Operations Research

Lecture 3: Duality Theory–Part I

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Outline

1. Motivation
2. The Dual Problem
3. The Duality Theorem
4. A Geometric View
5. Some Applications

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3. The Duality Theorem
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Motivation

- Duality theory can be motivated as an outgrowth of the Lagrange Multiplier Method, often used in calculus to minimize a function subject to equality constraints.
- For example, in order to solve the problem

$$\begin{aligned} \text{Min} \quad & x^2 + y^2 \\ s.t. \quad & x + y = 1. \end{aligned}$$

  We introduce a Lagrange multiplier p and form the Lagrangean $L(x, y, p)$ defined by

$$L(x, y, p) = x^2 + y^2 + p(1 - x - y).$$

- While keeping p fixed, we minimize the Lagrangean over all x and y, subject to no constraints, which can be done by setting $\partial L/\partial x$ and $\partial L/\partial y$ to zero.
- The optimal solution to this unconstrained problem is

$$x = y = \frac{p}{2},$$

  which depends on p. The constraint $x + y = 1$ gives us the additional relation $p = 1$, and the optimal solution to the original problem is $x = y = 1/2$.

Motivation

The main idea in the above example is the following:

- Instead of enforcing the hard constraint $x + y = 1$, we allow it to be violated and associate a Lagrange multiplier, or price, p with the amount $1 - x - y$ by which it is violated.
- This leads to the unconstrained minimization of

$$x^2 + y^2 + p(1 - x - y).$$

  When the price is properly chosen ($p = 1$, in our example), the optimal solution to the constrained problem is also optimal for the unconstrained problem.
- In particular, under that specific value of p, the presence or absence of the hard constraint does not affect the optimal cost.

Consider the following standard form LP (*Primal Problem*):

Primal Problem

$$\begin{aligned} \min_{\boldsymbol{x} \in \Re^n} \quad & \boldsymbol{c}'\boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} = \mathbf{b} \\ & \boldsymbol{x} \geq \mathbf{0} \end{aligned}$$

where $\boldsymbol{A} \in \Re^{m \times n}$.

Duality Theory

We penalize the above primal problem by introducing a vector $\boldsymbol{p}$, and form the following relaxed problem

Lagrange Dual Function

$$\begin{array}{rl} g(\boldsymbol{p}) = \underset{\boldsymbol{x}}{\text{Min}} & \boldsymbol{c}'\boldsymbol{x} + \boldsymbol{p}'(\boldsymbol{b} - \boldsymbol{A}\boldsymbol{x}) \\ \text{s.t.} & \boldsymbol{x} \geq \boldsymbol{0}. \end{array}$$

Suppose $\boldsymbol{x}^*$ is the optimal solution of the primal problem, we then have

$$g(\boldsymbol{p}) = \underset{\boldsymbol{x} \geq \boldsymbol{0}}{\text{Min}} \left\{ \boldsymbol{c}'\boldsymbol{x} + \boldsymbol{p}'(\boldsymbol{b} - \boldsymbol{A}\boldsymbol{x}) \right\} \leq \boldsymbol{c}'\boldsymbol{x}^* + \boldsymbol{p}' \underbrace{(\boldsymbol{b} - \boldsymbol{A}\boldsymbol{x}^*)}_{\boldsymbol{0}} = \boldsymbol{c}'\boldsymbol{x}^*, \ \forall \boldsymbol{p} \in \Re^m.$$

That is, $g(\boldsymbol{p})$ is a lower bound for the optimal value of the primal problem.

Duality Theory

We then pursue the maximum $g(\boldsymbol{p})$ over all the $\boldsymbol{p}$'s, which is termed the *dual* of the primal problem:

Dual Problem

$$\operatorname*{Max}_{\boldsymbol{p}} g(\boldsymbol{p}). \tag{1}$$

Noting that

$$\begin{aligned} g(\boldsymbol{p}) &= \operatorname*{Min}_{\boldsymbol{x}\geq\boldsymbol{0}} \left\{\boldsymbol{c}'\boldsymbol{x} + \boldsymbol{p}'(\boldsymbol{b} - \boldsymbol{A}\boldsymbol{x})\right\} \\ &= \boldsymbol{p}'\boldsymbol{b} + \operatorname*{Min}_{\boldsymbol{x}\geq\boldsymbol{0}} \left\{(\boldsymbol{c}' - \boldsymbol{p}'\boldsymbol{A})\boldsymbol{x}\right\} \\ &= \boldsymbol{p}'\boldsymbol{b} + \begin{cases} 0, & \text{if } \boldsymbol{c}' - \boldsymbol{p}'\boldsymbol{A} \geq \boldsymbol{0} \\ -\infty, & \text{otherwise.} \end{cases} \end{aligned}$$

Therefore, the dual problem

$$\operatorname*{Max}_{\boldsymbol{p}} g(\boldsymbol{p}) = \operatorname*{Max}_{\boldsymbol{p}} \operatorname*{Min}_{\boldsymbol{x} \geq \boldsymbol{0}} \left\{ \boldsymbol{c}'\boldsymbol{x} + \boldsymbol{p}'(\boldsymbol{b} - \boldsymbol{A}\boldsymbol{x}) \right\}$$

can be rewritten as

Dual Problem

$$\begin{aligned} \operatorname*{Max}_{\boldsymbol{p}} \quad & \boldsymbol{p}'\boldsymbol{b} \\ \text{s.t.} \quad & \boldsymbol{p}'\boldsymbol{A} \leq \boldsymbol{c}'. \end{aligned}$$

Motivation

We have established the following Primal-Dual system:

