Sensitivity Analysis

Based on

Chapter 6.1 – 6.3 of *Operations Research: Application and Algorithms*, 4th Edition.

Introduction

- Sensitivity Analysis (SA) and Duality are related and are two of most important topics in LP.
- SA is concerned with how changes in an LP's parameters affect the LP's optimal solution.
- In many applications, the values of an LP's parameters may change, for example, prices may change, demand is uncertain.
- The knowledge of SA often enables the analyst to determine from the original solution how changes in an LP's parameters change the optimal solution.

A Graphical Illustration of SA

Giapetto Example

		
Selling Price ($)	27	21
Raw materials cost ($)	10	9
Variable labor and overhead cost ($)	14	10
Finishing labor (hours)	2	1
Carpentry labor (hours)	1	1

Available finishing hours: 100

Available carpentry hours: 80

Max demand for soldiers: 40

x_1 = Number of soldiers produced each week

x_2 = Number of trains produced each week

$$\boxed{\begin{array}{ll} \max z = 3x_1 + 2x_2 & \\ \text{s.t.} & 2x_1 + x_2 \le 100 \\ & x_1 + x_2 \le 80 \\ & x_1 \qquad \le 40 \\ & x_1, x_2 \ge 0 \end{array}}$$

Standard Form

$$\boxed{\begin{array}{ll} \max z = 3x_1 + 2x_2 & \\ \text{s.t.} & 2x_1 + x_2 + s_1 = 100 \\ & x_1 + x_2 + s_2 = 80 \\ & x_1 + s_3 = 40 \\ & x_1, x_2, s_1, s_2, s_3 \ge 0 \end{array}}$$

Optimal Solution: z=180, x_1=20, x_2=60

$BV=\{x_1, x_2, s_3\}$, $NBV=\{s_1, s_2\}$

1. How would changes in the problem's objective function coefficients change this optimal solution?

Giapetto Example

$$\max\ z = 3x_1 + 2x_2 \quad (1)$$

$$\text{s.t.} \quad 2x_1 + x_2 \le 100 \quad (2)$$

$$x_1 + x_2 \le 80 \quad (3)$$

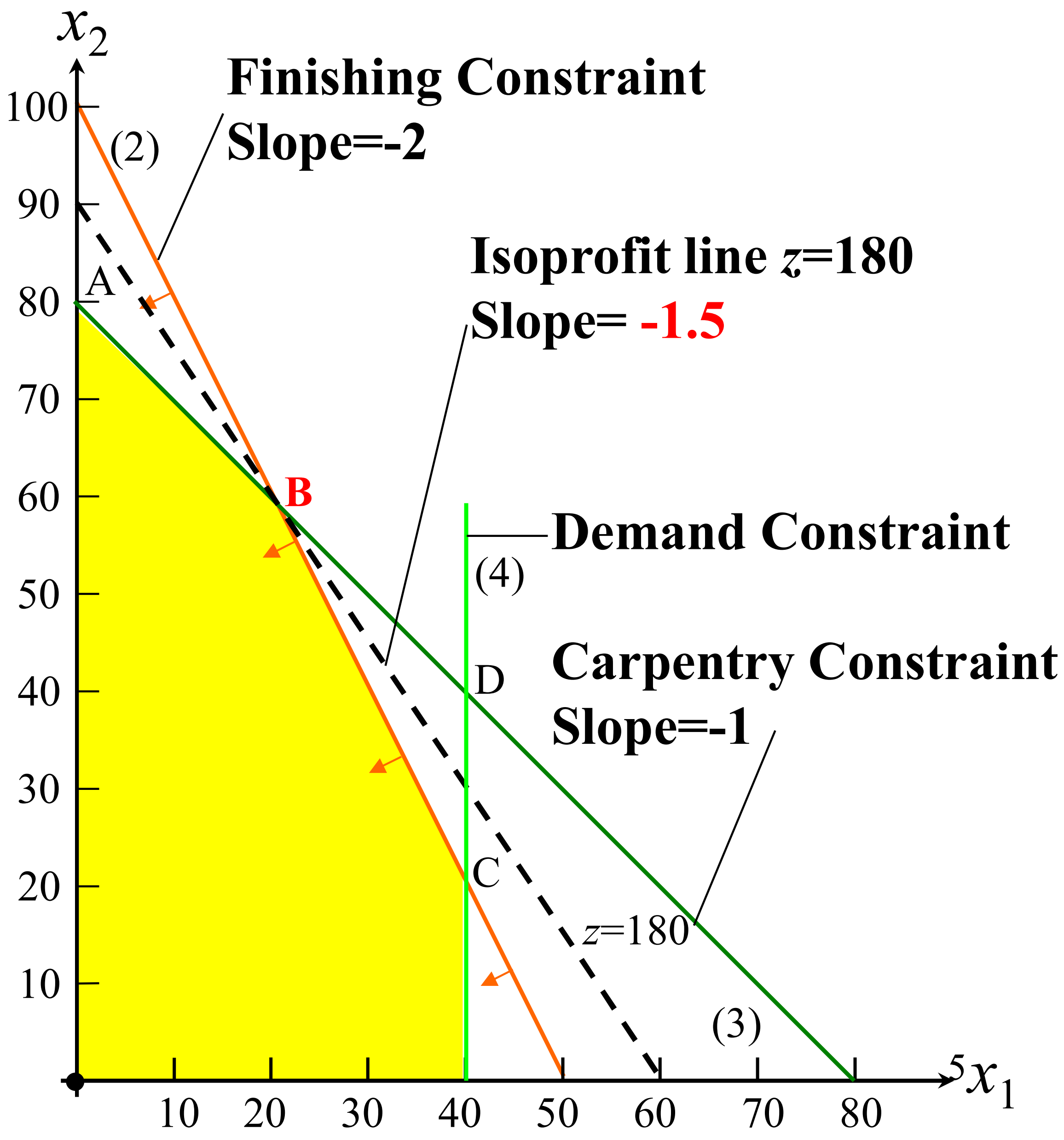
$$x_1 \le 40 \quad (4)$$

$$x_1 \ge 0 \quad (5)$$

$$x_2 \ge 0 \quad (6)$$

Optimal Point B:

$$\left.\begin{aligned} 2x_1 + x_2 &= 100 \\ x_1 + x_2 &= 80 \end{aligned}\right\} \Rightarrow \left\{\begin{aligned} x_1 &= 20 \\ x_2 &= 60 \end{aligned}\right.$$



If we write the objective function as Max $z = c_1 x_1 + 2x_2$, the slope of the isoprofit line is $-\frac{c_1}{2}$.

- Point A (0,80) is optimal if the isoprofit line is flatter than the carpentry constraint;
  i.e. $-c_1/2 \geq -1$ ($c_1 \leq 2$) because the carpentry constraint slope is -1.
- Point C (40,20) is optimal if the isoprofit line is steeper than the finishing constraint;
  i.e. $-c_1/2 \leq -2$ ($c_1 \geq 4$) because the finishing constraint slope is -2.
- Point B (20,60) remains optimal if isoprofit line is steeper than carpentry constraint but flatter than finishing constraint;
  i.e. $-2 \leq -c_1/2 \leq -1$ ($2 \leq c_1 \leq 4$) because between the slopes of the carpentry and finishing constraints.

