All-in at the River

Standard Code Library

Shanghai Jiao Tong University

Desprado2 fstqwq AntiLeaf

目录

1 数学

1.1 FWT

矩阵表示

or 形式 (子集卷积):

$$T_{ij} = [i|j = i] = [j \in i]$$

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

and 形式 (超集卷积):

$$T_{ij} = [i\&j = i] = [i \in j]$$

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

xor 形式 (T与自身互为逆矩阵):

$$T_{ij} = (-1)^{parity(i\&j)}$$

$$\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

```cpp
using ll = long long;

// or
void FWT(ll *a, int len, int inv) {
    for (int h = 1; h < len; h <<= 1) {
        for (int i = 0; i < len; i += (h << 1)) {
            for (int j = 0; j < h; ++j) {
                a[i + j + h] += a[i + j] * inv;
            }
        }
    }
}
// and
void FWT(ll *a, int len, int inv) {
    for (int h = 1; h < len; h <<= 1) {
        for (int i = 0; i < len; i += (h << 1)) {
            for (int j = 0; j < h; ++j) {
                a[i + j] += a[i + j + h] * inv;
            }
        }
    }
}
// xor
void FWT(ll *a, int len, int inv) {
    for (int h = 1; h < len; h <<= 1) {
        for (int i = 0; i < len; i += (h << 1)) {
            for (int j = 0; j < h; ++j) {
                ll x = a[i + j], y = a[i + j + h];
                a[i + j] = x + y, a[i + j + h] = x - y;
                if (inv == -1)
                    a[i + j] /= 2, a[i + j + h] /= 2;
            }
        }
    }
}
```

1.2 多项式

1.2.1 FFT

```cpp
using db = double;
using ll = long long;

int mod;

// init!
namespace FFT {
    const db pi = acos(-1.0);
    struct Comp {
        db x, y;
        Comp() {}
        Comp(db _x, db _y): x(_x), y(_y) {}
        Comp operator + (const Comp&rhs)const {
            return Comp(x+rhs.x, y+rhs.y);
        }
        Comp operator - (const Comp&rhs)const {
            return Comp(x-rhs.x, y-rhs.y);
        }
        Comp operator * (const Comp&rhs)const {
            return Comp(x*rhs.x-rhs.y*y, x*rhs.y+y*rhs.x);
        }
    };
    const Comp I(0, 1);
    Comp conj(const Comp&rhs) {
        return Comp(rhs.x, -rhs.y);
    }
    Comp exp_i(const db &x) {
        return Comp(cos(x), sin(x));
    }

    const int L = 21, N = 1 << L;
    Comp roots[N];
    int rev[N];
    struct _init {
        _init() {
            for(int i = 0; i < N; i++) {
                rev[i] = (rev[i>>1]>>1)|((i&1)<<L-1);
            }
            roots[1] = {1, 0};
            for(int i = 1; i < L; i++) {
                db angle = 2*pi/(1<<i+1);
                for(int j = (1<<i-1); j < (1<<i); j++) {
                    roots[j<<1] = roots[j];
                    roots[j<<1|1] = exp_i((j*2+1-(1<<i))*angle);
                }
            }
        }
    } init;
    inline void trans(Comp &a, Comp &b, const Comp &c) {
        Comp d = b * c;
        b = a - d;
        a = a + d;
    }
    void fft(Comp* a, int n, int op = 1) {
        assert((n & (n - 1)) == 0);
        int zeros = __builtin_ctz(n), shift = L - zeros;
        for(int i = 0; i < n; i++) {
            if(i < (rev[i]>>shift)) {
                swap(a[i], a[rev[i]>>shift]);
            }
        }
        for(int i = 1; i < n; i <<= 1)
            for(int j = 0; j < n; j += i * 2)
                for(int k = 0; k < i; k++)
                    trans(a[j+k], a[i+j+k], roots[i+k]);
        Comp r(1.0 / n, 0);
        if(op == -1) {
            reverse(a + 1, a + n);
            for(int i = 0; i < n; i++)
                a[i] = a[i] * r;
        }
```

```cpp
    }
    void dfft(Comp *a, Comp *b, int len) {
        for(int i = 0; i < len; ++i)
            a[i] = a[i] + I * b[i];
        fft(a, len, 1);
        for(int i = 0; i < len; ++i)
            b[i] = conj(a[i ? len - i : 0]);
        Comp r(0.5, 0);
        for(int i = 0; i < len; ++i) {
            Comp p = a[i], q = b[i];
            a[i] = (p + q) * r;
            b[i] = (q - p) * r * I;
        }
    }
    const int M = (1 << 15) - 1;
    vector<int> multiply_mod(const vector<int>& a, const vector<int>& b, bool eq = 0) {
        static Comp a0[N], a1[N], b0[N], b1[N];
        // (a0, a1) * (b0, b1)
        int la = a.size(), lb = b.size();
        int need = la + lb - 1, len = need > 1 ? 1 << (32 - __builtin_clz(need - 1)) : 1;
        for(int i = 0; i < len; i++) {
            a0[i] = Comp(i < la ? (a[i] >> 15) : 0, 0);
            a1[i] = Comp(i < la ? (a[i] & M) : 0, 0);
        }
        for(int i = 0; i < len; i++) {
            b0[i] = Comp(i < lb ? (b[i] >> 15) : 0, 0);
            b1[i] = Comp(i < lb ? (b[i] & M) : 0, 0);
        }
        dfft(a0, a1, len);
        dfft(b0, b1, len);
        vector<Comp> fa(len), fb(len);
        for(int i = 0; i < len; ++i) {
            fa[i] = a0[i] * b0[i] + I * a1[i] * b0[i];
            fb[i] = a0[i] * b1[i] + I * a1[i] * b1[i];
        }
        fft(fa.data(), len, -1);
        fft(fb.data(), len, -1);

        vector<int> res(need);
        for(int i = 0; i < need; i++) {
            ll x = ll(fa[i].x + 0.5) % mod, y = ll(fa[i].y + 0.5) % mod;
            ll z = ll(fb[i].x + 0.5) % mod, w = ll(fb[i].y + 0.5) % mod;
            res[i] = ((x<<30) + (y + z << 15) + w) % mod;
        }
        return res;
    }
}
using FFT::multiply_mod;

int main() {
    using namespace std;
    ios::sync_with_stdio(0), cin.tie(0), cout.tie(0);
    int n, m;
    cin >> n >> m >> mod;
    vector<int> f(n + 1), g(m + 1);
    for(auto &x : f) cin >> x;
    for(auto &x : g) cin >> x;
    vector<int> h = multiply_mod(f, g);
    for(auto &x : h) cout << x << " ";
    cout << endl;
}
```

1.2.2 NTT

```cpp
#include <bits/stdc++.h>

using ll = unsigned long long;
using Int = unsigned;

namespace Polynomial {

using Poly = std::vector<Int>;
constexpr Int P(998244353), G(3);
inline void inc(Int &x, Int y, Int mod = P) {
    (x += y) >= mod ? x -= mod : 0;
}

inline Int fpow(Int x, Int k = P - 2, Int mod = P) {
    Int r = 1;
    for (; k; k >>= 1, x = (ll) x * x % mod)
        if (k & 1) r = (ll) r * x % mod;
    return r;
}

const int MAXW = 1 << 22;
Int w[MAXW];
struct omegaGen {
    omegaGen() {
        Int x = fpow(G, (P - 1) / MAXW);
        for(int i = MAXW >> 1; i; i >>= 1) {
            w[i] = 1;
            for(int j = 1; j < i; ++j)
                w[i + j] = (ll) w[i + j - 1] * x % P;
            x = (ll) x * x % P;
        }
    }
} gen;

Poly &operator *= (Poly &a, Int b) {
    for(auto &x : a)
        x = (ll) x * b % P;
    return a;
}
Poly operator * (Poly a, Int b) { return a *= b; }
Poly operator * (Int a, Poly b) { return b * a; }
Poly &operator /= (Poly &a, Int b) { return a *= fpow(b); }
Poly operator / (Poly a, Int b) { return a /= b; }
Poly &operator += (Poly &a, Poly b) {
    a.resize(std::max(a.size(), b.size()));
    for(size_t i = 0; i < b.size(); i++)
        inc(a[i], b[i]);
    return a;
}
Poly operator + (Poly a, Poly b) { return a += b; }
Poly &operator -= (Poly &a, Poly b) {
    a.resize(std::max(a.size(), b.size()));
    for(size_t i = 0; i < b.size(); i++)
        inc(a[i], P - b[i]);
    return a;
}
Poly operator - (Poly a, Poly b) { return a - b; }
Poly operator - (Poly a) {
    for(auto &x : a)
        if(x) x = P - x;
    return a;
}
Poly &operator >>= (Poly &a, Int x) {
    if (x >= (Int)a.size()) {
        a.clear();
    } else {
        a.erase(a.begin(), a.begin() + x);
    }
    return a;
}
```

```
Poly &operator <<= (Poly &a, Int x) {
    a.insert(a.begin(), x, 0);
    return a;
}
Poly operator >> (Poly a, Int x) { return a >>= x; }
Poly operator << (Poly a, Int x) { return a <<= x; }

Poly &cdot(Poly &a, Poly b) {
    assert(a.size() == b.size());
    for (size_t i = 0; i < a.size(); i++)
        a[i] = (ll) a[i] * b[i] % P;
    return a;
}
Poly dot(Poly a, Poly b) { return cdot(a, b); }
void norm(Poly &a) {
    if (!a.empty()) {
        a.resize(1 << std::__lg(a.size() * 2 - 1));
    }
}

void dft(Int *a, int n) {
    //assert((n & n - 1) == 0);
    for(int k = n >> 1; k; k >>= 1) {
        for(int i = 0; i < n; i += k << 1) {
            for(int j = 0; j < k; j++) {
                Int x = a[i + j], y = a[i + j + k], ww = w[k + j];
                a[i + j] = x + y >= P ? x + y - P : x + y;
                a[i + j + k] = (ll) (x - y + P) * ww % P;
            }
        }
    }
}

void idft(Int *a, int n) {
    //assert((n & n - 1) == 0);
    for(int k = 1; k < n; k <<= 1) {
        for(int i = 0; i < n; i += k << 1) {
            for(int j = 0; j < k; j++) {
                Int x = a[i + j], y = (ll) a[i + j + k] * w[k + j] % P;
                if(x >= P) x -= P;
                a[i + j + k] = x - y + P;
                a[i + j] = x + y;
            }
        }
    }
    for (Int i = 0, inv = P - (P - 1) / n; i < n; i++)
        a[i] = (ll) a[i] * inv % P;
    std::reverse(a + 1, a + n);
}


void dft(Poly &a) { dft(a.data(), a.size()); }
void idft(Poly &a) { idft(a.data(), a.size()); }

Poly operator* (Poly a, Poly b) {
    if(a.empty() || b.empty()) return {};
    int len = a.size() + b.size() - 1;
    if(a.size() <= 16 || b.size() <= 16) {
        Poly c(len);
        for(size_t i = 0; i < a.size(); i++)
            for(size_t j = 0; j < b.size(); j++)
                c[i + j] = (c[i + j] + ll(a[i]) * b[j]) % P;
        return c;
    }
    int n = 1 << std::__lg(len - 1) + 1;
    if (a != b) {
        a.resize(n), b.resize(n);
        dft(a), dft(b);
        cdot(a, b);
    } else {
        a.resize(n), dft(a);
        cdot(a, a);
    }
    idft(a);
    a.resize(len);
    return a;
}

// return: inv(a) mod x ^ n, n = a.size()
// require n = 2 ^ k
// resize: n' = 1 << lg(2 * n - 1) (n >= 1)
Poly inverse(Poly a) {
    int n = a.size();
    assert((n & n - 1) == 0);
    if (n == 1) return {fpow(a[0])};
    int m = n >> 1;
    Poly b = inverse(Poly(a.begin(), a.begin() + m)), c = b;
    b.resize(n);
    dft(a), dft(b), cdot(a, b), idft(a);
    for (int i = 0; i < m; i++) a[i] = 0;
    for (int i = m; i < n; i++) a[i] = P - a[i];
    dft(a), cdot(a, b), idft(a);
    for (int i = 0; i < m; i++) a[i] = c[i];
    return a;
}

// return: q(len = n - m + 1), a = b * q + r
Poly operator/ (Poly a, Poly b) {
    int n = a.size(), m = b.size();
    if (n < m) return {0};
    int k = 1 << std::__lg(n - m << 1 | 1);
    std::reverse(a.begin(), a.end());
    std::reverse(b.begin(), b.end());
    a.resize(k), b.resize(k), b = inverse(b);
    a = a * b;
    a.resize(n - m + 1);
    std::reverse(a.begin(), a.end());
    return a;
}

// return: {q(len = n - m + 1), r(len = m - 1)}
// require: b.size() > 0
std::pair<Poly, Poly> operator% (Poly a, Poly b) {
    int m = b.size();
    Poly q = a / b;
    b = b * q;
    a.resize(m - 1);
    for (int i = 0; i < m - 1; i++) inc(a[i], P - b[i]);
    return {q, a};
}

Poly der(Poly a) {
    int sz = a.size();
    for(int i = 0; i + 1 < sz; i++)
        a[i] = (ll) (i + 1) * a[i + 1] % P;
    a.pop_back();
    return a;
}

std::vector<Int> inv = {1, 1};
void updateInv(Int n) {
    if (inv.size() <= n) {
        Int p = inv.size();
        inv.resize(n + 1);
        for (Int i = p; i <= n; i++)
            inv[i] = (ll) (P - P / i) * inv[P % i] % P;
    }
}

Poly integ(Poly a, Int c = 0) {
```

```
    int n = a.size();
    updateInv(n);
    a.resize(n + 1);
    for (int i = n - 1; i >= 0; i--)
        a[i + 1] = (ll) inv[i + 1] * a[i] % P;
    a[0] = c;
    return a;
}

// return: ln(a) mod x ^ n, n = a.size()
Poly ln(Poly a) {
    int n = a.size();
    norm(a);
    assert(a[0] == 1);
    a = inverse(a) * der(a);
    a.resize(n - 1);
    return integ(a);
}

// un-checked
Poly exp(Poly a) {
    int n = a.size();
    assert((n & n - 1) == 0);
    assert(a[0] == 0);
    if (n == 1) return {1};
    int m = n >> 1;
    Poly b = exp(Poly(a.begin(), a.begin() + m)), c;
    b.resize(n), c = ln(b);
    a.resize(n << 1), b.resize(n << 1), c.resize(n << 1);
    dft(a), dft(b), dft(c);
    for (int i = 0; i < n << 1; i++) a[i] = (ll(1) + P + a[i] - c[i]) * b[i] % P;
    idft(a);
    a.resize(n);
    return a;
}

// checked
Poly power(Poly a, Int k, Int kreal) {
    int n = a.size();
    long long d = 0;
    while (d < n && !a[d]) d++;
    if (d == n) return a;
    a >>= d;
    Int b = fpow(a[0]);
    norm(a *= b);
    a = exp(ln(a) * k) * fpow(b, P - 1 - k % (P - 1));
    a.resize(n);
    d *= kreal;
    for (int i = n - 1; i >= d; i--) a[i] = a[i - d];
    d = std::min(d, (long long) n);
    for(int i = d - 1; i >= 0; --i) a[i] = 0;
    return a;
}

// k1 = k % (P - 1), k2 = k % P
// kreal = min(k, P)
Poly power(Poly a, Int k1, Int k2, Int kreal) {
    int n = a.size();
    long long d = 0;
    while (d < n && !a[d]) d++;
    if (d == n) return a;
    a >>= d;
    Int b = fpow(a[0]);
    norm(a *= b);
    a = exp(ln(a) * k2) * fpow(b, P - 1 - k1 % (P - 1));
    a.resize(n);
    d *= kreal;
    for (int i = n - 1; i >= d; i--) a[i] = a[i - d];
    d = std::min(d, (long long) n);
    for(int i = d - 1; i >= 0; --i) a[i] = 0;
    return a;
}

// [x^n] f / g
Int divAt(Poly f, Poly g, ll n) {
    int len = std::max(f.size(), g.size());
    assert(len > 0);
    int m = 1 << std::__lg(2 * len - 1);
    len = m << 1;
    f.resize(len);
    for (; n; n >>= 1) {
        g.resize(len);
        dft(f), dft(g);
        for (int i = 0; i < len; ++i)
            f[i] = (ll) f[i] * g[i ^ 1] % P;
        for (int i = 0; i < m; ++i)
            g[i] = (ll) g[i << 1] * g[i << 1 | 1] % P;
        g.resize(m);
        idft(f), idft(g);
        for (int i = 0; i < m; ++i) f[i] = f[i * 2 + (n & 1)];
        fill(f.begin() + m, f.end(), 0);
    }
    return (ll) f[0] * fpow(g[0]) % P;
}

}  // namespace Polynomial

using namespace Polynomial;
using namespace std;

/*
int main() {
    ios::sync_with_stdio(0), cin.tie(0), cout.tie(0);
    int n;
    cin >> n;
    string s;
    cin >> s;
    Int k2 = 0, k1 = 0, kreal = 0;
    for(int i = 0, w = s.size(); i < s.size(); i++) {
        --w;
        inc(k2, (LL) (s[i] - '0') * fpow(10, w) % P);
        inc(k1, (LL) (s[i] - '0') * fpow(10, w, P - 1) % (P - 1), P - 1);
        if(kreal * 10 + (s[i] - '0') <= P) kreal = kreal * 10 + (s[i] - '0');
    }
    Poly a(n);
    for(int i = 0; i < n; ++i) cin >> a[i];
    Poly b = power(a, k1, k2, kreal);
    for(int i = 0; i < n; ++i) cout << b[i] << " \n"[i == n - 1];
} */

Int read() {
    int x;
    cin >> x;
    x %= 998244353;
    x += 998244353;
    x %= 998244353;
    return x;
}

int main() {
    ios::sync_with_stdio(0), cin.tie(0), cout.tie(0);
    ll n;
    int k;
    cin >> n >> k;
    Poly f(k + 1), a(k);
    f[0] = 1;
    for(int i = 1; i <= k; ++i) {
        f[i] = read();
        if(f[i]) f[i] = P - f[i];
```

```cpp
    }
    for(auto &x : a) {
        x = read();
    }
    Poly b = a * f;
    b.resize(k);
    Int ans = divAt(b, f, n);
    cout << ans << endl;
}
```

1.2.3 拉格朗日插值

$$f(x)=\sum_i f\left(x_i\right)\prod_{j\neq i}\frac{x-x_j}{x_i-x_j}$$

1.2.4 连通图计数

设大小为n的满足一个限制P的简单无向图数量为g_n，满足限制P且连通的简单无向图数量为f_n，如果已知$g_{1\dots n}$求f_n，可以得到递推式

$$f_n=g_n-\sum_{k=1}^{n-1}\binom{n-1}{k-1}f_k g_{n-k}$$

这个递推式的意义就是用任意图的数量减掉不连通的数量，而不连通的数量可以通过枚举1号点所在连通块大小来计算.
注意，由于$f_0=0$，因此递推式的枚举下界取0和1都是可以的.
推一推式子会发现得到一个多项式求逆，再仔细看看，其实就是一个多项式ln.

1.3 线性代数

1.3.1 高斯消元

辗转相除高斯消元
对于定义在环上而非域上的矩阵，利用辗转相除进行消元
下面例子中 mod 未必为素数，通过辗转相除消元求行列式

```cpp
using ll = long long;

int gauss(ll a[][N], int n) {
    int ans = 1;
    for (int j = 0; j < n; ++j) {
        for (int i = j + 1; i < n; ++i) {
            while (a[i][j]) {
                int t = a[j][j] / a[i][j];
                for (int k = j; k < m; ++k)
                    a[j][k] = (a[j][k] + (mod - t) * a[i][k]) % mod;
                swap(a[i], a[j]);
                ans = mod - ans;
            }
        }
    }
    for (int i = 0; i < n; ++i)
        ans = ans * a[i][i] % mod;
    return ans;
}
```

求秩与解线性方程组
分别维护行列指针，维护每一列所对应的有效行，有效行的总数即为秩
线性方程组有唯一解当且仅当 列满秩 且 系数矩阵的秩 等于 增广矩阵的秩

```cpp
// 消增广矩阵，注意第 m 行存放目标向量
// 若无解返回空 vector
// 若有无穷多解，则非主元置 0

vector<ll> gauss(ll a[][N], int n, int m) {
    vector<int> row(m, -1);  // 每个变元所对应的有效行
    vector<ll> ans(m, 0);

    int r = 0;
    for (int c = 0; c < m; ++c) { // 扫描每一列，用 r 维护
        int sig = -1;
        for (int i = r; i < n; ++i)
            if(a[i][c]) {
                sig = i; break;
            }
        if (sig == -1) continue; // 无效列

        row[c] = r;
        if (sig != r)
            swap(a[sig], a[r]);

        ll inv = fpow(a[r][c], mod - 2);
        for(int i = 0; i < n; ++i) {
            if (i == r) continue;
            ll del = inv * a[i][c] % mod;
            for (int j = c; j <= m; ++j)
                a[i][j] = (a[i][j] + (mod - del) * a[r][j]) % mod;
        }
        ++r;
    }

    if (r < n && a[r][m]) {
        cerr << "no solution!" << endl;
        return {};
    }

    for (int i = 0; i < m; ++i) { // ax = b
        if (row[i] != -1) {
            ans[i] = fpow(a[row[i]][i]) * a[row[i]][m] % mod;
        } else {
            ans[i] = 0; // 非主元置 0 即为合法解
        }
    }

    return ans;
}
```

矩阵求逆

维护一个矩阵 B，初始设为 n 阶单位矩阵，在消元的同时对 B 进行一样的操作，当把 A 消成单位矩阵时 B 就是逆矩阵.

1.3.2 矩阵树定理

无向图：设图G的基尔霍夫矩阵$L(G)$等于度数矩阵减去邻接矩阵，则G的生成树个数等于$L(G)$的任意一个代数余子式的值.

有向图：类似地定义$L_{in}(G)$等于**入度**矩阵减去邻接矩阵(i指向j有边，则$A_{i,j}=1$)，$L_{out}(G)$等于**出度**矩阵减去邻接矩阵.

则以i为根的内向树个数即为L_{out}的第i个主子式(即关于第i行第i列的余子式)，外向树个数即为L_{in}的第i个主子式.

(可以看出，只有无向图才满足$L(G)$的所有代数余子式都相等.)

BEST定理(有向图欧拉回路计数)：如果G是有向欧拉图，则G的欧拉回路的个数等于以一个任意点为根的内/外向树个数乘以$\prod_v(\deg(v)-1)!$.

并且在欧拉图里，无论以哪个结点为根，也无论内向树还是外向树，个数都是一样的.

另外无向图欧拉回路计数是NP问题.

1.3.3 线性基

```cpp
using ll = long long;

const int N = 70;
const int Lg = 60;

ll bs[N];

// find x in S s.t. k ^ x >= low and k ^ x is minimum,
// !!! return k ^ x, not x
// if not exist, return inf
ll lower(ll k, ll low) {

    // ll x = k ^ low; // expected value in S
    // ll res = 0; // value represented in S
    ll x = 0;
    ll ex = k ^ low;
    int lb = -1;

    // 在前缀可表示的范围内的寻找:
    // 对于范围内 low[i] = 0 的位,
    // 考虑另一分支的可行性
    for (int i = Lg - 1; i >= 0; i--) {
        if (((low >> i) & 1ll) == 0) {      // another
    ↪ branch
            int d = ((ex ^ x) >> i) & 1ll; // cur branch
            if (d || bs[i]) {
                lb = i;
            }
        }

        if (((ex ^ x) >> i) & 1ll) {
            if (bs[i]) {
                x ^= bs[i];
            } else {
                break;
            }
        }
    }

    if ((ex ^ x) == 0) // 可表示
        return k ^ x;

    if (lb == -1) // 不可表示, 且不可更大
        return inf;

    if ((((k ^ x ^ low) >> lb) & 1ll) == 0)
        x ^= bs[lb];

    for (int i = lb - 1; i >= 0; i--) {
        if (((k ^ x) >> i) & 1ll) {
            if (bs[i]) {
                x ^= bs[i];
            }
        }
    }

    return k ^ x;
}
```

1.4 反演与容斥

1.4.1 二项式反演

形式一:

$$f(n)=\sum_{i=m}^{n}\binom{n}{i}g(i) \iff g(i)g(n)=\sum_{i=m}^{n}(-1)^{n-i}\binom{n}{i}f(i)$$

形式二: (常用)

$$f(n)=\sum_{i=n}^{m}\binom{i}{n}g(i) \iff g(n)=\sum_{i=n}^{m}(-1)^{i-n}\binom{i}{n}f(i)$$

常见用法

钦定 (至少) k 个与恰好 k 个之间的转化

记号:

记 $f(n)$ 表示先钦定至少选 n 个, 再统计钦定情况如此的方案数之和, 其中会包含重复的方案数.

记 $g(n)$ 表示恰好选 n 个的方案数, 不会重复.