Primal-Dual System

$$\begin{bmatrix} \underset{\boldsymbol{x}}{\text{Min}} & \boldsymbol{c}'\boldsymbol{x} \\ \text{s.t.} & \boldsymbol{A}\boldsymbol{x} = \mathbf{b} \\ & \boldsymbol{x} \geq \mathbf{0}. \end{bmatrix} \geq \begin{bmatrix} \underset{\boldsymbol{p}}{\text{Max}} & \boldsymbol{p}'\boldsymbol{b} \\ \text{s.t.} & \boldsymbol{p}'\boldsymbol{A} \leq \boldsymbol{c}'. \end{bmatrix}$$

Motivation

- In the preceding example, we started with the equality constraint $\boldsymbol{Ax} = \boldsymbol{b}$ and we ended up with no constraints on the sign of the price vector $\boldsymbol{p}$.
- If the primal problem had instead inequality constraints of the form $\boldsymbol{Ax} \geq \boldsymbol{b}$, they could be replaced by $\boldsymbol{Ax} - \boldsymbol{s} = \boldsymbol{b}, \boldsymbol{s} \geq 0$. The equality constraint can be written in the form
$$[\boldsymbol{A} \mid -\boldsymbol{I}] \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{s} \end{bmatrix} = \boldsymbol{b},$$
which leads to the dual constraints
$$\boldsymbol{p}'[\boldsymbol{A} \mid -\boldsymbol{I}] \leq [\boldsymbol{c}' \mid -\boldsymbol{0}'],$$
or, equivalently,
$$\boldsymbol{p}'\boldsymbol{A} \leq \boldsymbol{c}', \boldsymbol{p} \geq \boldsymbol{0}.$$
- Also, if the vector $\boldsymbol{x}$ is free rather than sign-constrained, we use the fact
$$\operatorname*{Min}_{\boldsymbol{x}}(\boldsymbol{c}' - \boldsymbol{p}'\boldsymbol{A})\boldsymbol{x} = \begin{cases} 0, & \text{if } \boldsymbol{c}' - \boldsymbol{p}'\boldsymbol{A} = \boldsymbol{0}', \\ -\infty, & \text{otherwise.} \end{cases}$$
to end up with the constraint $\boldsymbol{p}'\boldsymbol{A} = \boldsymbol{c}'$ in the dual problem. These considerations motivate the general form of the dual problem.

Motivation

In summary, the construction of the dual of a primal minimization problem can be viewed as follows:

- We have a vector of parameters (dual variables) $\boldsymbol{p}$, and for every $\boldsymbol{p}$ we have a method for obtaining a lower bound on the optimal primal cost.
- The dual problem is a maximization problem that looks for the tightest such lower bound. For some vectors $\boldsymbol{p}$, the corresponding lower bound is equal to $-\infty$, and does not carry any useful information.
- Thus, we only need to maximize over those $\boldsymbol{p}$ that lead to nontrivial lower bounds, and this is what gives rise to the dual constraints.

Outline

1 Motivation

2 The Dual Problem

3 The Duality Theorem

4 A Geometric View

5 Some Applications

The Dual Problem

In general, we let $\boldsymbol{a}_i$ be the ith row of the matrix $\boldsymbol{A}$, and $\boldsymbol{A}_j$ the jth column, then we have the following recipe for the duality:

General Primal-Dual System

$$
\begin{array}{llll}
\min\limits_{\boldsymbol{x}} & \boldsymbol{c}'\boldsymbol{x} & \\
\text{s.t.} & \boldsymbol{a}_i'\boldsymbol{x} \geq b_i, & i \in M_1\\
 & \boldsymbol{a}_i'\boldsymbol{x} \leq b_i, & i \in M_2\\
 & \boldsymbol{a}_i'\boldsymbol{x} = b_i, & i \in M_3\\
 & x_j \geq 0, & j \in N_1\\
 & x_j \leq 0, & j \in N_2\\
 & x_j \text{ free}, & j \in N_3
\end{array}
\qquad
\begin{array}{llll}
\max\limits_{\boldsymbol{p}} & \boldsymbol{p}'\boldsymbol{b} & \\
\text{s.t.} & p_i \geq 0, & i \in M_1\\
 & p_i \leq 0, & i \in M_2\\
 & p_i \text{ free}, & i \in M_3\\
 & \boldsymbol{p}'\boldsymbol{A}_j \leq c_j, & j \in N_1\\
 & \boldsymbol{p}'\boldsymbol{A}_j \geq c_j, & j \in N_2\\
 & \boldsymbol{p}'\boldsymbol{A}_j = c_j, & j \in N_3
\end{array}
$$

The Dual Problem

Example (Duality Structure of LP)

	x_1	$\cdots$	x_n	
p_1	a_{11}	$\cdots$	a_{1n}	b_1
$\vdots$	$\vdots$	a_{ij}	$\vdots$	$\vdots$
p_m	a_{m1}	$\cdots$	a_{mn}	b_m
	c_1	$\cdots$	c_n	

The Dual Problem

Table 1: Duality Recipe for LP

Primal	Minimize	Maximize	Dual
Constraints	$\geq b_i$	≥ 0	Variables
	$\leq b_i$	≤ 0	
	$= b_i$	free	
Variables	≥ 0	$\leq c_j$	Constraints
	≤ 0	$\geq c_j$	
	$= 0$	free	

