Operations Research 2022

Problem Set 2

Lecturer: Wang Shuming TA: Mao Yuchen

1. Show the dual norm: $\|\boldsymbol{y}\|_\infty^* = \sup_{\|\boldsymbol{x}\|_\infty \le 1} \boldsymbol{y}^\top \boldsymbol{x} = \|\boldsymbol{y}\|_1$.

2. Let $f(\boldsymbol{x}) := \frac{1}{2}\boldsymbol{x}^\top \mathbf{M}\boldsymbol{x}$ with $\mathbf{M}$ symmetric, then show that $\nabla f(\boldsymbol{x}) = \mathbf{M}\boldsymbol{x}, \mathbf{H}_f(\boldsymbol{x}) = \mathbf{M}$.

3. Show if $f_i(\boldsymbol{x})$ is convex, then so is $g(\boldsymbol{x}) := \mathrm{Max}_i\, f_i(\boldsymbol{x})$.

4. **(EX 3.2) (Optimality conditions)** Consider the problem of minimizing $\boldsymbol{c}^\top \boldsymbol{x}$ over a polyhedron P. Prove the following:

   (a) A feasible solution $\boldsymbol{x}$ is optimal if and only if $\boldsymbol{c}^\top \boldsymbol{d} \ge 0$ for every feasible direction $\boldsymbol{d}$ at $\boldsymbol{x}$.

   (b) A feasible solution $\boldsymbol{x}$ is the unique optimal if and only if $\boldsymbol{c}^\top \boldsymbol{d} > 0$ for every nonzero feasible direction $\boldsymbol{d}$ at $\boldsymbol{x}$.

5. **(EX 3.7) (Optimality conditions)** Consider a feasible solution $\boldsymbol{x}$ to a standard form problem, and let $Z = \{i \mid x_i = 0\}$. Show that $\boldsymbol{x}$ is an optimal solution if and only if the linear programming problem

$$\begin{aligned} \text{Min} \quad & \boldsymbol{c}^\top \boldsymbol{d} \\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{d} = \mathbf{0} \\ & d_i \ge 0, \quad i \in Z \end{aligned}$$

   has an optimal cost of zero. (In this sense, deciding optimality is equivalent to solving a new linear programming problem.)

6. **(EX 3.4)** Consider the problem of the standard form polyhedron $P = \{\boldsymbol{x} \in \Re^n \mid \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b},\ \boldsymbol{D}\boldsymbol{x} \le \boldsymbol{f}, \boldsymbol{E}\boldsymbol{x} \le \mathbf{g}\}$. Let $\boldsymbol{x}^*$ be an element of P that satisfies $\boldsymbol{D}\boldsymbol{x}^* = \boldsymbol{f}, \boldsymbol{E}\boldsymbol{x}^* < \mathbf{g}$. Show that the set of feasible directions at the point $\boldsymbol{x}^*$ is the set

$$\{\boldsymbol{d} \in \Re^n \mid \boldsymbol{A}\boldsymbol{d} = \mathbf{0}, \boldsymbol{D}\boldsymbol{d} \le \mathbf{0}\}.$$

7. **(EX 4.6) (Duality in Chebychev approximation)** Let $\boldsymbol{A}$ be a $m \times n$ matrix and let $\boldsymbol{b}$ be a vector in $\Re^m$. We consider the problem of minimizing $\|\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}\|_\infty$ over all $\boldsymbol{x} \in \Re^n$. Here $\|\cdot\|_\infty$ is the vector norm defined by $\|\boldsymbol{y}\|_\infty = max_i|y_i|$. Let v be the value of the optimal cost.

(a) Let $\boldsymbol{p}$ be any vector in $\Re^m$ that satisfies $\sum_{i=1}^m |p_i| = 1$ and $\boldsymbol{p}^\top \boldsymbol{A} = \boldsymbol{0}$. Show that $\boldsymbol{p}^\top \boldsymbol{b} \leq v$.

(b) In order to obtain the best possible lower bound of the form consider in part (a), we form the linear programming problem

$$\begin{aligned} \text{Max} \quad & \boldsymbol{p}^\top \boldsymbol{b} \\ \text{s.t.} \quad & \boldsymbol{p}^\top \boldsymbol{A} = \boldsymbol{0} \\ & \sum_{i=1}^m |p_i| \leq 1. \end{aligned} \tag{1}$$

Show that the optimal cost in this problem is equal to v.

8. **(EX 4.7) (Duality in piecewise linear convex optimization)** Consider the problem of minimizing $max_{i=1,\cdots,m}(\boldsymbol{a}_i^\top \boldsymbol{x} - b_i)$ over all $\boldsymbol{x} \in \Re^n$. Let v be the value of optimal cost, assumed finite. Let $\boldsymbol{A}$ be the matrix with rows $\boldsymbol{a}_1, \cdots, \boldsymbol{a}_m$, and let $\boldsymbol{b}$ be the vector with components $\boldsymbol{b}_1, \cdots, \boldsymbol{b}_m$.

