1.1. Let N denote a nonnegative integer-valued random variable. Show that

$$E[N] = \sum_{k=1}^{\infty} P\{N \geq k\} = \sum_{k=0}^{\infty} P\{N > k\}$$

In general show that if X is nonnegative with distribution F, then

$$E[X] = \int_0^{\infty} \bar{F}(x)\, dx.$$

and

$$E[X^n] = \int_0^{\infty} n x^{n-1} \bar{F}(x)\, dx.$$

*proof. $E[N] = \sum_{k=1}^{\infty} P\{N \geq k\} = \sum_{k=1}^{\infty} P\{N>k \cup N=k\}$

$$= \sum_{k=1}^{\infty} P\{N>k\} + P\{N=k\} \quad \text{(exclusive)}.$$

$$= \sum_{k=1}^{\infty} P\{N>k\} + \sum_{k=1}^{\infty} P\{N=k\}.$$

Since N is a nonnegative integer-valued random variable.

then. $P\{N>0\} = \sum_{k=1}^{\infty} P\{N=k\}$.

Hence, $E[N] = \sum_{k=1}^{\infty} P\{N \geq k\} = \sum_{k=0}^{\infty} P\{N>k\}$.

Since X is nonegative, then.

$$EX = \int_0^{+\infty} x \, dF(x)$$

$$= \int_0^{+\infty} -x \, d(1-F(x))$$

$$= -[x(1-F(x))]\Big|_0^{+\infty} + \int_0^{+\infty} (1-F(x))\, dx.$$

$$= \int_0^{+\infty} \bar{F}(x)\, dx - \lim_{x\to+\infty} x(1-F(x))$$

Because the expectation of X exists.

$$\lim_{x\to+\infty} x(1-F(x)) = 0$$

$$EX = \int_0^{+\infty} \bar{F}(x)\, dx.$$

$$E[X^n] = \int_0^{+\infty} -x^n \, d(1-F(x))$$

$$= -[x^n(1-F(x))]\Big|_0^{+\infty} + \int_0^{+\infty} \bar{F}(x)\, dx^n.$$

$$= \int_0^{+\infty} n x^{n-1} \bar{F}(x)\, dx - \lim_{x\to\infty} x^n(1-F(x)).$$

Because the expectation of x^n exists,

$$E[X^n] = \int_0^{+\infty} n x^{n-1} \bar{F}(x)\, dx.$$

#

1.2. If X is a continuous random variable having distribution F show that.

a) $F(X)$ is uniformly distributed over $(0,1)$.

b) if U is a uniform $(0,1)$ random variable, then $F^{-1}(U)$ has distribution F, where $F^{-1}(x)$ is that value of y such that $F(y)=x$.

* proof. a). we suppose $Y=F(X)$, then

$$F_Y(x)=P(Y\le x)=P(F(X)\le x)=P(X\le F^{-1}(x))=F(F^{-1}(x))=x.$$

Hence. $Y=F(X)$ is uniformly distributed over $(0,1)$.

b). $U\sim U(0,1)$. We suppose $Y=F^{-1}(U)$, then

$$F_Y(x)=P(Y\le x)=P(F^{-1}(U)\le x)=P(U\le F(x))=F(x).$$

thus, $F^{-1}(U)$ has distribution F. #

1.3. Let X_n denote a binormal random variable with parameters (n, p_n), $n\geq 1$. If $np_n\to\lambda$ as $n\to\infty$, show that.

$$P\{X_n=i\}\to\frac{e^{-\lambda}\lambda^i}{i!}\quad\text{as}\quad n\to\infty.$$

* proof. Since $X_n\sim B(n, p_n)$. then we have.

