Manchester Metropolitan University

Faculty of Science and Engineering
Department of Computing and Mathematics

6G3Z3009 & 6G3Z3010
Foundation Mathematics

(2021 – 2022)

Lecture Notes and
Student Exercises

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FOUNDATION MATHEMATICS

6G3Z3009
& 6G3Z3010

UNIT GUIDE

1. Introduction

The aims of these two units are to prepare students for the mathematics encountered on the first year of degree courses; to provide fluent manipulative skills whilst also developing the conceptual and analytical abilities of the students; and to introduce students to the concept of differentiation.

These units run for ___ hours per week and are worth 15 credits each.

The notes contain example problems and solutions and additional online examples and exercises will also be supplied through Moodle.

You will need to purchase a simple scientific calculator, for example a Casio 83 GTX. You will be allowed to use your own calculator in tests and examinations as long as it conforms to Faculty regulations.

You will *not* need a calculator which has either a graphical display or a programming language.

We also highly recommend that students use the following resources:

Essential Reading/Resources :
HESTEM Sigma web-site, http://www.sigma-network.ac.uk/

Further Reading/Resources :

Croft, A. & Davison, R. (2006) Foundation Mathematics. 4th Ed, Prentice Hall.

Booth, D. J. (1998) Foundation Mathematics. Harlow: Addison-Wesley.

Mustoe, L. & Barry, M. D. J. (1997) Foundation Mathematics. Harlow: Addison-Wesley

Davis, M.E. and Edwards, C.H. (2001) Elementary Mathematical Modelling, Prentice-Hall

Various mathematical web-links, which can be accessed via Moodle

2. Teaching and Learning

We appreciate that some of you are less experienced in mathematics than others so we spend some time at the start of the unit on basic techniques which will be used throughout the unit. By the nature of the subject, mathematics courses build as they go on, using material from early in the course at later stages. The better your grasp of basic concepts, the easier it is going to be for you to make progress. Getting a good working knowledge of material as you go along is therefore important.

Learning mathematics is best done by doing mathematics and this is reflected in the organisation of classes. Typically, a class session will start with the lecturer developing some new ideas which you will then use in examples and exercises. Much of the time in class will be spent with you doing exercises.

Successful students work hard on the exercises in the classes and complete exercises in their own time. They are then in a position to bring any problems or difficulties that they have to the next class. We encourage students to ask questions and get any difficulties they may be having sorted out at an early opportunity. **Attendance at classes is very important and many students fail each year due to poor attendance.**

The pace at which material is covered will increase as the course goes on and you become more experienced learners.

You should note that the mathematics contained in the Numeracy and Data Analysis unit strongly underpins the material in this unit so it is vital that you gain proficiency in these fundamental skills.

3. Teaching Schedule

Part 1: 6G3Z3009 Foundation Mathematics 1

Week	Content	Chapter	Block
1.	Basic Algebra	1	III
2.	Solving linear equations; Transposition of formula	2, 3	
3.	Straight Lines, Intercepts, Simultaneous Equations	4	
4.	Quadratic formula and equations	5	
5.	Trigonometry	6	
6.	Consolidation & Revision		
7.	**EXAM 1**	1 − 6	

Part 2: 6G3Z3010 Foundation Mathematics 2

Week	Content	Chapter	Block
1.	Functions	7	IV
2.	Differentiation	8	
3.	Differentiation	8	
4.	Curve Sketching	9	
5.	Integration	10	
6.	Consolidation & Revision		
7.	**EXAM 2**	7 − 10	

4. Tutor contact information

Name: __________________

Office:__________________

Email:__________________

5. Assessments

The overall assessments comprise **two** examinations:

a) **EXAM 1**; This will take place on __________________. This exam will cover chapters 1 to 6 inclusively.

b) **EXAM 2;** This will take place on __________________. The exam will cover chapters 7 to 10 inclusively.

The pass mark for each examination is 40%. **You need to pass both examinations.**

If a student misses an exam for a valid reason, this will be considered at the summer exam board. Note, however, that full documentary evidence (e.g. a medical note) must be given to the Student Hub as soon as possible after the missed test/exam.

6. Referral Information

If you do not pass these Foundation Mathematics units, subject to your overall performance across the Foundation Year, you may be referred and thus required to take resit examinations.

There are no referrals / deferrals offered before the resit period.

Full details of any reassessment will be provided to you after the summer examination board.

Part 1

6G3Z3009 Foundation Mathematics 1

1. Basic Algebra
2. Solving Linear Equations
3. Transposition of Formula
4. Straight Lines, Intercepts, Simultaneous Equations
5. Quadratic Formula and Equations
6. Trigonometry

Chapter 1: Basic Algebra

Learning objectives:

By working through this chapter you should be able to:

- Create algebraic expressions from given contexts
- Use BODMAS
- Evaluate an algebraic expression by substitution, including expressions involving powers
- Simplify expressions by gathering terms and expanding brackets
- Factorise simple algebraic expressions
- Understand and apply laws of indices

1.1 What Is Algebra?

By now, you should already be familiar with arithmetic. In arithmetic, we only deal with numbers and mathematical operations on them like addition, subtraction, multiplication and division. The knowledge of arithmetic is very useful for solving simple problems but it will not be sufficient for more general and difficult problems. For example, if we know that a number and its half added together become 15, what is that number? You may well know the answer to this question, but without some basic knowledge of algebra it will probably take you some time to find it. So what is algebra? Basically, algebra is a branch of mathematics that has been developed to manipulate symbols. In algebra we use letters (English or Greek alphabets for example) to represent numbers or quantities of interest. In this chapter, first we will look at how to create simple algebraic expressions from given contexts and then look at some techniques that we can use to simplify them.

1.2 Creating Algebraic Expressions

Why do we use letters to stand for quantities?

Let's look at some examples

Example 1A

Errol buys ***x*** apples and ***y*** oranges. Apples are 16p each and oranges are 50p each. How much does he spend?

Total spent =

the number of apples that Errol buys times 16 pence plus
the number of oranges that Errol buys times 50 pence (1.1)

If we use the letters ***x*** and ***y*** to represent, respectively, the number of apples and the number of oranges that Errol buys, we can write the total spent as

$$16x + 50y \qquad \text{(units are pence)} \qquad (1.2)$$

Note (1.2) is much neater and easier to use than (1.1).

Example 1B

Suppose that Errol buys ***z*** pineapples as well and they are £2 each, the total cost is now

$$16x + 50y + 200z \qquad (1.3)$$

Note The £2 has to be expressed as 200p so that we are adding together pence throughout the whole expression. In general, we can only add things of the same type.

Example 1C

When internet calls at off-peak time are charged at 3p per minute then the cost of a call can be calculated from the formula

Cost (in p) = 3 × time on call (in minutes)

This ensures that if we double the time on call we will double the cost. For instance the cost of a 20 minute call is 60p. If we double the time to 40 minutes the call doubles to 120p.

In mathematics we could reduce this to

$$C = 3 \times t \qquad (1.4)$$

Remember that the symbols represent the cost and the time and the corresponding units are pence and minutes.

These are examples of the simplest form of **algebraic notation**. The abbreviation using symbols saves time and enables us to concentrate on the important issues associated with the formula. In fact in algebra when two values are placed side by side we assume they are to be multiplied so we would normally write the formula in (1.4) above as

$$C = 3t$$

Notes The quantities $\boldsymbol{x}$, $\boldsymbol{y}$ and $\boldsymbol{z}$ in (1.2) and (1.3) for cost are **variables**; they stand for any one of a range of possible values. A variable may be given a specific value when we want to work out a particular sum.

Similarly, in (1.4), the values for t and C will change from call to call and so they are also called variables. We tend to think we can choose the time to be any value within reason. The cost then depends on the value we have chosen. Because of this we call t the **independent** variable and C the **dependent** variable.

(1.2) defines a **function** for calculating the cost in terms of the variables $\boldsymbol{x}$ and $\boldsymbol{y}$, which works whatever values are given to $\boldsymbol{x}$ and $\boldsymbol{y}$.

We use variables and functions to describe general properties. This has huge advantages over specifying properties in a multitude of individual specific cases. From the general cases, we can establish patterns which can be used to solve other similar problems.

Because variables represent numbers, we manipulate expressions involving variables using the rules of arithmetic.

Exercise 1.1

1. Write algebraic expression (equation) for each the following:
   a) The number that is 4 more than x is 7
   b) The number that is twice x is 5.
   c) The number that is half of y is 8.
   d) The number that is 6 less than y is 4.
   e) The number that is a quarter of x is 2.

2. Determine the appropriate formulae in each of the following cases and write it down in abbreviated form using symbols. State clearly what the symbols represent and the corresponding units.
   a) The cost of a length of climber's rope is 90p per metre.
   b) The distance travelled on a journey in miles is equal to the average speed multiplied by the time taken.
   c) The concentration of a substance in solution is equal to the mass of the substance divided by the volume of the solution.
   d) The area of a triangle is equal to half the base times the perpendicular height.
   e) The cost in pence of an electricity bill is the standing charge plus 0.9 times the number of units used.

3. Evaluate the following algebraic expressions when $x = 10$ and $y = 5$.
   a) $x + y$ b) $x - y$
   c) xy d) $\frac{x}{y}$

4. Susan buys 7 apples at x pence each and 4 bananas at y pence each. She spends £1. Write an equation for this.

5. Jarred pays £x for a bus ticket. The following week the price has risen by £2. In total he spent £T on the two bus tickets. Write an equation expressing this information.

6. Fred runs x miles and John runs 3 times as far. John runs y miles. Write an equation that relates x and y.

1.3 *Substitution and Bodmas*

In the previous section, we have created a number of algebraic expressions. In example 1A, the cost for Errol to buy ***x*** apples and ***y*** oranges is represented by the algebraic expression $16x+50y$. If at one time he buys 5 apples and 4 oranges, how much will he pay? To work out this we simply replace ***x*** with number 5 and ***y*** with 4 and then the total cost will be

$$16*5+50*4=280 \text{ (pence) or 2.8 (pounds)}$$

This process of replacing letters (symbols) in an algebraic expression with actual numbers and then working out the final numerical value is called **evaluation by substitution**.

Example 1**D**

Evaluate $2x+4y+10$ when $x=1.4$, $y=2.2$.

Solution: Substituting $x=1.4$ and $y=2.2$ into $2x+4y+10$, we have
$2*1.4+4*2.2+10=2.8+8.8+10=21.6$.

In algebra we use letters or symbols to represent numbers **so the operations on them must follow the same rules that numbers obey.** The BODMAS rule tells us the order in which the mathematical operations must performed.

BIDMAS	**or**	**BODMAS**
Brackets		**B**rackets
Index		Exp**o**nents and roo**t**s
Division		**D**ivision
Multiplication		**M**ultiplication
Addition		**A**ddition
Subtraction		**S**ubtraction

Example 1E

a) To evaluate an expression such as

$11 - 3 + 14$

we would follow the **BODMAS** rule giving 22 as the answer.
Since addition and subtraction are of equal priority we scan the expression from left to right calculating as we go.
Similarly to find the value of $a + b - c$ we first add a to b and then subtract c.

b) When dealing with a numerical expression such as

$3 + 5 * 2$

we perform the multiplication before the addition (to get 13)
So when evaluating the algebraic expression $a + bc$ we would find the value of b times ***c*** first and then add the value of ***a***.

c) To override the normal rules of precedence we use **brackets**.
Thus to calculate $6 * (2 + 8)$ we perform the addition before the multiplication. This gives an answer of 60.
Likewise to find the value of $a(b + c)$ we first add ***b*** to ***c*** and then multiply the result by ***a***.

Since algebraic symbols only follow the rules of numbers if a rule does not work for numbers, it does not work for algebra either.

Example 1F

To add fractions we do not simply add the denominators and the numerators. This would give us the wrong answer. For instance,

$\frac{1}{2} + \frac{1}{3}$ does not equal $\frac{1+1}{2+3} = \frac{2}{5}$

so in algebra $\frac{a}{c} + \frac{b}{d}$ does not equal $\frac{a+b}{c+d}$

<u>Exercise 1.2</u>

Evaluate the following expressions when $w = 3, x = 5, y = 12$ and $z = 2$ without using a calculator

a)	$10x$	b)	$2y - 11$		
c	$w + 2z$	d)	$(w + 2)z$		
e)	$17 - 3w$	f)	$(x + 1)(y - 7)$		
g)	$3x - y + z$	h)	$5x + 3z - 3x$		
i)	$\frac{y}{w}$	j)	$\frac{x+3}{z}$	k)	$\frac{1}{x} + \frac{1}{w}$

1.4 Powers in Algebra

In Arithmetic, when a number is multiplied to itself several times, we can use a short-hand form called power or index to represent it. For example, $2\times2\times2\times2$ can be written as 2^4 (2 raised to the power 4). Similarly, in algebra we write a^2 for the product $a\times a$, a^3 for $a\times a\times a$, and so on. The calculation of a power takes precedence over multiplication and division.

Example 1G

To find the value of $5a^2$ we first calculate a^2 and then multiply by 5 so, if $a = 3$, $a^2 = 9$ and $5a^2 = 45$.

If we had wanted to perform the multiplication first, we would have written $(5a)^2$.
Since $a = 3$, $5a = 15$ and
$(5a)^2 = 15\times15 = 225$.

Take care here. Failing to make the calculations in the right order is a very common error.

Exercise 1.3

1. Find the value of the following expressions when $x = 3$ and $y = 2$.

a)	$3x^2$	b)	$(3x)^2$
c)	$3y^2$	d)	$(x+y)^2$
e)	xy^2	f)	$(xy)^2$

2. Write down the algebraic expression corresponding to the following instructions expressed in words.
   a) Multiply ***u*** by 6 and then cube the result.
   b) Halve the square of ***v***.
   c) Add ***p*** to twice ***q*** and square the result.
   d) Subtract ***a*** cubed from ***b*** squared.

1.5 Simplifying Algebraic Expressions-Gathering Terms and Expanding Brackets

Algebraic expressions can sometimes be shortened by combining terms. One instance of this can arise when the same term appears twice in a sum. Thus we can simplify

$a + a + 9$ to $2a + 9$

We can still simplify the expression if one or both of these terms are multiplied by a constant or if the terms are subtracted.

Example 1H

a) $2a = a + a$ and $3a = a + a + a \Rightarrow \quad 2a + 3a = 5a$

b) $6b = b + b + b + b + b + b$ and $4b = b + b + b + b \Rightarrow 6b - 4b = 2b$

We call this process **"gathering like terms".** Algebraic expressions may contain two or more sets of like terms and constants also.

Example 1I

In the expression $8x + 5y + 2 + 3x - y - 3$

$8x$ and $3x$ are like terms and 5y and y are like terms. Gathering like terms together we get

$8x + 3x + 5y - y + 2 - 3$

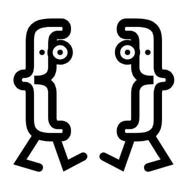
which simplifies to $11x + 4y - 1$.

Be careful not to combine unlike terms in this way. We cannot simplify $11x + 4y - 1$ any further.

<u>Exercise 1.4</u>

Simplify the following expressions by gathering like terms where possible.

a) $6p + 3$
b) $10q - 4q$
c) $7x + 4y - 2x + 3y$
d) $4z^2 + z$
e) $a + 3b + 2c + 4a - 5b$
f) $4s^2 + 8r^2 + 1 + s^2 - 3r^2 + 3$
g) $12u + 8v + 5w + 6u + 7v - 5w + 1$
h) $xy + xz + yz$



When creating algebraic expressions, frequently we use brackets to group some quantities together. To simplify certain algebraic expressions, such as, $1.2(2x + 4y)$, sometimes we need to remove all the brackets in them first. To do this, you need to use the number or letter in front of the brackets to multiply every term inside them. This process is called **expanding the brackets**. For example, after expanding the brackets, the above expression will become $2.4x + 4.8y$. Notice that the 1.2 multiplies every term in the brackets not just the first one. This remains true when there are more than two terms in the brackets and when the terms are negative.

Example 1J

Not
2p+q+r

a) $2(p+q+r) = 2p + 2q + 2r$

b) $5(2x^2 - y^2) = 10x^2 - 5y^2$

When there is a negative term outside the brackets, it is important to remember that the product of two negative terms is positive. This includes the case where there is only a subtraction sign preceding the brackets, as in example (d) below. In practice, this implies there is -1 before the brackets. Getting the signs wrong, when expanding brackets, is another very common error.
Once we have expanded two or more sets of brackets we may be able to gather like terms.

Example 1K

Not – 7y

c)
$$\begin{aligned} 9(2x+y) - 7(x-y) &= 18x + 9y - 7x + 7y \\ &= 18x - 7x + 9y + 7y \\ &= 11x + 16y \end{aligned}$$

d)
$$\begin{aligned} 13x - (x-7) &= 13x - x + 7 \\ &= 12x + 7 \end{aligned}$$

Exercise 1.5

Expand the brackets in the following expressions simplifying your answer where possible.

1. $4(x+y)$
2. $10(3a-5b)$
3. $6(u+v-20)$
4. $2(4a+3b)+3(a+3b)$
5. $5(2s+t)-2(2s-t)$
6. $8(p+2q-2r)-7(p-2q-3r)$
7. $4x+5y-(2x-3y+5)$
8. $6(x^2+y^2+1)-(x^2+y^2)-2(x^2-2y^2-1)$
9. $x^2(x^2-2x)$
10. $x^2(x^2-5x+2)$
11. $x(2x+1)$
12. $x(-x+1)-2x(3x+1)$
13. $2x^2(x-3)+5x(x^2+2x-1)$

1.6 Simple Factorisation

The reverse process of 'expanding the brackets' is called **factorising**. An example will suffice.

Example 1L

If we expand $10(2x+3y)$ we get $20x+30y$.

If, however, we start with the expression $20x+30y$ and rewrite it as the product of two factors

i.e. as $10(2x+3y)$,

we say the expression has been factorised.

We can only factorise terms when they have a **common factor**, for instance, we could not factorise $15x+14y$ because none of the factors of the first term, 3, 5 and $\boldsymbol{x}$ are factors of the second term.

Spotting the common factor is the essential first step in the factorisation process. Once the process is completed we can always check our answer by expanding the brackets to see if we obtain the original expression.

Example 1M

a) All the terms in the expression $3x+6y-18z$ have a factor 3.

To find the second factor we divide each term by 3 giving us $x+2y-6z$.

Hence we believe that

$$3x+6y-18z=3(x+2y-6z)$$

If we now expand the brackets we get $3x+6y-18z$ so our factorisation is indeed correct.

b) The term $2x$ is common to all the terms in the expression $8xy+6x^2$.

Dividing each of the terms by $2x$ we get $4y+3x$. Hence we think that $8xy+6x^2=2x(4y+3x)$.

<u>Exercise 1.6</u>

Factorise the following expressions.

1. $7u+21v$
2. $16x+24y$
3. z^2+3z
4. $8p-20q+12r$
5. $2a^2-11ab$
6. $9\text{xyz}+8\text{yz}$

Check your answers by expanding them.

