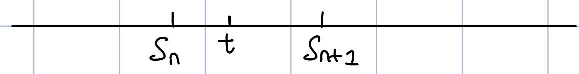
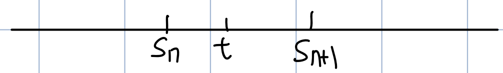
3.1. a) $N(t)<n$ 表示在时刻 t 前发生的更新次数小于 $n \Leftrightarrow$ 第 n 次更新是在时刻 t 后面发生的. $\Leftrightarrow S_n>t$，故等价

b)



在这种情况下. $N(t)\leq n$ 成立. 但 $S_n<t$. 故不等价

c).



此时. 有 $S_n<t$ 成立. 但 $N(t)=n$. 与 $N(t)>n$ 矛盾. 故不等价.

3.3. $X_{N(t)+1}$：表示是 t 时刻前最后一个更新与 t 时刻后第一个更新之间的时间.

$$P(X_{N(t)+1}\geq x)=\int_0^\infty p(X_{N(t)+1}\geq x \mid S_{N(t)}=s)\,p(S_{N(t)}=s)\,ds$$
$$=\int_0^\infty p(X_{N(t)+1}\geq x \mid X_{N(t)+1}>t-s)\,p(S_{N(t)}=s)\,ds.$$
$$=\int_0^\infty \frac{P(X_{N(t)+1}\geq x,\ X_{N(t)+1}>t-s)}{P(X_{N(t)+1}>t-s)}\,p(S_{N(t)}=s)\,ds$$
$$\leq\int_0^\infty \min[P(X_{N(t)+1}\geq x),\ P(X_{N(t)+1}>t+s)]\ p(S_{N(t)}=s)\,ds.$$
$$\leq\int_0^\infty P(X_{N(t)+1}\geq x)\cdot P(S_{N(t)}=s)\,ds.$$
$$=\overline{F}(x).$$

当 $F(x)=1-e^{-\lambda s}$，则 $P(X_{N(t)+1}\geq x)=\int_0^\infty \min\left[\frac{e^{-\lambda x}}{e^{-\lambda(t-s)}},1\right]dF_{S_{N(t)}}(s)=(1+\frac{x}{\lambda})e^{-\lambda x}-e^{-\lambda t}$

3.4. $$m(t)=E[N(t)]=\int_0^\infty E[N(t)\mid X_1=x]\cdot p(X_1=x)\,dx.$$

若 $t<x$. 则 $E[N(t)\mid X_1=x]=0$

若 $t\geq x$. 则 $E[N(t)\mid X_1=x]=1+E[N(t-x)]$

故 $$m(t)=\int_0^t f(x)+E[N(t-x)]f(x)\,dx$$
$$=F(t)+\int_0^t m(t-x)\,dF(x).$$

3.5. $$m(t)=E[N(t)]=\sum_{n=1}^\infty P(N(t)\geq n)$$
$$=\sum_{n=1}^\infty P(S_n\leq t)=\sum_{n=1}^\infty F_n(t).$$

故 $m(t)$ 由 $F_n(t)$ 唯一确定. 而 $F_n(t)$ 是 $F(t)$ 的 n 重卷积.
也是相互唯一确定的. 故 $m(t)$ 与 F 互相唯一确定.

3.6

$$\begin{aligned} E[N(s) \mid N(t)=n] &= \sum_{i=1}^{n} P[N(s) \geq i \mid N(t)=n] \\ &= \sum_{i=1}^{n} P[S_i \leq s \mid N(t)=n] \\ &= \sum_{i=1}^{n} P(U_{(i)} \leq s) \\ &= E[\sum_{i=1}^{n} I(U_{(i)} \leq s)] \\ &= E[\sum_{i=1}^{n} I(U_i \leq s)] \\ &= \sum_{i=1}^{n} P(U_i \leq s) = \frac{ns}{t} \end{aligned}$$

$$m(s) = E[N(s)] = E[E[N(s) \mid N(t)]] = \frac{s}{t} EN(t) = \frac{s}{t} m(t).$$

$\Rightarrow$ $m(s) = \lambda s$ 根据 3.5 知. $m(s)$ 唯一确定 F. 即
确定了这个 Renewal Process. 而符合 $m(s) = \lambda s$ 的
是 Poisson Process.

