4.1. Introduction and Examples.

$\{X_n, n=0,1,2,\cdots\}$ is a stochastic process

X_n takes on a finite or countable number of values.

for example. $\{0,1,2,\cdots,N\}$ or $\{\cdots,-2,-1,0,1,2,\cdots\}$

the set is called. state space.

$X_n = i$ is called. the process. is in state i in time n.

Transition Probability P_{ij}. the probability from state i to state j.

$$P_{ij} = P(X_{n+1}=j \mid X_n=i, X_{n-1}=i_{n-1}, \cdots, X_0=i_0).$$

Markov chain. :

Such a stochastic process defined above and $P_{ij} = P(X_{n+1}=j \mid X_n=i)$.

which means. the future state X_{n+1} just depend on the present. state X_n.

Markovian property.

Obviously. there are. some requirements. for P_{ij}.

① $P_{ij} \geq 0$.

② $\sum_{j=1}^{\infty} P_{ij} = 1$.

③ P_{ij} is fixed.

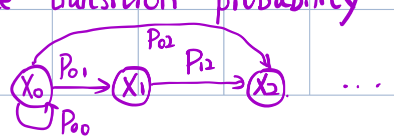
We. could. represent. transition probability into matrix form.

$$P = (P_{ij}) = \begin{bmatrix} P_{00} & P_{01} & P_{02} & \cdots \\ P_{10} & P_{11} & P_{12} & \cdots \\ \vdots & \vdots & \vdots & \\ P_{i0} & P_{i1} & P_{i2} & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix}$$

– from state X_0 to some state.

~ transition probability matrix.
(stochastic matrix).

Also. the transition probability could be represented into graph form.



node : the state.

directed edge : transition probability if exists.

Homogeneous. MC : $P_{nij} = P_{ij}$.

Nonhomogeneous. MC : P_{ij} depend on time n.

(转移概率随时间 n 变动).

Example 1. (s, S) inventory model with periodic Review.

character : * single item inventory system.

* periodic review 定期清点库存.

* Demand per period i.i.d. $\triangleq D_n$

D_n : 第 n 个周期的需求.

$\Pr(D_n = j) = \varphi_j \quad j = 0, 1, 2, \cdots$

* Review inventory position at the end of each period.

inventory position = 现有库存 + 订购库存 − 预售.

* Delivery of replenishment is instantaneous (订购瞬间到货). and available at the start of the next period.

(s, S) policy : $0 \le s < S$. if inventory position $\le s$, then we order up to S.

The situation of stock out (缺货):

Case I. (lost sale): Any demand exceeds hand inventory is lost.

Case II (backordering): Any demand exceeds hand inventory are satisfied after replenishment.

MC Model I (Inventory Level (start) $\bar{X}_n$).

For period n, $n=0,1,2,\cdots$. $\bar{X}_n$ inventory level at the beginning of the n_{th} period. (after delivery of any replenishment order and backordering fulfillment).

State Space: $\{s+1, s+2, \cdots, S\}$.

Replenishment Policy: $\bar{X}_{n+1} = \begin{cases} S & \bar{X}_n - D_n \le s \\ \bar{X}_n - D_n & \bar{X}_n - D_n > s \end{cases}$

即当期库存水平 − 当期需求 <s. 则下期库存补满至 S. 否则不需要补货.

Transition Probability Matrix.

$$P = (p_{ij}) = \begin{array}{c} \\ s+1 \\ s+2 \\ \vdots \\ S \end{array} \begin{array}{c} \begin{array}{cccc} s+1 & s+2 & \cdots & S \end{array} \\ \begin{bmatrix} \varphi_0 & 0 & \cdots & 1-\varphi_0 \\ \varphi_1 & \varphi_0 & \cdots & 1-\varphi_1-\varphi_0 \\ \vdots & \vdots & & \vdots \\ \varphi_{S-s-1} & \varphi_{S-s-2} & \cdots & 1-\sum_{i=1}^{S-s-1}\varphi_i \end{bmatrix} \end{array}$$

第 1 行中: $P_{s+1,s+1}$: 表示当期库存 s+1. 下期库存 s+1. 则只可能当期需求为 0.

$P_{s+1,S}$: 表示当期库存 s+1. 下期库存 S. 则 $s+1-D_n \le s$ 即 $D_n \ge 1$. 从而 $1-\varphi_0$.

