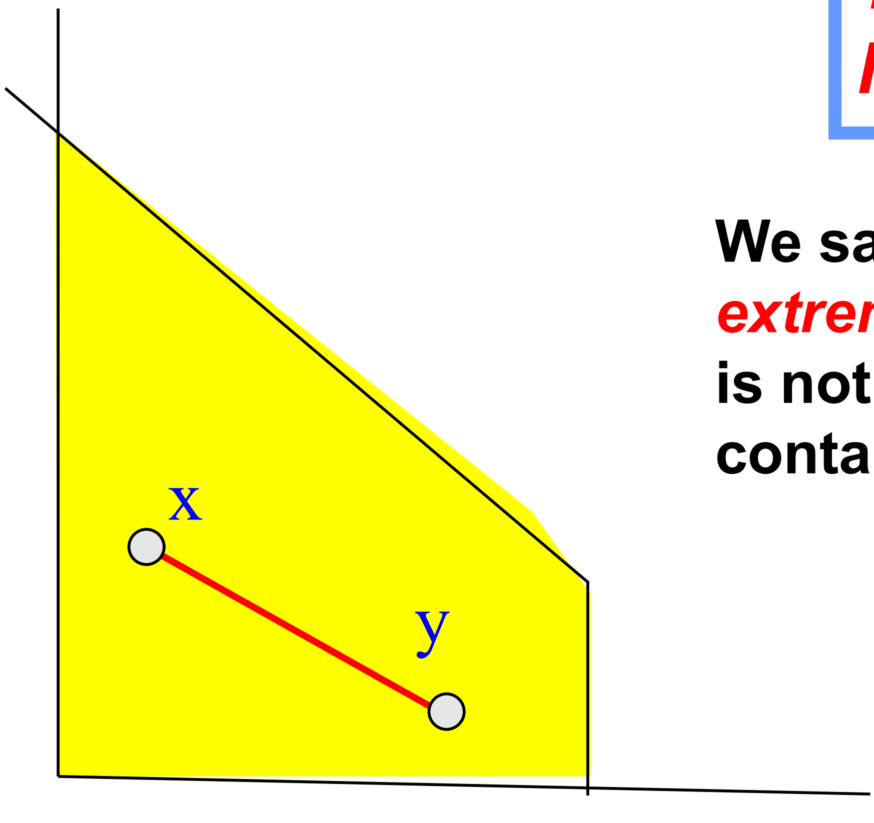
When is a locally optimal solution also globally optimal?

- For minimization problems
  - The feasible region is convex.
  - The objective function is convex.

1. Convexity set and Extreme Points



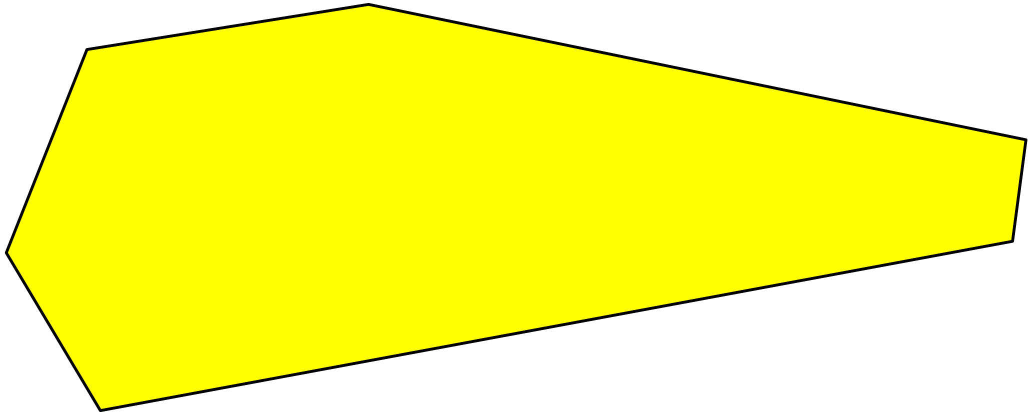
We say that a set S is *convex*, if for every two points x and y in S, and for every real number λ in [0,1], λx + (1-λ)y ∈ *S*.

The feasible region of a linear program is convex.

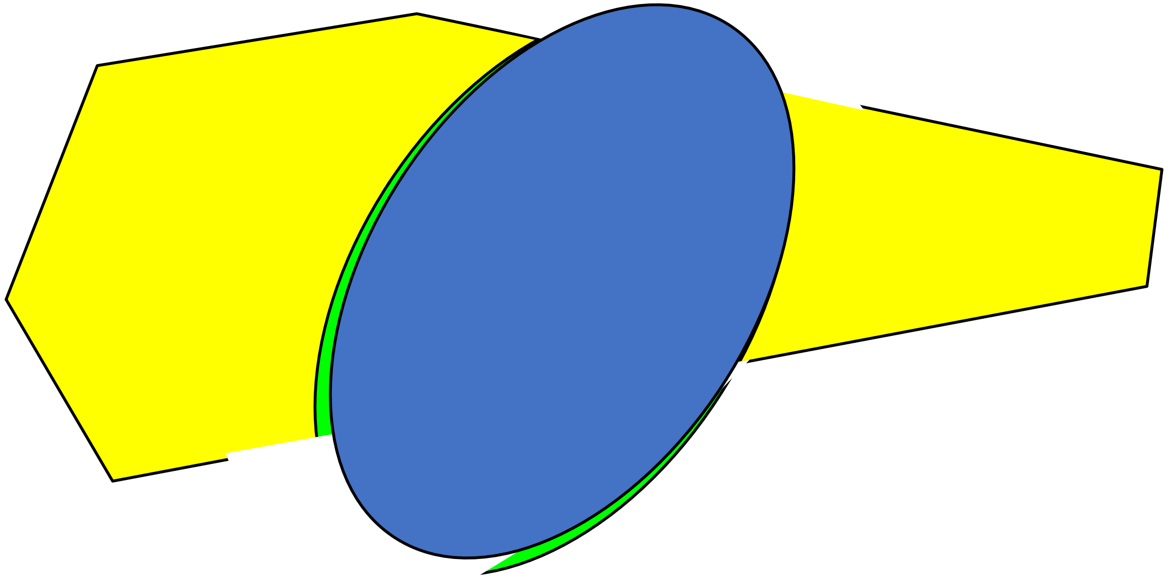
We say that an element *w* ∈ *S* is an *extreme point* (*vertex, corner point*), if *w* is not the midpoint of any line segment contained in S.

On convex feasible regions

- If all constraints are linear, then the feasible region is convex.