1.7 Further Expansions

We normally write the product of two or more sets of terms in brackets without the multiplication sign. Thus the expression $(x+2)\times(x+3)$ is written $(x+2)(x+3)$. When expanding such an expression involving two pairs of brackets we must multiply every term in the first bracket by every term in the second.

a. <u>The Box Method</u>

We can justify the process and determine the answer using a diagram of a rectangle. We can see that

$(x+2)(x+3)$ = length x height of rectangle
= the area of the rectangle
$= x^2+2x+3x+6$

	x	2
x	x^2	2x
3	3x	6

and gathering like terms

$(x+2)(x+3)=x^2+2x+3x+6$

Again when multiplying the terms we need to remember that the product of two negative terms is a positive one.

b. <u>The 'Smiley Face' Method</u>

$$(x+2)(x+3)$$

$= x^2+6+2x+3x$

and gathering like terms

$(x+2)(x+3)=x^2+5x+6$

c. <u>The 'Moon Method</u>

$$(x+2)(x+3)$$

$= x^2+2x+3x+6$

and gathering like terms

$(x+2)(x+3)=x^2+5x+6$

Example 1N

$(x-3)^2=(x-3)(x-3)$
$= x^2-3x-3x+9$
$= x^2-6x+9$

Notice that $(x - 3)^2$ does not equal $x^2 - 9$

Exercise 1.7

Expand the following

1. $(x+5)(x+2)$ 2. $(x-2)(x+1)$ 3. $(x-10)(x-1)$

4. $(y-4)(y+4)$ 5. $(z+5)^2$ 6. $(s-1)^2$

1.8 Further Factorisation

We described earlier how it was possible to expand an algebraic expression and to conduct the reverse process of factorisation. For instance we could expand the expression $u(3+v)$ to obtain $3u+uv$ and we could factorise $3u+uv$ back to $u(3+v)$. On the last page we considered the expansion of products of brackets. We showed that

$$(x+2)(x+3)=x^2+5x+6$$

and now, as you might expect, we discuss how we might factorise an expression such as x^2+5x+6 back to the original pair of brackets. This is not as easy a problem as one might expect. Sometimes the factorisation is not possible, at least using ordinary numbers. Even when factorisation is possible it is not always clear what the original brackets were. However, there are clues.

Consider the following expansion

$$(x+a)(x+b)=x^2+ax+bx+ab$$

We assume we are given the expression on the right and wish to find the constants a and b.

Note that the expression can be written $x^2+x(a+b)+ab$. The clues to the values of ***a*** and ***b*** are therefore

i) their sum is the term multiplying $\boldsymbol{x}$ and
ii) their product is the constant term.

Example 1O

To factorise x^2+6x+8

by clue (ii) the product of a and b is 8 so assuming they are integers the possible pairs of brackets are

$(x+1)(x+8)$, $(x+2)(x+4)$, $(x-1)(x-8)$ and $(x-2)(x-4)$

but by clue (i) the sum of ***a*** and ***b*** is 6 . This is only true for the second pair of brackets. Hence

$$x^2+6x+8=(x+2)(x+4)$$

Exercise 1.8

Factorise each of the following expressions.

1. $x^2 + 4x + 3$
2. $x^2 + 7x + 10$
3. $x^2 + 8x + 12$
4. $x^2 - 7x + 6$
5. $x^2 - x - 6$

Check your answers by expanding them.

1.9 Laws of Indices

The laws of indices listed below can be used for simplifying many algebraic expressions involving powers. For examples, the first law says that to multiply two numbers with the same base we can simply add the indices and the second law states that to divide two numbers with same base we can subtract the indices.

On the other hand, the sixth law gives the definition for negative powers.

According to this law, for example, 2^{-3} means $\frac{1}{2^3}$, that is $\frac{1}{8}$.

	Laws of indices
1.	$b^n \times b^m = b^{n+m}$, **e.g.** $x^2 \times x^7 = x^{2+7} = x^9$
2.	$b^n \div b^m = b^{n-m}$, **e.g.** $x^{11} \div x^6 = x^{11-6} = x^5$
3.	$(b^n)^m = b^{mn}$, **e.g.** $(x^3)^4 = x^{3\times 4} = x^{12}$
4.	$b^1 = b$, **i.e. power 1 can be omitted.**
5.	$b^0 = 1$, **this gives the definition for zero power.**
6.	$b^{-n} = \frac{1}{b^n}$ **e.g.** $x^{-8} = \frac{1}{x^8}$

Exercises 1.9

1. Use the laws of indices to simplify the following:

a) $a^3 \times a^2$ b) $a^3 \times a^{-1}$ c) $a^{31} \times a^{29}$

d) $a^6 \div a^4$ e) $a^9 \div a^2$ f) $(a^3)^4$

g) $a^3 \div a^{-2}$ h) $a^3 \times a^{-1} \times a^5$ i) $a^3 \times a^2 \div a^7$

j) $(a^{-2})^{-4}$

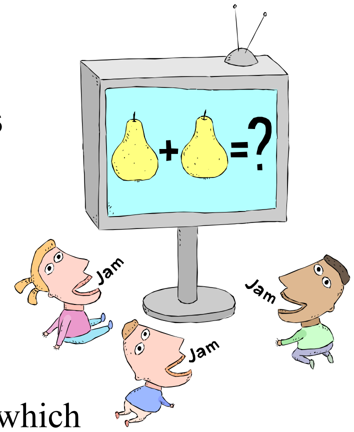
2. Simplify the following expressions as far as possible:

a) $2^2 \times 2^3$; b) $3^2 \times 3^3 \div 3^4$; c) $\frac{x^2 \times x^4}{x \times x^3}$

d) $\left(2^3\right)^4$; e) $\frac{x^3 \times \left(x^2\right)^5}{x^7}$; f) $\frac{\left(u^4\right)^{\frac{1}{2}}}{\left(u^{\frac{1}{2}}\right)^6 \times u^{-2}}$

Key points from Chapter

Chapter 2: Solving Linear Equations



Learning objectives:

By working through this chapter you should be able to:

- Solve linear equations in one unknown in which the unknown appears on either side or on both sides of the equation using the '*balance*' method.
- Solve linear equations involving fractions**.**

2.1 Introduction

Example 2A

A puzzle - I'm thinking of a number; 8 more than my number is 12. What number am I thinking of?

Let ***x*** stand for the number I'm thinking of, then we have two equal quantities, one of which involves ***x***, i.e. the information given above can be written as

$$x + 8 = 12 \qquad (2.1)$$

Note

This is an **equation** and we use the equals sign '=' to denote that the two expressions are equal or equivalent. An equation consist of an = sign in the middle and two "sides". In this equation the unknown x occurs only to the first power, it's called a **linear equation**. It is often helpful to number equations so that we can easily refer to the whole equation.
In order to keep equal quantities on both sides of an equation, we always do the same thing to both sides of the equation.

We can find ***x*** by subtracting 8 from both sides of (2.1) to give

$$x + 8 - 8 = 12 - 8$$

i.e. $x = 4$

Example 2B

Another puzzle - I'm thinking of a number; 4 times my number is equal to 20. What number am I thinking of?
Again, let ***x*** stand for the number I'm thinking of,
then the information given above can be written as

$$4x = 20 \qquad (2.2)$$

divide both sides of (2.2) by 4 and we get

$$x = 5$$

2.2 The 'Balance' Method for Solving Linear Equations

The guiding principle is simple but often forgotten. If the equation is to remain an equation then any change on one side must be matched by an equivalent change on the other. To change one side without changing the other to the same extent would mean that we could no longer use the equal sign.
There are many ways of making these changes. Here are two of the most important.

1. We can add (or subtract) the same term to both sides.

Example 2C
Suppose we have the equation
$$x-8=3$$
Adding 8 to both sides gives us
$$x=11$$

2. We can multiply (or divide) both sides by the same term

Example 2D
Suppose we have the equation
$$2a=16b$$
Dividing both sides by 2 gives us
$$a=8b$$

Example 2E
Suppose we have the equation
$$4x+5=25$$
Subtracting 5 from both sides gives
$$4x=20$$
Dividing both sides by 4 gives us
$$x=5$$

<u>Exercise 2.1</u>
Solve the following equations.

1. $x+5=20$
2. $x-3=14$
3. $2x=18$
4. $\dfrac{x}{3}=22$
5. $4+x=9$

2.3 The 'Balance' Method for Solving Advance Linear Equations

All the equations in exercise 2.1 could be solved using a single operation. Sadly most equations require two or more steps (sometimes many more steps) before the answer can be obtained. The steps often consist of eliminating unwanted terms.
We are now going to look at solving linear equations in one unknown in which the unknown appears on either side or on both sides of the equation.

Example 2F

Solve for ***x***

$$4x+2=x+14\,.$$

First we remove the +2 from the Left Hand side of the equation above by subtracting 2 from both sides

$$4x-x=14-2\,.$$

Now to remove the ***x*** from the Right Hand side we subtract ***x*** from both sides giving us the equation

$$3x=12\,.$$

Finally to remove the 3 multiplying ***x*** on the Left Hand side we divide both sides by 3. The solution of the equation is therefore

$$x=4\,.$$

Example 2G

Solve for ***x***

$$\frac{x}{7}-4=8$$

First we add 4 to both sides to give

$$\frac{x}{7}=12$$

and now we multiply both sides by 7 to obtain the answer

$$x=84$$

Remember that at every stage of the solution both sides of the equation must be treated in the same way.

<u>Exercise 2.2</u>

Solve the following equations.

1. $6x+1=10$
2. $3y-5=13$
3. $4z+3=3z+12$
4. $5u-8=4u+11$
5. $6+11t=8t+15$
6. $16-2s=4+4s$
7. $\dfrac{8+r}{3}=3r$
8. $\dfrac{1-q}{5}=\dfrac{1+q}{7}$

2.4 The ‘Balance’ Method for Solving Linear Equations Involving Brackets

Example 2H

The cost of jeans in Matalan is £10 less than in M&S and I can buy 2 pairs in Matalan for £5 more than I pay in M&S. How much do they cost in M&S?

Let ***x*** stand for the cost in M&S, then the cost in Matalan is $x-10$

Equating the costs of the purchases we get

$$2(x-10)=x+5 \qquad (2.3)$$

Notes

Step 1.

So that we can separate out the ***x’s*** from the numbers, we need to expand $2(x-10)$ in its equivalent form of $2x-20$

(2.3) can now be written as

$$2x-20=x+5 \qquad (2.4)$$

Step 2

We need the terms involving ***x*** in (2.4) all together on one side of the equality, so we subtract x from both sides so there is no ***x*** on the right hand side

$$2x-x-20=x-x+5$$

Collecting like terms we have

$$x-20=5$$

Adding 20 to both sides we get

$$x=25$$

2.5 The ‘Balance’ Method for Solving Linear Equations Involving Fractions

Example 2I

Consider Exercise 3.2 number 8

$$\frac{1-q}{5}=\frac{1+q}{7} \qquad (2.5)$$

Multiplying both sides by 5 gives

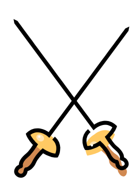
$$1-q=\frac{5(1+q)}{7} \qquad (2.6)$$

Multiplying both sides by 7 gives

$$7(1-q)=5(1+q) \qquad (2.7)$$

We could omit the middle step and go straight from equation (2.5) to (2.7). This method is sometimes known as **cross multiplication**

$$\frac{1-q}{5} = \frac{1+q}{7}$$



$$7(1-q)=5(1+q) \qquad (2.7)$$

Example 2J

To find x when

$$\frac{1+x}{1-x}=4$$

First we multiply both sides of the equation by $1-x$.

$$1+x=4(1-x)$$

Now multiply out the brackets on the Right Hand side

$$1+x=4-4x$$

Bring terms in $\boldsymbol{x}$ to the same side of the equation by adding $\boldsymbol{4x}$ to both sides

$$1+x+4x=4$$

Collect terms to get

$$1+5x=4$$

Subtract 1 from both sides of the equation to get

$$5x=3$$

Finally, divide both sides by 5 to get

$$x=\frac{3}{5}$$

Exercise 2.3

1. Find the value of $\boldsymbol{x}$ satisfying each of the following equations.

a) $14-3x=5(x-10)$

b) $3(x+2)=8-(2-x)$

c) $5=\dfrac{x+3}{x-3}$

Key points from Chapter 2

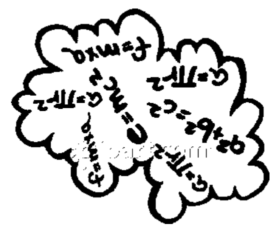
Chapter 3: Transposition of Formulae or Rearranging Equations

Learning objectives:

By working through this chapter you should be able to:

- Rearrange and Transpose formulas

3.1 Introduction



From solving equations we now turn to the exercise of rearranging equations which is similar because it uses the same techniques but instead of finding the numerical value of a variable, we find a formula. A **formula** is a way of calculating a quantity from other quantities.

Example 3A

Area of a rectangle = length × breadth

If we define variables in the obvious way ***A*** = area, ***L*** = length and b = breadth, we get

$$A = Lb$$

A is **the subject of the formula** i.e. the quantity we are calculating.

If we are given ***A*** and ***L***, we can solve an equation for b

e.g. Given ***A*** = 20, ***L*** = 5 then

$$20 = 5b$$

so $b = 4$

Rather than solve an equation for b every time we want to calculate the breadth of a rectangle from knowledge of its area and length, we can solve the problem algebraically to give a new formula which has b as its subject.

$$b = \frac{A}{L}$$ here we divided both sides by ***L***

This process is called **transposition** of a formula. Transposing a formula uses the same operations as solving an equation. In order to transpose a formula we need to know how the formula is put together.

Example 3B

Suppose we are given that

$$p = rq + s$$

and we wish to make ***q*** the subject. We first subtract ***s*** from each side of the equation to get

$$p - s = rq$$

and then divide both sides by ***r***. Then

$$q = \frac{p - s}{r}$$

Example 3C

The slowing down distance of a train, S, is given by the formula

$$S = \frac{v^2 - u^2}{2a}$$

where

u is the initial speed,
v is the final speed and
a is the acceleration (a constant)

Transpose this formula to make u the subject of the formula

$$S = \frac{v^2 - u^2}{2a}$$

Multiply both sides by $2a$

$$2aS = v^2 - u^2$$

Add u^2 to both sides and subtract $2aS$

$$u^2 = v^2 - 2aS$$

Take square roots of both sides

$$u = \sqrt{v^2 - 2aS}$$

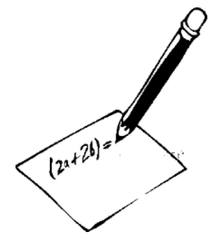
At every stage, we have done the same operation to both sides of the formula.

Exercises 3.1

Rearrange each of the following equations making x the subject.

1. $d = \frac{x}{t}$
2. $a = b + x$
3. $r = xt - u$
4. $r = st - x$
5. $p = \frac{1}{x}$
6. $v = \frac{x-1}{y}$
7. $a = \frac{bx}{d}$

3.2 Rearrangement Using Brackets



As equations get longer it usually becomes more difficult to rearrange them to obtain a new variable as the subject so some kind of strategy is needed. Our general strategy is to manipulate the equation so as to isolate the subject variable. If this variable appears in brackets, we may need to expand them at an early stage.

Example 3D
Suppose we know that

$$v = 2(3-u)$$

and we wish to make ***u*** the subject. First we expand the brackets to get

$$v = 6-2u$$

Adding ***2u*** to each side of the equation yields

$$v+2u=6$$

Now subtracting ***v*** from both sides and dividing by 2 we get

$$u=\frac{6-v}{2}$$

When the subject appears as part of a product in two or more terms it may be possible to isolate it by factorisation.

Example 3E
To make ***p*** the subject given that

$$q-rp=sp$$

we first bring the terms in ***p*** to the same side of the equation. Then

$$q=sp+rp$$

Now we can factorise

$$q=p(s+r)$$

and dividing by the term in brackets gives us

$$p=\frac{q}{s+r}$$

Exercises 3.2
All of these exercises are based on real-life problems

1. Given the equation $v = u + at$
   a) Transpose it to make ***a*** the subject
   b) Find the value of a when ***u*** = 20, ***v*** = 35 and ***t*** = 5

2. Given the formula $v = wr$
   a) Transpose the formula to make ***r*** the subject
   b) Find the value of ***r*** when ***v*** = 16 and ***w*** = 5

3. Using the formula $1.8C = F - 32$
   a) Transpose the formula to make ***F*** the subject
   b) Find the value of ***F*** when ***C*** = 40.

4. a) Transpose the formula $\frac{PV}{T} = C$ to make V the subject
   b) Calculate the value of V when C = 24, T = 36 and P = 144

5. Using the formula $E = IR + v$
   a) Rearrange the formula to make ***R*** the subject
   b) Calculate the value of ***R*** when ***E*** = 9.2, ***I*** = 2.0 and ***v*** = 3.6

6. Transpose the vector equation $r = q + vt$ to make **v** the subject

7. Given the formula $Ft = m(v - u)$
   a) Rearrange this formula to make ***u*** the subject
   b) Calculate ***u*** when ***F*** = 12, ***t*** = 2, ***m*** = 0.5 and ***v*** = 56

8. The equation $S = \frac{(u+v)t}{2}$ is used when an object moves with constant acceleration.
   a) Transpose the formula to make v the subject
   b) Find v when S = 84, u = 8 and t = 12

9. Given the equation $T = \frac{\lambda(x-5)}{5}$ which relates tension in a spring, T, to its length, x.
   a) Calculate T when λ= 1.5 and x = 8
   b) Transpose the formula to make x the subject.
   c) Use the formula from b) to find the value of x when T = 20 and λ = 4

10. The surface area of a cylinder is given by the formula $A = 2\pi r(r+h)$
    a) Calculate the value of A if r = 2.5cm and h = 4.2cm. Give your answer to the nearest whole number.
    b) Transpose the formula to make **h** the subject.
    c) Using the formula formed in b) determine the value for h correct to 2 dp when A = 32 cm^2 and r = 1.4cm

11. The sides of a right angled triangle are a, b, and c and $a = \sqrt{c^2 - b^2}$
   a) Calculate **a** given that c = 13m and b = 5m
   b) Transpose the formula to make **b** the subject.

   c) Calculate b given that **c** = 41cm and **a** = 9cm

12. The velocity of a particle is given in terms of its mass, **m**, and its kinetic energy, **E**, by the formula, $v = \sqrt{\frac{2E}{m}}$
   a) Calculate **v** (in ms^{-1}) given that **E** = 24J and **m** = 3kg.
   b) Rearrange the formula to make **m** the subject
   c) Find the value of **m** (in kg) given that **E** = 32J and **v** = 40 ms^{-1}
13. The period of oscillation, T(s), of a pendulum of length, l (m), is given by $T = 2\pi\sqrt{\frac{l}{g}}$
   a) Transpose this formula to make g the subject
   b) Calculate **g** to 2 decimal places, given T = 0.8s and **l** = 0.159m

14. The radius, **r**, of a cylinder is given in terms of its volume, **V**, and its height, **h**, by $r = \sqrt{\frac{V}{\pi h}}$
   a) Transpose this formula to make **h** the subject
   b) Calculate **h** to the nearest whole number, given **V** = 136m^3 and **r** = 3m

Exercises 3.3

1. Make $\boldsymbol{x}$ the subject in each of the following cases.

a) $y = a(6 + x)$

b) $z + 1 = 2(5 - x)$

c) $u + 3v = 3(u - v + x)$

d) $a + 2x = ax$

e) $3 - p^2x = r + q^2x$

f) $2(u - x) = s(t - x)$

More exercises are available in Foundation Mathematics by Croft and Davison Exercises 13.1

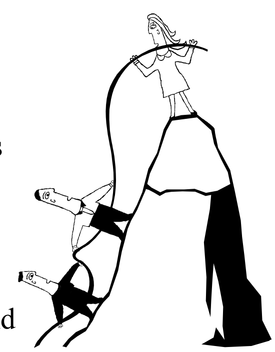
Key points from Chapter 3

Chapter 4: Straight Lines, Intercepts and Gradients, Simultaneous Equations

Learning objectives:

By working through this chapter you should be able to:

- Find a linear relationship between ordered pairs
- Plot straight lines from equations
- Find a distance on a rectangular grid
- Find gradients of lines.
- Find the intercept of lines
- Find the equation of a straight line, given two points
- Find the gradient and intercept of a given straight line
- Solve simultaneous equations graphically
- Solve simultaneous equations algebraically



4.1 Coordinates in Two Dimensions

The position of a point in a plane can be specified by giving its coordinates. The **x-coordinate is the horizontal distance** from the origin to the point. The **y-coordinate is the vertical distance** from the origin to the point.