$$\begin{aligned}P\{X_n=i\}&=\binom{n}{i}(p_n)^i(1-p_n)^{n-i}\\&=\frac{n!}{i!(n-i)!}(p_n)^i(1-p_n)^{n-i}\\&=\frac{n(n-1)\cdots(n-i+1)}{i!}(p_n)^i(1-p_n)^{n-i}\\&=\frac{1}{i!}\,1\cdot(1-\tfrac{1}{n})\cdots(1-\tfrac{i-1}{n})(np_n)^i\left[(1-p_n)^{\frac{1}{p_n}}\right]^{np_n-ip_n}\quad(*)\end{aligned}$$

Since $np_n\to\lambda$, $p_n\to 0$ as $n\to\infty$. Thus. $(1-p_n)^{\frac{1}{p_n}}\to e$ as $n\to\infty$. The formula $(*)$ has the limit $\frac{1}{i!}\lambda^i e^{\lambda}$ as $n\to\infty$. #

1.4. Compute the mean and variance of a binomial random variable with parameters n and p.

* proof. 我们设 X_i, $i=1,\cdots,n$ 是互相独立的重复试验. 试验发生的概率是 p. 则 $X=X_1+X_2+\cdots+X_n$ 是服从二项分布的随机变量. X_i 的期望 $EX_i=p$, 方差 $\mathrm{Var}\,X_i=EX_i^2-(EX_i)^2=p-p^2$

由于 X_i 之间相互独立. 故二项分布的期望:

$$EX=E\left(\sum_{i=1}^{n}X_i\right)=\sum_{i=1}^{n}EX_i=np.$$

方差:

$$\mathrm{Var}\,X=\mathrm{Var}\left(\sum_{i=1}^{n}X_i\right)=\sum_{i=1}^{n}\mathrm{Var}\,X_i=np(1-p).$$

#

1.5. Suppose that n independent trails – each of which results in either outcome $1,2,\cdots,r$ with respective probabilities $p_1,p_2,\cdots,p_r$ – are performed, $\sum_{i=1}^{r}p_i=1$ Let N_i denote the number of trails resulting in outcome i

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a) Compute the joint distribution of $N_1, \cdots, N_r$. This is called the multinomial distribution.

b) Compute $Cov(N_i, N_j)$.

c). Compute the mean and variance of the number of outcomes that do not occur.

*proof. a). 我们记 n_i 为试验结果为 i 的次数. 由于 $p_1+\cdots+p_r=1$. 因此 $n_1+\cdots+n_r=n$. 因此 $N_1,\cdots,N_r$ 的联合分布为.

$$P(N_1=n_1,\cdots,N_r=n_r)=\binom{n}{n_1}\binom{n-n_1}{n_2}\cdots\binom{n-n_1-\cdots-n_{r-1}}{n_r}p_1^{n_1}\cdots p_r^{n_r}$$
$$=\frac{n!}{n_1!(n-n_1)!}\frac{(n-n_1)!}{n_2!(n-n_1-n_2)!}\cdot\cdots\cdot\frac{(n-n_1-\cdots-n_{r-1})!}{n_r!\ 0!}p_1^{n_1}\cdots p_r^{n_r}$$
$$=\frac{n!}{n_1!\cdots n_r!}p_1^{n_1}\cdots p_r^{n_r}$$

b). 根据公式 $Cov(N_i,N_j)=E[N_iN_j]-EN_iEN_j$. 我们需要计算各项期望如下:

$$E[N_iN_j]=E[E[N_iN_j|N_j]]$$ (双期望公式. 证明见后面).
$$=E[N_j\,E[N_i|N_j]]$$
$$=E[N_j\cdot(n-N_j)\cdot\frac{p_i}{1-p_j}]$$
$$=\frac{p_i}{1-p_j}[nEN_j-EN_j^2]$$
$$=\frac{p_i}{1-p_j}[n\cdot np_j-[Var\,N_j+(EN_j)^2]]$$
$$=\frac{p_i}{1-p_j}[n^2p_j-np_j(1-p_j)-n^2p_j^2]$$
$$=n^2p_jp_i-np_ip_j$$