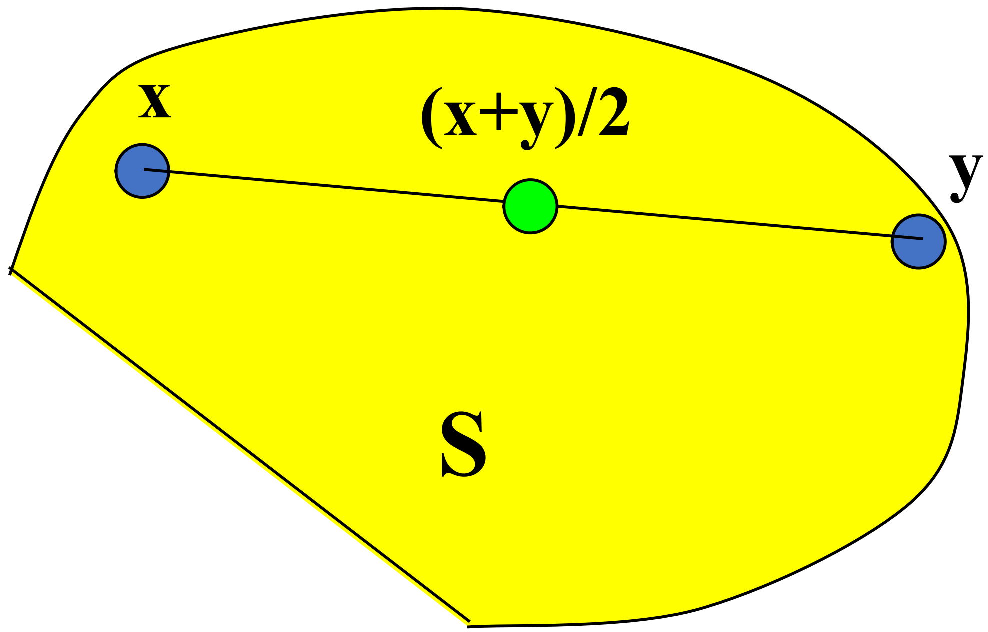


- The intersection of convex regions is convex.



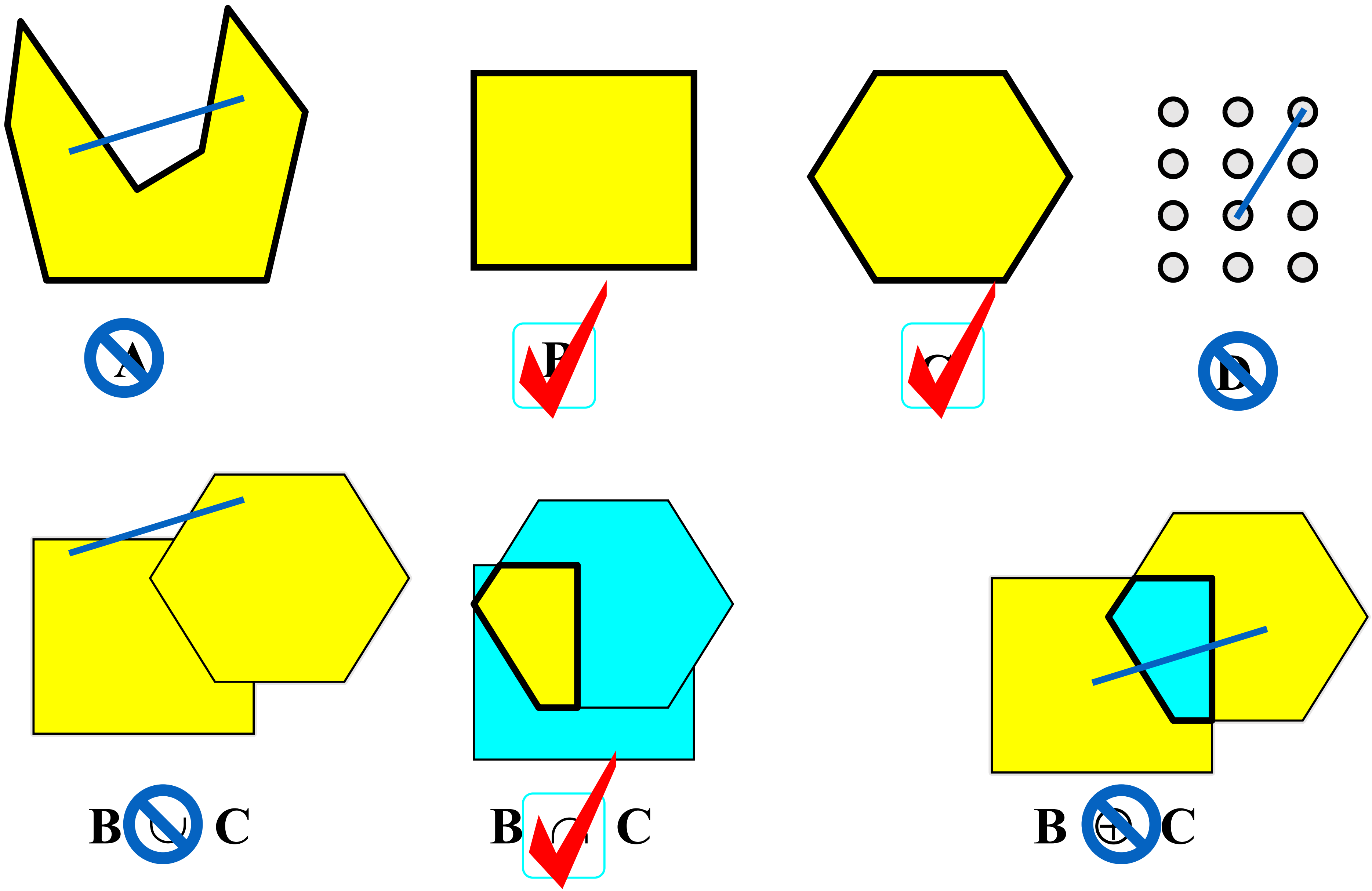
Recognizing convex sets

- Rule of thumb: suppose for all x, y ∈ S, the midpoint of x and y is in S. Then S is convex.



It is convex if the entire line segment is always in S.