The coordinates are written as a pair of numbers in brackets and separated by a comma. The x-coordinate is written first, followed by the y-coordinate.

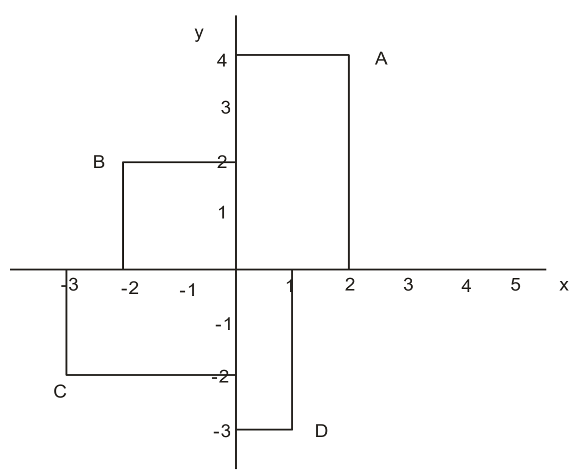


Figure 4.1

In Figure 4.1 A is (2, 4) B is (-2, 2) C is (-3,-2) and D is (1,-3)

Graphs

Graphs are a pictorial way of representing relations between variables. When we plot a graph we need the coordinates of several points. When we plot a graph from an equation we calculate the y values for a range of ***x*** values using the equation. To plot a straight line graph from an equation – we need the coordinates of a minimum of 2 points in order to fix a straight line. It is often a good idea to calculate a third point (but not more) as a check.

Note

Equations which have the form $y = mx + c$ always give a straight line as their graph.

When we plot a graph we need to decide on the range of ***x*** values that we are going to use. This determines the range of ***y***-values that will result and also the scales that we need to use on the axes so that the graph appears a reasonable size on the graph paper.

Exercises 4.1

Look at each set of points and find the relationship between the ***x*** and ***y***-coordinates.

1. (-2,-2), (-1,-1), (0,0), (1,1), (2,2)

x	-2	-1	0	1	2	
y	-2	-1	0	1	2	

2. (-2,0), (-1,1), (0,2), (1,3), (2,4)

x	-2	-1	0	1	2	
y	0	1	2	3	4	

3. (-2,-3), (-1,-2), (0,-1), (1, 0), (2, 1)

x	-2	-1	0	1	2	
y						

4. (-3,-6), (-2,-4), (-1,-2), (0, 0), (1, 2), (2, 4)

x	-3	-2	-1	0	1	2	
y	-6	-4	-2	0	2	4	

5. (-3,-5), (-2,-3), (-1,-1), (0, 1), (1, 3), (2, 5)

x	-3	-2	-1	0	1	2	
y	-5	-3	-1	1	3	5	

4.2 Plotting Straight Lines from Equations

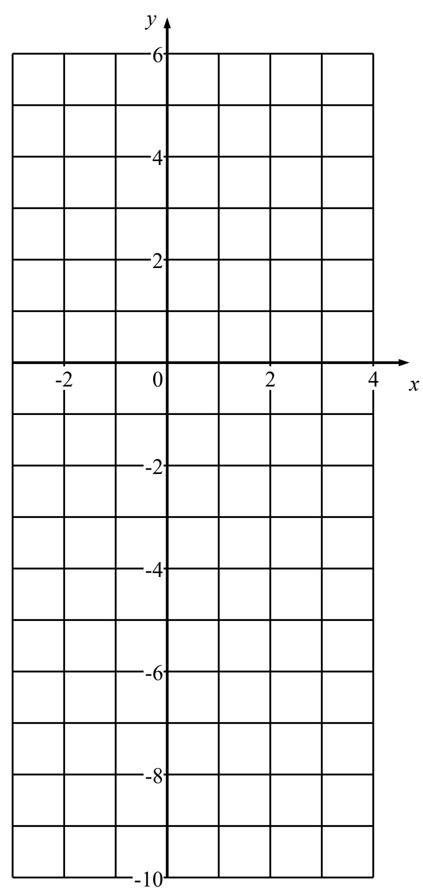
Example 4A

Draw the graph of $y = 2x - 3$

Choose suitable values of $\boldsymbol{x}$, they are determined from the context of the problem. If we are asked to plot a graph for values of $\boldsymbol{x}$ between 0 and 500 then these two values would be suitable. However, for the line whose equation is given above, the interesting part is for small positive and negative values of $\boldsymbol{x}$ so let's take $x = -2$ and $x = 3$, and plot the graph between $x = -3$ and $x = +4$.

Draw the graph of $y = 2x - 3$

x	y
-1	
0	
1	



Exercises 4.2

Plot the following straight line graphs on one set of axes. The values to use for **x** are suggested in the questions. Make a table of values for each line before plotting the points.

1. $y = 2x + 2$ use $x = -2$ and $x = 2$
2. $y = 3 - x$ use $x = -2$ and $x = 2$
3. $y = x + 5$ use $x = -3$ and $x = 1$
4. $y = 3x - 2$ use $x = -1$ and $x = 3$
5. $2y = 4x - 1$ use $x = -2$ and $x = 1$

$y = 2x + 2$

x	y
-2	
-1	
0	
1	
2	

$y = 3 - x$

x	y
-2	
-1	
0	
1	
2	

$y = x + 5$

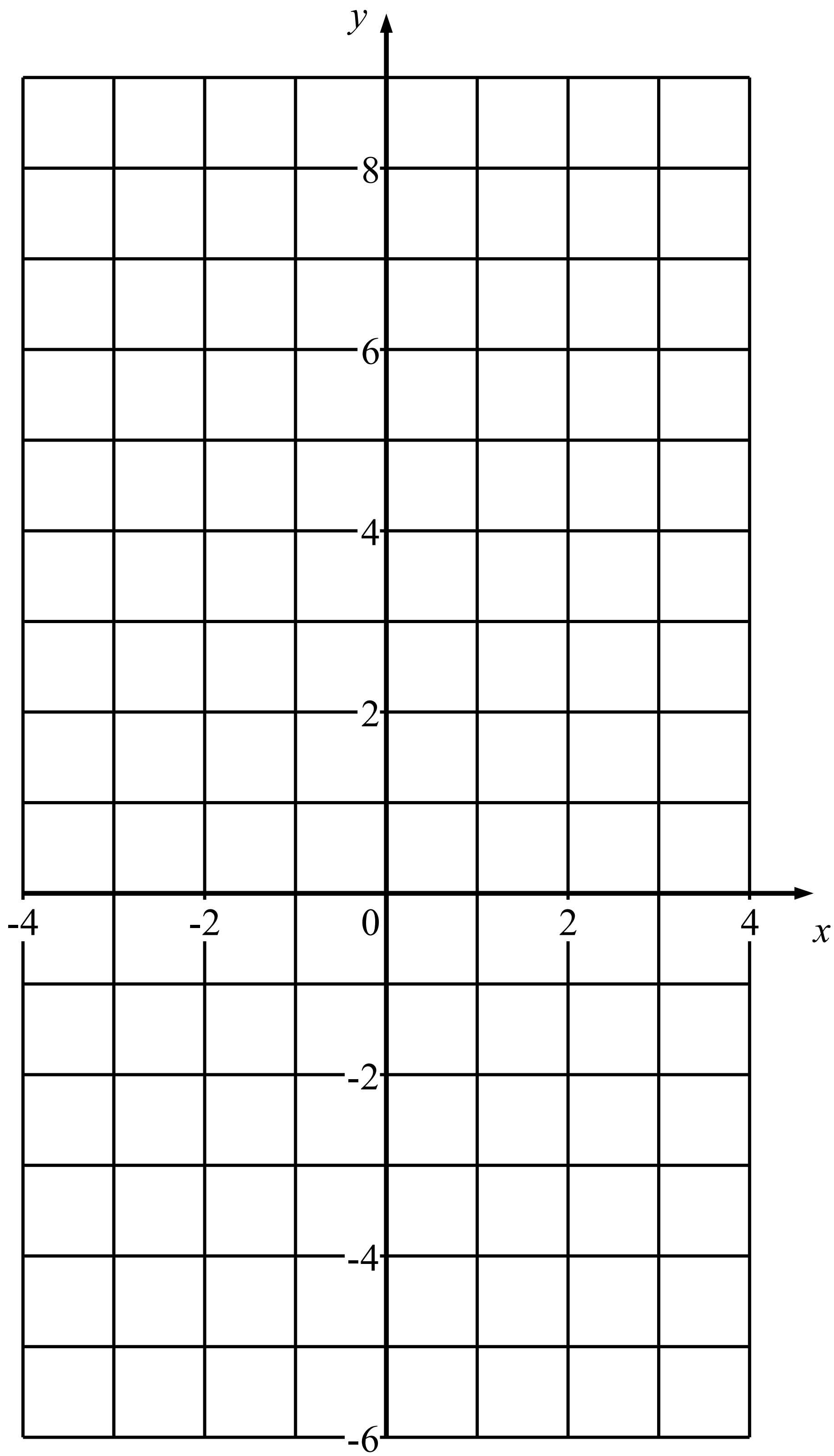
x	y
-3	
-2	
-1	
0	
1	

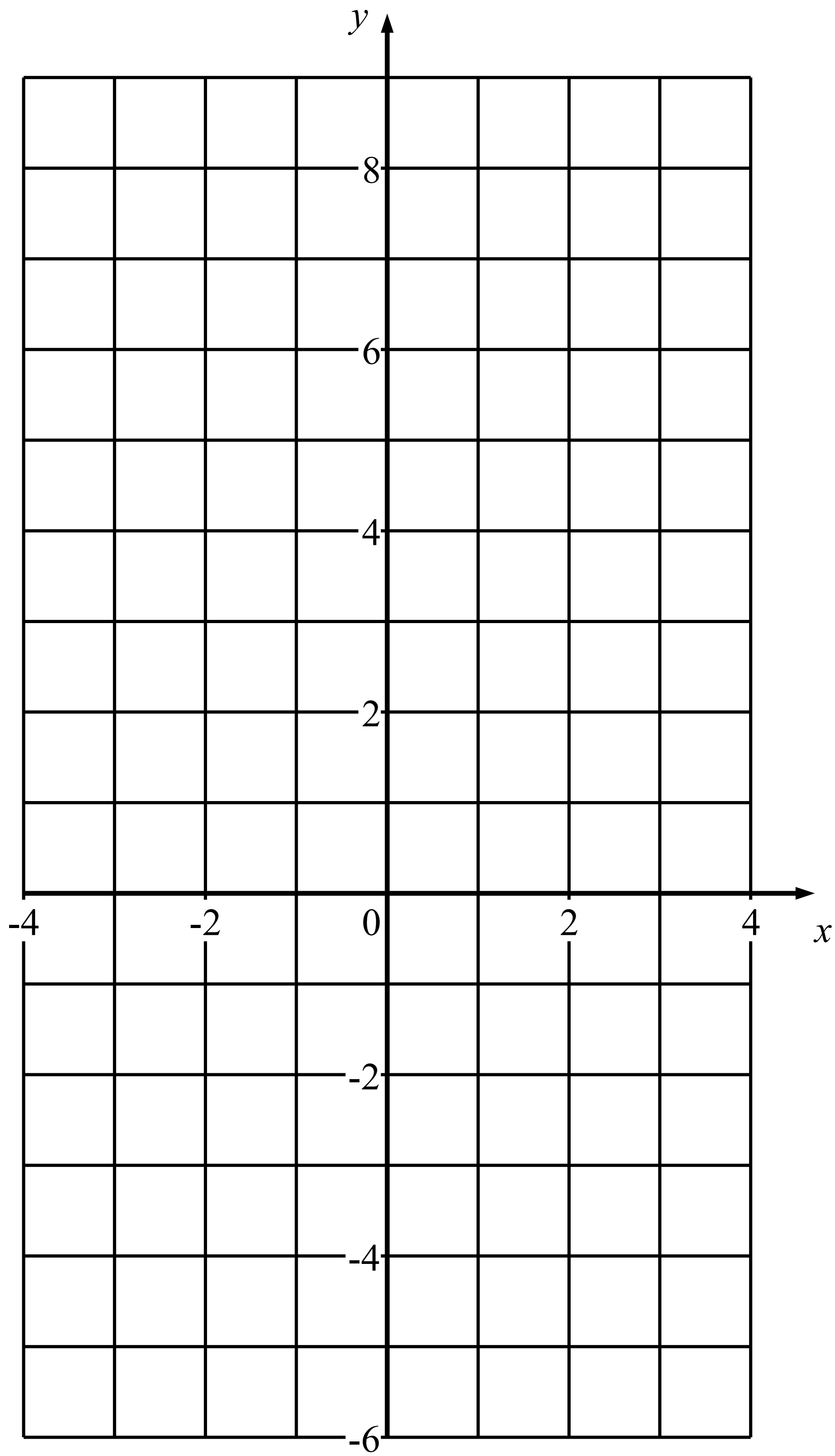
$y = 3x - 2$

x	y
-1	
0	
1	
2	
3	

$2y = 4x - 1$, $y = 2x - 1/2$

x	y
-2	
-1	
0	
1	


y
8
6
4
2
-4
-2
0
2
4
x
-2
-4
-6


y
8
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2
4
x
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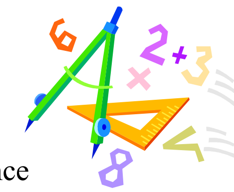
-4
-6

4.3 Distances on a Rectangular Grid



We can use Pythagoras' theorem to find the distance between two points if you know their co-ordinates.

For two points A and B in a x-y plane, if their co-ordinates are known to be (x_1, y_1) and (x_2, y_2) respectively, then the distance between A and B is

$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \tag{4.1}$$

Example 4B

Calculate the distance between the points A (-1, 3) and C (2, 7).

First, we form a right angle triangle with sides parallel to the x and y axes as in Figure 4.2.

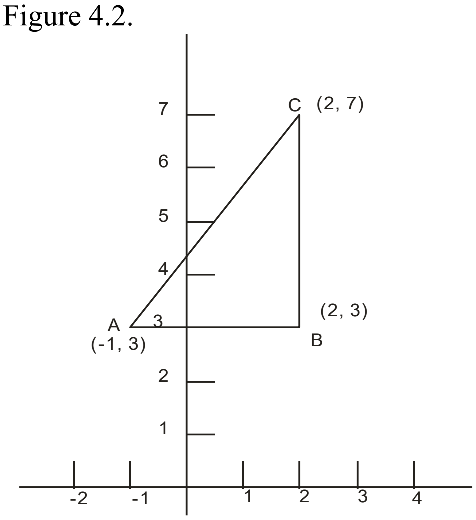


Figure 4.2

Pythagoras's Theorem states that the square of the hypotenuse of a right-angled triangle is equal to the sum of the squares of the other 2 sides.

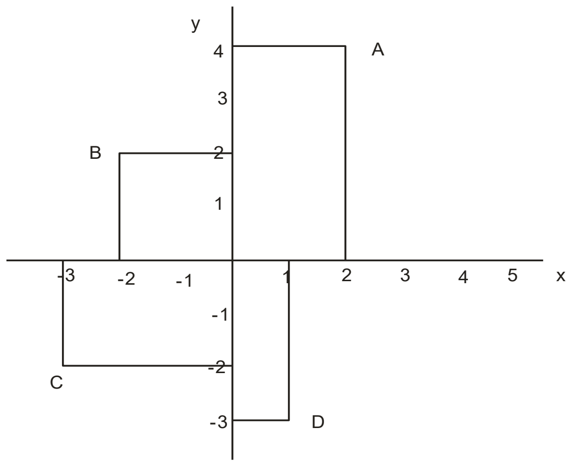
We apply Pythagoras's Theorem to our problem as follows:
In ΔABC

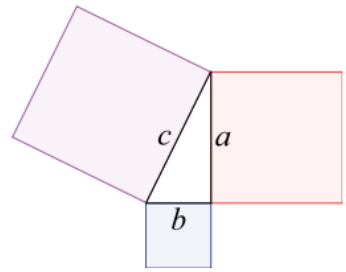
$$AC^2 = AB^2 + BC^2$$
$$AC^2 = [(2-(-1))^2 + (3-3)^2] + [(2-2)^2 + (7-3)^2]$$
$$AC^2 = (2-(-1))^2 + (7-3)^2$$
$$= (3)^2 + (4)^2$$
$$= 25$$

so $\quad AC = 5$

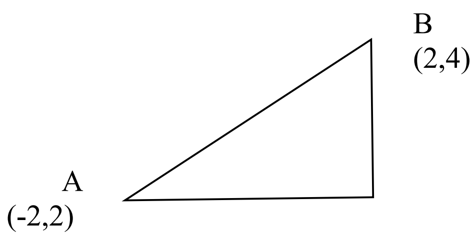
Exercise 4.3

With reference to Figure 4.1, calculate the distances AB, AC, AD, BC, BD, and CD.

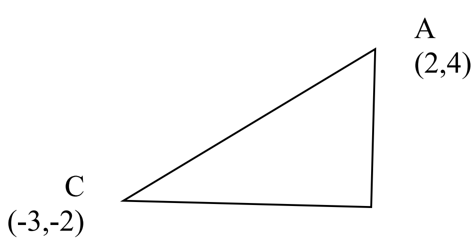




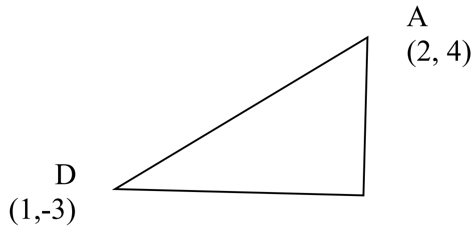
In Figure 4.1 A is (2, 4) B is (-2, 2) C is (-3,-2) and D is (1,-3)



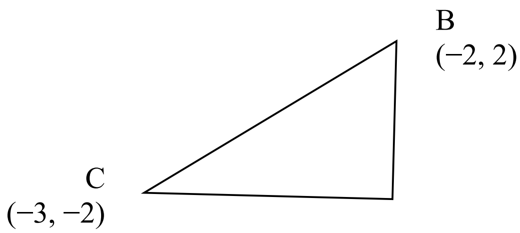
AB =



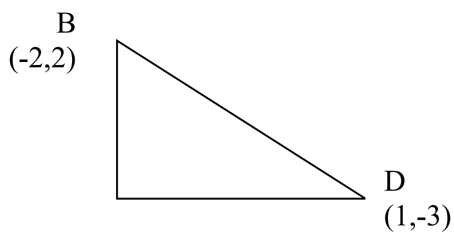
AC



AD =



BC =



BD =

Now find CD

4.4 Gradients and Intercepts of Straight Lines

The gradient of a straight line measures how steep its slope is and can be calculated by the **rate of increase** of ***y*** relative to ***x*** between any two points on the line

$$= \frac{\text{Increase in Vertical Distance}}{\text{Increase in Horizontal Distance}} = \frac{\text{Distance Up or Down}}{\text{Distance across}}$$

Given two points on the line, (x_1, y_1) and (x_2, y_2) say, then the gradient of the line is :

$$m = \frac{\text{change in } y}{\text{change in } x} = \frac{y_2 - y_1}{x_2 - x_1}. \qquad (4.2)$$

Example 4C

(i) If a straight line passes these two points (-1,-2) and (2,4), what the gradient of the line?

