OM9103: Stochastic Process

Lecture 10&11:
Continuous Time Markov Chains

A. Introduction

A.1 Definition

Consider a family of random variables $\{X(t), 0 \le t < \infty\}$ where the possible values of $X(t)$ are nonnegative integers. We will focus on the case where $\{X(t)\}$ is a Markov process with stationary or homogenous transition probabilities: for $t > 0$,

$$P_{ij}(t) = \Pr(X(t+s) = j \mid X(s) = i), \forall\ i, j = 1, 2, \ldots$$

which is independent of s. In other words, for all s, $0 \le u < s$,

$$\Pr(X(t+s) = j \mid X(s) = i, X(u) = x(u), 0 \le u < s)$$
$$= \Pr(X(t+s) = j \mid X(s) = i).$$

In other words, a continuous-time Markov chain is a stochastic process having the Markov property that the conditional distribution of the future state at time $t + s$, given the present state at s and all past states depends only on the present state and is independent of the past.

<u>Remarks:</u>

1. Stochastic models based on physical phenomena to prescribe the so-called infinitesimal probabilities related to the process;
2. Using the Markovian property, we will derive a system of differential equations satisfied by $P_{ij}(t)$ for all $t > 0$.

A.2 Sojourn Times

Let τ_i be the amount of time that the process stays in state i before making a transition into a different state, also known as <u>sojourn times</u>. Then the Markovian property implies that

$$\Pr(\tau_i > s+t \mid \tau_i > s) = \Pr(\tau_i > t),\ \forall s,t \ge 0.$$

Hence the random variables τ_i is momoryless and must be exponentially distributed.

The above property gives us a method of constructing a continuous-time Markov chain. Basically, it is a stochastic process having the properties that each time it enters state i:

(i). The amount of time it spends in that state before making a transition into a different state is exponentially distributed with rate, say, v_i; and

(ii). When the process leaves state i, it will next enter state j with some probability, call it P_{ij}, where $\sum_{j\neq i} P_{ij} = 1$.

Intuitively speaking, a continuous Markov chain is a stochastic process that moves from state to state in accordance with a (discrete-time) Markov chain, but is such that the amount of time it spends in each state, before proceeding to the next state, is exponentially distributed.

Remarks

- If $v_i = \infty$, we call it an *instantaneous state*;
- Throughout our discussion, we assume that $0 \le v_i < \infty$;
- When $v_i = 0$, we called an *absorbing state*.
- A continuous Markov chain is said to be *regular* if, with probability 1, the number of transitions in any finite length of time is finite.

Let

$$q_{ij} = v_i P_{ij}, \text{ for } i \neq j,$$

which is the rate when in state i that the process makes a transition into state j, hence known as *transition rate* from i to j.

B. Poisson Processes and Pure Birth Processes

B.1 Poisson Processes

A Poisson process is in fact a continuous Markov chain with the following properties:

(i). $P(X(t+h)–X(t) = 1|X(t) = x) = \lambda h + o(h)$ for all $x = 0, 1, \ldots$. Or alternatively,

$$\lim_{h \downarrow 0} \frac{P(X(t+h)-X(t)=1 \mid X(t)=x)}{h} = \lambda, \quad \forall x = 0,1,\ldots$$

(ii). $P(X(t+h) - X(t) = 0 \mid X(t) = x) = (1-\lambda h) + o(h)$ for all $x = 0, 1, \ldots$.

(iii). $X(0) = 0$.

Let $P_n(t) = \Pr(X(t) = n)$, $t > 0$, $n = 0, 1, \ldots$ Note that

$$
\begin{aligned}
&P_0(t+h) = \Pr(X(t+h) = 0) \\
&= \Pr(X(t+h) - X(t) = 0 | X(t) = 0) \Pr(X(t) = 0) \\
&= \Pr(X(h) = 0) \Pr(X(t) = 0) \\
&= P_0(h) P_0(t) = (1 - \lambda h) P_0(t) + o(h) \\
&\Rightarrow \lim_{h \to 0+} \frac{P_0(t+h) - P_0(t)}{h} = -\lambda P_0(t), \ \ \textit{i.e.,}\ P_0'(t) = -\lambda P_0(t) \\
&\Rightarrow P_0(t) = e^{-\lambda t} \text{ (with } P_0(0) = 1).
\end{aligned}
$$

By recursive argument, we can solve the above differential equation and get the following solutions:

$$P_n(t) = \frac{(\lambda t)^n}{n!} e^{-\lambda t}, \ n = 0, 1, \ldots$$

i.e., it is a Poisson distribution with parameter λt.

