All-in at the River

Standard Code Library

Shanghai Jiao Tong University

Desprado2 fstqwq AntiLeaf

目录

1 数学

1.1 FWT

矩阵表示

or 形式 (子集卷积):

$$T_{ij} = [i|j = i] = [j \in i]$$

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

and 形式 (超集卷积):

$$T_{ij} = [i\&j = i] = [i \in j]$$

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

xor 形式 (T与自身互为逆矩阵):

$$T_{ij} = (-1)^{parity(i\&j)}$$

$$\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

```cpp
using ll = long long;

// or
void FWT(ll *a, int len, int inv) {
    for (int h = 1; h < len; h <<= 1) {
        for (int i = 0; i < len; i += (h << 1)) {
            for (int j = 0; j < h; ++j) {
                a[i + j + h] += a[i + j] * inv;
            }
        }
    }
}
// and
void FWT(ll *a, int len, int inv) {
    for (int h = 1; h < len; h <<= 1) {
        for (int i = 0; i < len; i += (h << 1)) {
            for (int j = 0; j < h; ++j) {
                a[i + j] += a[i + j + h] * inv;
            }
        }
    }
}
// xor
void FWT(ll *a, int len, int inv) {
    for (int h = 1; h < len; h <<= 1) {
        for (int i = 0; i < len; i += (h << 1)) {
            for (int j = 0; j < h; ++j) {
                ll x = a[i + j], y = a[i + j + h];
                a[i + j] = x + y, a[i + j + h] = x - y;
                if (inv == -1)
                    a[i + j] /= 2, a[i + j + h] /= 2;
            }
        }
    }
}
```

1.2 多项式

1.2.1 FFT

```cpp
// 使用时一定要注意double的精度是否足够(极限大概是10 ^ 14)

const double pi = acos((double)-1.0);

// 手写复数类
// 支持加减乘三种运算
// += 运算符如果用的不多可以不重载
struct Complex {
    double a, b; // 由于long double精度和double几乎相同, 通
    ↪ 常没有必要用long double

    Complex(double a = 0.0, double b = 0.0) : a(a), b(b) {}

    Complex operator + (const Complex &x) const {
        return Complex(a + x.a, b + x.b);
    }

    Complex operator - (const Complex &x) const {
        return Complex(a - x.a, b - x.b);
    }

    Complex operator * (const Complex &x) const {
        return Complex(a * x.a - b * x.b, a * x.b + b *
    ↪ x.a);
    }

    Complex operator * (double x) const {
        return Complex(a * x, b * x);
    }

    Complex &operator += (const Complex &x) {
        return *this = *this + x;
    }

    Complex conj() const { // 共轭, 一般只有MTT需要用
        return Complex(a, -b);
    }
} omega[maxn], omega_inv[maxn];
const Complex ima = Complex(0, 1);

int fft_n; // 要在主函数里初始化

// FFT初始化
void FFT_init(int n) {
    fft_n = n;

    for (int i = 0; i < n; i++) // 根据单位根的旋转性质可以
    ↪ 节省计算单位根逆元的时间
        omega[i] = Complex(cos(2 * pi / n * i), sin(2 * pi
    ↪ / n * i));

    omega_inv[0] = omega[0];
    for (int i = 1; i < n; i++)
        omega_inv[i] = omega[n - i];
    // 当然不存单位根也可以, 只不过在FFT次数较多时很可能会
    ↪ 增大常数
}

// FFT主过程
void FFT(Complex *a, int n, int tp) {
    for (int i = 1, j = 0, k; i < n - 1; i++) {
        k = n;
        do
            j ^= (k >>= 1);
        while (j < k);

        if (i < j)
            swap(a[i], a[j]);
    }

    for (int k = 2, m = fft_n / 2; k <= n; k *= 2, m /= 2)
        for (int i = 0; i < n; i += k)
            for (int j = 0; j < k / 2; j++) {
                Complex u = a[i + j], v = (tp > 0 ? omega : omega_inv)[m * j] * a[i + j + k / 2];

                a[i + j] = u + v;
                a[i + j + k / 2] = u - v;
            }

    if (tp < 0)
        for (int i = 0; i < n; i++) {
            a[i].a /= n;
            a[i].b /= n; // 一般情况下是不需要的，只有MTT时才需要
        }
}
```

1.3 组合数行区间和

```
using ll = long long;

// 边界从 (s, x) 移动到 (s + 1, nx)
ll move(int x, int nx, int s, ll sum) {
    assert(x >= -1);
    ll res = (2 * sum + mod - C(s, x)) % mod;
    while (x + 1 <= nx) {
        x++;
        res = (res + C(s + 1, x)) % mod;
    }
    while (x > nx) {
        res = (res + mod - C(s + 1, x)) % mod;
        x--;
    }
    return res;
};

void proc(int k) {
    // 第 s 行的 左右边界
    auto le = [&](int s) -> int {
        // ...
    };
    auto ri = [&](int s) -> int {
        // ...
    };
    // 这里应该暴力计算首行和□当首行为 0 时也可以这样写
    ll lsum = move(-1, le(0), -1, 0);
    ll rsum = move(-1, ri(0), -1, 0);

    for (int s = 0; s <= k; s++) {
        if (le(s) < ri(s)) {
            // ... 第 s 行对答案的贡献
        }
        lsum = move(le(s), le(s + 1), s, lsum);
        rsum = move(ri(s), ri(s + 1), s, rsum);
    }
}
```

2 数论

2.1 常见预处理与快速幂

```cpp
// 预处理组合数
const int N = 2e5 + 7;
const ll mod = 998244353;

ll fac[N], ifac[N];
void init() {
    fac[0] = 1;
    for (int i = 1; i < N; ++i) fac[i] = i * fac[i-1] % mod;
    ifac[N - 1] = fpow(fac[N - 1], mod - 2);
    for (int i = N - 1; i; --i) ifac[i - 1] = i * ifac[i] % mod;
}

ll C(int n, int k) {
    if (k < 0 || k > n) return 0;
    return (fac[n] * ifac[k] % mod) * ifac[n - k] % mod;
}

// 线性求逆元，注意有效的 i < mod
ll inv[maxn];
void init() {
    inv[0] = 0, inv[1] = 1;
    for (int i = 2; i < N; ++i)
        inv[i] = inv[mod % i] * (mod - mod / i) % mod;
}

// 快速幂
ll fpow(ll a, ll k = mod - 2, ll p = mod) {
    ll res = 1; a %= p;
    for (; k; k >>= 1, a = a * a % p) {
        if (k & 1)
            res = res * a % p;
    }
    return res;
}
```

2.2 因数分解与素性判定

2.2.1 朴素因数分解

```cpp
// 素因数分解
int p[maxn], l[maxn], cnt2 = 0;
void Fact(int x) {
    cnt2 = 0;
    for (int i = 2; 1ll * i * i <= x; i++) {
        if(x % i == 0) {
            p[++cnt2] = i; l[cnt2] = 0;
            while(x % i == 0) {
                x /= i; ++l[cnt2];
            }
        }
    }
    if (x != 1) {  // 则此时x一定是素数，且为原本x的大于根号x的唯一素因子
        p[++cnt2] = x; l[cnt2] = 1;
    }
}

// vector ver. 无次数
void Fact(ll x, vector<int>& fact) {
    for (ll i = 2; 1ll * i * i <= x; ++i) {
        if(x % i == 0) {
            fact.push_back(i);
            while(x % i == 0) x /= i;
        }
    }
    if (x != 1) fact.push_back(x);
}
```

2.2.2 Miller-Rabin 与 Pollard-Rho

```cpp
#include<bits/stdc++.h>
using namespace std;
typedef long long ll;

ll randint(ll l, ll r) {
    static mt19937 eng(time(0));
    uniform_int_distribution<ll> dis(l, r);
    return dis(eng);
}

bool is_prime(ll x) {
    int s = 0; ll t = x - 1;
    if (x == 2) return true;
    if (x < 2 || !(x & 1)) return false;
    while (!(t & 1)) {   //将x分解成(2^s)*t的样子
        s++; t >>= 1;
    }
    ll lst[] = {2, 325, 9375, 28178, 450775, 9780504, 1795265022};
    for(ll a : lst) { //随便选一个素数进行测试
        if(a >= x) break;
        ll b = Pow(a, t, x); //先算出a^t
        for (int j = 1; j <= s; ++j) { //然后进行s次平方
            ll k = mul(b, b, x);        //求b的平方
            if (k == 1 && b != 1 && b != x - 1) //用二次探测判断
                return false;
            b = k;
        }
        if (b != 1)
            return false; //用费马小定律判断
    }
    return true; //如果进行多次测试都是对的，那么x就很有可能是素数
}

ll gcd(ll a, ll b) { return b == 0 ? a : gcd(b, a % b); }

// @author: Pecco
ll Pollard_Rho(ll n) {
    if (n == 4) return 2;
    if (is_prime(n)) return n;
    while (1) {
        ll c = randint(1, n - 1); // 生成随机的c
        auto f = [=](ll x) { return ((__int128)x * x + c) % n; }; // lll表示__int128，防溢出
        ll t = f(0), r = f(f(0));
        while (t != r) {
            ll d = __gcd(abs(t - r), n);
            if (d > 1)
                return d;
            t = f(t), r = f(f(r));
        }
    }
}


// 优化掉一个log
ll Pollard_Rho(ll n) {
    if (n == 4) return 2;
    if (is_prime(n)) return n;
    while (1) {
        ll c = randint(1, n - 1);
        auto f = [=](ll x) { return ((__int128)x * x + c) % n; };
        ll t = 0, r = 0, p = 1, q;
        do {
            for (int i = 0; i < 128; ++i) { // 令固定距离C=128
                t = f(t), r = f(f(r));
                if (t == r || (q = (lll)p * abs(t - r) % n) == 0) // 如果发现环□或者积即将为0□退出
                    break;
                p = q;
            }
            ll d = gcd(p, n);
            if (d > 1)
                return d;
        } while (t != r);
    }
}

vector<ll> factors;

void getfactors(ll n) {
    if (n == 1) return;
    if (is_prime(n)) { factors.push_back(n); return; } // 如果是质因子
    ll p = n;
    while (p == n)
        p = Pollard_Rho(n);
    getfactors(n / p), getfactors(p); //递归处理
}
```

2.3 筛法

2.3.1 线性筛

```cpp
const int maxn = 1000000 + 5;
bool isnt[maxn];
int prime[maxn];
int cnt = 0;

// 线性筛法 [1, n] 内素数
void Prime(int n) {
    isnt[1] = true;
    cnt = 0;
    for (int i = 2; i <= n; i++) {
        if (!isnt[i]) prime[++cnt] = i;
        for (int j = 1; j <= cnt; j++) {
            if (1ll * i * prime[j] > n) break;
            isnt[i * prime[j]] = 1;
            if (i % prime[j] == 0) break;
        }
    }
}

// 线性筛求积性函数
int phi[maxn], mu[maxn], d[maxn], D[maxn], q[maxn];
void Sieve(int n) {
    isnt[1] = true;
    phi[1] = 1;
    //mu[1] = 1;
    cnt = 0;
    for(int i = 2; i <= n; i++) {
        if (!isnt[i]) {
            prime[++cnt] = i;
            phi[i] = i - 1;
            //mu[i] = -1;
            // d[i] = 2; q[i] = 1;
            // D[i] = i + 1; q[i] = 1;
        }
        for (int j = 1; j <= cnt; j++) {
            int x = i * prime[j];
            if (x > n) break;
            isnt[x] = 1;
            if (i % prime[j] == 0) {
                phi[x] = phi[i] * prime[j];
                // mu[x] = 0;
                // d[x] = d[i] / (q[i] + 1) * (q[i] + 2), q[x] = q[i] + 1;
                // D[x] = D[i] / (prime[j] ^ (q[i] + 1) - 1) * (prime[j] ^ (q[i] + 2) - 1), q[x] = q[i] + 1;
                break;
            } else {
                phi[x] = phi[i] * (prime[j] - 1); // mu[x] = -mu[i]
                // d[x] = 2 * d[i], q[x] = 1;
                // D[x] = (prime[j] + 1) * D[i], q[x] = 1;
            }
        }
    }
}
```