Solution**:** taking (-1,-2) and (2,4) as point 1 and 2 respectively, then $x_1 = -1$, $y_1 = -2$, $x_2 = 2$, $y_2 = 4$. So the gradient

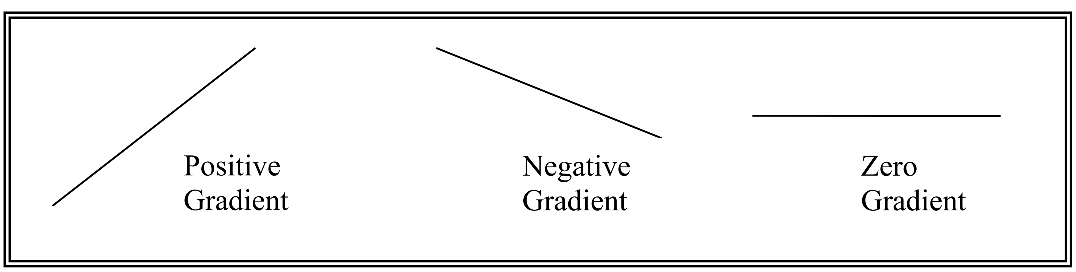
$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4 - (-2)}{2 - (-1)} = \frac{6}{3} = 2.$$

(ii) Work out the gradient for a straight line passing points (-2,3) and (3,-4.5).

Solution**:** taking (-2,3) and (3,-4.5) as point 1 and 2 respectively, then $x_1 = -2$, $y_1 = 3$, $x_2 = 3$, $y_2 = -4.5$. So the gradient

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-4.5 - 3}{3 - (-2)} = \frac{-7.5}{5} = -1.5$$

The sign of the gradient can positive, negative or zero.



- The **x-intercept** is defined as the point where the line crosses x-axis (when y=0)
- The **y-intercept** is defined as the point where the line crosses y-axis (when x=0)
- Lines with gradients m_1 *and* m_2
  - are parallel if $m_1 = m_2$ (4.3a)
  - are perpendicular if $m_1 \times m_2 = -1$ (4.3b)

The general equation of a straight line graph may be written as

$$y = mx + c\,, \tag{4.4}$$

Where m gives the gradient of the line and c is the y-intercept

Exercise 4.4

1. Write down the gradient and y-intercept for this graph

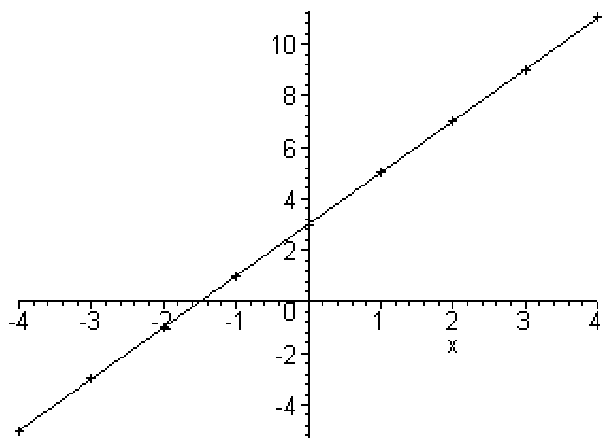


Figure 4.3 A graph of the straight line $y = 2x + 3$.

2. Write down the gradient and y-intercept for this graph

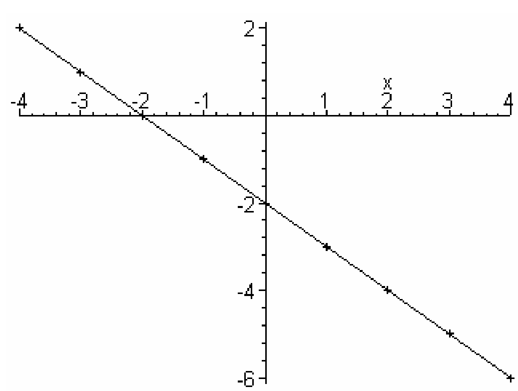


Figure 4.4 A graph of the straight line $y = -x - 2$.

3. Write down the gradient, *x*-intercept and *y*-intercept for this graph

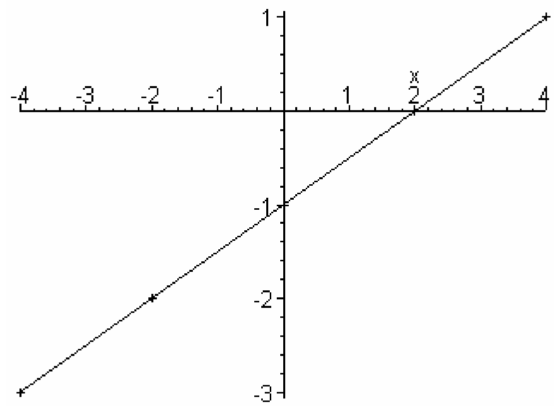


Figure 4.5 A graph of the line $y = \frac{1}{2}x - 1$.

4. Find the gradient and intercepts of each of the following straight lines. Find out if any of these lines are parallel or perpendicular. Rearrange the equations first if necessary

a)	$y = 5x + 2$	f)	$5y - 3x - 4 = 0$
b)	$y = -3x - 1$	g)	$2x + 4y + 5 = 0$
c)	$x + y = 3$	h)	$0.2y = 1.4x - 0.7$
d)	$2x - 4y = 8$	i)	$1.5x + 0.5y = 0.2$
e)	$x = 2y - 18$	j)	$\frac{y}{2} - 3x + 2 = 0$

5. Use graph paper to plot the following:

a) $y = 3x - 2$;   b) $y = 3 - 2x$;   c) $2y = \frac{1}{2}x + 3$.

Determine the gradient and y-intercept of each line.

6. The length of grass A is given by $L = 2t + 10$, and the length of grass B is given by $L = t + 40$; length measured in centimetres and time in days. Plot graphs on the same set of axes and use your graphs to find when the lengths of grass are equal.

7. Gasco have a standing charge of 10 pounds and charge 20p per unit for electricity. Elecom have a standing charge of 30 pounds and charge 10p per unit for electricity. Plot the graphs and determine which electrical supplier you should use based on the household consumption.

Example 4D

The total cost, C, of employing a plumber is £60 for the call-out and £30 per hour.
Write down the total cost of employing a plumber for t hours.
What is the value of the gradient and what is the value of the intercept of this straight line?

Time t hours	Cost, £C
1	£(60 + 30)
2	£(60+60)
3	£(60+90)
t	£(60+30t)

C = 60 + 30t
The gradient = rate of increase of C with respect to t
= cost per hour = 30£/h
The intercept = the cost when t=0 which is £60

Example 4E

The length of a spring, L (in cm), varies with the load, T (in N), that is applied.
The length of the spring under zero load is 10cm and it increases in length by 1cm for every 10N load that is applied.
A graph of extension against load is drawn. What is the gradient and intercept of the graph?

Load, T (in N),	Length L (in cm),
0	10cm
10	11cm = (10+1)cm
20	12cm= (10+2)cm
30	13cm= (10+3)cm
T	(10+0.1T)cm

L= 10 + 0.1T
The gradient
= rate of increase of Length with respect to Load
= 0.1 cm/N
The intercept = Length when the Load is zero which is 10cm

Exercise 4.5

1. Most electricity tariffs have two components, a standing quarterly charge and a cost per unit of electricity that is used. Scotweb charges £12 per quarter and 11p per unit used. Write down an equation for the cost, C, in terms of the units, u, used in a quarter.

2. A candle is initially 10cm long and it burns down at a rate of 1 cm per hour.
   A graph is drawn of the length of the candle, L, against t the time in hours that the candle has been burning.
   What is the gradient of the line ?
   What is the intercept?
   Write down the equation of the line.

3. John starts 500m away from home and gets 2m closer every second. Write down an equation that gives his distance from home. Define the variables that you use.
   For what range of times is this expression valid?

More exercises are available in Foundation Mathematics by Croft and Davison Exercises 18.1 and 18.2

4.5 Finding the Equation of a Straight Line through a Given Point and with a Given Gradient

In many practical problems, we can find the coordinates of isolated points on a straight line but we actually would like to know the equation of the line.

Example 4F
Find the equation of the straight line through the point (2, -1) with gradient 3

Using the equation (4.4), $y = mx + c$, we have.

$$y = 3x + c$$

As the line passes through the point (2, -1), means that $x = 2$, $y = -1$.
Substituting these values in the above equation, we have.
$-1 = 3(2) + c \quad \Rightarrow \quad -1 = 6 + c \quad \Rightarrow \quad c = -7$

Hence the required line is
$$y = 3x - 7$$

General case:

To find the equation of the straight line through the point $A(x_1, y_1)$ with gradient *m*.

From equation (4.4), as the line passes through the point (x_1, y_1) means that $x = x_1$ and $y = y_1$. Substituting these values in (4.4), we have

$$y_1 = mx_1 + c \Rightarrow c = y_1 - mx_1$$

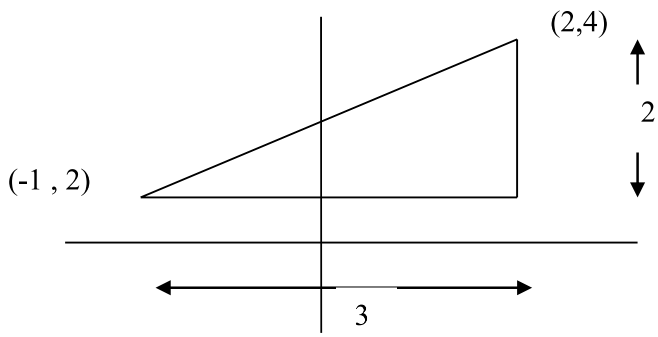
Hence the required line is

$$y = mx + y_1 - mx_1$$

$$y - y_1 = m(x - x_1) \qquad (4.5)$$

4.6 Finding the Equation of a Straight Line through Two Given Points

Example 4G
Find the equation of the straight line through the points (-1, 2) and (2,4)



Using equation (4.2) $m = \dfrac{y_2 - y_1}{x_2 - x_1}$, the gradient $m = \dfrac{2}{3}$

From equation (4.4), we have the following. he equation y= mx +c we have

$$y = mx + c \;\;\Rightarrow\;\; y = \frac{2}{3}x + c$$

As the line passes through the point (-1, 2) we know that when $x = -1$ and $y = 2$, substituting these values we have.

$$2 = \frac{2}{3}(-1) + c \;\Rightarrow\; 2 = -\frac{2}{3} + c \;\Rightarrow\; c = \frac{8}{3}$$

Hence the required line is

$$y = \frac{2}{3}x + \frac{8}{3} \;\Rightarrow\; 3y = 2x + 8$$

Alternatively use equation (4.5)

$$y - y_1 = m(x - x_1)$$

with $m = \dfrac{2}{3}$, $x = -1$ and $y = 2$ giving

$$y - 2 = \frac{2}{3}(x + 1) \quad \Rightarrow\; 3y - 6 = 2(x + 1) \;\Rightarrow\; 3y = 2x + 8$$

General case:

To find the equation of the straight line through the points $A(x_1, y_1)$ and $B(x_2, y_2)$

Using equations (4.2) and (4.5), the line is

$$y - y_1 = \frac{(y_2 - y_1)}{(x_2 - x_1)}\,(x - x_1) \;\Rightarrow\; \frac{(y - y_1)}{(y_2 - y_1)} = \frac{(x - x_1)}{(x_2 - x_1)} \qquad (4.6)$$

Example 4H

Find the equation of the line joining $(x_1, y_1) = (3,4)$ to $(x_2, y_2) = (5,-1)$.

Solution:

From equation (4.6)

$$\frac{(y - y_1)}{(y_2 - y_1)} = \frac{(x - x_1)}{(x_2 - x_1)} \Rightarrow \frac{(y-4)}{(-1-4)} = \frac{(x-3)}{(5-3)}$$

Note The difference is taken the same way round in each case.

$$\frac{(y-4)}{-5} = \frac{(x-3)}{2} \quad \Rightarrow \quad 2(y-4) = -5(x-3)$$

$$2y - 8 = -5x + 15 \quad \Rightarrow \quad 2y - 8 = -5x + 15 \Rightarrow \quad 2y = -5x + 23$$

Equation of line is $y = \frac{-5}{2}x + \frac{23}{2}$

Exercise 4.6

1 Find :

(a) the equation of the line with gradient 2 and passes through the point (1,6)
(b) the equation of the line with y-intercept -1 passing through $(2,7)$;
(c) the equation of the line passing through $(1,-2)$ and $(4,2)$.
(d) the equation of the line passing through (-1,2) which is parallel to the line with equation $2y = x + 4$
(e) the equation of the line passing through the origin and is perpendicular to the line $y = \frac{1}{2}x + 3$

4.7 Graphical interpretation

Unless they are parallel lines, two straight lines will intersect at just 1 point as is shown in Figure 4.6. We can read off where they intersect.
This is subject to 2 possible sources of error;

- inaccuracies in plotting the graph and
- inaccuracies in reading the point of intersection.

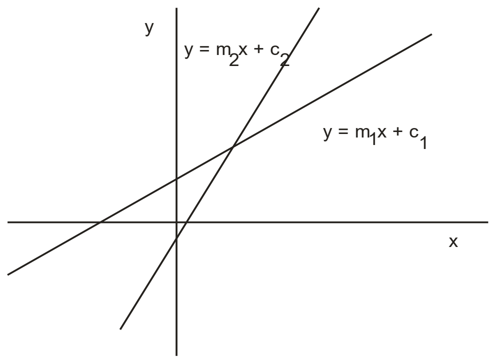


Figure 4.6

Example 4I

Find the point where the graphs $y = -\frac{4}{5}x + 44$ and $y = -\frac{1}{5}x + 26$ intersect.

<u>Solution</u> Construct a table of ***x*** and ***y*** values for each function :

The function $y = -\frac{4}{5}x + 44$:

x	0	5	10	20	30	40	50
y	44	40	36	28	20	12	4

The function $y = -\frac{1}{5}x + 26$:

x	0	5	10	20	30	40	50
y	26	25	24	22	20	18	16

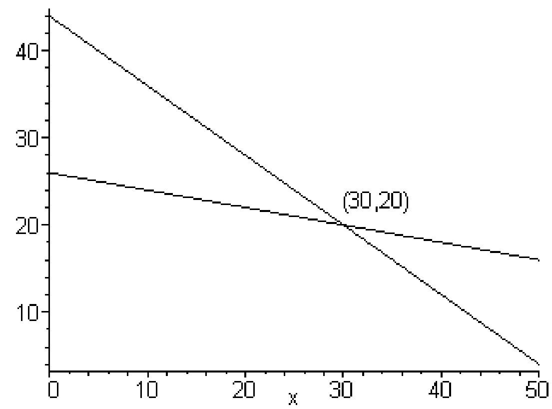


Figure 4.7 The straight lines $y = -\frac{4}{5}x + 44$ and $y = -\frac{1}{5}x + 26$.

Reading from the graph, the two lines intersect at the point $x = 30$ and $y = 20$ or the coordinates (30,20).

WARNING
Note that two straight lines need not cross (e.g. plot the lines $y = x$ and $y = x + 3$, which are parallel), or the lines may be the same (e.g. plot the lines $y = x - 1$ and $2y + 2 = 2x$ which are the same).

4.8 Calculating the intersection of two lines

Calculating where the graphs intersect gives us greater accuracy. Where the lines intersect both the x and y values are the same. We find x and y 'simultaneously'.

Example 4K

Solve the simultaneous equations given by

$$y = 10 - 2x \qquad (4.7)$$

$$y = x + 1 \qquad (4.8)$$

As each equation represents a straight line, when these two lines meet they must share the same ***x*** and ***y*** co-ordinates, i.e. the *x* and ***y*** co-ordinates must be equal respectively.

$\Rightarrow \quad 10 - 2x = x + 1$

$\Rightarrow \quad 10 - 2x - 1 = x$ -1

$\Rightarrow \quad 10 - 1 = x + 2x$ +2x

$\Rightarrow \quad 9 = 3x$ collect terms

$\Rightarrow \quad x = 3$ $\div$ 3

and hence, $y = 10 - 2(3) \quad \Rightarrow \quad y = 4$

So the solution for the simultaneous equations (4.7) and (4.8) is $x = 3$ and $y = 4$. You can check that $x = 3$ and $y = 4$ satisfy both equations.

Exercises 4.7

Solve the following pairs of simultaneous equations, question 1 has already been done for you:

1.	$y = 10 - 2x$ $y = x + 1$	6.	$2y = 10 + x$ $y = 3x + 5$
2.	$y = 2x + 1$ $y = 11 - 3x$	7.	$3y = 2x - 18$ $y = 16 - 3x$
3.	$y = 2x + 5$ $y = 11 - 4x$	8.	$y = 1 - x$ $2y = x + 5$
4.	$y = 4x + 7$ $y = 2x + 5$	9.	$2y = x - 1$ $y = -2 - x$
5.	$y = x + 2$ $y = -14 - 3x$	10.	$y = x - 3$ $-4y = 5x - 6$

Exercises 4.8

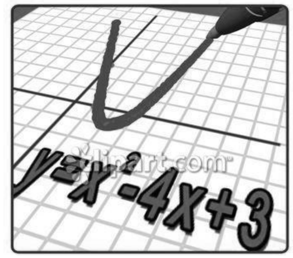
1. Scotpow and Mangen supply electricity to domestic consumers. Scotpow has a quarterly standing charge of £12 and a unit charge of 11p and Mangen has standing charge of £15 and a unit charge of 9.8p per unit. Write down equations for calculating the cost of u units of electricity from both suppliers. Use these equations to calculate the cost and the number of units used when the two companies charge the same.

2. On a farmyard there are chickens and sheep. In total there are 46 legs and 19 heads. How many chickens and sheep are there?

3. Cuthbert needs to hire a jack hammer to demolish a garage. Eurohammer and Jackhurt's rent the machines.
   Eurohammer charges £10 plus £4 per day and Jackhurt's charges £5.50 per day.
   Write expressions giving the total cost of hiring a machine for t days.
   Calculate the duration of hire that gives equal hiring costs.

More exercises are available in Foundation Mathematics by Croft and Davison Exercises 14.2

Key points from Chapter 4



Chapter 5: Quadratic Formulas and Equations, Polynomials

Learning objectives:

By working through this chapter you should be able to:

- Plot the graph of quadratic functions
- Use a given context to formulate quadratic expressions
- Find the roots of a quadratic equation using the quadratic formula
- Find the roots of a quadratic equation using factorisation
- Know the relationship between the number of solutions to a quadratic equation and the corresponding quadratic graph

5.1 Quadratic Functions

Quadratic functions involve terms in x^2, $\boldsymbol{x}$ and a constant. The general form for quadratic functions can be written as

$$y = ax^2 + bx + c$$

where ***a, b, c*** are all constants (numbers) and $a \neq 0$.

5.1.1 Plotting graphs of quadratics

To plot a graph of a quadratic we need a table of values of x and the corresponding y values. We evaluate each term in the quadratic for certain values of x. then add up the 3 parts to get a value of y.