B.2 Pure Birth Processes

This process is a natural generation of the Poisson process: allow the chance of an event occurring at a given instant of time to depend on the number of events which have already occurred.

A pure birth process is satisfying the following postulates:

(i). $\Pr(X(t+h) - X(t)) = 1 | X(t) = k) = \lambda_k h + o_{1,k}(h)$ $(h \downarrow 0)$;

(ii). $\Pr(X(t+h) - X(t)) = 0 | X(t) = k) = 1 - \lambda_k h + o_{2,k}(h)$ $(h \downarrow 0)$;

(iii). $\Pr(X(t+h) - X(t)) < 0 | X(t) = k) = 0, k \geq 0$;

(iv). $X(0) = 0$.

In this case, $X(t)$ no longer represents the population size, instead, it represents the number of births in the time interval $[0, t]$. Note that (*i*) and (*ii*) are equivalent to

$$P_{k,k+1}(h) = \lambda_k h + o_{1,k}(h);$$
$$P_{k,k}(h) = 1 - \lambda_k h + o_{2,k}(h).$$

Again, define $P_n(t) = \Pr(X(t) = n)$. Using the same technique as in the Poisson process, we can derive the following system of differential equations:

$$P_0'(t) = -\lambda_0 P_0(t),$$
$$P_n'(t) = -\lambda_n P_n(t) + \lambda_{n-1} P_{n-1}(t),\ n \geq 1.$$

with the boundary conditions:

$$P_0(0) = 1,\ P_n(0) = 0,\ \forall\ n \geq 1.$$

<u>Proposition 1</u>: The solution for a pure birth process is given by:

$$P_0(t) = e^{-\lambda_0 t},$$
$$P_n(t) = \lambda_{n-1} e^{-\lambda_n t} \int_0^t e^{\lambda_n x} P_{n-1}(x)dx,\ n = 1, 2, ..$$

Furthermore,

$$\sum_{n=0}^{\infty} P_n(t) = 1 \Leftrightarrow \sum_{n=0}^{\infty} \frac{1}{\lambda_n} = \infty.$$

<u>Proof</u>. Solution for $P_0(t)$ is obvious. For the general case, let us first define T_k as the time between k-th and (k-1)st birth. Then we must have the following,

$$P_n(t) = \Pr(\sum_{i=0}^{n-1} T_i \leq t < \sum_{i=0}^{n} T_i).$$

The random variables T_k are called "sojourn times" (i.e., waiting time) between births, and $S_k = \sum_{i=0}^{k-1} T_i$ is the time at which the k-th birth occurs. Since $P_0(t) = e^{-\lambda_0 t}$, it follows that

$$\Pr(T_0 \leq t) = 1 - \Pr(X(t) = 0) = 1 - e^{-\lambda_0 t},$$

i.e., T_0 has an exponential distribution with parameter λ_0.

It may be deduced from Postulates (i) – (iv) that T_k, $k > 0$, is also an exponential distribution with parameter λ_k and that T_i's are mutually independent.