那么, 对于 $i \ge n$, $g(i)$ 在 $f(n)$ 中被重复计算了 $\binom{i}{n}$ 次, 故 $f(n)=\sum_{i=n}^{m}\binom{i}{n}g(i)$, 其中 m 为数目上限

通常, f 较易求, 再通过反演即可求 g

1.4.2 单位根反演

$$[n|k]=\frac{1}{n}\sum_{i=0}^{n-1}\omega_n^{ik}$$

$$\sum_{i\ge 0}[x^{ik}]f(x)=\frac{1}{k}\sum_{j=0}^{k-1}f(\omega_k^j)$$

1.4.3 Min-Max容斥

设有数集 S, 有:

$$\max(S)=\sum_{\varnothing\ne T\subseteq S}(-1)^{|T|-1}\min(T)$$

$$\min(S)=\sum_{\varnothing\ne T\subseteq S}(-1)^{|T|-1}\max(T)$$

第 k 大数的 Min-Max 容斥

$$\max_k(S)=\sum_{T\subseteq S,|T|\ge k}(-1)^{|T|-k}\times\binom{|T|-1}{k-1}\times\min(T)$$

1.5 组合数

1.5.1 常用组合数

卡特兰数

$$C_n=\frac{1}{n+1}\binom{2n}{n}=\binom{2n}{n}-\binom{2n}{n-1}$$

- n个元素按顺序入栈, 出栈序列方案数
- 长为$2n$的合法括号序列数
- $n+1$个叶子的满二叉树个数

递推式:

$$C_n=\sum_{i=0}^{n-1}C_iC_{n-i-1}$$

$$C_n=C_{n-1}\frac{4n-2}{n+1}$$

普通生成函数:

$$C(x)=\frac{1-\sqrt{1-4x}}{2x}$$

扩展: 如果有n个左括号和m个右括号, 方案数为

$$\binom{n+m}{n}-\binom{n+m}{m-1}$$

施罗德数

$$S_n = S_{n-1} + \sum_{i=0}^{n-1} S_i S_{n-i-1}$$
$$(n+1)s_n = (6n-3)s_{n-1} - (n-2)s_{n-2}$$

其中S_n是(大)施罗德数, s_n是小施罗德数(也叫超级卡特兰数).
除了$S_0 = s_0 = 1$以外, 都有$S_i = 2s_i$.
施罗德数的组合意义:

- 从$(0,0)$走到(n,n), 每次可以走右, 上, 或者右上一步, 并且不能超过$y = x$这条线的方案数
- 长为n的括号序列, 每个位置也可以为空, 并且括号对数和空位置数加起来等于n的方案数
- 凸n边形的**任意**剖分方案数

(有些人会把大(而不是小)施罗德数叫做超级卡特兰数.)
默慈金数

$$M_{n+1} = M_n + \sum_{i=0}^{n-1} M_i M_{n-1-i} = \frac{(2n+3)M_n + 3nM_{n-1}}{n+3}$$

$$M_n = \sum_{i=0}^{\frac{n}{2}} \binom{n}{2i} C_i$$

在圆上的n个**不同的**点之间画任意条不相交(包括端点)的弦的方案数.
也等价于在网格图上, 每次可以走右上, 右下, 正右方一步, 且不能走到$y < 0$的位置, 在此前提下从$(0,0)$走到$(n,0)$的方案数.
扩展: 默慈金数画的弦不可以共享端点. 如果可以共享端点的话是A054726, 后面的表里可以查到.

1.5.2 斯特林数

1. **第一类斯特林数**
$\left[{n \atop k}\right]$表示n个元素划分成k个轮换的方案数.
递推式: $\left[{n \atop k}\right] = \left[{n-1 \atop k-1}\right] + (n-1)\left[{n-1 \atop k}\right]$.
求同一行: 分治FFT $O(n\log^2 n)$, 或者倍增$O(n\log n)$(每次都是$f(x) = g(x)g(x+d)$的形式, 可以用$g(x)$反转之后做一个卷积求出后者).

$$\sum_{k=0}^{n} \left[{n \atop k}\right] x^k = \prod_{i=0}^{n-1}(x+i)$$

求同一列: 用一个轮换的指数生成函数做k次幂

$$\sum_{n=0}^{\infty} \left[{n \atop k}\right] \frac{x^n}{n!} = \frac{\left(\ln(1-x)\right)^k}{k!} = \frac{x^k}{k!}\left(\frac{\ln(1-x)}{x}\right)^k$$

2. **第二类斯特林数**
$\left\{{n \atop k}\right\}$表示$n$个元素划分成$k$个子集的方案数.
递推式: $\left\{{n \atop k}\right\} = \left\{{n-1 \atop k-1}\right\} + k\left\{{n-1 \atop k}\right\}$.
求一个: 容斥, 狗都会做

$$\left\{{n \atop k}\right\} = \frac{1}{k!}\sum_{i=0}^{k}(-1)^i\binom{k}{i}(k-i)^n = \sum_{i=0}^{k}\frac{(-1)^i}{i!}\frac{(k-i)^n}{(k-i)!}$$

求同一行: FFT, 狗都会做
求同一列: 指数生成函数

$$\sum_{n=0}^{\infty}\left\{{n \atop k}\right\}\frac{x^n}{n!} = \frac{(e^x-1)^k}{k!} = \frac{x^k}{k!}\left(\frac{e^x-1}{x}\right)^k$$

普通生成函数

$$\sum_{n=0}^{\infty}\left\{{n \atop k}\right\}x^n = x^k\left(\prod_{i=1}^{k}(1-ix)\right)^{-1}$$

3. **幂的转换**
上升幂与普通幂的转换

$$x^{\overline{n}} = \sum_k \left[{n \atop k}\right] x^k$$

$$x^n = \sum_k \left\{{n \atop k}\right\}(-1)^{n-k}x^{\overline{k}}$$

下降幂与普通幂的转换

$$x^n = \sum_k \left\{{n \atop k}\right\} x^{\underline{k}} = \sum_k \binom{x}{k}\left\{{n \atop k}\right\}k!$$

$$x^{\underline{n}} = \sum_k \left[{n \atop k}\right](-1)^{n-k}x^k$$

另外, 多项式的**点值**表示的每项除以阶乘之后卷上e^{-x}乘上阶乘之后是牛顿插值表示, 或者不乘阶乘就是**下降幂**系数表示. 反过来的转换当然卷上e^x就行了. 原理是每次差分等价于乘以$(1-x)$, 展开之后用一次卷积取代多次差分.

4. **斯特林多项式(斯特林数关于斜线的性质)**
定义:

$$\sigma_n(x) = \frac{\left[{x \atop n}\right]}{x(x-1)\dots(x-n)}$$

$\sigma_n(x)$的最高次数是x^{n-1}. (所以作为唯一的特例, $\sigma_0(x) = \frac{1}{x}$不是多项式.)
斯特林多项式实际上非常神奇, 它与两类斯特林数都有关系.

$$\left[{n \atop n-k}\right] = n^{\underline{k+1}}\sigma_k(n)$$

$$\left\{{n \atop n-k}\right\} = (-1)^{k+1}n^{\underline{k+1}}\sigma_k(-(n-k))$$

不过它并不好求. 可以$O(k^2)$直接计算前几个点值然后插值, 或者如果要推式子的话可以用后面提到的二阶欧拉数.

1.5.3 欧拉数

1. **欧拉数**
$\left\langle{n \atop k}\right\rangle$: n个数的排列, 有k个上升的方案数.

$$\left\langle{n \atop k}\right\rangle = (n-k)\left\langle{n-1 \atop k-1}\right\rangle + (k+1)\left\langle{n-1 \atop k}\right\rangle$$

$$\left\langle{n \atop k}\right\rangle = \sum_{i=0}^{k+1}(-1)^i\binom{n+1}{i}(k+1-i)^n$$

$$\sum_{k=0}^{n-1}\left\langle{n \atop k}\right\rangle = n!$$

$$x^n = \sum_{k=0}^{n-1}\left\langle{n \atop k}\right\rangle\binom{x+k}{n}$$

$$k!\left\{{n \atop k}\right\} = \sum_{i=0}^{n-1}\left\langle{n \atop i}\right\rangle\binom{i}{n-k}$$

2. **二阶欧拉数**
$\left\langle\!\left\langle{n \atop k}\right\rangle\!\right\rangle$: 每个数都出现两次的多重排列, 并且每个数两次出现之间的数都比它要大. 在此前提下有k个上升的方案数.

$$\left\langle\!\left\langle{n \atop k}\right\rangle\!\right\rangle = (2n-k-1)\left\langle\!\left\langle{n-1 \atop k-1}\right\rangle\!\right\rangle + (k+1)\left\langle\!\left\langle{n-1 \atop k}\right\rangle\!\right\rangle$$

$$\sum_{k=0}^{n-1}\left\langle\!\left\langle{n \atop k}\right\rangle\!\right\rangle = (2n-1)!! = \frac{(2n)^{\underline{n}}}{2^n}$$

3. 二阶欧拉数与斯特林数的关系

$$\left\{ \begin{matrix} x \\ x-n \end{matrix} \right\} = \sum_{k=0}^{n-1} \left\langle\!\!\left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle\!\!\right\rangle \binom{x+n-k-1}{2n}$$

$$\left[\begin{matrix} x \\ x-n \end{matrix} \right] = \sum_{k=0}^{n-1} \left\langle\!\!\left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle\!\!\right\rangle \binom{x+k}{2n}$$

1.6 线性规划

1.6.1 对偶原理

给定一个原始线性规划:

$$\begin{aligned} \text{Minimize} \quad & \sum_{j=1}^{n} c_j x_j \\ \text{Subject to} \quad & \sum_{j=1}^{n} a_{ij} x_j \geq b_i, \\ & x_j \geq 0 \end{aligned}$$

定义它的对偶线性规划为:

$$\begin{aligned} \text{Maximize} \quad & \sum_{i=1}^{m} b_i y_i \\ \text{Subject to} \quad & \sum_{i=1}^{m} a_{ij} y_i \leq c_j, \\ & y_i \geq 0 \end{aligned}$$

用矩阵可以更形象地表示为:

$$\begin{array}{rr} \text{Minimize} & \mathbf{c}^T\mathbf{x} \\ \text{Subject to} & A\mathbf{x} \geq \mathbf{b}, \\ & \mathbf{x} \geq 0 \end{array} \iff \begin{array}{rr} \text{Maximize} & \mathbf{b}^T\mathbf{y} \\ \text{Subject to} & A^T\mathbf{y} \leq \mathbf{c}, \\ & \mathbf{y} \geq 0 \end{array}$$

1.7 杂项

1.7.1 约瑟夫环

```cpp
typedef long long ll;

ll k;

// 约瑟夫环
int main() {
    int T;
    cin >> T;
    ll n, m;
    for (int cas = 1; cas <= T; ++cas) {
        scanf("%lld %lld %lld", &n, &m, &k);
        ll res = (k-1) % (n-m+1);
        if (k == 1) res = (m - 1) % n;
        else for (ll i = n - m + 1, j, t; i < n; i = j) {
            if (i < k) {
                j = i + 1;
                res = (res + k) % j;
            } else {
                j = i + i / (k - 1);
                if (j > n) j = n;
                t = j - i;
                res -= j - t * k;
                if (res < 0) res += j;
                else res += res / (k - 1);
            }
        }
        printf("Case #%d: %lld\n", cas, res + 1);
    }
}
```

1.7.2 杨辉三角行区间和

```cpp
using ll = long long;

// 边界从 (s, x) 移动到 (s + 1, nx)
ll move(int x, int nx, int s, ll sum) {
    assert(x >= -1);
    ll res = (2 * sum + mod - C(s, x)) % mod;
    while (x + 1 <= nx) {
        x++;
        res = (res + C(s + 1, x)) % mod;
    }
    while (x > nx) {
        res = (res + mod - C(s + 1, x)) % mod;
        x--;
    }
    return res;
};

void proc(int k) {
    // 第 s 行的 左右边界
    auto le = [&](int s) -> int {
        // ...
    };
    auto ri = [&](int s) -> int {
        // ...
    };
    // 这里应该暴力计算首行和0当首行为 0 时也可以这样写
    ll lsum = move(-1, le(0), -1, 0);
    ll rsum = move(-1, ri(0), -1, 0);

    for (int s = 0; s <= k; s++) {
        if (le(s) < ri(s)) {
            // ... 第 s 行对答案的贡献
        }
        lsum = move(le(s), le(s + 1), s, lsum);
        rsum = move(ri(s), ri(s + 1), s, rsum);
    }
}
```

1.7.3 辛普森积分

```cpp
// Adaptive Simpson's method : double simpson::solve
 ↪ (double (*f) (double), double l, double r, double eps)
 ↪ : integrates f over (l, r) with error eps.

double area (double (*f)(double), double l, double r) {
    double m = l + (r - l) / 2;
    return (f(l) + 4 * f(m) + f(r)) * (r - l) / 6;
}

double solve (double (*f) (double), double l, double r,
 ↪ double eps, double a) {
    double m = l + (r - l) / 2;
    double left = area(f, l, m), right = area(f, m, r);
    if (fabs(left + right - a) <= 15 * eps)
        return left + right + (left + right - a) / 15.0;
    return solve(f, l, m, eps / 2, left) + solve(f, m, r,
 ↪ eps / 2, right);
}

double solve (double (*f) (double), double l, double r,
 ↪ double eps) {
    return solve(f, l, r, eps, area (f, l, r));
}
```

1.7.4 纳什均衡

首先定义纯策略和混合策略: 纯策略是指你一定会选择某个选项, 混合策略是指你对每个选项都有一个概率分布p_i, 你会以相应的概率选择这个选项.
考虑这样的游戏: 有几个人(当然也可以是两个)各自独立地做决定, 然后同时公布每个人的决定, 而每个人的收益和所有人的选择有关.
那么纳什均衡就是每个人都决定一个混合策略, 使得在其他人都是纯策略的情况下, 这个人最坏情况下(也就是说其他人的纯策略最针对他的时候)的收益是最大的. 也就是说, 收益函数对这个人的混合策略求一个偏导, 结果是0(因为是极大值).
纳什均衡点可能存在多个, 不过在一个双人**零和**游戏中, 纳什均衡点一定唯一存在.

1.7.5 康托展开

求排列的排名: 先对每个数都求出它后面有几个数比它小(可以用树状数组预处理), 记为c_i, 则排列的排名就是

$$\sum_{i=1}^{n} c_i (n-i)!$$

已知排名构造排列: 从前到后先分别求出c_i, 有了c_i之后再用一个平衡树(需要维护排名)倒序处理即可.

2 数论

2.1 常见预处理与快速幂

```cpp
// 预处理组合数
const int N = 2e5 + 7;
const ll mod = 998244353;

ll fac[N], ifac[N];
void init() {
    fac[0] = 1;
    for (int i = 1; i < N; ++i) fac[i] = i * fac[i-1] % mod;
    ifac[N - 1] = fpow(fac[N - 1], mod - 2);
    for (int i = N - 1; i; --i) ifac[i - 1] = i * ifac[i] % mod;
}

ll C(int n, int k) {
    if (k < 0 || k > n) return 0;
    return (fac[n] * ifac[k] % mod) * ifac[n - k] % mod;
}

// 线性求逆元，注意有效的 i < mod
ll inv[maxn];
void init() {
    inv[0] = 0, inv[1] = 1;
    for (int i = 2; i < N; ++i)
        inv[i] = inv[mod % i] * (mod - mod / i) % mod;
}

// 快速幂
ll fpow(ll a, ll k = mod - 2, ll p = mod) {
    ll res = 1; a %= p;
    for (; k; k >>= 1, a = a * a % p) {
        if (k & 1)
            res = res * a % p;
    }
    return res;
}
```

2.2 因数分解与素性判定

2.2.1 朴素因数分解

```cpp
// 素因数分解
int p[maxn], l[maxn], cnt2 = 0;
void Fact(int x) {
    cnt2 = 0;
    for (int i = 2; 1ll * i * i <= x; i++) {
        if(x % i == 0) {
            p[++cnt2] = i; l[cnt2] = 0;
            while(x % i == 0) {
                x /= i; ++l[cnt2];
            }
        }
    }
    if (x != 1) {  // 则此时x一定是素数，且为原本x的大于根号x的唯一素因子
        p[++cnt2] = x; l[cnt2] = 1;
    }
}

// vector ver. 无次数
void Fact(ll x, vector<int>& fact) {
    for (ll i = 2; 1ll * i * i <= x; ++i) {
        if(x % i == 0) {
            fact.push_back(i);
            while(x % i == 0) x /= i;
        }
    }
    if (x != 1) fact.push_back(x);
}
```

2.2.2 Miller-Rabin 与 Pollard-Rho

```cpp
typedef long long ll;

// 注意在 MR 里的 fpow 模数超过 int 范围，需要开 __int128

ll mul(ll a, ll b, ll p) {
    return __int128(a) * b % p;
}

ll fpow(ll a, ll k, ll p) {
    ll res = 1; a %= p;
    for (; k; k >>= 1, a = mul(a, a, p)) {
        if (k & 1)
            res = mul(res, a, p);
    }
    return res;
}


ll randint(ll l, ll r) {
    static mt19937 eng(time(0));
    uniform_int_distribution<ll> dis(l, r);
    return dis(eng);
}

bool is_prime(ll x) {
    int s = 0; ll t = x - 1;
    if (x == 2) return true;
    if (x < 2 || !(x & 1)) return false;
    while (!(t & 1)) {  //将x分解成(2^s)*t的样子
        s++; t >>= 1;
    }
    ll lst[] = {2, 325, 9375, 28178, 450775, 9780504, 1795265022};
    for(ll a : lst) { //随便选一个素数进行测试
        if(a >= x) break;
        ll b = fpow(a, t, x); //先算出a^t
        for (int j = 1; j <= s; ++j) { //然后进行s次平方
            ll k = mul(b, b, x);        //求b的平方
            if (k == 1 && b != 1 && b != x - 1) //用二次探测判断
                return false;
            b = k;
        }
        if (b != 1)
            return false; //用费马小定律判断
    }
    return true; //如果进行多次测试都是对的，那么x就很有可能是素数
}

ll gcd(ll a, ll b) { return b == 0 ? a : gcd(b, a % b); }

// @author: Pecco
ll Pollard_Rho(ll n) {
    if (n == 4) return 2;
    if (is_prime(n)) return n;
    while (1) {
        ll c = randint(1, n - 1); // 生成随机的c
        auto f = [=](ll x) { return ((__int128)x * x + c) % n; }; // lll表示__int128，防溢出
        ll t = f(0), r = f(f(0));
        while (t != r) {
            ll d = __gcd(abs(t - r), n);
            if (d > 1)
                return d;
```

```cpp
            t = f(t), r = f(f(r));
        }
    }
}


// 优化掉一个log
ll Pollard_Rho(ll n) {
    if (n == 4) return 2;
    if (is_prime(n)) return n;
    while (1) {
        ll c = randint(1, n - 1);
        auto f = [=](ll x) { return ((__int128)x * x + c) % n; };
        ll t = 0, r = 0, p = 1, q;
        do {
            for (int i = 0; i < 128; ++i) { // 令固定距离C=128
                t = f(t), r = f(f(r));
                if (t == r || (q = (lll)p * abs(t - r) % n) == 0) // 如果发现环或者积即将为0退出
                    break;
                p = q;
            }
            ll d = gcd(p, n);
            if (d > 1)
                return d;
        } while (t != r);
    }
}

vector<ll> factors;

// 将 n 进行素因子分解factors: 存放 n 的所有不去重素因子
void getfactors(ll n) {
    if (n == 1) return;
    if (is_prime(n)) { factors.push_back(n); return; } // 如果是质因子
    ll p = n;
    while (p == n)
        p = Pollard_Rho(n);
    getfactors(n / p), getfactors(p); //递归处理
}
```

2.3 筛法

2.3.1 线性筛

```cpp
const int maxn = 1000000 + 5;
bool isnt[maxn];
int prime[maxn];
int cnt = 0;

// 线性筛法 [1, n] 内素数
void Prime(int n) {
    isnt[1] = true;
    cnt = 0;
    for (int i = 2; i <= n; i++) {
        if (!isnt[i]) prime[++cnt] = i;
        for (int j = 1; j <= cnt; j++) {
            if (1ll * i * prime[j] > n) break;
            isnt[i * prime[j]] = 1;
            if (i % prime[j] == 0) break;
        }
    }
}

// 线性筛求积性函数
int phi[maxn], mu[maxn], d[maxn], D[maxn], q[maxn];
void Sieve(int n) {
    isnt[1] = true;
    phi[1] = 1;
    //mu[1] = 1;
    cnt = 0;
    for(int i = 2; i <= n; i++) {
        if (!isnt[i]) {
            prime[++cnt] = i;
            phi[i] = i - 1;
            //mu[i] = -1;
            // d[i] = 2; q[i] = 1;
            // D[i] = i + 1; q[i] = 1;
        }
        for (int j = 1; j <= cnt; j++) {
            int x = i * prime[j];
            if (x > n) break;
            isnt[x] = 1;
            if (i % prime[j] == 0) {
                phi[x] = phi[i] * prime[j];
                // mu[x] = 0;
                // d[x] = d[i] / (q[i] + 1) * (q[i] + 2), q[x] = q[i] + 1;
                // D[x] = D[i] / (prime[j] ^ (q[i] + 1) - 1) * (prime[j] ^ (q[i] + 2) - 1), q[x] = q[i] + 1;
                break;
            } else {
                phi[x] = phi[i] * (prime[j] - 1); // mu[x] = -mu[i]
                // d[x] = 2 * d[i], q[x] = 1;
                // D[x] = (prime[j] + 1) * D[i], q[x] = 1;
            }
        }
    }
}
```

2.3.2 Min25 筛

```cpp
using ll = long long;
using i128 = __int128;
//using i128 = int64_t;

const ll mod = 998244353;

namespace min25 {
    const int N = 1e6 + 10;
    ll po[40][N];
    inline ll fpow(ll e, ll k) {
        return po[e][k];
    }
    void precalc() {
        for(int e = 0; e < 40; ++e) {
            po[e][0] = 1;
            for(int i = 1; i < N; i++)
                po[e][i] = e * po[e][i - 1] % mod;
        }
    }
    ll n;
    int B;
    int _id[N * 2];
    inline int id(ll x) {
        return x <= B ? x : n / x + B;
    }
    inline int Id(ll x) {
        return _id[id(x)];
    }
    // f(p) = p - 1 = fh(p) - fg(p);
    inline ll fg(ll x) {
        // assert(x <= sqrt(n));
        return 1;
    }
    inline ll fh(ll x) {
        // assert(x <= sqrt(n));
```

```cpp
        return x;
    }
    // \sum_{i=2}^n fg(i)
    inline ll sg(ll x) {
        return (x - 1) % mod;
    }
    // \sum_{i=2}^n fh(i)
    inline ll sh(ll x) {
        return ((i128) x * (x + 1) / 2 + mod - 1) % mod;
    }
    // f(p^e)
    inline ll f(ll p, ll e) {
        //return (pe - pe / p) % mod;
        return (p + (mod - 2) * fpow(e, p)) % mod;
    }
    bitset<N> np;
    ll p[N>>2], pn;
    ll pg[N>>2], ph[N>>2];
    void sieve(ll sz) {
        for(int i = 2; i <= sz; i++) {
            if(!np[i]) {
                p[++pn] = i;
                pg[pn] = (pg[pn - 1] + fg(i)) % mod;
                ph[pn] = (ph[pn - 1] + fh(i)) % mod;
            }
            for(int j = 1; j <= pn && i * p[j] <= sz; j++)
↪ {
                np[i * p[j]] = 1;
                if(i % p[j] == 0) {
                    break;
                }
            }
        }
    }
    ll m;
    ll g[N * 2], h[N * 2];
    ll w[N * 2];

    void compress() {
        for (int i = 1; i <= m; i++) {
            g[i] = (h[i] + mod - g[i] + mod - g[i]) % mod;
        }
        for (int i = 1; i <= pn; i++) {
            pg[i] = (ph[i] + mod - pg[i] + mod - pg[i]) %
↪ mod;
        }
    }

    ll dfs_F(int k, ll n) {
        if (n < p[k] || n <= 1) return 0;
        ll res = g[Id(n)] + mod - pg[k - 1], pw2;
        for (int i = k; i <= pn && (pw2 = (ll) p[i] * p[i])
↪ <= n; ++i) {
            ll pw = p[i];
            for (int c = 1; pw2 <= n; ++c, pw = pw2, pw2 *=
↪ p[i])
                res = (res + ((ll) f(p[i], c) * dfs_F(i +
↪ 1, n / pw) + f(p[i], c + 1))) % mod;
        }
        return res;
    }

    void init(ll _n) {
        n = _n;
        B = sqrt(n) + 100;
        pn = 0;
        sieve(B);
        m = 0;
        for(ll i = 1, j; i <= n; i = j + 1) {
            j = n / (n / i);
            ll t = n / i;
            _id[id(t)] = ++m;
            w[m] = t;
            g[m] = sg(t);
            h[m] = sh(t);
            //printf("id: %lld, w: %lld, g: %lld, h:
↪ %lld\n", m, t, g[m], h[m]);
        }
        for (int j = 1; j <= pn; j++) {
            ll z = (ll) p[j] * p[j];
            for(int i = 1; i <= m && z <= w[i]; i++) {
                int k = Id(w[i] / p[j]);
                g[i] = (g[i] + (ll) (mod - fg(p[j])) *
↪ (g[k] - pg[j - 1] + mod)) % mod;
                h[i] = (h[i] + (ll) (mod - fh(p[j])) *
↪ (h[k] - ph[j - 1] + mod)) % mod;
            }
        }
        compress();

        /* 递推 min25
        for(int j = pn; j > 0; j--) {
            ll z = (ll) p[j] * p[j];
            for(int i = 1; i <= m && z <= w[i]; i++) {
                ll pe = p[j];
                for(int e = 1; pe * p[j] <= w[i]; e++, pe
↪ *= p[j]) {
                    g[i] = (g[i] + (ll) f(p[j], e) *
↪ (g[Id(w[i] / pe)] - pg[j] + mod) + f(p[j], e + 1)) %
↪ mod;
                }
            }
        } */
    }
    ll get(ll x) { // x == n / i
        if(x < 1) return 0;
        return (dfs_F(1, x) + 1) % mod;
    }
    ll get(ll l, ll r) {
        return get(r) - get(l - 1);
    }
}

void Solve() {
    long long n;
    scanf("%lld", &n);
    min25::init(n);
    long long res = min25::get(n);
    printf("%lld\n", res);
}

int main() {
    min25::precalc();
    int T;
    scanf("%d", &T);
    while(T--) {
        Solve();
    }
}
```