Example 5A Graph the quadratic $y = 2x^2 - x + 2$.<u>Solution</u> Construct a table of x and y values :

x	–4	–3	–2	–1	0	1	2	3	4
x^2	16	9	4	1	0	1	4	9	16
$2x^2$	32	18	8	2	0	2	8	18	32
$-x$	+4	+3	+2	+1	0	−1	−2	−3	−4
+2	+2	+2	+2	+2	+2	+2	+2	+2	+2
y	38	23	12	5	2	3	8	17	30

Example 5A, (continued)

Plot these points and sketch a curve which passes through the points.

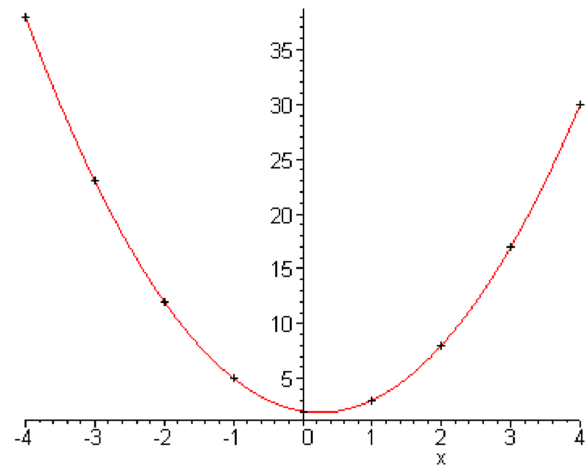


Figure 5.1 A graph of the quadratic $y = 2x^2 - x + 2$.

<u>Exercise 5.1</u> Draw graphs of the following quadratics

(a) $y = x^2 - x - 2$; (b) $y = 4 - x^2$.

<u>Exercises 5.2</u>

Use values of **x** from -3 to +3 to form a table of values so that you can plot the graph for all of the following:

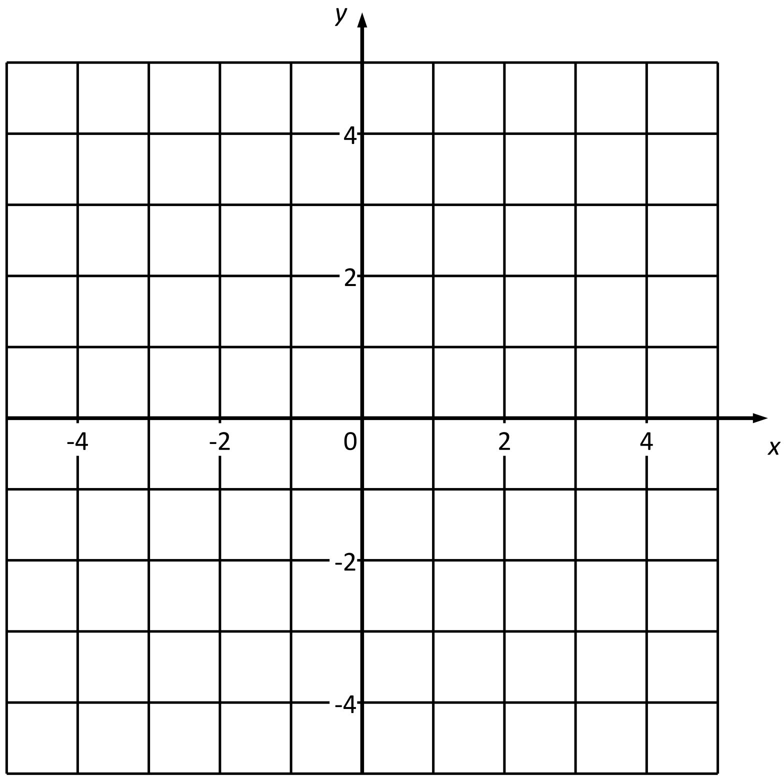
1. $y = x^2$ 5. $y = -x^2$
2. $y = x^2 + 1$ 6. $y = -x^2 + 1$
3. $y = 2x^2$ 7. $y = -2x^2$
4. $y = x^2 - 3$ 8. $y = -x^2 - 3$

a) How do the graphs 1 to 4 differ from the graphs 5 to 8?

b) How do the graphs 1 and 2 differ from each other and from 4.?

c) How does the graph of 3 differ from the graphs 1, 2 and 4?

Repeat parts b) and c) for graphs 5 to 8.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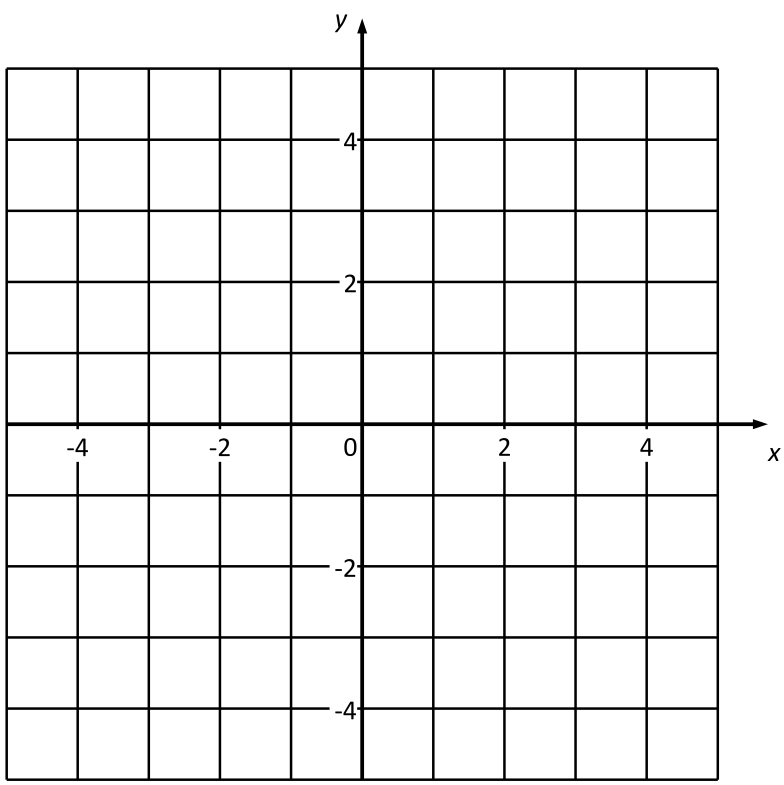
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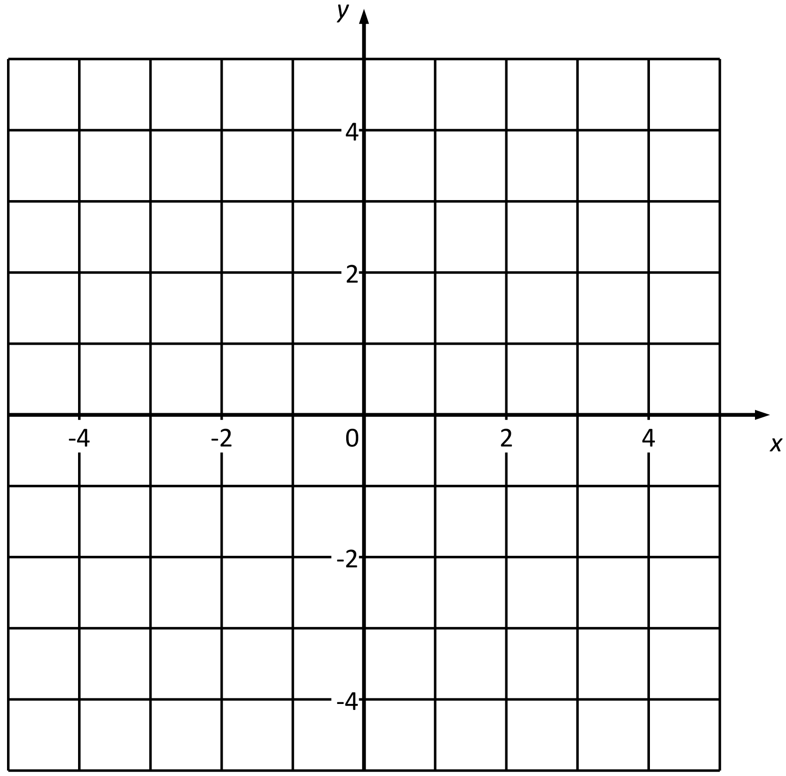
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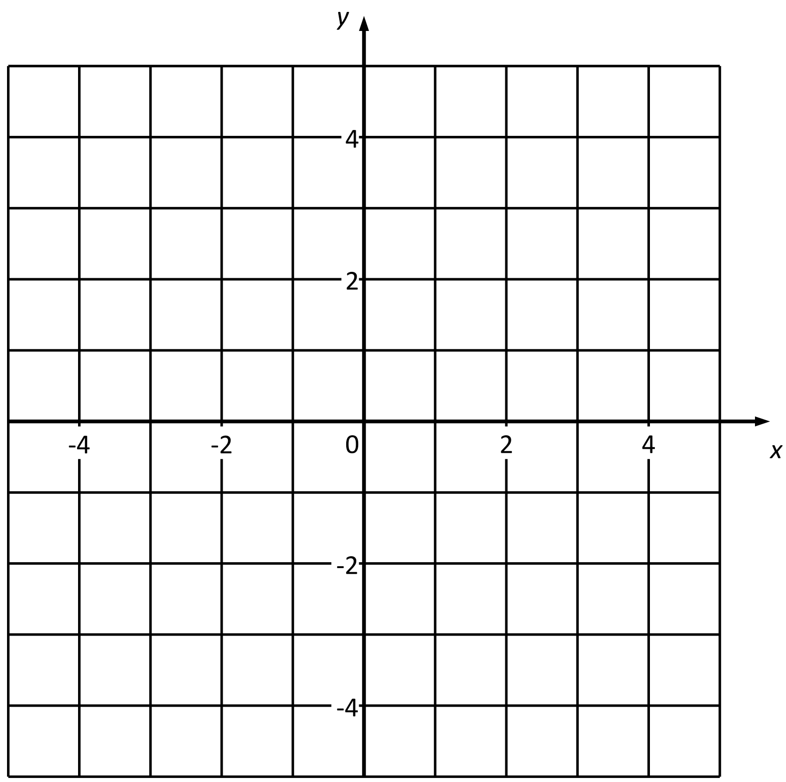
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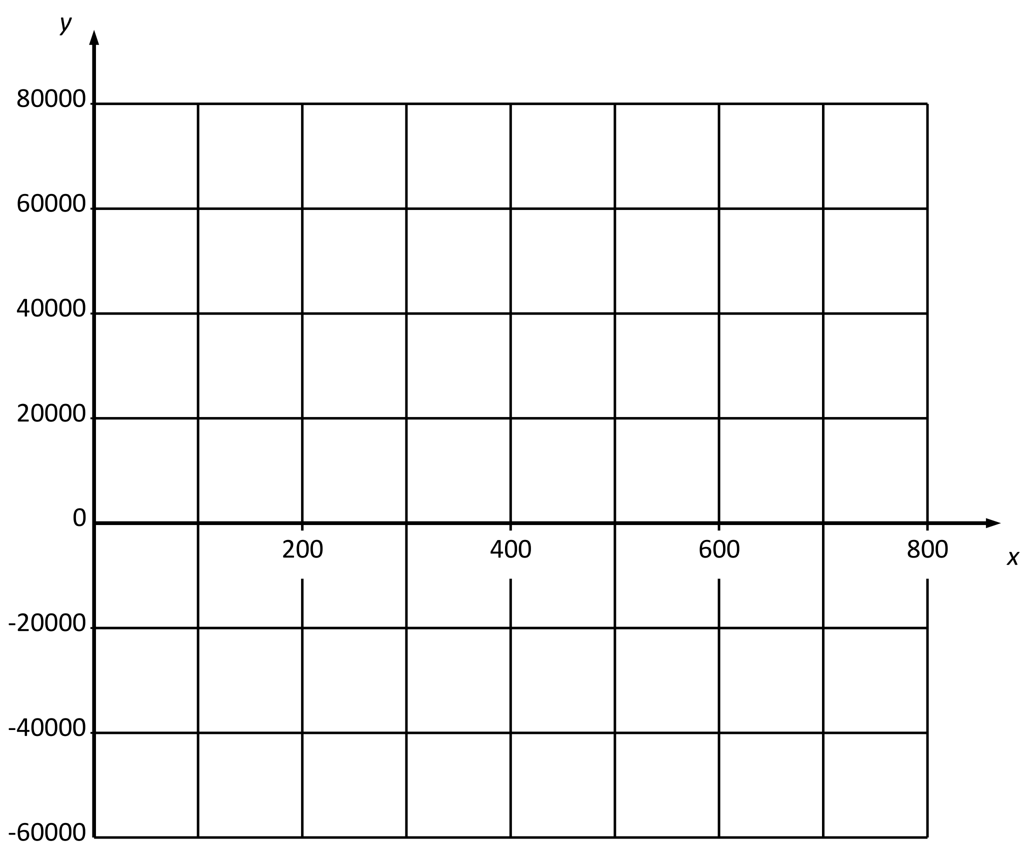
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5.2 Quadratic Equations

The general form for quadratic equations is

$$ax^2 + bx + c = 0$$

where a, b, c are numbers ($a \neq 0$) and $\boldsymbol{x}$ is the unknown that we would like to find. As the solutions of the quadratic equation make the value of the corresponding quadratic function $y = ax^2 + bx + c$ be zero, they are also called the **roots** of the quadratic function.

There are basically two methods one may use for finding the roots of a quadratic equation:

(a) the quadratic formula
or
(b) factorisation

5.2.1 Solving quadratic equations using the quadratic formula

A quadratic equation written in the general form

$$ax^2 + bx + c = 0,$$

can have solutions given by the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

The symbol '$\pm$' plus or minus means that we calculate the value of the formula

twice once with the '+' sign and once with the '-' sign.

Note Depending on the sign of the quantity $b^2 - 4ac$ (called **discriminant**), the quadratic equation $ax^2 + bx + c = 0$ can have either two, one or no solutions. If $b^2 - 4ac > 0$, the equation will have two distinct real solutions; if $b^2 - 4ac = 0$ then it has one or two repeated solution and if $b^2 - 4ac < 0$, the equation will have no real solutions as we cannot take square root of a negative number.

Example 5B

Solving $x^2 - 5x + 6 = 0$ using the formula:

Solution

In this case $a = 1, b = -5, c = 6$ and

$$x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(1)(6)}}{2(1)} = \frac{5 \pm \sqrt{1}}{2},$$

therefore $x = 3$ or $x = 2$.

Example 5C

Solve $x^2 - 6x + 9 = 0$ using the formula

Solution

In this case $a = 1, b = -6, c = 9$ and

$$x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(9)}}{2(1)} = \frac{6 \pm \sqrt{0}}{2},$$

therefore $x = 3$ is the only solution.

Example 5D

Solve $3x^2 - x + 2 = 0$ using the formula.

Solution

In this case $a = 3, b = -1, c = 2$ and

$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(3)(2)}}{2(3)} = \frac{1 \pm \sqrt{-23}}{6},$$

therefore there are no real solutions.

Exercises 5.3

Solve the following equations by using the quadratic formula

1. $2x^2 - 5x + 3 = 0$
2. $2x^2 + 13x - 7 = 0$
3. $2x^2 + x - 10 = 0$
4. $3x^2 + 4x + 1 = 0$
5. $3x^2 - x - 4 = 0$
6. $3x^2 - 11x + 6 = 0$
7. $4x^2 - 4x - 15 = 0$
8. $5x^2 + 12x - 9 = 0$
9. $-2x^2 - 5x + 18 = 0$
10. $6x^2 - 13x + 6 = 0$

5.2.2 Solving Quadratic Equations using Factorisation

To solve a quadratic equation, the first step is to write it in the form $ax^2 + bx + c = 0$.

The easy type when a=1, then we follow the following steps to solve the quadratic equation by factorisation

- We have two brackets (   )(   ) = 0
- If the sign of the constant term c is positive, the two brackets will have the same signs if not they will have different signs.
- For the same signs, we try to find two factors if you multiply them we get c and add them get b
- For the different signs, we try to find two factors if you multiply them we get c and subtract them get b
- The multiplication of two brackets equal zero, means one of them has two zero, so we get two values for x as roots

Example 5E

Solve $x^2 + 3x + 2 = 0$ **;**

Solution:

- (   )(   ) = 0
- Because the constant term is positive +2, (+)(+)
- The two factors that can be multiplied to get 2 and added to 3, are 1 and 2 (x + 1)(x +2)=0
- Either (x +1) = 0 or (x + 2) =0
- x = -1 or x = -2

Example 5F

Solve $x^2 - 5x + 6 = 0$

- **(   )(   ) = 0**
- $(x-2)(x-3)=0$ If you multiply the close terms, -2x and the outside terms, -3x, add them you get the middle term, -5x
- Either $(x-2)=0 \; or \; (x-3)=0$
- $x = 2 \quad or \quad x = 3$

Example 5G

Solve $x^2 + x - 6 = 0$**;**

$(x+3)(x-2)=0$ the close terms multiplication is 3x and the outside terms is -2x, subtract them you get the middle term, x

$(x+3)=0 \; or \; (x-2)=0$

$x = -3 \quad or \quad x = 2$

Example 5H

Solve $2x^2 + 5x + 2 = 0$.

In this case

- We have two brackets (   )(   )=0
- The signs distribute as in the previous examples $(\quad + \quad)(\quad + \quad) = 0$
- We will look at the constant term c = 2 as well as the coefficient of x^2, 2. We need to find two pairs of factors (1,2) and (1,2) that make the two numbers. They distributed in such way that the multiplications of the closed terms and the outside terms can be added to get the middle term

  $(2x+1)(x+2)=0$
- $2x+1=0 \quad or \quad x+2=0$
- $x = -\frac{1}{2} \quad or \quad x = -2$

5.2.3 Quadratic graphs and the number of solutions to a quadratic equation

To solve $x^2 - 10x + 30 = c$, first draw the graph of $y = x^2 - 10x + 30$ as shown in figure 5.2 below

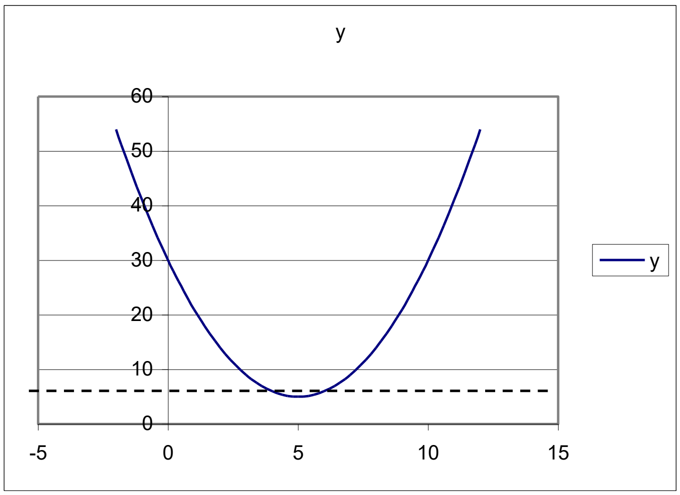


Figure 5.2

To solve $x^2 - 10x + 30 = c$, we look at where the graph of $y = x^2 - 10x + 30$ cuts the line $y = c$

- For $c = 5$, we get just one solution, namely, $x = 5$.
- We can solve for $\boldsymbol{x}$ and get two values of $\boldsymbol{x}$ as long as $c > 5$, and
- There is no solution when $c < 5$.

Generally for a quadratic equation $ax^2 + bx + c = 0$ if the graph of the corresponding quadratic function $y = ax^2 + bx + c$ cuts the x-axis twice, it has two real solutions and if the graph only cuts the x-axis once, then the equation has only one solution. Otherwise, the equation doesn't have any real solutions.