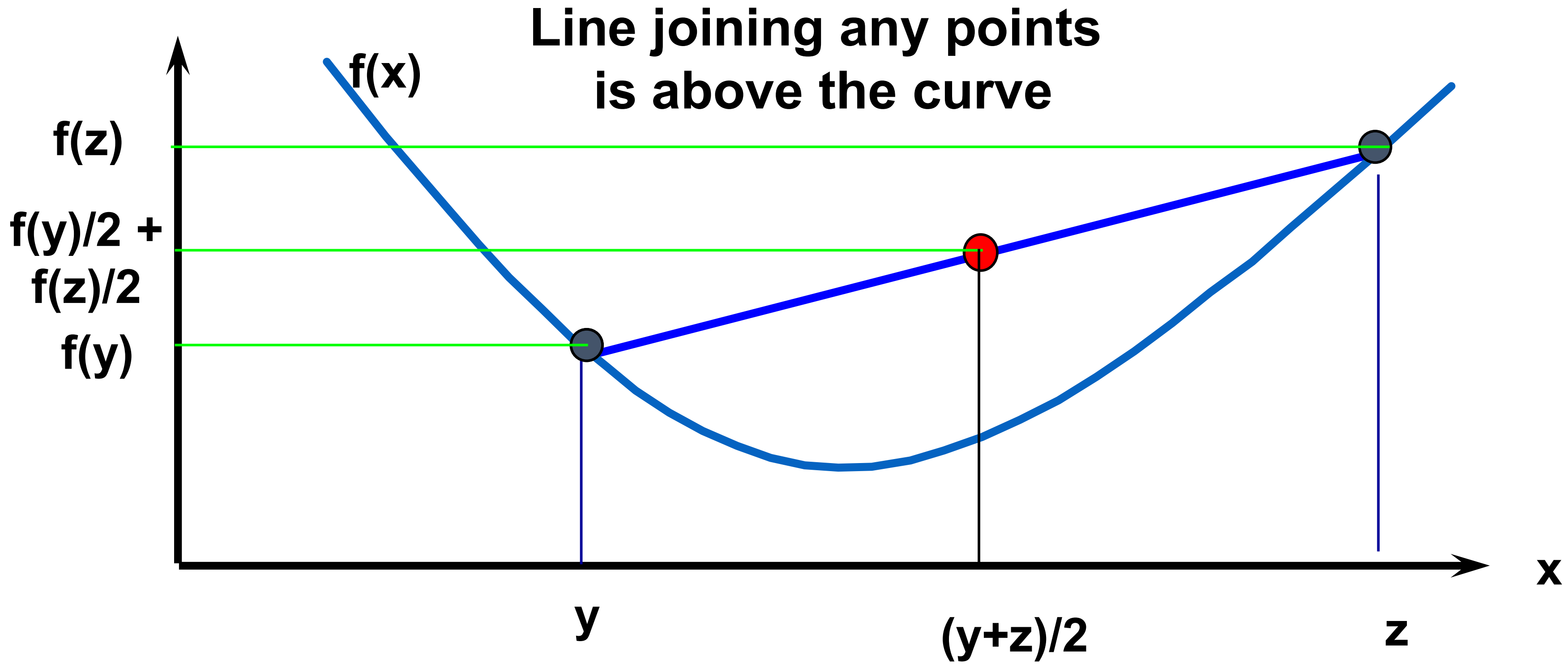
Which are convex?



2. Convex and Concave Functions

Convex Functions: $f(\lambda y + (1-\lambda)z) \leq \lambda f(y) + (1-\lambda)f(z)$
for every y and z and for $0 \leq \lambda \leq 1$.

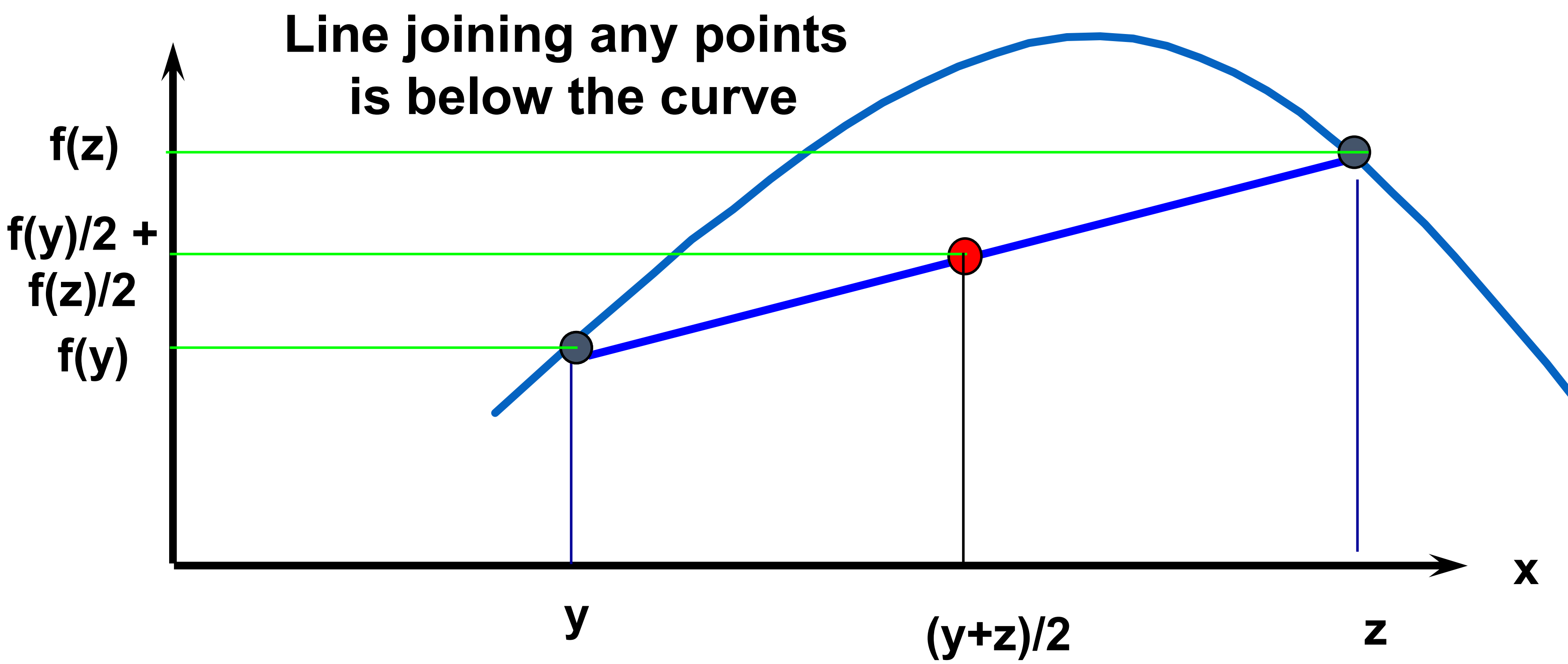
We say "*strict" convexity* if sign is "<" for $0 < \lambda < 1$.