2.3.2 Min25 筛

```cpp
using ll = long long;
using i128 = __int128;
//using i128 = int64_t;

const ll mod = 998244353;

namespace min25 {
    const int N = 1e6 + 10;
    ll po[40][N];
    inline ll fpow(ll e, ll k) {
        return po[e][k];
    }
    void precalc() {
        for(int e = 0; e < 40; ++e) {
            po[e][0] = 1;
            for(int i = 1; i < N; i++)
                po[e][i] = e * po[e][i - 1] % mod;
        }
    }
    ll n;
    int B;
    int _id[N * 2];
    inline int id(ll x) {
        return x <= B ? x : n / x + B;
    }
    inline int Id(ll x) {
        return _id[id(x)];
    }
    // f(p) = p - 1 = fh(p) - fg(p);
    inline ll fg(ll x) {
        // assert(x <= sqrt(n));
        return 1;
    }
    inline ll fh(ll x) {
        // assert(x <= sqrt(n));
        return x;
    }
    // \sum_{i=2}^n fg(i)
    inline ll sg(ll x) {
        return (x - 1) % mod;
    }
    // \sum_{i=2}^n fh(i)
    inline ll sh(ll x) {
        return ((i128) x * (x + 1) / 2 + mod - 1) % mod;
    }
    // f(p^e)
    inline ll f(ll p, ll e) {
        //return (pe - pe / p) % mod;
        return (p + (mod - 2) * fpow(e, p)) % mod;
    }
    bitset<N> np;
    ll p[N>>2], pn;
    ll pg[N>>2], ph[N>>2];
    void sieve(ll sz) {
        for(int i = 2; i <= sz; i++) {
            if(!np[i]) {
                p[++pn] = i;
                pg[pn] = (pg[pn - 1] + fg(i)) % mod;
                ph[pn] = (ph[pn - 1] + fh(i)) % mod;
            }
            for(int j = 1; j <= pn && i * p[j] <= sz; j++) {
                np[i * p[j]] = 1;
                if(i % p[j] == 0) {
                    break;
                }
            }
        }
    }
    ll m;
    ll g[N * 2], h[N * 2];
    ll w[N * 2];

    void compress() {
        for (int i = 1; i <= m; i++) {
            g[i] = (h[i] + mod - g[i] + mod - g[i]) % mod;
        }
        for (int i = 1; i <= pn; i++) {
            pg[i] = (ph[i] + mod - pg[i] + mod - pg[i]) % mod;
        }
    }

    ll dfs_F(int k, ll n) {
        if (n < p[k] || n <= 1) return 0;
        ll res = g[Id(n)] + mod - pg[k - 1], pw2;
        for (int i = k; i <= pn && (pw2 = (ll) p[i] * p[i]) <= n; ++i) {
            ll pw = p[i];
            for (int c = 1; pw2 <= n; ++c, pw = pw2, pw2 *= p[i])
                res = (res + ((ll) f(p[i], c) * dfs_F(i + 1, n / pw) + f(p[i], c + 1))) % mod;
        }
        return res;
    }

    void init(ll _n) {
        n = _n;
        B = sqrt(n) + 100;
        pn = 0;
        sieve(B);
        m = 0;
        for(ll i = 1, j; i <= n; i = j + 1) {
            j = n / (n / i);
            ll t = n / i;
            _id[id(t)] = ++m;
            w[m] = t;
            g[m] = sg(t);
            h[m] = sh(t);
            //printf("id: %lld, w: %lld, g: %lld, h: %lld\n", m, t, g[m], h[m]);
        }
        for (int j = 1; j <= pn; j++) {
            ll z = (ll) p[j] * p[j];
            for(int i = 1; i <= m && z <= w[i]; i++) {
                int k = Id(w[i] / p[j]);
                g[i] = (g[i] + (ll) (mod - fg(p[j])) * (g[k] - pg[j - 1] + mod)) % mod;
                h[i] = (h[i] + (ll) (mod - fh(p[j])) * (h[k] - ph[j - 1] + mod)) % mod;
            }
        }
        compress();

        /* 递推 min25
        for(int j = pn; j > 0; j--) {
            ll z = (ll) p[j] * p[j];
            for(int i = 1; i <= m && z <= w[i]; i++) {
                ll pe = p[j];
                for(int e = 1; pe * p[j] <= w[i]; e++, pe *= p[j]) {
                    g[i] = (g[i] + (ll) f(p[j], e) * (g[Id(w[i] / pe)] - pg[j] + mod) + f(p[j], e + 1)) % mod;
                }
            }
        } */
    }
    ll get(ll x) { // x == n / i
        if(x < 1) return 0;
        return (dfs_F(1, x) + 1) % mod;
    }
    ll get(ll l, ll r) {
        return get(r) - get(l - 1);
    }
}

void Solve() {
    long long n;
    scanf("%lld", &n);
    min25::init(n);
    long long res = min25::get(n);
    printf("%lld\n", res);
}

int main() {
    min25::precalc();
    int T;
    scanf("%d", &T);
    while(T--) {
        Solve();
    }
}
```

2.4 扩展欧几里得

```cpp
using i128 = __int128;

// ax + by = c
// 有解当且仅当 gcd(a, b) | c
// 要求 a, b 不全为 0
// 无合法性检查
void exgcd(i128 a, i128 b, i128 &x, i128 &y, i128 c = 1) {
    if (b == 0) {
        x = c / a;
        y = 0;
    } else {
        exgcd(b, a % b, x, y, c);
        i128 tmp = x;
        x = y;
        y = tmp - (a / b) * y;
    }
}
```

2.5 中国剩余定理

2.5.1 两个数的 crt

```cpp
// mul 表示慢速乘
ll crt(ll a1, ll p, ll a2, ll q) {
    ll ip = fpow(p, q-2, q);
    ll iq = fpow(q, p-2, p);
    ll n = p * q;
    return (mul(mul(a1, q, n), iq, n) + mul(mul(a2, p, n), ip, n)) % n;
}
```

2.5.2 excrt

```cpp
using ll = long long;

ll gcd(ll a, ll b) {
    return b == 0 ? a : gcd(b, a % b);
}

// x === a1 (mod b1), x === a2 (mod b2)
// 合法性检查⊠返回 -1 则为无解
pair<ll, ll> excrt(ll a1, ll b1, ll a2, ll b2) {
    ll g = gcd(b1, b2);
    ll lcm = (b1 / g) * b2;

    if ((a1 - a2) % g) return {-1, -1};

    i128 x, y;
    exgcd(b1, b2, x, y, a1 - a2);
    ll res = (a1 - b1 * x) % lcm;
    if (res < 0) res += lcm;
    return {res, lcm};
}
```

2.6 卢卡斯定理

2.6.1 模素数卢卡斯

```cpp
// 卢卡斯定理, 要求 p 为素数
ll lucas(ll n, ll m, ll p) {
    if (!m) return 1;
    return C(n % p, m % p, p) * lucas(n / p, m / p, p) % p;
}
```

2.6.2 扩展卢卡斯

```cpp
// 扩展卢卡斯定理

// 扩欧求逆元
ll INV(ll a, ll p) {
    ll x, y;
    exgcd(a, p, x, y);
    return (x % p + p) % p;
}

// 递归求解(n! / px) mod pk
ll F(ll n, ll p, ll pk) {
    if (n == 0) return 1;
    ll rou = 1; // 循环节
    ll rem = 1; // 余项
    for (ll i = 1; i <= pk; ++i) {
        if (i % p)
            rou = rou * i % pk;
    }
    rou = fpow(rou, n / pk, pk);
    for (ll i = pk * (n / pk); i <= n; ++i) {
        if (i % p)
            rem = rem * (i % pk) % pk; // 小心i炸int
    }
    return F(n / p, p, pk) * rou % pk * rem % pk;
}

// 素数p在n!中的次数
ll G(ll n, ll p) {
    if (n < p) return 0;
    return G(n / p, p) + (n / p);
}

ll C_pk(ll n, ll m, ll p, ll pk) {
    ll fz = F(n, p, pk), fm1 = INV(F(m, p, pk), pk),
       fm2 = INV(F(n - m, p, pk), pk);
    ll mi = fpow(p, G(n, p) - G(m, p) - G(n - m, p), pk);
    return fz * fm1 % pk * fm2 % pk * mi % pk;
}

ll exlucas(ll n, ll m, ll P) {
    Fact(P); // 素因子分解⊠见素因子分解.cpp
    for (int i = 1; i <= cnt2; ++i) {
        ll pk = 1;
        for (int j = 0; j < l[i]; ++j) {
            pk *= p[i];
        }
        bi[i] = pk, ai[i] = C_pk(n, m, p[i], pk);
    }
    return excrt(cnt2) % P;
}
```

2.7 原根与离散对数

2.7.1 原根

```cpp
// 得到 p 的原根
ll generator(ll p) {
    static ll rec, ans;
    if (p == rec)
        return ans;
    rec = p;
    vector<ll> fact;
    ll phi = p - 1, n = phi;
    for (ll i = 2; 1ll * i * i <= n; ++i) {
        if (n % i == 0) {
            fact.push_back(i);
            while (n % i == 0)
                n /= i;
        }
    }
    if (n > 1)
        fact.push_back(n);
    for (ll res = 2; res <= p; ++res) {
        bool ok = 1;
        for (ll factor : fact) {
            if (fpow(res, phi / factor, p) == 1) {
                ok = false;
                break;
            }
        }
        if (ok)
            return ans = res;
    }
    return ans = -1;
}
```

2.7.2 BSGS

```cpp
// a ^ k == b mod p
ll BSGS(ll a, ll b, ll p) { // p <= 1e9
    static ll rec;
    static map<ll, ll> mp;

    ll sq = (ll)ceil(sqrt(p));
    if (rec != p) {
        rec = p;
        mp.clear();
        ll le = 1, bs = fpow(a, sq, p);
        for (ll i = 1; i <= sq; ++i) {
            le = le * bs % p;
            if (le < 0)
                le += p;
            mp[le] = i * sq;
        }
    }

    ll ri = (b % p);
    if (ri < 0)
        ri += p;
    for (ll j = 0; j <= sq; ++j) {
        if (mp.count(ri)) {
            return mp[ri] - j;
        }
        ri = ri * a % p;
        if (ri < 0)
            ri += p;
    }
    return -1;
}

// x ^ a == b mod p
ll calc(ll a, ll b, ll p) {
    ll g = generator(p); // 求原根-见原根.cpp
    ll ga = fpow(g, a, p);
    ll c = BSGS(ga, b, p);
    ll res = fpow(g, c, p);
    return res;
}
```

2.8 杂项

2.8.1 大数整除小数取模

计算 $\frac{a}{b} \bmod p$
当 a 的本值太大无法表示时 可以计算 a 对 b * p 取模的结果 再除 b 模 p

$$\frac{a}{b} \bmod p = \frac{a \bmod b * p}{b} \bmod p$$

2.8.2 立方根复杂度求 mobius 函数

```
int getmu(int x) {
    int pr, cur = 0;
    for (int i = 1; i <= cnt; ++i) {
        cur = 0;
        while (x % prime[i] == 0) {
            ++cur; x /= prime[i];
        }
        if (cur > 1) return 0;
    }
    if (x == 1) return 1;
    int sq = sqrt(x) + 0.5;
    if (1ll * sq * sq == x) return 0;
    return 1;
}
```

2.8.3 直接求 euler 函数

```
// primes 为预处理的素数表
ll getPhi(ll x) {
    ll phi = x, num = x;
    for (int p : primes) {
        if (p > x / p) break;
        if (x % p == 0)
            phi = (phi / p) * (p - 1), x /= p;
        while (x % p == 0)
            x /= p;
    }
    if (x > 1)
        phi = phi / x * (x - 1);
    return phi;
}
```

3 图论

3.1 最小生成树

3.1.1 Boruvka算法

思想：每次选择连接每个连通块的最小边，把连通块缩起来.
每次连通块个数至少减半，所以迭代$O(\log n)$次即可得到最小生成树.
一种比较简单的实现方法：每次迭代遍历所有边，用并查集维护连通性和每个连通块的最小边权.
应用：最小异或生成树

3.1.2 动态最小生成树

动态最小生成树的离线算法比较容易，而在线算法通常极为复杂.
一个跑得比较快的离线做法是对时间分治，在每层分治时找出一定在/不在MST上的边，只带着不确定边继续递归.
简单起见，找确定边的过程用Kruskal算法实现，过程中的两种重要操作如下：

- Reduction：待修改边标为+INF，跑MST后把非树边删掉，减少无用边
- Contraction：待修改边标为-INF，跑MST后缩除待修改边之外的所有MST边，计算必须边

每轮分治需要Reduction-Contraction，借此减少不确定边，从而保证复杂度.
复杂度证明：假设当前区间有k条待修改边，n和m表示点数和边数，那么最坏情况下R-C的效果为$(n, m) \to (n, n+k-1) \to (k+1, 2k)$.

```cpp
// 全局结构体与数组定义
struct edge { //边的定义
    int u, v, w, id; // id表示边在原图中的编号
    bool vis; // 在Kruskal时用,记录这条边是否是树边
    bool operator < (const edge &e) const { return w < e.w;
    ↪ }
} e[20][maxn], t[maxn]; // 为了便于回滚,在每层分治存一个副
  ↪ 本


// 用于存储修改的结构体,表示第id条边的权值从u修改为v
struct A {
    int id, u, v;
} a[maxn];


int id[20][maxn]; // 每条边在当前图中的编号
int p[maxn], size[maxn], stk[maxn], top; // p和size是并查集
  ↪ 数组,stk是用来撤销的栈
int n, m, q; // 点数,边数,修改数


// 方便起见,附上可能需要用到的预处理代码
for (int i = 1; i <= n; i++) { // 并查集初始化
    p[i] = i;
    size[i] = 1;
}

for (int i = 1; i <= m; i++) { // 读入与预标号
    scanf("%d%d%d", &e[0][i].u, &e[0][i].v, &e[0][i].w);
    e[0][i].id = i;
    id[0][i] = i;
}

for (int i = 1; i <= q; i++) { // 预处理出调用数组
    scanf("%d%d", &a[i].id, &a[i].v);
    a[i].u = e[0][a[i].id].w;
    e[0][a[i].id].w = a[i].v;
}

for(int i = q; i; i--)
    e[0][a[i].id].w = a[i].u;

CDQ(1, q, 0, m, 0); // 这是调用方法


// 分治主过程 O(nlog^2n)
// 需要调用Reduction和Contraction
void CDQ(int l, int r, int d, int m, long long ans) { //
 ↪ CDQ分治
    if (l == r) { // 区间长度已减小到1,输出答案,退出
        e[d][id[d][a[l].id]].w = a[l].v;
        printf("%lld\n", ans + Kruskal(m, e[d]));
        e[d][id[d][a[l].id]].w=a[l].u;
        return;
    }

    int tmp = top;

    Reduction(l, r, d, m);
    ans += Contraction(l, r, d, m); // R-C

    int mid = (l + r) / 2;

    copy(e[d] + 1, e[d] + m + 1, e[d + 1] + 1);
    for (int i = 1; i <= m; i++)
        id[d + 1][e[d][i].id] = i; // 准备好下一层要用的数
 ↪ 组

    CDQ(l, mid, d + 1, m, ans);

    for (int i = l; i <= mid; i++)
        e[d][id[d][a[i].id]].w = a[i].v; // 进行左边的修改

    copy(e[d] + 1, e[d] + m + 1, e[d + 1] + 1);
    for (int i = 1; i <= m; i++)
        id[d + 1][e[d][i].id] = i; // 重新准备下一层要用的
 ↪ 数组

    CDQ(mid + 1, r, d + 1, m, ans);

    for (int i = top; i > tmp; i--)
        cut(stk[i]);//撤销所有操作
    top = tmp;
}


// Reduction(减少无用边):待修改边标为+INF,跑MST后把非树边删
 ↪ 掉,减少无用边
// 需要调用Kruskal
void Reduction(int l, int r, int d, int &m) {
    for (int i = l; i <= r; i++)
        e[d][id[d][a[i].id]].w = INF;//待修改的边标为INF

    Kruskal(m, e[d]);

    copy(e[d] + 1, e[d] + m + 1, t + 1);

    int cnt = 0;
    for (int i = 1; i <= m; i++)
        if (t[i].w == INF || t[i].vis){ // 非树边扔掉
            id[d][t[i].id] = ++cnt; // 给边重新编号
            e[d][cnt] = t[i];
        }

    for (int i = r; i >= l; i--)
        e[d][id[d][a[i].id]].w = a[i].u; // 把待修改的边改
 ↪ 回去