Exercises 5.4

Solve the following equations by factorisation:

1. $x^2 - 5x + 4 = 0$
2. $x^2 - 6x + 8 = 0$
3. $x^2 - 7x - 8 = 0$
4. $x^2 - 7x + 12 = 0$
5. $x^2 - 81 = 0$
6. $4x^2 - 9 = 0$
7. $2x^2 - 9x + 4 = 0$
8. $6x^2 + 11x + 3 = 0$
9. $3x^2 + x = 14$
10. $8x^2 - 14x = -3$

There are more exercises in Foundation Mathematics 4th Edition by Croft and Davison Exercises 14.3 question 2.

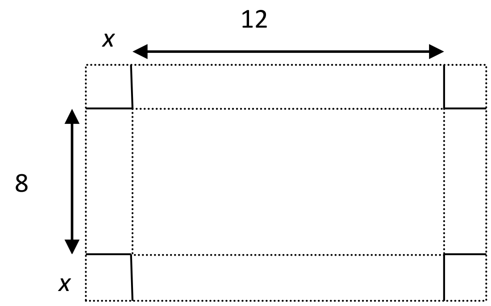
5.3 Applications of quadratic Equation

Quadratic expressions arise in many practical situations.

Example 5I

The area of a rectangular garden of length 12m and width 8m is being surrounded by a path of uniform width. If the area of the lot is 140 square meters, find the width of the path surrounding the garden.

Let ***x*** be the width of the path



The total area is

$$(2x+12)(2x+8)=140$$
$$4x^2+24x+16x+96=140$$
$$4x^2+40x+96-140=0$$
$$4x^2+40x-44=0$$
$$x^2+10x-11=0$$
$$(x+11)(x-1)=0$$
$$x=-11 \quad or \quad x=1$$

The length cannot be negative so, $x=1\,m$

Example 5I (continued)

(d) Plot a graph of the quadratic

$$y = x^2 + 10x - 11$$

The range of x-values goes from 0 to 5

x	-15	-10	-1	0	1	2	3	4	5
-11	-11	-11	-11	-11	-11	-11	-11	-11	-11
$10x$	-150	-100	-10	0	10	20	30	40	50
x^2	225	100	1	0	1	4	9	16	25
y	64	-11	-20	-11	0	13	28	45	64

To plot this graph, we need a set of axes in which x goes from 0 to 5 and y goes from -11 to 64

We now plot the points and join them up with a smooth curve as in Figure 5.3.

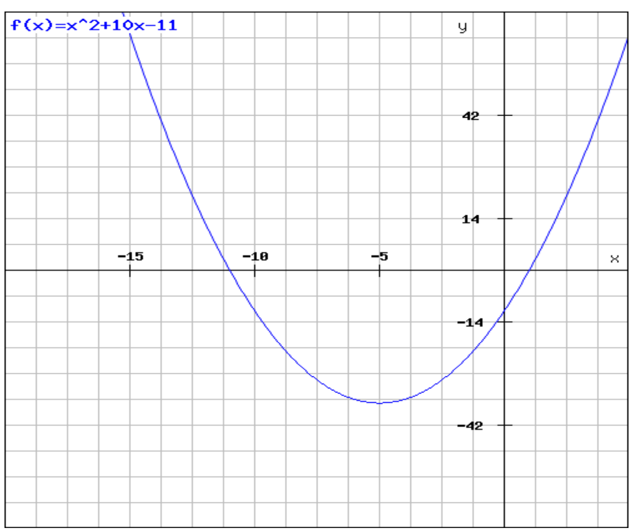


Figure 5.3

Exercise 5.5

1. A fence in a rectangular field has an enclosed area of 75 ft^2. The width of the field is to be 3 feet longer than the length of the field. What are the dimensions of the field?

2. Two consecutive even integers are such that the square of the smaller is four more than three times the larger. Find the integers.

3. The number of bacteria in a refrigerated food is given by $N(T) = 20T^2 - 20T - 120$, for $-2 \le T \le 14$ and where T is the temperature of the food in Celsius. At what temperature will the number of bacteria be zero?

Key points from Chapter 5

Chapter 6: Trigonometry

Learning objectives:

By working through this chapter, you should be able to:

- Understand the definition of trigonometric functions
- Use calculator to evaluate trigonometric functions
- Measure angles using degrees and radians
- Apply the knowledge of trigonometry to solve simple geometric problems

6.1 Introduction

The word *trigonometry* is very old. In fact the Ancient Greeks drew all this up. It allowed them to estimate the distance to the Sun!

Trigonometry is the branch of mathematics that deals with **triangles**, **circles**, **oscillations** and **waves**; it is absolutely crucial to much of geometry and physics. You'll often hear it described as if it was all about triangles, but I think that is missing much of the point. Waves and resonance are at the root of how matter works at the most fundamental level; they are behind how sound and light move, and probably also how minds and beauty work, on some level; so trigonometry turns out to be fundamental to pretty much everything. Any time you want to figure out anything to do with angles, or turning, or swinging, there's trigonometry involved.

The first thing to understand with trigonometry is why the mathematics of right-angled triangles should also be the mathematics of circles. Picture a line which can turn around one of its ends, like the hand of a clock. Obviously, the moving end of the line traces out a circle - it's like drawing with a compass. Now, consider how far this point is to the right or left of the centre point (we call this distance x), and how far above or below (which we'll call y). By attaching horizontal and vertical lines of lengths x and y to the ends of the first line we get a right-angled triangle, like this:

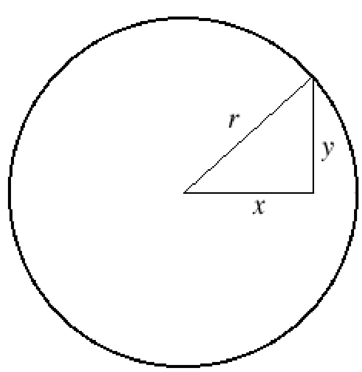


Figure 6.1

6.2 Definition of Trigonometric Functions

Trigonometry starts with a triangle, specifically a right-angled one (90°):

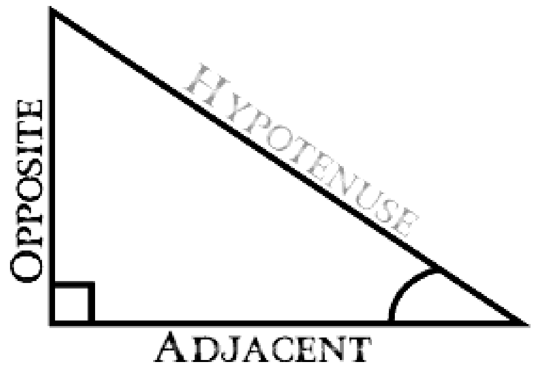


Figure 6.2

The **longest side** is always called the **hypotenuse**. This is always opposite to the right angle.
The side **opposite** a chosen angle is called the **opposite** side
The side **next to** the chosen angle is called the **adjacent** side.

Sin, Cos & Tan Functions

The ratios between sides of triangles are used to define trigonometric functions and in this chapter we will introduce three of them, namely sine, cosine and tangent functions (normally shortened to sin, cos and tan).

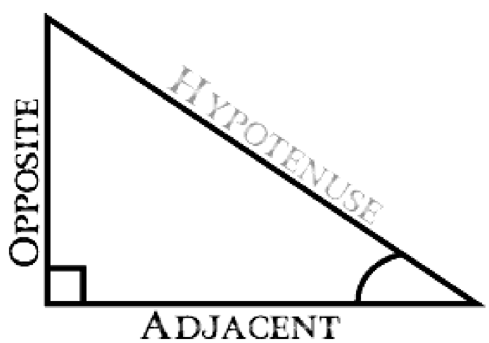


Figure 6.3 Definitions for Sine, Cosine and Tangent functions

Sin = **Opposite/hypotenuse**

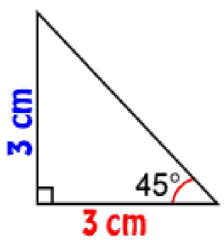
Cos = **Adjacent/hypotenuse**

Tan = **Opposite/adjacent**

Why's this important? If an angle is known, so too is the ratio between its sides. This means that if you know **how far** a mountain is from you, and you can *measure the angle* of elevation to its peak, you can work out **how tall** it is!

Let's have a look at ***tan*** in action. Below is a simple right-angle triangle with a 45° angle marked. Remember "sohcahtoa"!

Example 6.1



By definition , *tan 45°* = opposite/adjacent
=3/3

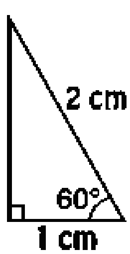
tan 45° = **1**

Tan 45° will **always** equal **1**, but only applies to right angle triangles.

Let's have a look at ***cos*** in action. Below is a right-angle triangle with a 60° angle marked and two sides. Again recall "sohcahtoa"!

Example 6.2

By definition, *cos 60°* = adjacent/hypotenuse



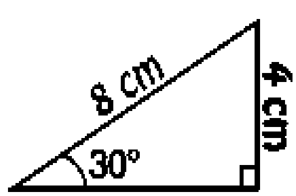
= 1/2

cos 60° = **0.5**

Cos 60° will **always** equal **0·5,** when applied to right angle triangles

Finally let's have a look at ***sin*** in use. Below is a right-angle triangle with a 30° angle marked and two sides. Again recall "sohcahtoa"!

Example 6.3



By definition, *sin 30°* = opposite/hypotenuse

= 4/8

sin 30° = **0.5**

Sin 30° will **always** equal **0·5**, when applied to right angle triangles. Now let's see how we use those pesky calculators...

6.3 Trigonometric Functions in the Calculator

Before the invention of the *electronic calculator*, mathematicians spent most of their time creating **tables of numbers** indicating huge numbers of values of **sin, cos** and **tan** for various angles.

Today, thanks to the physicists who invented the micro-processor, we can instantly find sin, cos and tan ratios after punching a few buttons. Type this:

[sin] [3] [0] [=]

You should get the answer 0·5

Some calculators expect you to type the number before you tell it what to do with it. In which case, try:

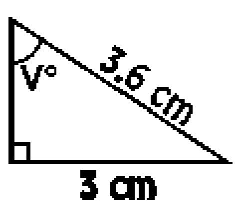
[3] [0] [sin]

Now can you try to work out cos 45° and tan 60° using your calculator?

6.4 Inverse Sin, Cos & Tan

From the above examples we know that if we were given the size of an angle we can work out the value of its trigonometric functions easily. Equally important, knowing a **ratio** between sides, we can use calculators to give us the *unknown angle*. Easy life!

Example 6.4



By definition, sin$V$° = opposite / hypotenuse

= 3 / 3·6

= **0·833**

We can find ***V*°** by telling our calculator that **0·833** is the **sin** of *V*°, and it will tell us *V* if we use ***inverse-sin***. Try this:

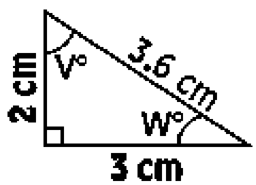
[inv] [sin] [0] [·] [8] [3] [3] [=]

Some calculators have a second-function button [2nd F] instead. Remember also that older ones will require the **number** to be put in *first*. Either way, you should be given the answer:

56.4° or close enough!

In the same way that we can find an unknown angle from the **sin** ratio (opposite/hypotenuse), we can do the same with **cos** and **tan.**

Example 6.5



By definition, $\cos V° = \text{adjacent} \div \text{hypotenuse}$
$= 2 \div 3{\cdot}6$
$= \mathbf{0{\cdot}556}$

We can find *V*° by telling our calculator that 0·556 is the cos of *V*°, and it will tell us *V* if we use *inverse-cos*. Try this:

[inv] [cos] [0] [·] [5] [5] [6] [=]
56.2° - close enough.

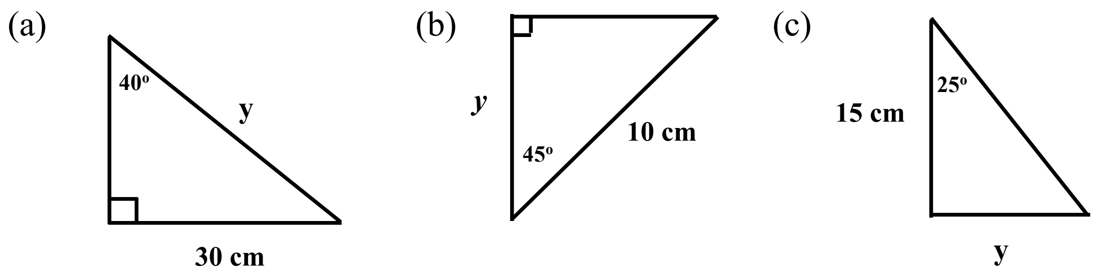
We can find *W*° in a similar way:
tan *W*° = opposite / adjacent = 2 / 3 = 0·666
W*° = inverse-tan 0·666 = *33·7°

Our answers are slightly inaccurate due to rounding errors.

6.5 Simple Application of Trigonometric Functions

Example 6.6

Find the value of y to one decimal place for each of the following (not to scale)



Solutions

(a) **y** is opposite the right angle, so y is the hypotenuse. The side opposite the 40° angle is 30 cm. We need to use the sides O and H. This is then SOH and we use the sin of angle 40° as follows.

$$\sin 40^\circ = \frac{O}{H}$$

$$\sin 40^\circ = \frac{30}{y}$$

Multiply both sides by y.

$$y \sin 40^\circ = 30$$

$$y = \frac{30}{\sin 40^\circ}$$

$$y = 46.67$$

To one decimal place y = 46.7 cm

(b) **y** is the side adjacent to the angle of 45°. The hypotenuse is known and is equal to 10cm. We are going to use A and H. This is then CAH and we use the cosine of the angle.

$$\cos 45^\circ = \frac{A}{H}$$

$$\cos 45^\circ = \frac{y}{10}$$

Multiply both sides by 10.

$$10\cos 45^\circ = y$$

$$y = 10\cos 45^\circ$$

$$y = 7.07$$

The length of y is 7.1 cm to one decimal place.

(c) Follow the same explanation as in (a) and (b) we need to use O and A, TOA

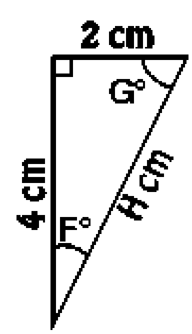
$$\tan 25^\circ = \frac{O}{A} = \frac{y}{15}$$

$$15\tan 25^\circ = y$$

$$y = 6.99$$

y =7.0 cm to one decimal place.

Exercises 6.1



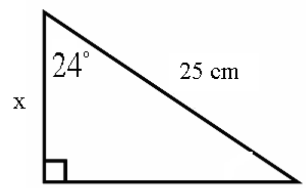
1. Find **H** using Pythagoras.
2. Find **tan F°** then inverse to find **F°**.
3. Find **sin G°** and consequently **G°**
4. Check that 90° + F° + G° = 180°
5. Finally check that cos F° when inversed gives you the same answer for F°

Write your answers to 2 decimal places. **HINT**: only round off the final answer!

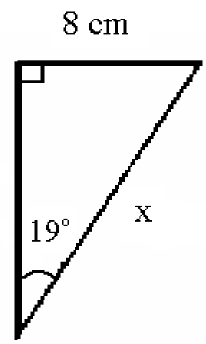
Exercises 6.2

1. Find the length of the side marked x in the following exercise, giving your answers correct to two decimal places.

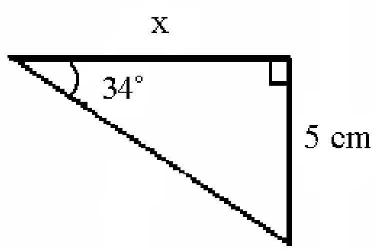
(a)



(b)



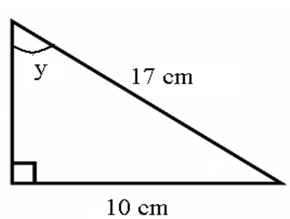
(c)



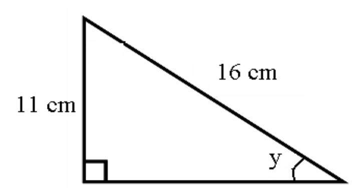
Calculate the length of the third side of the triangles in question 1 using Pythagoras's Theorem

2. Find angles y in the following exercises, give all your answers to the nearest degree.

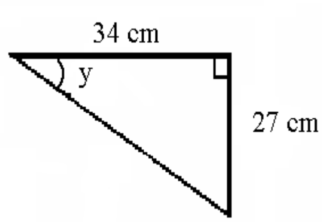
(a)



(b)



(c)



6.6 Units for Measuring Angles

There are two main units used to measure angles: the degree and the radian. Both units are defined with reference to a circle.

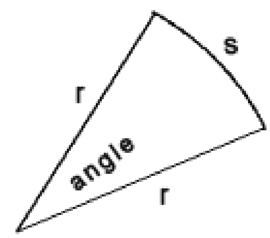
- Degree, the angle that is equivalent to a complete revolution is 360 degrees, denoted 360°

  1 complete revolution $= 360^\circ$

- Radian, the angle subtended at center by an arc whose length is one radius.

The size of an angle, in rad, is the length of the circle arc ***s*** divided by the circle radius ***r*.**

$$angle = \frac{s}{r} \quad \text{(rad).}$$



Measuring an angle in Radians

$$2\pi \ radians = 360^\circ \quad \rightarrow \pi \ radians = 180^\circ$$

Converting between Radians to Degrees

Because there are 2π radians in a circle:

To convert degrees to radians:

$$\theta^\circ = \theta \times \frac{\pi}{180} \textbf{(rad)}$$

To convert radians to degrees:

$$\phi^c = \phi \times \frac{180^\circ}{\pi}$$

Exercises 6.3

1. Convert the following angle measurements from degrees to radians. Express your answer exactly (in terms of π).

a) 180 degrees b) 90 degrees c) 45 degrees d) 137 degrees

2. Convert the following angle measurements from radians to degrees.

a) $\frac{\pi}{3}$ b) $\frac{\pi}{6}$ c) $\frac{\pi}{12}$ d) $\frac{3\pi}{4}$

Exercises 6.4

1. Set the mode on your calculator to degrees. Find

(a) $\sin 90^\circ$; (b) $\cos 90^\circ$; (c) $\tan(-90^\circ)$; (d) $\sin 45^\circ$; (e) $\cos 45^\circ$;

(f) $\sin^{-1} 0.5$; (g) $\tan^{-1} 0.2$; (h) $\sin 30^\circ$; (i) $\cos^{-1}(-0.1)$; (j) $\tan 0^\circ$.

2. Set the mode on your calculator to radians. Find

(a) $\sin \frac{\pi}{2}$; (b) $\cos\pi$; (c) $\tan 1.2$; (d) $\sin(-0.3)$; (e) $\cos 3$;

(f) $\sin 0.7$; (g) $\tan^{-1} 0.2$; (h) $\sin^{-1} 1$; (i) $\cos^{-1} 0.5$; (j) $\tan^{-1} 0$.

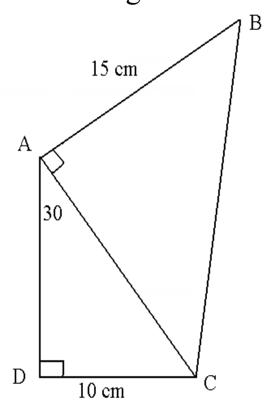
6.7 Further Application of Trigonometric Functions

Trigonometric questions may involve more than one triangle. Each triangle needs to be considered independently.