2.4 扩展欧几里得

```cpp
using i128 = __int128;

// ax + by = c
// 有解当且仅当 gcd(a, b) | c
// 要求 a, b 不全为 0
// 无合法性检查
```

```cpp
void exgcd(i128 a, i128 b, i128 &x, i128 &y, i128 c = 1) {
    if (b == 0) {
        x = c / a;
        y = 0;
    } else {
        exgcd(b, a % b, x, y, c);
        i128 tmp = x;
        x = y;
        y = tmp - (a / b) * y;
    }
}
```

2.5 扩展欧拉定理

$$a^b \equiv a^{b \bmod \phi(p)},\ (a,b)=1$$

$$a^b \equiv a^{b \bmod \phi(p)+\phi(p)},\ (a,b)\neq 1$$

2.6 中国剩余定理

2.6.1 两个数的 crt

```cpp
// mul 表示慢速乘
ll crt(ll a1, ll p, ll a2, ll q) {
    ll ip = fpow(p, q-2, q);
    ll iq = fpow(q, p-2, p);
    ll n = p * q;
    return (mul(mul(a1, q, n), iq, n) + mul(mul(a2, p, n),
    ↪ ip, n)) % n;
}
```

2.6.2 excrt

```cpp
using ll = long long;

ll gcd(ll a, ll b) {
    return b == 0 ? a : gcd(b, a % b);
}

// x === a1 (mod b1), x === a2 (mod b2)
// 合法性检查：返回 -1 则为无解
pair<ll, ll> excrt(ll a1, ll b1, ll a2, ll b2) {
    ll g = gcd(b1, b2);
    ll lcm = (b1 / g) * b2;

    if ((a1 - a2) % g) return {-1, -1};

    i128 x, y;
    exgcd(b1, b2, x, y, a1 - a2);
    ll res = (a1 - b1 * x) % lcm;
    if (res < 0) res += lcm;
    return {res, lcm};
}
```

2.7 卢卡斯定理

2.7.1 模素数卢卡斯

```cpp
// 卢卡斯定理，要求 p 为素数
ll lucas(ll n, ll m, ll p) {
    if (!m) return 1;
    return C(n % p, m % p, p) * lucas(n / p, m / p, p) % p;
}
```

2.7.2 扩展卢卡斯

```cpp
#include <bits/stdc++.h>
using namespace std;
using ll = long long;


// -p: 素因子
// -l: 次数
// return: 本质不同素因子个数
// 1-base
int Fact(int x, int *p, int *l);

ll gcd(ll a, ll b) {
    return b == 0 ? a : gcd(b, a % b);
}

void exgcd(ll a, ll b, ll &x, ll &y, ll c = 1);

// x === a1 (mod b1), x === a2 (mod b2)
// 合法性检查：返回 -1 则为无解
pair<ll, ll> excrt(ll a1, ll b1, ll a2, ll b2);


// -------------------------


// 扩展卢卡斯定理

// 扩欧求逆元
ll INV(ll a, ll p) {
    ll x, y;
    exgcd(a, p, x, y);
    return (x % p + p) % p;
}

// 递归求解(n! / px) mod pk
ll F(ll n, ll p, ll pk) {
    if (n == 0) return 1;
    ll rou = 1; // 循环节
    ll rem = 1; // 余项
    for (ll i = 1; i <= pk; ++i) {
        if (i % p)
            rou = rou * i % pk;
    }
    rou = fpow(rou, n / pk, pk);
    for (ll i = pk * (n / pk); i <= n; ++i) {
        if (i % p)
            rem = rem * (i % pk) % pk; // 小心i炸int
    }
    return F(n / p, p, pk) * rou % pk * rem % pk;
}

// 素数p在n!中的次数
ll G(ll n, ll p) {
    if (n < p) return 0;
    return G(n / p, p) + (n / p);
}

ll C_pk(ll n, ll m, ll p, ll pk) {
    ll fz = F(n, p, pk), fm1 = INV(F(m, p, pk), pk),
       fm2 = INV(F(n - m, p, pk), pk);
    ll mi = fpow(p, G(n, p) - G(m, p) - G(n - m, p), pk);
    return fz * fm1 % pk * fm2 % pk * mi % pk;
}

const int N = 107;

int ps[N], l[N];

ll exlucas(ll n, ll m, ll P) {
    int num = Fact(P, ps, l); // 素因子分解见素因子分
    ↪ 解.cpp
```

```cpp
    ll res = 0, mo = 1;
    for (int i = 1; i <= num; ++i) {
        ll pk = 1;
        for (int j = 0; j < l[i]; ++j) {
            pk *= ps[i];
        }

        ll x = C_pk(n, m, ps[i], pk);
        pair<ll, ll> pr = excrt(x, pk, res, mo);

        mo = pr.second;
        res = pr.first;
    }

    return res % P;
}

int main() {
    ll n, m, p; cin >> n >> m >> p;
    ll ans = exlucas(n, m, p);

    cout << ans << endl;
}
```

2.8 原根与离散对数

2.8.1 原根

```cpp
// 得到 p 的原根
ll generator(ll p) {
    static ll rec, ans;
    if (p == rec)
        return ans;
    rec = p;
    vector<ll> fact;
    ll phi = p - 1, n = phi;
    for (ll i = 2; 1ll * i * i <= n; ++i) {
        if (n % i == 0) {
            fact.push_back(i);
            while (n % i == 0)
                n /= i;
        }
    }
    if (n > 1)
        fact.push_back(n);
    for (ll res = 2; res <= p; ++res) {
        bool ok = 1;
        for (ll factor : fact) {
            if (fpow(res, phi / factor, p) == 1) {
                ok = false;
                break;
            }
        }
        if (ok)
            return ans = res;
    }
    return ans = -1;
}
```

2.8.2 BSGS

```cpp
// a ^ k == b mod p
ll BSGS(ll a, ll b, ll p) { // p <= 1e9
    static ll rec;
    static map<ll, ll> mp;

    ll sq = (ll)ceil(sqrt(p));
    if (rec != p) {
        rec = p;
        mp.clear();
        ll le = 1, bs = fpow(a, sq, p);
        for (ll i = 1; i <= sq; ++i) {
            le = le * bs % p;
            if (le < 0)
                le += p;
            mp[le] = i * sq;
        }
    }

    ll ri = (b % p);
    if (ri < 0)
        ri += p;
    for (ll j = 0; j <= sq; ++j) {
        if (mp.count(ri)) {
            return mp[ri] - j;
        }
        ri = ri * a % p;
        if (ri < 0)
            ri += p;
    }
    return -1;
}

// x ^ a == b mod p
ll calc(ll a, ll b, ll p) {
    ll g = generator(p); // 求原根-见原根.cpp
    ll ga = fpow(g, a, p);
    ll c = BSGS(ga, b, p);
    ll res = fpow(g, c, p);
    return res;
}
```

2.9 杂项

2.9.1 大数整除小数取模

计算 $\frac{a}{b} \bmod p$:
当 a 的本值太大无法表示时, 可以计算 a 对 b * p 取模的结果, 再除 b, 模 p

$$\frac{a}{b} \bmod p = \frac{a \bmod b*p}{b} \bmod p$$

2.9.2 立方根复杂度求 mobius 函数

```cpp
int getmu(int x) {
    int pr, cur = 0;
    for (int i = 1; i <= cnt; ++i) {
        cur = 0;
        while (x % prime[i] == 0) {
            ++cur; x /= prime[i];
        }
        if (cur > 1) return 0;
    }
    if (x == 1) return 1;
    int sq = sqrt(x) + 0.5;
    if (1ll * sq * sq == x) return 0;
    return 1;
}
```

2.9.3 直接求 euler 函数

```cpp
// primes 为预处理的素数表
ll getPhi(ll x) {
    ll phi = x, num = x;
    for (int p : primes) {
        if (p > x / p) break;
        if (x % p == 0)
```

```
            phi = (phi / p) * (p - 1), x /= p;
        while (x % p == 0)
            x /= p;
    }
    if (x > 1)
        phi = phi / x * (x - 1);
    return phi;
}
```

3 数据结构

3.1 点分治

```
vector<int> vec[N];
bool vis[N];

namespace DFZ {

    int sz[N], maxx[N];

    int rt, sum;
    void calcsz(int x, int dad) {
        //cerr << rt << "   " << sum << endl;
        maxx[x] = 0; sz[x] = 1;
        for (int y : vec[x]) {
            if(y == dad || vis[y]) continue;
            calcsz(y, x);
            sz[x] += sz[y];
            maxx[x] = max(sz[y], maxx[x]);
        }
        assert(sum - sz[x] >= 0);
        maxx[x] = max(maxx[x], sum - sz[x]);
        if (maxx[x] < maxx[rt]) rt = x;
    }

    void dfz(int x) {
        calcsz(x, 0); vis[rt] = 1;
        TREE::init();
        TREE::dfs(rt, 0, 0, 0, 1);
        ALL += TREE::gao();
        // calc mx, dis, build seg treedp

        int cen = rt, psum = sum;
        for (int y : vec[cen]) {
            if (vis[y]) continue;
            rt = cen;
            sum = (sz[y] > sz[cen] ? psum - sz[cen] :
 ↪ sz[y]);
            dfz(y);
        }
    }
}

DFZ::rt = 0; DFZ::maxx[0] = inf; DFZ::sum = n;
DFZ::dfz(1);
```

3.2 树链剖分

```
vector<int> vec[N];

struct cutTree {
    int dep[N], sz[N];
    int nd[N], fn[N], gn[N], clc;
    int top[N], son[N], fa[N];

    void init(int n) {
        clc = 0;
        for(int x = 1; x <= n; x++) son[x] = 0;
        for(int x = 1; x <= n; x++) fa[x] = 0;
    }
    // dfs1(rt, 1);
    void dfs1(int x, int d) {
        dep[x] = d, sz[x] = 1;
        for(int y : vec[x]) {
            if(y == fa[x]) continue;
            fa[y] = x; dfs1(y, d + 1); sz[x] += sz[y];
            if(!son[x] || sz[y] > sz[son[x]]) son[x] = y;
        }
    }
    // dfs2(rt, rt);
    void dfs2(int x, int an) {
        top[x] = an;
        nd[fn[x] = ++clc] = x;
        if(son[x]) dfs2(son[x], an);
        for(int y : vec[x]) {
            if(y == fa[x] || y == son[x]) continue;
            dfs2(y, y);
        }
        gn[x] = clc;
    }
    int lca(int x, int y) {
        while(top[x] != top[y]) {
            if(dep[top[x]] > dep[top[y]])
                swap(x, y);
            y = fa[top[y]];
        }
        return dep[x] < dep[y] ? x : y;
    }
    // anc[x][0] = x, !! 默认根为 1, 若 rt 非 1, 需要传参
    int up(int x, int k, int rt = 1) {
        if(k < 0) return -1;
        if(k >= dep[x]) return 0;
        while(k) {
            if(fn[x] - fn[top[x]] < k)
                k -= fn[x] - fn[top[x]] + 1, x =
 ↪ fa[top[x]];
            else x = nd[fn[x] - k], k = 0;
        }
        return x;
    }
    int query(int x, int y) {
        int res = 0;
        while(top[x] != top[y]) {
            if(dep[top[x]] > dep[top[y]]) swap(x, y);
            int l = fn[top[y]], r = fn[y];
            res += bit.query(r) - bit.query(l - 1);
            y = fa[top[y]];
        }
        if(dep[x] > dep[y]) swap(x, y);
        return res + bit.query(fn[y]) - bit.query(fn[x] -
 ↪ 1);
    }
} CT;
```

3.3 主席树

```
const int N = 100010;
const int M = N * 40;

struct node {
    int l, r;
    ll val;
} tr[M];

int rt[N], tot;

int clone(int k) {
    ++tot;
    tr[tot] = tr[k];
    return tot;
}

void up(int k) {
    tr[k].val = tr[tr[k].l].val ^ tr[tr[k].r].val;
}

void upd(int &k, int x, int l, int r, ll val) {
    k = clone(k);
    if(l == r) {
        tr[k].val ^= val;
```

```cpp
    } else {
        int m = ((l + r) >> 1);
        if (x <= m) upd(tr[k].l, x, l, m, val);
        else upd(tr[k].r, x, m + 1, r, val);
        up(k);
    }
}

/* 树形建主席树 */
void dfs(int x, int dad) {
    rt[x] = rt[dad];
    upd(rt[x], a[x], 1, n, val);
    for (auto y : ch[x]) {
        if(y == dad) continue;
        dfs(y, x);
    }
}
```

3.4 线段树合并

```cpp
// merge v to u

void merge(int &u, int v, int l, int r) {
    if (!v) return;
    else if (!u) u = v;
    else if (l == r) {
        // merge at leaf
    } else {
        int mid = ((l + r) >> 1);
        merge(tr[u].l, tr[v].l, l, mid);
        merge(tr[u].r, tr[v].r, mid + 1, r);
        up(u);
    }
}
```

3.5 zkw 线段树

```cpp
// 0-base

struct SegT {
    typedef pair<int, int> T;
    T ini = {0, 0};
    T combine(T u, T v) {
        return max(u, v);
    }
    int n; vector<T> arr;

    // local
    SegT(int sz): n{1} { while(n < sz) n <<= 1;
    arr.resize(n * 2, ini); };

    // global
    void init(int sz) {
        arr.clear(); for (n = 1; n < sz; n <<= 1);
    arr.resize(n * 2, ini);
    }

    void update(int i, T v) {
        for (arr[i += n] = v; i >>= 1; )
            arr[i] = combine(arr[i<<1], arr[i<<1|1]);
    }
    T query (int l, int r) {
        T resl = ini, resr = ini;
        for (l += n, r += n + 1; l < r; l >>= 1, r >>= 1) {
            if (l & 1) resl = combine(resl, arr[l++]);
            if (r & 1) resr = combine(resr, arr[--r]);
        }
        return combine(resl, resr);
    }
};
```

3.6 矩形面积并

```cpp
// 矩形面积并，标记永久化，注意要开八倍空间

typedef long long ll;

const int maxn = 100050;

struct node {
    int sum, lazy;
    void clear() {
        sum = 0; lazy = 0;
    }
} tr[maxn << 3];

void up(int k, int l, int r) {
    if (tr[k].lazy) {
        tr[k].sum = r - l + 1;
    } else {
        tr[k].sum = tr[k<<1].sum + tr[k<<1|1].sum;
    }
}

void build(int k, int l, int r) {
    tr[k].clear();

    if (l == r) {
        tr[k<<1].clear();
        tr[k<<1|1].clear();
        return;
    }

    int mid = ((l + r) >> 1);
    build(k<<1, l, mid);
    build(k<<1|1, mid + 1, r);
    up(k, l, r);
}

void upd(int k, int cl, int cr, int tag, int l, int r) {
    if (cl <= l && r <= cr) {
        tr[k].lazy += tag;
        up(k, l, r);
    } else {
        int mid = ((l + r) >> 1);
        if(cl <= mid) upd(k<<1, cl, cr, tag, l, mid);
        if(cr > mid) upd(k<<1|1, cl, cr, tag, mid + 1, r);
        up(k, l, r);
    }
}

// 查询的时候记一下祖先结点是否有 lazy > 0
```

4 图论

4.1 2-sat

4.2 Dinic 网络流

```cpp
using ll = long long;

// 前向链表多测注意清空，tot 初值要设为 1
struct edge {
    int to, pre;
    ll w;
} e[M];
int last[N], tot = 1;

void ine(int a, int b, ll w) {
    e[++tot] = (edge){b, last[a], w};
    last[a] = tot;
}

void add(int a, int b, ll w) {
    ine(a, b, w);
    ine(b, a, 0);
}

inline int sgn(ll v) {
    if (v == 0) return 0;
    return v > 0 ? 1 : -1;
}

namespace Dinic {

    int n, s, t;
    int lv[N], cur[M];  // lv，层数，cur，当前弧

    bool bfs()  {
        fill(lv + 1, lv + 1 + n, -1);
        lv[s] = 0;
        copy(last + 1, last + 1 + n, cur + 1);
        queue<int> q;
        q.push(s);
        while(!q.empty()) {
            int u = q.front(); q.pop();
            for(int i = cur[u]; i; i = e[i].pre) {
                int to = e[i].to;
                ll vol = e[i].w;
                if(vol > 0 && lv[to] == -1)
                    lv[to] = lv[u] + 1, q.push(to);
            }
        }
        return lv[t] != -1; // 如果汇点未访问过则不可达
    }

    ll dfs(int u = s, ll f = inf) {
        if(u == t)
            return f;
        for(int &i = cur[u]; i; i = e[i].pre) {
            int to = e[i].to;
            ll vol = e[i].w;
            if(vol > 0 && lv[to] == lv[u] + 1) {
                ll c = dfs(to, min(vol, f));
                if(sgn(c)) {
                    e[i].w -= c;
                    e[i ^ 1].w += c;    // 反向边
                    return c;
                }
            }
        }
        return 0; // 输出流量大小
    }

    ll dinic(int _n, int _s, int _t) {
        n = _n; s = _s; t = _t;
        ll ans = 0;
        while(bfs()) {
            ll f;
            while((f = dfs()) > 0)
                ans += f;
        }
        return ans;
    }
} // namespace Dinic

using Dinic::dinic;
```

4.3 Dijkstra 费用流

```cpp
#include <bits/stdc++.h>
#include <ext/pb_ds/priority_queue.hpp>
using ll = long long;
using namespace std;
typedef pair<ll, ll> P;

const int N = 5e3 + 7, M = 1e6 + 7;
const ll inf = 0x3f3f3f3f3f3f3f3f;

struct edge {
    int to, pre;
    ll w, c; // w: 流量 weight，c: 费用 cost
} e[M * 2];
int head[N], ecnt = 1; // 加入的第一条边编号为 2
void ine(int u, int v, ll w, ll c) {
    e[++ecnt] = {v, head[u], w, c};
    head[u] = ecnt;
}
void add(int u, int v, ll w, ll c) {
    ine(u, v, w, c); ine(v, u, 0, -c);
}
void init(int n) {
    fill(head + 1, head + 1 + n, 0);
    ecnt = 1;
}

ll h[N], dis[N]; // bool vis[N];
int pe[N]; // 父边

// spfa 只用来预处理 h，不跑流，也可以用来跑流，见注释，
// h 在 flow_dij 调用时清空
void spfa(int s, int n) {
    static bool inq[N];

    fill(dis + 1, dis + 1 + n, inf);
    fill(inq + 1, inq + 1 + n, false);

    queue<int> q;

    q.push(s);
    dis[s] = 0;

    while (!q.empty()) {
        int x = q.front();
        q.pop();
        inq[x] = false;

        for (int i = head[x]; i; i = e[i].pre)
            if (e[i].w > 0) {
                int y = e[i].to;

                if (dis[y] > dis[x] + e[i].c) {
                    // pe[y] = i; 若要跑流，则加上这一行
                    dis[y] = dis[x] + e[i].c;
                    if (!inq[y]) {
```

```cpp
                        q.push(y);
                        inq[y] = true;
                    }
                }
            }
        }
}

void dij(int s, int n) {
    fill(dis + 1, dis + 1 + n, inf);
    dis[s] = 0;
    // fill(vis + 1, vis + 1 + n, false);

    using pq_t = __gnu_pbds::priority_queue<P, greater<P>,
    ↪ __gnu_pbds::thin_heap_tag>;
    pq_t q;
    static pq_t::point_iterator it[N];

    for (int i = 1; i <= n; i++)
        it[i] = q.push({dis[i], i});

    while(!q.empty()) {
        auto tp = q.top();
        int x = tp.second;
        ll w = tp.first;
        q.pop();

        if(w != dis[x]) continue;
        // if(vis[x]) continue;
        // vis[x] = true;

        for(int i = head[x]; i; i = e[i].pre) {
            int y = e[i].to;
            if(e[i].w > 0 && dis[y] > dis[x] + h[x] - h[y]
    ↪ + e[i].c) {
                pe[y] = i;
                dis[y] = dis[x] + h[x] - h[y] + e[i].c;
                q.modify(it[y], {dis[y], y});
            }
        }
    }
}

P flow_dij(int s, int t, int n) {
    fill(h + 1, h + 1 + n, 0);
    /*
    spfa(s, n);
    for(int i = 1; i <= n; i++)
        h[i] = dis[i];
    // 如果初始有负权就像这样跑一遍 SPFA 预处理 h 数组
    */

    ll cost = 0, flow = 0;

    while (1) {
        dij(s, n);
        if(dis[t] >= inf) break;

        for (int i = 1; i <= n; i++) // 先更新 h 数组
            h[i] += dis[i];

        ll nowflow = inf;
        for (int x = t; x != s; x = e[pe[x] ^ 1].to)
            nowflow = min(nowflow, e[pe[x]].w);

        flow += nowflow;
        cost += nowflow * h[t]; // 计算流量和费用

        for (int x = t; x != s; x = e[pe[x] ^ 1].to) {
            e[pe[x]].w -= nowflow;
            e[pe[x] ^ 1].w += nowflow;
        } // 更新边容量
    }
    return {flow, cost};
}

// https://ac.nowcoder.com/acm/problem/222408
int main() {
    ios::sync_with_stdio(0), cin.tie(nullptr),
    ↪ cout.tie(nullptr);
    int n; cin >> n;
    vector<int> z(n + 1), v(n + 1);
    ll ans = 0;
    for(int i = 1; i <= n; i++) {
        int x, y; cin >> x >> y >> z[i] >> v[i];
        ans += x * x;
        ans += y * y;
        ans += z[i] * z[i];
    }
    int num = n * 2 + 2, s = num - 1, t = num;
    init(num);
    for(int i = 1; i <= n; i++)
        for(int j = 1; j <= n; j++)
            add(i, j + n, 1, 1ll * v[i] * v[i] * (j - 1) *
    ↪ (j - 1) + 2ll * v[i] * z[i] * (j - 1));
    for(int i = 1; i <= n; i++)
        add(s, i, 1, 0);
    for(int i = 1; i <= n; i++)
        add(n + i, t, 1, 0);
    ans += flow_dij(s, t, num).second;
    cout << ans << '\n';
    return 0;
}
```

4.4 KM

```cpp
// 1-base
// 主过程: 设置好 cost[N][N] 即可调用
// cost 可以为负数

using ll = long long;
#define prev prevv

const int N = 305;
const ll INF = 0x3f3f3f3f3f3f3f3f;

int n;
ll cost[N][N];
ll lx[N], ly[N];
int match[N];
ll slack[N];
int prev[N];
bool vy[N];

void augment(int root) {
    fill(vy + 1, vy + n + 1, false);
    fill(slack + 1, slack + n + 1, INF);
    int py;
    match[py = 0] = root;

    do {
        vy[py] = true;
        int x = match[py];
        ll delta = INF;
        int yy;
        for (int y = 1; y <= n; y++) {
            if (!vy[y]) {
                if (lx[x] + ly[y] - cost[x][y] < slack[y])
    ↪ {
                    slack[y] = lx[x] + ly[y] - cost[x][y];
                    prev[y] = py;
                }
                if (slack[y] < delta) {
```

```cpp
                    delta = slack[y];
                    yy = y;
                }
            }
        }
        for (int y = 0; y <= n; y++) {
            if (vy[y]) {
                lx[match[y]] -= delta;
                ly[y] += delta;
            } else {
                slack[y] -= delta;
            }
        }
        py = yy;
    } while (match[py] != -1);

    do {
        int pre = prev[py];
        match[py] = match[pre];
        py = pre;
    } while (py);
}

ll KM() {
    for (int i = 1; i <= n; i++) {
        lx[i] = ly[i] = -INF;
        match[i] = -1;
        for (int j = 1; j <= n; j++) {
            lx[i] = std::max(lx[i], cost[i][j]);
        }
    }
    ll answer = 0;
    for (int root = 1; root <= n; root++) {
        augment(root);
    }
    for (int i = 1; i <= n; i++) {
        answer += lx[i];
        answer += ly[i];
        // printf("%d %d\n", match[i], i);
    }
    return answer;
}
```

4.5 带花树

```cpp
// 带花树: 一般图最大匹配
const int maxn = 505;

struct Match {
    int n, father[maxn], vst[maxn], match[maxn], pre[maxn],
 ↪ Type[maxn], times;
    vector<int> edges[maxn];
    queue<int> Q;

    void ine(int x, int y) { edges[x].push_back(y); }
    void ine2(int x, int y) {
        ine(x, y);
        ine(y, x);
    }

    void init(int num) {
        times = 0;
        n = num;
        for (int i = 0; i <= n; ++i)
            edges[i].clear(), vst[i] = 0, match[i] = 0,
 ↪ pre[i] = 0;
    }

    int LCA(int x, int y) {
        times++;
        x = father[x], y = father[y]; //已知环位置
        while (vst[x] != times) {
            if (x) {
                vst[x] = times;
                x = father[pre[match[x]]];
            }
            swap(x, y);
        }
        return x;
    }

    void blossom(int x, int y, int lca) {
        while (father[x] != lca) {
            pre[x] = y;
            y = match[x];
            if (Type[y] == 1) {
                Type[y] = 0;
                Q.push(y);
            }
            father[x] = father[y] = father[lca];
            x = pre[y];
        }
    }

    int Augument(int s) {
        for (int i = 0; i <= n; ++i)
            father[i] = i, Type[i] = -1;
        Q = queue<int>();
        Type[s] = 0;
        Q.push(s); //仅入队o型点
        while (!Q.empty()) {
            int Now = Q.front();
            Q.pop();
            for (int Next : edges[Now]) {
                if (Type[Next] == -1) {
                    pre[Next] = Now;
                    Type[Next] = 1; //标记为i型点
                    if (!match[Next]) {
                        for (int to = Next, from = Now; to;
 ↪ from = pre[to]) {
                            match[to] = from;
                            swap(match[from], to);
                        }
                        return true;
                    }
                    Type[match[Next]] = 0;
                    Q.push(match[Next]);
                } else if (Type[Next] == 0 && father[Now] !
 ↪ = father[Next]) {
                    int lca = LCA(Now, Next);
                    blossom(Now, Next, lca);
                    blossom(Next, Now, lca);
                }
            }
        }
        return false;
    }

    void gao() {
        int res = 0; // 最大匹配数
        for (int i = n; i >= 1; --i)
            if (!match[i])
                res += Augument(i);
        printf("%d\n", res);
        for (int i = 1; i <= n; ++i)
            printf("%d ", match[i]);
        printf("\n");
    }
} G;

int main() {
    int n, m;
    cin >> n >> m;
```

```
    G.init(n);
    for (int i = 1, x, y; i <= m; i++) {
        scanf("%d %d", &x, &y);
        G.ine2(x, y);
    }
    G.gao();
    return 0;
}
```

4.6 Hall 定理

霍尔定理

二分图中, 左侧点集 X 存在最大匹配的充要条件是:
X 中的任意 k 个点至少与 Y 中 k 个点相邻.
推论: 正则二分图存在完美匹配 (正则图 指每个点度数相等的图)

5 字符串

5.1 后缀自动机

```
// 在字符集比较小的时候可以直接开go数组，否则需要用map或者
    ↪ 哈希表替换
// 注意!!!结点数要开成串长的两倍

// 全局变量与数组定义
int last, len[maxn], fa[maxn], go[maxn][26], sam_cnt;
int c[maxn], q[maxn]; // 用来桶排序

// 在主函数开头加上这句初始化
last = sam_cnt = 1;

// 以下是按val进行桶排序的代码
for (int i = 1; i <= sam_cnt; i++)
    c[len[i] + 1]++;
for (int i = 1; i <= n; i++)
    c[i] += c[i - 1]; // 这里n是串长
for (int i = 1; i <= sam_cnt; i++)
    q[++c[len[i]]] = i;

//加入一个字符 均摊O(1)
void extend(int c) {
    int p = last, np = ++sam_cnt;
    len[np] = len[p] + 1;

    while (p && !go[p][c]) {
        go[p][c] = np;
        p = fa[p];
    }

    if (!p)
        fa[np] = 1;
    else {
        int q = go[p][c];

        if (len[q] == len[p] + 1)
            fa[np] = q;
        else {
            int nq = ++sam_cnt;
            len[nq] = len[p] + 1;
            memcpy(go[nq], go[q], sizeof(go[q]));

            fa[nq] = fa[q];
            fa[np] = fa[q] = nq;

            while (p && go[p][c] == q){
                go[p][c] = nq;
                p = fa[p];
            }
        }
    }

    last = np;
}
```

5.2 广义后缀自动机

5.2.1 对 Trie 建自动机

以 bfs 遍历 Trie 上的每一个结点，以 父结点 在 SAM 中对应的结点为 last 调用 insert 即可.