```

```cpp
    m=cnt;
}


// Contraction(缩必须边):待修改边标为-INF,跑MST后缩除待修改边之外的所有树边
// 返回缩掉的边的总权值
// 需要调用Kruskal
long long Contraction(int l, int r, int d, int &m) {
    long long ans = 0;

    for (int i = l; i <= r; i++)
        e[d][id[d][a[i].id]].w = -INF; // 待修改边标为-INF

    Kruskal(m, e[d]);
    copy(e[d] + 1, e[d] + m + 1, t + 1);

    int cnt = 0;
    for (int i = 1; i <= m ; i++) {

        if (t[i].w != -INF && t[i].vis) { // 必须边
            ans += t[i].w;
            mergeset(t[i].u, t[i].v);
        }
        else { // 不确定边
            id[d][t[i].id]=++cnt;
            e[d][cnt]=t[i];
        }
    }

    for (int i = r ; i >= l; i--) {
        e[d][id[d][a[i].id]].w = a[i].u; // 把待修改的边改回去
        e[d][id[d][a[i].id]].vis = false;
    }

    m = cnt;

    return ans;
}


// Kruskal算法 O(mlogn)
// 方便起见,这里直接沿用进行过缩点的并查集,在过程结束后撤销即可
long long Kruskal(int m, edge *e) {
    int tmp = top;
    long long ans = 0;

    sort(e + 1, e + m + 1); // 比较函数在结构体中定义过了

    for (int i = 1; i <= m; i++) {
        if (findroot(e[i].u) != findroot(e[i].v)) {
            e[i].vis = true;
            ans += e[i].w;
            mergeset(e[i].u, e[i].v);
        }
        else
            e[i].vis = false;
    }

    for(int i = top; i > tmp; i--)
        cut(stk[i]); // 撤销所有操作
    top = tmp;

    return ans;
}


// 以下是并查集相关函数
int findroot(int x) { // 因为需要撤销,不写路径压缩
    while (p[x] != x)
        x = p[x];

    return x;
}

void mergeset(int x, int y) { // 按size合并,如果想跑得更快就写一个按秩合并
    x = findroot(x); // 但是按秩合并要再开一个栈记录合并之前的秩
    y = findroot(y);

    if (x == y)
        return;

    if (size[x] > size[y])
        swap(x, y);

    p[x] = y;
    size[y] += size[x];
    stk[++top] = x;
}

void cut(int x) { // 并查集撤销
    int y = x;

    do
        size[y = p[y]] -= size[x];
    while (p[y]! = y);

    p[x] = x;
}
```

3.2 费用流

3.2.1 SPFA费用流

```cpp
constexpr int maxn = 20005, maxm = 200005;

struct edge {
    int to, prev, cap, w;
} e[maxm * 2];

int last[maxn], cnte, d[maxn], p[maxn]; // 记得把last初始化成-1, 不然会死循环
bool inq[maxn];

void spfa(int s) {

    memset(d, -63, sizeof(d));
    memset(p, -1, sizeof(p));

    queue<int> q;

    q.push(s);
    d[s] = 0;

    while (!q.empty()) {
        int x = q.front();
        q.pop();
        inq[x] = false;

        for (int i = last[x]; ~i; i = e[i].prev)
            if (e[i].cap) {
                int y = e[i].to;

                if (d[x] + e[i].w > d[y]) {
                    p[y] = i;
                    d[y] = d[x] + e[i].w;
                    if (!inq[y]) {
                        q.push(y);
                        inq[y] = true;
                    }
                }
            }
    }
}

int mcmf(int s, int t) {
    int ans = 0;

    while (spfa(s), d[t] > 0) {
        int flow = 0x3f3f3f3f;
        for (int x = t; x != s; x = e[p[x] ^ 1].to)
            flow = min(flow, e[p[x]].cap);

        ans += flow * d[t];

        for (int x = t; x != s; x = e[p[x] ^ 1].to) {
            e[p[x]].cap -= flow;
            e[p[x] ^ 1].cap += flow;
        }
    }

    return ans;
}

void add(int x, int y, int c, int w) {
    e[cnte].to = y;
    e[cnte].cap = c;
    e[cnte].w = w;

    e[cnte].prev = last[x];
    last[x] = cnte++;
}

void addedge(int x, int y, int c, int w) {
    add(x, y, c, w);
    add(y, x, 0, -w);
}
```

3.2.2 Dijkstra费用流

有的地方也叫原始-对偶费用流.
原理和求多源最短路的Johnson算法是一样的，都是给每个点维护一个势h_u，使得对任何有向边$u \to v$都满足$w + h_u - h_v \geq 0$.
如果有负费用则从s开始跑一遍SPFA初始化，否则可以直接初始化$h_u = 0$.
每次增广时得到的路径长度就是$d_{s,t} + h_t$，增广之后让所有$h_u = h'_u + d'_{s,u}$，直到$d_{s,t} = \infty$(最小费用最大流)或$d_{s,t} \geq 0$(最小费用流)为止.
注意最大费用流要转成取负之后的最小费用流，因为Dijkstra求的是最短路.

```
struct edge {
    int to, cap, prev, w;
} e[maxe * 2];

int last[maxn], cnte;

long long d[maxn], h[maxn];
int p[maxn];

bool vis[maxn];
int s, t;

void Adde(int x, int y, int z, int w) {
    e[cnte].to = y;
    e[cnte].cap = z;
    e[cnte].w = w;
    e[cnte].prev = last[x];
    last[x] = cnte++;
}

void addedge(int x, int y, int z, int w) {
    Adde(x, y, z, w);
    Adde(y, x, 0, -w);
}

void dijkstra() {
    memset(d, 63, sizeof(d));
    memset(vis, 0, sizeof(vis));

    priority_queue<pair<long long, int> > heap;

    d[s] = 0;
    heap.push(make_pair(0ll, s));

    while (!heap.empty()) {
        int x = heap.top().second;
        heap.pop();

        if (vis[x])
            continue;

        vis[x] = true;
        for (int i = last[x]; ~i; i = e[i].prev)
            if (e[i].cap > 0 && d[e[i].to] > d[x] + e[i].w
    ↪ + h[x] - h[e[i].to]) {
                d[e[i].to] = d[x] + e[i].w + h[x] -
    ↪ h[e[i].to];
                p[e[i].to] = i;
                heap.push(make_pair(-d[e[i].to], e[i].to));
            }
    }
}

pair<long long, long long> mcmf() {
    /*
    spfa();
    for (int i = 1; i <= t; i++)
        h[i] = d[i];
    // 如果初始有负权就像这样跑一遍SPFA预处理
    */

    long long flow = 0, cost = 0;

    while (dijkstra(), d[t] < 0x3f3f3f3f) {
        for (int i = 1; i <= t; i++)
            h[i] += d[i];

        int a = 0x3f3f3f3f;

        for (int x = t; x != s; x = e[p[x] ^ 1].to)
            a = min(a, e[p[x]].cap);

        flow += a;
        cost += (long long)a * h[t];

        for (int x = t; x != s; x = e[p[x] ^ 1].to) {
            e[p[x]].cap -= a;
            e[p[x] ^ 1].cap += a;
        }

    }

    return make_pair(flow, cost);
}
```

```
// 记得初始化
memset(last, -1, sizeof(last));
```

4 字符串

4.1 后缀自动机

```cpp
// 在字符集比较小的时候可以直接开go数组, 否则需要用map或者
  ↪ 哈希表替换
// 注意!!!结点数要开成串长的两倍

// 全局变量与数组定义
int last, len[maxn], fa[maxn], go[maxn][26], sam_cnt;
int c[maxn], q[maxn]; // 用来桶排序

// 在主函数开头加上这句初始化
last = sam_cnt = 1;

// 以下是按val进行桶排序的代码
for (int i = 1; i <= sam_cnt; i++)
    c[len[i] + 1]++;
for (int i = 1; i <= n; i++)
    c[i] += c[i - 1]; // 这里n是串长
for (int i = 1; i <= sam_cnt; i++)
    q[++c[len[i]]] = i;

//加入一个字符 均摊O(1)
void extend(int c) {
    int p = last, np = ++sam_cnt;
    len[np] = len[p] + 1;

    while (p && !go[p][c]) {
        go[p][c] = np;
        p = fa[p];
    }

    if (!p)
        fa[np] = 1;
    else {
        int q = go[p][c];

        if (len[q] == len[p] + 1)
            fa[np] = q;
        else {
            int nq = ++sam_cnt;
            len[nq] = len[p] + 1;
            memcpy(go[nq], go[q], sizeof(go[q]));

            fa[nq] = fa[q];
            fa[np] = fa[q] = nq;

            while (p && go[p][c] == q){
                go[p][c] = nq;
                p = fa[p];
            }
        }
    }

    last = np;
}
```

Theoretical Computer Science Cheat Sheet

Definitions	
$f(n) = O(g(n))$	iff $\exists$ positive c, n_0 such that $0 \le f(n) \le cg(n)$ $\forall n \ge n_0$.
$f(n) = \Omega(g(n))$	iff $\exists$ positive c, n_0 such that $f(n) \ge cg(n) \ge 0$ $\forall n \ge n_0$.
$f(n) = \Theta(g(n))$	iff $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$.
$f(n) = o(g(n))$	iff $\lim_{n\to\infty} f(n)/g(n) = 0$.
$\lim\limits_{n\to\infty} a_n = a$	iff $\forall \epsilon > 0$, $\exists n_0$ such that $\lvert a_n - a\rvert < \epsilon$, $\forall n \ge n_0$.
$\sup S$	least $b \in \mathbb{R}$ such that $b \ge s$, $\forall s \in S$.
$\inf S$	greatest $b \in \mathbb{R}$ such that $b \le s$, $\forall s \in S$.
$\liminf\limits_{n\to\infty} a_n$	$\lim\limits_{n\to\infty} \inf\{a_i \mid i \ge n, i \in \mathbb{N}\}$.
$\limsup\limits_{n\to\infty} a_n$	$\lim\limits_{n\to\infty} \sup\{a_i \mid i \ge n, i \in \mathbb{N}\}$.
$\binom{n}{k}$	Combinations: Size k subsets of a size n set.
$\left[{n \atop k}\right]$	Stirling numbers (1st kind): Arrangements of an n element set into k cycles.
$\left\{{n \atop k}\right\}$	Stirling numbers (2nd kind): Partitions of an n element set into k non-empty sets.
$\left\langle{n \atop k}\right\rangle$	1st order Eulerian numbers: Permutations $\pi_1\pi_2\ldots\pi_n$ on $\{1, 2, \ldots, n\}$ with k ascents.
$\left\langle\!\left\langle{n \atop k}\right\rangle\!\right\rangle$	2nd order Eulerian numbers.
C_n	Catalan Numbers: Binary trees with $n+1$ vertices.

Series

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}, \qquad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}, \qquad \sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}.$$

In general:

$$\sum_{i=1}^{n} i^m = \frac{1}{m+1}\left[(n+1)^{m+1} - 1 - \sum_{i=1}^{n}\left((i+1)^{m+1} - i^{m+1} - (m+1)i^m\right)\right]$$

$$\sum_{i=1}^{n-1} i^m = \frac{1}{m+1}\sum_{k=0}^{m}\binom{m+1}{k}B_k n^{m+1-k}.$$

Geometric series:

$$\sum_{i=0}^{n} c^i = \frac{c^{n+1}-1}{c-1}, \quad c \ne 1, \qquad \sum_{i=0}^{\infty} c^i = \frac{1}{1-c}, \qquad \sum_{i=1}^{\infty} c^i = \frac{c}{1-c}, \quad |c| < 1,$$

$$\sum_{i=0}^{n} ic^i = \frac{nc^{n+2} - (n+1)c^{n+1} + c}{(c-1)^2}, \quad c \ne 1, \qquad \sum_{i=0}^{\infty} ic^i = \frac{c}{(1-c)^2}, \quad |c| < 1.$$

Harmonic series:

$$H_n = \sum_{i=1}^{n} \frac{1}{i}, \qquad \sum_{i=1}^{n} iH_i = \frac{n(n+1)}{2}H_n - \frac{n(n-1)}{4}.$$

$$\sum_{i=1}^{n} H_i = (n+1)H_n - n, \qquad \sum_{i=1}^{n}\binom{i}{m}H_i = \binom{n+1}{m+1}\left(H_{n+1} - \frac{1}{m+1}\right).$$

1. $\displaystyle\binom{n}{k} = \frac{n!}{(n-k)!k!}$, **2.** $\displaystyle\sum_{k=0}^{n}\binom{n}{k} = 2^n$, **3.** $\displaystyle\binom{n}{k} = \binom{n}{n-k}$,

4. $\displaystyle\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}$, **5.** $\displaystyle\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$,

6. $\displaystyle\binom{n}{m}\binom{m}{k} = \binom{n}{k}\binom{n-k}{m-k}$, **7.** $\displaystyle\sum_{k=0}^{n}\binom{r+k}{k} = \binom{r+n+1}{n}$,