Example 6.7

Using the diagram shown,

a) Find the length of AC
b) calculate angle BCA to the nearest degree.



a) Using triangle ADC, AC is the hypotenuse, DC is the opposite the angle 30°. Use O and H means use SOH.

$$\sin 30^\circ = \frac{10}{AC}$$

$$AC = \frac{10}{\sin 30^\circ}$$

Using your calculator, you should find that AC=20 cm

b) We now use triangle ABC. AC, which we have just found is adjacent to the required angle, BCA. AB is opposite angle BCA. We need to use O and A means use TOA.

$$\tan BCA = \frac{15}{20}$$

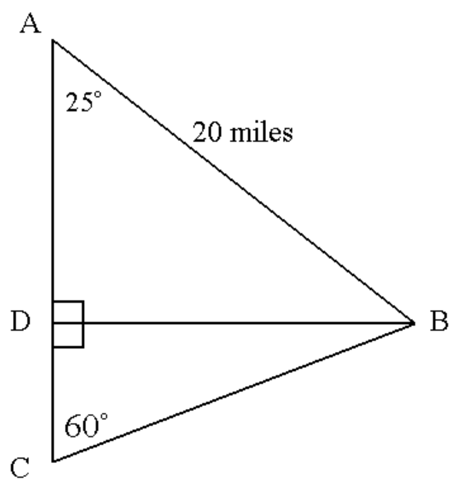
$$\tan^{-1}\frac{15}{20} = BCA$$

Calculator: $15 \div 20 = INV$ tan 36.869898

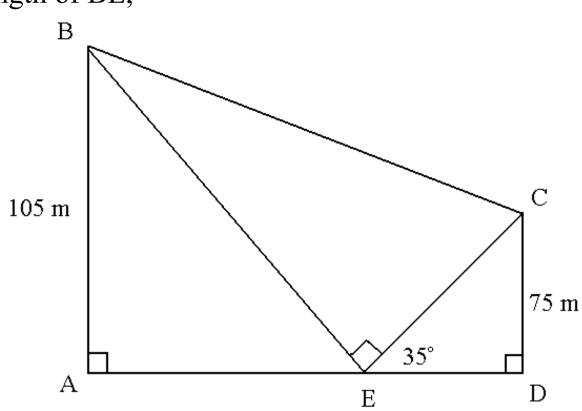
To the nearest degree, angle BCA is equal to 37°

Exercises 6.5

1. The diagram below is of a motorway system linking four towns marked A, B, C and D. The distance between A and B is 20 miles, angle DAB = 25° and angle DCB = 60°. Calculate correct to one decimal place,
   a) the distance AD,
   b) the distance BD,
   c) the distance DC
   d) the total distance AC.



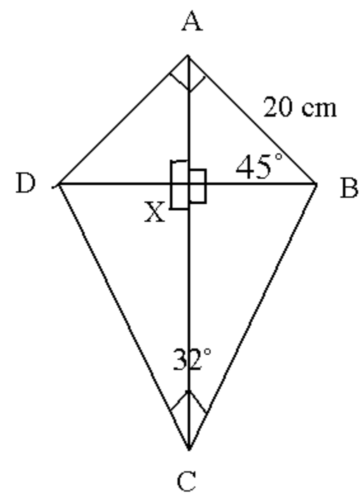
2. The diagram below shows two pylons marked AB and CD, where AB is of height 105 m. and CD is of height 75 m. Angle CED is 35° and AED is a straight line. Find, correct to the nearest whole number,
   a) the length of CE,
   b) the size of angle AEB.
   c) the length of BE,
   d) the size of angle ECB



3. The diagram below is of a child's kite in which AC is a line of symmetry. If AB is 20 cm and angle BCD is 32°. Find,

a) the length of AX,
b) the length of BX
c) the length of CX
d) the total length of the kite, AC.

Give all your answers correct to the nearest whole number.



Key points from Chapter 6

Part 2

6G3Z3010 Foundation Mathematics 2

Chapter 7: Functions

Learning objectives:

By working through this chapter, you should be able to:

- Understand the formal definition of functions
- Use a number of ways to represent functions
- Recognise and establish certain types and properties of a range of functions
- find the inverse of a function
- Calculate the value of an exponential to any base by using a calculator
- Know what is meant by the exponential function
- Solve simple problems by applying exponential functions to growth and decay models
- Know the definition and the laws of logarithms
- Calculate logarithms to any base by using a calculator
- Solve simple problems using logarithms

7.1 Introduction

In the previous chapters, we have come across a number of different functions such as linear functions, quadratic functions and polynomials, but we haven't given a formal definition for them. Mathematically, a function is a mechanism or rule that associates every element of a set with just one element of another set (which may or may not be the same as the first set).

The concept of function appears quite often even in our daily life.

Example 7A

1. A social security number uniquely identifies the person.
2. The income tax rate varies with income.
   Given an income, we can assign a tax rate.

Given two sets A and B, a function, f, is a rule which maps (or associates) every element in A to just one element in B. Every time we put a particular element of A into the function we get the same element of B out but we can get different members of B when we input different members of A.

The set A in the definition above is called the **domain** of the function and B is the **codomain** of the function. If f is a function then it covers the domain (maps every element of the domain) and it is single valued.

We can usefully picture the domain A as a set of inputs and the codomain B as a set which contains all the possible outputs from *f*. Suppose we take an element of A, let's call it *a*, and input it to *f* then an element of B is the output from *f*, let's call it *b*, **the image of a under the function *f***.

In order to link up *f*, a and b we use the notation, $f(a) = b$. Note that $f(a)$ is a member of the codomain i.e. an output from *f*.

Example 7B

Let *f* be the function from the set of natural numbers, N, to N that maps each natural number x to x^2. Then the domain and codomain of f are N, the image of, say 3, under this function is 9, and the range of f is the set of squares, i.e. { 1, 4, 9, 16,…}

Exercises 7.1

Input the given numbers and find the output number in every case.

	Function	Input	Output
1.	$f(x) = 2x + 7$	-3, 5, 8	
2.	$f(x) = 4x - 1$	-2, 1, 5	
3.	$f(x) = -x + 3$	-3, 0, 9	
4.	$f(x) = 2x^2 - 4$	-5, 2, 4	
5.	$f(x) = 3x^2 + 5$	-1, 3, 6	

7.2 Representations of Functions

There are a number of different ways we can use to represent functions. The most common one is to use algebraic expressions as found in exercises 7.1 and previous chapters. Functions, depending on the situation, can also be represented by other methods such as graphs, mapping diagrams and ordered pairs.

Pictorial representations of functions: it helps us to understand the effects of a function if we can represent it by a picture.

Mapping diagrams: Given that *f* is a function with domain A = {1, 2, 3, 4} and codomain B = {5, 6, 7, 8} with $f(1) = 5$, $f(2) = 6$, $f(3) = 8$, $f(4) = 8$, we can represent *f* by the **mapping diagram** shown in Figure B1.

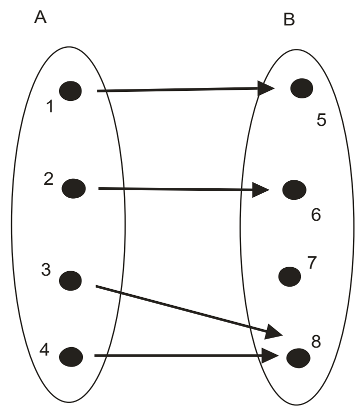


Figure 7.1

We put the domain on the left and codomain on the right and join inputs to their corresponding output.

Ordered pairs: The same function f can be represented by the **ordered pairs** (1, 5), (2, 6), (3, 8), (4, 8). In each case the first part of the ordered pair is an element of the domain and the second part is an element of the codomain. We can think of them as an input-output pair.

If we extend ordered pairs to include functions which have domain and codomain equal to the real numbers the pairs are of the form $(x, f(x))$ and we can plot $f(x)$ against x to get the graph of f.

Properties of functions

One-to-one function (injection)

A function f is said to be **one-to-one (injective),** if and only if whenever $f(x) = f(y)$, $x = y$. In other words given a value in the co-domain there is only one element in the domain that is mapped to it. Graphically, the function f is 1-1 if a horizontal line crosses the curve $y = f(x)$ only once for all y values.

Example 7C

The function f_1 is defined by

$$f_1 : N \rightarrow N$$
$$f_1(x) = x^2$$

The function f_1 from the set of natural numbers N to N is a one-to-one function. Every output can be reached from only 1 input.

Note that f_2 defined by

$$f_2 : Z \rightarrow N$$
$$f_2(x) = x^2$$

is a different function because it has a different domain even though the rule and the codomain are the same.

f_2 is not one-to-one because for example

$$f_2(1) = f_2(-1) = 1$$

i.e. we can reach the output 1 from 2 different inputs.

Onto function (surjection)

A function f from a set A to a set B is said to be onto(surjective), if and only if for every element y of B , there is an element x in A such that $f(x) = y$, that is, f is onto if and only if $f(A) = B$. If f is onto then the range of f is equal to the codomain, or every member of the codomain can be reached through f from some element of the domain.

Example 7D

Let E be the set of positive even numbers, and define f_3 by

$$f_3 : N \to E$$
$$f_3(x) = 2x$$

f_3 is an onto function because every positive even number is double some natural number

However, the function

$$f_4 : N \to N$$
$$f_4(x) = 2x$$

is not onto, because, for example, nothing in N can be mapped to 3 by this function.

Note We only need to find one element of the codomain that cannot be reached from anywhere in the domain.

Bijection

A function is a **bijection** , if it is both onto and one-to-one.

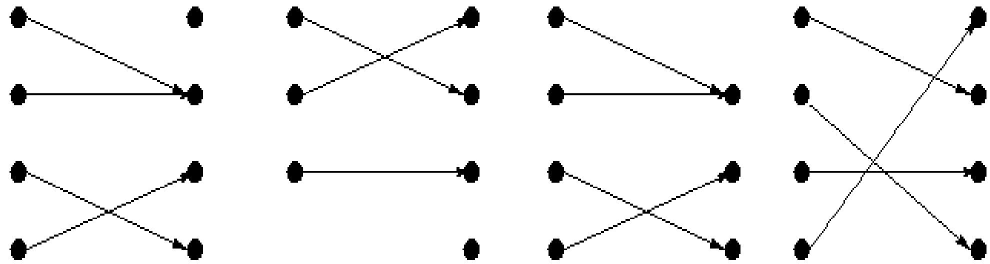
Example 7E

$$f_3 : N \to E$$
$$f_3(x) = 2x$$

f_3 is both one-to-one and onto. So it is a bijection.

Note: Every bijection has a function called the inverse function.

These concepts are illustrated in the figure below. In each figure below, the points on the left are in the domain and the ones on the right are in the codomain, and arrows show the $(x, f(x))$ relation.



A function | A one-to-one function (Not onto) | An onto function (Not one-to-one) | A bijection

7.3 Inverse Function

If you give an input, say *x*, to a function, it will generate an output, say *y*. Sometimes, we need to reverse the process by inputting *y* to a function and the output will be *x* and the function, if exists for doing this job is called **the inverse function**. But not every function has an inverse function and below is the formal definition of the inverse function.

Let f : A → B be a bijection, then there is a function f^{-1} : B → A such that $f^{-1} \circ f(x) = x$ for all *x* in A, f^{-1} is called **the inverse function of f**

> **VERY IMPORTANT**
>
> **The inverse function of *f* only exists if *f* is a bijection (i.e. it is both onto and one-to-one)**

Note When the inverse function f^{-1} is applied after the function *f*, its effect is to undo the effect of *f* and get us back to where we started. To construct a rule for f^{-1} we note that if $y = f(x)$ then $f^{-1}(y) = x$.

A useful technique for finding the formula for an inverse function is to let $y = f(x)$ and rearrange to make *x* the subject.

Example 7F

The function f is defined by $f(x) = 5x + 4,\ x \in R$.

Find $f^{-1}(x)$.

Solution

To find $f^{-1}(x)$, let y = 5x +4.

Then rearranging we get $x = \dfrac{y-4}{5}$.

This tells us what is done to input y so the inverse function acting on x gives $f^{-1}(x) = \dfrac{x-4}{5}$

Exercises 7.2

Find the inverse functions for:

(a) $y = f(x) = x + 1$

(b) $y = f(x) = 2x + 3$

(c) $y = f(x) = 7x + 1$

(d) $y = f(x) = 5 - 3x$

(e) $y = \dfrac{x}{3} - 1$

(f) $y = f(x) = \sqrt{x-2},\ x \text{ is a real number}, x > 3$

.

7.4 Exponential Functions

7.4.1 Growth Functions

Example 7G

Consider the following table for cell division

Time in days x	0	1	2	3	4	5	6	
Number of cells y	1	2	4	8	16	32	64	

The y values can be rewritten as powers of 2

Time in days x	0	1	2	3	4	5	6	x
Number of cells y	1	2	4	8	16	32	64	
y	2^0	2^1	2^2	2^3	2^4	2^5	2^6	2^x

If we plot these points we get the graph below

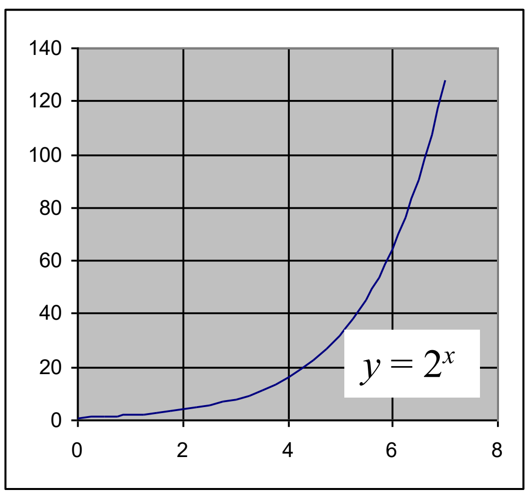


Figure 7.2

Example 7H

The population of bacteria on an agar plate doubles every hour. Initially it is 200, what is the population after 1, 2,…,10 hours.

Time (h)	Population	
0	200	$(200\text{x}2^{0})$
1	400	$(200\text{x}2^{1})$
2	800	$(200\text{x}2^{2})$
3	1600	$(200\text{x}2^{3})$
4	3200	$(200\text{x}2^{4})$
5	6400	$(200\text{x}2^{5})$
6	12800	$(200\text{x}2^{6})$
7	25600	$(200\text{x}2^{7})$
8	51200	$(200\text{x}2^{8})$
9	102400	$(200\text{x}2^{9})$
10	204800	$(200\text{x}2^{10})$
…	…	…
t		?

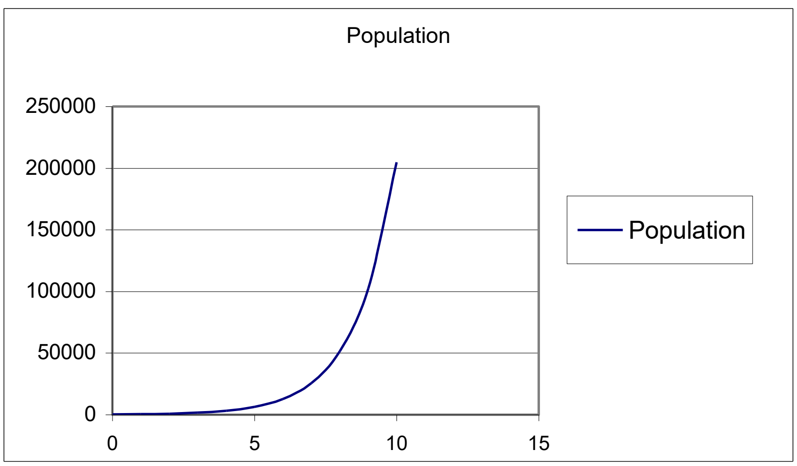


Figure 7.3

These process cannot be described by either linear or quadratic models. In example 7G the number of cells after $\boldsymbol{x}$ days is given by $y = 2^x$. Every time $\boldsymbol{x}$ increases by 1, $\boldsymbol{y}$ is multiplied by 2.

In example 7H the population, $\boldsymbol{P}$, is given by $P = 200 \times 2^t$ where $\boldsymbol{t}$ is measured in hours from the start of the experiment. Every time $\boldsymbol{t}$ increases by 1, $\boldsymbol{P}$ is multiplied by 2. It is simple to calculate 2^t for integer $\boldsymbol{t}$. For other values of $\boldsymbol{t}$ we need to use a calculator using the x^y button.

These types of functions where the variable is the power or exponent of a fixed number are called exponential functions. The number that is raised to a variable power is called the base of the exponential function. In these two examples, the exponential functions are used to describe the process called the exponential growth.

Example 7I

The earth's population of human beings is doubling every 30 years. In 1995 the population was 6×10^9. What was the population in 2005?

It is easy to calculate the population in 2025, 2055 and 2085.

Year	Population (10^9)
1995	6
2025	12
2055	24
2085	48

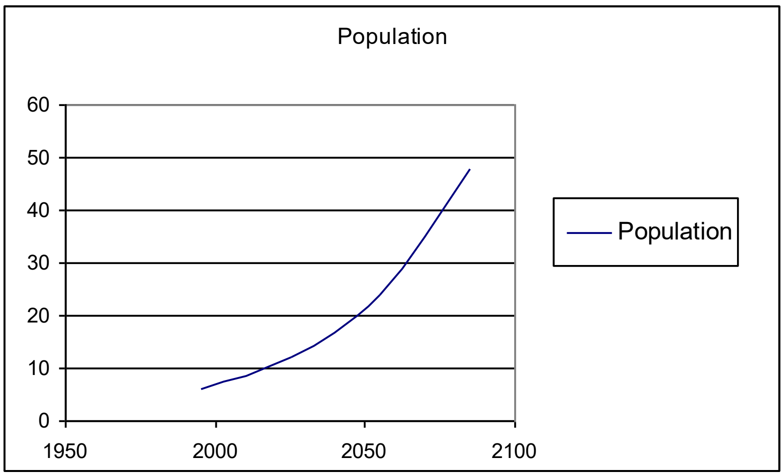


Figure 7.4

Let P be the population in year t after 1995 then the power of 2 must equal 1 when t = 30 so

$$P = 6 \times 2^{\frac{t}{30}} \ (\times 10^9)$$

Put t = 30 and $2^{\frac{t}{30}} = 2^1 = 2$

We can calculate P for any value of t.

To calculate the population in 2005 we put t = 10

then $P = 6 \times 2^{\frac{10}{30}} = 6 \times 2^{\frac{1}{3}} = 6 \times 1.260 = 7.560 \ (\times 10^9)$

In Figure 7.5, the graphs for the exponential growth functions with different bases $y = 2^x$, $y = 3^x$, $y = 4^x$ and $y = 5^x$ are shown. A label has been put to the graph of the function $y = 2^x$, can you label each of the remaining graphs with their functions?