Now define $Q_n(t) = e^{\lambda_n t} P_n(t)$, $n \geq 0$. Then from the second differential equation, it follows that for $n \geq 1$,

$$Q_n'(t) = \lambda_n e^{\lambda_n t} P_n(t) + e^{\lambda_n t} P_n'(t) = e^{\lambda_n t}(\lambda_n P_n(t) + P_n'(t)) = e^{\lambda_n t} \lambda_{n-1} P_{n-1}(t)$$

Therefore, since $Q_n(0) = 0$ ($n \geq 1$), we then have

$$Q_n(t) = \int_0^t e^{\lambda_n x} \lambda_{n-1} P_{n-1}(x) dx.$$

Hence, we get

$$P_n(t) = \lambda_{n-1} e^{-\lambda_n t} \int_0^t e^{\lambda_n x} P_{n-1}(x) dx .$$

The proof of the second part is quite tedious; hence omitted. A complete proof can be found from the following reference:

W. Feller. *An Introduction to Probability Theory and Its Applications* Vol. 1, 2nd Edition, Wiley, NY, 1957. (p.406).

B.3 The Yule Process

The Yule process is an example of a pure birth process that arises in physics and biology. It describes the growth of a population in which each member has a probability βh + o(h) of giving birth to a new member during an internal of time length h ($\beta > 0$).

Assuming independence and no interaction among members, the binomial Theorem indicates that

$$\begin{aligned}&\Pr(X(t+h) - X(t) = 1 \mid X(t) = n) \\ &= \binom{n}{1}(\beta h + o(h))(1 - \beta h + o(h))^{n-1} + o(h) = n\beta h + o_n(h)\end{aligned}$$

This implies that $\lambda_n = n\beta$. For $X(0) = 1$, the system of differential equations becomes:

$$P_n'(t) = -\beta(nP_n(t) - (n-1)P_{n-1}(t)),\ n = 1, 2, \ldots$$

with the initial conditions: $P_1(0) = 1$, $P_n(0) = 0$ for $n > 1$. The solution is

$$P_n(t) = e^{-\beta t}(1 - e^{-\beta t})^{n-1},\ n \geq 1$$

which is a geometric distribution with $p = e^{-\beta t}$.

Note that the moment generating function for the Yule process with $X(0) = 1$ is given by

$$f(s) = \sum_{n=1}^{\infty} P_n(t) s^n = s e^{-\beta t} \sum_{n=1}^{\infty} [(1 - e^{-\beta t})s]^{n-1} = \frac{s e^{-\beta t}}{1 - (1 - e^{-\beta t})s}.$$

The reason to derive this m.g.f. is that it allows us to find $P_n(t)$ under a general case $X(0) = N$. Note that

$$P_{Nn}(t) = \Pr(X(t) = n \mid X(0) = N).$$

Let

$$f_N(s) = \sum_{n=1}^{\infty} P_{Nn}(t) s^n .$$

Then, by independence, it follows that

$$f_N(s) = (f(s))^N = \left[\frac{se^{-\beta t}}{1-(1-e^{-\beta t})s} \right]^N .$$

Using the fact that $(1-x)^{-N} = \sum_{m=0}^{\infty} \binom{m+N-1}{m} x^m$, we have the following expansion of $f_N(s)$:

$$f_N(s) = \sum_{n=N}^{\infty} \binom{n-1}{n-N} (e^{-\beta t})^N (1-e^{-\beta t})^{n-N} s^n .$$

Therefore,

$$P_{Nn}(t) = \binom{n-1}{n-N} e^{-N\beta t} (1-e^{-\beta t})^{n-N}, \; n = N, N+1, \ldots$$

C. Birth and Death Processes

One of the obvious generalization of the pure birth processes is to allow $X(t)$ to decrease as well as increase. This implies that at time t, the process is in state n, it may, after a random sojourn (waiting) time, move to either of the neighboring states n+1 or n-1.

C.1 Postulates

Definition: A birth and death process $\{X(t)\}$ satisfies the following postulates:

(i). $P_{i,i+1}(h) = \lambda_i h + o(h)$, $\forall\, i \geq 0$ and $h \downarrow 0$;
(ii). $P_{i,i-1}(h) = \mu_i h + o(h)$, $\forall\, i \geq 1$ and $h \downarrow 0$;
(iii). $P_{i,i}(h) = 1 - (\lambda_i + \mu_i)h + o(h)$, $\forall\, i \geq 0$ and $h \downarrow 0$;
(iv). $P_{ij}(0) = \delta_{ij}$;
(v). $\mu_0 = 0$, $\lambda_0 > 0$, μ_i, $\lambda_i > 0$, $i = 1, 2, \ldots$

The matrix

$$A = \begin{pmatrix} -\lambda_0 & \lambda_0 & 0 & \cdots \\ \mu_1 & -(\lambda_1+\mu_1) & \lambda_1 & \cdots \\ 0 & \mu_2 & -(\lambda_2+\mu_2) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

is called the infinitesimal generator of the process. The parameters λ_i and μ_i are called, respectively, the infinitesimal birth and death rates.