由于 $N_i\sim B(n,p_i)$ $N_j\sim B(n,p_j)$. 因此

$EN_i=np_i$. $Var\,N_i=np_i(1-p_i)$

$EN_j=np_j$ $Var\,N_j=np_j(1-p_j)$

因此协方差 $Cov(N_i,N_j)=n^2p_jp_i-np_ip_j-n^2p_ip_j=-np_ip_j$

双期望公式证明: $E[E[X|Y]]=\int E[X|y]f_Y(y)dy$

$$=\int[\int x f(x|y)dx]f_Y(y)dy$$
$$=\iint x\cdot[f(x|y)\cdot f_Y(y)]dxdy$$
$$=\iint x f(x,y)dydx$$
$$=\int x\int f(x,y)dy\,dx$$
$$=\int x f_X(x)dx=EX.$$

c). 我们记示性函数 I_j 如下:

$$I_j = \begin{cases} 1 & \text{如果结果 } j \text{ 未出现} \\ 0 & \text{otherwise.} \end{cases}$$

则 $E I_j = P\{$结果 j 在 n 次试验中均未出现$\}$

$$= (1-P_j)^n$$

$$\mathrm{Var}\, I_j = E I_j^2 - (E I_j)^2 = (1-P_j)^n - (1-P_j)^{2n}$$

$E[I_j I_i] = P\{$结果 i 和结果 j 在 n 次试验中均未出现$\}$

$$= (1-P_i-P_j)^n$$

则 $\mathrm{Cov}(I_i, I_j) = E I_j I_i - E I_j\, E I_i = (1-P_i-P_j)^n - [(1-P_j)(1-P_i)]^n$

从而 记 I 为在 n 次试验中未出现的结果个数. 有.

$$EI = E\sum_{i=1}^{r} I_i = \sum_{i=1}^{r} E I_j = \sum_{i=1}^{r}(1-P_i)^n$$

$$\mathrm{Var}\, I = \mathrm{Var}\sum_{i=1}^{r} I_i = \sum_{i=1}^{r}\mathrm{Var}(I_i) + 2\sum_{i<j}\mathrm{Cov}(I_i, I_j)$$

$$= \sum_{i=1}^{r}[(1-P_i)^n - (1-P_i)^{2n}] + 2\sum_{i<j}[(1-P_i-P_j)^n - (1-P_j)^n(1-P_i)^n]$$

$$= \sum_{i=1}^{r}[(1-P_i)^n(1-(1-P_i)^n)] + \sum_{i\neq j}[(1-P_i-P_j)^n - (1-P_j)^n(1-P_i)^n]$$

#

1.6 Let $X_1, X_2, \cdots$ be independent and identically distributed continuous random variables. We say that a record occurs at time n, $n>0$ and has values X_n if $X_n > \max(X_1, \cdots, X_{n-1})$, where $X_0 = -\infty$.

a) Let N_n denote the total number of records that have occurred up to (and including) time n. Compute $E(N_n)$ and $\mathrm{Var}(N_n)$.

b). Let $T = \min\{n, n>1$ and a record occurs at $n\}$. Compute $P(T>n)$ and show that $P(T<\infty) = 1$ and $E(T) = \infty$

c). Let T_y denote the time of the first record value greater than y. That is

$$T_y = \min\{n : X_n > y\}$$

show that T_y is independent of X_{T_y}. That is the time of the first value greater than y is independent of that value.

*proof a) 我们记 $I_j = \begin{cases} 1 & \text{若在 } j \text{ 时刻有记录} \\ 0 & \text{otherwise} \end{cases}$.

由于 $X_1, X_2, \cdots$ 独立同分布. 故 $E(I_j) = P(X_j = \max\{X_1, \cdots, X_j\}) = \frac{1}{j}$.

