2022-2023 Autumn Semester

Operation Research

Assignment 3

-Extension of Revised Simplex Algorithm(Big M & Two Phase)-

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2022 年 9 月 19 日

目录

1 Problem 1

1.1 Question:

Use either the Big-M method or two-phase method to solve the following LP problem.

$$
\begin{array}{lll}
\min & z = & 2x_1 + 3x_2 \\
s.t. & & \frac{1}{2}x_1 + \frac{1}{4}x_2 \quad \leq \quad 4 \\
& & x_1 + 3x_2 \quad \geq \quad 36 \\
& & x_1 + x_2 \quad = \quad 10 \\
& & x_1, x_2 \quad \geq \quad 0
\end{array}
\tag{1}
$$

1.2 Solution 1 (Revised Simplex Method - Big M):

Step 1 Convert LP to Canonical Form

We need to add the slack variable s_1、the excess variable s_2 and the artificial variable a_2, a_3 to obtain the Canonical Form as below.

$$
\begin{array}{lll}
\min & z = & 2x_1 + 3x_2 + Ma_2 + Ma_3 \\
& & \frac{1}{2}x_1 + \frac{1}{4}x_2 \quad +s_1 \quad = 4 \\
s.t. & & x_1 + 3x_2 \quad -s_2 + a_2 \quad = 36 \\
& & x_1 + x_2 \quad +a_3 \quad = 10 \\
& & x_1, x_2, s_1, s_2, a_2, a_3 \geq 0
\end{array}
\tag{2}
$$

Step 2 Obtain a bfs

We choose basic variables $BV = \{s_1, a_2, a_3\}$ and nonbasic variables $NBV = \{x_1, x_2, s_2\}$, then we can obtain a basic fessible solution $x_1 = 0, x_2 = 0, s_1 = 4, s_2 = 0, a_2 = 36, a_3 = 10$.

Step 3 Determine if the current bfs is optimal and choose the entering variable

Since we have decided the basic variables and nonbasic variables, we can know

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad , \quad C_{BV} = \begin{bmatrix} 0 & M & M \end{bmatrix};$$
$$N = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & 0 \\ 1 & 3 & -1 \\ 1 & 1 & 0 \end{bmatrix} \quad , \quad C_{NBV} = \begin{bmatrix} 2 & 3 & 0 \end{bmatrix}.$$

According to these matrices and vectors, we could calculate $C_{NBV} - C_{BV}B^{-1}N$, and the result is

$$C_{NBV} - C_{BV}B^{-1}N = \begin{bmatrix} 2-2M & 3-4M & M \end{bmatrix}. \tag{3}$$

Hence, the bfs obtained initially is not the optimal solution and we choose x_2 as the entering variable.

Step 4 The Ratio Test for determining the leaving variable

Because we choose x_2 as entering variable, we need to calculate $B^{-1}a_2$ and $B^{-1}b$ respectively.

$$B^{-1}a_2 = \begin{bmatrix} \frac{1}{4} \\ 3 \\ 1 \end{bmatrix}, \quad B^{-1}b = \begin{bmatrix} 4 \\ 36 \\ 10 \end{bmatrix}. \tag{4}$$

So the ratio test result is shown in table 2.

表 1: Result of ratio test

variables	result
s_1	16
a_2	12
a_3	10

In light of the result, we choose a_3 as the leaving variable. And then we repeat the Step 3 - Step 4.

Step 5 Determine if the current bfs is optimal and choose the entering variable

Now we need to update the basic variables and nonbasic variables as $BV = \{s_1, a_2, x_2\}, NBV = x_1, a_3, s_2$.By the new variables, we can update correspond matrices and vectors as below.

$$B = \begin{bmatrix} 1 & 0 & \frac{1}{4} \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \quad , \quad C_{BV} = \begin{bmatrix} 0 & M & 3 \end{bmatrix};$$
$$N = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix} \quad , \quad C_{NBV} = \begin{bmatrix} 2 & M & 0 \end{bmatrix}.$$

Using what we learned in the course, we can get the inverse matrix of B quickly.

$$E = \begin{bmatrix} 1 & 0 & -\frac{1}{4} \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}, \; B_{new}^{-1} = EB_{old}^{-1} = \begin{bmatrix} 1 & 0 & -\frac{1}{4} \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}$$

According to the updated matrices and vectors, we could calculate $C_{NBV} - C_{BV}B^{-1}N$, and the result is

$$C_{NBV} - C_{BV}B^{-1}N = \begin{bmatrix} 2M - 1 & 4M - 3 & M \end{bmatrix}. \tag{5}$$

Hence, the procedure should stop. Obviously, the artificial variable a_2 is in basic variables, so the LP has no optimal solution.