Alternatively, we can use the following simple definition: a continuous-time Markov chain with states 0, 1, … for which $q_{ij} = 0$ whenever $|i - j| > 1$ is called a *birth and death process*.

Using the Markovian property of the process, we can derive the Chapman-Kolmogorov equation:

$$P_{ij}(t+s) = \sum_{k=0}^{\infty} P_{ik}(t)P_{kj}(s).$$

In order obtain the probability of $X(t) = n$, we must specify the distribution of the initial states: $q_i = \Pr(X(0) = i)$, for $i = 0, 1, 2\ldots$. Then, it follows that

$$\Pr(X(t) = n) = \sum_{i=0}^{\infty} q_i P_{in}(t).$$

C.2 Sojourn Times of Birth and Death Processes

Let τ_i be the sojourn time of the process $X(t)$ in state i. Define $G_i(t) = \Pr(\tau_i \geq t)$.

Proposition 2. $G_i(t) = \exp(-(\lambda_i + \mu_i)t)$, that is, it is an exponential distribution with parameters $(\lambda_i + \mu_i)$.

Proof. First, note that for small $h > 0$,

$$G_i(h) = P_{ii}(h) + o(h).$$

Now note that by Markovian property, for small $h > 0$,

$$\begin{aligned}
&G_i(t+h) = G_i(t)G_i(h) = G_i(t)(P_{ii}(h) + o(h)) \\
&= G_i(t)[1 - (\lambda_i + \mu_i)h + o(h)] \\
&\Rightarrow \frac{G_i(t+h) - G(t)}{h} = -(\lambda_i + \mu_i)G_i(t) + o(1) \\
&\Rightarrow G_i\,'(t) = -(\lambda_i + \mu_i)G_i(t).
\end{aligned}$$

This proves the result.

According to Postulates 1 and 2, during a time duration of length h, a transition occurs from i to i+1 with probability $\mu_i h + o(h)$. It follows intuitively that given that a transition occurs at time t, the probability that this transition is to state i+1 is $\lambda_i/(\lambda_i + \mu_i)$ and to state i-1 is $\mu_i/(\lambda_i + \mu_i)$. This is in fact true' but the proof is omitted here.

With the above discussion, we will have the following fresh description of $X(t)$:

- the process sojourns in a given state i for a random length of time whose distribution function is exponential with parameter $(\lambda_i + \mu_i)$;
- after leaving state i, the process enters either state i+1 or state i-1 with probabilities $\lambda_i/(\lambda_i + \mu_i)$ and $\mu_i/(\lambda_i + \mu_i)$, respectively.

Therefore, the motion is analogous to a random walk except transition occurs at random times rather than at fixed time periods.

Remarks

(1) Some textbooks actually define the birth and death process as follows:

$$P_{01} = 1;$$

$$P_{i,i+1} = \frac{\lambda_i}{\lambda_i + \mu_i}, \quad i \geq 1;$$

$$P_{i,i-1} = \frac{\mu_i}{\lambda_i + \mu_i}, \quad i \geq 1.$$

(2) The traditional procedure for constructing a birth and death process is to prescribe the birth and death parameters $\{\lambda_i, \mu_i; i = 0, 1, \ldots, \infty\}$ and to build the path structure by building utilizing the above description concerning the sojourn (waiting) times and the conditional transition probabilities of the various states.