因此 $E(N_n) = E(\sum_{i=1}^{n} I_i) = \sum_{i=1}^{n} E(I_i) = \sum_{i=1}^{n}\frac{1}{i}$

$Var(N_n) = Var(\sum_{i=1}^{n} I_i) = \sum_{i=1}^{n} Var(I_i) = \sum_{i=1}^{n} \frac{1}{i}(1-\frac{1}{i})$.

b). $P(T>n) = P(X_1 = \max\{X_1,\cdots,X_n\}) = \frac{1}{n}$ (相当于 $X_1,\cdots,X_n$ 是独立同分布事件. 现求其中 X_1 最大的概率. $X_1,\cdots,X_n$ 最大是等可能的).

$E[T] = \sum_{i=2}^{\infty} i\cdot p(T=i) = \sum_{i=2}^{\infty} i\text{-}1 = \sum_{i=1}^{\infty} i = \infty$.

$P(T<\infty) = \lim_{n\to\infty} P(T<n) = \lim_{n\to\infty} \frac{n-1}{n} = 1$.

c). $P[X_{T_y}>x,\ T_y=n] = P(X_n>x,\ X_1<y,\ X_2<y,\cdots,X_{n-1}<y,\ X_n>y)$

$$= \begin{cases} P(X_1<y,\cdots,X_{n-1}<y,\ X_n>x) & x>y \\ P(X_1<y,\cdots,X_{n-1}<y,\ X_n>y) & x\le y \end{cases}$$

$$= \begin{cases} [F(y)]^{n-1}\cdot\bar{F}(x) & x>y \\ [F(y)]^{n-1}\bar{F}(y) & x\le y \end{cases}$$

$P(T_y=n) = P(X_1<y,\ X_2<y,\cdots,X_{n-1}<y,\ X_n>y) = [F(y)]^{n-1}\bar{F}(y)$.

$$P(X_{T_y}>x) = P(X_n>x \mid T_y=n) = \begin{cases} 1 & x\le y \\ \frac{\bar{F}(x)}{\bar{F}(y)} & x>y \end{cases}$$

$\Rightarrow P(X_{T_y}>x,\ T_y=n) = P(X_{T_y}>x)\cdot P(T_y=n)$.

因此 T_y 和 X_{T_y} 独立.

#

1.9. A round-robin tournament of n contestants is one in wich each of the $\binom{n}{2}$ pairs of contestants plays each other exactly once, with the outcome of any play being that one of the contestants wins and the other loses. Suppose the players are initially numbered $1,2,\cdots,n$. The permutation $i_1,\cdots,i_n$ is called a Hamiltonian permutation if i_1 beats i_2, i_2 beats i_3, $\cdots$ and i_{n-1} beats i_n. Show that there is an outcome of the round-robin for which the number of Hamiltonians is at least $\frac{n!}{2^{n-1}}$.

proof. 我们令 X 表示 n 个参赛者时从一个特定参赛者出发的 Hamiltonian 置换的数量. 其期望记为 E_n. 则一场循环赛后. 总的 Hamiltonian 置换数量的期望为 nE_n.

我们可以假设知道 $n-1$ 个参赛者的情况. 那么第 n 个参赛者加入后. 若他赢 3 - 场(与 $n-1$ 个参赛者相比). 则有 $\binom{n-1}{1}(\frac{1}{2})^1(\frac{1}{2})^{n-2}$ 的概率. 此时. Hamiltonian 置换的数量为 $n-1$ 个参赛者的情形.

因此: $E_n = \sum_{k=1}^{n-1}\binom{n-1}{k}(\frac{1}{2})^k(\frac{1}{2})^{n-1-k}\,k\,E_{n-1}$

$= \sum_{k=1}^{n-1}\frac{(n-2)!}{(k-1)!(n-k-1)!}(\frac{1}{2})^{k-1}(\frac{1}{2})^{n-k-1}E_{n-1}\cdot\frac{n-1}{2}$

$= \frac{n-1}{2} E_{n-1}$.

