

## Appendix A. Phase Plane Analysis

We discuss here, only very briefly, general autonomous second-order ordinary differential equations of the form

$$\frac{dx}{dt} = f(x, y), \quad \frac{dy}{dt} = g(x, y). \quad (\text{A.1})$$

We present the basic results which are required in the main text. There are many books which discuss phase plane analysis in varying depth, such as Jordan and Smith (1999) and Guckenheimer and Holmes (1983). A good, short and practical exposition of the qualitative theory of ordinary differential equation systems, including phase plane techniques, is given by Odell (1980). *Phase curves* or *phase trajectories* of (A.1) are solutions of

$$\frac{dx}{dy} = \frac{f(x, y)}{g(x, y)}. \quad (\text{A.2})$$

Through any point  $(x_0, y_0)$  there is a unique curve except at *singular points*  $(x_s, y_s)$  where

$$f(x_s, y_s) = g(x_s, y_s) = 0.$$

Let  $x \rightarrow x - x_s$ ,  $y \rightarrow y - y_s$ ; then,  $(0, 0)$  is a singular point of the transformed equation. Thus, without loss of generality we now consider (A.2) to have a singular point at the origin; that is,

$$f(x, y) = g(x, y) = 0 \quad \Rightarrow \quad x = 0, y = 0. \quad (\text{A.3})$$

If  $f$  and  $g$  are analytic near  $(0, 0)$  we can expand  $f$  and  $g$  in a Taylor series and, retaining only the linear terms, we get

$$\frac{dx}{dy} = \frac{ax + by}{cx + dy}, \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix}_{(0,0)} \quad (\text{A.4})$$

which defines the matrix  $A$  and the constants  $a$ ,  $b$ ,  $c$  and  $d$ . The linear form is equivalent to the system

$$\frac{dx}{dt} = ax + by, \quad \frac{dy}{dt} = cx + dy. \quad (\text{A.5})$$

Solutions of (A.5) give the parametric forms of the phase curves;  $t$  is the parametric parameter.

Let  $\lambda_1$  and  $\lambda_2$  be the eigenvalues of  $A$  defined in (A.4); that is,

$$\begin{vmatrix} a - \lambda & b \\ c & d - \lambda \end{vmatrix} = 0 \quad \Rightarrow \quad \lambda_1, \lambda_2 = \frac{1}{2}(a + d \pm [(a + d)^2 - 4 \det A]^{1/2}). \quad (\text{A.6})$$

Solutions of (A.5) are then

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 \mathbf{v}_1 \exp[\lambda_1 t] + c_2 \mathbf{v}_2 \exp[\lambda_2 t], \quad (\text{A.7})$$

where  $c_1$  and  $c_2$  are arbitrary constants and  $\mathbf{v}_1, \mathbf{v}_2$  are the eigenvectors of  $A$  corresponding to  $\lambda_1$  and  $\lambda_2$  respectively; they are given by

$$\mathbf{v}_i = (1 + p_i^2)^{-1/2} \begin{pmatrix} 1 \\ p_i \end{pmatrix}, \quad p_i = \frac{\lambda_i - a}{b}, \quad b \neq 0, \quad i = 1, 2. \quad (\text{A.8})$$

Elimination of  $t$  in (A.7) gives the phase curves in the  $(x, y)$  plane.

The form (A.7) is for distinct eigenvalues. If the eigenvalues are equal the solutions are proportional to  $(c_1 + c_2 t) \exp[\lambda t]$ .

### *Catalogue of (Linear) Singularities in the Phase Plane*

(i)  $\lambda_1, \lambda_2$  real and distinct:

- (a)  $\lambda_1$  and  $\lambda_2$  have the same sign. Typical eigenvectors  $\mathbf{v}_1$  and  $\mathbf{v}_2$  are illustrated in Figure A.1(a). Suppose  $\lambda_2 < \lambda_1 < 0$ . Then, from (A.7), for example, for  $c_2 = 0, c_1 \neq 0$ ,