(3) There may be several Markov processes that posses the same infinitesimal generator. For the birth and death process, a sufficient condition that there exists a unique Markov process with transition probability function $P_{ij}(t)$ for which the infinitesimal relations, the unity and the Chapman-Kolmogorov equation hold is that

$$\sum_{n=0}^{\infty} \pi_n \sum_{k=0}^{n} \frac{1}{\lambda_k \pi_k} = \infty,$$

where

$$\pi_0 = 1, \pi_n = \frac{\lambda_0 \lambda_1 \cdots \lambda_{n-1}}{\mu_1 \mu_2 \cdots \mu_n}, n \geq 1.$$

C.3 Examples

Example 1 (**Multi-server Queueing System M/M/s**): Customers arrive at an s-server service station in accordance with a Poisson process having rate λ. Each customer, upon arrival, goes directly into service if any of the servers are free, and if not, then the customer joins the queue. The successive service times are assumed to be independent exponential random variables having mean $1/\mu$. Let $X(t)$ be the number of customers in the system at time t, then $\{X(t),\ t \geq 0\}$ is a birth and death process with

$$\lambda_n = \lambda, n \geq 0; \quad \mu_n = \begin{cases} n\mu & 1 \leq n \leq s \\ s\mu & n > s \end{cases}$$

Example 2 (**Linear Growth Model with Immigration**): A birth and death process with the following parameters

$$\lambda_n = a + \lambda n,\ n \geq 0; \quad \mu_n = \mu n,\ n \geq 1$$

is called **a linear growth model with immigration**, where,

$\lambda > 0$ is the individual birth rate;

$a > 0$ is the rate of immigration into the population;

$\mu > 0$ is the individual death rate.

Remark

- The Yule process corresponds to the pure birth process case of linear growth model with immigration: $a = 0$ and $\mu = 0$.

D. Kolmogorov Differential Equations

D.1 General Kolmogorov Differential Equations

As usual, assume that $X(t)$ is a Markov process on the states 0, 1, 2, …, with the stationary transition probabilities,

$$P_{ij}(t) = \Pr(X(t+s) = j \mid X(s) = i).$$

Based on the fact that the probability of two or more transitions in time t is o(t), we have the following limiting results:

$$\lim_{t\downarrow 0} \frac{1-P_{ii}(t)}{t} = v_i; \quad \lim_{t\downarrow 0} \frac{P_{ij}(t)}{t} = q_{ij} \ \ (i \neq j).$$

The corresponding Chapman-Kolmogorov equation is

$$P_{ij}(t+s) = \sum_{k=0}^{\infty} P_{ik}(t) P_{kj}(s).$$

Theorem 1 (**Kolmogorov Backward Equation)**: For all i, j, and $t \geq 0$,

$$P_{ij}'(t) = \sum_{k \neq i} q_{ik} P_{kj}(t) - v_i P_{ij}(t).$$

Proof: refer to textbook, p. 240 – 241.

Theorem 2 (**Kolmogorov Forward Equation)**: Under suitable regularity conditions,

$$P_{ij}'(t) = \sum_{k \neq j} q_{kj} P_{ik}(t) - v_j P_{ij}(t).$$

Proof (omitted).

If we define r_{ij} as follows:

$$r_{ij} = \begin{cases} q_{ij} & \text{if } i \neq j \\ -v_i & \text{if } i = j \end{cases}$$

then, using the matrix format, the Kolmogorov backward equations and forward equations are given by

$$\boldsymbol{P}'(t) = \boldsymbol{R}\boldsymbol{P}(t), \ \ \boldsymbol{P}'(t) = \boldsymbol{P}(t)\boldsymbol{R},$$

respectively. Intuitively, the above expression suggests the following solution:

$$\boldsymbol{P}(t) = e^{\boldsymbol{R}t} = \sum_{i=0}^{\infty} \frac{(\boldsymbol{R}t)^i}{i!},$$

which is valid as long as v_i is bounded. But it is inefficient in actually computing $\boldsymbol{P}(t)$. A more efficient method is as follows:

$$e^{\boldsymbol{R}t} = \lim_{n\to\infty}(\boldsymbol{I} + \boldsymbol{R}t/n)^n$$

D.2 Kolmogorov Equations of Birth and Death Processes

The backward Kolmogorov differential equations for the birth and death process are given by:

$$\begin{cases} P_{0j}\,'(t) = -\lambda_0 P_{0j}(t) + \lambda_0 P_{1j}(t) \\ P_{ij}\,'(t) = \mu_i P_{i-1,j}(t) - (\lambda_i + \mu_i)P_{ij}(t) + \lambda_i P_{i+1,j}(t),\ i \ge 1 \end{cases}$$

with boundary condition $P_{ij}(0) = \delta_{ij}$.