其余行中基本类似，$P_{s,S}$：① 当期没有需求

② 当期 $S-D_n \le s$，即 $D_n \ge S-s$.

故 $P_{s,S} = \varphi_0 + \varphi_{S-s} + \varphi_{S-s+1} + \cdots$

$= 1 - \sum_{i=1}^{S-s-1} \varphi_i$

MC Model II (Inventory Level (end). Y_n)

Y_n: inventory level at the end of the nth period just before ordering.

Case I (lost sale).

State Space: $\{0, 1, 2, \cdots, S\}$.

$$Y_{n+1} = \begin{cases} (S-D_{n+1})^+ & Y_n \le s \\ (Y_n - D_{n+1})^+ & Y_n > s \end{cases}$$

上期最后库存 $\le s$. 则当期一开始补满至 S，当期最后库存为 $\begin{cases} S-D_{n+1} & D_{n+1} \le S \\ 0 & D_{n+1} > S \end{cases}$，即 $(S-D_{n+1})^+$.

同理，上期最后库存 $> s$. 则不需补货. 当期初始为 Y_n

其余同上.

Transition Probability Matrix $Q = (Q_{ij})$

If $i \le s$. 则 $Q_{ij} = \begin{cases} \sum_{i=S}^{\infty} \varphi_i & j=0,\ i=1,2,\cdots s \\ \varphi_{S-j} & j=1,2,\cdots,S,\ i=1,2,\cdots,s. \end{cases}$

下期初始库存为 S. 最终库存为 0. 则 $D_{n+1} \ge S$. 否则最终库存即为 $S-D_{n+1}$

If $i > s$, 則 $Q_{ij} = \begin{cases} \sum_{k=i}^{\infty} \varphi_k & j=0 \\ \varphi_{i-j} & j=1,2,\cdots,i \\ 0 & j=i+1,\cdots,S \end{cases}$

Case II (Backordering):

State Space: $\{\cdots,-2,-1,0,\cdots S\}$.

$Y_{n+1} = \begin{cases} S-D_{n+1} & Y_n \le s \\ Y_n - D_{n+1} & Y_n > s \end{cases}$

Transition Probability Matrix. $Q = (Q_{ij})$.

If $i \le s$. $Q_{ij} = \phi_{S-j}$

If $i > s$ $Q_{ij} = \begin{cases} 0 & j=i+1,\cdots,S \\ \phi_{i-j} & j \le i. \end{cases}$

Example 2. M/G/1 Queue.

M/G/1:

arrival ~ Poisson (λ).

service ~ General G.

Single server.

$X(t)$: # of customers in system at time t.

$\{X(t)\}$ is non Markovian

Embedded MC.

Suppose we only observe the system at time when a customer departs.

X_n: # of customers after departure of nth customer

* $n=0,1,2,\cdots$ events not equally distributed.
* time between epochs are random

$\{\bar{X}_n, n=0,1\cdots\}$ embedded MC.

$\bar{X}_n=0 \Leftrightarrow$ queue is empty after the n_{th} departure

* system empty until arrival of $(n+1)_{th}$ customer
* Y_n : # of customers arriving while the $(n+1)_{th}$ customer is in service
* # of customers in system after its departure is Y_n

$$\bar{X}_{n+1} = \begin{cases} Y_n & \bar{X}_n=0 \quad \text{(第n个人离开. 系统为空)}. \\ X_n+Y_n-1 & \bar{X}_n \geq 1 \quad \text{(第n个人离开后. 系统不为空)} \end{cases}$$

transition probability P_{ij}

$$\varphi_j = P(Y_n=j) = \int_0^\infty P(Y_n=j, T=t)\, dG(t)$$
$$= \int_0^\infty \frac{(\lambda t)^j}{j!} e^{-\lambda t}\, dG(t).$$

(T 为第 n+1 个人的服务时间. arrival ~ Poisson Process).

$$P_{ij} = \begin{cases} \varphi_j & i=0 \\ \varphi_{j-i+1} & i\geq 1,\ j\geq i-1 \\ 0 & i\geq 1,\ j<i-1 \end{cases}$$

Example 3. G/M/1 Queue.

arrival ~ general distribution G

service ~ Poisson distribution μ.

single server.

$\bar{X}_n$: # of customers in system at the n_{th} arrival

$\{\bar{X}_n, n=0,1,\cdots\}$ embedded MC.