5.2.2 对多模式串建自动机

```
// 多串创建 sam 的方法:
// 在插入每个串前, 设置 last = 1, 然后一路 last =
    ↪ extend(last, c)

int extend(int p, int c) {
    int np = 0;

    if (!go[p][c]) {
        np = ++sam_cnt;
        len[np] = len[p] + 1;
        while (p && !go[p][c]) {
            go[p][c] = np;
            p = fa[p];
        }
    }

    if (!p)
        fa[np] = 1;
    else {
        int q = go[p][c];

        if (len[q] == len[p] + 1) {
            if (np)
                fa[np] = q;
            else
                return q;
        }
        else {
            int nq = ++sam_cnt;
            len[nq] = len[p] + 1;
            memcpy(go[nq], go[q], sizeof(go[q]));

            fa[nq] = fa[q];
            fa[q] = nq;
            if (np)
                fa[np] = nq;

            while (p && go[p][c] == q){
                go[p][c] = nq;
                p = fa[p];
            }

            if (!np)
                return nq;
        }
    }

    return np;
}
```

5.3 AC 自动机

```
using namespace std;
const int maxn = 500000 + 5;

const int rt = 0; // 默认 trie 树根为 0

int tr[maxn][26], tot = rt; // 初始化时要设置 tot = rt
int e[maxn];                // 标记字符串结尾
int fail[maxn];

void insert(char *s) {
    int p = rt; // from root
    for (int i = 0; s[i]; i++) {
        int k = s[i] - 'a';
        if (!tr[p][k])
            tr[p][k] = ++tot;   // 根节点为0
```

```cpp
            p = tr[p][k];
        }
        e[p]++;  // 尾部标记
    }

    void build() {
        queue<int> q;
        fill(fail, fail + 1 + tot, 0);

        for (int i = 0; i < 26; i++)
            if (tr[rt][i])
                q.push(tr[rt][i]);

        while (!q.empty()) {
            int k = q.front(); q.pop();
            for (int i = 0; i < 26; i++) {
                if (tr[k][i]) {
                    fail[tr[k][i]] = tr[fail[k]][i];
                    q.push(tr[k][i]); //入队
                } else
                    tr[k][i] = tr[fail[k]][i]; // trie图
            }
        }
    }

    int query(char *t) {
        int p = rt, ans = 0;
        for (int i = 0; t[i]; i++) {
            p = tr[p][t[i] - 'a'];
            for (int j = p; j && (e[j] != -1); j = fail[j]) {
                ans += e[j];
                e[j] = -1;  // 防止重复遍历
            }
        }
        return ans;
    }
```

5.4 后缀数组

5.4.1 倍增

```cpp
// h[i] = lcp(sa[i], sa[i - 1])
// sa[i] = 排名为 i 的后缀下标
// rk[i] = 后缀 i 的排名
// 需要将 s[n] 设为一个比一切字符大的数才能正确地输出 height
// 不需要清空

template<int MAXN> class SA {
    public:
    int n, sa[MAXN], rk[MAXN], h[MAXN];

    void init() {
        // 不需要 init
    }

    void compute(int *s, int n, int m) {
        int i, p, w, j, k;
        this->n = n;
        if (n == 1) {
            sa[0] = rk[0] = h[0] = 0; return;
        }
        memset(cnt, 0, m * sizeof(int));
        for (i = 0; i < n; ++i) ++cnt[rk[i] = s[i]];
        for (i = 1; i < m; ++i) cnt[i] += cnt[i - 1];
        for (i = n - 1; ~i; --i) sa[--cnt[rk[i]]] = i;
        for (w = 1; w < n; w <<= 1, m = p) {
            for (p = 0, i = n - 1; i >= n - w; --i) id[p++] = i;
            for (i = 0; i < n; ++i)
                if (sa[i] >= w) id[p++] = sa[i] - w;
            memset(cnt, 0, m * sizeof(int));
            for (i = 0; i < n; ++i) ++cnt[px[i] = rk[id[i]]];
            for (i = 1; i < m; ++i) cnt[i] += cnt[i - 1];
            for (i = n - 1; ~i; --i) sa[--cnt[px[i]]] = id[i];
            memcpy(old_rk, rk, n * sizeof(int));
            for (i = p = 1, rk[sa[0]] = 0; i < n; ++i)
                rk[sa[i]] = cmp(sa[i], sa[i-1], w) ? p - 1 : p++;
        }
        for (i = 0; i < n; ++i) rk[sa[i]] = i;
        for (i = k = h[rk[0]] = 0; i < n; h[rk[i++]] = k)
            if (rk[i])
                for (k > 0 ? --k : 0, j = sa[rk[i] - 1]; s[i + k] == s[j + k]; ++k) {}
    }
    private:
    int old_rk[MAXN], id[MAXN], px[MAXN], cnt[MAXN];
    bool cmp(int x, int y, int w) {
        return old_rk[x] == old_rk[y] && old_rk[x + w] == old_rk[y + w];
    }
};
```

5.4.2 SAIS

```cpp
const int BUFFER_SIZE = 1u << 26 | 1;
char buf[BUFFER_SIZE], *buf_ptr = buf;

#define alloc(x, type, len)      \
    type *x = (type *)buf_ptr;   \
    buf_ptr += (len) * sizeof(type);

#define clear_buf() \
    memset(buf, 0, buf_ptr - buf), buf_ptr = buf;

template <int MAXN> class SuffixArray {
#define ltype true
#define stype false

public:
    int sa[MAXN], rk[MAXN], hei[MAXN];

    void compute(int n, int m, int *s) {
        sais(n, m, s, sa);
        for (int i = 0; i < n; ++i) rk[sa[i]] = i;
        for (int i = 0, h = 0; i < n; i++) {
            if (rk[i]) {
                int j = sa[rk[i] - 1];
                while (s[i + h] == s[j + h]) ++h;
                hei[rk[i]] = h;
            } else {
                h = 0;
            }
            if (h) --h;
        }
    }

private:
    int lbuc[MAXN], sbuc[MAXN];

    void induce(int n, int m, int *s, bool *type, int *sa, int *buc,
                int *lbuc, int *sbuc)
    {
        memcpy(lbuc + 1, buc, m * sizeof(int));
        memcpy(sbuc + 1, buc + 1, m * sizeof(int));

        sa[lbuc[s[n - 1]]++] = n - 1;
```

```cpp
        for (int i = 0; i < n; i++) {
            int t = sa[i] - 1;
            if (t >= 0 && type[t] == ltype)
                sa[lbuc[s[t]]++] = t;
        }

        for (int i = n - 1; i >= 0; i--) {
            int t = sa[i] - 1;
            if (t >= 0 && type[t] == stype)
                sa[--sbuc[s[t]]] = t;
        }
    }

    void sais(int n, int m, int *s, int *sa) {
        alloc(type, bool, n + 1);
        alloc(buc, int , m + 1);

        type[n] = false;

        for (int i = n - 1; i >= 0; i--) {
            ++buc[s[i]];
            type[i] = s[i] > s[i + 1] || (s[i] == s[i + 1] && type[i + 1] == ltype);
        }
        for (int i = 1; i <= m; i++) {
            buc[i] += buc[i - 1];
            sbuc[i] = buc[i];
        }
        memset(rk, -1, n * sizeof(int));

        alloc(lms, int, n + 1);

        int n1 = 0;
        for (int i = 0; i < n; i++) {
            if (!type[i] && (i == 0 || type[i - 1]))
                lms[rk[i] = n1++] = i;
        }

        lms[n1] = n;
        memset(sa, -1, n * sizeof(int));

        for (int i = 0; i < n1; i++) sa[--sbuc[s[lms[i]]]] = lms[i];
        induce(n, m, s, type, sa, buc, lbuc, sbuc);

        int m1 = 0;
        alloc(s1, int, n + 1);

        for (int i = 0, t = -1; i < n; i++) {
            int r = rk[sa[i]];
            if (r != -1) {
                int len = lms[r + 1] - sa[i] + 1;
                m1 += t == -1 || len != lms[rk[t] + 1] - t + 1 ||
                    memcmp(s + t, s + sa[i], len * sizeof(int)) != 0;
                s1[r] = m1;
                t = sa[i];
            }
        }

        alloc(sa1, int, n + 1);

        if (n1 == m1) {
            for (int i = 0; i < n1; i++)
                sa1[s1[i] - 1] = i;
        } else {
            sais(n1, m1, s1, sa1);
        }

        memset(sa, -1, n * sizeof(int));
        memcpy(sbuc + 1, buc + 1, m * sizeof(int));

        for (int i = n1 - 1; i >= 0; i--) {
            int t = lms[sa1[i]];
            sa[--sbuc[s[t]]] = t;
        }
        induce(n, m, s, type, sa, buc, lbuc, sbuc);
    }
#undef stype
#undef rtype
};
```

5.5 马拉车

```cpp
/* manacher */
const int maxn = 300007;
// s[0]: 特殊值□s[1, 3 .. tot]: 分隔值
int s[maxn], p[maxn]; // p 为半径
//int L[maxn], R[maxn]; // 包含点i的回文串□回文中心最左□右
int n, tot;

void manacher() {
    int right = 0, idx = 0;
    for(int i = 1; i <= tot; i++) {
        if(i < right)
            p[i] = min(p[2 * idx - i], right - i);
        else
            p[i] = 1;

        while(i + p[i] <= tot && s[i + p[i]] == s[i - p[i]])
            p[i]++;

        p[i]--;

        if(i + p[i] > right) {
            right = i + p[i];
            idx = i;
        }
    }
}
```

6 动态规划

6.1 数位 DP

```cpp
// 记忆化搜索
struct cmp {
    bool operator()(const state& a, const state& b) const {
        // ...
    }
};

struct _hash {
    size_t operator()(const state& st) const {
        size_t res = st.cnt[0];
        for(int i = 1; i < 10; ++i) {
            res *= 19260817;
            res += st.cnt[i];
        }
        return res;
    }
};
unordered_map <state, ll, _hash, cmp> dp[20][20];

// -pos: 搜到的位置
// -st: 当前状态
// -lead: 是否有前导 0
// -limit: 是否有最高位限制
ll dfs(int pos, state st, int lead, int limit){
    // 边界情况
    if(pos < 0 /* && ... */) return 0;
    // 记忆化搜索
    int wd = st.w[st.d];
    if((!limit) && (!lead) && dp[pos][wd].count(st)) return dp[pos][wd][st];

    ll res = 0;
    // 最高位最大值
    int cur = limit ? a[pos] : 9;
    for(int i = 0; i <= cur; ++i) {
        // 有前导0且当前位也是0
        if((!i) && lead) res += dfs(pos-1, st, 1, limit&&(i==cur));
        // 有前导0且当前位非0 (出现最高位)
        else if(i && lead) res += dfs(pos-1, st.add(i), 0, limit&&(i==cur));
        else res += dfs(pos-1, st.add(i), 0, limit&&(i==cur));
    }
    // 没有前导0和最高限制时可以直接记录当前dp值以便下次搜到同样的情况可以直接使用
    if(!limit&&!lead) dp[pos][wd][st] = res;
    return res;
}

ll gao(ll x) {
    memset(a, 0, sizeof(a));
    int len=0;
    while(x) a[len++]=x%10,x/=10;
    // init st
    return dfs(len-1, st, 1, 1);
}

int main()
{
    int T;
    ll l, r; int d;
    cin >> T;
    while(T--) {
        scanf("%lld %lld %d", &l, &r, &d);
        ll ans = gao(r, d) - gao(l - 1, d);
        printf("%lld\n", ans);
    }
}
```