Note that even the current basis (point B) remains optimal, but the optimal objective value changes as the c_1 changes. For example, if $c_1=4$, then total profit = 4(20) +2(60) = \$200 instead of \$180.

For example, $c_1 = 1 < 2$

$$\max \quad z = 1x_1 + 2x_2 \qquad (1)$$

s.t.

$$2x_1 + x_2 \le 100 \qquad (2)$$

$$x_1 + x_2 \le 80 \qquad (3)$$

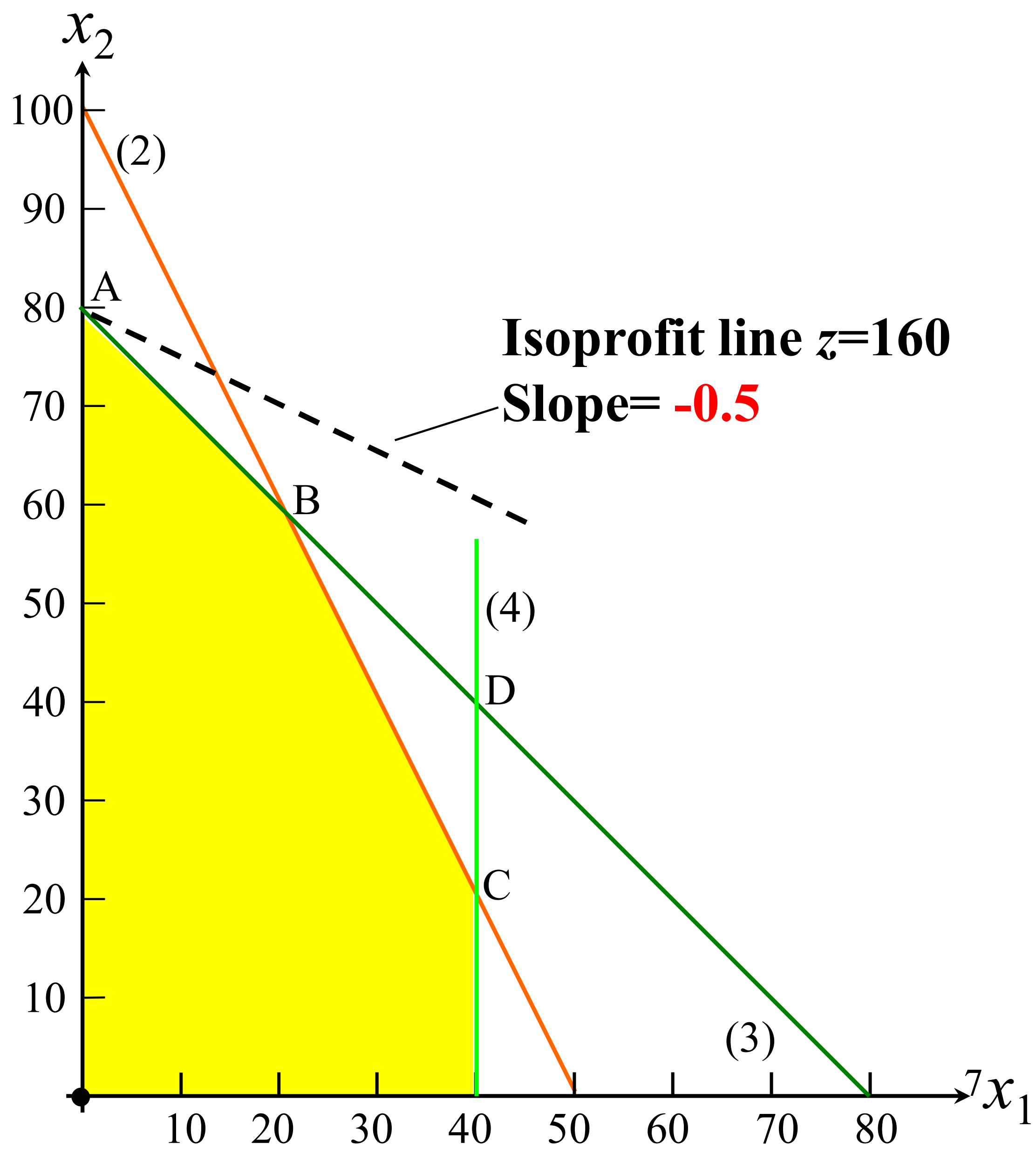
$$x_1 \le 40 \qquad (4)$$

$$x_1 \ge 0 \qquad (5)$$

$$x_2 \ge 0 \qquad (6)$$

Point A:

$$\left.\begin{array}{l} x_1 + x_2 = 80 \\ x_1 = 0 \end{array}\right\} \Rightarrow \left\{\begin{array}{l} x_1 = 0 \\ x_2 = 80 \end{array}\right.$$



For example, $c_1 = 6 > 4$

$$\max \quad z = 6x_1 + 2x_2 \qquad (1)$$

s.t.