8. $\displaystyle\sum_{k=0}^{n}\binom{k}{m} = \binom{n+1}{m+1}$, **9.** $\displaystyle\sum_{k=0}^{n}\binom{r}{k}\binom{s}{n-k} = \binom{r+s}{n}$,

10. $\displaystyle\binom{n}{k} = (-1)^k\binom{k-n-1}{k}$, **11.** $\displaystyle\left\{{n \atop 1}\right\} = \left\{{n \atop n}\right\} = 1$,

12. $\displaystyle\left\{{n \atop 2}\right\} = 2^{n-1} - 1$, **13.** $\displaystyle\left\{{n \atop k}\right\} = k\left\{{n-1 \atop k}\right\} + \left\{{n-1 \atop k-1}\right\}$,

14. $\displaystyle\left[{n \atop 1}\right] = (n-1)!$, **15.** $\displaystyle\left[{n \atop 2}\right] = (n-1)!H_{n-1}$, **16.** $\displaystyle\left[{n \atop n}\right] = 1$, **17.** $\displaystyle\left[{n \atop k}\right] \ge \left\{{n \atop k}\right\}$,

18. $\displaystyle\left[{n \atop k}\right] = (n-1)\left[{n-1 \atop k}\right] + \left[{n-1 \atop k-1}\right]$, **19.** $\displaystyle\left\{{n \atop n-1}\right\} = \left[{n \atop n-1}\right] = \binom{n}{2}$, **20.** $\displaystyle\sum_{k=0}^{n}\left[{n \atop k}\right] = n!$, **21.** $\displaystyle C_n = \frac{1}{n+1}\binom{2n}{n}$,

22. $\displaystyle\left\langle{n \atop 0}\right\rangle = \left\langle{n \atop n-1}\right\rangle = 1$, **23.** $\displaystyle\left\langle{n \atop k}\right\rangle = \left\langle{n \atop n-1-k}\right\rangle$, **24.** $\displaystyle\left\langle{n \atop k}\right\rangle = (k+1)\left\langle{n-1 \atop k}\right\rangle + (n-k)\left\langle{n-1 \atop k-1}\right\rangle$,

25. $\displaystyle\left\langle{0 \atop k}\right\rangle = \begin{cases}1 & \text{if } k = 0,\\ 0 & \text{otherwise}\end{cases}$ **26.** $\displaystyle\left\langle{n \atop 1}\right\rangle = 2^n - n - 1$, **27.** $\displaystyle\left\langle{n \atop 2}\right\rangle = 3^n - (n+1)2^n + \binom{n+1}{2}$,

28. $\displaystyle x^n = \sum_{k=0}^{n}\left\langle{n \atop k}\right\rangle\binom{x+k}{n}$, **29.** $\displaystyle\left\langle{n \atop m}\right\rangle = \sum_{k=0}^{m}\binom{n+1}{k}(m+1-k)^n(-1)^k$, **30.** $\displaystyle m!\left\{{n \atop m}\right\} = \sum_{k=0}^{n}\left\langle{n \atop k}\right\rangle\binom{k}{n-m}$,

31. $\displaystyle\left\langle{n \atop m}\right\rangle = \sum_{k=0}^{n}\left\{{n \atop k}\right\}\binom{n-k}{m}(-1)^{n-k-m}k!$, **32.** $\displaystyle\left\langle\!\left\langle{n \atop 0}\right\rangle\!\right\rangle = 1$, **33.** $\displaystyle\left\langle\!\left\langle{n \atop n}\right\rangle\!\right\rangle = 0 \quad \text{for } n \ne 0$,

34. $\displaystyle\left\langle\!\left\langle{n \atop k}\right\rangle\!\right\rangle = (k+1)\left\langle\!\left\langle{n-1 \atop k}\right\rangle\!\right\rangle + (2n-1-k)\left\langle\!\left\langle{n-1 \atop k-1}\right\rangle\!\right\rangle$, **35.** $\displaystyle\sum_{k=0}^{n}\left\langle\!\left\langle{n \atop k}\right\rangle\!\right\rangle = \frac{(2n)^{\underline{n}}}{2^n}$,

36. $\displaystyle\left\{{x \atop x-n}\right\} = \sum_{k=0}^{n}\left\langle\!\left\langle{n \atop k}\right\rangle\!\right\rangle\binom{x+n-1-k}{2n}$, **37.** $\displaystyle\left\{{n+1 \atop m+1}\right\} = \sum_{k}\binom{n}{k}\left\{{k \atop m}\right\} = \sum_{k=0}^{n}\left\{{k \atop m}\right\}(m+1)^{n-k}$,

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Identities Cont.

38. $\left[{n+1 \atop m+1}\right] = \sum_k \left[{n \atop k}\right]\binom{k}{m} = \sum_{k=0}^{n} \left[{k \atop m}\right] n^{\underline{n-k}} = n! \sum_{k=0}^{n} \frac{1}{k!}\left[{k \atop m}\right],$

39. $\left[{x \atop x-n}\right] = \sum_{k=0}^{n} \left\langle\!\!\left\langle {n \atop k} \right\rangle\!\!\right\rangle \binom{x+k}{2n},$

40. $\left\{{n \atop m}\right\} = \sum_k \binom{n}{k}\left\{{k+1 \atop m+1}\right\}(-1)^{n-k},$

41. $\left[{n \atop m}\right] = \sum_k \left[{n+1 \atop k+1}\right]\binom{k}{m}(-1)^{m-k},$

42. $\left\{{m+n+1 \atop m}\right\} = \sum_{k=0}^{m} k\left\{{n+k \atop k}\right\},$

43. $\left[{m+n+1 \atop m}\right] = \sum_{k=0}^{m} k(n+k)\left[{n+k \atop k}\right],$

44. $\binom{n}{m} = \sum_k \left\{{n+1 \atop k+1}\right\}\left[{k \atop m}\right](-1)^{m-k},$

45. $(n-m)!\binom{n}{m} = \sum_k \left[{n+1 \atop k+1}\right]\left\{{k \atop m}\right\}(-1)^{m-k}, \quad \text{for } n \ge m,$

46. $\left\{{n \atop n-m}\right\} = \sum_k \binom{m-n}{m+k}\binom{m+n}{n+k}\left[{m+k \atop k}\right],$

47. $\left[{n \atop n-m}\right] = \sum_k \binom{m-n}{m+k}\binom{m+n}{n+k}\left\{{m+k \atop k}\right\},$

48. $\left\{{n \atop \ell+m}\right\}\binom{\ell+m}{\ell} = \sum_k \left\{{k \atop \ell}\right\}\left\{{n-k \atop m}\right\}\binom{n}{k},$

49. $\left[{n \atop \ell+m}\right]\binom{\ell+m}{\ell} = \sum_k \left[{k \atop \ell}\right]\left[{n-k \atop m}\right]\binom{n}{k}.$

Trees

Every tree with n vertices has $n-1$ edges.

Kraft inequality: If the depths of the leaves of a binary tree are $d_1, \ldots, d_n$:
$$\sum_{i=1}^{n} 2^{-d_i} \le 1,$$
and equality holds only if every internal node has 2 sons.

Recurrences

Master method:
$$T(n) = aT(n/b) + f(n), \quad a \ge 1, b > 1$$

If $\exists \epsilon > 0$ such that $f(n) = O(n^{\log_b a - \epsilon})$ then
$$T(n) = \Theta(n^{\log_b a}).$$

If $f(n) = \Theta(n^{\log_b a})$ then
$$T(n) = \Theta(n^{\log_b a} \log_2 n).$$

If $\exists \epsilon > 0$ such that $f(n) = \Omega(n^{\log_b a + \epsilon})$, and $\exists c < 1$ such that $af(n/b) \le cf(n)$ for large n, then
$$T(n) = \Theta(f(n)).$$

Substitution (example): Consider the following recurrence
$$T_{i+1} = 2^{2^i} \cdot T_i^2, \quad T_1 = 2.$$

Note that T_i is always a power of two. Let $t_i = \log_2 T_i$. Then we have
$$t_{i+1} = 2^i + 2t_i, \quad t_1 = 1.$$

Let $u_i = t_i/2^i$. Dividing both sides of the previous equation by 2^{i+1} we get
$$\frac{t_{i+1}}{2^{i+1}} = \frac{2^i}{2^{i+1}} + \frac{t_i}{2^i}.$$

Substituting we find
$$u_{i+1} = \tfrac{1}{2} + u_i, \qquad u_1 = \tfrac{1}{2},$$

which is simply $u_i = i/2$. So we find that T_i has the closed form $T_i = 2^{i2^{i-1}}$. Summing factors (example): Consider the following recurrence
$$T(n) = 3T(n/2) + n, \quad T(1) = 1.$$

Rewrite so that all terms involving T are on the left side
$$T(n) - 3T(n/2) = n.$$

Now expand the recurrence, and choose a factor which makes the left side "telescope"
$$1\big(T(n) - 3T(n/2) = n\big)$$
$$3\big(T(n/2) - 3T(n/4) = n/2\big)$$
$$\vdots \quad \vdots \quad \vdots$$
$$3^{\log_2 n - 1}\big(T(2) - 3T(1) = 2\big)$$

Let $m = \log_2 n$. Summing the left side we get $T(n) - 3^m T(1) = T(n) - 3^m = T(n) - n^k$ where $k = \log_2 3 \approx 1.58496$. Summing the right side we get
$$\sum_{i=0}^{m-1} \frac{n}{2^i} 3^i = n \sum_{i=0}^{m-1} \left(\tfrac{3}{2}\right)^i.$$

Let $c = \frac{3}{2}$. Then we have
$$\begin{aligned} n\sum_{i=0}^{m-1} c^i &= n\left(\frac{c^m - 1}{c - 1}\right) \\ &= 2n(c^{\log_2 n} - 1) \\ &= 2n(c^{(k-1)\log_c n} - 1) \\ &= 2n^k - 2n, \end{aligned}$$

and so $T(n) = 3n^k - 2n$. Full history recurrences can often be changed to limited history ones (example): Consider
$$T_i = 1 + \sum_{j=0}^{i-1} T_j, \quad T_0 = 1.$$

Note that
$$T_{i+1} = 1 + \sum_{j=0}^{i} T_j.$$

Subtracting we find
$$\begin{aligned} T_{i+1} - T_i &= 1 + \sum_{j=0}^{i} T_j - 1 - \sum_{j=0}^{i-1} T_j \\ &= T_i. \end{aligned}$$

And so $T_{i+1} = 2T_i = 2^{i+1}$.

Generating functions:

1. Multiply both sides of the equation by x^i.
2. Sum both sides over all i for which the equation is valid.
3. Choose a generating function $G(x)$. Usually $G(x) = \sum_{i=0}^{\infty} x^i g_i$.
3. Rewrite the equation in terms of the generating function $G(x)$.
4. Solve for $G(x)$.
5. The coefficient of x^i in $G(x)$ is g_i.

Example:
$$g_{i+1} = 2g_i + 1, \quad g_0 = 0.$$

Multiply and sum:
$$\sum_{i\ge0} g_{i+1}x^i = \sum_{i\ge0} 2g_i x^i + \sum_{i\ge0} x^i.$$

We choose $G(x) = \sum_{i\ge0} x^i g_i$. Rewrite in terms of $G(x)$:
$$\frac{G(x) - g_0}{x} = 2G(x) + \sum_{i\ge0} x^i.$$

Simplify:
$$\frac{G(x)}{x} = 2G(x) + \frac{1}{1-x}.$$

Solve for $G(x)$:
$$G(x) = \frac{x}{(1-x)(1-2x)}.$$

Expand this using partial fractions:
$$\begin{aligned} G(x) &= x\left(\frac{2}{1-2x} - \frac{1}{1-x}\right) \\ &= x\left(2\sum_{i\ge0} 2^i x^i - \sum_{i\ge0} x^i\right) \\ &= \sum_{i\ge0}(2^{i+1} - 1)x^{i+1}. \end{aligned}$$

So $g_i = 2^i - 1$.

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$\pi \approx 3.14159$, $e \approx 2.71828$, $\gamma \approx 0.57721$, $\phi = \frac{1+\sqrt{5}}{2} \approx 1.61803$, $\hat{\phi} = \frac{1-\sqrt{5}}{2} \approx -.61803$

i	2^i	p_i
1	2	2
2	4	3
3	8	5
4	16	7
5	32	11
6	64	13
7	128	17
8	256	19
9	512	23
10	1,024	29
11	2,048	31
12	4,096	37
13	8,192	41
14	16,384	43
15	32,768	47
16	65,536	53
17	131,072	59
18	262,144	61
19	524,288	67
20	1,048,576	71
21	2,097,152	73
22	4,194,304	79
23	8,388,608	83
24	16,777,216	89
25	33,554,432	97
26	67,108,864	101
27	134,217,728	103
28	268,435,456	107
29	536,870,912	109
30	1,073,741,824	113
31	2,147,483,648	127
32	4,294,967,296	131

Pascal's Triangle

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
1 9 36 84 126 126 84 36 9 1
1 10 45 120 210 252 210 120 45 10 1

General

Bernoulli Numbers ($B_i = 0$, odd $i \neq 1$):

$$B_0 = 1,\ B_1 = -\tfrac{1}{2},\ B_2 = \tfrac{1}{6},\ B_4 = -\tfrac{1}{30},\ B_6 = \tfrac{1}{42},\ B_8 = -\tfrac{1}{30},\ B_{10} = \tfrac{5}{66}.$$

Change of base, quadratic formula:

$$\log_b x = \frac{\log_a x}{\log_a b}, \qquad \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Euler's number e:

$$e = 1 + \tfrac{1}{2} + \tfrac{1}{6} + \tfrac{1}{24} + \tfrac{1}{120} + \cdots$$

$$\lim_{n\to\infty}\left(1 + \frac{x}{n}\right)^n = e^x.$$

$$\left(1 + \tfrac{1}{n}\right)^n < e < \left(1 + \tfrac{1}{n}\right)^{n+1}.$$

$$\left(1 + \tfrac{1}{n}\right)^n = e - \frac{e}{2n} + \frac{11e}{24n^2} - O\left(\frac{1}{n^3}\right).$$

Harmonic numbers:

$$1,\ \tfrac{3}{2},\ \tfrac{11}{6},\ \tfrac{25}{12},\ \tfrac{137}{60},\ \tfrac{49}{20},\ \tfrac{363}{140},\ \tfrac{761}{280},\ \tfrac{7129}{2520},\ \ldots$$

$$\ln n < H_n < \ln n + 1,$$

$$H_n = \ln n + \gamma + O\left(\frac{1}{n}\right).$$

Factorial, Stirling's approximation:

1, 2, 6, 24, 120, 720, 5040, 40320, 362880, ...