3.7

由 Renewal 方程:

$$m(t) = F(t) + \int_0^t m(t-x)\, dF(x).$$

$$\Rightarrow m(t) = t + \int_0^t m(t-x)\, dx$$

$$= t + \int_0^t m(y)\, dy$$

两边求导: $m'(t) = 1 + m(t)$.

$$\Rightarrow (e^{-t} m(t))' = e^{-t}$$

$$e^{-t} m(t) = -e^{-t} + C$$

由于 $m(0) = 0 \Rightarrow C = 1$

故 $m(t) = e^t - 1$.

$N(1)+1$ 是超过 1 之后所有更新点的数量.

故 $E[N(1)+1] = 1+M(1) = e$ 即为题目所求.

3.10 (a). $\lim_{M\to\infty}\left(\frac{S_1+\cdots+S_M}{N_1+\cdots+N_M}\right) = \lim_{M\to\infty}\frac{\sum_{i=1}^{N_1+\cdots+N_M} X_M}{N_1+\cdots+N_M}$

根据强大数定律:

$\lim_{M\to\infty}\left(\frac{S_1+\cdots+S_M}{N_1+\cdots+N_M}\right) = EX_1$

(b) 由于 $N_1, N_2, \cdots$ 是 i.i.d 的随机变量. $EN_1<\infty$.

$S_1,\cdots,S_M$ 也是 i.i.d 的. 且 $ES_1 = EN_1\cdot EX_1<\infty$.

故由强大数定律:

$\lim_{M\to\infty}\left(\frac{S_1+\cdots+S_M}{N_1+\cdots+N_M}\right) = \lim_{M\to\infty}\frac{S_1+\cdots+S_M}{M}\cdot\frac{M}{N_1+\cdots+N_M}$

$= \frac{E(S_1)}{EN_1}$

(c). $EX_1 = \frac{ES_1}{EN_1} \Rightarrow ES_1 = E\sum_{i=1}^{N_1}X_1 = EN_1\,EX_1$. (wald's equation).

3.11. a) 令 X_i 为第 i 次与第 i+1 次选择间的天数.

$$X_i = \begin{cases} 2 & X_i \text{ 选择 } 1 \\ 4 & X_i \text{ 选择 } 2 \\ 8 & X_i \text{ 选择 } 3. \end{cases}$$

stopping time. $N = \min\{n: X_n=2\}$.

若 $X_n=2$, 则不存在 X_{n+1}. 即停止观测.

b) $E[T] = E[\sum_{i=1}^{N}X_i] = EN\cdot EX$.

而 $EX = \frac{1}{3}\times(2+4+8) = \frac{14}{3}$. $EN = \sum_{n=1}^{\infty} n\times(\frac{2}{3})^{n-1}\times\frac{1}{3}$

$= \lim_{n\to\infty} 3-(3+n)\times(\frac{2}{3})^n = 3$.

故 $E[T] = 14$

c). $E[\sum_{i=1}^{N}X_i \mid N=n] = E[\sum_{i=1}^{n}X_i \mid X_1,\cdots,X_{n-1}\neq 2, X_n=2]$

$= (n-1)\,E[X_i \mid X_i\neq 2] + E[X \mid X=2]$

$= (n-1)\times 6 + 2 = 6n-4.$

$$E\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} EX_i = \frac{14}{3}n$$

d) $E\left[\sum_{i=1}^{N} X_i\right] = E[6N-4] = 6EN-4 = 6\times 3-4 = 14$

3.13 我们可以把该问题抽象为如下一个 Reward Renewal Process. 将每次重新回到 state 1 作为更新点. 则 X 表示每一轮所用的时间. 是 i.i.d 的. 当在 state i 时. 系统 "on". 值为 1. 否则系统 "off". 则

$$\lim_{t\to\infty} P(\text{process is in state } i \text{ at time } t)$$

$$= \frac{E(\text{state } i)}{EX} = \frac{\int_0^\infty \bar{F}_i(t)\,dt}{\int_0^\infty \overline{F_1 * F_2 * \cdots * F_n * H}(t)\,dt}.$$

3.17 把 $g = h + h*m$ 代入方程.

$$h + (h + h*m)*F = h + h*F + h*\sum_{n=2}^{\infty} F_n(x) = h + h*\sum_{n=1}^{\infty} F_n(x) = g.$$

故 $g = h + h*m$ 是 renewal type equation 的一个解.