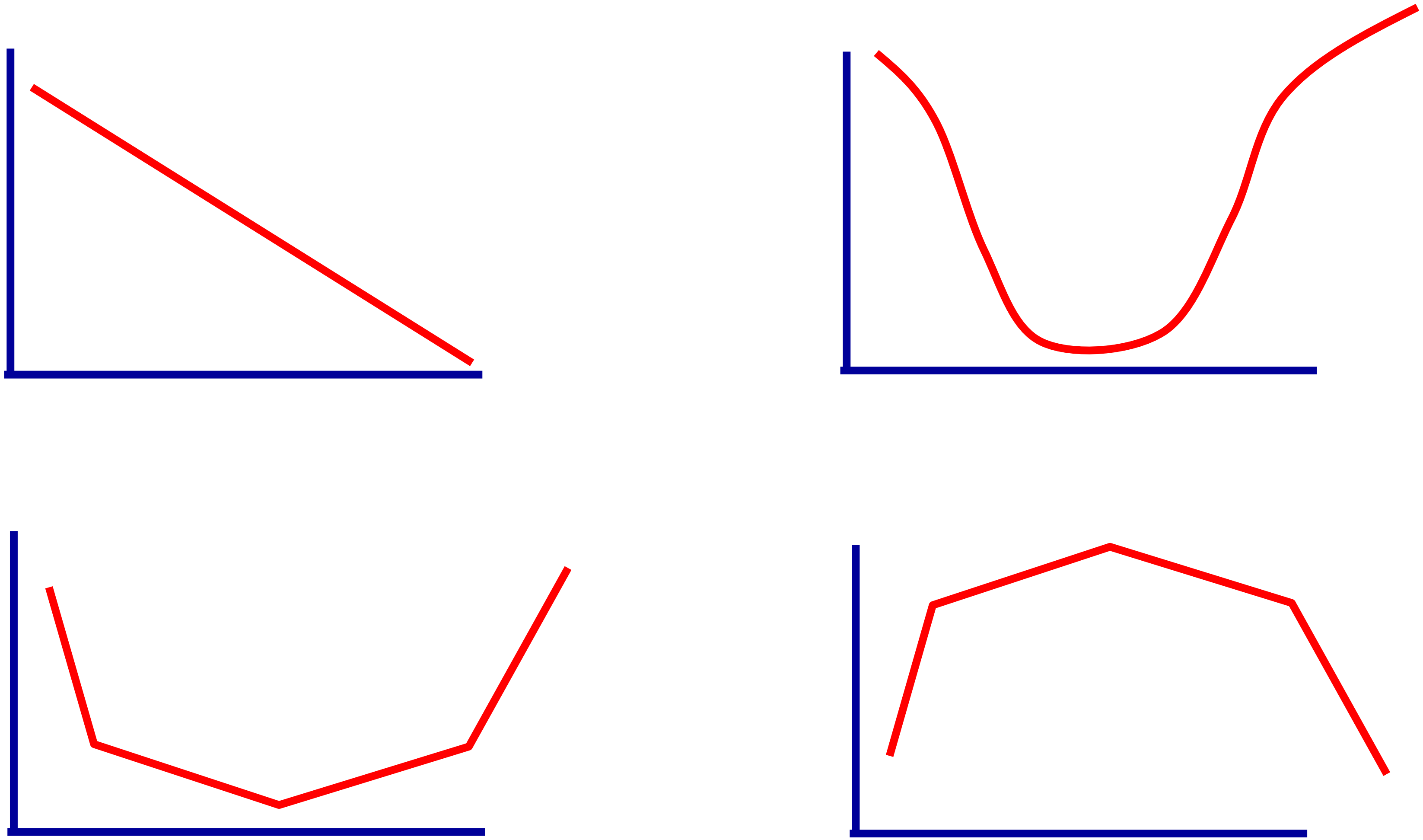
Concave Functions

Concave Functions: $f(\lambda y + (1-\lambda)z) \geq \lambda f(y) + (1-\lambda)f(z)$
for every y and z and for $0 \leq \lambda \leq 1$.

We say "*strict" concavity* if sign is "<" for $0 < \lambda < 1$.

