in rof is $\mathbf{c}_{\mathrm{BV}}\mathbf{B}^{-1}\mathbf{b} = [0 \quad 10 \quad 10]\begin{bmatrix} 48 \\ 20 \\ 8 \end{bmatrix} = 280$.

The basic varaibles x_1, x_3, s_1 have zero coefficients in rof.

Framework of the Simplex Algorithm for a Max LP

Step 1 Convert the LP to canonical form.

Step 2 Obtain a initial bfs from canonical form. This is easy if all the constraints are ≤ with nonnegative rhs. Then the slack variable s_i may be used as the basic variable for constraint *i*. If no bfs is readily apparent, then use the Big M method or Two-phase method to find an initial bfs.

Step 3 If all nonbasic variables have nonpositive coefficients in the revised objective function, then the current bfs is optimal, and STOP. Otherwise, choose the entering variable, which is with the (most) positive coefficient in the revised objective function, to enter the basis.

Step 4 With respect to the entering variable, use the ratio test to determine the leaving variable (tie may be broken arbitrarily).

Step 5 Find the new adjacent bfs using the new set of basic variables (switching the entering and leaving variables). Go back to step 3.

Example

Decision Variables

x_1=number of desks produced

x_2=number of tables produced

x_3=number of chairs produced

LP: $\max z = 60x_1 + 30x_2 + 20x_3$

$$
\begin{array}{lrrrll}
\text{s.t.} & 8x_1 + & 6x_2 + & x_3 & \le 48 & \text{(lumber constraint)} \\
 & 4x_1 + & 2x_2 + & 1.5x_3 & \le 20 & \text{(finishing constraint)} \\
 & 2x_1 + & 1.5x_2 + & 0.5x_3 & \le 8 & \text{(carpentry constraint)} \\
 & & x_2 & & \le 5 & \text{(table demand constraint)} \\
 & & & x_1, x_2, x_3 & \ge 0 &
\end{array}
$$

Step 1- Convert the LP to Canonical Form

Canonical Form

		Basic Variables
rof	$z = 60x_1 + 30x_2 + 20x_3$	
Row 1	$8x_1 + 6x_2 + x_3 + s_1 = 48$	$s_1=48$
Row 2	$4x_1 + 2x_2 + 1.5x_3 + s_2 = 20$	$s_2=20$
Row 3	$2x_1 + 1.5x_2 + 0.5x_3 + s_3 = 8$	$s_3=8$
Row 4	$x_2 + s_4 = 5$	$s_4=5$

Canonical Form: A system of linear equations in which each equation has a variable with a coefficient 1 in that equation and a zero coefficient in all other equations.

If the rhs of each constraint in a canonical form is **nonnegative**, a bfs can be obtained by *inspection (most of cases, it cannot)*.

Step 2 - Obtain a bfs

- Each basic variable may be associated with the constraint of the canonical form in which the basic variable has a coefficient of 1.
- The initial bfs has $BV=\{s_1, s_2, s_3, s_4\}$, $NBV=\{x_1, x_2, x_3\}$
- For this initial bfs $s_1=48$, $s_2=20$, $s_3=8$, $s_4=5$, $x_1=x_2=x_3=0$. $z=0$.
- If the initial BFS is not ready, use Big-M method or Two-Phase method.

Step 3 – Determine if the current bfs is optimal and choose the entering variable

• To do this, we try to determine if there is any way z can be increased by increasing some nonbasic variable from its current value of zero while holding all other nonbasic variables at their current values of zero.

$$z=60x_1 + 30x_2 + 20x_3$$

For each nonbasic variable, we can use the equation above to determine if increasing a nonbasic variable (while holding other nonbasic variables to zero) will increase z.

• Increasing any of the nonbasic variables will cause an increase in z; *thus not optimal*. However, increasing x_1 causes the greatest rate of increase in z.

If x_1 increases from its current value of zero, it will have to become a basic variable. For this reason, x_1 is called the entering variable.

Optimality Rule – A canonical form is optimal (for a max problem) if each nonbasic variable has a **nonpositive coefficient** in the rof.