(a) Consider any vector $\boldsymbol{p} \in \Re^m$ that satisfies $\boldsymbol{p}^\top \boldsymbol{A} = \boldsymbol{0}$, $\boldsymbol{p} \geq \boldsymbol{0}$, and $\sum_{i=1}^m p_i = 1$. Show that $-\boldsymbol{p}^\top \boldsymbol{b} \leq v$.

(b) In order to obtain the best possible lower bound of the form considered in part (a), we form the linear programming problem

$$\begin{aligned} \text{Max} \quad & -\boldsymbol{p}^\top \boldsymbol{b} \\ \text{s.t.} \quad & \boldsymbol{p}^\top \boldsymbol{A} = \boldsymbol{0} \\ & \boldsymbol{p}^\top \boldsymbol{e} = 1 \\ & \boldsymbol{p} \geq \boldsymbol{0}, \end{aligned} \tag{2}$$

where $\boldsymbol{e}$ is the vector with all components equal to 1. Show that the optimal cost in the problem is equal to v.

9. **(EX 4.10) (Saddle points of the Lagrangean)** Consider the standard form problem of minimizing $\boldsymbol{c}^\top \boldsymbol{x}$ subject to $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ and $\boldsymbol{x} \geq \boldsymbol{0}$. We define the *Lagrangean* by

$$L(\boldsymbol{x}, \boldsymbol{p}) = \boldsymbol{c}^\top \boldsymbol{x} + \boldsymbol{p}^\top (\boldsymbol{b} - \boldsymbol{A}\boldsymbol{x}).$$

Consider the following "game": player 1 choose some $\boldsymbol{x} \geq \boldsymbol{0}$, and player 2 choose some $\boldsymbol{p}$; then, player 1 pays to player 2 the amount $L(\boldsymbol{x}, \boldsymbol{p})$. Player 1 would like to minimize $L(\boldsymbol{x}, \boldsymbol{p})$, while player 2 would like to maximize it.

A pair $(\boldsymbol{x}^*, \boldsymbol{p}^*)$, with $\boldsymbol{x}^* \geq \boldsymbol{0}$, is called an *equilibrium* point (or a *saddle point*, or a *Nash equilibrium*) if

$$L(\boldsymbol{x}^*, \boldsymbol{p}) \leq L(\boldsymbol{x}^*, \boldsymbol{p}^*) \leq L(\boldsymbol{x}, \boldsymbol{p}^*), \quad \forall \boldsymbol{x} \geq \boldsymbol{0},\ \forall \boldsymbol{p}.$$

(Thus, we have an equilibrium if no player is able to improve her performance by unilaterally modifying her choice.)

Show that a pair $(\boldsymbol{x}^*, \boldsymbol{p}^*)$ is an equilibrium if and only if $\boldsymbol{x}^*$ and $\boldsymbol{p}^*$ are optimal solutions to the standard form problem under consideration and its dual, respectively.

10. **EX 4.20 (Strict complementary slackness)**

    (a) Consider the following linear programming problem and its dual

    $$\begin{array}{rlrl} \text{Min} & \boldsymbol{c}^\top \boldsymbol{x} & \text{Max} & \boldsymbol{p}^\top \boldsymbol{b} \\ \text{s.t.} & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} & \text{s.t.} & \boldsymbol{p}^\top \boldsymbol{A} \leq \boldsymbol{c}^\top \\ & \boldsymbol{x} \geq \boldsymbol{0}, & & \boldsymbol{p} \text{ is free,} \end{array}$$

    and assume that both problems have an optimal solution. Fix some j. Suppose that every optimal solution to the primal satisfies $x_j = 0$. Show that there exists an optimal solution $\mathbf{p}$ to the dual such that $\mathbf{p}^\top \boldsymbol{A}_j < c_j$ (Here, $\boldsymbol{A}_j$ is the j th column of $\boldsymbol{A}$.) *Hint*: Let d be the optimal cost. Consider the problem of minimizing $-x_j$ subject to $\boldsymbol{A}\boldsymbol{x} = \mathbf{b}, \boldsymbol{x} \geq \mathbf{0}$, and $-\mathbf{c}^\top \boldsymbol{x} \geq -d$, and form its dual.

    (b) Show that there exist optimal solutions $\boldsymbol{x}$ and $\boldsymbol{p}$ to the primal and to the dual, respectively, such that for every j we have either $x_j > 0$ or $\boldsymbol{p}^\top \boldsymbol{A}_j < c_j$. *Hint*: Use part (a) for each j, and then take the average of the vectors obtained.