1.3 Solution 2 (Revised Simplex Method - Two Phase):

Step 1 Convert LP to Canonical Form

We need to add the slack variable s_1、the excess variable s_2 and the artificial variable a_2, a_3 to obtain the Canonical Form as below.

$$\begin{aligned} \min \quad z = \quad & Ma_2 + Ma_3 \\ & \frac{1}{2}x_1 + \frac{1}{4}x_2 \quad +s_1 \qquad = 4 \\ s.t. \qquad & x_1 + 3x_2 \quad -s_2 + a_2 \quad = 36 \\ & x_1 + x_2 \qquad +a_3 \qquad = 10 \\ & x_1, x_2, s_1, s_2, a_2, a_3 \geq 0 \end{aligned} \tag{6}$$

Step 2 Obtain a bfs

We choose basic variables $BV = \{s_1, a_2, a_3\}$ and nonbasic variables $NBV = \{x_1, x_2, s_2\}$, then we can obtain a basic fessible solution $x_1 = 0, x_2 = 0, s_1 = 4, s_2 = 0, a_2 = 36, a_3 = 10$.

Step 3 Determine if the current bfs is optimal and choose the entering variable

Since we have decided the basic variables and nonbasic variables, we can know

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad C_{BV} = \begin{bmatrix} 0 & M & M \end{bmatrix};$$
$$N = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & 0 \\ 1 & 3 & -1 \\ 1 & 1 & 0 \end{bmatrix}, \quad C_{NBV} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}.$$

According to these matrices and vectors, we could calculate $C_{NBV} - C_{BV}B^{-1}N$, and the result is

$$C_{NBV} - C_{BV}B^{-1}N = \begin{bmatrix} -2M & -4M & M \end{bmatrix}. \tag{7}$$

Hence, the bfs obtained initially is not the optimal solution and we choose x_2 as the entering variable.

Step 4 The Ratio Test for determining the leaving variable

Because we choose x_2 as entering variable, we need to calculate $B^{-1}a_2$ and $B^{-1}b$ respectively.

$$B^{-1}a_2 = \begin{bmatrix} \frac{1}{4} \\ 3 \\ 1 \end{bmatrix}, \quad B^{-1}b = \begin{bmatrix} 4 \\ 36 \\ 10 \end{bmatrix}. \tag{8}$$

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According to the updated matrices and vectors, we could calculate $C_{NBV} - C_{BV}B^{-1}N$, and the result is

$$C_{NBV} - C_{BV}B^{-1}N = \begin{bmatrix} 2M & 4M & M \end{bmatrix}. \tag{9}$$

Hence, the procedure should stop. Obviously, the artificial variable a_2 is in basic variables, so the LP has no optimal solution.

1.4 Solution 3: Graphical Method

We can also solve this LP problem by graphic method. The result is shown in picture 1. Apparently, the fessible region is empty.

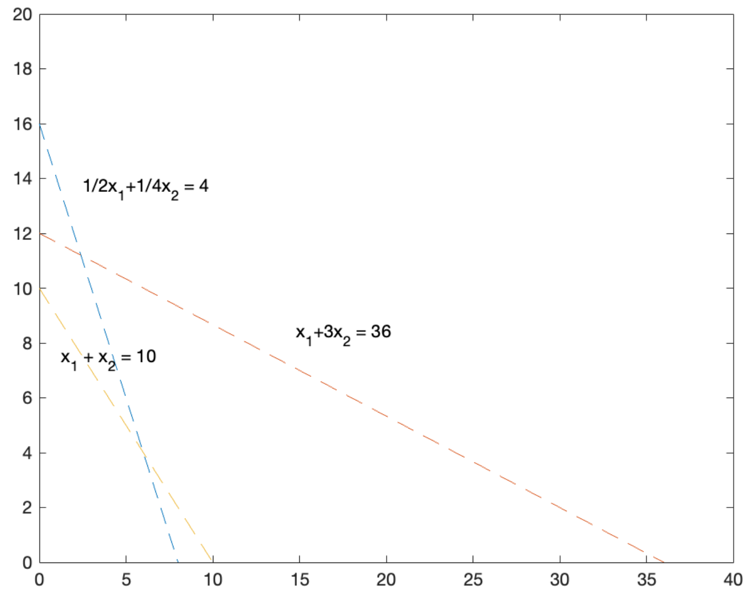

图 1: Graphical solution for LP