State Space: $\{0,1,2,\cdots\}$.

transition probability P_{ij}:

$$\varphi_j = P(Y_n=j) = \int_0^\infty P(Y_n=j \mid T=t)\, dG(t)$$
$$= \int_0^\infty \frac{(\lambda t)^j}{j!} e^{-\lambda t}\, dG(t)$$

(Y_n: 第n个顾客与第n+1个顾客到达期间服务的人数，T为两次到达间的长度)

$$\bar{X}_{n+1} = \begin{cases} \bar{X}_n - \bar{Y}_n + 1, & \bar{X}_n \geq \bar{Y}_n - 1 \\ 0 & \bar{X}_n < \bar{Y}_n - 1 \end{cases}$$

$$P_{ij} = \begin{cases} 0 & i \geq 0, \ j = i+2, \cdots \\ \varphi_{i-j+1} & i \geq 0, \ j = 0, \cdots, i+1 \end{cases}$$

Example 4. Simple Random walk.

$\bar{X}_i$ 只有两种取值 -1, 1. $P(\bar{X}_i = 1) = p$, $P(\bar{X}_i = -1) = 1-p$.

Denote $S_n = \sum_{i=1}^n \bar{X}_i$, $S_0 = 0$ simple random walk.

State Space: $\{\cdots, -2, -1, 0, 1, 2, \cdots\}$.

transition probability P_{ij}:

$$P_{ij} = \begin{cases} p & j-i=1 \\ 1-p & j-i=-1 \\ 0 & \text{otherwise} \end{cases}$$

4.2 Some Definitions and Properties

B.1. Multistep transition. Chapman - kolmogrov equation.

Multistep Transition

P_{ij} — 1-step transition.

$P_{ij}^{(n)}$ — n-step transition $P_{ij}^{(n)} = P(X_{m+n}=j \mid X_m = i)$ $n=0,1\dots$

when $n=0$. $P_{ij}^{(0)} = 1(i=j)$,

when $n=1$, $P_{ij}^{(1)} = P_{ij}$

By the definition. $(n+1)$-step transition

$$P_{ij}^{(n+1)} = \sum_{k=0}^{\infty} P(X_{m+n+1}=j \mid X_{m+n}=k)\, P(X_{m+n}=k \mid X_m = i)$$

$$= \sum_{k=0}^{\infty} P_{kj} P_{ik}^{(n)}.$$

the transition probability matrix.

$$P^{(n+1)} = P^{(n)} \cdot P = P \cdot P^{(n)}$$

CK Equation:

$$P^{(n+m)} = P^{(n)} \cdot P^{(m)}, \quad P^{(n)} = (P^{(1)})^n$$

B.2. Communication. Class. Irreducible

Communication & Accessable

☆ Def (Accessable) state j is accessable from state i if $\exists$ some $n \geq 0$ such that $P_{ij}^{(n)} > 0$

☆ Def (Communication). We say state i and state j communicate $(i \leftrightarrow j)$ if satisfied

1). i is accessable from j

2) j is accessable from i

Prop. Communication is an equivalence relation, since

1) $i \leftrightarrow i$

2) $i \leftrightarrow j \Rightarrow j \leftrightarrow i$

3) $i \leftrightarrow j,\ j \leftrightarrow k \Rightarrow i \leftrightarrow k$.

proof. since $i \leftrightarrow j$. $\exists\ k.\ m.$ s.t.

$P_{ij}^{(k)} > 0.\ P_{ji}^{(m)} > 0. \Rightarrow j \leftrightarrow i$

since $i \leftrightarrow j$. $j \leftrightarrow k$. $\exists\ k_1, m_1, k_2, m_2$. s.t.

$P_{ij}^{(m_1)} > 0,\ P_{jk}^{(k_1)} > 0,\ P_{ji}^{(m_2)} > 0,\ P_{kj}^{(k_2)} > 0$

$\Rightarrow P_{ik}^{(m_1+k_1)} = \sum_{l=0}^{\infty} P_{il}^{(m_1)} P_{lk}^{(k_1)} > P_{ij}^{(m_1)} P_{jk}^{(k_1)} > 0$

$P_{ki}^{(m_2+k_2)} = \sum_{l=0}^{\infty} P_{kl}^{(k_2)} P_{li}^{(m_2)} > P_{kj}^{(k_2)} P_{ji}^{(m_2)} > 0$

$\Rightarrow i \leftrightarrow k$.