$$2x_1 + x_2 \le 100 \qquad (2)$$

$$x_1 + x_2 \le 80 \qquad (3)$$

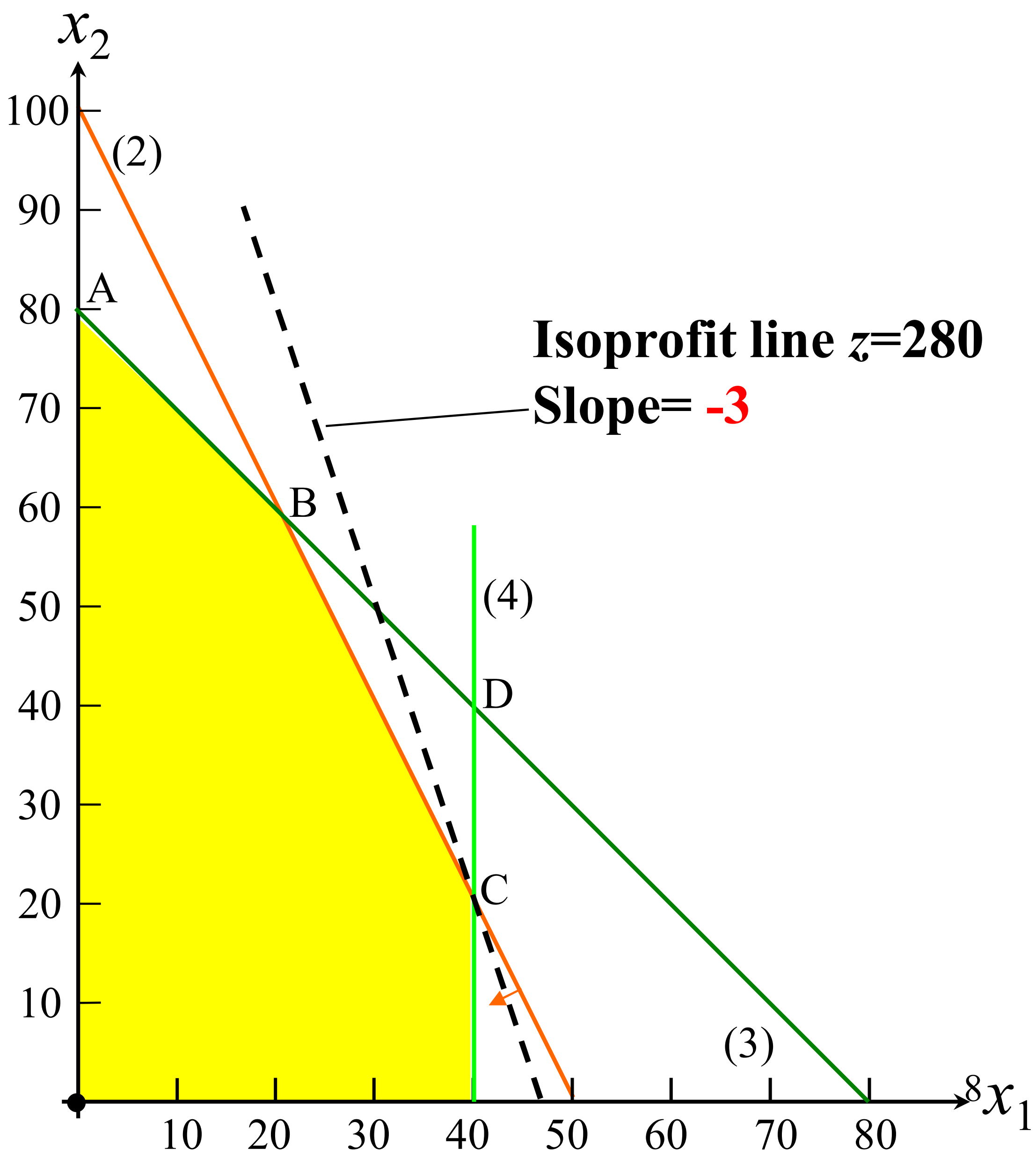
$$x_1 \le 40 \qquad (4)$$

$$x_1 \ge 0 \qquad (5)$$

$$x_2 \ge 0 \qquad (6)$$

Point C:

$$\left.\begin{array}{l} 2x_1 + x_2 = 100 \\ x_1 = 40 \end{array}\right\} \Rightarrow \left\{\begin{array}{l} x_1 = 40 \\ x_2 = 20 \end{array}\right.$$



2. Will a change in the rhs of a constraint make the current basis no longer optimal?

$$\max \quad z = 3x_1 + 2x_2 \qquad (1)$$

s.t.

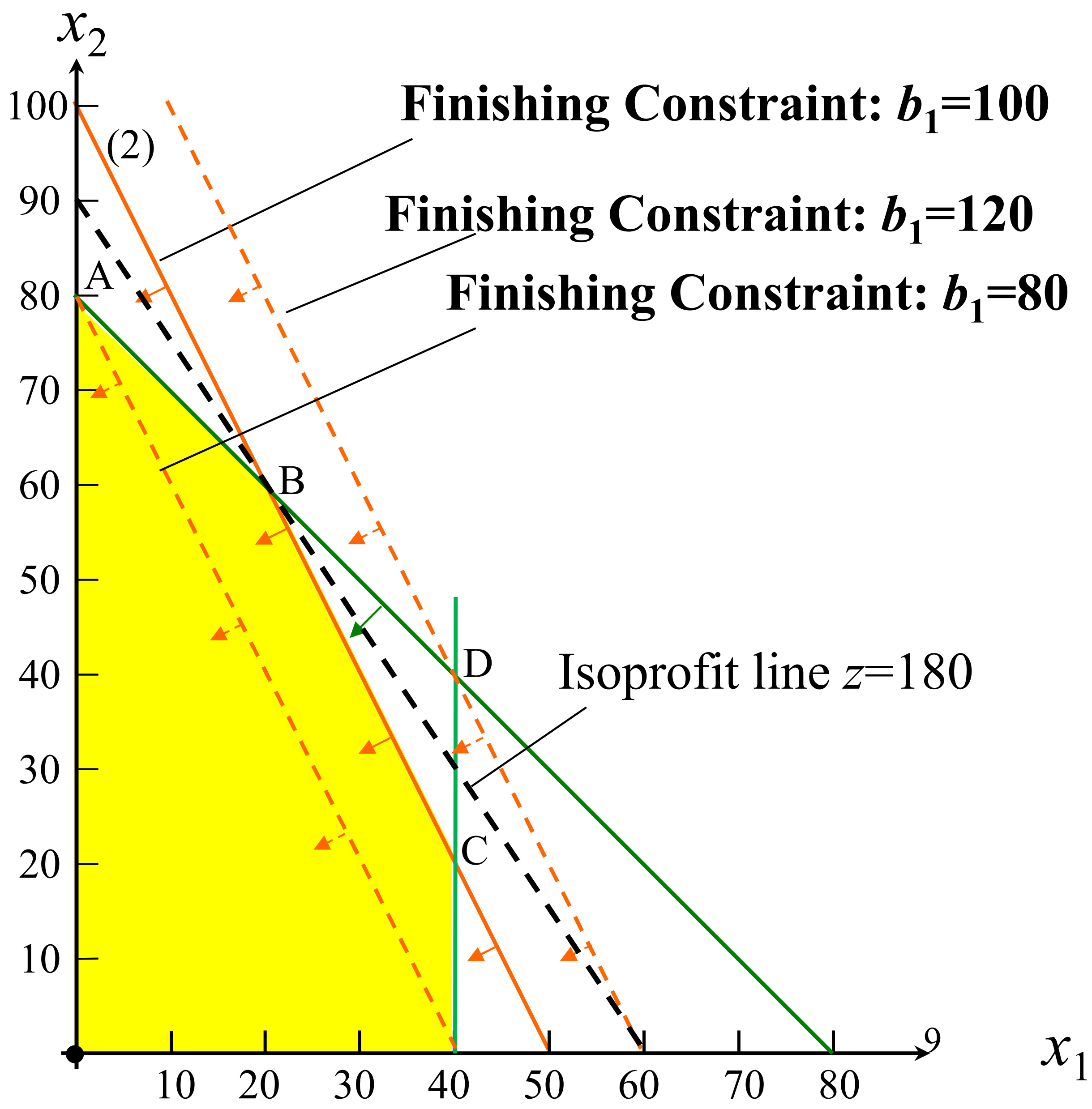
$$2x_1 + x_2 \le b_1 \qquad (2)$$

$$x_1 + x_2 \le 80 \qquad (3)$$

$$x_1 \le 40 \qquad (4)$$

$$x_1 \ge 0 \qquad (5)$$

$$x_2 \ge 0 \qquad (6)$$



(1). Binding constraint: finishing constraint (b_1)

- A change in b_1 shifts the finishing constraint parallel to its current position.
- The current optimal solution (Point B) is where the carpentry and finishing constraints are binding.
- If we change the value of b_1, then as long as the point where the finishing and carpentry constraints are binding remains feasible, the optimal solution will still occur where these constraints intersect.
- If $b_1 > 120$, x_1 will be greater than 40 and will violate demand constraint.
- If $b_1 < 80$, x_1 will be less than 0 and will violate sign restriction.
- Therefore, if $80 \leq b_1 \leq 120$, the current basis remains optimal.
- But the decision variable values and z-value will change.