For the sake of completeness, here is a simple proof. Note that

$$\begin{aligned} P_{ij}(t+h) &= \sum_{k=0}^{\infty} P_{ik}(h)P_{kj}(h) \\ &= P_{i,i-1}(h)P_{i-1,j}(t) + P_{i,i}(h)P_{ij}(t) + P_{i,i+1}(h)P_{i+1,j}(t) + o(h). \end{aligned}$$

Since $P_{i,i-1}(h) = \mu_i h + o(h)$, $P_{i,i}(h) = 1 - (\lambda_i + \mu_i)h + o(h)$, $P_{i,i+1}(h) = \lambda_i h + o(h)$, it follows that

$$\begin{aligned} &P_{ij}(t+h) = \mu_i h P_{i-1,j}(t) + (1-(\lambda_i+\mu_i)h)P_{ij}(t) + \lambda_i h P_{i+1,j}(t) + o(h) \\ &\Rightarrow P_{ij}\,'(t) = \lim_{h\downarrow 0}\frac{P_{ij}(t+h) - P_{ij}(t)}{h} = \mu_i P_{i-1,j}(t) - (\lambda_i+\mu_i)P_{ij}(t) + \lambda_i P_{i+1,j}(t) \end{aligned}$$

as required.

The forward Kolmogorov differential equations for the birth and death process are given by:

$$\begin{cases} P_{i0}\,'(t) = -\lambda_0 P_{i0}(t) + \mu_1 P_{i1}(t) \\ P_{ij}\,'(t) = \lambda_{j-1}P_{i,j-1}(t) - (\lambda_j + \mu_j)P_{ij}(t) + \mu_{j+1}P_{i,j+1}(t),\ (j \ge 1) \end{cases}$$

with initial condition $P_{ij}(0) = \delta_{ij}$.

A set of sufficient conditions for deriving the above forward Kolmogorov differential equations is given as follows:

$$\frac{P_{kj}(h)}{h} = o(1),\quad \forall k \ne j, j-1, j+1.$$

E. Limiting Probabilities

E.1 General Results

Note that a continuous Markov chain is a semi-Markov process with

$$F_{ij}(t) = 1 - e^{-v_i t}$$

it follows that if the discrete-time Markov chain with transition probabilities P_{ij} is irreducible and positive recurrent, then the limiting probabilities $P_j = \lim_{t\to\infty} P_{ij}(t)$ are given by

$$P_j = \frac{\pi_j / v_j}{\sum_i \pi_i / v_i}$$

where the π_j's are the unique nonnegative solution of the following system:

$$\pi_j = \sum_i \pi_i P_{ij}; \quad \sum_j \pi_j = 1.$$

Consequently, P_j's are the unique nonnegative solution of

$$v_j P_j = \sum_i v_i P_i P_{ij}, \quad \sum_j P_j = 1,$$

or, equivalently, using $q_{ij} = v_i P_{ij}$,

$$v_j P_j = \sum_i v_i q_{ij}, \quad \sum_j P_j = 1.$$

Remarks:

- The last equality is also known as *balance equations* because of the following facts: (1) $v_j P_j$ is the rate at which the process leaves state j; and (2) $\sum_i P_i q_{ij}$ is the rate at which the process enters state j.
- When the continuous-time Markov chain is irreducible and $P_j > 0$ for all j, we say that the chain is ***ergodic***.