7 几何

```cpp
#include <bits/stdc++.h>
#include <vector>
#define mp make_pair
#define fi first
#define se second
#define pb push_back
using namespace std;
using db = double;

mt19937 eng(time(0));

const db eps = 1e-6;
const db pi = acos(-1);
int sgn(db k) {
    if (k > eps)
        return 1;
    else if (k < -eps)
        return -1;
    return 0;
}
// -1: < | 0: == | 1: >
int cmp(db k1, db k2) { return sgn(k1 - k2); }
// k3 in [k1, k2]
int inmid(db k1, db k2, db k3) { return sgn(k1 - k3) * sgn(k2 - k3) <= 0; }
// 点 (x, y)
struct point {
    db x, y;
    point operator+(const point &k1) const {
        return (point){k1.x + x, k1.y + y};
    }
    point operator-(const point &k1) const {
        return (point){x - k1.x, y - k1.y};
    }
    point operator*(db k1) const { return (point){x * k1, y * k1}; }
    point operator/(db k1) const { return (point){x / k1, y / k1}; }
    int operator==(const point &k1) const {
        return cmp(x, k1.x) == 0 && cmp(y, k1.y) == 0;
    }
    // 逆时针旋转 k1 弧度
    point rotate(db k1) {
        return (point){x * cos(k1) - y * sin(k1), x * sin(k1) + y * cos(k1)};
    }
    // 逆时针旋转 90 度
    point rotleft() { return (point){-y, x}; }
    // 优先比较 x 坐标
    bool operator<(const point k1) const {
        int a = cmp(x, k1.x);
        if (a == -1)
            return 1;
        else if (a == 1)
            return 0;
        else
            return cmp(y, k1.y) == -1;
    }
    // 模长
    db abs() { return sqrt(x * x + y * y); }
    // 模长的平方
    db abs2() { return x * x + y * y; }
    // 与点 k1 的距离
    db dis(point k1) { return ((*this) - k1).abs(); }
    // 化为单位向量, require: abs() > 0
    point unit() {
        db w = abs();
        return (point){x / w, y / w};
    }
    // 读入
    void scan() {
        double k1, k2;
        scanf("%lf%lf", &k1, &k2);
        x = k1;
        y = k2;
    }
    // 输出
    void print() { printf("%.11lf %.11lf\n", x, y); }
    // 方向角 atan2(y, x)
    db getw() { return atan2(y, x); }
    // 将向量对称到 (-pi, pi] 半平面中
    point getdel() {
        if (sgn(x) == -1 || (sgn(x) == 0 && sgn(y) == -1))
            return (*this) * (-1);
        else
            return (*this);
    }
    // (-pi, 0] -> 0, (0, pi] -> 1
    int getP() const { return sgn(y) == 1 || (sgn(y) == 0 && sgn(x) == -1); }
};

/* 点与线段的位置关系及交点 */

// k3 在 矩形 [k1, k2] 中
int inmid(point k1, point k2, point k3) {
    return inmid(k1.x, k2.x, k3.x) && inmid(k1.y, k2.y, k3.y);
}
db cross(point k1, point k2) { return k1.x * k2.y - k1.y * k2.x; }
db dot(point k1, point k2) { return k1.x * k2.x + k1.y * k2.y; }
// 从 k1 转到 k2 的方向角
db rad(point k1, point k2) { return atan2(cross(k1, k2), dot(k1, k2)); }
// k1 k2 k3 逆时针 1 顺时针 -1 否则 0
int clockwise(point k1, point k2, point k3) {
    return sgn(cross(k2 - k1, k3 - k1));
}
// 按 (-pi, pi] 顺序进行极角排序
int cmpangle(point k1, point k2) {
    return k1.getP() < k2.getP() ||
           (k1.getP() == k2.getP() && sgn(cross(k1, k2)) > 0);
}
// 点 q 在线段 k1, k2 上
int onS(point k1, point k2, point q) {
    return inmid(k1, k2, q) && sgn(cross(k1 - q, k2 - k1)) == 0;
}
// q 到直线 k1,k2 的投影
point proj(point k1, point k2, point q) {
    point k = k2 - k1;
    return k1 + k * (dot(q - k1, k) / k.abs2());
}
// q 关于直线 k1, k2 的镜像
point reflect(point k1, point k2, point q) { return proj(k1, k2, q) * 2 - q; }
// 判断 直线 (k1, k2) 和 直线 (k3, k4) 是否相交
int checkLL(point k1, point k2, point k3, point k4) {
    return cmp(cross(k3 - k1, k4 - k1), cross(k3 - k2, k4 - k2)) != 0;
}
// 求直线 (k1, k2) 和 直线 (k3, k4) 的交点
point getLL(point k1, point k2, point k3, point k4) {
    db w1 = cross(k1 - k3, k4 - k3), w2 = cross(k4 - k3, k2 - k3);
    return (k1 * w2 + k2 * w1) / (w1 + w2);
}
int intersect(db l1, db r1, db l2, db r2) {
```

```cpp
    if (l1 > r1)
        swap(l1, r1);
    if (l2 > r2)
        swap(l2, r2);
    return cmp(r1, l2) != -1 && cmp(r2, l1) != -1;
}
// 线段与线段相交判断 (非严格相交)
int checkSS(point k1, point k2, point k3, point k4) {
    return intersect(k1.x, k2.x, k3.x, k4.x) &&
           intersect(k1.y, k2.y, k3.y, k4.y) &&
           sgn(cross(k3 - k1, k4 - k1)) * sgn(cross(k3 - k2, k4 - k2)) <= 0 &&
           sgn(cross(k1 - k3, k2 - k3)) * sgn(cross(k1 - k4, k2 - k4)) <= 0;
}
// 点 q 到 直线 (k1, k2) 的距离
db disLP(point k1, point k2, point q) {
    return fabs(cross(k1 - q, k2 - q)) / k1.dis(k2);
}
// 点 q 到 线段 (k1, k2) 的距离
db disSP(point k1, point k2, point q) {
    point k3 = proj(k1, k2, q);
    if (inmid(k1, k2, k3))
        return q.dis(k3);
    else
        return min(q.dis(k1), q.dis(k2));
}
// 线段 (k1, k2) 到 线段 (k3, k4) 的距离
db disSS(point k1, point k2, point k3, point k4) {
    if (checkSS(k1, k2, k3, k4))
        return 0;
    else
        return min(min(disSP(k1, k2, k3), disSP(k1, k2, k4)),
                   min(disSP(k3, k4, k1), disSP(k3, k4, k2)));
}

/* 直线与半平面交 */

// 直线 p[0] -> p[1]
struct line {
    point p[2];
    line(point k1, point k2) {
        p[0] = k1;
        p[1] = k2;
    }
    point &operator[](int k) { return p[k]; }
    // k 严格位于直线左侧 / 半平面 p[0] -> p[1]
    int include(point k) { return sgn(cross(p[1] - p[0], k - p[0])) > 0; }
    // 方向向量
    point dir() { return p[1] - p[0]; }
    // 向左平移 d, 默认为 eps
    line push(db d = eps) {
        point delta = (p[1] - p[0]).rotleft().unit() * d;
        return {p[0] + delta, p[1] + delta};
    }
};
// 直线与直线交点
point getLL(line k1, line k2) { return getLL(k1[0], k1[1], k2[0], k2[1]); }
// 两直线平行
int parallel(line k1, line k2) { return sgn(cross(k1.dir(), k2.dir())) == 0; }
// 平行且同向
int sameDir(line k1, line k2) {
    return parallel(k1, k2) && sgn(dot(k1.dir(), k2.dir())) == 1;
}
// 同向则左侧优先，否则按极角排序，用于半平面交
int operator<(line k1, line k2) {
    if (sameDir(k1, k2))
        return k2.include(k1[0]);
    return cmpangle(k1.dir(), k2.dir());
}
// k3 (半平面) 包含 k1, k2 的交点, 用于半平面交
int checkpos(line k1, line k2, line k3) { return k3.include(getLL(k1, k2)); }
// 求半平面交, 半平面是逆时针方向, 输出按照逆时针
vector<line> getHL(vector<line> L) {
    sort(L.begin(), L.end());
    deque<line> q;
    for (int i = 0; i < (int)L.size(); i++) {
        if (i && sameDir(L[i], L[i - 1]))
            continue;
        while (q.size() > 1 &&
               !checkpos(q[q.size() - 2], q[q.size() - 1], L[i]))
            q.pop_back();
        while (q.size() > 1 && !checkpos(q[1], q[0], L[i]))
            q.pop_front();
        q.push_back(L[i]);
    }
    while (q.size() > 2 && !checkpos(q[q.size() - 2], q[q.size() - 1], q[0]))
        q.pop_back();
    while (q.size() > 2 && !checkpos(q[1], q[0], q[q.size() - 1]))
        q.pop_front();
    vector<line> ans;
    for (int i = 0; i < q.size(); i++)
        ans.push_back(q[i]);
    return ans;
}
db closepoint(vector<point> &A, int l,
              int r) { // 最近点对 , 先要按照 x 坐标排序
    if (r - l <= 5) {
        db ans = 1e20;
        for (int i = l; i <= r; i++)
            for (int j = i + 1; j <= r; j++)
                ans = min(ans, A[i].dis(A[j]));
        return ans;
    }
    int mid = l + r >> 1;
    db ans = min(closepoint(A, l, mid), closepoint(A, mid + 1, r));
    vector<point> B;
    for (int i = l; i <= r; i++)
        if (abs(A[i].x - A[mid].x) <= ans)
            B.push_back(A[i]);
    sort(B.begin(), B.end(), [](point k1, point k2) { return k1.y < k2.y; });
    for (int i = 0; i < B.size(); i++)
        for (int j = i + 1; j < B.size() && B[j].y - B[i].y < ans; j++)
            ans = min(ans, B[i].dis(B[j]));
    return ans;
}

/* 圆基础操作 */

// 圆 (o, r)
struct circle {
    point o;
    db r;
    void scan() {
        o.scan();
        scanf("%lf", &r);
    }
    int inside(point k) { return cmp(r, o.dis(k)); }
};

// 两圆位置关系 (两圆公切线数量)
```

```cpp
int checkposCC(circle k1, circle k2) {
    if (cmp(k1.r, k2.r) == -1)
        swap(k1, k2);
    db dis = k1.o.dis(k2.o);
    int w1 = cmp(dis, k1.r + k2.r), w2 = cmp(dis, k1.r - k2.r);
    if (w1 > 0)
        return 4;
    else if (w1 == 0)
        return 3;
    else if (w2 > 0)
        return 2;
    else if (w2 == 0)
        return 1;
    else
        return 0;
}
// 直线与圆交点，沿 k2->k3 方向给出，相切给出两个
vector<point> getCL(circle k1, point k2, point k3) {
    point k = proj(k2, k3, k1.o);
    db d = k1.r * k1.r - (k - k1.o).abs2();
    if (sgn(d) == -1)
        return {};
    point del = (k3 - k2).unit() * sqrt(max((db)0.0, d));
    return {k - del, k + del};
}
// 两圆交点，沿圆 k1 逆时针给出，相切给出两个
vector<point> getCC(circle k1, circle k2) {
    int pd = checkposCC(k1, k2);
    if (pd == 0 || pd == 4)
        return {};
    db a = (k2.o - k1.o).abs2(), cosA = (k1.r * k1.r + a - k2.r * k2.r) /
                                           (2 * k1.r * sqrt(max(a, (db)0.0)));
    db b = k1.r * cosA, c = sqrt(max((db)0.0, k1.r * k1.r - b * b));
    point k = (k2.o - k1.o).unit(), m = k1.o + k * b, del = k.rotleft() * c;
    return {m - del, m + del};
}
// 点到圆的切点，沿圆 k1 逆时针给出，注意未判位置关系
vector<point> TangentCP(circle k1, point k2) {
    db a = (k2 - k1.o).abs(), b = k1.r * k1.r / a,
       c = sqrt(max((db)0.0, k1.r * k1.r - b * b));
    point k = (k2 - k1.o).unit(), m = k1.o + k * b, del = k.rotleft() * c;
    return {m - del, m + del};
}
// 外公切线
vector<line> TangentoutCC(circle k1, circle k2) {
    int pd = checkposCC(k1, k2);
    if (pd == 0)
        return {};
    if (pd == 1) {
        point k = getCC(k1, k2)[0];
        return {(line){k, k}};
    }
    if (cmp(k1.r, k2.r) == 0) {
        point del = (k2.o - k1.o).unit().rotleft().getdel();
        return {(line){k1.o - del * k1.r, k2.o - del * k2.r},
                (line){k1.o + del * k1.r, k2.o + del * k2.r}};
    } else {
        point p = (k2.o * k1.r - k1.o * k2.r) / (k1.r - k2.r);
        vector<point> A = TangentCP(k1, p), B = TangentCP(k2, p);
        vector<line> ans;
        for (int i = 0; i < A.size(); i++)
            ans.push_back((line){A[i], B[i]});
        return ans;
    }
}
// 内公切线
vector<line> TangentinCC(circle k1, circle k2) {
    int pd = checkposCC(k1, k2);
    if (pd <= 2)
        return {};
    if (pd == 3) {
        point k = getCC(k1, k2)[0];
        return {(line){k, k}};
    }
    point p = (k2.o * k1.r + k1.o * k2.r) / (k1.r + k2.r);
    vector<point> A = TangentCP(k1, p), B = TangentCP(k2, p);
    vector<line> ans;
    for (int i = 0; i < A.size(); i++)
        ans.push_back((line){A[i], B[i]});
    return ans;
}
// 所有公切线
vector<line> TangentCC(circle k1, circle k2) {
    int flag = 0;
    if (k1.r < k2.r)
        swap(k1, k2), flag = 1;
    vector<line> A = TangentoutCC(k1, k2), B = TangentinCC(k1, k2);
    for (line k : B)
        A.push_back(k);
    if (flag)
        for (line &k : A)
            swap(k[0], k[1]);
    return A;
}
// 圆 k1 与三角形 k2 k3 k1.o 的有向面积交
db getarea(circle k1, point k2, point k3) {
    point k = k1.o;
    k1.o = k1.o - k;
    k2 = k2 - k;
    k3 = k3 - k;
    int pd1 = k1.inside(k2), pd2 = k1.inside(k3);
    vector<point> A = getCL(k1, k2, k3);
    if (pd1 >= 0) {
        if (pd2 >= 0)
            return cross(k2, k3) / 2;
        return k1.r * k1.r * rad(A[1], k3) / 2 + cross(k2, A[1]) / 2;
    } else if (pd2 >= 0) {
        return k1.r * k1.r * rad(k2, A[0]) / 2 + cross(A[0], k3) / 2;
    } else {
        int pd = cmp(k1.r, disSP(k2, k3, k1.o));
        if (pd <= 0)
            return k1.r * k1.r * rad(k2, k3) / 2;
        return cross(A[0], A[1]) / 2 +
               k1.r * k1.r * (rad(k2, A[0]) + rad(A[1], k3)) / 2;
    }
}
// 多边形与圆面积交
db getarea(vector<point> A, circle c) {
    int n = A.size();
    if (n <= 2)
        return 0.0;
    A.push_back(A[0]);
    db res = 0.0;
    for (int i = 0; i < n; i++) {
        point k1 = A[i], k2 = A[i + 1];
        res += getarea(c, k1, k2);
    }
    return fabs(res);
```

```
}
// 三角形外接圆
circle getcircle(point k1, point k2, point k3) {
    db a1 = k2.x - k1.x, b1 = k2.y - k1.y, c1 = (a1 * a1 +
    ↪ b1 * b1) / 2;
    db a2 = k3.x - k1.x, b2 = k3.y - k1.y, c2 = (a2 * a2 +
    ↪ b2 * b2) / 2;
    db d = a1 * b2 - a2 * b1;
    point o =
        (point){k1.x + (c1 * b2 - c2 * b1) / d, k1.y + (a1
    ↪ * c2 - a2 * c1) / d};
    return (circle){o, k1.dis(o)};
}
// 最小圆覆盖
circle getScircle(vector<point> A) {
    shuffle(A.begin(), A.end(), eng);
    circle ans = (circle){A[0], 0};
    for (int i = 1; i < A.size(); i++)
        if (ans.inside(A[i]) == -1) {
            ans = (circle){A[i], 0};
            for (int j = 0; j < i; j++)
                if (ans.inside(A[j]) == -1) {
                    ans.o = (A[i] + A[j]) / 2;
                    ans.r = ans.o.dis(A[i]);
                    for (int k = 0; k < j; k++)
                        if (ans.inside(A[k]) == -1)
                            ans = getcircle(A[i], A[j],
    ↪ A[k]);
                }
        }
    return ans;
}

/* 多边形 */

// 多边形有向面积
db area(vector<point> A) {
    db ans = 0;
    for (int i = 0; i < A.size(); i++)
        ans += cross(A[i], A[(i + 1) % A.size()]);
    return ans / 2;
}
// 判断是否为逆时针凸包
int checkconvex(vector<point> A) {
    int n = A.size();
    A.push_back(A[0]);
    A.push_back(A[1]);
    for (int i = 0; i < n; i++)
        if (sgn(cross(A[i + 1] - A[i], A[i + 2] - A[i])) ==
    ↪ -1)
            return 0;
    return 1;
}
// 点与简单多边形位置关系: 2 内部 1 边界 0 外部
int contain(vector<point> A, point q) {
    int pd = 0;
    A.push_back(A[0]);
    for (int i = 1; i < A.size(); i++) {
        point u = A[i - 1], v = A[i];
        if (onS(u, v, q))
            return 1;
        if (cmp(u.y, v.y) > 0)
            swap(u, v);
        if (cmp(u.y, q.y) >= 0 || cmp(v.y, q.y) < 0)
            continue;
        if (sgn(cross(u - v, q - v)) < 0)
            pd ^= 1;
    }
    return pd << 1;
}
// flag=0 不严格 flag=1 严格
vector<point> ConvexHull(vector<point> A, int flag = 1) {
    int n = A.size();
    if (n == 1)
        return A;
    if (n == 2) {
        if (A[0] == A[1])
            return {A[0]};
        else
            return A;
    }
    vector<point> ans(n * 2);
    sort(A.begin(), A.end());
    int now = -1;
    for (int i = 0; i < A.size(); i++) {
        while (now > 0 &&
               sgn(cross(ans[now] - ans[now - 1], A[i] -
    ↪ ans[now - 1])) < flag)
            now--;
        ans[++now] = A[i];
    }
    int pre = now;
    for (int i = n - 2; i >= 0; i--) {
        while (now > pre &&
               sgn(cross(ans[now] - ans[now - 1], A[i] -
    ↪ ans[now - 1])) < flag)
            now--;
        ans[++now] = A[i];
    }
    ans.resize(now);
    return ans;
}
// 凸包直径
db convexDiameter(vector<point> A) {
    int now = 0, n = A.size();
    db ans = 0;
    for (int i = 0; i < A.size(); i++) {
        now = max(now, i);
        while (1) {
            db k1 = A[i].dis(A[now % n]), k2 =
    ↪ A[i].dis(A[(now + 1) % n]);
            ans = max(ans, max(k1, k2));
            if (k2 > k1)
                now++;
            else
                break;
        }
    }
    return ans;
}
// 直线切凸包, 保留 k1,k2,p 逆时针的所有点
vector<point> convexcut(vector<point> A, point k1, point
    ↪ k2) {
    int n = A.size();
    A.push_back(A[0]);
    vector<point> ans;
    for (int i = 0; i < n; i++) {
        int w1 = clockwise(k1, k2, A[i]), w2 =
    ↪ clockwise(k1, k2, A[i + 1]);
        if (w1 >= 0)
            ans.push_back(A[i]);
        if (w1 * w2 < 0)
            ans.push_back(getLL(k1, k2, A[i], A[i + 1]));
    }
    return ans;
}
// 多边形 A 和 直线 (线段) k1->k2 严格相交, 注释部分为线段
int checkPoS(vector<point> A, point k1, point k2) {
    struct ins {
        point m, u, v;
        int operator<(const ins &k) const { return m < k.m;
    ↪ }
    };
    vector<ins> B;
```

```
    // if (contain(A,k1)==2||contain(A,k2)==2) return 1;
    vector<point> poly = A;
    A.push_back(A[0]);
    for (int i = 1; i < A.size(); i++)
        if (checkLL(A[i - 1], A[i], k1, k2)) {
            point m = getLL(A[i - 1], A[i], k1, k2);
            if (inmid(A[i - 1], A[i], m) /*&&inmid(k1,k2,m)*/)
                B.push_back((ins){m, A[i - 1], A[i]});
        }
    if (B.size() == 0)
        return 0;
    sort(B.begin(), B.end());
    int now = 1;
    while (now < B.size() && B[now].m == B[0].m)
        now++;
    if (now == B.size())
        return 0;
    int flag = contain(poly, (B[0].m + B[now].m) / 2);
    if (flag == 2)
        return 1;
    point d = B[now].m - B[0].m;
    for (int i = now; i < B.size(); i++) {
        if (!(B[i].m == B[i - 1].m) && flag == 2)
            return 1;
        int tag = sgn(cross(B[i].v - B[i].u, B[i].m + d - B[i].u));
        if (B[i].m == B[i].u || B[i].m == B[i].v)
            flag += tag;
        else
            flag += tag * 2;
    }
    // return 0;
    return flag == 2;
}
int checkinp(point r, point l, point m) {
    if (cmpangle(l, r)) {
        return cmpangle(l, m) && cmpangle(m, r);
    }
    return cmpangle(l, m) || cmpangle(m, r);
}
// 快速检查线段是否和多边形严格相交
int checkPosFast(vector<point> A, point k1, point k2) {
    if (contain(A, k1) == 2 || contain(A, k2) == 2)
        return 1;
    if (k1 == k2)
        return 0;
    A.push_back(A[0]);
    A.push_back(A[1]);
    for (int i = 1; i + 1 < A.size(); i++)
        if (checkLL(A[i - 1], A[i], k1, k2)) {
            point now = getLL(A[i - 1], A[i], k1, k2);
            if (inmid(A[i - 1], A[i], now) == 0 || inmid(k1, k2, now) == 0)
                continue;
            if (now == A[i]) {
                if (A[i] == k2)
                    continue;
                point pre = A[i - 1], ne = A[i + 1];
                if (checkinp(pre - now, ne - now, k2 - now))
                    return 1;
            } else if (now == k1) {
                if (k1 == A[i - 1] || k1 == A[i])
                    continue;
                if (checkinp(A[i - 1] - k1, A[i] - k1, k2 - k1))
                    return 1;
            } else if (now == k2 || now == A[i - 1])
                continue;
            else
                return 1;
        }
    return 0;
}

/* 普通凸包中的二分 */

// 求经过点 x 切凸包 A 的两个切点, 返回下标. 方向: A 上 [fi, se] 为点 x
// 能看到的区域. 需要保证 x 严格在凸包 A 外侧, A 的点数 >= 3 需要保证 A
// 是严格凸包, 即无三点共线
pair<int, int> getTangentCoP(const vector<point> &A, point x) {
    int sz = A.size();
    assert(sz >= 3);
    int res[2];
    int flag = 1;
    if (clockwise(A[sz - 1], A[0], x) == -1)
        flag = -1;
    int l = 0, r = sz - 1, ans = 0;
    while (l < r) {
        int mid = ((l + r) >> 1);
        if (clockwise(A[mid], A[mid + 1], x) == flag &&
            clockwise(A[0], A[mid + 1], x) == flag)
            ans = mid + 1, l = mid + 1;
        else
            r = mid;
    }
    res[0] = ans;
    l = ans, r = sz - 1, ans = sz - 1;
    while (l < r) {
        int mid = ((l + r) >> 1);
        if (clockwise(A[mid], A[mid + 1], x) == flag)
            ans = mid, r = mid;
        else
            l = mid + 1;
    }
    res[1] = ans;
    if (flag == -1)
        swap(res[0], res[1]);
    return {res[0], res[1]};
}

// 判断点是否在凸多边形 A 内部, flag = 1 严格, 0 不严格
bool containCoP(const vector<point> &A, point x, int flag = 1) {
    int sz = A.size();
    assert(sz >= 3);
    if (!flag && (onS(A[0], A[1], x) || onS(A[sz - 1], A[0], x)))
        return 1;
    if (!(clockwise(A[0], A[1], x) == 1 && clockwise(A[sz - 1], A[0], x) == 1))
        return 0;
    int l = 1, r = sz - 1, ans = 1;
    while (l < r) {
        int mid = l + r >> 1;
        if (clockwise(A[0], A[mid], x) == 1)
            ans = mid, l = mid + 1;
        else
            r = mid;
    }
    return clockwise(A[ans], A[ans + 1], x) >= flag;
}

/* 上下凸包中的二分 */

// 拆分凸包成上下凸壳 凸包尽量都随机旋转一个角度来避免出现 相同横坐标
// 尽量特判只有一个点的情况 凸包逆时针
void getUDP(vector<point> A, vector<point> &U, vector<point> &D) {
```

```cpp
    db l = 1e100, r = -1e100;
    for (int i = 0; i < A.size(); i++)
        l = min(l, A[i].x), r = max(r, A[i].x);
    int wherel, wherer;
    for (int i = 0; i < A.size(); i++)
        if (cmp(A[i].x, l) == 0)
            wherel = i;
    for (int i = A.size(); i; i--)
        if (cmp(A[i - 1].x, r) == 0)
            wherer = i - 1;
    U.clear();
    D.clear();
    int now = wherel;
    while (1) {
        D.push_back(A[now]);
        if (now == wherer)
            break;
        now++;
        if (now >= A.size())
            now = 0;
    }
    now = wherel;
    while (1) {
        U.push_back(A[now]);
        if (now == wherer)
            break;
        now--;
        if (now < 0)
            now = A.size() - 1;
    }
}
// 需要保证凸包点数大于等于 3, 2 内部 ,1 边界 ,0 外部
int containCoP(const vector<point> &U, const vector<point> &D, point k) {
    db lx = U[0].x, rx = U[U.size() - 1].x;
    if (k == U[0] || k == U[U.size() - 1])
        return 1;
    if (cmp(k.x, lx) == -1 || cmp(k.x, rx) == 1)
        return 0;
    int where1 =
        lower_bound(U.begin(), U.end(), (point){k.x, -1e100}) - U.begin();
    int where2 =
        lower_bound(D.begin(), D.end(), (point){k.x, -1e100}) - D.begin();
    int w1 = clockwise(U[where1 - 1], U[where1], k),
        w2 = clockwise(D[where2 - 1], D[where2], k);
    if (w1 == 1 || w2 == -1)
        return 0;
    else if (w1 == 0 || w2 == 0)
        return 1;
    return 2;
}
// d 是方向 , 输出上方切点和下方切点
pair<point, point> getTangentCow(const vector<point> &U, const vector<point> &D,
                                 point d) {
    if (sgn(d.x) < 0 || (sgn(d.x) == 0 && sgn(d.y) < 0))
        d = d * (-1);
    point whereU, whereD;
    if (sgn(d.x) == 0)
        return mp(U[0], U[U.size() - 1]);
    int l = 0, r = U.size() - 1, ans = 0;
    while (l < r) {
        int mid = l + r >> 1;
        if (sgn(cross(U[mid + 1] - U[mid], d)) <= 0)
            l = mid + 1, ans = mid + 1;
        else
            r = mid;
    }
    whereU = U[ans];
    l = 0, r = D.size() - 1, ans = 0;
    while (l < r) {
        int mid = l + r >> 1;
        if (sgn(cross(D[mid + 1] - D[mid], d)) >= 0)
            l = mid + 1, ans = mid + 1;
        else
            r = mid;
    }
    whereD = D[ans];
    return mp(whereU, whereD);
}
// 先检查 contain, 逆时针给出
pair<point, point> getTangentCoP(const vector<point> &U, const vector<point> &D,
                                 point k) {
    db lx = U[0].x, rx = U[U.size() - 1].x;
    if (k.x < lx) {
        int l = 0, r = U.size() - 1, ans = U.size() - 1;
        while (l < r) {
            int mid = l + r >> 1;
            if (clockwise(k, U[mid], U[mid + 1]) == 1)
                l = mid + 1;
            else
                ans = mid, r = mid;
        }
        point w1 = U[ans];
        l = 0, r = D.size() - 1, ans = D.size() - 1;
        while (l < r) {
            int mid = l + r >> 1;
            if (clockwise(k, D[mid], D[mid + 1]) == -1)
                l = mid + 1;
            else
                ans = mid, r = mid;
        }
        point w2 = D[ans];
        return mp(w1, w2);
    } else if (k.x > rx) {
        int l = 1, r = U.size(), ans = 0;
        while (l < r) {
            int mid = l + r >> 1;
            if (clockwise(k, U[mid], U[mid - 1]) == -1)
                r = mid;
            else
                ans = mid, l = mid + 1;
        }
        point w1 = U[ans];
        l = 1, r = D.size(), ans = 0;
        while (l < r) {
            int mid = l + r >> 1;
            if (clockwise(k, D[mid], D[mid - 1]) == 1)
                r = mid;
            else
                ans = mid, l = mid + 1;
        }
        point w2 = D[ans];
        return mp(w2, w1);
    } else {
        int where1 =
            lower_bound(U.begin(), U.end(), (point){k.x, -1e100}) - U.begin();
        int where2 =
            lower_bound(D.begin(), D.end(), (point){k.x, -1e100}) - D.begin();
        if ((k.x == lx && k.y > U[0].y) ||
            (where1 && clockwise(U[where1 - 1], U[where1], k) == 1)) {
            int l = 1, r = where1 + 1, ans = 0;
            while (l < r) {
                int mid = l + r >> 1;
                if (clockwise(k, U[mid], U[mid - 1]) == 1)
                    ans = mid, l = mid + 1;
                else
                    r = mid;
```

```cpp
            }
            point w1 = U[ans];
            l = where1, r = U.size() - 1, ans = U.size() - 1;
            while (l < r) {
                int mid = l + r >> 1;
                if (clockwise(k, U[mid], U[mid + 1]) == 1)
                    l = mid + 1;
                else
                    ans = mid, r = mid;
            }
            point w2 = U[ans];
            return mp(w2, w1);
        } else {
            int l = 1, r = where2 + 1, ans = 0;
            while (l < r) {
                int mid = l + r >> 1;
                if (clockwise(k, D[mid], D[mid - 1]) == -1)
                    ans = mid, l = mid + 1;
                else
                    r = mid;
            }
            point w1 = D[ans];
            l = where2, r = D.size() - 1, ans = D.size() - 1;
            while (l < r) {
                int mid = l + r >> 1;
                if (clockwise(k, D[mid], D[mid + 1]) == -1)
                    l = mid + 1;
                else
                    ans = mid, r = mid;
            }
            point w2 = D[ans];
            return mp(w1, w2);
        }
    }
}

// 三维计算几何

struct P3 {
    db x, y, z;
    P3 operator+(P3 k1) { return (P3){x + k1.x, y + k1.y, z + k1.z}; }
    P3 operator-(P3 k1) { return (P3){x - k1.x, y - k1.y, z - k1.z}; }
    P3 operator*(db k1) { return (P3){x * k1, y * k1, z * k1}; }
    P3 operator/(db k1) { return (P3){x / k1, y / k1, z / k1}; }
    db abs2() { return x * x + y * y + z * z; }
    db abs() { return sqrt(x * x + y * y + z * z); }
    P3 unit() { return (*this) / abs(); }
    int operator<(const P3 k1) const {
        if (cmp(x, k1.x) != 0)
            return x < k1.x;
        if (cmp(y, k1.y) != 0)
            return y < k1.y;
        return cmp(z, k1.z) == -1;
    }
    int operator==(const P3 k1) {
        return cmp(x, k1.x) == 0 && cmp(y, k1.y) == 0 && cmp(z, k1.z) == 0;
    }
    void scan() {
        double k1, k2, k3;
        scanf("%lf%lf%lf", &k1, &k2, &k3);
        x = k1;
        y = k2;
        z = k3;
    }
};
P3 cross(P3 k1, P3 k2) {
    return (P3){k1.y * k2.z - k1.z * k2.y, k1.z * k2.x - k1.x * k2.z,
                k1.x * k2.y - k1.y * k2.x};
}
db dot(P3 k1, P3 k2) { return k1.x * k2.x + k1.y * k2.y + k1.z * k2.z; }
// p=(3,4,5),l=(13,19,21),theta=85 ans=(2.83,4.62,1.77)
P3 turn3D(db k1, P3 l, P3 p) {
    l = l.unit();
    P3 ans;
    db c = cos(k1), s = sin(k1);
    ans.x = p.x * (l.x * l.x * (1 - c) + c) +
            p.y * (l.x * l.y * (1 - c) - l.z * s) +
            p.z * (l.x * l.z * (1 - c) + l.y * s);
    ans.y = p.x * (l.x * l.y * (1 - c) + l.z * s) +
            p.y * (l.y * l.y * (1 - c) + c) +
            p.z * (l.y * l.z * (1 - c) - l.x * s);
    ans.z = p.x * (l.x * l.z * (1 - c) - l.y * s) +
            p.y * (l.y * l.z * (1 - c) + l.x * s) +
            p.z * (l.x * l.x * (1 - c) + c);
    return ans;
}
typedef vector<P3> VP;
typedef vector<VP> VVP;
db Acos(db x) { return acos(max(-(db)1, min(x, (db)1))); }
// 球面距离 , 圆心原点 , 半径 1
db Odist(P3 a, P3 b) {
    db r = Acos(dot(a, b));
    return r;
}
db r;
P3 rnd;
vector<db> solve(db a, db b, db c) {
    db r = sqrt(a * a + b * b), th = atan2(b, a);
    if (cmp(c, -r) == -1)
        return {0};
    else if (cmp(r, c) <= 0)
        return {1};
    else {
        db tr = pi - Acos(c / r);
        return {th + pi - tr, th + pi + tr};
    }
}
vector<db> jiao(P3 a, P3 b) {
    // dot(rd+x*cos(t)+y*sin(t),b) >= cos(r)
    if (cmp(Odist(a, b), 2 * r) > 0)
        return {0};
    P3 rd = a * cos(r), z = a.unit(), y = cross(z, rnd).unit(),
       x = cross(y, z).unit();
    vector<db> ret = solve(-(dot(x, b) * sin(r)), -(dot(y, b) * sin(r)),
                           -(cos(r) - dot(rd, b)));
    return ret;
}
db norm(db x, db l = 0, db r = 2 * pi) { // change x into [l,r)
    while (cmp(x, l) == -1)
        x += (r - l);
    while (cmp(x, r) >= 0)
        x -= (r - l);
    return x;
}
db disLP(P3 k1, P3 k2, P3 q) {
    return (cross(k2 - k1, q - k1)).abs() / (k2 - k1).abs();
}
db disLL(P3 k1, P3 k2, P3 k3, P3 k4) {
    P3 dir = cross(k2 - k1, k4 - k3);
    if (sgn(dir.abs()) == 0)
        return disLP(k1, k2, k3);
```

```
    return fabs(dot(dir.unit(), k1 - k2));
}
VP getFL(P3 p, P3 dir, P3 k1, P3 k2) {
    db a = dot(k2 - p, dir), b = dot(k1 - p, dir), d = a -
  ↪ b;
    if (sgn(fabs(d)) == 0)
        return {};
    return {(k1 * a - k2 * b) / d};
}
VP getFF(P3 p1, P3 dir1, P3 p2, P3 dir2) { // 返回一条线
    P3 e = cross(dir1, dir2), v = cross(dir1, e);
    db d = dot(dir2, v);
    if (sgn(abs(d)) == 0)
        return {};
    P3 q = p1 + v * dot(dir2, p2 - p1) / d;
    return {q, q + e};
}
// 3D Covex Hull Template
db getV(P3 k1, P3 k2, P3 k3, P3 k4) { // get the Volume
    return dot(cross(k2 - k1, k3 - k1), k4 - k1);
}
db rand_db() { return 1.0 * rand() / RAND_MAX; }
VP convexHull2D(VP A, P3 dir) {
    P3 x = {(db)rand(), (db)rand(), (db)rand()};
    x = x.unit();
    x = cross(x, dir).unit();
    P3 y = cross(x, dir).unit();
    P3 vec = dir.unit() * dot(A[0], dir);
    vector<point> B;
    for (int i = 0; i < A.size(); i++)
        B.push_back((point){dot(A[i], x), dot(A[i], y)});
    B = ConvexHull(B);
    A.clear();
    for (int i = 0; i < B.size(); i++)
        A.push_back(x * B[i].x + y * B[i].y + vec);
    return A;
}
namespace CH3 {
VVP ret;
set<pair<int, int>> e;
int n;
VP p, q;
void wrap(int a, int b) {
    if (e.find({a, b}) == e.end()) {
        int c = -1;
        for (int i = 0; i < n; i++)
            if (i != a && i != b) {
                if (c == -1 || sgn(getV(q[c], q[a], q[b],
  ↪ q[i])) > 0)
                    c = i;
            }
        if (c != -1) {
            ret.push_back({p[a], p[b], p[c]});
            e.insert({a, b});
            e.insert({b, c});
            e.insert({c, a});
            wrap(c, b);
            wrap(a, c);
        }
    }
}
VVP ConvexHull3D(VP _p) {
    p = q = _p;
    n = p.size();
    ret.clear();
    e.clear();
    for (auto &i : q)
        i = i + (P3){rand_db() * 1e-4, rand_db() * 1e-4,
  ↪ rand_db() * 1e-4};
    for (int i = 1; i < n; i++)
        if (q[i].x < q[0].x)
            swap(p[0], p[i]), swap(q[0], q[i]);
    for (int i = 2; i < n; i++)
        if ((q[i].x - q[0].x) * (q[1].y - q[0].y) >
            (q[i].y - q[0].y) * (q[1].x - q[0].x))
            swap(q[1], q[i]), swap(p[1], p[i]);
    wrap(0, 1);
    return ret;
}
} // namespace CH3
VVP reduceCH(VVP A) {
    VVP ret;
    map<P3, VP> M;
    for (VP nowF : A) {
        P3 dir = cross(nowF[1] - nowF[0], nowF[2] -
  ↪ nowF[0]).unit();
        for (P3 k1 : nowF)
            M[dir].pb(k1);
    }
    for (pair<P3, VP> nowF : M)
        ret.pb(convexHull2D(nowF.se, nowF.fi));
    return ret;
}
//  把一个面变成 ( 点 , 法向量 ) 的形式
pair<P3, P3> getF(VP F) {
    return mp(F[0], cross(F[1] - F[0], F[2] -
  ↪ F[0]).unit());
}
// 3D Cut 保留 dot(dir,x-p)>=0 的部分
VVP ConvexCut3D(VVP A, P3 p, P3 dir) {
    VVP ret;
    VP sec;
    for (VP nowF : A) {
        int n = nowF.size();
        VP ans;
        int dif = 0;
        for (int i = 0; i < n; i++) {
            int d1 = sgn(dot(dir, nowF[i] - p));
            int d2 = sgn(dot(dir, nowF[(i + 1) % n] - p));
            if (d1 >= 0)
                ans.pb(nowF[i]);
            if (d1 * d2 < 0) {
                P3 q = getFL(p, dir, nowF[i], nowF[(i + 1)
  ↪ % n])[0];
                ans.push_back(q);
                sec.push_back(q);
            }
            if (d1 == 0)
                sec.push_back(nowF[i]);
            else
                dif = 1;
            dif |= (sgn(dot(dir, cross(nowF[(i + 1) % n] -
  ↪ nowF[i],
                                       nowF[(i + 1) % n] -
  ↪ nowF[i]))) == -1);
        }
        if (ans.size() > 0 && dif)
            ret.push_back(ans);
    }
    if (sec.size() > 0)
        ret.push_back(convexHull2D(sec, dir));
    return ret;
}
db vol(VVP A) {
    if (A.size() == 0)
        return 0;
    P3 p = A[0][0];
    db ans = 0;
    for (VP nowF : A)
        for (int i = 2; i < nowF.size(); i++)
            ans += abs(getV(p, nowF[0], nowF[i - 1],
  ↪ nowF[i]));
    return ans / 6;
}
```

```
VVP init(db INF) {
    VVP pss(6, VP(4));
    pss[0][0] = pss[1][0] = pss[2][0] = {-INF, -INF, -INF};
    pss[0][3] = pss[1][1] = pss[5][2] = {-INF, -INF, INF};
    pss[0][1] = pss[2][3] = pss[4][2] = {-INF, INF, -INF};
    pss[0][2] = pss[5][3] = pss[4][1] = {-INF, INF, INF};
    pss[1][3] = pss[2][1] = pss[3][2] = {INF, -INF, -INF};
    pss[1][2] = pss[5][1] = pss[3][3] = {INF, -INF, INF};
    pss[2][2] = pss[4][3] = pss[3][1] = {INF, INF, -INF};
    pss[5][0] = pss[4][0] = pss[3][0] = {INF, INF, INF};
    return pss;
}
```