We choose the **entering variable** (in a max problem) to be the nonbasic variable with the positive coefficient in the rof (ties broken arbitrarily).

Step 4 – The Ratio Test for determining the leaving variable

- We desire to make entering variable x_1 as large as possible but as we do, the current basic variables (s_1,s_2,s_3,s_4) will change value. Thus, increasing x_1 may cause a basic variable to become negative (i.e., the basic solution is not feasible).

From row 1, we see that s_1=48-8x_1. Since $s_1 \geq 0$, $x_1 \leq 48/8 = 6$

From row 2, we see that s_2=20-4x_1. Since $s_2 \geq 0$, $x_1 \leq 20/4 = 5$

From row 3, we see that s_3=8-2x_1. Since $s_3 \geq 0$, $x_1 \leq 8/2 = 4$

From row 4, we see that s_4=5. For any x_1, s_4 will always be ≥ 0

✓ This means to keep all the basic variables nonnegative, the largest we can make x_1 is min{6,5,4} = 4.

Step 4 – The Ratio Test for determining the leaving variable

- If the entering variable has a nonpositive coefficient in a revised constraint (such as x_1 in constraint 4), the basic variable in the constraint will remain positive for all values of the entering variable.
- If the entering variable has a positive coefficient in a row, the row's basic variable became negative when the entering variable exceeded the ratio (*)
- **The Ratio Test:** When entering a variable into the basis, compute the ratio

$$\frac{\text{rhs of row}}{\text{coefficient of entering variable in row}} \qquad (*)$$

for every revised constraint in which the entering variable has a positive coefficient. (if negative or 0, do not do Ratio test; put NONE in ratio column.)

- **The constraint with the smallest ratio is called the winner to the ratio test**.
- **The smallest ratio is the largest value of the entering variable** that will keep all the current basic variables nonnegative.

Step 5 – Find a new bfs: Pivot in the entering variable

Always make the entering variable a basic variable in a constraint that wins the ratio test (ties may be broken arbitrarily).

$BV=\{s_1,s_2,x_1,s_4\}$ and $NBV=\{s_3,x_2,x_3\}$, yielding the bfs $s_1=16$, $s_2=4$, $x_1=4$, $s_4=5$, $s_3=x_2=x_3=0$. $z=240$.

The procedure going from one bfs to a better **adjacent bfs** is called an **iteration** (or pivot) of the simplex algorithm.

Begin iteration 2 – with the new current bfs

- Return to **Step 3** to determine if the current bfs is optimal.

Rearranging the objective function

$$z = 240 - 15x_2 + 5x_3 - 30s_3$$

$BV=\{s_1,x_3,x_1,s_4\}$ and $NBV=\{s_3,s_2,x_2\}$, yielding the bfs $s_1=24$, $x_3=8$, $x_1=2$, $s_4=5$, $s_2=s_3=x_2=0$. $z=280$

Now, we need to determine if the current bfs is optimal. Rearranging the objective function:

$$z = 280 - 5x_2 - 10s_2 - 10s_3$$

Increasing x_2, s_2, or s_3 while holding other *NBV*s to zero will not cause the value of *z* to increase. The solution at the end of iteration 2 is therefore optimal.

Revised Simplex Algorithm: Updating B^{-1}

- The revised simplex method is based directly on the matrix form of the simplex method. However, the difference is that the revised simplex method incorporates a key improvement into the matrix form. Instead of needing to invert the new basis matrix **B** after each iteration, which is computationally expensive for large matrices, the revised simplex method uses a much more efficient procedure to simply updates $\mathbf{B}^{-1}$ from one iteration to the next. Next, we describe this procedure.

- x_k = entering basic variable;
- a'_{ik} = revised coefficient of x_k in current row (i), for $i = 1,2, \ldots, m$, i.e., revised coefficient column vector $(a'_{1k}, a'_{2k}, \ldots, a'_{mk})^{-1}$ corresponding to x_k in $\mathbf{B}_{\text{old}}^{-1}$;
- r = index of row containing the leaving basic variable.