注意到 $E_1 = 1$，故 $E_n = \frac{(n-1)!}{2^{n-1}}$，$nE_n = \frac{n!}{2^{n-1}}$

由于期望值为 $\frac{n!}{2^{n-1}}$，故存在一场循环赛使 Hamiltonian 置换数量至少为 $\frac{n!}{2^{n-1}}$

#

1.10. Consider a round-robin tournament having n contestants, and let k, $k<n$, be a positive integer such that $\binom{n}{k}(1-(\frac{1}{2})^k)^{n-k} < 1$. Show that it is possible for the tournament outcome to be such that for every set of k contestants there is a constentant who beat every member of this set.

proof. 令 S 表示由 k 个参赛者组成的任意一组合，$A(S)$ 表示"在 S^c 中没有人打败了 S 中任何一人"这一事件

$P(A(S)) = P($"$n-k$个参赛者与这k个参赛者的比赛中至少输一场"$)$

$= [1-(\frac{1}{2})^k]^{n-k}$

故 $P(\bigcup_S A(S)) \le \sum_S P(A(S)) = \binom{n}{k}[1-(\frac{1}{2})^k]^{n-k} < 1$.

所以有可能出现在一场循环赛中，任取k个人，总有一个选手打败了这k个人。

#

1.12. If $P[0 \le X \le a] = 1$. Show that

$$Var(X) \le \frac{a^2}{4}.$$

proof. $Var(X) = EX^2 - (EX)^2 = \int x^2 \, dF(x) - (\int x \, dF(x))^2$

$$= x^2 F(x)\Big|_0^a - \int 2xF(x)\,dx - (xF(x)\Big|_0^a - \int F(x)\,dx)^2$$

$$= a^2 - \int 2xF(x)\,dx - a^2 - [\int F(x)\,dx]^2 + 2a\int F(x)\,dx$$

$$= -2\int (x-a)F(x)\,dx - [\int F(x)\,dx]^2$$

$$= -\frac{2}{a}\int_0^a 1\,dx \cdot \int_0^a (x-a)F(x)\,dx - [\int F(x)\,dx]^2$$

$$\le -\frac{2}{a}[\int_0^a (x-a)\,dx \cdot \int_0^a F(x)\,dx] - [\int F(x)\,dx]^2$$

$$= a\int_0^a F(x)\,dx - [\int_0^a F(x)\,dx]^2 \le \frac{a^2}{4}$$

#

1.15. Let F be a continuous distribution function and let U be a uniform $(0,1)$ random variable

a) If $X = F^{-1}(U)$, show that X has distribution function F

b). show that $-\log(U)$ is an exponential random variable. with mean 1.

proof. a) $F_X(x) = P(X \le x) = P(F^{-1}(U) \le x)$

$= P(U \le F(x)) = F_U(F(x)) = F(x).$

b). 令 $F(x) = e^{x}$. 则 $F^{-1}(U) = -\log(U)$. 由 a) 知

$-\log U \sim e^{x}$. 为指数分布. 其均值为 1.

#

1.16. Let $f(x)$ and $g(x)$ be probability density function, and suppose that for some constant c, $f(x) \le cg(x)$. for all x. Suppose we can generate random variables having density function g, and consider the following algorithm.

Step 1: Generate Y, a random variable having density function g

Step 2: Generate U, a uniform $(0,1)$ random variable.

Step 3: If $U \le \frac{f(Y)}{cg(Y)}$ set $X=Y$. Otherwise, go back to Step 1.

Assuming that successively generated random variables are independent, show that:

a) X has density function f.

b). the number of iterations of the algorithm needed to generate X is a geometric random variable with mean c.