a) $P(t) = P(t \mid S_{N(t)}=0)\,\bar{F}(t) + \int_0^t P(t \mid S_{N(t)}=y)\,\bar{F}(t-y)\,dm(y).$

$$h(t) = P(t \mid S_{N(t)}=0)\times\bar{F}(t) = \frac{P(X>t)}{\bar{F}(t)}\bar{F}(t). = P(X>t)$$

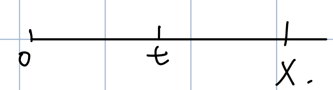
$$h(t-y) = P(t \mid S_{N(t)}=y)\,\bar{F}(t-y) = \frac{P(X>t-y)}{\bar{F}(t-y)}\bar{F}(t-y) = P(X>t-y).$$

因此符合 Renewal type equation. 从而

$$\lim_{t\to\infty} g(t) = \frac{\int_0^\infty P(X>t)\,dx}{\int_0^\infty \bar{F}(t)\,dt} = \frac{E[\text{system on}]}{EX}$$

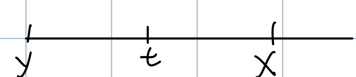
b). $g(t) = E[A(t)] = E[A(t) \mid S_{N(t)}=0]\,\bar{F}(t) + \int_0^t E[A(t) \mid S_{N(t)}=y]\,\bar{F}(t-y)\,dm(y)$

令 $h(t) = E[A(t) \mid S_{N(t)}=0]\,\bar{F}(t)$



$= E[t \mid X>t]\,\bar{F}(t)$

$h(t-y) = E[A(t) \mid S_{N(t)}=y]\,\bar{F}(t-y)$



$= E[t-y \mid X>t-y]\,\bar{F}(t-y)$

因此 $g(t)$ 是 Renewal Type Equation 的解.

$$\int_0^\infty h(t)\,dt = \int_0^\infty t\int_t^\infty dF(s)\,dt = \int_0^\infty \int_0^s t\,dt\,dF(s) = \frac{1}{2}EX^2$$

$$\lim_{t\to\infty} g(t) = \frac{\int_0^\infty h(t)\,dt}{EX} = \frac{EX^2}{2EX}$$

#

3.25. a). $m_D(t) = E[N_D(t)] = \sum_{i=1}^{n} P(N_D(t)\geq i)$

$$= \sum_{i=1}^{n} P(S_{Di}\leq t).$$

$$= P(X_1\leq t) + \sum_{i=2}^{\infty} P(S_{Di}\leq t)$$

$$= G(t) + \sum_{i=2}^{\infty} P(S_{Di}\leq t).$$

$$= G(t) + \sum_{i=2}^{\infty} F_{n-1} * G(t)$$

$$= G(t) + \sum_{i=2}^{\infty} \int_0^t F_{i-1}(t-x)\,dG(t)$$

$$= G(t) + \int_0^t \sum_{i=1}^{\infty} F_i(t-x)\,dG(t)$$

$$= G(t) + \int_0^t m(t-x)\,dG(t)$$

b) $P(S_{N(t)}\leq s) = \sum_{n=0}^{\infty} P(S_n\leq s,\ S_{n+1}>t)$

$$= P(X_1>t) + P(X_1\leq s,\ S_2>t) + \sum_{n=2}^{\infty} P(S_n\leq s,\ S_{n+1}>t)$$

其中 $P(X_1\leq s,\ S_2>t) = \int_0^\infty P(X_1\leq s.\ S_2>t \mid X_1=y)\,dG(y)$

$$= \int_0^s P(S_2>t\mid X_1=y)\,dG(y)$$

$$= \int_0^s \bar F(t-y)\,dG(y)$$

$$\sum_{n=2}^{\infty} P(S_n\leq s,\ S_{n+1}>t) = \sum_{n=2}^{\infty}\int_0^\infty P(S_n\leq s,\ S_{n+1}>t \mid S_n=y)\,d\,F_{n-1}*G(y)$$

$$= \sum_{n=2}^{\infty}\int_0^s P(S_{n+1}>t\mid S_n=y)\,d\,F_{n-1}*G(y)$$

$$= \sum_{n=2}^{\infty}\int_0^s \bar F(t-y)\,d\,F_{n-1}*G(y)$$

$$= \int_0^s \bar F(t-y)\,d\sum_{n=1}^{\infty} F_n * G(y)$$

故 $P(S_{N(t)} \le s) = \bar{G}(t) + \int_0^s \bar{F}(t-y)\, d\, m_D(y)$

$P(S_{N(t)}=0) = \bar{G}(t)$. $\quad P(S_{N(t)}=s) = \bar{F}(t-s)\, d\, m_D(s)$.