- $\mathbf{E}\mathbf{a}'_k = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & & -\frac{a'_{1k}}{a'_{rk}} & 0 & & 0 \\ 0 & 1 & & -\frac{a'_{2k}}{a'_{rk}} & 0 & & 0 \\ \vdots & 0 & \cdots & \vdots & & \cdots & \vdots \\ \vdots & \vdots & & \frac{1}{a'_{rk}} & 1 & & \vdots \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & & -\frac{a'_{mk}}{a'_{rk}} & 0 & & 1 \end{bmatrix} \begin{bmatrix} a'_{1k} \\ a'_{2k} \\ \vdots \\ a'_{rk} \\ a'_{(r+1)k} \\ \vdots \\ a'_{mk} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

Finding $\mathbf{B}_{\text{new}}^{-1} = \mathbf{E}\mathbf{B}_{\text{old}}^{-1}$

- This procedure can be expressed in terms of matrix as

$$\mathbf{B}_{\text{new}}^{-1} = \mathbf{E}\mathbf{B}_{\text{old}}^{-1}$$

where matrix $\mathbf{E} = [\mathbf{U}_1, \mathbf{U}_2, \dots, \mathbf{U}_{r-1}, \boldsymbol{\eta}, \mathbf{U}_{r+1}, \dots, \mathbf{U}_m]$. The m elements of column vector $\mathbf{U}_i$ are 0 expect for a 1 in the ith position. That is, matrix $\mathbf{E}$ is an identify matrix except that its rth column is replaced by the vector

$$\boldsymbol{\eta} = \begin{bmatrix} \eta_1 \\ \eta_2 \\ \vdots \\ \eta_m \end{bmatrix}, \text{ where } \eta_m = \begin{cases} -\dfrac{a'_{ik}}{a'_{rk}} & if\ i \neq r \\ \dfrac{1}{a'_{rk}} & if\ i = r. \end{cases}$$

The Revised Simplex Algorithm for a min LP

Method 1: convert to max problem

Method 2

A simple modification of the simplex algorithm for solving max LPs can be used directly to solve min LPs.

<u>Modify Step 3 of the simplex algorithm as follows:</u>

If all nonbasic variables in the rof have nonnegative coefficients, the current bfs is optimal. If any nonbasic variable has a negative coefficient in the rof, choose the variable with the (most) negative coefficient in the revised objective function as the entering variable.

Special cases:
Alternative Optimal Solutions
Unbounded LPs

Alternative Optimal Solutions

For some LPs, more than one extreme point (optimal solution) is optimal -- Multiple or alternative optimal solutions

<u>The simplex algorithm can determine whether an LP has alternative optimal solutions</u>

If there is no nonbasic variable with a zero coefficient in the optimal rof, the LP has a unique optimal solution.

If there is a nonbasic variable with a zero coefficient in the optimal rof, it is possible that the LP may have alternative optimal solutions.

Alternative Optimal Solutions

Solve the following LP using simplex method:

$$\begin{aligned} &\max z = 2x_1 + 2x_2 \\ &\text{s.t.} \quad x_1 + x_2 \le 6 \\ &\qquad 2x_1 + x_2 \le 13 \\ &\qquad\quad x_1, x_2 \ge 0 \end{aligned}$$

Standard form:

$$\begin{aligned} &\max z = 2x_1 + 2x_2 \\ &\text{s.t.} \quad x_1 + x_2 + s_1 = 6 \\ &\qquad 2x_1 + x_2 + s_2 = 13 \\ &\qquad\quad x_1, x_2, s_1, s_2 \ge 0 \end{aligned}$$

Unbounded LPs

For some LPs, there exist points in the feasible region for which z assumes arbitrarily large (in max problem) or arbitrarily small (in min problem) values. When this occurs, we say the LP is unbounded.

An unbounded LP occurs in a max problem if there is a nonbasic variable with a positive coefficient in the rof and there is no constraint that limits how large we can make this nonbasic variable.

Specially, an unbounded LP for a max problem occurs when a variable with a positive coefficient in the rof has a non-positive coefficient in each revised constraint – the ratio test fails!