The Dual Problem

In matrix notation, we can present the primal and the dual in more compact forms:

$$\begin{array}{rl} \text{Min} & \boldsymbol{c}'\boldsymbol{x} \\ \text{s.t.} & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{array} \qquad \begin{array}{rl} \text{Max} & \boldsymbol{p}'\boldsymbol{b} \\ \text{s.t.} & \boldsymbol{p}'\boldsymbol{A} \leq \boldsymbol{c}' \end{array}$$

and

$$\begin{array}{rl} \text{Min} & \boldsymbol{c}'\boldsymbol{x} \\ \text{s.t.} & \boldsymbol{A}\boldsymbol{x} \geq \boldsymbol{b} \end{array} \qquad \begin{array}{rl} \text{Max} & \boldsymbol{p}'\boldsymbol{b} \\ \text{s.t.} & \boldsymbol{p}'\boldsymbol{A} = \boldsymbol{c}' \\ & \boldsymbol{p} \geq \boldsymbol{0} \end{array}$$

The Dual Problem

Example (Find the dual)

Primal:

$$\begin{aligned} \text{Min} \quad & x_1 + 2x_2 + 3x_3 \\ \text{s.t.} \quad & -x_1 + 3x_2 = 5 \\ & 2x_1 - x_2 + 3x_3 \geq 6 \\ & x_3 \leq 4 \\ & x_1 \geq 0 \\ & x_2 \leq 0 \\ & x_3 \text{ free} \end{aligned}$$

Example (Find the dual)

Primal:

$$\begin{aligned} \text{Min} \quad & -5x_1 - 6x_2 - 4x_3 \\ \text{s.t.} \quad & x_1 - 2x_2 \geq -1 \\ & -3x_1 + x_2 \leq -2 \\ & -3x_2 - x_3 = -3 \\ & x_1 \text{ free} \\ & x_2 \geq 0 \\ & x_3 \leq 0 \end{aligned}$$

- **The dual of the dual = The primal**

The Dual Problem

Theorem (The Dual of the Dual is the Primal,4.1)

If we transform the dual into an equivalent minimization problem and then form its dual, we obtain a problem equivalent to the original problem.

- A compact statement that is often used to describe Theorem 4.1 is that "the dual of the dual is the primal."
- Any linear programming problem can be manipulated into one of several equivalent forms, for example, by introducing slack variables or by using the difference of two nonnegative variables to replace a single free variable.
- Each equivalent form leads to a somewhat different form for the dual problem. Nevertheless, the examples that follow indicate that the duals of equivalent problems are equivalent.

The Dual Problem

Example 4.2: Consider the primal problem shown on the left and its dual shown on the right:

$$
\begin{array}{rl}
\text{minimize} & \mathbf{c}'\mathbf{x} \\
\text{subject to} & \mathbf{A}\mathbf{x} \geq \mathbf{b} \\
 & \mathbf{x} \text{ free},
\end{array}
\qquad\qquad
\begin{array}{rl}
\text{maximize} & \mathbf{p}'\mathbf{b} \\
\text{subject to} & \mathbf{p} \geq \mathbf{0} \\
 & \mathbf{p}'\mathbf{A} = \mathbf{c}'.
\end{array}
$$

We transform the primal problem by introducing surplus variables and then obtain its dual:

$$
\begin{array}{rl}
\text{minimize} & \mathbf{c}'\mathbf{x} + \mathbf{0}'\mathbf{s} \\
\text{subject to} & \mathbf{A}\mathbf{x} - \mathbf{s} = \mathbf{b} \\
 & \mathbf{x} \text{ free} \\
 & \mathbf{s} \geq \mathbf{0},
\end{array}
\qquad\qquad
\begin{array}{rl}
\text{maximize} & \mathbf{p}'\mathbf{b} \\
\text{subject to} & \mathbf{p} \text{ free} \\
 & \mathbf{p}'\mathbf{A} = \mathbf{c}' \\
 & -\mathbf{p} \leq \mathbf{0}.
\end{array}
$$

The Dual Problem

- Alternatively, if we take the original primal problem and replace $\boldsymbol{x}$ by signconstrained variables, we obtain the following pair of problems:

$$\begin{array}{rl} \text{minimize} & \mathbf{c}'\mathbf{x}^+ - \mathbf{c}'\mathbf{x}^- \\ \text{subject to} & \mathbf{A}\mathbf{x}^+ - \mathbf{A}\mathbf{x}^- \geq \mathbf{b} \\ & \mathbf{x}^+ \geq \mathbf{0} \\ & \mathbf{x}^- \geq \mathbf{0}, \end{array} \qquad \begin{array}{rr} \text{maximize} & \mathbf{p}'\mathbf{b} \\ \text{subject to} & \mathbf{p} \geq \mathbf{0} \\ & \mathbf{p}'\mathbf{A} \leq \mathbf{c}' \\ & -\mathbf{p}'\mathbf{A} \leq -\mathbf{c}'. \end{array}$$

- Note that we have three equivalent forms of the primal. We observe that the constraint $\boldsymbol{p} \geq \mathbf{0}$ is equivalent to the constraint $-\boldsymbol{p} \leq \mathbf{0}$.
- Furthermore, the constraint $\boldsymbol{p}'\boldsymbol{A} = \boldsymbol{c}'$ is equivalent to the two constraints $\boldsymbol{p}'\boldsymbol{A} \leq \boldsymbol{c}'$ and $-\boldsymbol{p}'\boldsymbol{A} \leq -\boldsymbol{c}'$. Thus, the duals of the three variants of the primal problem are also equivalent.

The Dual Problem

Theorem (Equivalence of the Primals vs. Equivalence of the duals, 4.2)

Suppose that we have transformed a linear programming problem Π_1 to another linear programming problem Π_2, by a sequence of transformations of the following types:

1. *Replace a free variable with the difference of two nonnegative variables.*
2. *Replace an inequality constraint by an equality constraint involving a nonnegative slack variable.*
3. *If some row of the matrix $\boldsymbol{A}$ in a feasible standard form problem is a linear combination of the other rows, eliminate the corresponding equality constraint.*

Then, the duals of Π_1 and Π_2 are equivalent, i.e., they are either both infeasible, or they have the same optimal cost.