E.2 Limiting Probabilities of Birth and Death Processes

For a general birth and death process, we can derive a closed-form solution for the limiting distribution, that is, we have the following:

$$\lambda_0 P_0 = \mu_1 P_1$$
$$(\lambda_j + \mu_j) P_j = \lambda_{j-1} P_{j-1} + \mu_{j+1} P_{j+1}, \; j \ge 1$$

Proposition 3. The following distribution of the general birth and death process is

$$P_0 = \frac{1}{\sum_{k=0}^{\infty} \theta_k}$$
$$P_j = \theta_j P_0 = \frac{\theta_j}{\sum_{k=0}^{\infty} \theta_k}, \; j \ge 1$$

where

$$\theta_0 = 1, \theta_j = \frac{\lambda_0\lambda_1 \cdots \lambda_{j-1}}{\mu_1\mu_2 \cdots \mu_j}, \ j \geq 1.$$

Proof. Just check that the given sequence $\{\pi_k\}$ satisfies the two required equations.

Remark:

- It is evident that the existence of the limiting distribution requires the following condition:

$$\sum_{k=1}^{\infty} \theta_k = \sum_{k=1}^{\infty} \frac{\lambda_0\lambda_1 \cdots \lambda_{k-1}}{\mu_1\mu_2 \cdots \mu_k} < \infty.$$

E.3 Examples

Example (**Linear Growth with Immigration** - cont.)

Assume $\lambda < \mu$. Then

$$\theta_0 = 1, \theta_1 = \frac{a}{\mu}, \theta_2 = \frac{a(a+\lambda)}{\mu(2\mu)}, \cdots,$$

$$\theta_k = \frac{a(a+\lambda)\cdots(a+(k-1)\lambda)}{k!\,\mu^k} = \binom{\frac{a}{\lambda}+k-1}{k}\left(\frac{\lambda}{\mu}\right)^k.$$

Using the expression,

$$(1-x)^{-N} = \sum_{k=0}^{\infty}\binom{N+k-1}{k}x^k, |x|<1,$$

we have

$$\sum_{k=0}^{\infty}\theta_k = \left(1-\frac{\lambda}{\mu}\right)^{-\frac{a}{\lambda}}.$$

Therefore,

$$P_0 = \left(1-\frac{\lambda}{\mu}\right)^{\frac{a}{\lambda}},$$

$$P_k = \left(\frac{\lambda}{\mu}\right)^k \frac{\frac{a}{\lambda}(\frac{a}{\lambda}+1)\cdots(\frac{a}{\lambda}+k-1)}{k!}\left(1-\frac{\lambda}{\mu}\right)^{\frac{a}{\lambda}}, k \geq 1.$$

Example (**Logistic Process**):

Suppose we consider a population whose size $X(t)$ ranges between two fixed integers N and M ($N < M$) for all $t \geq 0$. We assume that the birth and death rates per individual at time t are given by

$$\lambda = \alpha(M - X(t)),\ \mu = \beta(X(t) - N),$$

and that the individual members of the population act independent of each other. The resulting birth and death rates for the population then become

$$\lambda_n = \alpha n(M - n),\ \mu_n = \beta n(n - N).$$

Too see this, we observe that if the population size $X(t)$ is n, then each of the n individuals has an infinitesimal birth rate λ so that $\lambda_n = \alpha n(M - n)$. Similar rationale follows for μ_n.

The stationary distribution is

$$P_{N+m} = \frac{c}{N+m}\binom{M-N}{m}\left(\frac{\alpha}{\beta}\right)^m,\ m = 0,1,\ldots,M-N,$$

where c is a constant such that $\sum_m P_{N+m} = 1.$ The result follows from noting that

$$\begin{aligned}
\theta_{N+m} &= \frac{\lambda_N \lambda_{N+1} \cdots \lambda_{N+m+1}}{\mu_{N+1}\mu_{N+2}\cdots\mu_{M+m}} \\
&= \frac{\alpha^m N(N+1)\cdots(N+m-1)(M-N)\cdots(M-N-m+1)}{\beta^m (N+1)\cdots(N+m)m!} \\
&= \frac{N}{N+m}\binom{M-N}{m}\left(\frac{\alpha}{\beta}\right)^m.
\end{aligned}$$