Unbounded LPs

Solving the following LP using the simplex method:

Standard form

$$\begin{aligned} \max\ & z = x_1 + 2x_2 \\ \text{s.t.}\ & -x_1 + x_2 \le 2 \\ & -2x_1 + x_2 \le 1 \\ & x_1, x_2 \ge 0 \end{aligned}$$

$$\begin{aligned} \max\ & z = x_1 + 2x_2 \\ \text{s.t.}\ & -x_1 + x_2 + s_1 = 2 \\ & -2x_1 + x_2 + s_2 = 1 \\ & x_1, x_2, s_1, s_2 \ge 0 \end{aligned}$$

We would like s_2 to enter the basis but there is no revised constraint in which s_2 has a positive coefficient. Therefore, the **LP is unbounded**.

The Big-M Method

The Big M Method

- Recall that the (revised) simplex algorithm requires a starting bfs. In all the LPs we have considered so far, we have found starting bfs by using the slack variables as our basic variables.
- If an LP has ≥ or = constraints, however, a starting bfs may not be readily apparent. In such a case, we need another method to solve the problem.
- Big-M method: a version of the simplex algorithm that first finds a bfs by adding "*artificial* variables" to the problem.
- The objective function of the original LP must be modified to ensure that the artificial variables are all equal to 0 at the conclusions of the revised simplex algorithm.

Example

Consider this LP

$$
\begin{aligned}
\text{Min } & z = 2x_1 + 3x_2 \\
\text{s.t. } & 0.5x_1 + 0.25x_2 \le 4 \\
& x_1 + 3x_2 \ge 20 \\
& x_1 + x_2 = 10 \\
& x_1, x_2 \ge 0
\end{aligned}
$$

Converting to standard form:

$$
\begin{aligned}
\text{Min } & z = 2x_1 + 3x_2 \\
\text{s.t. } & 0.5x_1 + 0.25x_2 + s_1 = 4 \\
& x_1 + 3x_2 - e_2 = 20 \\
& x_1 + x_2 = 10 \\
& x_1, x_2, s_1, e_2 \ge 0
\end{aligned}
$$

The LP in standard form has z and s_1 which could be used as basic variables. But constraint 2 would violate sign restriction and constraint 3 has no readily apparent basic variable.

Artificial variables

In order to use the simplex algorithm, a bfs is needed.

The idea of the Big M method is to create artificial variables to provide an initial bfs and then eliminate them from the final solution.

The variables will be labeled a_i according to the row i in which they are used as follows:

$$\boxed{\begin{array}{lllllll}
\text{Row 0: } z = & 2x_1 + & 3x_2 & & & & \\
\text{Row 1: } & 0.5x_1 + & 0.25x_2 & + s_1 & & & = 4 \\
\text{Row 2: } & x_1 + & 3x_2 & & - e_2 + a_2 & & = 20 \\
\text{Row 3: } & x_1 + & x_2 & & & + a_3 & = 10
\end{array}}$$

Now, we have a bfs: s_1=4, a_2=20, a_3 =10.

Artificial variables (cont'd)

To make sure that the optimal solution to the problem with artificial variables solves the original LP, all artificial variables in the optimal solution must be equal to 0.

To accomplish this, in a min LP, a term Ma_i is added to the objective function for each artificial variable a_i. For a max LP, the term $-Ma_i$ is added to the objective function for each a_i. M represents some very large number.

The previous example is modified as follow:

$$
\boxed{
\begin{array}{lllllllll}
\min\ z = & 2x_1 + & 3x_2 & & & + Ma_2 & + Ma_3 & & \\
\text{s.t.} & 0.5x_1 + & 0.25x_2 & + s_1 & & & & = & 4 \\
 & x_1 + & 3x_2 & & - e_2 & + a_2 & & = & 20 \\
 & x_1 + & x_2 & & & & + a_3 & = & 10 \\
 & \multicolumn{8}{l}{x_1, x_2, s_1, e_2, a_2, a_3 \geq 0}
\end{array}
}
$$

Description of the Big M Method

Step 1 Modify the constraints so that the right-hand side of each constraint is nonnegative. Identify each constraint that is now an = or ≥ constraint.