(2) nonbinding constraint: soldiers demand constraint $x_1 \le b_3$.

If we change b_3 from 40 to $40 + \Delta$, it can be shown that current basis remains optimal for $\Delta \ge -20$. Note that as b_3 changes (as long as $\Delta \ge$ -20), the optimal solution is still the point where the finishing and carpentry constraints are binding. We can find the new values of the decision variables by solving

$$2x_1 + x_2 = 100 \text{ and } x_1 + x_2 = 80.$$

This yields $x_1 = 20$ and $x_2 = 60$.

➢In a constraint with positive slack (or excess) in an LP's optimal solution (in this instance s_3=20), i.e., the nonbinding constraint, if we change the rhs of the constraint to a value in the range where the current basis remains optimal, the optimal solution is unchanged.

Shadow Prices

It is important to determine how a constraint's rhs (available resources) changes the optimal *z*-value. Define

The **shadow price** *for the ith constraint* of an LP to be the amount by which the optimal *z*-value is improved (improvement means increase in a max problem and decrease in a min problem) if the rhs of the *i*th constraint is increased by 1.

This definition applied only if the change in the rhs of the *i*th constraint leaves the current basis optimal (i.e., binding constraints) .

For example, if 110 finishing hours are available, then Δb_i =10, and the new *z*-value = 180 +10(1)=190.

Whenever the slack or excess variable for a constraint is positive in an LP's optimal solution (in this instance s_3=20) (i.e., the constraint is nonbinding), the shadow price of the constraint = 0.

Sensitivity Analysis based on the derived formulas

Our discussion focuses on max problems. The modifications for min problems are straightforward.

Summary of a Simplex Algorithm

$$z = \mathbf{c}_{BV}(\mathbf{B}^{-1}\mathbf{b} - \mathbf{B}^{-1}\mathbf{N}\mathbf{x}_{NBV}) + \mathbf{c}_{NBV}\mathbf{x}_{NBV}$$

$$= \mathbf{c}_{BV}\mathbf{B}^{-1}\mathbf{b} - (\mathbf{c}_{BV}\mathbf{B}^{-1}\mathbf{N} - \mathbf{c}_{NBV})\mathbf{x}_{NBV}$$

$$\mathbf{x}_{BV} + \mathbf{B}^{-1}\mathbf{N}\mathbf{x}_{NBV} = \mathbf{B}^{-1}\mathbf{b}$$

- x_j column in constraints=$\mathbf{B}^{-1}\mathbf{a}_j$
- rhs of constraints=$\mathbf{B}^{-1}\mathbf{b}$
- reduced cost for x_j (- coefficient of x_j in rof) $\overline{c}_j = \mathbf{c}_{BV}\mathbf{B}^{-1}\mathbf{a}_j - c_j$(definition)
- rhs of rof = $\mathbf{c}_{BV}\mathbf{B}^{-1}\mathbf{b}$

Important Observation

The mechanisms of SA hinge on the following important observation:

A simplex algorithm (for a max problem) for a set of basic variables BV is **optimal** iff each constraint has a nonnegative rhs and each variable has a nonpositive coefficient in rof.

Mathematically, we have two conditions:

"Feasibility condition": $\mathbf{B}^{-1}\mathbf{b} \geq 0$ (Note $\mathbf{x}_{BV} = \mathbf{B}^{-1}\mathbf{b}, \mathbf{x}_{BV} \geq 0$)

"Optimality condition": $\overline{c}_j = \mathbf{c}_{BV}\mathbf{B}^{-1}\mathbf{a}_j - c_j \geq 0, \forall j \in NBV$

Sensitivity Analysis

Six types of changes in an LP's parameters :

1: Changing the objective function coefficient of a nonbasic variable

2: Changing the objective function coefficient of a basic variable

3: Changing the rhs of a constraint

4: Changing the column of a nonbasic variable

5: Adding a new variable or activity

Example

$$\max z = 60x_1 + 30x_2 + 20x_3 + 0s_1 + 0s_2 + 0s_3 \qquad \text{(Total revenue)}$$

$$\begin{aligned}
\text{s.t.} \quad & 8x_1 + 6x_2 + x_3 + s_1 = 48 && \text{(Lumber constraint)} \\
& 4x_1 + 2x_2 + 1.5x_3 + s_2 = 20 && \text{(Finishing constraint)} \\
& 2x_1 + 1.5x_2 + 0.5x_3 + s_3 = 8 && \text{(Carpentry constraint)} \\
& x_1, x_2, x_3, s_1, s_2, s_3 \geq 0 &&
\end{aligned}$$

x_1=number of desks manufactured

x_2=number of tables manufactured

x_3=number of chairs manufactured

Optimal solution

$$\mathbf{B} = \begin{bmatrix} 1 & 1 & 8 \\ 0 & 1.5 & 4 \\ 0 & 0.5 & 2 \end{bmatrix} \quad \mathbf{B}^{-1} = \begin{bmatrix} 1 & 2 & -8 \\ 0 & 2 & -4 \\ 0 & -0.5 & 1.5 \end{bmatrix} \quad \mathbf{x}_{BV} = \begin{pmatrix} s_1 \\ x_3 \\ x_1 \end{pmatrix} = \begin{pmatrix} 24 \\ 8 \\ 2 \end{pmatrix}$$

$$z^* = 280 - 10s_2 - 10s_3 - 5x_2$$

1. Changing the objective function coefficient of a nonbasic variable

- Since $\mathbf{B}$ and $\mathbf{b}$ are unchanged, $\mathbf{B}^{-1}\mathbf{b} \geq 0$ (Note $\mathbf{x}_{BV} = \mathbf{B}^{-1}\mathbf{b} \geq 0$)
  feasibility is not affected;
- $\mathbf{c}_{BV}$ is unchanged but c_j is changed (hence $\overline{c}_j$ may change),
  optimality is affected in this case.
- When is BV still optimal?

> If the objective function coefficient of a nonbasic varaible x_j is changed, the current basis remains optimal if $\overline{c}_j \geq 0$.
>
> If $\overline{c}_j < 0$, then the current basis is no longer optimal, and x_j will be a basic variable in the new optimal solution.

Example (cont'd)

- Consider nonbasic variable x_2. Currently c_2=30
- For what values of c_2 would BV={s_1,x_3,x_1} remain optimal?

Let Δ denote the amount by which we have changed c_2.