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n\left(1 + \Theta\left(\frac{1}{n}\right)\right).$$

Ackermann's function and inverse:

$$a(i,j) = \begin{cases} 2^j & i = 1 \\ a(i-1, 2) & j = 1 \\ a(i-1, a(i, j-1)) & i, j \geq 2 \end{cases}$$

$$\alpha(i) = \min\{j \mid a(j,j) \geq i\}.$$

Binomial distribution:

$$\Pr[X = k] = \binom{n}{k} p^k q^{n-k}, \qquad q = 1 - p,$$

$$\mathrm{E}[X] = \sum_{k=1}^{n} k \binom{n}{k} p^k q^{n-k} = np.$$

Poisson distribution:

$$\Pr[X = k] = \frac{e^{-\lambda}\lambda^k}{k!}, \quad \mathrm{E}[X] = \lambda.$$

Normal (Gaussian) distribution:

$$p(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2}, \quad \mathrm{E}[X] = \mu.$$

The "coupon collector": We are given a random coupon each day, and there are n different types of coupons. The distribution of coupons is uniform. The expected number of days to pass before we to collect all n types is

$$nH_n.$$

Probability

Continuous distributions: If

$$\Pr[a < X < b] = \int_a^b p(x)\,dx,$$

then p is the probability density function of X. If

$$\Pr[X < a] = P(a),$$

then P is the distribution function of X. If P and p both exist then

$$P(a) = \int_{-\infty}^{a} p(x)\,dx.$$

Expectation: If X is discrete

$$\mathrm{E}[g(X)] = \sum_x g(x)\Pr[X = x].$$

If X continuous then

$$\mathrm{E}[g(X)] = \int_{-\infty}^{\infty} g(x)p(x)\,dx = \int_{-\infty}^{\infty} g(x)\,dP(x).$$

Variance, standard deviation:

$$\mathrm{VAR}[X] = \mathrm{E}[X^2] - \mathrm{E}[X]^2,$$

$$\sigma = \sqrt{\mathrm{VAR}[X]}.$$

For events A and B:

$$\Pr[A \vee B] = \Pr[A] + \Pr[B] - \Pr[A \wedge B]$$

$$\Pr[A \wedge B] = \Pr[A] \cdot \Pr[B],$$

iff A and B are independent.

$$\Pr[A|B] = \frac{\Pr[A \wedge B]}{\Pr[B]}$$

For random variables X and Y:

$$\mathrm{E}[X \cdot Y] = \mathrm{E}[X] \cdot \mathrm{E}[Y],$$

if X and Y are independent.

$$\mathrm{E}[X + Y] = \mathrm{E}[X] + \mathrm{E}[Y],$$

$$\mathrm{E}[cX] = c\,\mathrm{E}[X].$$

Bayes' theorem:

$$\Pr[A_i|B] = \frac{\Pr[B|A_i]\Pr[A_i]}{\sum_{j=1}^{n}\Pr[A_j]\Pr[B|A_j]}.$$

Inclusion-exclusion:

$$\Pr\left[\bigvee_{i=1}^{n} X_i\right] = \sum_{i=1}^{n}\Pr[X_i] + \sum_{k=2}^{n}(-1)^{k+1}\sum_{i_i < \cdots < i_k}\Pr\left[\bigwedge_{j=1}^{k} X_{i_j}\right].$$

Moment inequalities:

$$\Pr\left[|X| \geq \lambda\,\mathrm{E}[X]\right] \leq \frac{1}{\lambda},$$

$$\Pr\left[\left|X - \mathrm{E}[X]\right| \geq \lambda \cdot \sigma\right] \leq \frac{1}{\lambda^2}.$$

Geometric distribution:

$$\Pr[X = k] = pq^{k-1}, \qquad q = 1 - p,$$

$$\mathrm{E}[X] = \sum_{k=1}^{\infty} kpq^{k-1} = \frac{1}{p}.$$

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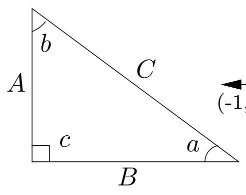
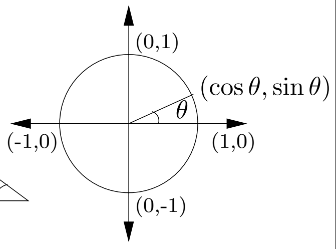
Trigonometry





Pythagorean theorem:
$$C^2 = A^2 + B^2.$$

Definitions:
$$\sin a = A/C, \quad \cos a = B/C,$$
$$\csc a = C/A, \quad \sec a = C/B,$$
$$\tan a = \frac{\sin a}{\cos a} = \frac{A}{B}, \quad \cot a = \frac{\cos a}{\sin a} = \frac{B}{A}.$$

Area, radius of inscribed circle:
$$\tfrac{1}{2}AB, \quad \frac{AB}{A+B+C}.$$

Identities:
$$\sin x = \frac{1}{\csc x}, \qquad \cos x = \frac{1}{\sec x},$$
$$\tan x = \frac{1}{\cot x}, \qquad \sin^2 x + \cos^2 x = 1,$$
$$1 + \tan^2 x = \sec^2 x, \qquad 1 + \cot^2 x = \csc^2 x,$$
$$\sin x = \cos\left(\tfrac{\pi}{2} - x\right), \qquad \sin x = \sin(\pi - x),$$
$$\cos x = -\cos(\pi - x), \qquad \tan x = \cot\left(\tfrac{\pi}{2} - x\right),$$
$$\cot x = -\cot(\pi - x), \qquad \csc x = \cot\tfrac{x}{2} - \cot x,$$
$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y,$$
$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y,$$
$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y},$$
$$\cot(x \pm y) = \frac{\cot x \cot y \mp 1}{\cot x \pm \cot y},$$
$$\sin 2x = 2 \sin x \cos x, \qquad \sin 2x = \frac{2 \tan x}{1 + \tan^2 x},$$
$$\cos 2x = \cos^2 x - \sin^2 x, \qquad \cos 2x = 2\cos^2 x - 1,$$
$$\cos 2x = 1 - 2\sin^2 x, \qquad \cos 2x = \frac{1 - \tan^2 x}{1 + \tan^2 x},$$
$$\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}, \qquad \cot 2x = \frac{\cot^2 x - 1}{2 \cot x},$$
$$\sin(x+y)\sin(x-y) = \sin^2 x - \sin^2 y,$$
$$\cos(x+y)\cos(x-y) = \cos^2 x - \sin^2 y.$$

Euler's equation:
$$e^{ix} = \cos x + i \sin x, \qquad e^{i\pi} = -1.$$

Matrices

Multiplication:
$$C = A \cdot B, \quad c_{i,j} = \sum_{k=1}^{n} a_{i,k} b_{k,j}.$$

Determinants: $\det A \neq 0$ iff A is non-singular.
$$\det A \cdot B = \det A \cdot \det B,$$
$$\det A = \sum_{\pi} \prod_{i=1}^{n} \operatorname{sign}(\pi) a_{i,\pi(i)}.$$

2×2 and 3×3 determinant:
$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc,$$
$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = g \begin{vmatrix} b & c \\ e & f \end{vmatrix} - h \begin{vmatrix} a & c \\ d & f \end{vmatrix} + i \begin{vmatrix} a & b \\ d & e \end{vmatrix}$$
$$= \begin{matrix} aei + bfg + cdh \\ - ceg - fha - ibd. \end{matrix}$$

Permanents:
$$\operatorname{perm} A = \sum_{\pi} \prod_{i=1}^{n} a_{i,\pi(i)}.$$

Hyperbolic Functions

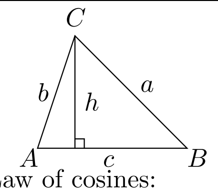
Definitions:
$$\sinh x = \frac{e^x - e^{-x}}{2}, \qquad \cosh x = \frac{e^x + e^{-x}}{2},$$
$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}, \qquad \operatorname{csch} x = \frac{1}{\sinh x},$$
$$\operatorname{sech} x = \frac{1}{\cosh x}, \qquad \coth x = \frac{1}{\tanh x}.$$

Identities:
$$\cosh^2 x - \sinh^2 x = 1, \qquad \tanh^2 x + \operatorname{sech}^2 x = 1,$$
$$\coth^2 x - \operatorname{csch}^2 x = 1, \qquad \sinh(-x) = -\sinh x,$$
$$\cosh(-x) = \cosh x, \qquad \tanh(-x) = -\tanh x,$$
$$\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y,$$
$$\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y,$$
$$\sinh 2x = 2 \sinh x \cosh x,$$
$$\cosh 2x = \cosh^2 x + \sinh^2 x,$$
$$\cosh x + \sinh x = e^x, \qquad \cosh x - \sinh x = e^{-x},$$
$$(\cosh x + \sinh x)^n = \cosh nx + \sinh nx, \quad n \in \mathbb{Z},$$
$$2\sinh^2 \tfrac{x}{2} = \cosh x - 1, \qquad 2\cosh^2 \tfrac{x}{2} = \cosh x + 1.$$

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$
0	0	1	0
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
$\frac{\pi}{2}$	1	0	∞

... in mathematics you don't understand things, you just get used to them.
– J. von Neumann

More Trig.



Law of cosines:
$$c^2 = a^2 + b^2 - 2ab\cos C.$$

Area:
$$A = \tfrac{1}{2}hc,$$
$$= \tfrac{1}{2}ab\sin C,$$
$$= \frac{c^2 \sin A \sin B}{2 \sin C}.$$

Heron's formula:
$$A = \sqrt{s \cdot s_a \cdot s_b \cdot s_c},$$
$$s = \tfrac{1}{2}(a+b+c),$$
$$s_a = s - a,$$
$$s_b = s - b,$$
$$s_c = s - c.$$

More identities:
$$\sin\tfrac{x}{2} = \sqrt{\frac{1 - \cos x}{2}},$$
$$\cos\tfrac{x}{2} = \sqrt{\frac{1 + \cos x}{2}},$$
$$\tan\tfrac{x}{2} = \sqrt{\frac{1 - \cos x}{1 + \cos x}},$$
$$= \frac{1 - \cos x}{\sin x},$$
$$= \frac{\sin x}{1 + \cos x},$$
$$\cot\tfrac{x}{2} = \sqrt{\frac{1 + \cos x}{1 - \cos x}},$$
$$= \frac{1 + \cos x}{\sin x},$$
$$= \frac{\sin x}{1 - \cos x},$$
$$\sin x = \frac{e^{ix} - e^{-ix}}{2i},$$
$$\cos x = \frac{e^{ix} + e^{-ix}}{2},$$
$$\tan x = -i\frac{e^{ix} - e^{-ix}}{e^{ix} + e^{-ix}},$$
$$= -i\frac{e^{2ix} - 1}{e^{2ix} + 1},$$
$$\sin x = \frac{\sinh ix}{i},$$
$$\cos x = \cosh ix,$$
$$\tan x = \frac{\tanh ix}{i}.$$

Theoretical Computer Science Cheat Sheet

Number Theory

The Chinese remainder theorem: There exists a number C such that:

$$C \equiv r_1 \bmod m_1$$
$$\vdots \quad \vdots \quad \vdots$$
$$C \equiv r_n \bmod m_n$$

if m_i and m_j are relatively prime for $i \neq j$.

Euler's function: $\phi(x)$ is the number of positive integers less than x relatively prime to x. If $\prod_{i=1}^{n} p_i^{e_i}$ is the prime factorization of x then

$$\phi(x) = \prod_{i=1}^{n} p_i^{e_i-1}(p_i - 1).$$

Euler's theorem: If a and b are relatively prime then

$$1 \equiv a^{\phi(b)} \bmod b.$$

Fermat's theorem:

$$1 \equiv a^{p-1} \bmod p.$$

The Euclidean algorithm: if $a > b$ are integers then

$$\gcd(a, b) = \gcd(a \bmod b, b).$$

If $\prod_{i=1}^{n} p_i^{e_i}$ is the prime factorization of x then

$$S(x) = \sum_{d|x} d = \prod_{i=1}^{n} \frac{p_i^{e_i+1} - 1}{p_i - 1}.$$

Perfect Numbers: x is an even perfect number iff $x = 2^{n-1}(2^n - 1)$ and $2^n - 1$ is prime.
Wilson's theorem: n is a prime iff

$$(n-1)! \equiv -1 \bmod n.$$

Möbius inversion:

$$\mu(i) = \begin{cases} 1 & \text{if } i = 1. \\ 0 & \text{if } i \text{ is not square-free.} \\ (-1)^r & \text{if } i \text{ is the product of} \\ & r \text{ distinct primes.} \end{cases}$$

If

$$G(a) = \sum_{d|a} F(d),$$

then

$$F(a) = \sum_{d|a} \mu(d) G\Big(\frac{a}{d}\Big).$$

Prime numbers:

$$p_n = n \ln n + n \ln\ln n - n + n\frac{\ln\ln n}{\ln n} + O\Big(\frac{n}{\ln n}\Big),$$

$$\pi(n) = \frac{n}{\ln n} + \frac{n}{(\ln n)^2} + \frac{2!n}{(\ln n)^3} + O\Big(\frac{n}{(\ln n)^4}\Big).$$

Graph Theory

Definitions:

Loop	An edge connecting a vertex to itself.
Directed	Each edge has a direction.
Simple	Graph with no loops or multi-edges.
Walk	A sequence $v_0 e_1 v_1 \dots e_\ell v_\ell$.
Trail	A walk with distinct edges.
Path	A trail with distinct vertices.
Connected	A graph where there exists a path between any two vertices.
Component	A maximal connected subgraph.
Tree	A connected acyclic graph.
Free tree	A tree with no root.
DAG	Directed acyclic graph.
Eulerian	Graph with a trail visiting each edge exactly once.
Hamiltonian	Graph with a cycle visiting each vertex exactly once.
Cut	A set of edges whose removal increases the number of components.
Cut-set	A minimal cut.
Cut edge	A size 1 cut.
k-Connected	A graph connected with the removal of any $k-1$ vertices.
k-Tough	$\forall S \subseteq V, S \neq \emptyset$ we have $k \cdot c(G - S) \leq \lvert S \rvert$.
k-Regular	A graph where all vertices have degree k.
k-Factor	A k-regular spanning subgraph.
Matching	A set of edges, no two of which are adjacent.
Clique	A set of vertices, all of which are adjacent.
Ind. set	A set of vertices, none of which are adjacent.
Vertex cover	A set of vertices which cover all edges.
Planar graph	A graph which can be embeded in the plane.
Plane graph	An embedding of a planar graph.

$$\sum_{v \in V} \deg(v) = 2m.$$

If G is planar then $n - m + f = 2$, so

$$f \leq 2n - 4, \quad m \leq 3n - 6.$$

Any planar graph has a vertex with degree ≤ 5.