Step 2 Convert to standard form (add a slack variable for ≤ constraints; subtract an excess variable for ≥ constraints; $x_i=x_i'-x_i''$ for urs x_i).

Step 3 For each ≥ or = constraint, add artificial variables, a_i and sign restriction $a_i \geq 0$.

Step 4 Let M denote a very large positive number. Add (for each artificial variable a_i) Ma_i to min problem objective functions or $-Ma_i$ to max problem objective functions.

Step 5 Solve the transformed problem by the revised simplex algorithm.

The solutions for the original LP

The big-M method:

- If all artificial variables in the optimal solution equal zero, the solution is optimal.
- If any artificial variables are positive in the optimal solution, the original problem is infeasible.

Description of The Two-phase Simplex Method

Phase I:

Steps 1 – 3： the same as the Big-M method

Step 4: Min the sum of all artificial variables

Step 5: Solve the transformed problem by the simplex algorithm.

If all artificial variables in the optimal solution of Phase I equal zero, the original LP is feasible and go to Phase II. Otherwise, the original LP is infeasible.

Phase II:

Step 1: Remove all artificial variables that are nonbasic variables, i.e., setting artificial variables to zero, (and the nonbasic variables that have a positive coefficient in the optimal solution of Phase I) from the final format of the Phase I model.

Step 2: Solve the problem with the original objective function and the revised constraints from Phase I by the revised simplex algorithm.

Example: Infeasible LP

Use the Big M method or two-phase method to solve the following LP.

$$
\begin{aligned}
\min \quad & z = 2x_1 + 3x_2 \\
\text{s.t.} \quad & \tfrac{1}{2}x_1 + \tfrac{1}{4}x_2 \le 4 \\
& x_1 + 3x_2 \ge 36 \\
& x_1 + x_2 = 10 \\
& x_1, x_2 \ge 0
\end{aligned}
$$

The optimal solution for the transformed LP is z=30+6M, s_1=3/2, a_2=6, x_2=10, a_3=e_2=x_1=0. Thus, the original LP has no feasible solution.

Final words

The big M method or two-phase method is a simplex method with a method to find an initial bfs for any LP or judge the infeasibility.

$$\text{Big } M \begin{cases} \text{find a bfs} \\ \text{(all artificial variables=0)} & \begin{cases} \text{unique optimal solution} \\ \text{alterntive optimal solution} \\ \text{unbounded} \end{cases} \\ \text{no feasible solution (at least one artificial variable} \neq 0) \end{cases}$$

Issues of the Big M Method and Two-Phase simplex method

1. It is difficult to determine how large M should be (Generally chosen to be at least 100 times the largest coefficient in the objective function).
2. Introduction of such large numbers causes round off errors and other computational difficulties (numerical instability).

This motivates the need for the **Two-Phase Simplex Method**

Degeneracy and Convergence

- Theoretically, the simplex algorithm can fail to find the optimal solution to an LP.
- LPs arising from actual applications seldom exhibit this unpleasant behavior.
- Now, discuss the situation in which the simplex can fail.

Fundamentals

Consider the following relationship for a Max LP:

z-value for new bfs = *z*-value for current bfs + (coefficient of entering variable in the rof of current bfs) $\times$ (value of entering variable in new bfs)

Recall: (coefficient of entering variable in the rof) > 0 and
(value of entering variable in new bfs) >= 0.

Two Facts:

(1) If (value of entering variable in new bfs) > 0, then
(*z*-value for new bfs) > (*z*-value for current bfs);

(2) If (value of entering variable in new bfs) = 0, then
(*z*-value for new bfs) = (*z*-value for current bfs).

Definitions

- An LP is nondegenerate if in each of the LP's bfs's, all of the basic variables are positive.