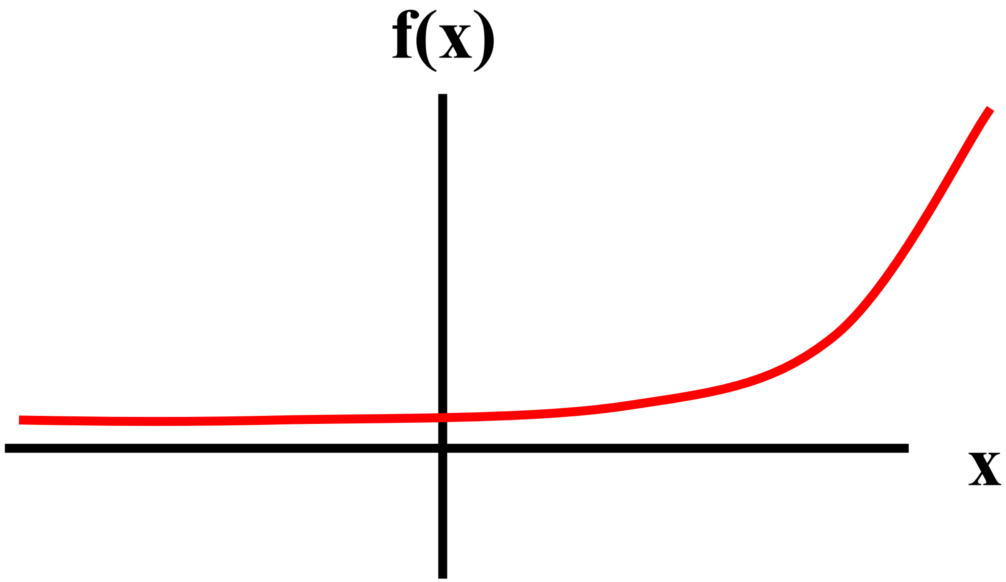


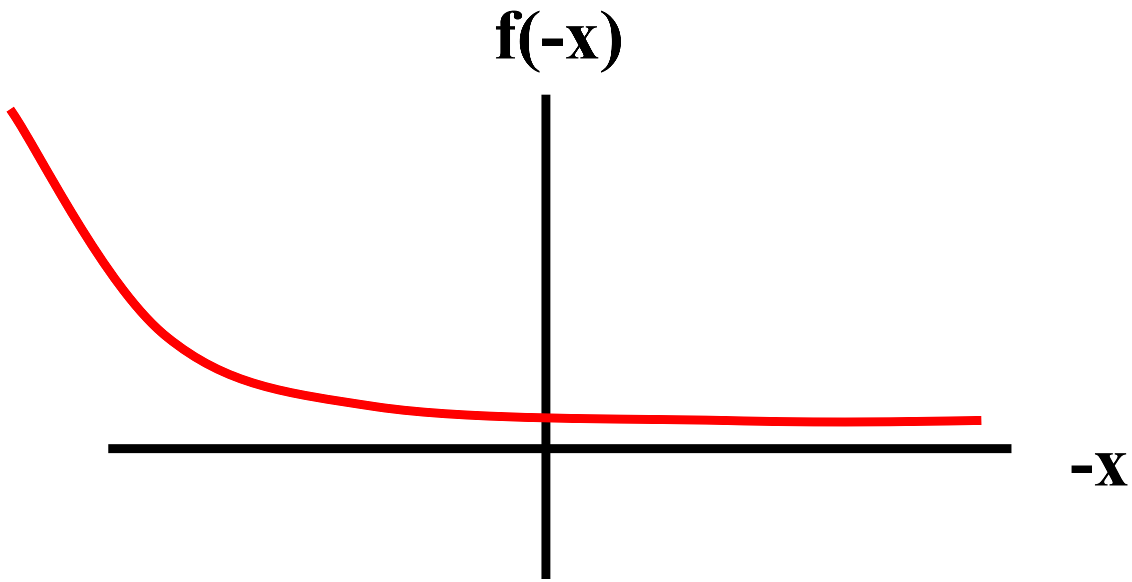
Classify as convex or concave or both or neither.



Properties on convex functions (1)

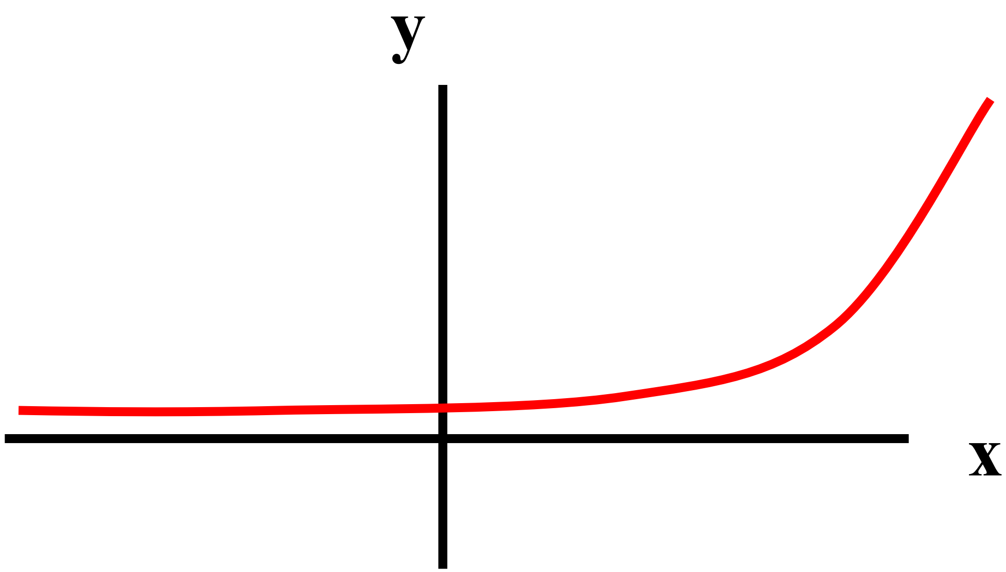
If f(x) is convex, then f(-x) is convex.

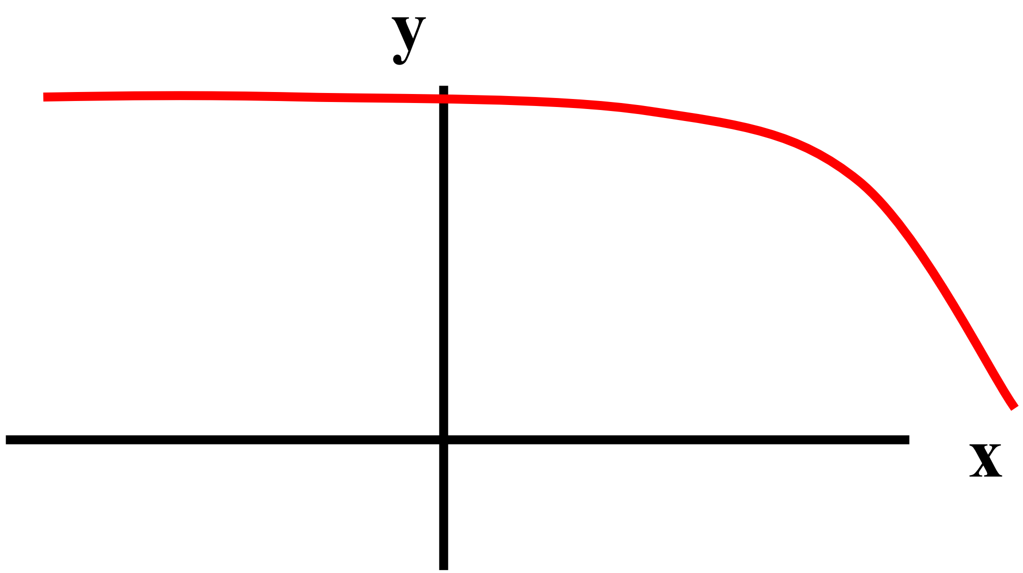




Properties on convex functions (2)