Notation:

$E(G)$	Edge set
$V(G)$	Vertex set
$c(G)$	Number of components
$G[S]$	Induced subgraph
$\deg(v)$	Degree of v
$\Delta(G)$	Maximum degree
$\delta(G)$	Minimum degree
$\chi(G)$	Chromatic number
$\chi_E(G)$	Edge chromatic number
G^c	Complement graph
K_n	Complete graph
K_{n_1,n_2}	Complete bipartite graph
$\mathrm{r}(k, \ell)$	Ramsey number

Geometry

Projective coordinates: triples (x, y, z), not all x, y and z zero.

$$(x, y, z) = (cx, cy, cz) \quad \forall c \neq 0.$$

Cartesian	Projective
(x, y)	$(x, y, 1)$
$y = mx + b$	$(m, -1, b)$
$x = c$	$(1, 0, -c)$

Distance formula, L_p and L_∞ metric:

$$\sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2},$$

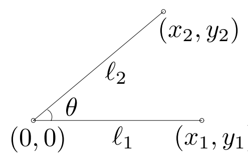
$$\big[|x_1 - x_0|^p + |y_1 - y_0|^p\big]^{1/p},$$

$$\lim_{p \to \infty} \big[|x_1 - x_0|^p + |y_1 - y_0|^p\big]^{1/p}.$$

Area of triangle (x_0, y_0), (x_1, y_1) and (x_2, y_2):

$$\tfrac{1}{2} \operatorname{abs} \begin{vmatrix} x_1 - x_0 & y_1 - y_0 \\ x_2 - x_0 & y_2 - y_0 \end{vmatrix}.$$

Angle formed by three points:



$$\cos\theta = \frac{(x_1, y_1) \cdot (x_2, y_2)}{\ell_1 \ell_2}.$$

Line through two points (x_0, y_0) and (x_1, y_1):

$$\begin{vmatrix} x & y & 1 \\ x_0 & y_0 & 1 \\ x_1 & y_1 & 1 \end{vmatrix} = 0.$$

Area of circle, volume of sphere:

$$A = \pi r^2, \qquad V = \tfrac{4}{3}\pi r^3.$$

If I have seen farther than others, it is because I have stood on the shoulders of giants.
– Issac Newton

Theoretical Computer Science Cheat Sheet

π

Wallis' identity:
$$\pi = 2 \cdot \frac{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdots}{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdots}$$
Brouncker's continued fraction expansion:
$$\tfrac{\pi}{4} = 1 + \cfrac{1^2}{2 + \cfrac{3^2}{2 + \cfrac{5^2}{2 + \cfrac{7^2}{2 + \cdots}}}}$$
Gregrory's series:
$$\tfrac{\pi}{4} = 1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \tfrac{1}{9} - \cdots$$
Newton's series:
$$\tfrac{\pi}{6} = \frac{1}{2} + \frac{1}{2 \cdot 3 \cdot 2^3} + \frac{1 \cdot 3}{2 \cdot 4 \cdot 5 \cdot 2^5} + \cdots$$
Sharp's series:
$$\tfrac{\pi}{6} = \frac{1}{\sqrt{3}}\left(1 - \frac{1}{3^1 \cdot 3} + \frac{1}{3^2 \cdot 5} - \frac{1}{3^3 \cdot 7} + \cdots\right)$$
Euler's series:
$$\tfrac{\pi^2}{6} = \tfrac{1}{1^2} + \tfrac{1}{2^2} + \tfrac{1}{3^2} + \tfrac{1}{4^2} + \tfrac{1}{5^2} + \cdots$$
$$\tfrac{\pi^2}{8} = \tfrac{1}{1^2} + \tfrac{1}{3^2} + \tfrac{1}{5^2} + \tfrac{1}{7^2} + \tfrac{1}{9^2} + \cdots$$
$$\tfrac{\pi^2}{12} = \tfrac{1}{1^2} - \tfrac{1}{2^2} + \tfrac{1}{3^2} - \tfrac{1}{4^2} + \tfrac{1}{5^2} - \cdots$$

Partial Fractions

Let $N(x)$ and $D(x)$ be polynomial functions of x. We can break down $N(x)/D(x)$ using partial fraction expansion. First, if the degree of N is greater than or equal to the degree of D, divide N by D, obtaining
$$\frac{N(x)}{D(x)} = Q(x) + \frac{N'(x)}{D(x)},$$
where the degree of N' is less than that of D. Second, factor $D(x)$. Use the following rules: For a non-repeated factor:
$$\frac{N(x)}{(x-a)D(x)} = \frac{A}{x-a} + \frac{N'(x)}{D(x)},$$
where
$$A = \left[\frac{N(x)}{D(x)}\right]_{x=a}.$$
For a repeated factor:
$$\frac{N(x)}{(x-a)^m D(x)} = \sum_{k=0}^{m-1} \frac{A_k}{(x-a)^{m-k}} + \frac{N'(x)}{D(x)},$$
where
$$A_k = \frac{1}{k!}\left[\frac{d^k}{dx^k}\left(\frac{N(x)}{D(x)}\right)\right]_{x=a}.$$

The reasonable man adapts himself to the world; the unreasonable persists in trying to adapt the world to himself. Therefore all progress depends on the unreasonable.
– George Bernard Shaw

Calculus

Derivatives:

1. $\dfrac{d(cu)}{dx} = c\dfrac{du}{dx}$, **2.** $\dfrac{d(u+v)}{dx} = \dfrac{du}{dx} + \dfrac{dv}{dx}$, **3.** $\dfrac{d(uv)}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$,

4. $\dfrac{d(u^n)}{dx} = nu^{n-1}\dfrac{du}{dx}$, **5.** $\dfrac{d(u/v)}{dx} = \dfrac{v\left(\frac{du}{dx}\right) - u\left(\frac{dv}{dx}\right)}{v^2}$, **6.** $\dfrac{d(e^{cu})}{dx} = ce^{cu}\dfrac{du}{dx}$,

7. $\dfrac{d(c^u)}{dx} = (\ln c)c^u\dfrac{du}{dx}$, **8.** $\dfrac{d(\ln u)}{dx} = \dfrac{1}{u}\dfrac{du}{dx}$,

9. $\dfrac{d(\sin u)}{dx} = \cos u\dfrac{du}{dx}$, **10.** $\dfrac{d(\cos u)}{dx} = -\sin u\dfrac{du}{dx}$,

11. $\dfrac{d(\tan u)}{dx} = \sec^2 u\dfrac{du}{dx}$, **12.** $\dfrac{d(\cot u)}{dx} = \csc^2 u\dfrac{du}{dx}$,

13. $\dfrac{d(\sec u)}{dx} = \tan u \sec u\dfrac{du}{dx}$, **14.** $\dfrac{d(\csc u)}{dx} = -\cot u \csc u\dfrac{du}{dx}$,

15. $\dfrac{d(\arcsin u)}{dx} = \dfrac{1}{\sqrt{1-u^2}}\dfrac{du}{dx}$, **16.** $\dfrac{d(\arccos u)}{dx} = \dfrac{-1}{\sqrt{1-u^2}}\dfrac{du}{dx}$,

17. $\dfrac{d(\arctan u)}{dx} = \dfrac{1}{1+u^2}\dfrac{du}{dx}$, **18.** $\dfrac{d(\operatorname{arccot} u)}{dx} = \dfrac{-1}{1+u^2}\dfrac{du}{dx}$,

19. $\dfrac{d(\operatorname{arcsec} u)}{dx} = \dfrac{1}{u\sqrt{1-u^2}}\dfrac{du}{dx}$, **20.** $\dfrac{d(\operatorname{arccsc} u)}{dx} = \dfrac{-1}{u\sqrt{1-u^2}}\dfrac{du}{dx}$,

21. $\dfrac{d(\sinh u)}{dx} = \cosh u\dfrac{du}{dx}$, **22.** $\dfrac{d(\cosh u)}{dx} = \sinh u\dfrac{du}{dx}$,

23. $\dfrac{d(\tanh u)}{dx} = \operatorname{sech}^2 u\dfrac{du}{dx}$, **24.** $\dfrac{d(\coth u)}{dx} = -\operatorname{csch}^2 u\dfrac{du}{dx}$,

25. $\dfrac{d(\operatorname{sech} u)}{dx} = -\operatorname{sech} u \tanh u\dfrac{du}{dx}$, **26.** $\dfrac{d(\operatorname{csch} u)}{dx} = -\operatorname{csch} u \coth u\dfrac{du}{dx}$,

27. $\dfrac{d(\operatorname{arcsinh} u)}{dx} = \dfrac{1}{\sqrt{1+u^2}}\dfrac{du}{dx}$, **28.** $\dfrac{d(\operatorname{arccosh} u)}{dx} = \dfrac{1}{\sqrt{u^2-1}}\dfrac{du}{dx}$,

29. $\dfrac{d(\operatorname{arctanh} u)}{dx} = \dfrac{1}{1-u^2}\dfrac{du}{dx}$, **30.** $\dfrac{d(\operatorname{arccoth} u)}{dx} = \dfrac{1}{u^2-1}\dfrac{du}{dx}$,

31. $\dfrac{d(\operatorname{arcsech} u)}{dx} = \dfrac{-1}{u\sqrt{1-u^2}}\dfrac{du}{dx}$, **32.** $\dfrac{d(\operatorname{arccsch} u)}{dx} = \dfrac{-1}{|u|\sqrt{1+u^2}}\dfrac{du}{dx}$.

Integrals:

1. $\displaystyle\int cu\,dx = c\int u\,dx$, **2.** $\displaystyle\int (u+v)\,dx = \int u\,dx + \int v\,dx$,

3. $\displaystyle\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1$, **4.** $\displaystyle\int \frac{1}{x}dx = \ln x$, **5.** $\displaystyle\int e^x\,dx = e^x$,

6. $\displaystyle\int \frac{dx}{1+x^2} = \arctan x$, **7.** $\displaystyle\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$,

8. $\displaystyle\int \sin x\,dx = -\cos x$, **9.** $\displaystyle\int \cos x\,dx = \sin x$,

10. $\displaystyle\int \tan x\,dx = -\ln|\cos x|$, **11.** $\displaystyle\int \cot x\,dx = \ln|\cos x|$,

12. $\displaystyle\int \sec x\,dx = \ln|\sec x + \tan x|$, **13.** $\displaystyle\int \csc x\,dx = \ln|\csc x + \cot x|$,

14. $\displaystyle\int \arcsin\tfrac{x}{a}dx = \arcsin\tfrac{x}{a} + \sqrt{a^2-x^2}, \quad a > 0$,

Theoretical Computer Science Cheat Sheet

Calculus Cont.