The Primal Problem

$$
\begin{aligned}
R(\boldsymbol{x}, \boldsymbol{s}, \boldsymbol{d}) = \operatorname*{Max}_{\boldsymbol{y}} \quad & \boldsymbol{r}^\top \mathbf{A} \boldsymbol{y} + \boldsymbol{p}^\top \mathbf{D} \boldsymbol{y} \\
\text{s.t.} \quad & \mathbf{W}_1 \boldsymbol{y} \leq \mathbf{V}_2 \boldsymbol{d}, \\
& \mathbf{W}_2 \boldsymbol{y} \leq \mathbf{Q}_1 \boldsymbol{s}, \\
& \mathbf{W}_3 \boldsymbol{y} \leq \mathbf{Q}_2 \boldsymbol{x}, \\
& \boldsymbol{y} \geq \mathbf{0},
\end{aligned}
$$

The Dual Problem

$$
\begin{aligned}
R(\boldsymbol{x}, \boldsymbol{s}, \boldsymbol{d}) = \underset{\boldsymbol{g},\boldsymbol{h},\boldsymbol{u}}{\mathrm{Min}} \quad & \boldsymbol{g}^\top \mathbf{V}_2 \boldsymbol{d} + \boldsymbol{h}^\top \mathbf{Q}_1 \boldsymbol{s} + \boldsymbol{u}^\top \mathbf{Q}_2 \boldsymbol{x} \\
\text{s.t.} \quad & \boldsymbol{g}^\top \mathbf{W}_1 + \boldsymbol{h}^\top \mathbf{W}_2 + \boldsymbol{u}^\top \mathbf{W}_3 \geq \boldsymbol{r}^\top \mathbf{A} + \boldsymbol{p}^\top \mathbf{D} \\
& \boldsymbol{g}, \boldsymbol{h}, \boldsymbol{u} \geq \mathbf{0}.
\end{aligned}
$$

The Primal Problem

$$
\begin{aligned}
R(\boldsymbol{x}, \boldsymbol{s}, \boldsymbol{d}) = \max_{\boldsymbol{y}} \quad & \sum_{i \in [I]} \sum_{j \in [J]} \sum_{k \in [K]} \sum_{l \in [L]} (r_{ijk} + p_{jlk})\, y_{ijlk} \\
\text{s.t.} \quad & \sum_{i \in [I]} y_{ijlk} \leq V_{lk} d_{jl}, \forall j \in [J], k \in [K], l \in [L] \\
& \sum_{j \in [J]} \sum_{l \in [L]} y_{ijlk} \leq s_{ik} q_k, \forall i \in [I], k \in [K] \\
& \sum_{j \in [J]} y_{ijlk} \leq x_{il} B_{ik} q_k, \forall i \in [I], k \in [K], l \in [L] \\
& \boldsymbol{y} \geq \mathbf{0}.
\end{aligned}
$$

The Dual Problem

$R(\boldsymbol{x}, \boldsymbol{s}, \boldsymbol{d})$ can also be reformulated as the following minimization problem:

$$
\begin{aligned}
\min_{\boldsymbol{g},\boldsymbol{h},\boldsymbol{u}} \quad & \sum_{k\in[K]}\sum_{j\in[J]}\sum_{l\in[L]} g_{jlk}V_{lk}d_{jl} + \sum_{i\in[I]}\sum_{k\in[K]} h_{ik}s_{ik}q_k + \sum_{i\in[I]}\sum_{l\in[L]}\sum_{k\in[K]} u_{ilk}x_{il}B_{ik}q_k \\
\text{s.t.} \quad & g_{jlk} + h_{ik} + u_{ilk} \geq r_{ijk} + p_{jkl}, \forall j \in [J], k \in [K], l \in [L], i \in [I] \\
& \boldsymbol{g}, \boldsymbol{h}, \boldsymbol{u} \geq \mathbf{0}.
\end{aligned}
$$

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The Duality Theorem

Theorem (Weak duality, 4.3)

If $\boldsymbol{x}$ is a feasible solution to the primal problem (minimization) and $\boldsymbol{p}$ is a feasible solution to the dual problem (maximization), then

$$\boldsymbol{c}'\boldsymbol{x} \geq \boldsymbol{p}'\boldsymbol{b}.$$

The Duality Theorem

Corollary (4.1)

Let the primal problem be of an objective of minimization, we then have:

- *If the optimal cost in the primal problem is $-\infty$ (unbounded), then the dual problem must be infeasible.*
- *If the optimal cost in the dual problem is $+\infty$ (unbounded), then the primal problem must be infeasible.*

The Duality Theorem

Corollary (4.2)

Let $\boldsymbol{x}$ and $\boldsymbol{p}$ be feasible solutions to the primal and dual problems, respectively, and suppose that $\mathbf{c}'\boldsymbol{x} = \boldsymbol{p}'\boldsymbol{b}$. Then, $\boldsymbol{x}$ and $\boldsymbol{p}$ are optimal solutions to the primal and the dual, respectively.

Polyhedra in Standard Form and BFS

Theorem (2.4, Construction of BS)

Consider the constraints $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ and $\boldsymbol{x} \geq \boldsymbol{0}$ and assume that the $m \times n$ matrix $\boldsymbol{A}$ has linearly independent rows. A vector $\boldsymbol{x} \in \mathbb{R}^n$ is a basic solution (BS) if and only if

(a) *we have $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$,*

and there exist indices $B(1), \ldots, B(m)$ such that:

(b) *The columns $\boldsymbol{A}_{B(1)}, \ldots, \boldsymbol{A}_{B(m)}$ are linearly independent;*

(c) *If $i \neq B(1), \ldots, B(m)$, then $x_i = 0$.*

In view of Theorem 2.4, all basic solutions to a standard form polyhedron can be constructed according to the following procedure.

