Operations Research 2022

Problem Set 1

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CH2. The Geometry of Linear Programming

1. **(EX 2.2)** Let $f : \mathbb{R}^n \mapsto \mathbb{R}$ be a convex function and let c be some constant. Show that the set $S = \{\boldsymbol{x} \in \mathbb{R}^n \mid f(\boldsymbol{x}) \leq c\}$ is convex.

2. **(EX 2.3) (Basic feasible solution in standard form polyhedra with upper bonds)** Consider a polyhedron defined by the constraints $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ and $\boldsymbol{0} \leq \boldsymbol{x} \leq \boldsymbol{u}$, and assume that the matrix $\boldsymbol{A}$ has linearly independent rows. Provide a procedure analogous to the one in Section 2.3 for constructing basic solutions,and prove an analog of Theorem 2.4.

3. **(EX 2.6) (Caratheodory's theorem)** Let $\boldsymbol{A}_1, \boldsymbol{A}_2, \cdots, \boldsymbol{A}_n$ be a collection of vectors in $\mathbb{R}^m$.

   (i) Let
   $$C = \left\{ \sum_{i=1}^{n} \lambda_i \boldsymbol{A}_i \mid \lambda_1, \lambda_2, \cdots, \lambda_n \geq 0 \right\}.$$
   Show that any element of $\boldsymbol{C}$ can be expressed in the form $\sum_{i=1}^{n} \lambda_i \boldsymbol{A}_i$,with $\lambda_i \leq 0$, and with at most m of the coefficients λ_i being nonzero. *hint:* Consider the polyhedron
   $$\Lambda = \left\{ (\lambda_1, \lambda_2, \cdots, \lambda_n) \in \mathbb{R}^n \mid \sum_{i=1}^{n} \lambda_i \boldsymbol{A}_i = \boldsymbol{y}, \lambda_1, \lambda_2, \cdots, \lambda_n \geq 0 \right\}.$$

   (ii) Let P be the convex hull of the vectors $\boldsymbol{A}_i$:
   $$P = \left\{ \sum_{i=1}^{n} \lambda_i \boldsymbol{A}_i \mid \sum_{i=1}^{n} \lambda_i = 1, \lambda_1, \lambda_2, \cdots, \lambda_n \geq 0 \right\}.$$
   Show that any element of P can be expressed in the form $\sum_{i=1}^{n} \lambda_i \boldsymbol{A}_i$, where $\sum_{i=1}^{n} \lambda_i = 1$, and $\lambda_i \geq 0$ for all i, with at most $m+1$ of the coefficients λ_i being nonzero.

4. **(EX 2.7)** Suppose that $\{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{a}_i^\top \boldsymbol{x} \geq \boldsymbol{b}_i, i = 1, \cdots, m\}$ and $\{\boldsymbol{x} \in \mathbb{R}^n \mid \mathbf{g}_i^\top \boldsymbol{x} \geq h_i, i = 1, \cdots, k\}$ are two representations of the same nonempty polyhedron. Suppose that the vectors $\boldsymbol{a}_1, \cdots, \boldsymbol{a}_m$ span $\mathbb{R}^n$. Show that the same must be true for the vectors $\mathbf{g}_1, \cdots, \mathbf{g}_k$.

5. **(EX 2.16)** Consider the set $\{\boldsymbol{x} \in \mathbb{R}^n \mid x_1 = \cdots = x_{n-1} = 0, 0 \leq x_n \leq 1\}$. Could this be the feasible set of a problem in standard form?

6. **(EX 2.22)** Let P and Q be polyhedron in $\mathbb{R}^n$. Let $P + Q = \{\boldsymbol{x} + \boldsymbol{y} \mid \boldsymbol{x} \in P, \boldsymbol{y} \in Q\}$.

(i) Show that $P + Q$ is a polyhedron.

(ii) Show that every extreme point of $P + Q$ is the sum of an extreme point of P and an extreme point of Q.

7. Show that the **convex hull** $\operatorname{conv}\{\boldsymbol{x}_1, \boldsymbol{x}_2, \cdots, \boldsymbol{x}_n\}$ is the smallest **convex set** that contains the set $\{\boldsymbol{x}_1, \boldsymbol{x}_2, \cdots, \boldsymbol{x}_n\}$.

8. Show that the **conic hull** $\operatorname{cone}\{\boldsymbol{x}_1, \boldsymbol{x}_2, \cdots, \boldsymbol{x}_n\}$ is the smallest **convex cone** that contains the set $\{\boldsymbol{x}_1, \boldsymbol{x}_2, \cdots, \boldsymbol{x}_n\}$.

9. **(Optional)** Show that any polyhedral cone $\boldsymbol{C} := \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{A}\boldsymbol{x} \geq \boldsymbol{0}\}$ can be generated by a finite number of vectors: there exists a set of finite vectors $\{\boldsymbol{b}_1, \boldsymbol{b}_2, \cdots, \boldsymbol{b}_m\} \subseteq \mathbb{R}^n$ such that $\boldsymbol{C} = \operatorname{cone}\{\boldsymbol{b}_1, \boldsymbol{b}_2, \cdots, \boldsymbol{b}_m\}$, *i.e.*,

$$\{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{A}\boldsymbol{x} \geq \boldsymbol{0}\} = \left\{ \sum_{i=1}^{m} \lambda_i \boldsymbol{b}_i \mid \lambda_i \geq 0 \right\}.$$

10. We know the optimality condition of LP tells us that "for a general LP, (i) the optimal cost is either $-\infty$, or (ii) there exists an **extreme point** that is optimal solution." Please use the procedure of Simplex Algorithm (SA) to specify the condition under which the SA will output the optimal basic feasible solution (BFS) which is an extreme point, and the condition under which SA will output an extreme ray which corresponds to the unbounded case (optimal cost being $-\infty$).

11. **(Optional)** Consider two polyhedrons P_1 and P_2 with

$$P_1 = \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{A}_1\boldsymbol{x} \leq \boldsymbol{b}_1\}, \quad P_2 = \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{A}_2\boldsymbol{x} \leq \boldsymbol{b}_2\}.$$

How do we know $P_1 \subseteq P_2$ is correct?