    (c) Consider now the following linear programming problem and its dual:

    $$\begin{array}{rlrl} \text{Min} & \boldsymbol{c}^\top \boldsymbol{x} & \text{Max} & \boldsymbol{p}^\top \boldsymbol{b} \\ \text{s.t.} & \boldsymbol{A}\boldsymbol{x} \geq \boldsymbol{b} & \text{s.t.} & \boldsymbol{p}^\top \boldsymbol{A} \leq \boldsymbol{c}^\top \\ & \boldsymbol{x} \geq \boldsymbol{0}, & & \boldsymbol{p} \geq \boldsymbol{0} \end{array}$$

    Assume that both problems have an optimal solution. Show that there exist optimal solutions to the primal and to the dual, respectively, that satisfy *complementary slackness*, that is:

    i. For every j we have either $x_j > 0$ or $\boldsymbol{p}^\top \boldsymbol{A}_j < c_j$

    ii. For every i, we have either $\boldsymbol{a}_i^\top \boldsymbol{x} > b_i$ or $p_i > 0$. (Here, $\boldsymbol{a}_i^\top$ is the i th row of $\boldsymbol{A}$.) *Hint*: Convert the primal to the standard form and apply part(b).

    (d) Consider the linear programming problem

    $$\begin{array}{rl} \text{Min} & 5x_1 + 5x_2 \\ \text{s.t.} & x_1 + x_2 \geq 2 \\ & 2x_1 - x_2 \geq 0 \\ & x_1, x_2 \geq 0 \end{array}$$

Does the optimal primal solution $(2/3, 4/3)$, together with the corresponding dual optimal solution, satisfy strict complementary slackness? Determine all primal and dual optimal solutions and identify the set of *all* strictly complementary pairs.

11. **(EX 4.21) (Clark's theorem)** Consider the following pair of linear programming problems:

$$\begin{array}{llll} \text{(P)} & \text{Min} & \boldsymbol{c}^\top \boldsymbol{x} \\ & \text{s.t.} & \boldsymbol{A}\boldsymbol{x} \geq \boldsymbol{b} \\ & & \boldsymbol{x} \geq \boldsymbol{0} \end{array} \qquad \begin{array}{llll} \text{(D)} & \text{Max} & \boldsymbol{p}^\top \boldsymbol{b} \\ & \text{s.t.} & \boldsymbol{p}^\top \boldsymbol{A} \leq \boldsymbol{c}^\top \\ & & \boldsymbol{p} \geq \boldsymbol{0} \end{array}$$

Suppose that at least one of these two problems has a feasible solution. Prove that the set of feasible solutions to at least one of the two problems is unbounded. *Hint:* Interpret boundedness of a set in terms of the finiteness of the optimal cost of some linear programming problem.

12. **(EX 4.27)** Let $\boldsymbol{A}$ be a given matrix. Show that the following two statements are equivalent.

    (a) Every vector such that $\boldsymbol{A}\boldsymbol{x} \geq \boldsymbol{0}$ and $\boldsymbol{x} \geq \boldsymbol{0}$ must satisfy $x_1 = 0$.

    (b) there exists some $\boldsymbol{p}$ such that $\boldsymbol{p}^\top \boldsymbol{A} \leq \boldsymbol{0}$, $\boldsymbol{p} \geq \boldsymbol{0}$, and $\boldsymbol{p}^\top \boldsymbol{A}_1 < \boldsymbol{0}$, where $\boldsymbol{A}_1$ is the first column of $\boldsymbol{A}$.

13. **(EX 4.50) (Optimality conditions)** We are interested in the problem of deciding whether a polyhedron

$$Q = \{\boldsymbol{x} \in \Re^n \mid \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}, \boldsymbol{D}\boldsymbol{x} \geq \boldsymbol{d}, \boldsymbol{x} \geq \boldsymbol{0}\}$$

is nonempty. We assume that the polyhedron $P = \{\boldsymbol{x} \in \Re^n \mid \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}\}$ is nonempty and bounded. For any vector $\boldsymbol{p}$, of the same dimension as $\boldsymbol{d}$, we define

$$g(\boldsymbol{p}) = -\boldsymbol{p}^\top \boldsymbol{d} + \operatorname*{Max}_{\boldsymbol{x} \in P} \boldsymbol{p}^\top \boldsymbol{D}\boldsymbol{x}.$$

    (a) Show that if Q is nonempty, then $g(\boldsymbol{p}) \geq 0$ for all $\boldsymbol{p} \geq \boldsymbol{0}$.

    (b) Show that if Q is empty, then there exists some $\boldsymbol{p} \geq \boldsymbol{0}$, such that $g(\boldsymbol{p}) < 0$.

    (c) If Q is empty, what is the minimum of $g(\boldsymbol{p})$ over all $\boldsymbol{p} \geq \boldsymbol{0}$.

14. Prove the equivalence between the Min Max problem and the Max Min problem in the zero-sum game using strong duality of LP.