Then $c_2 = 30 + \Delta$.

$$BV = \{s_1, x_3, x_1\}, \mathbf{B} = \begin{bmatrix} 1 & 1 & 8 \\ 0 & 1.5 & 4 \\ 0 & 0.5 & 2 \end{bmatrix}, \mathbf{B}^{-1} = \begin{bmatrix} 1 & 2 & -8 \\ 0 & 2 & -4 \\ 0 & -0.5 & 1.5 \end{bmatrix}$$

$$\mathbf{c}_{BV}\mathbf{B}^{-1} = \begin{bmatrix} 0 & 20 & 60 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & -8 \\ 0 & 2 & -4 \\ 0 & -0.5 & 1.5 \end{bmatrix} = \begin{bmatrix} 0 & 10 & 10 \end{bmatrix}$$

Then $\overline{c}_2 = \mathbf{c}_{BV}\mathbf{B}^{-1}\mathbf{a}_2 - c_2$

$$= \begin{bmatrix} 0 & 10 & 10 \end{bmatrix} \begin{bmatrix} 6 \\ 2 \\ 1.5 \end{bmatrix} - (30 + \Delta)$$

$$= 35 - (30 + \Delta)$$

$$= 5 - \Delta.$$

Thus $\overline{c}_2 \geq 0$ holds, and BV remains optimal if $5 - \Delta \geq 0$ or $\Delta \leq 5$.

Thus, if the price of tables is decreased or increased by \$5 or less, BV remains optimal. Thus for $c_2 \leq 30+5=35$, BV remains optimal. Also the z-value remains the same (\$280).

Example (cont'd)

- If c_2=30, then z=280-10s_2-10s_3-5x_2.

This tells us that each table that Dakota manufactures will decrease revenue by $5 (in other words, the reduced cost for table is 5), and so x_2=0. If we increase the price of tables by more than $5, each table would now increase Dakota's revenue. Thus, as before, for Δ>5, the current basis is no longer optimal.

Conclusion: The **reduced cost for a nonbasic variable** (in a max problem) is the maximum amount by which the variable's objective function coefficient can be increased before the current basis is no longer optimal.

Example (cont'd)

- Consider the case when c_2>30, e.g. c_2=40.
- We know that BV will now be suboptimal.

Then $\bar{c}_2 = \mathbf{c}_{BV}\mathbf{B}^{-1}\mathbf{a}_2 - c_2 = 35 - 40 = -5$

Thus, $\bar{c}_2 < 0$ and x_2 can now enter the basis!

Perform an iteration of the simplex method.

- When c_2=40, the optimal solution to the LP changes to z=288, x_1=0, x_2=1.6, x_3=11.2, and s_1=27.2, s_2=0, s_3=0.
- The increase in the price of tables causes the company to manufacture tables instead of desks

2. Changing the objective function coefficient of a basic variable

- Since $\mathbf{B}$ (hence $\mathbf{B}^{-1}$) and $\mathbf{b}$ are unchanged, feasibility is not affected;
- $\mathbf{c}_{BV}$ is changed, hence $\overline{c}_j$ may change, optimality is affected in this case.
- When is BV still optimal?

> If the objective function coefficient of a basic varaible x_j is changed, the current basis remains optimal if $\overline{c}_i \geq 0, \forall x_i \in NBV$.
>
> If $\overline{c}_i < 0$ for any varaible $x_{i,}$ then the current basis is no longer optimal.

Note: If the current basis remains optimal, then the values of the decision varaibles do not change because $\mathbf{x}_{BV}$=$\mathbf{B}^{-1}\mathbf{b}$ remains unchanged.

However, the optimal z-value (z=$\mathbf{c}_{BV}\mathbf{B}^{-1}\mathbf{b}$) does change.

Example (cont'd)

- Consider changing c_1 from its current value of c_1=60.
- For what values of c_1 would BV={s_1,x_3,x_1} remain optimal?

Let Δ denote the amount by which we have changed c_1. Then $c_1 = 60+\Delta$.

$$BV = \{s_1, x_3, x_1\}, \mathbf{B} = \begin{bmatrix} 1 & 1 & 8 \\ 0 & 1.5 & 4 \\ 0 & 0.5 & 2 \end{bmatrix}, \mathbf{B}^{-1} = \begin{bmatrix} 1 & 2 & -8 \\ 0 & 2 & -4 \\ 0 & -0.5 & 1.5 \end{bmatrix}$$

$$\mathbf{c}_{BV}\mathbf{B}^{-1} = \begin{bmatrix} 0 & 20 & 60+\Delta \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & -8 \\ 0 & 2 & -4 \\ 0 & -0.5 & 1.5 \end{bmatrix} = \begin{bmatrix} 0 & 10-0.5\Delta & 10+1.5\Delta \end{bmatrix}$$

We can now compute the new rof. Since s_1, x_3, and x_1 are basic variables, their coefficients in rof must still be zero.

Example (cont’d)

For nonbasic varaibles x_2, s_2, and s_3, we have:

x_2: $\overline{c}_2 = \mathbf{c}_{BV}\mathbf{B}^{-1}\mathbf{a}_2 - c_2$

$$= \begin{bmatrix} 0 & 10-0.5\Delta & 10+1.5\Delta \end{bmatrix} \begin{bmatrix} 6 \\ 2 \\ 1.5 \end{bmatrix} - 30 = 5 + 1.25\Delta$$

s_2: $\overline{c}_5 = \mathbf{c}_{BV}\mathbf{B}^{-1}\mathbf{a}_5 - c_5$

$$= \begin{bmatrix} 0 & 10-0.5\Delta & 10+1.5\Delta \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - 0 = 10 - 0.5\Delta$$

s_3: $\overline{c}_6 = \mathbf{c}_{BV}\mathbf{B}^{-1}\mathbf{a}_6 - c_6$

$$= \begin{bmatrix} 0 & 10-0.5\Delta & 10+1.5\Delta \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - 0 = 10 + 1.5\Delta$$

BV remains optimal iff the following hold

$\overline{c}_2 \geq 0 \Rightarrow 5 + 1.25\Delta \geq 0 \Rightarrow \Delta \geq -4$

$\overline{c}_5 \geq 0 \Rightarrow 10 - 0.5\Delta \geq 0 \Rightarrow \Delta \leq 20$

$\overline{c}_6 \geq 0 \Rightarrow 10 + 1.5\Delta \geq 0 \Rightarrow \Delta \geq -20/3$

$\therefore BV$ remains optimal iff $-4 \leq \Delta \leq 20$,

i.e., $56 \leq c_1 \leq 80$.

If BV remains optimal, the values of the basic variables are unchanged.

However, the optimal z-value may change:

e.g. $c_1 = 70 \Rightarrow z = 70x_1 + 30x_2 + 20x_3$

$= 300$

Example (cont'd)

- When the current BV is no longer optimal.

Consider $c_1 = 100$, that is Δ=40.

Compute the reduced cost (in rof):

$x_1 : \overline{c}_1 = 0$ (basic variable)

$x_2 : \overline{c}_2 = 5 + 1.25\Delta = 55$

$x_3 : \overline{c}_3 = 0$ (basic variable)

$s_1 : \overline{c}_4 = 0$ (basic variable)

$s_2 : \overline{c}_5 = 10 - 0.5\Delta = -10$

$s_3 : \overline{c}_6 = 10 + 1.5\Delta = 70$

$$\mathbf{c}_{BV}\mathbf{B}^{-1} = \begin{bmatrix} 0 & 10-0.5\Delta & 10+1.5\Delta \end{bmatrix} = \begin{bmatrix} 0 & -10 & 70 \end{bmatrix}$$

constant in rof $= \mathbf{c}_{BV}\mathbf{B}^{-1}\mathbf{b}$

$$= \begin{bmatrix} 0 & -10 & 70 \end{bmatrix} \begin{bmatrix} 48 \\ 20 \\ 8 \end{bmatrix} = 360$$

If c_1=100, BV={s_1, x_3, x_1} is now suboptimal.