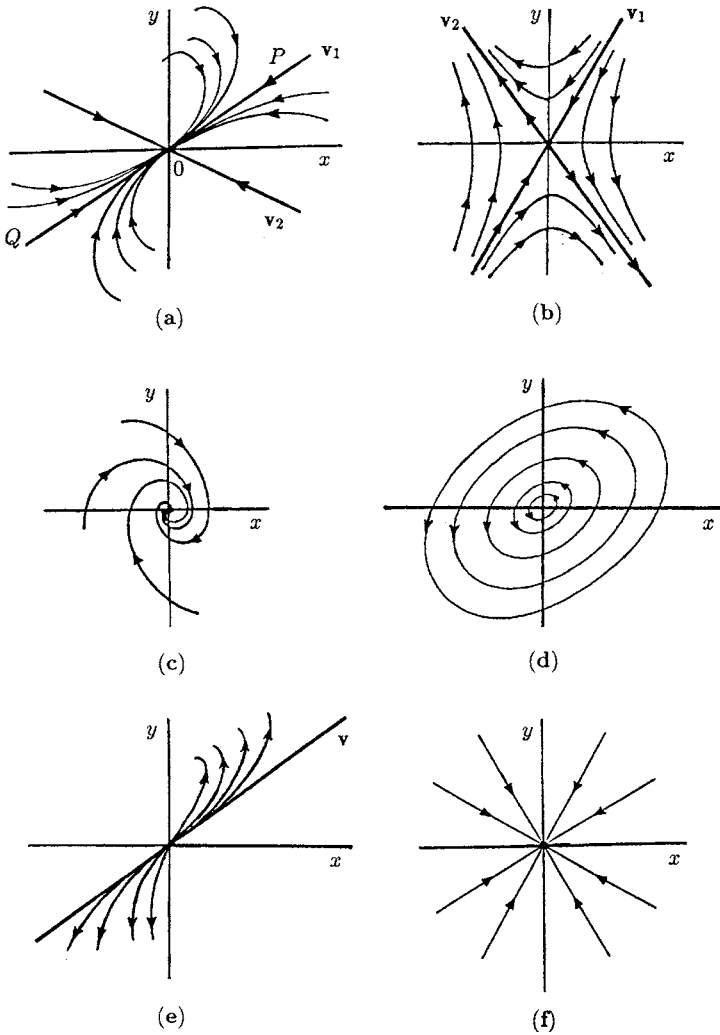
$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 \mathbf{v}_1 \exp[\lambda_1 t],$$

so the solution in the phase plane simply moves along  $\mathbf{v}_1$  towards the origin as  $t \rightarrow \infty$  in the direction shown in Figure A.1(a) — along  $PO$  if  $c_1 > 0$  and along  $QO$  if  $c_1 < 0$ .

From (A.7) every solution tends to  $(0, 0)$  as  $t \rightarrow \infty$  since, with  $\lambda_2 < \lambda_1 < 0$ ,  $\exp[\lambda_2 t] = o(\exp[\lambda_1 t])$  as  $t \rightarrow \infty$  and so

$$\begin{pmatrix} x \\ y \end{pmatrix} \sim c_1 \mathbf{v}_1 \exp[\lambda_1 t] \quad \text{as } t \rightarrow \infty.$$

Thus, close enough to the origin all solutions tend to zero along  $\mathbf{v}_1$  as shown in Figure A.1(a). This is called a *node* (Type I) singularity. With  $\lambda_1 \leq \lambda_2 < 0$  it is a stable node since all trajectories tend to  $(0, 0)$  as  $t \rightarrow \infty$ . If  $\lambda_1 > \lambda_2 > 0$  it is an unstable node; here  $(x, y) \rightarrow (0, 0)$  as  $t \rightarrow -\infty$ .