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Definitions	
$f(n) = O(g(n))$	iff $\exists$ positive c, n_0 such that $0 \le f(n) \le cg(n)\ \forall n \ge n_0$.
$f(n) = \Omega(g(n))$	iff $\exists$ positive c, n_0 such that $f(n) \ge cg(n) \ge 0\ \forall n \ge n_0$.
$f(n) = \Theta(g(n))$	iff $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$.
$f(n) = o(g(n))$	iff $\lim_{n\to\infty} f(n)/g(n) = 0$.
$\lim\limits_{n\to\infty} a_n = a$	iff $\forall \epsilon > 0$, $\exists n_0$ such that $\lvert a_n - a\rvert < \epsilon$, $\forall n \ge n_0$.
$\sup S$	least $b \in \mathbb{R}$ such that $b \ge s$, $\forall s \in S$.
$\inf S$	greatest $b \in \mathbb{R}$ such that $b \le s$, $\forall s \in S$.
$\liminf\limits_{n\to\infty} a_n$	$\lim\limits_{n\to\infty} \inf\{a_i \mid i \ge n, i \in \mathbb{N}\}$.
$\limsup\limits_{n\to\infty} a_n$	$\lim\limits_{n\to\infty} \sup\{a_i \mid i \ge n, i \in \mathbb{N}\}$.
$\binom{n}{k}$	Combinations: Size k subsets of a size n set.
$\left[n \atop k \right]$	Stirling numbers (1st kind): Arrangements of an n element set into k cycles.
$\left\{ n \atop k \right\}$	Stirling numbers (2nd kind): Partitions of an n element set into k non-empty sets.
$\left\langle n \atop k \right\rangle$	1st order Eulerian numbers: Permutations $\pi_1\pi_2\ldots\pi_n$ on $\{1, 2, \ldots, n\}$ with k ascents.
$\left\langle\!\left\langle n \atop k \right\rangle\!\right\rangle$	2nd order Eulerian numbers.
C_n	Catalan Numbers: Binary trees with $n+1$ vertices.

Series

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}, \quad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}, \quad \sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}.$$

In general:

$$\sum_{i=1}^{n} i^m = \frac{1}{m+1}\left[(n+1)^{m+1} - 1 - \sum_{i=1}^{n}\left((i+1)^{m+1} - i^{m+1} - (m+1)i^m\right)\right]$$

$$\sum_{i=1}^{n-1} i^m = \frac{1}{m+1}\sum_{k=0}^{m}\binom{m+1}{k}B_k n^{m+1-k}.$$

Geometric series:

$$\sum_{i=0}^{n} c^i = \frac{c^{n+1}-1}{c-1}, \quad c \ne 1, \quad \sum_{i=0}^{\infty} c^i = \frac{1}{1-c}, \quad \sum_{i=1}^{\infty} c^i = \frac{c}{1-c}, \quad |c| < 1,$$

$$\sum_{i=0}^{n} ic^i = \frac{nc^{n+2} - (n+1)c^{n+1} + c}{(c-1)^2}, \quad c \ne 1, \quad \sum_{i=0}^{\infty} ic^i = \frac{c}{(1-c)^2}, \quad |c| < 1.$$

Harmonic series:

$$H_n = \sum_{i=1}^{n}\frac{1}{i}, \quad \sum_{i=1}^{n} iH_i = \frac{n(n+1)}{2}H_n - \frac{n(n-1)}{4}.$$

$$\sum_{i=1}^{n} H_i = (n+1)H_n - n, \quad \sum_{i=1}^{n}\binom{i}{m}H_i = \binom{n+1}{m+1}\left(H_{n+1} - \frac{1}{m+1}\right).$$

1. $\displaystyle\binom{n}{k} = \frac{n!}{(n-k)!k!}$, **2.** $\displaystyle\sum_{k=0}^{n}\binom{n}{k} = 2^n$, **3.** $\displaystyle\binom{n}{k} = \binom{n}{n-k}$,

4. $\displaystyle\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}$, **5.** $\displaystyle\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$,

6. $\displaystyle\binom{n}{m}\binom{m}{k} = \binom{n}{k}\binom{n-k}{m-k}$, **7.** $\displaystyle\sum_{k=0}^{n}\binom{r+k}{k} = \binom{r+n+1}{n}$,

8. $\displaystyle\sum_{k=0}^{n}\binom{k}{m} = \binom{n+1}{m+1}$, **9.** $\displaystyle\sum_{k=0}^{n}\binom{r}{k}\binom{s}{n-k} = \binom{r+s}{n}$,

10. $\displaystyle\binom{n}{k} = (-1)^k\binom{k-n-1}{k}$, **11.** $\displaystyle\left\{ n \atop 1 \right\} = \left\{ n \atop n \right\} = 1$,

12. $\displaystyle\left\{ n \atop 2 \right\} = 2^{n-1} - 1$, **13.** $\displaystyle\left\{ n \atop k \right\} = k\left\{ n-1 \atop k \right\} + \left\{ n-1 \atop k-1 \right\}$,

14. $\displaystyle\left[n \atop 1 \right] = (n-1)!$, **15.** $\displaystyle\left[n \atop 2 \right] = (n-1)!H_{n-1}$, **16.** $\displaystyle\left[n \atop n \right] = 1$, **17.** $\displaystyle\left[n \atop k \right] \ge \left\{ n \atop k \right\}$,

18. $\displaystyle\left[n \atop k \right] = (n-1)\left[n-1 \atop k \right] + \left[n-1 \atop k-1 \right]$, **19.** $\displaystyle\left\{ n \atop n-1 \right\} = \left[n \atop n-1 \right] = \binom{n}{2}$, **20.** $\displaystyle\sum_{k=0}^{n}\left[n \atop k \right] = n!$, **21.** $\displaystyle C_n = \frac{1}{n+1}\binom{2n}{n}$,

22. $\displaystyle\left\langle n \atop 0 \right\rangle = \left\langle n \atop n-1 \right\rangle = 1$, **23.** $\displaystyle\left\langle n \atop k \right\rangle = \left\langle n \atop n-1-k \right\rangle$, **24.** $\displaystyle\left\langle n \atop k \right\rangle = (k+1)\left\langle n-1 \atop k \right\rangle + (n-k)\left\langle n-1 \atop k-1 \right\rangle$,

25. $\displaystyle\left\langle 0 \atop k \right\rangle = \begin{cases} 1 & \text{if } k = 0, \\ 0 & \text{otherwise} \end{cases}$ **26.** $\displaystyle\left\langle n \atop 1 \right\rangle = 2^n - n - 1$, **27.** $\displaystyle\left\langle n \atop 2 \right\rangle = 3^n - (n+1)2^n + \binom{n+1}{2}$,

28. $\displaystyle x^n = \sum_{k=0}^{n}\left\langle n \atop k \right\rangle\binom{x+k}{n}$, **29.** $\displaystyle\left\langle n \atop m \right\rangle = \sum_{k=0}^{m}\binom{n+1}{k}(m+1-k)^n(-1)^k$, **30.** $\displaystyle m!\left\{ n \atop m \right\} = \sum_{k=0}^{n}\left\langle n \atop k \right\rangle\binom{k}{n-m}$,

31. $\displaystyle\left\langle n \atop m \right\rangle = \sum_{k=0}^{n}\left\{ n \atop k \right\}\binom{n-k}{m}(-1)^{n-k-m}k!$, **32.** $\displaystyle\left\langle\!\left\langle n \atop 0 \right\rangle\!\right\rangle = 1$, **33.** $\displaystyle\left\langle\!\left\langle n \atop n \right\rangle\!\right\rangle = 0 \quad \text{for } n \ne 0$,

34. $\displaystyle\left\langle\!\left\langle n \atop k \right\rangle\!\right\rangle = (k+1)\left\langle\!\left\langle n-1 \atop k \right\rangle\!\right\rangle + (2n-1-k)\left\langle\!\left\langle n-1 \atop k-1 \right\rangle\!\right\rangle$, **35.** $\displaystyle\sum_{k=0}^{n}\left\langle\!\left\langle n \atop k \right\rangle\!\right\rangle = \frac{(2n)^{\underline{n}}}{2^n}$,

36. $\displaystyle\left\{ x \atop x-n \right\} = \sum_{k=0}^{n}\left\langle\!\left\langle n \atop k \right\rangle\!\right\rangle\binom{x+n-1-k}{2n}$, **37.** $\displaystyle\left\{ n+1 \atop m+1 \right\} = \sum_{k}\binom{n}{k}\left\{ k \atop m \right\} = \sum_{k=0}^{n}\left\{ k \atop m \right\}(m+1)^{n-k}$,

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Identities Cont.

38. $\left[{n+1 \atop m+1}\right] = \sum_k \left[{n \atop k}\right]\binom{k}{m} = \sum_{k=0}^{n} \left[{k \atop m}\right] n^{\underline{n-k}} = n! \sum_{k=0}^{n} \frac{1}{k!}\left[{k \atop m}\right]$,

39. $\left[{x \atop x-n}\right] = \sum_{k=0}^{n} \left\langle\!\!\left\langle {n \atop k} \right\rangle\!\!\right\rangle \binom{x+k}{2n}$,

40. $\left\{{n \atop m}\right\} = \sum_k \binom{n}{k}\left\{{k+1 \atop m+1}\right\}(-1)^{n-k}$,

41. $\left[{n \atop m}\right] = \sum_k \left[{n+1 \atop k+1}\right]\binom{k}{m}(-1)^{m-k}$,

42. $\left\{{m+n+1 \atop m}\right\} = \sum_{k=0}^{m} k \left\{{n+k \atop k}\right\}$,

43. $\left[{m+n+1 \atop m}\right] = \sum_{k=0}^{m} k(n+k)\left[{n+k \atop k}\right]$,

44. $\binom{n}{m} = \sum_k \left\{{n+1 \atop k+1}\right\}\left[{k \atop m}\right](-1)^{m-k}$,

45. $(n-m)!\binom{n}{m} = \sum_k \left[{n+1 \atop k+1}\right]\left\{{k \atop m}\right\}(-1)^{m-k}$, for $n \ge m$,

46. $\left\{{n \atop n-m}\right\} = \sum_k \binom{m-n}{m+k}\binom{m+n}{n+k}\left[{m+k \atop k}\right]$,

47. $\left[{n \atop n-m}\right] = \sum_k \binom{m-n}{m+k}\binom{m+n}{n+k}\left\{{m+k \atop k}\right\}$,

48. $\left\{{n \atop \ell+m}\right\}\binom{\ell+m}{\ell} = \sum_k \left\{{k \atop \ell}\right\}\left\{{n-k \atop m}\right\}\binom{n}{k}$,

49. $\left[{n \atop \ell+m}\right]\binom{\ell+m}{\ell} = \sum_k \left[{k \atop \ell}\right]\left[{n-k \atop m}\right]\binom{n}{k}$.

Trees

Every tree with n vertices has $n-1$ edges.

Kraft inequality: If the depths of the leaves of a binary tree are $d_1, \ldots, d_n$:

$$\sum_{i=1}^{n} 2^{-d_i} \le 1,$$

and equality holds only if every internal node has 2 sons.

Recurrences

Master method:

$$T(n) = aT(n/b) + f(n), \quad a \ge 1, b > 1$$

If $\exists \epsilon > 0$ such that $f(n) = O(n^{\log_b a - \epsilon})$ then

$$T(n) = \Theta(n^{\log_b a}).$$

If $f(n) = \Theta(n^{\log_b a})$ then

$$T(n) = \Theta(n^{\log_b a} \log_2 n).$$

If $\exists \epsilon > 0$ such that $f(n) = \Omega(n^{\log_b a + \epsilon})$, and $\exists c < 1$ such that $af(n/b) \le cf(n)$ for large n, then

$$T(n) = \Theta(f(n)).$$

Substitution (example): Consider the following recurrence

$$T_{i+1} = 2^{2^i} \cdot T_i^2, \quad T_1 = 2.$$

Note that T_i is always a power of two. Let $t_i = \log_2 T_i$. Then we have

$$t_{i+1} = 2^i + 2t_i, \quad t_1 = 1.$$

Let $u_i = t_i/2^i$. Dividing both sides of the previous equation by 2^{i+1} we get

$$\frac{t_{i+1}}{2^{i+1}} = \frac{2^i}{2^{i+1}} + \frac{t_i}{2^i}.$$

Substituting we find

$$u_{i+1} = \tfrac{1}{2} + u_i, \qquad u_1 = \tfrac{1}{2},$$

which is simply $u_i = i/2$. So we find that T_i has the closed form $T_i = 2^{i2^{i-1}}$.

Summing factors (example): Consider the following recurrence

$$T(n) = 3T(n/2) + n, \quad T(1) = 1.$$

Rewrite so that all terms involving T are on the left side

$$T(n) - 3T(n/2) = n.$$

Now expand the recurrence, and choose a factor which makes the left side "telescope"

$$
\begin{aligned}
1\big(T(n) - 3T(n/2) &= n\big)\\
3\big(T(n/2) - 3T(n/4) &= n/2\big)\\
\vdots \quad \vdots \quad &\vdots\\
3^{\log_2 n - 1}\big(T(2) - 3T(1) &= 2\big)
\end{aligned}
$$

Let $m = \log_2 n$. Summing the left side we get $T(n) - 3^m T(1) = T(n) - 3^m = T(n) - n^k$ where $k = \log_2 3 \approx 1.58496$. Summing the right side we get

$$\sum_{i=0}^{m-1} \frac{n}{2^i} 3^i = n \sum_{i=0}^{m-1} \left(\tfrac{3}{2}\right)^i.$$

Let $c = \frac{3}{2}$. Then we have

$$
\begin{aligned}
n \sum_{i=0}^{m-1} c^i &= n\left(\frac{c^m - 1}{c - 1}\right)\\
&= 2n(c^{\log_2 n} - 1)\\
&= 2n(c^{(k-1)\log_c n} - 1)\\
&= 2n^k - 2n,
\end{aligned}
$$

and so $T(n) = 3n^k - 2n$. Full history recurrences can often be changed to limited history ones (example): Consider

$$T_i = 1 + \sum_{j=0}^{i-1} T_j, \quad T_0 = 1.$$

Note that

$$T_{i+1} = 1 + \sum_{j=0}^{i} T_j.$$

Subtracting we find

$$
\begin{aligned}
T_{i+1} - T_i &= 1 + \sum_{j=0}^{i} T_j - 1 - \sum_{j=0}^{i-1} T_j\\
&= T_i.
\end{aligned}
$$

And so $T_{i+1} = 2T_i = 2^{i+1}$.

Generating functions:

1. Multiply both sides of the equation by x^i.
2. Sum both sides over all i for which the equation is valid.
3. Choose a generating function $G(x)$. Usually $G(x) = \sum_{i=0}^{\infty} x^i g_i$.
3. Rewrite the equation in terms of the generating function $G(x)$.
4. Solve for $G(x)$.
5. The coefficient of x^i in $G(x)$ is g_i.

Example:

$$g_{i+1} = 2g_i + 1, \quad g_0 = 0.$$

Multiply and sum:

$$\sum_{i \ge 0} g_{i+1} x^i = \sum_{i \ge 0} 2g_i x^i + \sum_{i \ge 0} x^i.$$

We choose $G(x) = \sum_{i \ge 0} x^i g_i$. Rewrite in terms of $G(x)$:

$$\frac{G(x) - g_0}{x} = 2G(x) + \sum_{i \ge 0} x^i.$$

Simplify:

$$\frac{G(x)}{x} = 2G(x) + \frac{1}{1-x}.$$

Solve for $G(x)$:

$$G(x) = \frac{x}{(1-x)(1-2x)}.$$

Expand this using partial fractions:

$$
\begin{aligned}
G(x) &= x\left(\frac{2}{1-2x} - \frac{1}{1-x}\right)\\
&= x\left(2\sum_{i \ge 0} 2^i x^i - \sum_{i \ge 0} x^i\right)\\
&= \sum_{i \ge 0} (2^{i+1} - 1)x^{i+1}.
\end{aligned}
$$

So $g_i = 2^i - 1$.

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$\pi \approx 3.14159$, $e \approx 2.71828$, $\gamma \approx 0.57721$, $\phi = \frac{1+\sqrt{5}}{2} \approx 1.61803$, $\hat{\phi} = \frac{1-\sqrt{5}}{2} \approx -.61803$

i	2^i	p_i
1	2	2
2	4	3
3	8	5
4	16	7
5	32	11
6	64	13
7	128	17
8	256	19
9	512	23
10	1,024	29
11	2,048	31
12	4,096	37
13	8,192	41
14	16,384	43
15	32,768	47
16	65,536	53
17	131,072	59
18	262,144	61
19	524,288	67
20	1,048,576	71
21	2,097,152	73
22	4,194,304	79
23	8,388,608	83
24	16,777,216	89
25	33,554,432	97
26	67,108,864	101
27	134,217,728	103
28	268,435,456	107
29	536,870,912	109
30	1,073,741,824	113
31	2,147,483,648	127
32	4,294,967,296	131

Pascal's Triangle

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
1 9 36 84 126 126 84 36 9 1
1 10 45 120 210 252 210 120 45 10 1

General

Bernoulli Numbers ($B_i = 0$, odd $i \neq 1$):

$$B_0 = 1,\ B_1 = -\tfrac{1}{2},\ B_2 = \tfrac{1}{6},\ B_4 = -\tfrac{1}{30},\ B_6 = \tfrac{1}{42},\ B_8 = -\tfrac{1}{30},\ B_{10} = \tfrac{5}{66}.$$

Change of base, quadratic formula:

$$\log_b x = \frac{\log_a x}{\log_a b}, \qquad \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Euler's number e:

$$e = 1 + \tfrac{1}{2} + \tfrac{1}{6} + \tfrac{1}{24} + \tfrac{1}{120} + \cdots$$

$$\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x.$$

$$\left(1 + \tfrac{1}{n}\right)^n < e < \left(1 + \tfrac{1}{n}\right)^{n+1}.$$

$$\left(1 + \tfrac{1}{n}\right)^n = e - \frac{e}{2n} + \frac{11e}{24n^2} - O\left(\frac{1}{n^3}\right).$$

Harmonic numbers:

$$1,\ \tfrac{3}{2},\ \tfrac{11}{6},\ \tfrac{25}{12},\ \tfrac{137}{60},\ \tfrac{49}{20},\ \tfrac{363}{140},\ \tfrac{761}{280},\ \tfrac{7129}{2520},\ \ldots$$

$$\ln n < H_n < \ln n + 1,$$

$$H_n = \ln n + \gamma + O\left(\frac{1}{n}\right).$$

Factorial, Stirling's approximation:

1, 2, 6, 24, 120, 720, 5040, 40320, 362880, ...