Enter s_2 into the basis to get new optimal solution.

- Thus, if c_1=100, the optimal solution to the LP changes to z=400, x_1=4, x_2=0, x_3=0, and s_1=16, s_2=4, s_3=0.
- The increase in the price of desks causes the company to manufacture desks only.

3. Changing the rhs of a constraint

- Since **b** does not appear in the optimality condition, changing the rhs of a constraint does not affect optimalilty but affects feasibility;
- Feasibility requires $\mathbf{x}_{BV} = \mathbf{B}^{-1}\mathbf{b} \geq 0$.
- When is BV still optimal?

> If the rhs of a constraint is changed, then the current basis remains optimal if the rhs of each constraint remains nonnegative.
>
> If the rhs of any constraint becomes negative, then the current basis is infeasible, and a new optimal solutoin must be found (Use the Dual Simplex Method).

Note: Changing **b** will change the values of basic variables and the optimal z-value.

Example (cont'd)

- Consider changing b_2 from its current value b_2 =20
- For what values of b_2 would BV={s_1,x_3,x_1} remain optimal?

Let Δ denote the amount by which we have changed b_2.

Then $b_2 = 20 + \Delta$.

$$BV = \{s_1, x_3, x_1\}, \mathbf{B}^{-1} = \begin{bmatrix} 1 & 2 & -8 \\ 0 & 2 & -4 \\ 0 & -0.5 & 1.5 \end{bmatrix}$$

$$\mathbf{B}^{-1}\mathbf{b} = \begin{bmatrix} 1 & 2 & -8 \\ 0 & 2 & -4 \\ 0 & -0.5 & 1.5 \end{bmatrix} \begin{bmatrix} 48 \\ 20+\Delta \\ 8 \end{bmatrix} = \begin{bmatrix} 24+2\Delta \\ 8+2\Delta \\ 2-0.5\Delta \end{bmatrix}$$

For BVto remain optimal, $\mathbf{B}^{-1}\mathbf{b} \geq 0$

$$24 + 2\Delta \geq 0 \Rightarrow \Delta \geq -12$$

$$8 + 2\Delta \geq 0 \Rightarrow \Delta \geq -4$$

$$2 - 0.5\Delta \geq 0 \Rightarrow \Delta \leq 4$$

$$\Rightarrow -4 \leq \Delta \leq 4$$

$$\Rightarrow 16 \leq b_2 \leq 24$$

Even if BV remains optimal, the values of

$$\mathbf{x}_{BV} = \mathbf{B}^{-1}(\text{new } \mathbf{b})$$

$$e.g.\ b_2 = 22, \mathbf{x}_{BV} = \mathbf{B}^{-1} \begin{bmatrix} 48 \\ 22 \\ 8 \end{bmatrix} = \begin{bmatrix} 28 \\ 12 \\ 1 \end{bmatrix}$$

$$z = \mathbf{c}_{BV}\mathbf{B}^{-1}(\text{new } \mathbf{b}) = \mathbf{c}_{BV}\mathbf{x}_{BV} = 300$$

Example (cont'd)

- What if BV is no longer optimal?

Let $b_2 = 30 > 24$.

$$\mathbf{B}^{-1}\mathbf{b} = \begin{bmatrix} 1 & 2 & -8 \\ 0 & 2 & -4 \\ 0 & -0.5 & 1.5 \end{bmatrix} \begin{bmatrix} 48 \\ 30 \\ 8 \end{bmatrix} = \begin{bmatrix} 44 \\ 28 \\ -3 \end{bmatrix}$$

$$\text{constant in rof} = \mathbf{c}_{BV}\mathbf{B}^{-1}\mathbf{b} = \begin{bmatrix} 0 & 10 & 10 \end{bmatrix} \begin{bmatrix} 44 \\ 28 \\ -3 \end{bmatrix} = 380.$$

Since we have no available bfs, use Dual Simplex Algorithm to get the new optimal solution.

4. Changing the column of a nonbasic variable

- Does not affect feasibility ($\mathbf{x}_{BV} = \mathbf{B}^{-1}\mathbf{b} \geq 0$) but affects optimalilty.
- When is BV still optimal?

> If the column of a nonbasic variable x_j is changed, the current basis remains optimal if $\bar{c}_j \geq 0$. If $\bar{c}_j < 0$, then the current basis is no longer optimal and x_j will be a basic variable in the new optimal tableau.

Note: If the column of a basic variable is changed, then it is usually difficult to determine whether the current basis is optimal ($\mathbf{B}$ and $\mathbf{c}_{BV}$ are changed and both optimality and feasibility conditions are affected). As always, the current basis would remain optimal if the optimality and the feasibility conditions are both satisfied.

Example (cont'd)

- Suppose the price of tables increases from \$30 to \$43 and that due to changes in technology the table now requires 5 board ft of lumber, 2 finishing hours, and 2 carpentry hours.
- Would this change the optimal solution to the LP?

$c_2 = 43, \mathbf{a}_2 = \begin{bmatrix} 5 \\ 2 \\ 2 \end{bmatrix}$. Before $c_2 = 30, \mathbf{a}_2 = \begin{bmatrix} 6 \\ 2 \\ 1.5 \end{bmatrix}$

Simply use $\overline{c}_2 = \mathbf{c}_{BV}\mathbf{B}^{-1}\mathbf{a}_2 - c_2$ to calculate the reduced cost of x_2 in rof:

$$\overline{c}_2 = \mathbf{c}_{BV}\mathbf{B}^{-1}\mathbf{a}_2 - c_2 = \begin{bmatrix} 0 & 10 & 10 \end{bmatrix} \begin{bmatrix} 5 \\ 2 \\ 2 \end{bmatrix} - 43 = -3 < 0$$

Since $\overline{c}_2 < 0, BV = \{s_1, x_3, x_1\}$ is no longer optimal.

Apply the simplex algorithm with $BV = \{s_1, x_3, x_1\}$

5. Adding a new activity (addition of a new variable)

- In many practical situations, opportunities arise to undertake new activities.
- Does not affect feasibility but affects opimality.
- To determine whether a new activity x_j will cause the current to be no longer optimal, calculate $\overline{c}_j = \mathbf{c}_{BV}\mathbf{B}^{-1}\mathbf{a}_j - c_j$.

> If a new column (corresponding to a new variable x_j) is added to an LP, then the current basis remains optimal if $\overline{c}_j \geq 0$. If $\overline{c}_j < 0$, then the current basis is no longer optimal and x_j will become a basic variable.

Example (cont'd)

- Suppose the company decides to start making stools. The price of a stool is \$15 and requires 1 board ft of lumber, 1 finishing hour, and 1 carpentry hour.
- Should the company manufacture stools?

Let $x_4 = \#$ of stools manufactured. Then $c_4 = 15, \mathbf{a}_4 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ (added column).