#

Class.

☆ Def (class). Communication decomposes the state space into a number of classes:

* $\forall$ 2 state in the same class communicate
* $\forall$ 2 state in different classes do not communicate.

Irreducibility

不可约的.

☆ Def (Irreducible). We say MC irreducible if all states communicate.

B3. Recurrence and Transience.

复现 瞬态.

We denote $f_{ij} = P(\exists n,\ X_n = j \mid X_0 = i)$.

$f_{ij} > 0 \Leftrightarrow j$ is accessable from i

$f_{ij} = P(\exists n,\ X_n = j \mid X_0 = i)$

$$= P(\exists n,\ X_n = j,\ X_k \neq j,\ k=1,2,\cdots,n-1 \mid X_0 = i)$$

$$= \sum_{n=1}^{\infty} P(X_n = j,\ X_k \neq j,\ k=1,2,\cdots,n-1 \mid X_0 = i).$$

$$= \sum_{n=1}^{\infty} f_{ij}^{(n)}$$

$$f_{ij}^{(n)} = P(X_n = j,\ X_k \neq j,\ k=1,2,\cdots,n-1 \mid X_0 = i).$$

$f_{ij}^{(n)}$：表示从 i 状态出发，经过 n 次转移，第一次到达 j 的概率.

f_{ij}：表示从状态 i 出发 第一次 到达状态 j 的总概率.

Prop：$\forall i,j,\ n \geq 0,\quad P_{ij}^{(n)} = \sum_{k=0}^{n} f_{ij}^{(k)} P_{jj}^{(n-k)}$

proof. $f_{ij}^{(n)} = P(X_n = j,\ X_k \neq j,\ k=1,\cdots,n-1 \mid X_0 = i)$

通俗的来看，$P_{ij}^{(n)}$ 是指从 i 状态经过 n 次 transition 后，最后在状态 j. 其中这 n 次 transition 中，可能第 k 次 transition 后是状态 i 第 1 次来到状态 j $(1 \leq k \leq n)$. 概率为 $f_{ij}^{(k)}$，在剩余的 n-k 次 transition，需从状态 j 到状态 j，概率为 $P_{jj}^{(n-k)}$. 因此 $P_{ij}^{(n)} = \sum_{k=0}^{n} f_{ij}^{(k)} P_{jj}^{(n-k)}$

☆ Def (Recurrence and Transience).

Given that we are in state j. The probability that we will return to state j again is equal to f_{jj}

* if $f_{jj} = 1$, we say state j recurrent
* if $f_{jj} < 1$, we say state j transient, which means it will never return to j with probability $1 - f_{jj}$

Prob(state j is visited n times) $= (f_{jj})^{n-1}(1 - f_{jj})$

is a geometric distribution with mean $\frac{1}{1-f_{jj}} < \infty$.

the expected number of transitions into state j

Prop. $f_{jj} = 1 \iff \sum_{n=1}^{\infty} P_{jj}^{(n)} = \infty \iff$ recurrent

Corollary. Recurrence is a class property, which means.

i recurrent. $i \leftrightarrow j \Rightarrow j$ recurrent.

proof. Since $i \leftrightarrow j$. $\exists$ m. n. such that $P_{ij}^{(m)} > 0$.

$P_{ji}^{(n)} > 0$. $\sum_{s=0}^{\infty} P_{jj}^{(s)} \geq \sum_{s=0}^{\infty} P_{jj}^{(s+m+n)} \geq \sum_{s=0}^{\infty} P_{ji}^{(n)} P_{ii}^{(s)} P_{ij}^{(m)}$

$= P_{ji}^{(n)} P_{ij}^{(m)} \sum_{s=0}^{\infty} P_{ii}^{(s)}$. 而 i recurrent.

故 $\sum_{s=0}^{\infty} P_{ii}^{(s)} = \infty$. 进一步 $\sum_{s=0}^{\infty} P_{jj}^{(s)} = \infty$. j recurrent.

#

☆ Def (Null and Positive recurrent).

$N_j(t)$: # of transitions into j by time t.