15. $\int \arccos\frac{x}{a}dx = \arccos\frac{x}{a} - \sqrt{a^2-x^2}, \quad a>0,$

16. $\int \arctan\frac{x}{a}dx = x\arctan\frac{x}{a} - \frac{a}{2}\ln(a^2+x^2), \quad a>0,$

17. $\int \sin^2(ax)dx = \frac{1}{2a}\big(ax - \sin(ax)\cos(ax)\big),$

18. $\int \cos^2(ax)dx = \frac{1}{2a}\big(ax + \sin(ax)\cos(ax)\big),$

19. $\int \sec^2 x\,dx = \tan x,$

20. $\int \csc^2 x\,dx = -\cot x,$

21. $\int \sin^n x\,dx = -\frac{\sin^{n-1}x\cos x}{n} + \frac{n-1}{n}\int \sin^{n-2}x\,dx,$

22. $\int \cos^n x\,dx = \frac{\cos^{n-1}x\sin x}{n} + \frac{n-1}{n}\int \cos^{n-2}x\,dx,$

23. $\int \tan^n x\,dx = \frac{\tan^{n-1}x}{n-1} - \int \tan^{n-2}x\,dx, \quad n\neq 1,$

24. $\int \cot^n x\,dx = -\frac{\cot^{n-1}x}{n-1} - \int \cot^{n-2}x\,dx, \quad n\neq 1,$

25. $\int \sec^n x\,dx = \frac{\tan x\sec^{n-1}x}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2}x\,dx, \quad n\neq 1,$

26. $\int \csc^n x\,dx = -\frac{\cot x\csc^{n-1}x}{n-1} + \frac{n-2}{n-1}\int \csc^{n-2}x\,dx, \quad n\neq 1,$

27. $\int \sinh x\,dx = \cosh x,$

28. $\int \cosh x\,dx = \sinh x,$

29. $\int \tanh x\,dx = \ln|\cosh x|,$

30. $\int \coth x\,dx = \ln|\sinh x|,$

31. $\int \operatorname{sech} x\,dx = \arctan\sinh x,$

32. $\int \operatorname{csch} x\,dx = \ln\left|\tanh\frac{x}{2}\right|,$

33. $\int \sinh^2 x\,dx = \frac{1}{4}\sinh(2x) - \frac{1}{2}x,$

34. $\int \cosh^2 x\,dx = \frac{1}{4}\sinh(2x) + \frac{1}{2}x,$

35. $\int \operatorname{sech}^2 x\,dx = \tanh x,$

36. $\int \operatorname{arcsinh}\frac{x}{a}dx = x\operatorname{arcsinh}\frac{x}{a} - \sqrt{x^2+a^2}, \quad a>0,$

37. $\int \operatorname{arctanh}\frac{x}{a}dx = x\operatorname{arctanh}\frac{x}{a} + \frac{a}{2}\ln|a^2-x^2|,$

38. $\int \operatorname{arccosh}\frac{x}{a}dx = \begin{cases} x\operatorname{arccosh}\dfrac{x}{a} - \sqrt{x^2+a^2}, & \text{if } \operatorname{arccosh}\frac{x}{a} > 0 \text{ and } a>0, \\ x\operatorname{arccosh}\dfrac{x}{a} + \sqrt{x^2+a^2}, & \text{if } \operatorname{arccosh}\frac{x}{a} < 0 \text{ and } a>0, \end{cases}$

39. $\int \frac{dx}{\sqrt{a^2+x^2}} = \ln\left(x+\sqrt{a^2+x^2}\right), \quad a>0,$

40. $\int \frac{dx}{a^2+x^2} = \frac{1}{a}\arctan\frac{x}{a}, \quad a>0,$

41. $\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\arcsin\frac{x}{a}, \quad a>0,$

42. $\int (a^2-x^2)^{3/2}dx = \frac{x}{8}(5a^2-2x^2)\sqrt{a^2-x^2} + \frac{3a^4}{8}\arcsin\frac{x}{a}, \quad a>0,$

43. $\int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin\frac{x}{a}, \quad a>0,$

44. $\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right|,$

45. $\int \frac{dx}{(a^2-x^2)^{3/2}} = \frac{x}{a^2\sqrt{a^2-x^2}},$

46. $\int \sqrt{a^2\pm x^2}\,dx = \frac{x}{2}\sqrt{a^2\pm x^2} \pm \frac{a^2}{2}\ln\left|x+\sqrt{a^2\pm x^2}\right|,$

47. $\int \frac{dx}{\sqrt{x^2-a^2}} = \ln\left|x+\sqrt{x^2-a^2}\right|, \quad a>0,$

48. $\int \frac{dx}{ax^2+bx} = \frac{1}{a}\ln\left|\frac{x}{a+bx}\right|,$

49. $\int x\sqrt{a+bx}\,dx = \frac{2(3bx-2a)(a+bx)^{3/2}}{15b^2},$

50. $\int \frac{\sqrt{a+bx}}{x}\,dx = 2\sqrt{a+bx} + a\int \frac{1}{x\sqrt{a+bx}}dx,$

51. $\int \frac{x}{\sqrt{a+bx}}\,dx = \frac{1}{\sqrt{2}}\ln\left|\frac{\sqrt{a+bx}-\sqrt{a}}{\sqrt{a+bx}+\sqrt{a}}\right|, \quad a>0,$

52. $\int \frac{\sqrt{a^2-x^2}}{x}\,dx = \sqrt{a^2-x^2} - a\ln\left|\frac{a+\sqrt{a^2-x^2}}{x}\right|,$

53. $\int x\sqrt{a^2-x^2}\,dx = -\frac{1}{3}(a^2-x^2)^{3/2},$

54. $\int x^2\sqrt{a^2-x^2}\,dx = \frac{x}{8}(2x^2-a^2)\sqrt{a^2-x^2} + \frac{a^4}{8}\arcsin\frac{x}{a}, \quad a>0,$

55. $\int \frac{dx}{\sqrt{a^2-x^2}} = -\frac{1}{a}\ln\left|\frac{a+\sqrt{a^2-x^2}}{x}\right|,$

56. $\int \frac{x\,dx}{\sqrt{a^2-x^2}} = -\sqrt{a^2-x^2},$

57. $\int \frac{x^2\,dx}{\sqrt{a^2-x^2}} = -\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\arcsin\frac{x}{a,} \quad a>0,$

58. $\int \frac{\sqrt{a^2+x^2}}{x}\,dx = \sqrt{a^2+x^2} - a\ln\left|\frac{a+\sqrt{a^2+x^2}}{x}\right|,$

59. $\int \frac{\sqrt{x^2-a^2}}{x}\,dx = \sqrt{x^2-a^2} - a\arccos\frac{a}{|x|}, \quad a>0,$

60. $\int x\sqrt{x^2\pm a^2}\,dx = \frac{1}{3}(x^2\pm a^2)^{3/2},$

61. $\int \frac{dx}{x\sqrt{x^2+a^2}} = \frac{1}{a}\ln\left|\frac{x}{a+\sqrt{a^2+x^2}}\right|,$

Theoretical Computer Science Cheat Sheet

Calculus Cont.

62. $\displaystyle\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\arccos\frac{a}{|x|}, \quad a>0,$

63. $\displaystyle\int \frac{dx}{x^2\sqrt{x^2\pm a^2}} = \mp\frac{\sqrt{x^2\pm a^2}}{a^2x},$

64. $\displaystyle\int \frac{x\,dx}{\sqrt{x^2\pm a^2}} = \sqrt{x^2\pm a^2},$

65. $\displaystyle\int \frac{\sqrt{x^2\pm a^2}}{x^4}\,dx = \mp\frac{(x^2+a^2)^{3/2}}{3a^2x^3},$

66. $$\int \frac{dx}{ax^2+bx+c} = \begin{cases} \dfrac{1}{\sqrt{b^2-4ac}}\ln\left|\dfrac{2ax+b-\sqrt{b^2-4ac}}{2ax+b+\sqrt{b^2-4ac}}\right|, & \text{if } b^2>4ac, \\ \dfrac{2}{\sqrt{4ac-b^2}}\arctan\dfrac{2ax+b}{\sqrt{4ac-b^2}}, & \text{if } b^2<4ac, \end{cases}$$

67. $$\int \frac{dx}{\sqrt{ax^2+bx+c}} = \begin{cases} \dfrac{1}{\sqrt{a}}\ln\left|2ax+b+2\sqrt{a}\sqrt{ax^2+bx+c}\right|, & \text{if } a>0, \\ \dfrac{1}{\sqrt{-a}}\arcsin\dfrac{-2ax-b}{\sqrt{b^2-4ac}}, & \text{if } a<0, \end{cases}$$

68. $$\int \sqrt{ax^2+bx+c}\,dx = \frac{2ax+b}{4a}\sqrt{ax^2+bx+c} + \frac{4ax-b^2}{8a}\int\frac{dx}{\sqrt{ax^2+bx+c}},$$

69. $$\int \frac{x\,dx}{\sqrt{ax^2+bx+c}} = \frac{\sqrt{ax^2+bx+c}}{a} - \frac{b}{2a}\int\frac{dx}{\sqrt{ax^2+bx+c}},$$

70. $$\int \frac{dx}{x\sqrt{ax^2+bx+c}} = \begin{cases} \dfrac{-1}{\sqrt{c}}\ln\left|\dfrac{2\sqrt{c}\sqrt{ax^2+bx+c}+bx+2c}{x}\right|, & \text{if } c>0, \\ \dfrac{1}{\sqrt{-c}}\arcsin\dfrac{bx+2c}{|x|\sqrt{b^2-4ac}}, & \text{if } c<0, \end{cases}$$

71. $\displaystyle\int x^3\sqrt{x^2+a^2}\,dx = \left(\tfrac{1}{3}x^2-\tfrac{2}{15}a^2\right)(x^2+a^2)^{3/2},$

72. $\displaystyle\int x^n\sin(ax)\,dx = -\tfrac{1}{a}x^n\cos(ax)+\tfrac{n}{a}\int x^{n-1}\cos(ax)\,dx,$

73. $\displaystyle\int x^n\cos(ax)\,dx = \tfrac{1}{a}x^n\sin(ax)-\tfrac{n}{a}\int x^{n-1}\sin(ax)\,dx,$

74. $\displaystyle\int x^n e^{ax}\,dx = \frac{x^n e^{ax}}{a}-\tfrac{n}{a}\int x^{n-1}e^{ax}\,dx,$

75. $\displaystyle\int x^n\ln(ax)\,dx = x^{n+1}\left(\frac{\ln(ax)}{n+1}-\frac{1}{(n+1)^2}\right),$

76. $\displaystyle\int x^n(\ln ax)^m\,dx = \frac{x^{n+1}}{n+1}(\ln ax)^m - \frac{m}{n+1}\int x^n(\ln ax)^{m-1}\,dx.$

$x^1 =$	$x^{\underline{1}}$	$=$	$x^{\overline{1}}$
$x^2 =$	$x^{\underline{2}}+x^{\underline{1}}$	$=$	$x^{\overline{2}}-x^{\overline{1}}$
$x^3 =$	$x^{\underline{3}}+3x^{\underline{2}}+x^{\underline{1}}$	$=$	$x^{\overline{3}}-3x^{\overline{2}}+x^{\overline{1}}$
$x^4 =$	$x^{\underline{4}}+6x^{\underline{3}}+7x^{\underline{2}}+x^{\underline{1}}$	$=$	$x^{\overline{4}}-6x^{\overline{3}}+7x^{\overline{2}}-x^{\overline{1}}$
$x^5 =$	$x^{\underline{5}}+15x^{\underline{4}}+25x^{\underline{3}}+10x^{\underline{2}}+x^{\underline{1}}$	$=$	$x^{\underline{5}}-15x^{\overline{4}}+25x^{\overline{3}}-10x^{\overline{2}}+x^{\overline{1}}$

$x^{\overline{1}} =$	x^1	$x^{\underline{1}} =$	x^1
$x^{\overline{2}} =$	x^2+x^1	$x^{\underline{2}} =$	x^2-x^1
$x^{\overline{3}} =$	$x^3+3x^2+2x^1$	$x^{\underline{3}} =$	$x^3-3x^2+2x^1$
$x^{\overline{4}} =$	$x^4+6x^3+11x^2+6x^1$	$x^{\underline{4}} =$	$x^4-6x^3+11x^2-6x^1$
$x^{\overline{5}} =$	$x^5+10x^4+35x^3+50x^2+24x^1$	$x^{\underline{5}} =$	$x^5-10x^4+35x^3-50x^2+24x^1$

Finite Calculus

Difference, shift operators:

$$\Delta f(x) = f(x+1)-f(x),$$
$$\mathrm{E}\,f(x) = f(x+1).$$

Fundamental Theorem:

$$f(x)=\Delta F(x) \Leftrightarrow \sum f(x)\delta x = F(x)+C.$$
$$\sum_a^b f(x)\delta x = \sum_{i=a}^{b-1} f(i).$$

Differences:

$\Delta(cu)=c\Delta u, \qquad \Delta(u+v)=\Delta u+\Delta v,$

$\Delta(uv)=u\Delta v+\mathrm{E}\,v\Delta u,$

$\Delta(x^{\underline{n}})=nx^{\underline{n-1}},$

$\Delta(H_x)=x^{\underline{-1}}, \qquad \Delta(2^x)=2^x,$

$\Delta(c^x)=(c-1)c^x, \qquad \Delta\binom{x}{m}=\binom{x}{m-1}.$

Sums:

$\sum cu\,\delta x = c\sum u\,\delta x,$

$\sum (u+v)\,\delta x = \sum u\,\delta x+\sum v\,\delta x,$

$\sum u\Delta v\,\delta x = uv-\sum \mathrm{E}\,v\Delta u\,\delta x,$

$\sum x^{\underline{n}}\,\delta x = \frac{x^{\underline{n+1}}}{m+1}, \qquad \sum x^{\underline{-1}}\,\delta x = H_x,$

$\sum c^x\,\delta x = \frac{c^x}{c-1}, \qquad \sum\binom{x}{m}\,\delta x = \binom{x}{m+1}.$

Falling Factorial Powers:

$x^{\underline{n}} = x(x-1)\cdots(x-n+1), \quad n>0,$

$x^{\underline{0}} = 1,$

$x^{\underline{n}} = \dfrac{1}{(x+1)\cdots(x+|n|)}, \quad n<0,$

$x^{\underline{n+m}} = x^{\underline{m}}(x-m)^{\underline{n}}.$

Rising Factorial Powers:

$x^{\overline{n}} = x(x+1)\cdots(x+n-1), \quad n>0,$

$x^{\overline{0}} = 1,$

$x^{\overline{n}} = \dfrac{1}{(x-1)\cdots(x-|n|)}, \quad n<0,$

$x^{\overline{n+m}} = x^{\overline{m}}(x+m)^{\overline{n}}.$

Conversion:

$x^{\underline{n}} = (-1)^n(-x)^{\overline{n}} = (x-n+1)^{\overline{n}}$
$\quad = 1/(x+1)^{\overline{-n}},$

$x^{\overline{n}} = (-1)^n(-x)^{\underline{n}} = (x+n-1)^{\underline{n}}$
$\quad = 1/(x-1)^{\underline{-n}},$

$$x^n = \sum_{k=1}^n \left\{ n \atop k \right\} x^{\underline{k}} = \sum_{k=1}^n \left\{ n \atop k \right\} (-1)^{n-k} x^{\overline{k}},$$

$$x^{\underline{n}} = \sum_{k=1}^n \left[n \atop k \right] (-1)^{n-k} x^k,$$

$$x^{\overline{n}} = \sum_{k=1}^n \left[n \atop k \right] x^k.$$

Theoretical Computer Science Cheat Sheet

Series

Taylor's series:

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2}f''(a) + \cdots = \sum_{i=0}^{\infty} \frac{(x-a)^i}{i!} f^{(i)}(a).$$

Expansions:

$\dfrac{1}{1-x}$	$= 1 + x + x^2 + x^3 + x^4 + \cdots$	$= \sum_{i=0}^{\infty} x^i,$
$\dfrac{1}{1-cx}$	$= 1 + cx + c^2x^2 + c^3x^3 + \cdots$	$= \sum_{i=0}^{\infty} c^i x^i,$
$\dfrac{1}{1-x^n}$	$= 1 + x^n + x^{2n} + x^{3n} + \cdots$	$= \sum_{i=0}^{\infty} x^{ni},$
$\dfrac{x}{(1-x)^2}$	$= x + 2x^2 + 3x^3 + 4x^4 + \cdots$	$= \sum_{i=0}^{\infty} i x^i,$
$x^k \dfrac{d^n}{dx^n}\left(\dfrac{1}{1-x}\right)$	$= x + 2^n x^2 + 3^n x^3 + 4^n x^4 + \cdots$	$= \sum_{i=0}^{\infty} i^n x^i,$
e^x	$= 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots$	$= \sum_{i=0}^{\infty} \dfrac{x^i}{i!},$
$\ln(1+x)$	$= x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 - \cdots$	$= \sum_{i=1}^{\infty} (-1)^{i+1} \dfrac{x^i}{i},$
$\ln \dfrac{1}{1-x}$	$= x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 + \cdots$	$= \sum_{i=1}^{\infty} \dfrac{x^i}{i},$
$\sin x$	$= x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots$	$= \sum_{i=0}^{\infty} (-1)^i \dfrac{x^{2i+1}}{(2i+1)!},$
$\cos x$	$= 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots$	$= \sum_{i=0}^{\infty} (-1)^i \dfrac{x^{2i}}{(2i)!},$
$\tan^{-1} x$	$= x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \cdots$	$= \sum_{i=0}^{\infty} (-1)^i \dfrac{x^{2i+1}}{(2i+1)},$
$(1+x)^n$	$= 1 + nx + \frac{n(n-1)}{2}x^2 + \cdots$	$= \sum_{i=0}^{\infty} \binom{n}{i} x^i,$
$\dfrac{1}{(1-x)^{n+1}}$	$= 1 + (n+1)x + \binom{n+2}{2}x^2 + \cdots$	$= \sum_{i=0}^{\infty} \binom{i+n}{i} x^i,$
$\dfrac{x}{e^x - 1}$	$= 1 - \frac{1}{2}x + \frac{1}{12}x^2 - \frac{1}{720}x^4 + \cdots$	$= \sum_{i=0}^{\infty} \dfrac{B_i x^i}{i!},$
$\dfrac{1}{2x}(1 - \sqrt{1-4x})$	$= 1 + x + 2x^2 + 5x^3 + \cdots$	$= \sum_{i=0}^{\infty} \dfrac{1}{i+1}\binom{2i}{i} x^i,$
$\dfrac{1}{\sqrt{1-4x}}$	$= 1 + x + 2x^2 + 6x^3 + \cdots$	$= \sum_{i=0}^{\infty} \binom{2i}{i} x^i,$
$\dfrac{1}{\sqrt{1-4x}}\left(\dfrac{1-\sqrt{1-4x}}{2x}\right)^n$	$= 1 + (2+n)x + \binom{4+n}{2}x^2 + \cdots$	$= \sum_{i=0}^{\infty} \binom{2i+n}{i} x^i,$
$\dfrac{1}{1-x} \ln \dfrac{1}{1-x}$	$= x + \frac{3}{2}x^2 + \frac{11}{6}x^3 + \frac{25}{12}x^4 + \cdots$	$= \sum_{i=1}^{\infty} H_i x^i,$
$\dfrac{1}{2}\left(\ln \dfrac{1}{1-x}\right)^2$	$= \frac{1}{2}x^2 + \frac{3}{4}x^3 + \frac{11}{24}x^4 + \cdots$	$= \sum_{i=2}^{\infty} \dfrac{H_{i-1} x^i}{i},$
$\dfrac{x}{1-x-x^2}$	$= x + x^2 + 2x^3 + 3x^4 + \cdots$	$= \sum_{i=0}^{\infty} F_i x^i,$
$\dfrac{F_n x}{1 - (F_{n-1} + F_{n+1})x - (-1)^n x^2}$	$= F_n x + F_{2n} x^2 + F_{3n} x^3 + \cdots$	$= \sum_{i=0}^{\infty} F_{ni} x^i.$

Ordinary power series:

$$A(x) = \sum_{i=0}^{\infty} a_i x^i.$$

Exponential power series:

$$A(x) = \sum_{i=0}^{\infty} a_i \frac{x^i}{i!}.$$

Dirichlet power series:

$$A(x) = \sum_{i=1}^{\infty} \frac{a_i}{i^x}.$$

Binomial theorem:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k.$$

Difference of like powers:

$$x^n - y^n = (x-y) \sum_{k=0}^{n-1} x^{n-1-k} y^k.$$

For ordinary power series:

$$\alpha A(x) + \beta B(x) = \sum_{i=0}^{\infty} (\alpha a_i + \beta b_i) x^i,$$

$$x^k A(x) = \sum_{i=k}^{\infty} a_{i-k} x^i,$$

$$\frac{A(x) - \sum_{i=0}^{k-1} a_i x^i}{x^k} = \sum_{i=0}^{\infty} a_{i+k} x^i,$$

$$A(cx) = \sum_{i=0}^{\infty} c^i a_i x^i,$$

$$A'(x) = \sum_{i=0}^{\infty} (i+1) a_{i+1} x^i,$$

$$xA'(x) = \sum_{i=1}^{\infty} i a_i x^i,$$

$$\int A(x)\,dx = \sum_{i=1}^{\infty} \frac{a_{i-1}}{i} x^i,$$

$$\frac{A(x) + A(-x)}{2} = \sum_{i=0}^{\infty} a_{2i} x^{2i},$$

$$\frac{A(x) - A(-x)}{2} = \sum_{i=0}^{\infty} a_{2i+1} x^{2i+1}.$$

Summation: If $b_i = \sum_{j=0}^{i} a_i$ then

$$B(x) = \frac{1}{1-x} A(x).$$

Convolution:

$$A(x)B(x) = \sum_{i=0}^{\infty} \left(\sum_{j=0}^{i} a_j b_{i-j} \right) x^i.$$

God made the natural numbers;
all the rest is the work of man.
– Leopold Kronecker

Theoretical Computer Science Cheat Sheet

Series

Expansions:

$$\frac{1}{(1-x)^{n+1}}\ln\frac{1}{1-x} = \sum_{i=0}^{\infty}(H_{n+i}-H_n)\binom{n+i}{i}x^i,$$

$$\left(\frac{1}{x}\right)^{\overline{-n}} = \sum_{i=0}^{\infty}\left\{ {i \atop n} \right\}x^i,$$

$$x^{\overline{n}} = \sum_{i=0}^{\infty}\left[{n \atop i} \right]x^i,$$

$$(e^x-1)^n = \sum_{i=0}^{\infty}\left\{ {i \atop n} \right\}\frac{n!x^i}{i!},$$

$$\left(\ln\frac{1}{1-x}\right)^n = \sum_{i=0}^{\infty}\left[{i \atop n} \right]\frac{n!x^i}{i!},$$

$$x\cot x = \sum_{i=0}^{\infty}\frac{(-4)^iB_{2i}x^{2i}}{(2i)!},$$

$$\tan x = \sum_{i=1}^{\infty}(-1)^{i-1}\frac{2^{2i}(2^{2i}-1)B_{2i}x^{2i-1}}{(2i)!},$$

$$\zeta(x) = \sum_{i=1}^{\infty}\frac{1}{i^x},$$

$$\frac{1}{\zeta(x)} = \sum_{i=1}^{\infty}\frac{\mu(i)}{i^x},$$

$$\frac{\zeta(x-1)}{\zeta(x)} = \sum_{i=1}^{\infty}\frac{\phi(i)}{i^x},$$

$$\zeta(x) = \prod_p\frac{1}{1-p^{-x}},$$

$$\zeta^2(x) = \sum_{i=1}^{\infty}\frac{d(i)}{x^i} \quad \text{where } d(n)=\textstyle\sum_{d|n}1,$$

$$\zeta(x)\zeta(x-1) = \sum_{i=1}^{\infty}\frac{S(i)}{x^i} \quad \text{where } S(n)=\textstyle\sum_{d|n}d,$$

$$\zeta(2n) = \frac{2^{2n-1}|B_{2n}|}{(2n)!}\pi^{2n}, \quad n\in\mathbb{N},$$

$$\frac{x}{\sin x} = \sum_{i=0}^{\infty}(-1)^{i-1}\frac{(4^i-2)B_{2i}x^{2i}}{(2i)!},$$

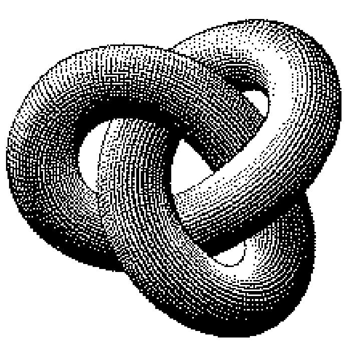
$$\left(\frac{1-\sqrt{1-4x}}{2x}\right)^n = \sum_{i=0}^{\infty}\frac{n(2i+n-1)!}{i!(n+i)!}x^i,$$

$$e^x\sin x = \sum_{i=1}^{\infty}\frac{2^{i/2}\sin\frac{i\pi}{4}}{i!}x^i,$$

$$\sqrt{\frac{1-\sqrt{1-x}}{x}} = \sum_{i=0}^{\infty}\frac{(4i)!}{16^i\sqrt{2}(2i)!(2i+1)!}x^i,$$

$$\left(\frac{\arcsin x}{x}\right)^2 = \sum_{i=0}^{\infty}\frac{4^ii!^2}{(i+1)(2i+1)!}x^{2i}.$$

Escher's Knot



Stieltjes Integration

If G is continuous in the interval $[a,b]$ and F is nondecreasing then

$$\int_a^b G(x)\,dF(x)$$

exists. If $a\le b\le c$ then

$$\int_a^c G(x)\,dF(x) = \int_a^b G(x)\,dF(x) + \int_b^c G(x)\,dF(x).$$

If the integrals involved exist

$$\int_a^b \big(G(x)+H(x)\big)\,dF(x) = \int_a^b G(x)\,dF(x) + \int_a^b H(x)\,dF(x),$$

$$\int_a^b G(x)\,d\big(F(x)+H(x)\big) = \int_a^b G(x)\,dF(x) + \int_a^b G(x)\,dH(x),$$

$$\int_a^b c\cdot G(x)\,dF(x) = \int_a^b G(x)\,d\big(c\cdot F(x)\big) = c\int_a^b G(x)\,dF(x),$$

$$\int_a^b G(x)\,dF(x) = G(b)F(b) - G(a)F(a) - \int_a^b F(x)\,dG(x).$$

If the integrals involved exist, and F possesses a derivative F' at every point in $[a,b]$ then

$$\int_a^b G(x)\,dF(x) = \int_a^b G(x)F'(x)\,dx.$$

Cramer's Rule

If we have equations:

$$\begin{aligned} a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n &= b_1 \\ a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n &= b_2 \\ \vdots \qquad\quad \vdots \qquad\qquad\quad \vdots \quad & \\ a_{n,1}x_1 + a_{n,2}x_2 + \cdots + a_{n,n}x_n &= b_n \end{aligned}$$

Let $A=(a_{i,j})$ and B be the column matrix (b_i). Then there is a unique solution iff $\det A\neq 0$. Let A_i be A with column i replaced by B. Then

$$x_i = \frac{\det A_i}{\det A}.$$

00	47	18	76	29	93	85	34	61	52
86	11	57	28	70	39	94	45	02	63
95	80	22	67	38	71	49	56	13	04
59	96	81	33	07	48	72	60	24	15
73	69	90	82	44	17	58	01	35	26
68	74	09	91	83	55	27	12	46	30
37	08	75	19	92	84	66	23	50	41
14	25	36	40	51	62	03	77	88	99
21	32	43	54	65	06	10	89	97	78
42	53	64	05	16	20	31	98	79	87

Fibonacci Numbers

$1,1,2,3,5,8,13,21,34,55,89,\ldots$

Definitions:

$$F_i = F_{i-1}+F_{i-2}, \quad F_0=F_1=1,$$

$$F_{-i} = (-1)^{i-1}F_i,$$

$$F_i = \tfrac{1}{\sqrt{5}}\left(\phi^i - \hat{\phi}^i\right),$$

Cassini's identity: for $i>0$:

$$F_{i+1}F_{i-1} - F_i^2 = (-1)^i.$$

Additive rule:

$$F_{n+k} = F_kF_{n+1} + F_{k-1}F_n,$$

$$F_{2n} = F_nF_{n+1} + F_{n-1}F_n.$$

Calculation by matrices:

$$\begin{pmatrix} F_{n-2} & F_{n-1} \\ F_{n-1} & F_n \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^n.$$

The Fibonacci number system: Every integer n has a unique representation

$$n = F_{k_1} + F_{k_2} + \cdots + F_{k_m},$$

where $k_i \ge k_{i+1}+2$ for all i, $1\le i<m$ and $k_m\ge 2$.

Improvement makes strait roads, but the crooked roads without Improvement, are roads of Genius.
– William Blake (The Marriage of Heaven and Hell)