从而 $E[A_D(t)] = E[A_D(t) \mid S_{N(t)}=0]\, \bar{G}(t) + \int_0^t E[A_D(t) \mid S_{N(t)}=y]\, \bar{F}(t-y)\, d\, m(y)$

令 $h(t) = E[A_D(t) \mid S_{N(t)}=0]\, \bar{F}(t)$

$= E[t \mid X>t]\, \bar{F}(t)$.

$h(t-y) = E[A_D(t) \mid S_{N(t)}=y]\, \bar{F}(t-y)$

$= E[t-y \mid X>t-y]\, \bar{F}(t-y)$.

则 h 非负. 非增. 且.

$\int_0^\infty h(t)\, dt = \int_0^\infty t \int_t^{+\infty} dF(s)\, dt$

$= \int_0^\infty \int_0^s t\, dt\, dF(s) = \frac{1}{2} EX^2 < \infty$.

故由 KRT 定理:

$\lim_{t\to\infty} E[A_D(t)] = \lim_{t\to\infty} E[A_D(t) \mid S_{N(t)}=0]\, \bar{G}(t) - h(t) + h(t) + \int_0^t h(t-y)\, d\, m(y)$

$= \frac{1}{\mu} \int_0^\infty h(x)\, dx = \frac{EX^2}{2EX} = \frac{\int_0^\infty x^2\, dF(x)}{2\int_0^\infty x\, dF(x)}$.

c). $E[G] = \int_0^\infty \bar{G}(t)\, dt$

$= t\bar{G}(t) \Big|_0^\infty - \int_0^\infty t\, d\, \bar{G}(t)$

$= t\bar{G}(t) \Big|_0^\infty + \int_0^\infty t\, dG(t)$

$= \lim_{t\to\infty} t\bar{G}(t) + E[G]$

$\Rightarrow \lim_{t\to\infty} t\bar{G}(t) = 0$

3.29 a). ① 若 $F(x \le A)$. 则 汽车因为 $f(x)$ 而更换. 则.

$cost = C_1 + C_2$.

② 若 $F(x > A)$, 则 汽车更换时还良好. 则

$cost = C_1 - R(A)$.

故. $E[cost] = \int_0^A C_1 + C_2\, d\, F(x) + \int_0^A C_1 - R(A)\, d\, \bar{F}(x)$

$$= C_1 F(A) + C_2 F(A) + [C_1 - R(A)] \bar{F}(A)$$

$$= C_1 (F(A) + \bar{F}(A)) + C_2 F(A) - R(A)\bar{F}(A).$$

$$= C_1 + C_2 F(A) - R(A)\bar{F}(A)$$

则 $E[time] = \int_0^A x \, dF(x) + \int_A^\infty A \, dF(x)$

$$= \int_0^A x \, dF(x) + A\bar{F}(A).$$

因此. long-run average cost per time unit

$$= \frac{E[cost]}{E[time]} = \frac{C_1 + C_2 F(A) - R(A)\bar{F}(A)}{\int_0^A x \, dF(x) + A\bar{F}(A)}.$$

b). 若 car fails 才进行更新. 则在一个周期中.

汽车更换次数服从几何分布. 其中.

汽车没有 fail 就更换的概率 $\int_0^A dF(x) = F(A)$

汽车 fail 的概率是 $1 - F(A)$.

因此平均更换次数 $\frac{1}{F(A)}$.

则 $E[cost] = E[(N-1)(C_1 - R(A)) + C_1 + C_2]$

$$= E[N(C_1 - R(A)) + C_2 + R(A)]$$

$$= \frac{C_1 - R(A)}{F(A)} + C_2 + R(A)$$

$E[time] = E[E[time \mid N = n]]$

$$= E[(N-1)A + \int_0^A x \, dF(x)]$$

$$= \frac{\bar{F}(A)}{F(A)} A + \int_0^A x \, dF(x)$$

故 long-run average cost per time unit

$$= \frac{E[cost]}{E[time]} = \frac{C_1 - R(A) + C_2 F(A) + R(A) F(A)}{\bar{F}(A) \cdot A + \int_0^A x \, dF(x) \cdot F(A)}$$