If j is recurrence, $X_0 = j$, then $\{N_j(t)\}$ is Renewal Process. its interarrival distribution $\{f_{jj}^{(n)}, n \geq 1\}$

Expected number of transitions needed to return to state j is

$$\mu_{jj} = \begin{cases} \infty & j \text{ is transient} \\ \sum_{n=1}^{\infty} n f_{jj}^{(n)} & j \text{ is recurrent} \end{cases}$$

If j is recurrent, $\mu_{jj} = \infty \Rightarrow$ null recurrent, $\mu_{jj} < \infty$

$\Rightarrow$ positive recurrent.

Null and Positive recurrent are class properties.

Example a MC has three states : 0, 1, 2.

its transition probability matrix is.

$$P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

obviously. $0 \leftrightarrow 1 \leftrightarrow 2$. hence just 1 class

$\{0, 1, 2\}$. they are positive recurrent

Example. a MC has four states : 0, 1, 2, 3

its transition probability matrix is.

$$P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

classes : $\{0, 1\}$ $\{2\}$ $\{3\}$.

$\{0, 1\}$ – positive recurrent

$\{2\}$ – transient 2 可能转移后回不来

$\{3\}$ – positive recurrent

Example (Null recurrence)

Simple Random walk

$1 - P = \frac{1}{2}$ $-1 - (1-P) = \frac{1}{2}$

state $0, \pm 1, \pm 2, \dots$ null recurrence (类有无穷多项)

B.4 Periodicity.

☆ Def (period $d(i)$, aperiodic)

A state i of a MC has period $d(i)$ if

* $P_{ii}^{(n)} > 0$, $\forall n$ divisibe of $d(i)$.

* $d(i)$ is the largest integer having the property.

if $d(i) = 1$, then state i is called. aperiodic.

(即 $d(i)$ 的倍数 n. 有 $P_{ii}^{(n)} > 0$. 回到 i 的概率 > 0)

Periodicity is a class property.

☆ Def (Ergodic).

A state which is positive recurrent and aperiodic is called ergodic

A MC is ergodic if all states are ergodic.

Prop. If a MC just has finite state, then $\exists$ at least 1 state that is positive recurrent.

proof. 假设 MC 有 N 个 states $\{1, 2, \cdots, N\}$, 若所有 states 是 transient 或 null positive. 则每个状态只返回至多有限次. 从而分别记为 $T_1, \cdots, T_N$. 则 $T_1 + \cdots + T_N + 1$ 次 transition 将在上述状态之外, 矛盾.

#

Prop. Consider a MC with finite state space, then state i is either positive recurrence or transient.

4.3. Limiting Behaviour.

We fix a state j and denote $N_j(t)$: # of transitions into state j by time t. If $X_0 = j$, then $\{N_j(t) : t \geq 0\}$

is a renewal process with interarrival distribution

$P(X=n) = f_{jj}^{(n)}$ for $n = 1, 2, \cdots$.

If we assume $N_j(t)$ is a lattice with period $d(j)$. then by blackwell's theorem. $\lim_{n\to\infty} P_{jj}^{(nd(j))} = \frac{d(j)}{\mu_{jj}}$

long-run fraction of time in state j (start from j). is $\frac{1}{\mu_{jj}}$

If $X_0 = i \neq j$. then $\{N_j(t) : t \geq 0\}$ is a delayed renewal process. If j is aperiodic. then by Blackwell's theorem. $\lim_{n\to\infty} P_{jj}^{(n)} = \frac{1}{\mu_{jj}}$ (long-run fraction of time in state j from state i).

We now denote $\pi_j = \frac{1}{\mu_{jj}}$ as limiting distribution

$\pi_j > 0 \iff \mu_{jj} < \infty \iff j$ is positive recurrent

Stationary Distribution

We fix X_0 and suppose $p(X_0 = j) = \alpha_j$. $j = 0, 1, 2, \cdots$. then

$p(X_1 = j) = \sum_{i=0}^{\infty} p(X_1 = j \mid X_0 = i)\, p(X_0 = i) = \sum_{i=0}^{\infty} P_{ij}\alpha_i$

If $p(X_1 = j) = \sum_{i=0}^{\infty} P_{ij}\alpha_i = \alpha_j$, $j = 1, 2, \cdots$, then α is stationary distribution. 即一步转移后，状态概率没变。

Prop. (1) $P'\alpha = \alpha$ (2) $e^T\alpha = 1$. (3) $\alpha \geq 0$

Thm Consider an Irreducible, Aperiodic MC.