Then $\overline{c}_4 = \mathbf{c}_{BV}\mathbf{B}^{-1}\mathbf{a}_4 - c_4 = \begin{bmatrix} 0 & 10 & 10 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - 15 = 5.$

Since $\overline{c}_4 \geq 0, BV = \{s_1, x_3, x_1\}$ is still optimal. The reduced cost is \$5. This means that each stool would decrease revenue by \$5. Therefore, the company should not manufacture stools.

Duality

In this lecture you will learn:

- Finding the dual of an LP
- Some basic duality theory, (describing the relation between LP and its dual)
- Shadow price and dual variables

Based on Chapter 6.5 – 6.8 of *Operations Research: Applications and Algorithms*, 4th Edition.

Primal and Dual

Associate with any LP is another LP, called the *dual*. The relation between an LP and its dual gives us interesting economic and sensitivity analysis insights.

When taking the dual of an LP, the given LP is referred to as the *primal*.

If the primal is a max problem, then the dual will be a min problem and vice versa.

Let us define (arbitrarily) the variables for a max problem to be $z, x_1, x_2, \ldots, x_n$,

and the variables for a min problem to be $w, y_1, y_2, \ldots, y_m$.

Finding the dual of an arbitrary LP

$$\begin{array}{ll} \max & z = 2x_1 + x_2 \\ \text{st} & x_1 + x_2 = 2 \quad \leftarrow y_1 \\ & 2x_1 - x_2 \geq 3 \quad \leftarrow y_2 \\ & x_1 - x_2 \leq 1 \quad \leftarrow y_3 \\ & x_1 \geq 0, x_2 \text{ urs} \end{array}$$

primal (dual)		**dual (primal)**	
Objective	MAX	MIN	Objective
Variables	≥ 0	$\geq c_i$	Constraints
	≤ 0	$\leq c_i$	
	urs	$= c_i$	
Constraints	$\leq b_i$	≥ 0	Variables
	$\geq b_i$	≤ 0	
	$= b_i$	urs	

min problem (dual)

$$\begin{array}{ll} \min & w = 2y_1 + 3y_2 + y_3 \\ \text{st} & y_1 + 2y_2 + y_3 \geq 2 \quad \leftarrow x_1 \\ & y_1 - y_2 - y_3 = 1 \quad \leftarrow x_2 \\ & y_1 \text{ urs}, y_2 \leq 0, y_3 \geq 0 \end{array}$$

The i^{th} *dual constraint* corresponds to the i^{th} *primal variable* x_i.
Similarly, *dual variable* y_i is associated with the i^{th} *primal constraint*.

Dual Theorems

Relations between the primal and dual problems.

The dual of a dual is the primal itself.

To simplify the exposition, we assume that the primal is a normal max problem and then its dual is a normal min problem.

Primal

$$\begin{aligned} \max\ z &= \mathbf{cx} \\ \text{s.t.}\quad \mathbf{Ax} &\le \mathbf{b} \\ \mathbf{x} &\ge \mathbf{0} \end{aligned}$$

Dual

$$\begin{aligned} \min\ w &= \mathbf{b}^T\mathbf{y} \\ \text{s.t.}\quad \mathbf{A}^T\mathbf{y} &\ge \mathbf{c}^T \\ \mathbf{y} &\ge \mathbf{0} \end{aligned}$$

Weak Duality

Theorem 8.1 (Weak Duality): If $\mathbf{x}$ is a feasible solution to the primal (max) problem and $\mathbf{y}$ is a feasible solution to the dual problem, then

$$\mathbf{cx} \le \mathbf{b}^T\mathbf{y} \quad \text{(i.e., } z\text{-value for } \mathbf{x} \le w\text{-value for } \mathbf{y}\text{)}$$

Corollary 8.1: Let $\mathbf{x}'$ and $\mathbf{y}'$ be feasible solutions to the primal and the dual, respectively, and suppose that $\mathbf{cx}' = \mathbf{b}^T\mathbf{y}'$, Then $\mathbf{x}'$ and $\mathbf{y}'$ are the optimal solutions to the primal and the dual, respectively.

Corollary 8.2:

(a) If the primal is unbounded, then the dual is infeasible.

(b) If the dual is unbounded, then the primal is infeasible.

The Dual Theorem or Strong Duality

Theorem 8.2: Suppose BV is an optimal basis for the primal. Then $\mathbf{c}_{BV}\boldsymbol{B}^{-1}$ is an optimal solution to the dual. Also the respective optimal objective function values are equal.

Remark:

(a) Optimal solution to the dual is $\mathbf{c}_{BV}\boldsymbol{B}^{-1}$, (thus, could be directly calculated).

(b) Optimal objective value of dual = optimal objective value of primal = $\mathbf{c}_{BV}\boldsymbol{B}^{-1}\mathbf{b}$

(c) The optimal solution to dual can be found in the optimal rof and revised constraints for the primal.

Shadow prices and Dual variables

- The **shadow price** for the i^{th} constraint of an LP (sensitivity analysis) is defined to be the amount by which the optimal z-value is improved if the rhs of the i^{th} constraint is increased by 1. This definition applied only if the change in the rhs of the ith constraint leaves the current basis optimal.
- The main result is

  the shadow price of the i^{th} constraint of a *max* problem is the optimal value of the i^{th} dual variable.

  The shadow price of the i^{th} constraint of a *min* problem is – (the optimal value of the i^{th} dual variable).

- When the slack or excess variable for a constraint is positive in an LP's optimal solution
  - the constraint is nonbinding
  - the shadow price of the constraint = 0
    the optimal dual variable = 0
    the reduced cost of the slack or excess variable = 0
- When the slack or excess variable for a constraint is zero in an LP's optimal solution,
  - the constraint is binding
  - the shadow price and optimal dual variable can be found from the reduced cost of the slack or excess variable as shown in the tables

Read the optimal dual solution from the optimal rof – if primal is a max LP

- The Dual Theorem: the optimal value of the ith dual variable (y_i^*) is the ith element of $\mathbf{c}_{BV}\mathbf{B}^{-1}$.

Constraints in primal	$\leq$	$\geq$	$=$
Dual variable y_i	≥ 0	≤ 0	urs
Associated variable in primal	slack variable s_i	excess variable e_i	artificial variable a_i
Optimal value of dual variable y_i	reduced cost of s_i in optimal rof	– (reduced cost of e_i in optimal rof)	(reduced cost of a_i in optimal rof) – M
Reasons: $\bar{c}_j = \mathbf{c}_{BV}\mathbf{B}^{-1}\mathbf{a}_j - c_j$	reduced cost of s_i = $\mathbf{c}_{BV}\mathbf{B}^{-1}(0,\ldots,1,..0)'-0$ = i^{th} element of $\mathbf{c}_{BV}\mathbf{B}^{-1}$	reduced cost of e_i = $\mathbf{c}_{BV}\mathbf{B}^{-1}(0,\ldots,-1,..0)'-0$ = – (i^{th} element of $\mathbf{c}_{BV}\mathbf{B}^{-1}$)	reduced cost of a_i = $\mathbf{c}_{BV}\mathbf{B}^{-1}(0,\ldots,1,..0)'-(-M)$ = i^{th} element of $\mathbf{c}_{BV}\mathbf{B}^{-1}$ $+M$
Shadow price	≥ 0	≤ 0	urs
Value of shadow price	reduced cost of s_i in optimal rof	– (reduced cost of e_i in optimal rof)	(reduced cost of a_i in optimal rof) – M

The shadow price of the i^{th} constraint = the optimal value of the i^{th} dual variable (y_i^*).