proof. a). 设 X 的密度函数为 $f_X(x)$. 则

$$f_X(x) = P\{Y=x \mid U \le \frac{f(Y)}{cg(Y)}\}$$

$$= P(Y=x,\ U \le \frac{f(x)}{cg(x)}) \times \frac{1}{P(U \le \frac{f(Y)}{cg(Y)})}$$

$$= g(x) \cdot P(U \le \frac{f(x)}{cg(x)}) \times \frac{1}{P(U \le \frac{f(Y)}{cg(Y)})}$$

$$= g(x) \cdot \frac{f(x)}{cg(x)} \times \frac{1}{P(U \le \frac{f(Y)}{cg(Y)})}$$

$$= \frac{1}{c} \frac{1}{P(U \le \frac{f(Y)}{cg(Y)})} \cdot f(x).$$

而 $P(U \le \frac{f(Y)}{cg(Y)}) = \int_{-\infty}^{+\infty} \frac{f(y)}{cg(y)} g(y)dy = \frac{1}{c}\int_{-\infty}^{+\infty} f(y)dy = \frac{1}{c}$

故 $f_X(x) = f(x).$

b). 由于每次迭代都相互独立. 每次成功的概率为

$$P(U \le \frac{f(Y)}{cg(Y)}) = \frac{1}{c}$$

故服从分布 $Ge(\frac{1}{c})$. 均值为 c.

#

1.22. the conditional variance of X given Y, is defined by

$$Var(X|Y) = E[(X-E(X|Y))^2|Y]$$

Prove the conditional variance formula, namely.

$$Var(X) = E[Var(X|Y)] + Var[E[X|Y]]$$

Use this to obtain $var(X)$ in Example 1.5 (B) and check your result by differentiating the generating function.

prove. $Var(X) = E(X-EX)^2 = E[X-E(X|Y)+E(X|Y)-EX]^2$

$$= E(X-E(X|Y))^2 + E(E(X|Y)-EX)^2 + 2E\{(X-E(X|Y))(E(X|Y)-EX)\}$$

其中 $E(X-E(X|Y))^2 = E\{E[(X-E(X|Y))^2|Y]\}$

$$= E\{E(x^2|Y)-2(E(X|Y))^2+(E(X|Y))^2\}$$

$$= E\{E(X^2|Y)-(E(X|Y))^2\} = E[Var(X|Y)]$$

$$E(E(X|Y)-EX)^2 = E(E(X|Y)-E(E(X|Y)))^2$$

$$= Var(E(X|Y))$$

$$E[(X-E(X|Y))(E(X|Y)-EX)] = E\{E[(X-E(X|Y))(E(X|Y)-EX)|Y]\}$$

$$= E\{(E(X|Y))^2-(E(X|Y))^2-EX\,E[X|Y]+E(X|Y)EX\}$$

$$=0$$

故 $Var(X) = E[Var(X|Y)] + Var(E(X|Y))$.

#

1.25 Consider a gambler who on each gamble is equally likely to either win or lose 1 unit Starting with i show that expected time until the gambler's fortune is either 0 or k is $i(k-i)$, $i=0, \cdots, k$.

1.39. A particle moves along the following graph so that at each step it is equally likely to move to any of its neighbours

$$0 - 1 - 2 - \cdots - n-1 - n.$$

Starting at 0 show that the expected number of steps it takes to reach n is n^2.

proof. 令 T_i 为从 $i-1$ 到 i 所需要的步数

$i=1$. 从 0 到 1. 此时显然 $T_1 = 1$.

$i>1$. 则从 $i-1$ 到 i 有两种可能.

① 从 $i-1$ 跳到 i, 概率 $\frac{1}{2}$. 步数为 1.

② 从 $i-1$ 跳到 $i-2$. 再从 $i-2$ 跳到 i, 概率为 $\frac{1}{2}$. 步数 $1+T_{i-1}+T_i$

故 $ET_i = \frac{1}{2} \times 1 + \frac{1}{2} \times (1 + ET_{i-1} + ET_i)$

$\Rightarrow ET_i = 2 + ET_{i-1}$.

而从 0 到 n 的平均步数为 $\sum_{i=0}^{n-1} T_i = 1 + \sum_{i=1}^{n-1} T_i = 1+3+5+\cdots+2n-1$

$= \frac{1}{2} \times (2n-1+1) \times n = n^2$

#