(1) If all states are either null recurrent or transient,

$P_{ij}^{n} \to 0$ as $n \to \infty$, then there does not exist stationary distribution.

(2) If all states are positive recurrent (ergodic MC).

then $\pi_j = \lim_{n\to\infty} P_{ij}^{(n)} > 0$, π_j is stationary distribution.

hence. $\pi_j = \sum_{i=0}^{\infty} P_{ij} \pi_i \quad \forall j, \quad \sum \pi_i = 1$

(For Finite State space, ergodic MC = aperiodic + positive recurrent + irreducible)

Example (s-S) Inventory System with Periodic Review.

For MC Model I.

Demand D is i.i.d. $P(D_n=j) = \varphi_j$, $j = 0, 1, \cdots$

X_n: inventory level at the beginning of period n.

State Space: $[s+1, \cdots, S]$.

If we suppose $\varphi_0 > 0 \Rightarrow P_{ii} > 0$, then the MC is an Ergodic MC.

the limit distribution.

$$\pi_j = \sum_{i=s+1}^{S} t_{ij} \pi_i, \quad j = s+1, \cdots, S. \quad \sum_{i=s+1}^{S} \pi_i = 1.$$

$$P_{ij} = \begin{bmatrix} \varphi_0 & 0 & \cdots & 1-\varphi_0 \\ \varphi_1 & \varphi_0 & \cdots & 1-\varphi_0-\varphi_1 \\ \vdots & \vdots & & \vdots \\ \varphi_{S-s-1} & \varphi_{S-s-2} & \cdots & 1-\sum_{i=1}^{S-s-1} \varphi_i \end{bmatrix}$$

故 $\pi_S = \sum_{i=s+1}^{S} t_{iS} \pi_i$

$$= \sum_{i=s+1}^{S} (1 - \sum_{k=0}^{i-s-1} \varphi_k) \pi_i + \varphi_0 \pi_S$$

$$\pi_j = \sum_{i=j}^{S} \varphi_{i-j} \pi_i \quad j = s+1, s+2, \cdots, S-1$$

$$\sum_{j=s+1}^{S} \pi_j = 1.$$

上述有一个冗余方程：我们抛掉第一个，用后

两个联立求解。

Example 4.

Consider a MC with state space S and transition matrix P.

P is doubly stochastic：

* $\sum_{i\in S} P_{ij} = 1$. $\forall j \in S$. (column). 即各行和、各列和

* $\sum_{j\in S} P_{ij} = 1$, $\forall i \in S$, (row) 均为 1.

then its stationary distribution is

$$\pi_j = \sum_{i\in S} P_{ij}\, \pi_i \qquad \forall j \in S.$$

$\pi_j = c$, $\forall j \in S$ is a solution of above equation.

If MC is irreducible and aperiodic, then

* $c = \frac{1}{|S|}$, if $|S| < \infty$, MC is ergodic.

* $c = 0$, if $|S| = \infty$, MC is null recurrent or transient.

Example Age of a Renewal Process.

We suppose that an item is put in use at time 1.

the life of the item is $j \overset{iid}{\sim} P_j$.

then $\{P_j\}$ is aperiodic and $\sum j P_j < \infty$.

Now let X_n = age of item in use at time n.

$\{X_n, n\geq 0\}$ is a MC.

$$P_{i,j} = \begin{cases} 1 - P_{i,i+1} = 1 - P(D\geq i+1 \mid D\geq i) = \dfrac{P_i}{\sum_{j=i} P_j} & j = 1. \\ P_{i,i+1} = 1 - \dfrac{P_i}{\sum_{j=i} P_j} & j = i+1 \end{cases}$$

$$\pi_j = \sum_{i=1}^{\infty} P_{ij}\, \pi_i$$

$$\pi_1 = \sum_{i=1}^{\infty} P_{i,1}\, \pi_i = \sum_{i=1}^{\infty} \frac{P_i}{\sum_{j=i} P_j}\, \pi_i .$$

$$\pi_j = P_{j-1,j}\,\pi_{j-1} = \left(1 - \frac{P_{j-1}}{\sum_{k=j-1} P_k}\right)$$

$$\sum_j \pi_j = 1$$

$$\Rightarrow \pi_1 = \frac{1}{EX}, \quad \pi_j = \frac{P(X \geq j)}{EX}.$$