Procedure to Construct BSs of Standard Polyhedra

1. Choose m linearly independent columns $\boldsymbol{A}_{B(1)}, \ldots, \boldsymbol{A}_{B(m)}$.
2. Let $x_i = 0$ for all $i \neq B(1), \ldots, B(m)$.
3. Solve the system of m equations $\boldsymbol{Ax} = \boldsymbol{b}$ for the unknowns $x_{B(1)}, \ldots, x_{B(m)}$

Q: is the solution obtained a BFS?

Polyhedra in Standard Form and BFS

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1. Choose m linearly independent columns $\boldsymbol{A}_{B(1)}, \ldots, \boldsymbol{A}_{B(m)}$.
2. Let $x_i = 0$ for all $i \neq B(1), \ldots, B(m)$.
3. Solve the system of m equations $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ for the unknowns $x_{B(1)}, \ldots, x_{B(m)}$

Q: is the solution obtained a BFS?

- If a basic solution constructed according to this procedure is nonnegative, then it is feasible, and it is a basic feasible solution.
- Conversely, since every BFS is a BS, it can be obtained from this procedure.

The Duality Theorem

Theorem (Strong Duality, 4.4)

If a linear programming problem has an optimal solution (finite optimum), so does its dual, and the respective optimal are equal.

Proof. Standard form LP.

Corollary: If the optimal cost is finite, then there exists an optimal basis such that $\mathbf{B}^{-1}\boldsymbol{b} \geq \mathbf{0}$ and $\boldsymbol{c}^\top - \boldsymbol{c}_B^\top \mathbf{B}^{-1}\mathbf{A} \geq \mathbf{0}^\top$.

Theorem (Complementary Slackness, 4.5)

Let $\boldsymbol{x}$ and $\boldsymbol{p}$ be feasible solutions to the primal (min.) and the dual (max.) problem, respectively. The vectors $\boldsymbol{x}$ and $\boldsymbol{p}$ are optimal solutions for the two respective problems if and only if

$$p_i(\boldsymbol{a}_i'\boldsymbol{x} - b_i) = 0, \ \forall i;$$

$$(c_j - \boldsymbol{p}'\boldsymbol{A}_j)x_j = 0, \ \forall j.$$

Proof.

$$\sum_i p_i \boldsymbol{a}_i^\top \boldsymbol{x} = \left[\sum_i p_i \boldsymbol{a}_i^\top\right] \boldsymbol{x} = \left[\sum_i p_i \boldsymbol{a}_i\right]^\top \boldsymbol{x} = \left[\mathbf{A}^\top \boldsymbol{p}\right]^\top \boldsymbol{x}$$

The Duality Theorem

- The first complementary slackness condition is automatically satisfied by every feasible solution to a problem in standard form.
- If the primal problem is not in standard form and has a constraint like $\boldsymbol{a}_i'\boldsymbol{x} \geq b_i$, the corresponding complementary slackness condition asserts that the dual variable p_i is zero unless the constraint is active.
- An intuitive explanation: a constraint which is not active at an optimal solution can be removed from the problem without affecting the optimal cost (WHY?), and there is no point in associating a nonzero price with such a constraint.

- If the primal problem is in standard form and a nondegenerate optimal BFS is known, the complementary slackness conditions determine a unique solution to the dual problem.

Definition (Degeneracy, 2.10)

A basic solution (BS) $\boldsymbol{x} \in \mathbb{R}^n$ is said to be degenerate if more than n of the constraints are active at $\boldsymbol{x}$.

Definition (Degeneracy in Standard Form Polyhedra, 2.11)

Consider a standard form polyhedron $P = \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}\}$ and a basic solution (BS) $\boldsymbol{x}$. Let m be the number of rows of $\boldsymbol{A}$. The vector $\boldsymbol{x}$ is a **degenerate basic solution** if more than $n - m$ of the components of $\boldsymbol{x}$ are zeros.

The Duality Theorem

Proposition (Recovering the Optimal Dual Solution)

Consider the primal problem (min) in standard form with an optimal nondegenerate BFS $\boldsymbol{x}^*$ and the associated basis $\boldsymbol{B}$. Then the optimal dual solution is

$$\boldsymbol{p}' = \boldsymbol{c}'_B \boldsymbol{B}^{-1}.$$

Proof.

- Suppose that x_j is a basic variable in a nondegenerate optimal basic feasible solution to a primal problem in standard form.
- Then, the complementary slackness condition $(c_j - \boldsymbol{p}'\boldsymbol{A}_j)x_j = 0$ yields $\boldsymbol{p}'\boldsymbol{A}_j = c_j$ for every such j. Since the basic columns $\boldsymbol{A}_j$ are linearly independent, we obtain a system of equations for $\boldsymbol{p}$ which has a unique solution, namely, $\boldsymbol{p}' = \boldsymbol{c}'_B \boldsymbol{B}^{-1}$.

REMARK: If we are given a degenerate optimal basic feasible solution to the primal, complementary slackness may be of very little help in determining an optimal solution to the dual problem (Exercise 4.17).