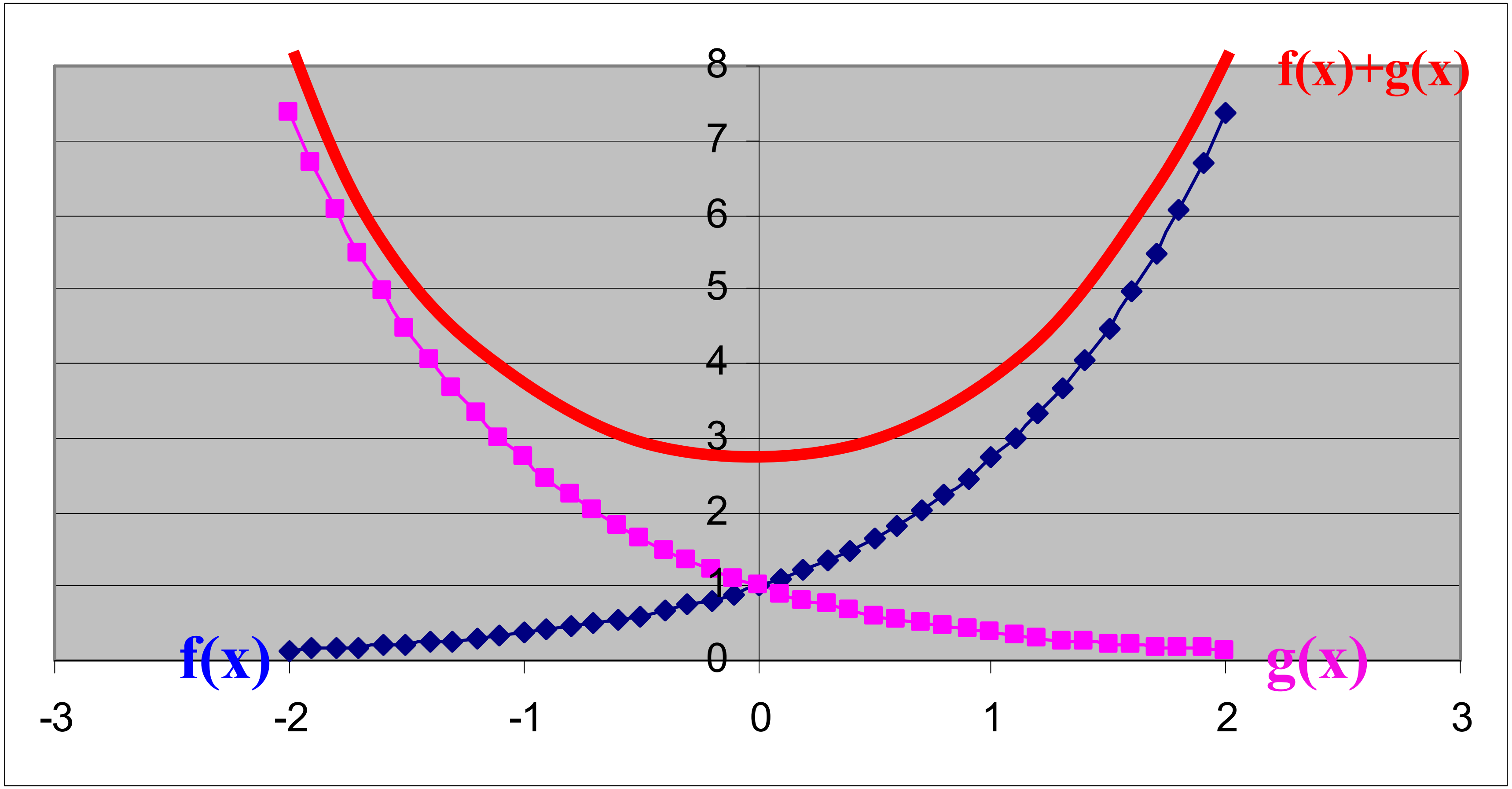
If f(x) is convex, then K - f(x) is concave.





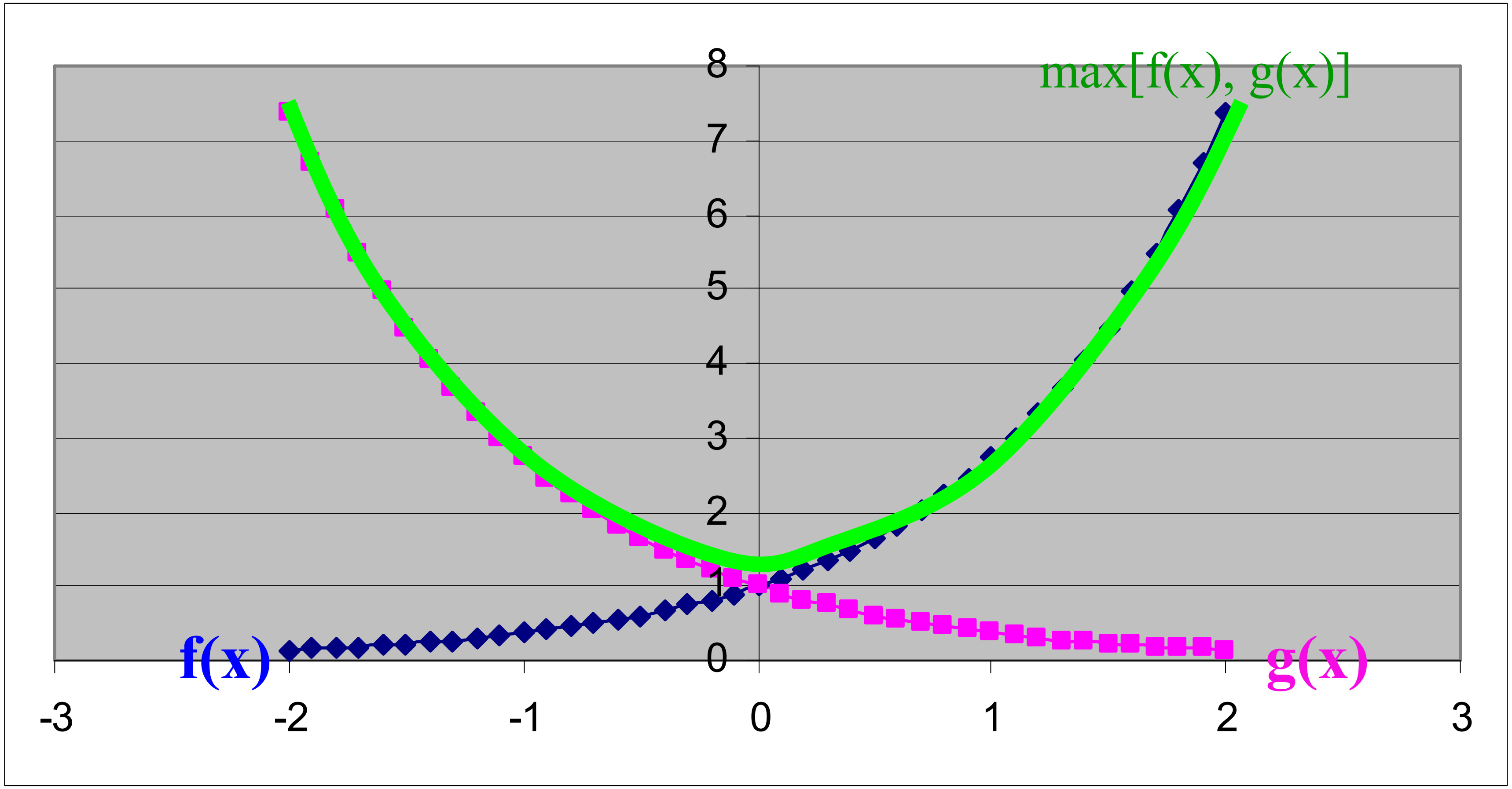
Properties on convex functions (3)

If f(x) is convex and g(x) is convex, then so is f(x) + g(x).



Properties on convex functions (4)

If f(x) is convex and g(x) is convex, then so is max [f(x), g(x)].