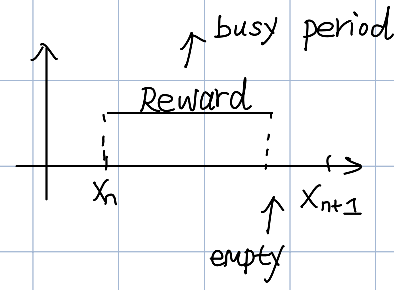
$$= \frac{C_1 - R(A)\bar{F}(A) + C_2 F(A)}{\bar{F}(A) \cdot A + F(A) \int_0^A x \, dF(x)}.$$

3.33 a) 我们定义如下一个 Renewal Process. 更新点是每次 server 进入忙时的时间点（即 server 在结束忙时后迎来第一个顾客的时间点），在 work time 获得 reward.

则 V 是一个周期内平均的 Reward.

因此. $V = \lim_{t\to\infty} \frac{\int_0^t V(s)ds}{t}$

$= \frac{E[\text{reward}]}{E[\text{cycle time}]}$.



故. V 存在且为常数概率为 1.

类似地. 我们考虑 $n=N$. 为一个周期内顾客数量. 则

$$W_Q = \lim_{n\to\infty} \frac{D_1+\cdots+D_n}{n} = \lim_{N\to\infty} \frac{D_1+\cdots D_N}{N}$$

$$= E[D] = \frac{E[\text{time}]}{E[\text{customers}]}$$

故 W_Q 存在，为常数概率为 1.

b) 令 T 是一个周期的时长. N 为人数.

则 $V = \frac{E[\int_0^T V(s)ds]}{ET}$, $W_D = \frac{E[D_1+\cdots+D_N]}{E[N]}$

而 $\text{reward} = \int_0^T V(s)\,ds$

$$= \sum_{i=1}^{N} [D_iY_i + \int_0^{Y_i} (Y_i-t)dt]$$

$$= \sum_{i=1}^{N} D_iY_i + \sum_{i=1}^{N} \frac{Y_i^2}{2}$$

故 $E\int_0^T V(s)\,ds = E[\sum_{i=1}^{N} D_iY_i] + \frac{1}{2}E[Y^2]\cdot E[N]$

而 $\frac{E[\sum_{i=1}^{N} D_iY_i]}{EN} = \lim_{n\to\infty} \frac{E[D_1Y_1+\cdots+D_nY_n]}{n}$

$$= \lim_{n\to\infty} \frac{\sum_{i=1}^{n} ED_i\,EY}{n}$$

$$= EY\cdot W_Q.$$

故 $E\int_0^T V(s)\,ds = EY\cdot W_Q\cdot EN + \frac{1}{2}E[Y^2]\,EN$.

$$V = \frac{E\int_0^T V(s)ds}{ET} = EY\cdot W_Q\cdot\lambda + \frac{\lambda}{2}E[Y^2]$$

3.36. $\lim_{t\to\infty} \int_0^t \frac{r(X(s))ds}{t} = \frac{E[\int_0^T r(X(s))ds]}{ET}$

$= \frac{E[\sum_j r(j)\cdot(\text{在T中出现j的次数})]}{ET}$

$= \sum_j r(j) \cdot \frac{E[\text{在T中出现j的次数}]}{ET}$

$= \sum_j r(j)\cdot P_j.$

3.20. a) E[times until HHTHHTT] $=2^7$ since overlap $=0$

b) E[times until HHTT] $= 2^4$ since overlap $=0$

E[times until HTHT] = E[times until HT] + 2^4

$= 2^2 + 2^4$

故 HTHT 需要更多时间发生.

3.2 则每次更新后，下次还有更新的概率为 $F(\infty)$. 故总的更新过程可以看成一个几何分布. $P(1+N(\infty)=n) = (F(\infty))^{n-1}(1-F(\infty))$

3.12. KRT ⇒ Blackwell's.

令 $h(t) = \begin{cases} 1 & t \le s \\ 0 & t > s. \end{cases}$

则 $h(t)$ 非负. 非增，$\int_0^\infty h(t)dt = \int_0^s 1\,dt = s$

从而 $\lim_{t\to\infty} \int_0^t h(t-x)\,dm(x) = \lim_{t\to\infty} \int_{t-s}^t 1\,dm(x)$

$= \lim_{t\to\infty} m(t) - m(t-s).$

$= \frac{1}{\mu}\int_0^\infty h(x)\,dx = \frac{s}{\mu}$