Example 4.6:

- Consider a problem in standard form and its dual:

$$
\begin{array}{lrcrcrcl}
\text{minimize} & 13x_1 & + & 10x_2 & + & 6x_3 & & \\
\text{subject to} & 5x_1 & + & x_2 & + & 3x_3 & = & 8 \\
 & 3x_1 & + & x_2 & & & = & 3 \\
 & \multicolumn{7}{l}{x_1, x_2, x_3 \geq 0,}
\end{array}
\qquad
\begin{array}{lrcrcl}
\text{maximize} & 8p_1 & + & 3p_2 & & \\
\text{subject to} & 5p_1 & + & 3p_2 & \leq & 13 \\
 & p_1 & + & p_2 & \leq & 10 \\
 & 3p_1 & & & \leq & 6.
\end{array}
$$

- As will be verified shortly, the vector $\boldsymbol{x}^* = (1, 0, 1)$ is a nondegenerate optimal solution to the primal problem. Assuming this to be the case, we use the complementary slackness conditions to construct the optimal solution to the dual.

- The condition $p_i(\boldsymbol{a}_i'\boldsymbol{x}^* - b_i) = 0$ is automatically satisfied for each i, since the primal is in standard form. The condition $(c_j - \boldsymbol{p}'\boldsymbol{A}_j)x_j^* = 0$ is clearly satisfied for $j = 2$, because $x_2^* = 0$. However, since $x_1^* > 0$ and $x_3^* > 0$, we obtain

$$5p_1 + 3p_2 = 13,$$

and

$$3p_1 = 6$$

which we can solve to obtain $p_1 = 2$ and $p_2 = 1$. Note that this is a dual feasible solution whose cost is equal to 19, which is the same as the cost of $\boldsymbol{x}^*$. This verifies that $\boldsymbol{x}^*$ is indeed an optimal solution as claimed earlier.

The Duality Theorem

- **REMARK:** We finally mention that if the primal constraints are of the form $\boldsymbol{A}\boldsymbol{x} \geq \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}$, and the primal problem has an optimal solution, then there exist optimal solutions to the primal and the dual which satisfy Strict Complementary Slackness. That is, a variable in one problem is nonzero if and only if the corresponding constraint in the other problem is active (Exercise 4.20).

Duality Theory

Theorem (Optimality Condition: A Dual View)

Consider a basic feasible solution (BFS) $\boldsymbol{x}$ associated with a basis matrix $\boldsymbol{B}$, and let $\overline{\mathbf{c}}$ be the corresponding vector of reduced costs. If

$$\overline{\boldsymbol{c}} := \boldsymbol{c}^\top - \boldsymbol{c}_B^\top \boldsymbol{B}^{-1} \boldsymbol{A} \geq \mathbf{0}^\top,$$

then $\boldsymbol{x}$ is optimal.

- We have the equal objective functions $\boldsymbol{c}^\top \boldsymbol{x} = \boldsymbol{p}^\top \boldsymbol{b}$
- The dual feasibility implies optimality!

How about the other situations when the finite optimum is not available? Consider the following example:

Example

Primal:

$$\begin{aligned} \text{Min} \quad & x_1 + 2x_2 \\ \text{s.t.} \quad & x_1 + x_2 = 1 \\ & 2x_1 + 2x_2 = 3 \end{aligned}$$

which is infeasible whose dual is

$$\begin{aligned} \text{Max} \quad & p_1 + 3p_2 \\ \text{s.t.} \quad & p_1 + 2p_2 = 1 \\ & p_1 + 2x_2 = 2 \end{aligned}$$

which is also infeasible.

Table 2: Different possibilities for the primal and the dual

Primal/Dual	Finite optimum	Unbounded	Infeasible
Finite optimum	$\star$	–	–
Unbounded	–	–	$\star$
Infeasible	–	$\star$	$\star$

Outline

1. Motivation
2. The Dual Problem
3. The Duality Theorem
4. A Geometric View
5. Some Applications

A Geometric View

- We now develop a geometric view that allows us to visualize pairs of primal and dual vectors without having to draw the dual feasible set.
- We consider the primal problem

$$\begin{array}{rl} \text{Min} & \boldsymbol{c}'\boldsymbol{x} \\ \text{s.t.} & \boldsymbol{a}_i'\boldsymbol{x} \geq b_i, i = 1, \ldots, m, \end{array}$$

where the dimension of $\boldsymbol{x}$ is equal to n. We assume that the vectors $\boldsymbol{a}_i$ span $\Re^n$. The corresponding dual problem is

$$\begin{array}{rl} \text{Max} & \boldsymbol{p}'\boldsymbol{b} \\ \text{s.t.} & \displaystyle\sum_{i=1}^{m} p_i \boldsymbol{a}_i = \boldsymbol{c} \\ & \boldsymbol{p} \geq \boldsymbol{0}. \end{array}$$

A Geometric View

- Let $\boldsymbol{I}$ be a subset of $\{1, \ldots, m\}$ of cardinality n, such that the vectors $\boldsymbol{a}_i, i \in \boldsymbol{I}$, are linearly independent. The system $\boldsymbol{a}_i'\boldsymbol{x} = b_i, i \in \boldsymbol{I}$, has a unique solution, denoted by $\boldsymbol{x}^{\boldsymbol{I}}$, which is a BS to the primal problem. We assume, that $\boldsymbol{x}^{\boldsymbol{I}}$ is nondegenerate, that is, $\boldsymbol{a}_i'\boldsymbol{x} \neq b_i, i \notin \boldsymbol{I}$.
- Let $\boldsymbol{p} \in \Re^m$ be a dual vector (not necessarily dual feasible), and let us consider what is required for $\boldsymbol{x}^{\boldsymbol{I}}$ and $\boldsymbol{p}$ to be optimal solutions to the primal and the dual problem, respectively. We need:
  (a) $\boldsymbol{a}_i'\boldsymbol{x}^I \geq b_i$, for all i (primal feasibility)
  (b) $p_i = 0$, for all $i \notin I$ (complementary slackness)
  (c) $\sum_{i=1}^{m} p_i\boldsymbol{a}_i = \boldsymbol{c}$ (dual feasibility)
  (d) $\boldsymbol{p} \geq \boldsymbol{0}$ (dual feasibility)