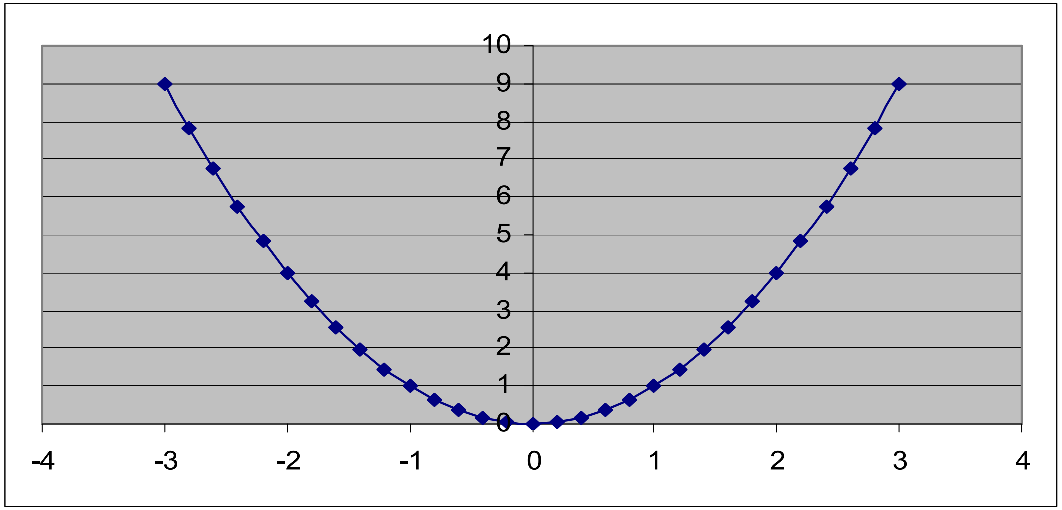


Properties on convex functions (5)

If $f(x)$ is a twice differentiable function of one variable, and if $f''(x) > 0$ for all x, then $f(x)$ is convex.