  For a nondegenerate LP, Fact 1 implies that each iteration of the simplex will *strictly increase z*, and consequently, it is impossible to encounter the same bfs twice. So for solving a nondegenerate LP, the simplex algorithm is guaranteed to find the optimal solution in a finite number of iterations.

- **Definition**: Any bfs that has at least one basic variable equal to zero is a degenerate bfs.

  An LP is degenerate if it has at least one bfs in which a basic variable is equal to zero.

Degeneracy and Convergence

- For an LP with n decision variables to be degenerate, **(n+1) or more of the LP's constraints** (including the sign restrictions as constraints) must be binding at an extreme point.
- Such an extreme point (at which three or more constraints are binding) corresponds to more than one set of basic variables.
- Such an extreme point corresponds to a **degenerate bfs.**

Example: degenerate LP

Solve the following LP using the simplex method.

$$\begin{aligned}
\max\ & z = 5x_1 + 2x_2 \\
\text{s.t.}\ & x_1 + x_2 \le 6 \\
& x_1 - x_2 \le 0 \\
& x_1, x_2 \ge 0
\end{aligned}$$

Converting to standard form:

$$\begin{aligned}
\max\ & z = 5x_1 + 2x_2 \\
\text{s.t.}\ & x_1 + x_2 + s_1 \qquad = 6 \\
& x_1 - x_2 \qquad + s_2 = 0 \\
& x_1, x_2, s_1, s_2 \ge 0
\end{aligned}$$

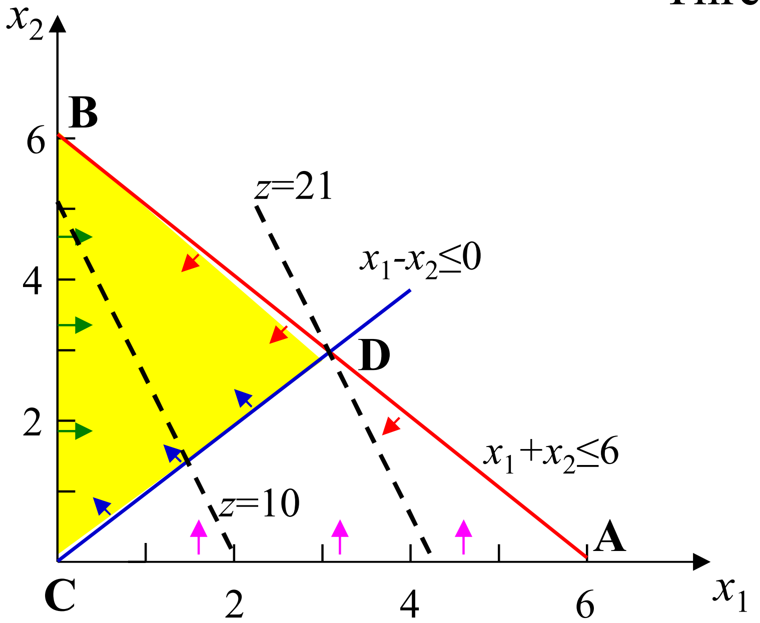
Example (cont'd)

- Iteration 1: $z = 0 + 5x_1 + 2x_2$
  BV: $s_1 = 6$, $s_2 = 0$ (degenerate BFS)
  entering variable: x_1, leaving variable: s_2
- Iteration 2: $z = 0 + 7x_2 - 5s_2$
  BV: $s_1 = 6$, $x_1 = 0$ (degenerate BFS)
  entering variable: x_2, leaving variable: s_1
- Iteration 3: $z = 21 - 3.5s_1 - 1.5s_2$
  BV: $x_2 = 3$, $x_1 = 3$ (optimal BFS)

Inefficiency

If an LP has many degenerate bfs (or a bfs with many basic variables equal to zero), then the simplex algorithm is often very inefficient.

Example: Solve the previous example using the graphical method



The extreme points of the feasible region are B, C, and D. Three sets of basic variables correspond to extreme point C.