A Geometric View

- Given the complementary slackness condition (b), condition (c) becomes

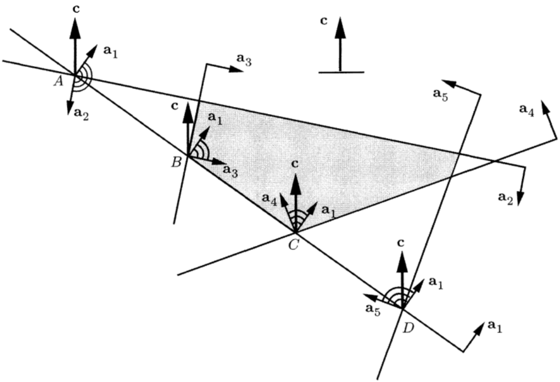
$$\sum_{i \in I} p_i \boldsymbol{a}_i = \boldsymbol{c}.$$

- Since the vectors $\boldsymbol{a}_i, i \in I$, are linearly independent, the latter equation has a unique solution that we denote by $\boldsymbol{p}^I$. In fact, it is readily seen that the vectors, $\boldsymbol{a}_i, i \in I$, form a basis for the dual problem (which is in standard form) and $\boldsymbol{p}^I$ is the associated BS.

- For the vector $\boldsymbol{p}^I$ to be dual feasible, we also need it to be nonnegative. We conclude that once the complementary slackness condition (b) is enforced, feasibility of the resulting dual vector $\boldsymbol{p}^I$ is equivalent to $\boldsymbol{c}$ being a nonnegative linear combination of the vectors $\boldsymbol{a}_i, i \in I$, associated with the active primal constraints.

A Geometric View

This allows us to visualize dual feasibility without having to draw the dual feasible set; see the Figure given below.



A Geometric View

- Consider a primal problem with two variables and five inequality constraints ($n = 2, m = 5$), and suppose that no two of the vectors $\boldsymbol{a}_i$ are collinear. Every two-element subset I of $\{1, 2, 3, 4, 5\}$ determines basic solutions $\boldsymbol{x}^I$ and $\boldsymbol{p}^I$ of the primal and the dual, respectively.
- If $I = \{1, 2\}$, x^I is primal infeasible (point A) and $\boldsymbol{p}^I$ is dual infeasible, because $\boldsymbol{c}$ cannot be expressed as a nonnegative linear combination of the vectors $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$.
- If $I = \{1, 3\}$, x^I is primal feasible (point B) and $\boldsymbol{p}^I$ is dual infeasible.
- If $I = \{1, 4\}$, x^I is primal feasible (point C) and $\boldsymbol{p}^I$ is dual feasible, because $\boldsymbol{c}$ can be expressed as a nonnegative linear combination of the vectors $\boldsymbol{a}_1$ and $\boldsymbol{a}_4$. In particular, x^I and $\boldsymbol{p}^I$ are optimal.
- If $I = \{1, 5\}$, x^I is primal infeasible (point D) and $\boldsymbol{p}^I$ is dual feasible.

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Example (Optimal Dual Variables as Shadow Prices)

Let's consider the diet problem: Suppose that there are n different foods and m different nutrients, and that we are given the following table with the nutritional content of a unit of each food:

Table 3: Foods & Nutrients

	food 1	$\cdots$	food n
nutrient 1	a_{11}	$\cdots$	a_{1n}
$\vdots$	$\vdots$	a_{ij}	$\vdots$
nutrient m	a_{m1}	$\cdots$	a_{mn}

Example (Optimal Dual Variables as Shadow Prices, cont'd)

- Let $\boldsymbol{c}$ be the cost vector with each component c_j be the cost of the jth food.
- Let $\boldsymbol{A}$ be the $m \times n$ matrix with entries a_{ij}, where the jth column $\boldsymbol{A}_j$ of this matrix represents the nutritional content of the jth food.
- Let $\boldsymbol{b}$ be a vector with the requirements of an ideal diet or, equivalently, a specification of the nutritional contents of an "ideal food".
- $\boldsymbol{x}$ is the decision vector with each x_j be the quantity of the jth food to be used to make the diet.

Example (Optimal Dual Variables as Shadow Prices, cont'd)

Now our diet problem is to minimize the total food cost by choosing the optimal quantities of all the foods such that the nutritional requirement. The problem can be expressed as the following standard form LP:

$$\begin{aligned} \text{Min} \quad & \boldsymbol{c}'\boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{aligned}$$

whose dual is

$$\begin{aligned} \text{Max} \quad & \boldsymbol{p}'\boldsymbol{b} \\ \text{s.t.} \quad & \boldsymbol{p}'\boldsymbol{A} \leq \boldsymbol{c}' \end{aligned}$$

Example (Optimal Dual Variables as Shadow Prices, cont'd)

- p_i can be interpreted as a "fair" price of the per unit of the ith nutrient.
- A unit of jth food has a value of c_j in the food market, but it also has a value of $\boldsymbol{p}'\boldsymbol{A}_j$ if priced in the nutrient market.
- Every food j which is used ($x_j > 0$) to synthesize the idea food, should be consistently priced at the two market ($c_j = \boldsymbol{p}'\boldsymbol{A}_j$). (Complementary Slackness)
- The strong duality $\boldsymbol{c}'\boldsymbol{x}^* = \boldsymbol{b}'\boldsymbol{p}^*$ asserts that when prices are chosen appropriately, the two accounting methods should give the same result.