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \left(1 + \Theta\left(\frac{1}{n}\right)\right).$$

Ackermann's function and inverse:

$$a(i,j) = \begin{cases} 2^j & i = 1 \\ a(i-1, 2) & j = 1 \\ a(i-1, a(i, j-1)) & i, j \geq 2 \end{cases}$$

$$\alpha(i) = \min\{j \mid a(j,j) \geq i\}.$$

Binomial distribution:

$$\Pr[X = k] = \binom{n}{k} p^k q^{n-k}, \qquad q = 1 - p,$$

$$\mathrm{E}[X] = \sum_{k=1}^{n} k \binom{n}{k} p^k q^{n-k} = np.$$

Poisson distribution:

$$\Pr[X = k] = \frac{e^{-\lambda}\lambda^k}{k!}, \quad \mathrm{E}[X] = \lambda.$$

Normal (Gaussian) distribution:

$$p(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2}, \quad \mathrm{E}[X] = \mu.$$

The "coupon collector": We are given a random coupon each day, and there are n different types of coupons. The distribution of coupons is uniform. The expected number of days to pass before we to collect all n types is

$$nH_n.$$

Probability

Continuous distributions: If

$$\Pr[a < X < b] = \int_a^b p(x)\,dx,$$

then p is the probability density function of X. If

$$\Pr[X < a] = P(a),$$

then P is the distribution function of X. If P and p both exist then

$$P(a) = \int_{-\infty}^{a} p(x)\,dx.$$

Expectation: If X is discrete

$$\mathrm{E}[g(X)] = \sum_x g(x) \Pr[X = x].$$

If X continuous then

$$\mathrm{E}[g(X)] = \int_{-\infty}^{\infty} g(x)p(x)\,dx = \int_{-\infty}^{\infty} g(x)\,dP(x).$$

Variance, standard deviation:

$$\mathrm{VAR}[X] = \mathrm{E}[X^2] - \mathrm{E}[X]^2,$$

$$\sigma = \sqrt{\mathrm{VAR}[X]}.$$

For events A and B:

$$\Pr[A \vee B] = \Pr[A] + \Pr[B] - \Pr[A \wedge B]$$

$$\Pr[A \wedge B] = \Pr[A] \cdot \Pr[B],$$

iff A and B are independent.

$$\Pr[A|B] = \frac{\Pr[A \wedge B]}{\Pr[B]}$$

For random variables X and Y:

$$\mathrm{E}[X \cdot Y] = \mathrm{E}[X] \cdot \mathrm{E}[Y],$$

if X and Y are independent.

$$\mathrm{E}[X + Y] = \mathrm{E}[X] + \mathrm{E}[Y],$$

$$\mathrm{E}[cX] = c\,\mathrm{E}[X].$$

Bayes' theorem:

$$\Pr[A_i|B] = \frac{\Pr[B|A_i]\Pr[A_i]}{\sum_{j=1}^{n} \Pr[A_j]\Pr[B|A_j]}.$$

Inclusion-exclusion:

$$\Pr\left[\bigvee_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} \Pr[X_i] + \sum_{k=2}^{n} (-1)^{k+1} \sum_{i_i < \cdots < i_k} \Pr\left[\bigwedge_{j=1}^{k} X_{i_j}\right].$$

Moment inequalities:

$$\Pr\left[|X| \geq \lambda\,\mathrm{E}[X]\right] \leq \frac{1}{\lambda},$$

$$\Pr\left[\left|X - \mathrm{E}[X]\right| \geq \lambda \cdot \sigma\right] \leq \frac{1}{\lambda^2}.$$

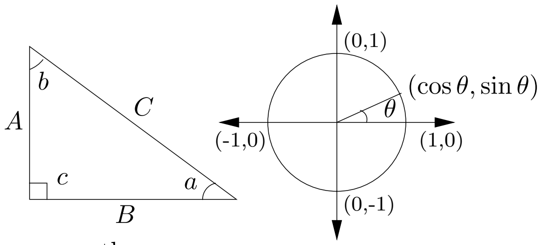
Geometric distribution:

$$\Pr[X = k] = pq^{k-1}, \qquad q = 1 - p,$$

$$\mathrm{E}[X] = \sum_{k=1}^{\infty} kpq^{k-1} = \frac{1}{p}.$$

Theoretical Computer Science Cheat Sheet

Trigonometry



Pythagorean theorem:
$$C^2 = A^2 + B^2.$$

Definitions:
$$\sin a = A/C, \quad \cos a = B/C,$$
$$\csc a = C/A, \quad \sec a = C/B,$$
$$\tan a = \frac{\sin a}{\cos a} = \frac{A}{B}, \quad \cot a = \frac{\cos a}{\sin a} = \frac{B}{A}.$$

Area, radius of inscribed circle:
$$\tfrac{1}{2}AB, \quad \frac{AB}{A+B+C}.$$

Identities:
$$\sin x = \frac{1}{\csc x}, \qquad \cos x = \frac{1}{\sec x},$$
$$\tan x = \frac{1}{\cot x}, \qquad \sin^2 x + \cos^2 x = 1,$$
$$1 + \tan^2 x = \sec^2 x, \qquad 1 + \cot^2 x = \csc^2 x,$$
$$\sin x = \cos\left(\tfrac{\pi}{2} - x\right), \qquad \sin x = \sin(\pi - x),$$
$$\cos x = -\cos(\pi - x), \qquad \tan x = \cot\left(\tfrac{\pi}{2} - x\right),$$
$$\cot x = -\cot(\pi - x), \qquad \csc x = \cot\tfrac{x}{2} - \cot x,$$
$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y,$$
$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y,$$
$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y},$$
$$\cot(x \pm y) = \frac{\cot x \cot y \mp 1}{\cot x \pm \cot y},$$
$$\sin 2x = 2 \sin x \cos x, \qquad \sin 2x = \frac{2 \tan x}{1 + \tan^2 x},$$
$$\cos 2x = \cos^2 x - \sin^2 x, \qquad \cos 2x = 2\cos^2 x - 1,$$
$$\cos 2x = 1 - 2\sin^2 x, \qquad \cos 2x = \frac{1 - \tan^2 x}{1 + \tan^2 x},$$
$$\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}, \qquad \cot 2x = \frac{\cot^2 x - 1}{2 \cot x},$$
$$\sin(x+y)\sin(x-y) = \sin^2 x - \sin^2 y,$$
$$\cos(x+y)\cos(x-y) = \cos^2 x - \sin^2 y.$$

Euler's equation:
$$e^{ix} = \cos x + i \sin x, \qquad e^{i\pi} = -1.$$

sseiden@acm.org
http://www.csc.lsu.edu/~seiden

Matrices

Multiplication:
$$C = A \cdot B, \quad c_{i,j} = \sum_{k=1}^{n} a_{i,k} b_{k,j}.$$

Determinants: $\det A \neq 0$ iff A is non-singular.
$$\det A \cdot B = \det A \cdot \det B,$$
$$\det A = \sum_{\pi} \prod_{i=1}^{n} \operatorname{sign}(\pi) a_{i,\pi(i)}.$$

2×2 and 3×3 determinant:
$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc,$$
$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = g \begin{vmatrix} b & c \\ e & f \end{vmatrix} - h \begin{vmatrix} a & c \\ d & f \end{vmatrix} + i \begin{vmatrix} a & b \\ d & e \end{vmatrix}$$
$$= \begin{array}{c} aei + bfg + cdh \\ - ceg - fha - ibd. \end{array}$$

Permanents:
$$\operatorname{perm} A = \sum_{\pi} \prod_{i=1}^{n} a_{i,\pi(i)}.$$

Hyperbolic Functions

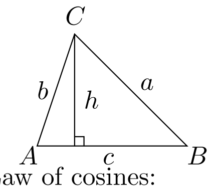
Definitions:
$$\sinh x = \frac{e^x - e^{-x}}{2}, \qquad \cosh x = \frac{e^x + e^{-x}}{2},$$
$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}, \qquad \operatorname{csch} x = \frac{1}{\sinh x},$$
$$\operatorname{sech} x = \frac{1}{\cosh x}, \qquad \coth x = \frac{1}{\tanh x}.$$

Identities:
$$\cosh^2 x - \sinh^2 x = 1, \qquad \tanh^2 x + \operatorname{sech}^2 x = 1,$$
$$\coth^2 x - \operatorname{csch}^2 x = 1, \qquad \sinh(-x) = -\sinh x,$$
$$\cosh(-x) = \cosh x, \qquad \tanh(-x) = -\tanh x,$$
$$\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y,$$
$$\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y,$$
$$\sinh 2x = 2 \sinh x \cosh x,$$
$$\cosh 2x = \cosh^2 x + \sinh^2 x,$$
$$\cosh x + \sinh x = e^x, \qquad \cosh x - \sinh x = e^{-x},$$
$$(\cosh x + \sinh x)^n = \cosh nx + \sinh nx, \quad n \in \mathbb{Z},$$
$$2\sinh^2 \tfrac{x}{2} = \cosh x - 1, \qquad 2\cosh^2 \tfrac{x}{2} = \cosh x + 1.$$

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$
0	0	1	0
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
$\frac{\pi}{2}$	1	0	∞

... in mathematics you don't understand things, you just get used to them.
– J. von Neumann

More Trig.



Law of cosines:
$$c^2 = a^2 + b^2 - 2ab\cos C.$$

Area:
$$A = \tfrac{1}{2}hc,$$
$$= \tfrac{1}{2}ab\sin C,$$
$$= \frac{c^2 \sin A \sin B}{2 \sin C}.$$

Heron's formula:
$$A = \sqrt{s \cdot s_a \cdot s_b \cdot s_c},$$
$$s = \tfrac{1}{2}(a+b+c),$$
$$s_a = s - a,$$
$$s_b = s - b,$$
$$s_c = s - c.$$

More identities:
$$\sin\tfrac{x}{2} = \sqrt{\frac{1 - \cos x}{2}},$$
$$\cos\tfrac{x}{2} = \sqrt{\frac{1 + \cos x}{2}},$$
$$\tan\tfrac{x}{2} = \sqrt{\frac{1 - \cos x}{1 + \cos x}},$$
$$= \frac{1 - \cos x}{\sin x},$$
$$= \frac{\sin x}{1 + \cos x},$$
$$\cot\tfrac{x}{2} = \sqrt{\frac{1 + \cos x}{1 - \cos x}},$$
$$= \frac{1 + \cos x}{\sin x},$$
$$= \frac{\sin x}{1 - \cos x},$$
$$\sin x = \frac{e^{ix} - e^{-ix}}{2i},$$
$$\cos x = \frac{e^{ix} + e^{-ix}}{2},$$
$$\tan x = -i\frac{e^{ix} - e^{-ix}}{e^{ix} + e^{-ix}},$$
$$= -i\frac{e^{2ix} - 1}{e^{2ix} + 1},$$
$$\sin x = \frac{\sinh ix}{i},$$
$$\cos x = \cosh ix,$$
$$\tan x = \frac{\tanh ix}{i}.$$

Theoretical Computer Science Cheat Sheet

Number Theory

The Chinese remainder theorem: There exists a number C such that:

$$\begin{aligned} C &\equiv r_1 \bmod m_1 \\ \vdots \quad &\vdots \quad \vdots \\ C &\equiv r_n \bmod m_n \end{aligned}$$

if m_i and m_j are relatively prime for $i \neq j$.

Euler's function: $\phi(x)$ is the number of positive integers less than x relatively prime to x. If $\prod_{i=1}^{n} p_i^{e_i}$ is the prime factorization of x then

$$\phi(x) = \prod_{i=1}^{n} p_i^{e_i - 1}(p_i - 1).$$

Euler's theorem: If a and b are relatively prime then

$$1 \equiv a^{\phi(b)} \bmod b.$$

Fermat's theorem:

$$1 \equiv a^{p-1} \bmod p.$$

The Euclidean algorithm: if $a > b$ are integers then

$$\gcd(a, b) = \gcd(a \bmod b, b).$$

If $\prod_{i=1}^{n} p_i^{e_i}$ is the prime factorization of x then

$$S(x) = \sum_{d|x} d = \prod_{i=1}^{n} \frac{p_i^{e_i+1} - 1}{p_i - 1}.$$

Perfect Numbers: x is an even perfect number iff $x = 2^{n-1}(2^n - 1)$ and $2^n - 1$ is prime.

Wilson's theorem: n is a prime iff

$$(n-1)! \equiv -1 \bmod n.$$

Möbius inversion:

$$\mu(i) = \begin{cases} 1 & \text{if } i = 1. \\ 0 & \text{if } i \text{ is not square-free.} \\ (-1)^r & \text{if } i \text{ is the product of} \\ & r \text{ distinct primes.} \end{cases}$$

If

$$G(a) = \sum_{d|a} F(d),$$

then

$$F(a) = \sum_{d|a} \mu(d) G\Big(\frac{a}{d}\Big).$$

Prime numbers:

$$p_n = n \ln n + n \ln \ln n - n + n \frac{\ln \ln n}{\ln n} + O\Big(\frac{n}{\ln n}\Big),$$

$$\pi(n) = \frac{n}{\ln n} + \frac{n}{(\ln n)^2} + \frac{2!n}{(\ln n)^3} + O\Big(\frac{n}{(\ln n)^4}\Big).$$

Graph Theory

Definitions:

Loop	An edge connecting a vertex to itself.
Directed	Each edge has a direction.
Simple	Graph with no loops or multi-edges.
Walk	A sequence $v_0 e_1 v_1 \ldots e_\ell v_\ell$.
Trail	A walk with distinct edges.
Path	A trail with distinct vertices.
Connected	A graph where there exists a path between any two vertices.
Component	A maximal connected subgraph.
Tree	A connected acyclic graph.
Free tree	A tree with no root.
DAG	Directed acyclic graph.
Eulerian	Graph with a trail visiting each edge exactly once.
Hamiltonian	Graph with a cycle visiting each vertex exactly once.
Cut	A set of edges whose removal increases the number of components.
Cut-set	A minimal cut.
Cut edge	A size 1 cut.
k-Connected	A graph connected with the removal of any $k-1$ vertices.
k-Tough	$\forall S \subseteq V, S \neq \emptyset$ we have $k \cdot c(G - S) \leq \lvert S \rvert$.
k-Regular	A graph where all vertices have degree k.
k-Factor	A k-regular spanning subgraph.
Matching	A set of edges, no two of which are adjacent.
Clique	A set of vertices, all of which are adjacent.
Ind. set	A set of vertices, none of which are adjacent.
Vertex cover	A set of vertices which cover all edges.
Planar graph	A graph which can be embeded in the plane.
Plane graph	An embedding of a planar graph.

$$\sum_{v \in V} \deg(v) = 2m.$$

If G is planar then $n - m + f = 2$, so

$$f \leq 2n - 4, \quad m \leq 3n - 6.$$

Any planar graph has a vertex with degree ≤ 5.

Notation:

$E(G)$	Edge set
$V(G)$	Vertex set
$c(G)$	Number of components
$G[S]$	Induced subgraph
$\deg(v)$	Degree of v
$\Delta(G)$	Maximum degree
$\delta(G)$	Minimum degree
$\chi(G)$	Chromatic number
$\chi_E(G)$	Edge chromatic number
G^c	Complement graph
K_n	Complete graph
K_{n_1,n_2}	Complete bipartite graph
$\mathrm{r}(k, \ell)$	Ramsey number

Geometry

Projective coordinates: triples (x, y, z), not all x, y and z zero.

$$(x, y, z) = (cx, cy, cz) \quad \forall c \neq 0.$$

Cartesian	Projective
(x, y)	$(x, y, 1)$
$y = mx + b$	$(m, -1, b)$
$x = c$	$(1, 0, -c)$

Distance formula, L_p and L_∞ metric:

$$\sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2},$$

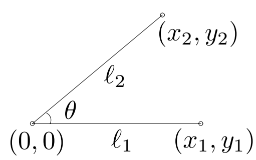
$$\big[|x_1 - x_0|^p + |y_1 - y_0|^p\big]^{1/p},$$

$$\lim_{p \to \infty} \big[|x_1 - x_0|^p + |y_1 - y_0|^p\big]^{1/p}.$$

Area of triangle (x_0, y_0), (x_1, y_1) and (x_2, y_2):

$$\tfrac{1}{2} \operatorname{abs} \begin{vmatrix} x_1 - x_0 & y_1 - y_0 \\ x_2 - x_0 & y_2 - y_0 \end{vmatrix}.$$

Angle formed by three points:



$$\cos\theta = \frac{(x_1, y_1) \cdot (x_2, y_2)}{\ell_1 \ell_2}.$$

Line through two points (x_0, y_0) and (x_1, y_1):

$$\begin{vmatrix} x & y & 1 \\ x_0 & y_0 & 1 \\ x_1 & y_1 & 1 \end{vmatrix} = 0.$$

Area of circle, volume of sphere:

$$A = \pi r^2, \qquad V = \tfrac{4}{3}\pi r^3.$$

If I have seen farther than others, it is because I have stood on the shoulders of giants.
– Issac Newton

Theoretical Computer Science Cheat Sheet

π

Wallis' identity:

$$\pi = 2 \cdot \frac{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdots}{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdots}$$

Brouncker's continued fraction expansion:

$$\tfrac{\pi}{4} = 1 + \cfrac{1^2}{2 + \cfrac{3^2}{2 + \cfrac{5^2}{2 + \cfrac{7^2}{2 + \cdots}}}}$$

Gregrory's series:

$$\tfrac{\pi}{4} = 1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \tfrac{1}{9} - \cdots$$

Newton's series:

$$\tfrac{\pi}{6} = \frac{1}{2} + \frac{1}{2 \cdot 3 \cdot 2^3} + \frac{1 \cdot 3}{2 \cdot 4 \cdot 5 \cdot 2^5} + \cdots$$

Sharp's series:

$$\tfrac{\pi}{6} = \frac{1}{\sqrt{3}}\left(1 - \frac{1}{3^1 \cdot 3} + \frac{1}{3^2 \cdot 5} - \frac{1}{3^3 \cdot 7} + \cdots\right)$$

Euler's series:

$$\tfrac{\pi^2}{6} = \tfrac{1}{1^2} + \tfrac{1}{2^2} + \tfrac{1}{3^2} + \tfrac{1}{4^2} + \tfrac{1}{5^2} + \cdots$$
$$\tfrac{\pi^2}{8} = \tfrac{1}{1^2} + \tfrac{1}{3^2} + \tfrac{1}{5^2} + \tfrac{1}{7^2} + \tfrac{1}{9^2} + \cdots$$
$$\tfrac{\pi^2}{12} = \tfrac{1}{1^2} - \tfrac{1}{2^2} + \tfrac{1}{3^2} - \tfrac{1}{4^2} + \tfrac{1}{5^2} - \cdots$$

Partial Fractions

Let $N(x)$ and $D(x)$ be polynomial functions of x. We can break down $N(x)/D(x)$ using partial fraction expansion. First, if the degree of N is greater than or equal to the degree of D, divide N by D, obtaining

$$\frac{N(x)}{D(x)} = Q(x) + \frac{N'(x)}{D(x)},$$

where the degree of N' is less than that of D. Second, factor $D(x)$. Use the following rules: For a non-repeated factor:

$$\frac{N(x)}{(x-a)D(x)} = \frac{A}{x-a} + \frac{N'(x)}{D(x)},$$

where

$$A = \left[\frac{N(x)}{D(x)}\right]_{x=a}.$$

For a repeated factor:

$$\frac{N(x)}{(x-a)^m D(x)} = \sum_{k=0}^{m-1} \frac{A_k}{(x-a)^{m-k}} + \frac{N'(x)}{D(x)},$$

where

$$A_k = \frac{1}{k!}\left[\frac{d^k}{dx^k}\left(\frac{N(x)}{D(x)}\right)\right]_{x=a}.$$

The reasonable man adapts himself to the world; the unreasonable persists in trying to adapt the world to himself. Therefore all progress depends on the unreasonable.
– George Bernard Shaw

Calculus

Derivatives:

1. $\dfrac{d(cu)}{dx} = c\dfrac{du}{dx}$, **2.** $\dfrac{d(u+v)}{dx} = \dfrac{du}{dx} + \dfrac{dv}{dx}$, **3.** $\dfrac{d(uv)}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$,

4. $\dfrac{d(u^n)}{dx} = nu^{n-1}\dfrac{du}{dx}$, **5.** $\dfrac{d(u/v)}{dx} = \dfrac{v\left(\frac{du}{dx}\right) - u\left(\frac{dv}{dx}\right)}{v^2}$, **6.** $\dfrac{d(e^{cu})}{dx} = ce^{cu}\dfrac{du}{dx}$,

7. $\dfrac{d(c^u)}{dx} = (\ln c)c^u\dfrac{du}{dx}$, **8.** $\dfrac{d(\ln u)}{dx} = \dfrac{1}{u}\dfrac{du}{dx}$,

9. $\dfrac{d(\sin u)}{dx} = \cos u\dfrac{du}{dx}$, **10.** $\dfrac{d(\cos u)}{dx} = -\sin u\dfrac{du}{dx}$,

11. $\dfrac{d(\tan u)}{dx} = \sec^2 u\dfrac{du}{dx}$, **12.** $\dfrac{d(\cot u)}{dx} = \csc^2 u\dfrac{du}{dx}$,

13. $\dfrac{d(\sec u)}{dx} = \tan u \sec u\dfrac{du}{dx}$, **14.** $\dfrac{d(\csc u)}{dx} = -\cot u \csc u\dfrac{du}{dx}$,

15. $\dfrac{d(\arcsin u)}{dx} = \dfrac{1}{\sqrt{1-u^2}}\dfrac{du}{dx}$, **16.** $\dfrac{d(\arccos u)}{dx} = \dfrac{-1}{\sqrt{1-u^2}}\dfrac{du}{dx}$,

17. $\dfrac{d(\arctan u)}{dx} = \dfrac{1}{1+u^2}\dfrac{du}{dx}$, **18.** $\dfrac{d(\operatorname{arccot} u)}{dx} = \dfrac{-1}{1+u^2}\dfrac{du}{dx}$,

19. $\dfrac{d(\operatorname{arcsec} u)}{dx} = \dfrac{1}{u\sqrt{1-u^2}}\dfrac{du}{dx}$, **20.** $\dfrac{d(\operatorname{arccsc} u)}{dx} = \dfrac{-1}{u\sqrt{1-u^2}}\dfrac{du}{dx}$,

21. $\dfrac{d(\sinh u)}{dx} = \cosh u\dfrac{du}{dx}$, **22.** $\dfrac{d(\cosh u)}{dx} = \sinh u\dfrac{du}{dx}$,

23. $\dfrac{d(\tanh u)}{dx} = \operatorname{sech}^2 u\dfrac{du}{dx}$, **24.** $\dfrac{d(\coth u)}{dx} = -\operatorname{csch}^2 u\dfrac{du}{dx}$,

25. $\dfrac{d(\operatorname{sech} u)}{dx} = -\operatorname{sech} u \tanh u\dfrac{du}{dx}$, **26.** $\dfrac{d(\operatorname{csch} u)}{dx} = -\operatorname{csch} u \coth u\dfrac{du}{dx}$,

27. $\dfrac{d(\operatorname{arcsinh} u)}{dx} = \dfrac{1}{\sqrt{1+u^2}}\dfrac{du}{dx}$, **28.** $\dfrac{d(\operatorname{arccosh} u)}{dx} = \dfrac{1}{\sqrt{u^2-1}}\dfrac{du}{dx}$,

29. $\dfrac{d(\operatorname{arctanh} u)}{dx} = \dfrac{1}{1-u^2}\dfrac{du}{dx}$, **30.** $\dfrac{d(\operatorname{arccoth} u)}{dx} = \dfrac{1}{u^2-1}\dfrac{du}{dx}$,

31. $\dfrac{d(\operatorname{arcsech} u)}{dx} = \dfrac{-1}{u\sqrt{1-u^2}}\dfrac{du}{dx}$, **32.** $\dfrac{d(\operatorname{arccsch} u)}{dx} = \dfrac{-1}{|u|\sqrt{1+u^2}}\dfrac{du}{dx}$.

Integrals:

1. $\displaystyle\int cu\,dx = c\int u\,dx$, **2.** $\displaystyle\int (u+v)\,dx = \int u\,dx + \int v\,dx$,

3. $\displaystyle\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1$, **4.** $\displaystyle\int \frac{1}{x}dx = \ln x$, **5.** $\displaystyle\int e^x\,dx = e^x$,

6. $\displaystyle\int \frac{dx}{1+x^2} = \arctan x$, **7.** $\displaystyle\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$,

8. $\displaystyle\int \sin x\,dx = -\cos x$, **9.** $\displaystyle\int \cos x\,dx = \sin x$,

10. $\displaystyle\int \tan x\,dx = -\ln|\cos x|$, **11.** $\displaystyle\int \cot x\,dx = \ln|\cos x|$,

12. $\displaystyle\int \sec x\,dx = \ln|\sec x + \tan x|$, **13.** $\displaystyle\int \csc x\,dx = \ln|\csc x + \cot x|$,

14. $\displaystyle\int \arcsin\tfrac{x}{a}dx = \arcsin\tfrac{x}{a} + \sqrt{a^2 - x^2}, \quad a > 0$, Calculus Cont.

15. $\int \arccos\frac{x}{a}dx = \arccos\frac{x}{a} - \sqrt{a^2-x^2}, \quad a>0,$

16. $\int \arctan\frac{x}{a}dx = x\arctan\frac{x}{a} - \frac{a}{2}\ln(a^2+x^2), \quad a>0,$

17. $\int \sin^2(ax)dx = \frac{1}{2a}\big(ax - \sin(ax)\cos(ax)\big),$

18. $\int \cos^2(ax)dx = \frac{1}{2a}\big(ax + \sin(ax)\cos(ax)\big),$

19. $\int \sec^2 x\,dx = \tan x,$

20. $\int \csc^2 x\,dx = -\cot x,$

21. $\int \sin^n x\,dx = -\frac{\sin^{n-1}x\cos x}{n} + \frac{n-1}{n}\int \sin^{n-2}x\,dx,$

22. $\int \cos^n x\,dx = \frac{\cos^{n-1}x\sin x}{n} + \frac{n-1}{n}\int \cos^{n-2}x\,dx,$

23. $\int \tan^n x\,dx = \frac{\tan^{n-1}x}{n-1} - \int \tan^{n-2}x\,dx, \quad n\neq 1,$

24. $\int \cot^n x\,dx = -\frac{\cot^{n-1}x}{n-1} - \int \cot^{n-2}x\,dx, \quad n\neq 1,$

25. $\int \sec^n x\,dx = \frac{\tan x\sec^{n-1}x}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2}x\,dx, \quad n\neq 1,$

26. $\int \csc^n x\,dx = -\frac{\cot x\csc^{n-1}x}{n-1} + \frac{n-2}{n-1}\int \csc^{n-2}x\,dx, \quad n\neq 1,$

27. $\int \sinh x\,dx = \cosh x,$

28. $\int \cosh x\,dx = \sinh x,$

29. $\int \tanh x\,dx = \ln|\cosh x|,$

30. $\int \coth x\,dx = \ln|\sinh x|,$

31. $\int \operatorname{sech} x\,dx = \arctan\sinh x,$

32. $\int \operatorname{csch} x\,dx = \ln\left|\tanh\frac{x}{2}\right|,$

33. $\int \sinh^2 x\,dx = \frac{1}{4}\sinh(2x) - \frac{1}{2}x,$

34. $\int \cosh^2 x\,dx = \frac{1}{4}\sinh(2x) + \frac{1}{2}x,$

35. $\int \operatorname{sech}^2 x\,dx = \tanh x,$

36. $\int \operatorname{arcsinh}\frac{x}{a}dx = x\operatorname{arcsinh}\frac{x}{a} - \sqrt{x^2+a^2}, \quad a>0,$

37. $\int \operatorname{arctanh}\frac{x}{a}dx = x\operatorname{arctanh}\frac{x}{a} + \frac{a}{2}\ln|a^2-x^2|,$

38. $$\int \operatorname{arccosh}\frac{x}{a}dx = \begin{cases} x\operatorname{arccosh}\dfrac{x}{a} - \sqrt{x^2+a^2}, & \text{if } \operatorname{arccosh}\frac{x}{a} > 0 \text{ and } a>0, \\ x\operatorname{arccosh}\dfrac{x}{a} + \sqrt{x^2+a^2}, & \text{if } \operatorname{arccosh}\frac{x}{a} < 0 \text{ and } a>0, \end{cases}$$

39. $\int \frac{dx}{\sqrt{a^2+x^2}} = \ln\left(x + \sqrt{a^2+x^2}\right), \quad a>0,$

40. $\int \frac{dx}{a^2+x^2} = \frac{1}{a}\arctan\frac{x}{a}, \quad a>0,$

41. $\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\arcsin\frac{x}{a}, \quad a>0,$

42. $\int (a^2-x^2)^{3/2}dx = \frac{x}{8}(5a^2-2x^2)\sqrt{a^2-x^2} + \frac{3a^4}{8}\arcsin\frac{x}{a}, \quad a>0,$

43. $\int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin\frac{x}{a}, \quad a>0,$

44. $\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right|,$

45. $\int \frac{dx}{(a^2-x^2)^{3/2}} = \frac{x}{a^2\sqrt{a^2-x^2}},$

46. $\int \sqrt{a^2\pm x^2}\,dx = \frac{x}{2}\sqrt{a^2\pm x^2} \pm \frac{a^2}{2}\ln\left|x + \sqrt{a^2\pm x^2}\right|,$

47. $\int \frac{dx}{\sqrt{x^2-a^2}} = \ln\left|x + \sqrt{x^2-a^2}\right|, \quad a>0,$

48. $\int \frac{dx}{ax^2+bx} = \frac{1}{a}\ln\left|\frac{x}{a+bx}\right|,$

49. $\int x\sqrt{a+bx}\,dx = \frac{2(3bx-2a)(a+bx)^{3/2}}{15b^2},$

50. $\int \frac{\sqrt{a+bx}}{x}dx = 2\sqrt{a+bx} + a\int \frac{1}{x\sqrt{a+bx}}dx,$

51. $\int \frac{x}{\sqrt{a+bx}}dx = \frac{1}{\sqrt{2}}\ln\left|\frac{\sqrt{a+bx}-\sqrt{a}}{\sqrt{a+bx}+\sqrt{a}}\right|, \quad a>0,$

52. $\int \frac{\sqrt{a^2-x^2}}{x}dx = \sqrt{a^2-x^2} - a\ln\left|\frac{a+\sqrt{a^2-x^2}}{x}\right|,$

53. $\int x\sqrt{a^2-x^2}\,dx = -\frac{1}{3}(a^2-x^2)^{3/2},$

54. $\int x^2\sqrt{a^2-x^2}\,dx = \frac{x}{8}(2x^2-a^2)\sqrt{a^2-x^2} + \frac{a^4}{8}\arcsin\frac{x}{a}, \quad a>0,$

55. $\int \frac{dx}{\sqrt{a^2-x^2}} = -\frac{1}{a}\ln\left|\frac{a+\sqrt{a^2-x^2}}{x}\right|,$

56. $\int \frac{x\,dx}{\sqrt{a^2-x^2}} = -\sqrt{a^2-x^2},$

57. $\int \frac{x^2\,dx}{\sqrt{a^2-x^2}} = -\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\arcsin\frac{x}{a}, \quad a>0,$

58. $\int \frac{\sqrt{a^2+x^2}}{x}dx = \sqrt{a^2+x^2} - a\ln\left|\frac{a+\sqrt{a^2+x^2}}{x}\right|,$

59. $\int \frac{\sqrt{x^2-a^2}}{x}dx = \sqrt{x^2-a^2} - a\arccos\frac{a}{|x|}, \quad a>0,$

60. $\int x\sqrt{x^2\pm a^2}\,dx = \frac{1}{3}(x^2\pm a^2)^{3/2},$

61. $\int \frac{dx}{x\sqrt{x^2+a^2}} = \frac{1}{a}\ln\left|\frac{x}{a+\sqrt{a^2+x^2}}\right|,$

Theoretical Computer Science Cheat Sheet

Calculus Cont.