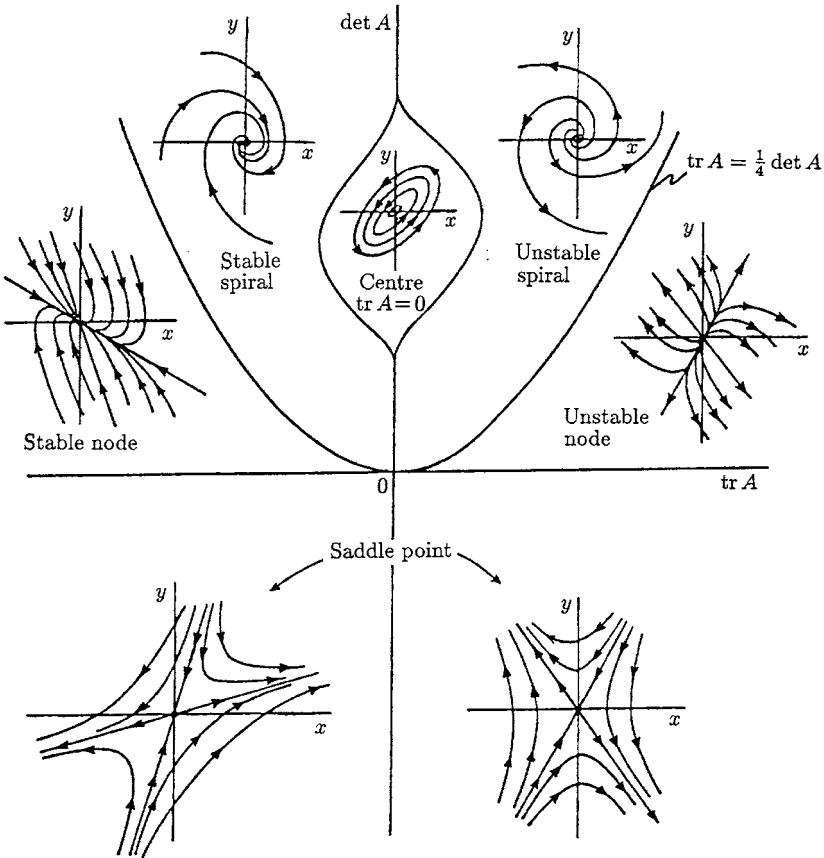


**Figure A.1.** Typical examples of the basic linear singularities of the phase plane solutions of (A.4). (a) Node (Type I): these can be stable (as shown) or unstable. (b) Saddle point: these are always unstable. (c) Spiral: these can be stable or unstable. (d) Centre: this is neutrally stable. (e) Node (Type II): these can be stable or unstable. (f) Star: these can be stable or unstable.

- (b)  $\lambda_1$  and  $\lambda_2$  have different signs. Suppose, for example,  $\lambda_1 < 0 < \lambda_2$  then  $\mathbf{v}_1 \exp[\lambda_1 t] \mathbf{v}_1 \rightarrow 0$  along  $\mathbf{v}_1$  as  $t \rightarrow \infty$  while  $\mathbf{v}_2 \exp[\lambda_2 t] \rightarrow 0$  along  $\mathbf{v}_2$  as  $t \rightarrow -\infty$ .

There are thus different directions on  $\mathbf{v}_1$  and  $\mathbf{v}_2$ : the solutions near  $(0, 0)$  are as shown in Figure A.1(b). This is a *saddle point* singularity. It is always *unstable*: except strictly along  $\mathbf{v}_1$  any small perturbation from  $(0, 0)$  grows exponentially.

- (ii)  $\lambda_1, \lambda_2$  complex:  $\lambda_1, \lambda_2 = \alpha \pm i\beta, \beta \neq 0$ . Solutions (A.7) here involve  $\exp[\alpha t]$   $\exp[\pm i\beta t]$  which implies an oscillatory approach to  $(0, 0)$ .
- (a)  $\alpha \neq 0$ . Here we have a *spiral*, which is stable if  $\alpha < 0$  and unstable if  $\alpha > 0$ ; Figure A.1(c) illustrates a spiral singularity.
- (b)  $\alpha = 0$ . In this case the phase curves are ellipses. This singularity is called a *centre* and is illustrated in Figure A.1(d). Centres are not stable in the usual sense; a small perturbation from one phase curve does not die out in the sense of returning to the original unperturbed curve. The perturbation simply gives another solution. In the case of centre singularities, determined by the linear approximation to  $f(x, y)$  and  $g(x, y)$ , we must look at the higher-order (than linear) terms to determine whether or not it is really a spiral and hence whether it is stable or unstable.



**Figure A.2.** Summary diagram showing how  $\text{tr } A$  and  $\det A$ , where  $A$  is the linearisation matrix given by (A.4), determine the type of phase plane singularity for (A.1). Here  $\det A = f_x g_y - f_y g_x$ ,  $\text{tr } A = f_x + g_y$ , where the partial derivatives are evaluated at the singularities, the solutions of  $f(x, y) = g(x, y) = 0$ .

- (iii)  $\lambda_1 = \lambda_2 = \lambda$ . Here the eigenvalues are *not* distinct.
- (a) In general, solutions now involve terms like  $t \exp[\lambda t]$  and there is only one eigenvector  $\mathbf{v}$  along which the solutions tend to  $(0, 0)$ . The  $t$  in  $t \exp[\lambda t]$  modifies the solution away from  $(0, 0)$ . It is called a *node* (Type II) singularity, an illustration of which is given in Figure A.1(e).
  - (b) If the solutions do not contain the  $t \exp[\lambda t]$  term we have a *star* singularity, which may be stable or unstable, depending on the sign of  $\lambda$ . Trajectories in the vicinity of a star singularity are shown in Figure A.1(f).

The singularity depends on  $a$ ,  $b$ ,  $c$  and  $d$  in the matrix  $A$  in (A.4). Figure A.2 summarises the results in terms of the trace and determinant of  $A$ .

If the system (A.1) possesses a confined set (that is, a domain on the boundary  $\partial B$  of which the vector  $(dx/dt, dy/dt)$  points into the domain) enclosing a single singular point which is an unstable spiral or node then any phase trajectory cannot tend to the singularity with time, nor can it leave the confined set. The *Poincaré–Bendixson theorem* says that as  $t \rightarrow \infty$  the trajectory will tend to a limit cycle solution. This is the simplest application of the theorem. If the sole singularity is a saddle point a limit cycle cannot exist; see, for example, Jordan and Smith (1999) for a proof of the theorem, its general application and some practical illustrations.