Example (Two-Person Zero-Sum Game)

Payoff Matrix. Scissor (S), Rock (R), Paper (P)
Payment ($\pm$) is from row player to column player:

$$A = \begin{array}{c} \\ P \\ S \\ R \end{array} \begin{array}{c} \begin{array}{ccc} P & S & R \end{array} \\ \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} \end{array}$$

Example (Two-Person Zero-Sum Game)

Generalization: given an $m \times n$ matrix A (Payoff matrix).

- Row player (rowboy) selects a strategy $i \in \{1, 2, \ldots, m\}$.
- Col player (colgirl) selects a strategy $j \in \{1, 2, \ldots, n\}$.
- a_{ij}: given the strategy pair (i, j) (picked by boy and girl, resp.), rowboy pays colgirl a_{ij} dollars.

- The row indices of A represent deterministic strategies for rowboy, while column indices of A represent deterministic strategies for colgirl.
- Deterministic strategies can be bad.

Example (Two-Person Zero-Sum Game)

Randomized Strategy:

- Suppose rowboy picks i with probability y_i.
- Suppose colgirl picks j with probability x_j.
- Now, $\boldsymbol{y} = [y_1, y_2, \ldots, y_m]'$ and $\boldsymbol{x} = [x_1, x_2, \ldots, x_n]'$ are rowboy's and colgirl's decisions, respectively.
  $\sum_{i=1}^m y_i = 1, y_i \geq 0; \quad \sum_{j=1}^n x_j = 1, x_j \geq 0.$

If rowboy uses random strategy $\boldsymbol{y}$ and colgirl uses $\boldsymbol{x}$, then expected payoff from rowboy to colgirl is

$$\sum_{i=1}^{m} \sum_{j=1}^{n} y_i a_{ij} x_j = \boldsymbol{y}' \boldsymbol{A} \boldsymbol{x}.$$

Example (Two-Person Zero-Sum Game)

Now:

- Rowboy's problem:

$$\boldsymbol{y} : \boldsymbol{y}'\boldsymbol{A}\boldsymbol{x} \longrightarrow \operatorname*{Min}_{\boldsymbol{y}} \left[\operatorname*{Max}_{\boldsymbol{x}} \boldsymbol{y}'\boldsymbol{A}\boldsymbol{x} \right]$$

- Colgirl's problem:

$$\boldsymbol{x} : \boldsymbol{y}'\boldsymbol{A}\boldsymbol{x} \longrightarrow \operatorname*{Max}_{\boldsymbol{x}} \left[\operatorname*{Min}_{\boldsymbol{y}} \boldsymbol{y}'\boldsymbol{A}\boldsymbol{x} \right]$$

- H.W.

Example (An Ultra-Conservative Investor (Vanderbei 2007))

Consider the historical return on investment data:

Year	US 3-Month T-Bills	US Gov. Long Bonds	S&P 500	Wilshire 5000	NASDAQ Composite	Lehman Bros. Corp. Bonds	EAFE	Gold
1973	1.075	0.942	0.852	0.815	0.698	1.023	0.851	1.677
1974	1.084	1.020	0.735	0.716	0.662	1.002	0.768	1.722
1975	1.061	1.056	1.371	1.385	1.318	1.123	1.354	0.760
1976	1.052	1.175	1.236	1.266	1.280	1.156	1.025	0.960
1977	1.055	1.002	0.926	0.974	1.093	1.030	1.181	1.200
1978	1.077	0.982	1.064	1.093	1.146	1.012	1.326	1.295
1979	1.109	0.978	1.184	1.256	1.307	1.023	1.048	2.212
1980	1.127	0.947	1.323	1.337	1.367	1.031	1.226	1.296
1981	1.156	1.003	0.949	0.963	0.990	1.073	0.977	0.688
1982	1.117	1.465	1.215	1.187	1.213	1.311	0.981	1.084
1983	1.092	0.985	1.224	1.235	1.217	1.080	1.237	0.872
1984	1.103	1.159	1.061	1.030	0.903	1.150	1.074	0.825
1985	1.080	1.366	1.316	1.326	1.333	1.213	1.562	1.006
1986	1.063	1.309	1.186	1.161	1.086	1.156	1.694	1.216
1987	1.061	0.925	1.052	1.023	0.959	1.023	1.246	1.244
1988	1.071	1.086	1.165	1.179	1.165	1.076	1.283	0.861
1989	1.087	1.212	1.316	1.292	1.204	1.142	1.105	0.977
1990	1.080	1.054	0.968	0.938	0.830	1.083	0.766	0.922
1991	1.057	1.193	1.304	1.342	1.594	1.161	1.121	0.958
1992	1.036	1.079	1.076	1.090	1.174	1.076	0.878	0.926
1993	1.031	1.217	1.100	1.113	1.162	1.110	1.326	1.146
1994	1.045	0.889	1.012	0.999	0.968	0.965	1.078	0.990

- We can view this as a payoff matrix in a game between Fate and the Investor
- The columns represent pure strategies for our conservative investor.
- The rows represent how history might repeat itself.

Fate's conspiracy

- For next year (1995), Fate won't just repeat a previous year but, rather, will present some mixture of these previous years.
- Likewise, the investor won't put all of her money into one asset. Instead she will put a certain fraction into each.

Investor's Optimal Asset Mix:

- US 3-MONTH T-BILLS: 93.9%
- NASDAQ COMPOSITE: 5.0%
- EAFE: 1.1%

Fate's Mix:

- 1992: 28.1%
- 1993: 7.8%
- 1994: 64.1%

The value of the game is the investor's expected return: 4:10%.