62. $\displaystyle\int \frac{dx}{x\sqrt{x^2-a^2}} = \tfrac{1}{a}\arccos\tfrac{a}{|x|}, \quad a>0,$

63. $\displaystyle\int \frac{dx}{x^2\sqrt{x^2\pm a^2}} = \mp\frac{\sqrt{x^2\pm a^2}}{a^2x},$

64. $\displaystyle\int \frac{x\,dx}{\sqrt{x^2\pm a^2}} = \sqrt{x^2\pm a^2},$

65. $\displaystyle\int \frac{\sqrt{x^2\pm a^2}}{x^4}\,dx = \mp\frac{(x^2+a^2)^{3/2}}{3a^2x^3},$

66. $$\int \frac{dx}{ax^2+bx+c} = \begin{cases} \dfrac{1}{\sqrt{b^2-4ac}}\ln\left|\dfrac{2ax+b-\sqrt{b^2-4ac}}{2ax+b+\sqrt{b^2-4ac}}\right|, & \text{if } b^2>4ac,\\[2ex] \dfrac{2}{\sqrt{4ac-b^2}}\arctan\dfrac{2ax+b}{\sqrt{4ac-b^2}}, & \text{if } b^2<4ac,\end{cases}$$

67. $$\int \frac{dx}{\sqrt{ax^2+bx+c}} = \begin{cases} \dfrac{1}{\sqrt{a}}\ln\left|2ax+b+2\sqrt{a}\sqrt{ax^2+bx+c}\right|, & \text{if } a>0,\\[2ex] \dfrac{1}{\sqrt{-a}}\arcsin\dfrac{-2ax-b}{\sqrt{b^2-4ac}}, & \text{if } a<0,\end{cases}$$

68. $$\int\sqrt{ax^2+bx+c}\,dx = \frac{2ax+b}{4a}\sqrt{ax^2+bx+c}+\frac{4ax-b^2}{8a}\int\frac{dx}{\sqrt{ax^2+bx+c}},$$

69. $$\int\frac{x\,dx}{\sqrt{ax^2+bx+c}} = \frac{\sqrt{ax^2+bx+c}}{a}-\frac{b}{2a}\int\frac{dx}{\sqrt{ax^2+bx+c}},$$

70. $$\int \frac{dx}{x\sqrt{ax^2+bx+c}} = \begin{cases} \dfrac{-1}{\sqrt{c}}\ln\left|\dfrac{2\sqrt{c}\sqrt{ax^2+bx+c}+bx+2c}{x}\right|, & \text{if } c>0,\\[2ex] \dfrac{1}{\sqrt{-c}}\arcsin\dfrac{bx+2c}{|x|\sqrt{b^2-4ac}}, & \text{if } c<0,\end{cases}$$

71. $\displaystyle\int x^3\sqrt{x^2+a^2}\,dx = (\tfrac{1}{3}x^2-\tfrac{2}{15}a^2)(x^2+a^2)^{3/2},$

72. $\displaystyle\int x^n\sin(ax)\,dx = -\tfrac{1}{a}x^n\cos(ax)+\tfrac{n}{a}\int x^{n-1}\cos(ax)\,dx,$

73. $\displaystyle\int x^n\cos(ax)\,dx = \tfrac{1}{a}x^n\sin(ax)-\tfrac{n}{a}\int x^{n-1}\sin(ax)\,dx,$

74. $\displaystyle\int x^ne^{ax}\,dx = \frac{x^ne^{ax}}{a}-\tfrac{n}{a}\int x^{n-1}e^{ax}\,dx,$

75. $\displaystyle\int x^n\ln(ax)\,dx = x^{n+1}\left(\frac{\ln(ax)}{n+1}-\frac{1}{(n+1)^2}\right),$

76. $\displaystyle\int x^n(\ln ax)^m\,dx = \frac{x^{n+1}}{n+1}(\ln ax)^m-\frac{m}{n+1}\int x^n(\ln ax)^{m-1}\,dx.$

$$\begin{array}{lclcl}
x^1 = & x^{\underline{1}} & = & x^{\overline{1}}\\
x^2 = & x^{\underline{2}}+x^{\underline{1}} & = & x^{\overline{2}}-x^{\overline{1}}\\
x^3 = & x^{\underline{3}}+3x^{\underline{2}}+x^{\underline{1}} & = & x^{\overline{3}}-3x^{\overline{2}}+x^{\overline{1}}\\
x^4 = & x^{\underline{4}}+6x^{\underline{3}}+7x^{\underline{2}}+x^{\underline{1}} & = & x^{\overline{4}}-6x^{\overline{3}}+7x^{\overline{2}}-x^{\overline{1}}\\
x^5 = & x^{\underline{5}}+15x^{\underline{4}}+25x^{\underline{3}}+10x^{\underline{2}}+x^{\underline{1}} & = & x^{\underline{5}}-15x^{\overline{4}}+25x^{\overline{3}}-10x^{\overline{2}}+x^{\overline{1}}
\end{array}$$

$$\begin{array}{lclc}
x^{\overline{1}} = & x^1 & x^{\underline{1}} = & x^1\\
x^{\overline{2}} = & x^2+x^1 & x^{\underline{2}} = & x^2-x^1\\
x^{\overline{3}} = & x^3+3x^2+2x^1 & x^{\underline{3}} = & x^3-3x^2+2x^1\\
x^{\overline{4}} = & x^4+6x^3+11x^2+6x^1 & x^{\underline{4}} = & x^4-6x^3+11x^2-6x^1\\
x^{\overline{5}} = & x^5+10x^4+35x^3+50x^2+24x^1 & x^{\underline{5}} = & x^5-10x^4+35x^3-50x^2+24x^1
\end{array}$$

Finite Calculus

Difference, shift operators:

$$\Delta f(x) = f(x+1)-f(x),$$
$$\mathrm{E}\,f(x) = f(x+1).$$

Fundamental Theorem:

$$f(x)=\Delta F(x) \Leftrightarrow \sum f(x)\delta x = F(x)+C.$$
$$\sum_a^b f(x)\delta x = \sum_{i=a}^{b-1} f(i).$$

Differences:

$\Delta(cu)=c\Delta u, \qquad \Delta(u+v)=\Delta u+\Delta v,$

$\Delta(uv)=u\Delta v+\mathrm{E}\,v\Delta u,$

$\Delta(x^{\underline{n}})=nx^{\underline{n-1}},$

$\Delta(H_x)=x^{\underline{-1}}, \qquad \Delta(2^x)=2^x,$

$\Delta(c^x)=(c-1)c^x, \qquad \Delta\binom{x}{m}=\binom{x}{m-1}.$

Sums:

$\sum cu\,\delta x = c\sum u\,\delta x,$

$\sum(u+v)\,\delta x = \sum u\,\delta x+\sum v\,\delta x,$

$\sum u\Delta v\,\delta x = uv-\sum \mathrm{E}\,v\Delta u\,\delta x,$

$\sum x^{\underline{n}}\,\delta x = \frac{x^{\underline{n+1}}}{m+1}, \qquad \sum x^{\underline{-1}}\,\delta x = H_x,$

$\sum c^x\,\delta x = \frac{c^x}{c-1}, \qquad \sum\binom{x}{m}\,\delta x = \binom{x}{m+1}.$

Falling Factorial Powers:

$x^{\underline{n}} = x(x-1)\cdots(x-n+1), \quad n>0,$

$x^{\underline{0}} = 1,$

$x^{\underline{n}} = \dfrac{1}{(x+1)\cdots(x+|n|)}, \quad n<0,$

$x^{\underline{n+m}} = x^{\underline{m}}(x-m)^{\underline{n}}.$

Rising Factorial Powers:

$x^{\overline{n}} = x(x+1)\cdots(x+n-1), \quad n>0,$

$x^{\overline{0}} = 1,$

$x^{\overline{n}} = \dfrac{1}{(x-1)\cdots(x-|n|)}, \quad n<0,$

$x^{\overline{n+m}} = x^{\overline{m}}(x+m)^{\overline{n}}.$

Conversion:

$x^{\underline{n}} = (-1)^n(-x)^{\overline{n}} = (x-n+1)^{\overline{n}}$

$\quad = 1/(x+1)^{\overline{-n}},$

$x^{\overline{n}} = (-1)^n(-x)^{\underline{n}} = (x+n-1)^{\underline{n}}$

$\quad = 1/(x-1)^{\underline{-n}},$

$$x^n = \sum_{k=1}^n \left\{ n \atop k \right\} x^{\underline{k}} = \sum_{k=1}^n \left\{ n \atop k \right\}(-1)^{n-k}x^{\overline{k}},$$

$$x^{\underline{n}} = \sum_{k=1}^n \left[n \atop k \right](-1)^{n-k}x^k,$$

$$x^{\overline{n}} = \sum_{k=1}^n \left[n \atop k \right]x^k.$$

Theoretical Computer Science Cheat Sheet

Series

Taylor's series:

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2}f''(a) + \cdots = \sum_{i=0}^{\infty} \frac{(x-a)^i}{i!} f^{(i)}(a).$$

Expansions:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots = \sum_{i=0}^{\infty} x^i,$$

$$\frac{1}{1-cx} = 1 + cx + c^2x^2 + c^3x^3 + \cdots = \sum_{i=0}^{\infty} c^i x^i,$$

$$\frac{1}{1-x^n} = 1 + x^n + x^{2n} + x^{3n} + \cdots = \sum_{i=0}^{\infty} x^{ni},$$

$$\frac{x}{(1-x)^2} = x + 2x^2 + 3x^3 + 4x^4 + \cdots = \sum_{i=0}^{\infty} ix^i,$$

$$x^k \frac{d^n}{dx^n}\left(\frac{1}{1-x}\right) = x + 2^n x^2 + 3^n x^3 + 4^n x^4 + \cdots = \sum_{i=0}^{\infty} i^n x^i,$$

$$e^x = 1 + x + \tfrac{1}{2}x^2 + \tfrac{1}{6}x^3 + \cdots = \sum_{i=0}^{\infty} \frac{x^i}{i!},$$

$$\ln(1+x) = x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 - \cdots = \sum_{i=1}^{\infty} (-1)^{i+1}\frac{x^i}{i},$$

$$\ln \frac{1}{1-x} = x + \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 + \tfrac{1}{4}x^4 + \cdots = \sum_{i=1}^{\infty} \frac{x^i}{i},$$

$$\sin x = x - \tfrac{1}{3!}x^3 + \tfrac{1}{5!}x^5 - \tfrac{1}{7!}x^7 + \cdots = \sum_{i=0}^{\infty} (-1)^i \frac{x^{2i+1}}{(2i+1)!},$$

$$\cos x = 1 - \tfrac{1}{2!}x^2 + \tfrac{1}{4!}x^4 - \tfrac{1}{6!}x^6 + \cdots = \sum_{i=0}^{\infty} (-1)^i \frac{x^{2i}}{(2i)!},$$

$$\tan^{-1} x = x - \tfrac{1}{3}x^3 + \tfrac{1}{5}x^5 - \tfrac{1}{7}x^7 + \cdots = \sum_{i=0}^{\infty} (-1)^i \frac{x^{2i+1}}{(2i+1)},$$

$$(1+x)^n = 1 + nx + \tfrac{n(n-1)}{2}x^2 + \cdots = \sum_{i=0}^{\infty} \binom{n}{i} x^i,$$

$$\frac{1}{(1-x)^{n+1}} = 1 + (n+1)x + \tbinom{n+2}{2}x^2 + \cdots = \sum_{i=0}^{\infty} \binom{i+n}{i} x^i,$$

$$\frac{x}{e^x - 1} = 1 - \tfrac{1}{2}x + \tfrac{1}{12}x^2 - \tfrac{1}{720}x^4 + \cdots = \sum_{i=0}^{\infty} \frac{B_i x^i}{i!},$$

$$\frac{1}{2x}(1 - \sqrt{1-4x}) = 1 + x + 2x^2 + 5x^3 + \cdots = \sum_{i=0}^{\infty} \frac{1}{i+1}\binom{2i}{i} x^i,$$

$$\frac{1}{\sqrt{1-4x}} = 1 + x + 2x^2 + 6x^3 + \cdots = \sum_{i=0}^{\infty} \binom{2i}{i} x^i,$$

$$\frac{1}{\sqrt{1-4x}}\left(\frac{1-\sqrt{1-4x}}{2x}\right)^n = 1 + (2+n)x + \tbinom{4+n}{2}x^2 + \cdots = \sum_{i=0}^{\infty} \binom{2i+n}{i} x^i,$$

$$\frac{1}{1-x}\ln\frac{1}{1-x} = x + \tfrac{3}{2}x^2 + \tfrac{11}{6}x^3 + \tfrac{25}{12}x^4 + \cdots = \sum_{i=1}^{\infty} H_i x^i,$$

$$\frac{1}{2}\left(\ln \frac{1}{1-x}\right)^2 = \tfrac{1}{2}x^2 + \tfrac{3}{4}x^3 + \tfrac{11}{24}x^4 + \cdots = \sum_{i=2}^{\infty} \frac{H_{i-1}x^i}{i},$$

$$\frac{x}{1-x-x^2} = x + x^2 + 2x^3 + 3x^4 + \cdots = \sum_{i=0}^{\infty} F_i x^i,$$

$$\frac{F_n x}{1 - (F_{n-1} + F_{n+1})x - (-1)^n x^2} = F_n x + F_{2n}x^2 + F_{3n}x^3 + \cdots = \sum_{i=0}^{\infty} F_{ni} x^i.$$

Ordinary power series:

$$A(x) = \sum_{i=0}^{\infty} a_i x^i.$$

Exponential power series:

$$A(x) = \sum_{i=0}^{\infty} a_i \frac{x^i}{i!}.$$

Dirichlet power series:

$$A(x) = \sum_{i=1}^{\infty} \frac{a_i}{i^x}.$$

Binomial theorem:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k.$$

Difference of like powers:

$$x^n - y^n = (x-y)\sum_{k=0}^{n-1} x^{n-1-k} y^k.$$

For ordinary power series:

$$\alpha A(x) + \beta B(x) = \sum_{i=0}^{\infty} (\alpha a_i + \beta b_i) x^i,$$

$$x^k A(x) = \sum_{i=k}^{\infty} a_{i-k} x^i,$$

$$\frac{A(x) - \sum_{i=0}^{k-1} a_i x^i}{x^k} = \sum_{i=0}^{\infty} a_{i+k} x^i,$$

$$A(cx) = \sum_{i=0}^{\infty} c^i a_i x^i,$$

$$A'(x) = \sum_{i=0}^{\infty} (i+1) a_{i+1} x^i,$$

$$xA'(x) = \sum_{i=1}^{\infty} i a_i x^i,$$

$$\int A(x)\,dx = \sum_{i=1}^{\infty} \frac{a_{i-1}}{i} x^i,$$

$$\frac{A(x) + A(-x)}{2} = \sum_{i=0}^{\infty} a_{2i} x^{2i},$$

$$\frac{A(x) - A(-x)}{2} = \sum_{i=0}^{\infty} a_{2i+1} x^{2i+1}.$$

Summation: If $b_i = \sum_{j=0}^{i} a_i$ then

$$B(x) = \frac{1}{1-x} A(x).$$

Convolution:

$$A(x)B(x) = \sum_{i=0}^{\infty} \left(\sum_{j=0}^{i} a_j b_{i-j} \right) x^i.$$

God made the natural numbers;
all the rest is the work of man.
– Leopold Kronecker

Theoretical Computer Science Cheat Sheet

Series

Expansions:

$$\frac{1}{(1-x)^{n+1}}\ln\frac{1}{1-x} = \sum_{i=0}^{\infty}(H_{n+i}-H_n)\binom{n+i}{i}x^i, \qquad \left(\frac{1}{x}\right)^{\overline{-n}} = \sum_{i=0}^{\infty}\left\{{i \atop n}\right\}x^i,$$

$$x^{\overline{n}} = \sum_{i=0}^{\infty}\left[{n \atop i}\right]x^i, \qquad (e^x-1)^n = \sum_{i=0}^{\infty}\left\{{i \atop n}\right\}\frac{n!x^i}{i!},$$

$$\left(\ln\frac{1}{1-x}\right)^n = \sum_{i=0}^{\infty}\left[{i \atop n}\right]\frac{n!x^i}{i!}, \qquad x\cot x = \sum_{i=0}^{\infty}\frac{(-4)^i B_{2i}x^{2i}}{(2i)!},$$

$$\tan x = \sum_{i=1}^{\infty}(-1)^{i-1}\frac{2^{2i}(2^{2i}-1)B_{2i}x^{2i-1}}{(2i)!}, \qquad \zeta(x) = \sum_{i=1}^{\infty}\frac{1}{i^x},$$

$$\frac{1}{\zeta(x)} = \sum_{i=1}^{\infty}\frac{\mu(i)}{i^x}, \qquad \frac{\zeta(x-1)}{\zeta(x)} = \sum_{i=1}^{\infty}\frac{\phi(i)}{i^x},$$

$$\zeta(x) = \prod_p \frac{1}{1-p^{-x}},$$

$$\zeta^2(x) = \sum_{i=1}^{\infty}\frac{d(i)}{x^i} \quad \text{where } d(n)=\textstyle\sum_{d|n}1,$$

$$\zeta(x)\zeta(x-1) = \sum_{i=1}^{\infty}\frac{S(i)}{x^i} \quad \text{where } S(n)=\textstyle\sum_{d|n}d,$$

$$\zeta(2n) = \frac{2^{2n-1}|B_{2n}|}{(2n)!}\pi^{2n}, \quad n\in\mathbb{N},$$

$$\frac{x}{\sin x} = \sum_{i=0}^{\infty}(-1)^{i-1}\frac{(4^i-2)B_{2i}x^{2i}}{(2i)!},$$

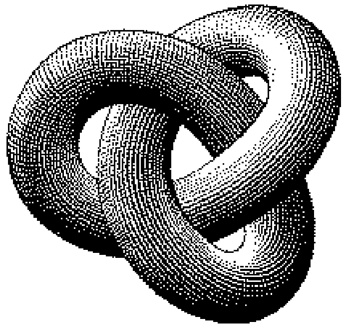
$$\left(\frac{1-\sqrt{1-4x}}{2x}\right)^n = \sum_{i=0}^{\infty}\frac{n(2i+n-1)!}{i!(n+i)!}x^i,$$

$$e^x\sin x = \sum_{i=1}^{\infty}\frac{2^{i/2}\sin\frac{i\pi}{4}}{i!}x^i,$$

$$\sqrt{\frac{1-\sqrt{1-x}}{x}} = \sum_{i=0}^{\infty}\frac{(4i)!}{16^i\sqrt{2}(2i)!(2i+1)!}x^i,$$

$$\left(\frac{\arcsin x}{x}\right)^2 = \sum_{i=0}^{\infty}\frac{4^i i!^2}{(i+1)(2i+1)!}x^{2i}.$$

Escher's Knot



Stieltjes Integration

If G is continuous in the interval $[a,b]$ and F is nondecreasing then

$$\int_a^b G(x)\,dF(x)$$

exists. If $a\le b\le c$ then

$$\int_a^c G(x)\,dF(x) = \int_a^b G(x)\,dF(x) + \int_b^c G(x)\,dF(x).$$

If the integrals involved exist

$$\int_a^b \big(G(x)+H(x)\big)\,dF(x) = \int_a^b G(x)\,dF(x) + \int_a^b H(x)\,dF(x),$$

$$\int_a^b G(x)\,d\big(F(x)+H(x)\big) = \int_a^b G(x)\,dF(x) + \int_a^b G(x)\,dH(x),$$

$$\int_a^b c\cdot G(x)\,dF(x) = \int_a^b G(x)\,d\big(c\cdot F(x)\big) = c\int_a^b G(x)\,dF(x),$$

$$\int_a^b G(x)\,dF(x) = G(b)F(b) - G(a)F(a) - \int_a^b F(x)\,dG(x).$$

If the integrals involved exist, and F possesses a derivative F' at every point in $[a,b]$ then

$$\int_a^b G(x)\,dF(x) = \int_a^b G(x)F'(x)\,dx.$$

Cramer's Rule

If we have equations:

$$\begin{aligned} a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n &= b_1\\ a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n &= b_2\\ \vdots \qquad\quad \vdots \qquad\qquad\quad \vdots \quad&\\ a_{n,1}x_1 + a_{n,2}x_2 + \cdots + a_{n,n}x_n &= b_n \end{aligned}$$

Let $A=(a_{i,j})$ and B be the column matrix (b_i). Then there is a unique solution iff $\det A\ne 0$. Let A_i be A with column i replaced by B. Then

$$x_i = \frac{\det A_i}{\det A}.$$

00	47	18	76	29	93	85	34	61	52
86	11	57	28	70	39	94	45	02	63
95	80	22	67	38	71	49	56	13	04
59	96	81	33	07	48	72	60	24	15
73	69	90	82	44	17	58	01	35	26
68	74	09	91	83	55	27	12	46	30
37	08	75	19	92	84	66	23	50	41
14	25	36	40	51	62	03	77	88	99
21	32	43	54	65	06	10	89	97	78
42	53	64	05	16	20	31	98	79	87

Fibonacci Numbers

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots$$

Definitions:

$$F_i = F_{i-1}+F_{i-2}, \quad F_0=F_1=1,$$

$$F_{-i} = (-1)^{i-1}F_i,$$

$$F_i = \frac{1}{\sqrt{5}}\left(\phi^i - \hat{\phi}^i\right),$$

Cassini's identity: for $i>0$:

$$F_{i+1}F_{i-1} - F_i^2 = (-1)^i.$$

Additive rule:

$$F_{n+k} = F_kF_{n+1} + F_{k-1}F_n,$$

$$F_{2n} = F_nF_{n+1} + F_{n-1}F_n.$$

Calculation by matrices:

$$\begin{pmatrix} F_{n-2} & F_{n-1}\\ F_{n-1} & F_n\end{pmatrix} = \begin{pmatrix} 0 & 1\\ 1 & 1\end{pmatrix}^n.$$

The Fibonacci number system: Every integer n has a unique representation

$$n = F_{k_1} + F_{k_2} + \cdots + F_{k_m},$$

where $k_i \ge k_{i+1}+2$ for all i, $1\le i<m$ and $k_m\ge 2$.

Improvement makes strait roads, but the crooked roads without Improvement, are roads of Genius.
– William Blake (The Marriage of Heaven and Hell)