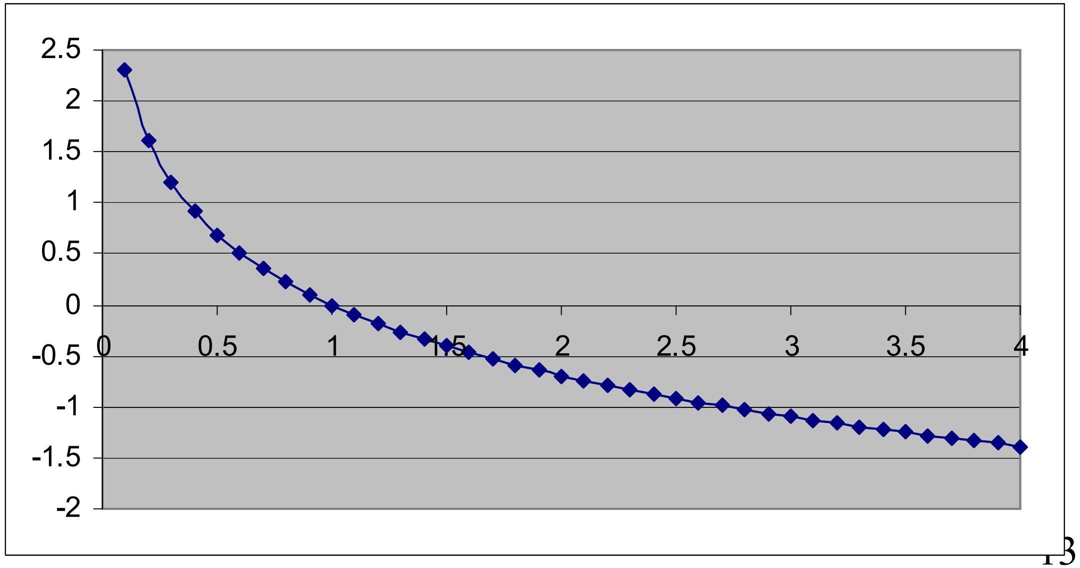
$$f(x) = x^2$$

$$f'(x) = 2x,\ f''(x) = 2$$



$$f(x) = -\ln x \text{ for } x > 0$$

$$f'(x) = -\frac{1}{x},\ f''(x) = \frac{1}{x^2}$$

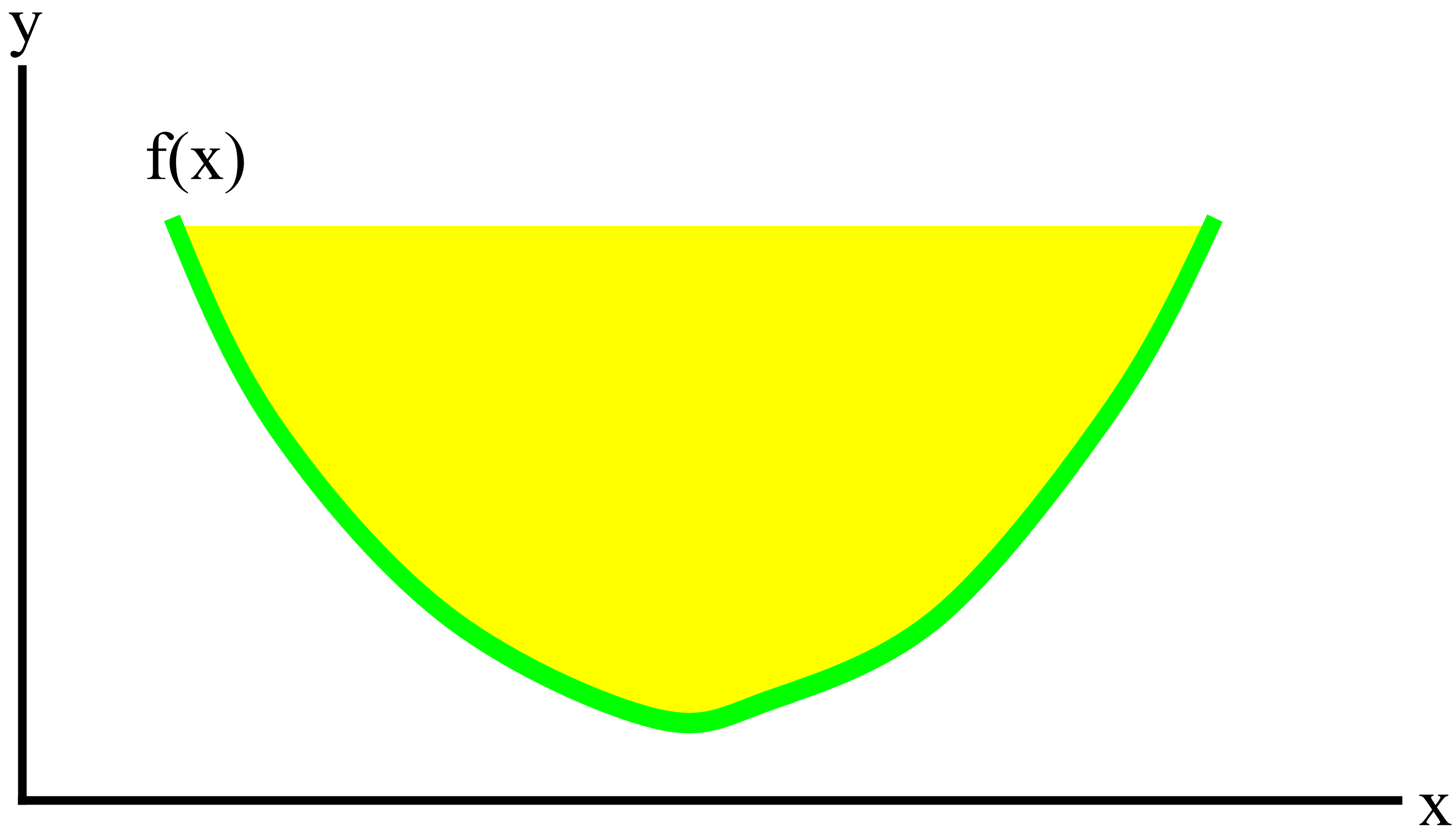


What functions are convex?

- $f(x) = 4x + 7$ all linear functions
- $f(x) = 4x^2 – 13$ some quadratic functions
- $f(x) = e^x$
- $f(x) = 1/x$ for $x > 0$
- $f(x) = |x|$
- $f(x) = - \mathit{ln}(x)$ for $x > 0$

3. Convex functions vs. convex sets

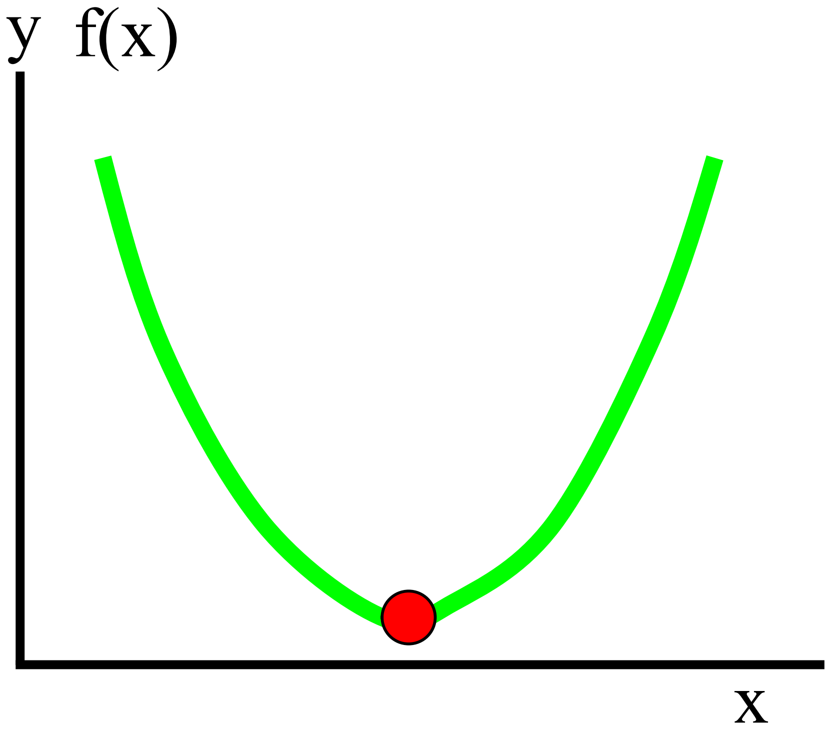
If $y = f(x)$ is convex, then $\{(x,y) : f(x) \leq y\}$ is a convex set.



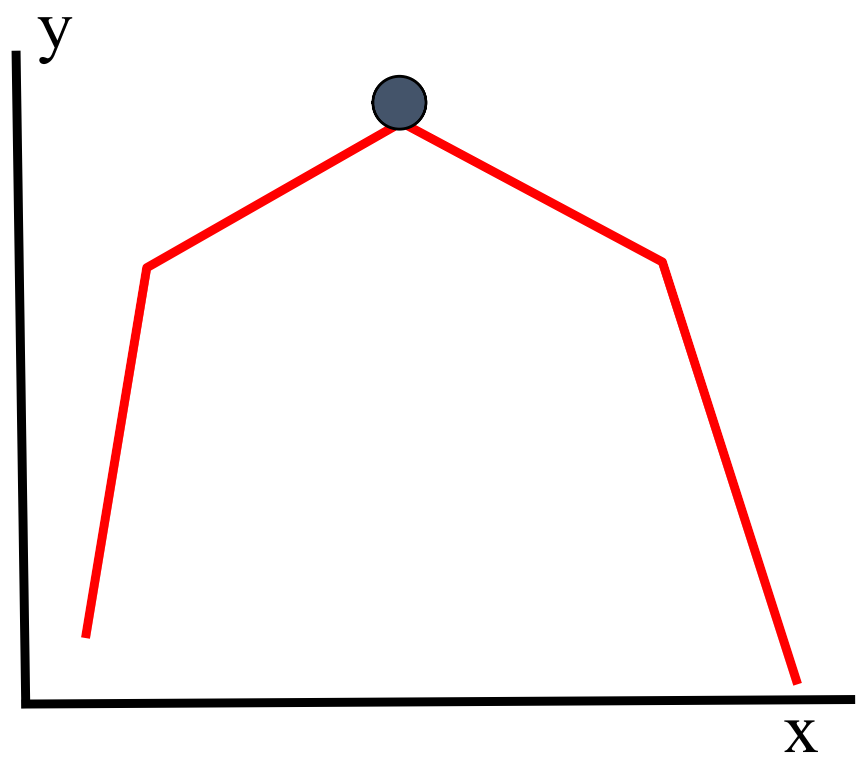
4. Local Minimum or Maximum Property

- A local min (max) of a convex (concave) function on a convex feasible region is also a global min.
- Strict convexity (concavity) implies that the global minimum is unique.
- The following NLPs can be solved
  - Minimization Problems with a convex objective function and linear constraints
  - Maximization Problems with a concave objective function and linear constraints

There is a unique local minimum for the function below. The local minimum is a global minimum



There is a unique local maximum for the function below. The local maximum is a global minimum.



More on local optimality

- The techniques for non-linear optimization minimization usually find local optima.
- This is useful when a locally optimal solution is a globally optimal solution.

- Conclusion: if you solve an NLP, try to find out how good the local optimal solutions are.