Basic variables	BFS	Extreme point
x_1,x_2	x_1=x_2=3, s_1=s_2=0	D
x_1,s_1	x_1=0, s_1=6, x_2=s_2=0	C
x_1,s_2	x_1=6, s_2=-6, x_2=s_1=0	infeasible
x_2,s_1	x_2=0, s_1=6, x_1=s_2=0	C
x_2,s_2	x_2=6, s_2=6, x_1=s_1=0	B
s_1,s_2	s_1=6, s_2=0, x_1=x_2=0	C

Cycling

- Fact 2 implies that the simplex algorithm may have problem for solving a degenerate LP. It is possible that the algorithm will encounter the same bfs at least twice. This occurrence may cause cycling. If cycling occurs then the algorithm can loop, or cycle, forever among a set of bfs and never get to the optimal solution!

The simplex method can be modified to ensure that cycling will never occur. For example, apply the so called anti-cycling rules, e.g., Bland's rule.

Bland’s rule

Bland’s rule showed that cycling can be avoided by applying the following rules (assume that slack and excess variables are numbered $x_{n+1}, x_{n+2}, \ldots$):

- Choose as the entering variable (in a max problem) the variable with a positive coefficient in rof that has the smallest subscript.
- (Tie breaking for the leaving basic variable) If there is a tie in the ratio test, then break the tie by choosing the winner of the ratio test so that the variable leaving the basis has the smallest subscript.

Large-scale LP problems

1. Modeling language
2. Solvers
3. Interior-point methods, which can not do sensitive analysis
4. Decomposition methods; column generations; Bender's decompositions
5. Self-designed algorithms (based on the special structure of the LP models)

Using LINDO and Excel Solver to Solve LPs

- LINDO: Linear Interactive and Discrete Optimizer (1986)
- A user-friendly computer package
- Linear, integer, and quadratic programming problems

An Example

Decision Variables

x_1=number of desks produced

x_2=number of tables produced

x_3=number of chairs produced

LP: $\max z = 60x_1 + 30x_2 + 20x_3$

$$
\begin{array}{lrrrll}
\text{s.t.} & 8x_1 + & 6x_2 + & x_3 & \leq 48 & \text{(lumber constraint)} \\
& 4x_1 + & 2x_2 + & 1.5x_3 & \leq 20 & \text{(finishing constraint)} \\
& 2x_1 + & 1.5x_2 + & 0.5x_3 & \leq 8 & \text{(carpentry constraint)} \\
& & x_2 & & \leq 5 & \text{(table demand constraint)} \\
& & & x_1, x_2, x_3 & \geq 0 &
\end{array}
$$

LINDO – input

Input the following lines in a Model Window:

```
max 60 x1 + 30 x2 + 20 x3
st
   8 x1 + 6 x2 + x3 < 48
   4 x1 + 2 x2 + 1.5 x3 < 20
   2 x1 + 1.5 x2 + 0.5x3 < 8
   x2 < 5
```

- New from the File menu
- Save from the File menu
- Solve command from the Solve menu
- Reports – Tableau: ART (artificial variable in row 1 is *z*.)

LINDO

- Start all models with MAX or MIN
- Variable Names: Limited to 8 characters
- Recognized Operators: +, -, >, <, =, >=, <=

  Note that LINDO interprets the < symbol as meaning "less-than-or-equal-to" rather than "strictly less than". If you prefer, you may alternatively enter <= in place of the < character.
- Parentheses not recognized
- Right-hand Side Syntax: Only constant values
- Left-hand Side Syntax: Only variables and their coefficients
- Case Sensitivity: LINDO has none
- LINDO assumes that all variables are nonnegative, so the nonnegative constraints need not be input to the computer.

  For unrestricted variables: FREE x1

  For integer variables: GIN x1

Excel Solver

- To activate the Excel Solver for the first time, select Tools and then select Add-Ins, Check the Solver Add-in box.
- The key to solving an LP on a spreadsheet is to set up a spreadsheet that tracks everything of interest.
- See 4_ExcelSolver.xls
  - Solver and infeasible LPs
  - Solver and unbounded LPs
- Other functions: =MMULT and =MINVERSE.
- Reading assignment: Section 4.17 (=SUMPRODUCT)