Read the optimal dual solution from the optimal rof – if primal is a min LP

Constraints in primal	$\leq$	$\geq$	$=$
Dual variable y_i	≤ 0	≥ 0	urs
Associated variable in primal	slack variable s_i	excess variable e_i	artificial variable a_i
Optimal value of dual variable y_i	reduced cost of s_i in optimal rof	– (reduced cost of e_i in optimal rof)	(reduced cost of a_i in optimal rof) + M
Reasons: $\bar{c}_j = \mathbf{c}_{BV}\mathbf{B}^{-1}\mathbf{a}_j - c_j$	reduced cost of s_i = $\mathbf{c}_{BV}\mathbf{B}^{-1}(0,\ldots,1,..0)'-0$ = i^{th} element of $\mathbf{c}_{BV}\mathbf{B}^{-1}$	reduced cost of e_i = $\mathbf{c}_{BV}\mathbf{B}^{-1}(0,\ldots,-1,..0)'-0$ = – (i^{th} element of $\mathbf{c}_{BV}\mathbf{B}^{-1}$)	reduced cost of a_i = $\mathbf{c}_{BV}\mathbf{B}^{-1}(0,\ldots,1,..0)'-(M)$ = i^{th} element of $\mathbf{c}_{BV}\mathbf{B}^{-1} - M$
Shadow price	≥ 0	≤ 0	urs
Value of shadow price	– (reduced cost of s_i in optimal rof)	reduced cost of e_i in optimal rof	– (reduced cost of a_i in optimal rof) – M

The shadow price of the i^{th} constraint = – (the optimal value of the i^{th} dual variable) = - y_i^*.

Example (cont'd)

Primal

$$
\begin{aligned}
\max\ z &= 60x_1 + 30x_2 + 20x_3 \\
\text{s.t.}\quad & 8x_1 + 6x_2 + x_3 \le 48 \text{ (Lumber constraint)} \\
& 4x_1 + 2x_2 + 1.5x_3 \le 20 \text{ (Finishing constraint)} \\
& 2x_1 + 1.5x_2 + 0.5x_3 \le 8 \text{ (Carpentry constraint)} \\
& x_2 \le 5 \text{ (table demand constraint)} \\
& x_1, x_2, x_3 \ge 0
\end{aligned}
$$

Dual

$$
\begin{aligned}
\min\ w &= 48y_1 + 20y_2 + 8y_3 + 5y_4 \\
\text{s.t.}\quad & 8y_1 + 4y_2 + 2y_3 \ge 60 \quad \text{(Desk constraint)} \\
& 6y_1 + 2y_2 + 1.5y_3 + y_4 \ge 30 \quad \text{(Table constraint)} \\
& y_1 + 1.5y_2 + 0.5y_3 \ge 20 \quad \text{(Chair constraint)} \\
& y_1, y_2, y_3, y_4 \ge 0
\end{aligned}
$$

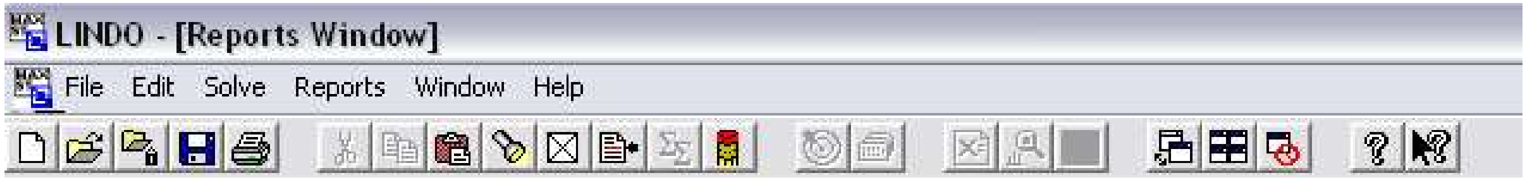
The optimal dual objective function value and dual solution is w=280, y_1=0, y_2=10, y_3=10, y_4=0.

Optimal rof and revised constraints for the primal LP

$$z = 280 - 5x_2 - 10s_2 - 10s_3$$

$$
\begin{pmatrix} s_1 \\ x_3 \\ x_1 \\ s_4 \end{pmatrix} + \begin{pmatrix} -2 \\ -2 \\ 1.25 \\ 1 \end{pmatrix} x_2 + \begin{pmatrix} 2 \\ 2 \\ -0.5 \\ 0 \end{pmatrix} s_2 + \begin{pmatrix} -8 \\ -4 \\ 1.5 \\ 0 \end{pmatrix} s_3 = \begin{pmatrix} 24 \\ 8 \\ 2 \\ 5 \end{pmatrix}. \text{ Thus, } \begin{pmatrix} s_1^* \\ x_3^* \\ x_1^* \\ s_4^* \end{pmatrix} = \begin{pmatrix} 24 \\ 8 \\ 2 \\ 5 \end{pmatrix}
$$

LINDO – output




```
LP OPTIMUM FOUND AT STEP      2

      OBJECTIVE FUNCTION VALUE

      1)       280.0000

 VARIABLE        VALUE          REDUCED COST
       X1         2.000000          0.000000
       X2         0.000000          5.000000
       X3         8.000000          0.000000

      ROW   SLACK OR SURPLUS     DUAL PRICES
       2)        24.000000          0.000000
       3)         0.000000         10.000000
       4)         0.000000         10.000000
       5)         5.000000          0.000000

NO. ITERATIONS=       2

RANGES IN WHICH THE BASIS IS UNCHANGED:

                          OBJ COEFFICIENT RANGES
VARIABLE         CURRENT        ALLOWABLE        ALLOWABLE
                   COEF         INCREASE         DECREASE
      X1        60.000000       20.000000         4.000000
      X2        30.000000        5.000000         INFINITY
      X3        20.000000        2.500000         5.000000

                          RIGHTHAND SIDE RANGES
     ROW         CURRENT        ALLOWABLE        ALLOWABLE
                   RHS          INCREASE         DECREASE
       2        48.000000        INFINITY        24.000000
       3        20.000000        4.000000         4.000000
       4         8.000000        2.000000         1.333333
       5         5.000000        INFINITY         5.000000
```

Indicate that LINDO found the optimal solution after two iterations of the simplex algorithm

$z^* = 280.0$

Reduced cost: coefficient of a variable in rof for min problem or negative of it for max problem. For each basic variable it must be 0. For a nonbasic variable x_j, the reduced cost is the amount by which the optimal z-value is increased (decreased) if x_j is increased by 1 unit (and all other nonbasic variables remain equal to 0) for a min (max) problem.

LINDO outputs the Shadow Prices as “Dual Prices”.

Sensitivity analysis