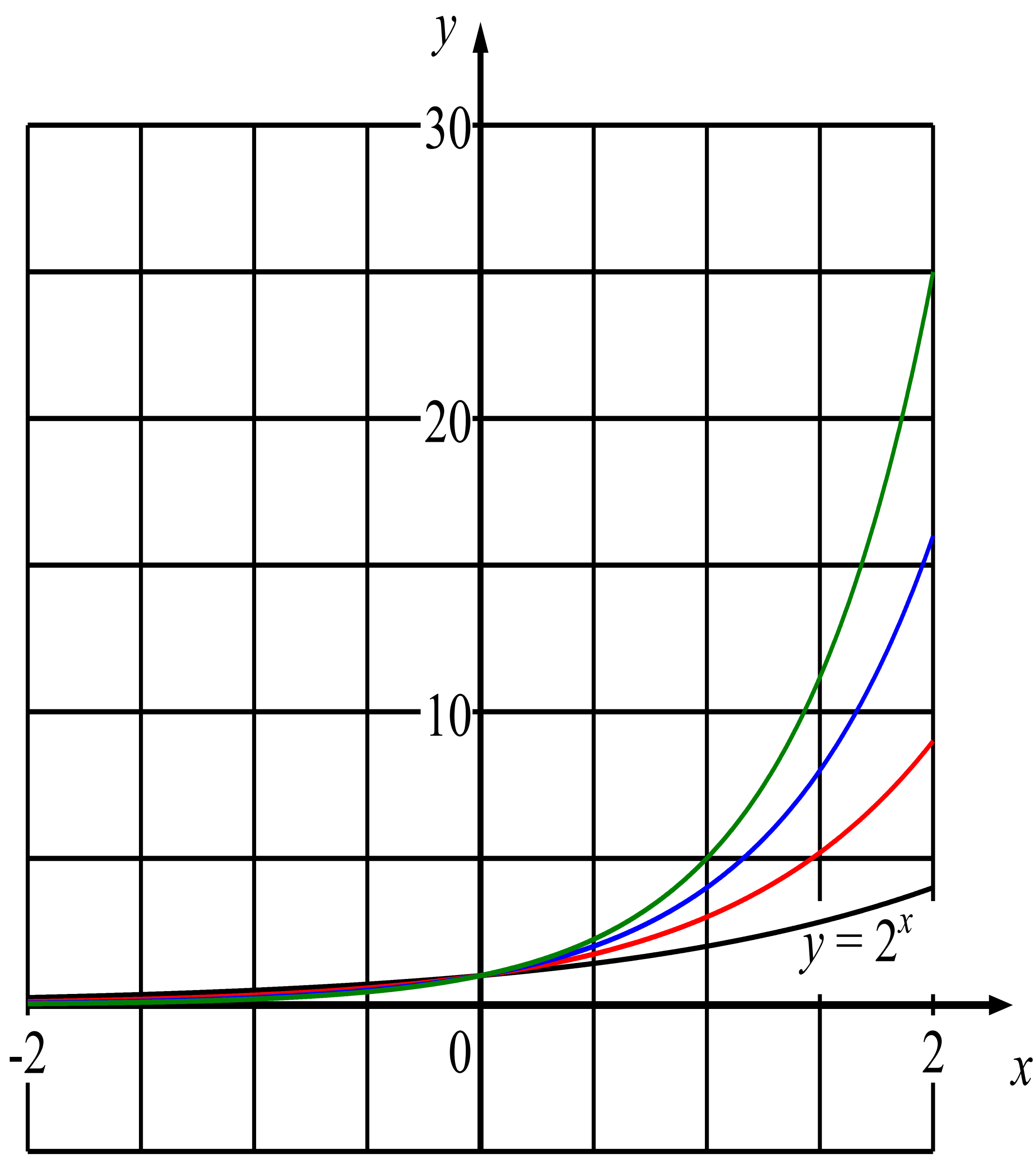


Figure 7.5

7.4.2 Decay Functions

Example 7J

Consider the following table for radioactive decay

Time in days x	0	1	2	3	4	5	
Radio activity y	1	1/2	1/4	1/8	1/16	1/32	

The y values can be rewritten as powers of 2

Time in days x	0	1	2	3	4	5	x
Radio activity y	1	1/2	1/4	1/8	1/16	1/32	
y	2^0	2^{-1}	2^{-2}	2^{-3}	2^{-4}	2^{-5}	2^{-x}

If we plot these points we get the graph below

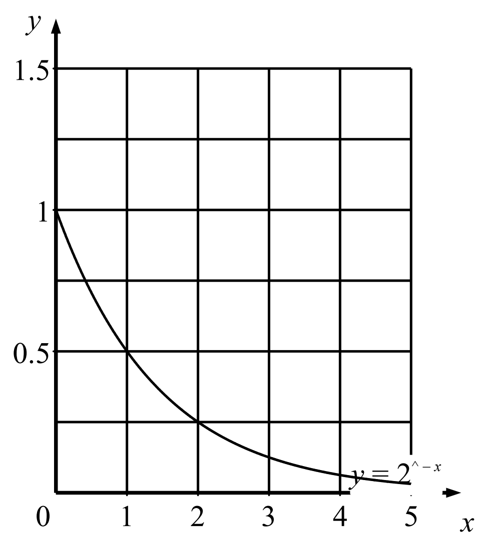


Figure 7.6
Graph of $y = 2^{-x}$

In this example, as there is a minus sign before the variable ***x*** on the power, the value of the function will decrease exponentially when x increases. This process is called exponential decay.

In figure 7.7, the graphs for the delay functions $y = 2^{-x}$, $y = 3^{-x}$, $y = 4^{-x}$ and $y = 5^{-x}$ are shown. Can you label the graph for each of these functions?

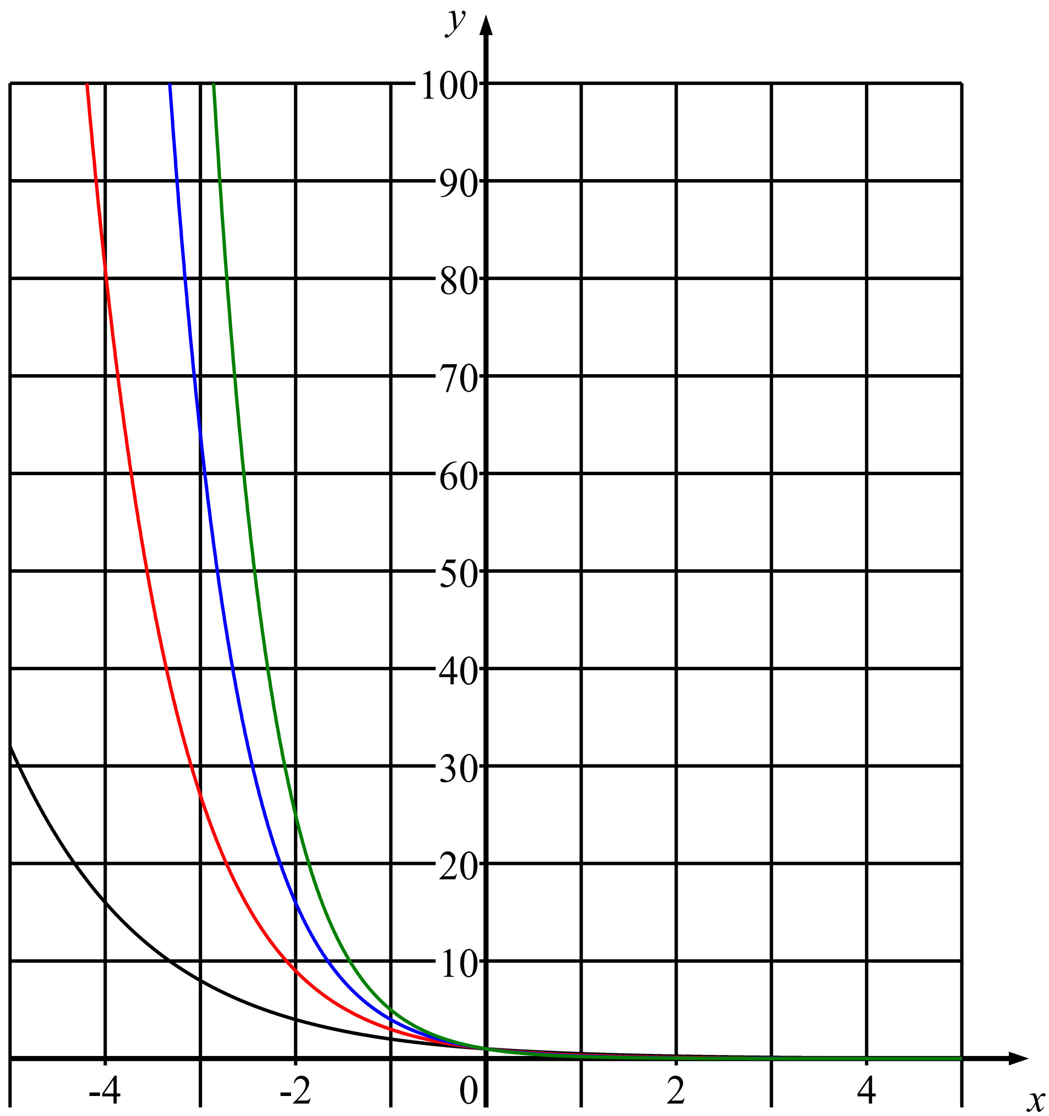


Figure 7.7

7.4.3 Exponential Functions using Number e as the Base

Many growth and decay processes are defined in terms of a number which is called ***e***. Number e is like π in that it has no exact decimal equivalent – ***e*** is about 2.7182818284590452… and arises naturally in mathematics. It is the value of '***a***' in $y = a^x$, which has gradient = 1 at the point (0,1). Your calculator has a button for raising ***e*** to any power. It is the e^x (or exp ***x***) button. The exponential function using number e as the base (e^x) is often referred to as the exponential function. It is usual to express physical growth and decay models in terms of the exponential function.

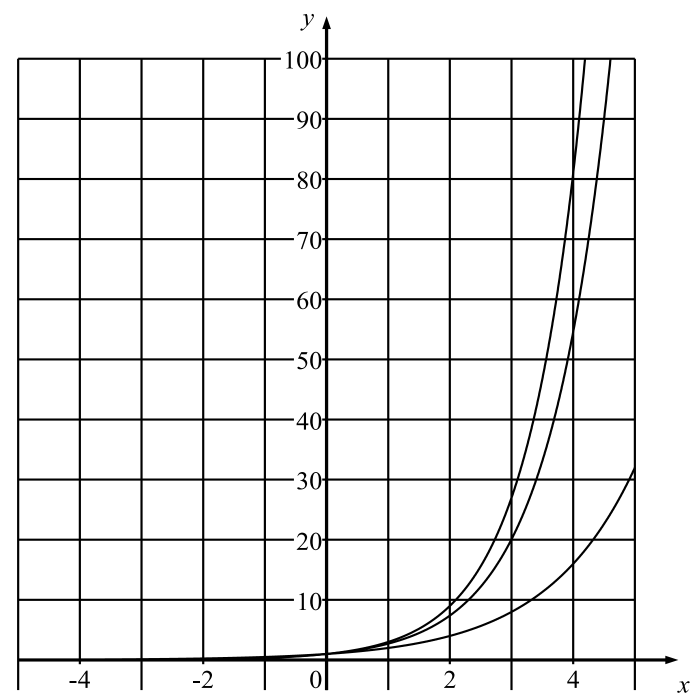


Figure 7.8 Showing the graphs of $y = 2^x$, $y = e^x$ and $y = 3^x$

Which one is which?

$e = 2.718281728459045235$...........

This is the value of ***a*** in $y = a^x$ which has gradient =1 at the point (0, 1)

Example 7K
Graph the exponential function $y = e^x$.

Solution Construct a table of x and y values:

x	-3	-2	-1	0	1	2	3
y	0.05	0.14	0.37	1	2.72	7.39	20.09

Plot these points and sketch a curve which passes through the points.

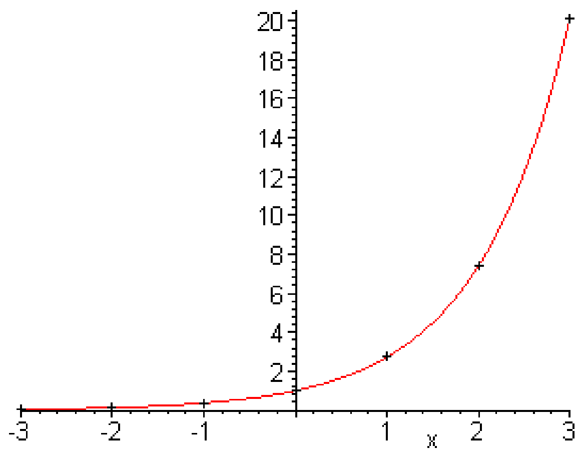


Figure 7.9
A graph of the exponential function $y = e^x$.

Exercises 7.3

Use your calculator to evaluate the following to 3 decimal places:

1. a) $10^{1.3}$ b) $10^{0.6}$

2. a) $2^{12.3}$ b) $2^{3.2}$ c) $2^{-0.2}$

3. a) $7^{-0.4}$ b) $7^{2.3}$

4. a) $16^{0.25}$ b) $16^{-0.75}$

Exercises 7.4

1. A laptop computer loses its value each month after it is purchased. Its value as a function of time, in months, is modelled by.

$$V(m) = 800(0.84)^m$$

What is the value of the laptop after 3 months? (2dp)

2. A baker records the internal temperature of a pie that has been left to cool on a counter. An equation that models this situation is

$$T(t) = 68(0.5)^{\frac{t}{10}}$$

where T is the temperature in degrees Celsius and t is the time in minutes. Determine the temperature, to the nearest degree, of the pie after 15 minutes.

3. A programmer learns to type. The number of words per minute that can be typed after t hours of practice is given by

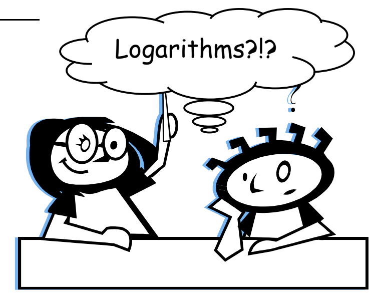
$$W(t) = 60(1 - e^{-0.005t})$$

What speed has been achieved after (i) 40h (ii) 100h (iii) 200h? (3dp)
What is the greatest speed that is achieved ever?

There are more exercises in Foundation Mathematics 4th Edition by Croft and Davison Exercises 19.2 questions 1 to 3

7.5 Logarithms

7.5.1 What is a Logarithm?



Example 7L

Population, ***P***, of bugs increases by a factor of 10 every day.
so $P = P_0 \times 10^t$
where t is measured in days.

How long does it take for the population to increase by a factor of 50?

We need to find the ***t*** so that $10^t = 50$
In other words, what power of 10 is equal to 50?

We have to answer this type of 'what's the power?' question whenever we want to know when a quantity which is subject to exponential growth reaches a particular value.
The answer to 'what power of 10 is equal to ***x***?' is
the logarithm to base 10 of ***x***, written as $\mathbf{log_{10}x}$.

In our case we want the logarithm to base 10 of 50 or $\log_{10} 50$.

Generally, to solve an exponential equation in the form $a^x = b$ where both a and b are positive numbers, we introduce a new operation called logarithm so its solution can be written as $x = \log_a b$. From this definition, for example, as
$2^5 = 32$, so $\log_2 32 = 5$ All the positive numbers other than 1 can be used as the base for logarithms but the most commonly used bases are number 10 and ***e***.

7.5.2 Logarithms to Base 10 (common logarithms)

Number	Power or index notation	Logarithm value
0.0001	10^{-4}	$\log_{10}(10^{-4}) = -4$
0.001	10^{-3}	$\log_{10}(10^{-3}) = -3$
0.01	10^{-2}	$\log_{10}(10^{-2}) = -2$
0.1	$10^{-1} = 1/10$	$\log_{10}(10^{-1}) = -1$
1	$10^0 = 1$	$\log_{10}(10^0) = 0$
10	$10^1 = 10$	$\log_{10}(10^1) = 1$
100	10^2	$\log_{10}(10^2) = 2$
1000	10^3	$\log_{10}(10^3) = 3$
10000	10^4	$\log_{10}(10^4) = 4$

Note

$\log_{10}(10) = \log_{10}(10^1) = 1$ and $\log_{10}(1) = \log_{10}(10^0) = 0$.

Logarithms to base 10 are often called **common logarithms** and most scientific calculators has a 'log' button which is used to calculate common logarithms.

Example 7M
Graph the logarithmic function $y = \log_{10} x$.

Solution Construct a table of x and y values:

x	0	0.01	0.1	1	10	100	1000
y	Error	–2	–1	0	1	2	3

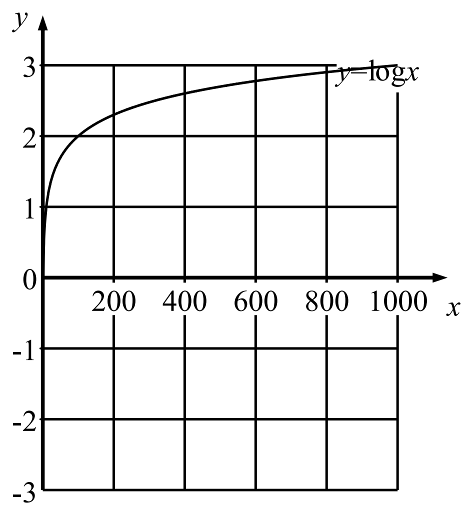


Figure 7.10: A graph of the logarithmic function $y = \log_{10} x$.

Exercises 7.5
Evaluate the following common logarithms using a calculator

1.	a) log 4	b) log 40	c) log 400	
2.	a) log 14	b) log 140	c) log 1400	
3.	a) log 6	b) log 0.6	c) log 0.06	d) log 0.006

What pattern do you notice?

7.5.3 Logarithms to Base e or Natural Logarithms)

We have met number ***e*** in the previous sections which is approximately equal to 2.7182818284590452… . If $y = e^x$ then $x = \log_e y$ which is a logarithm to base or natural logarithm. Also, because $e^1 = e$ and $e^0 = 1$, we have $\log_e(e) = 1$, $\log_e(1) = 0$. We often use the notation $y = \ln x$ for the natural logarithm so the 'ln' button on your calculator is for the evaluation of natural logarithms.

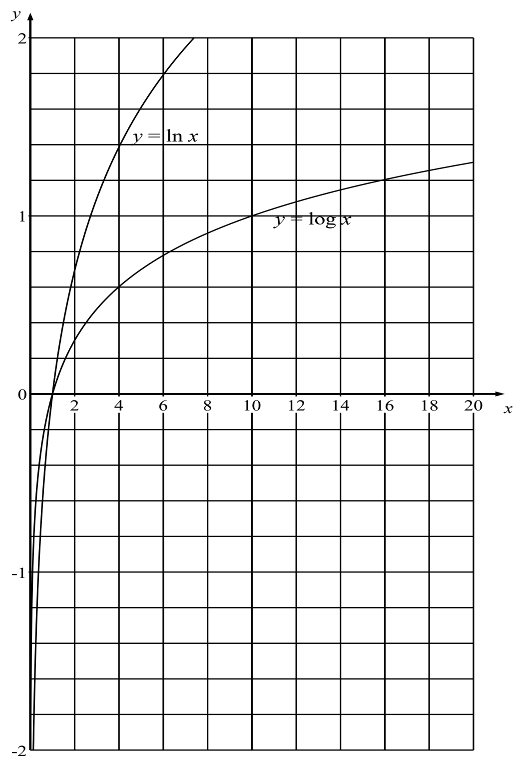


figure 7.11: showing the graphs of $y = \ln x$ and $y = \log x$.

Exercises 7.6

Evaluate the following natural logarithms using a calculator

1.	a) ln 5	b) ln 4.5	c) ln 10
2.	a) ln 1.5	b) ln 2	c) ln 0.6

7.5.4 Laws of Logarithms

Example 7N

Number	Power or index notation		Logarithm value
8	2^3	→	$\log_2 8 = 3$
32	2^5	→	$\log_2 32 = 5$
(8×32)	$2^3 \times 2^5 = 2^{(3+5)} = 2^8$	→	$\log_2(8 \times 32) = 8$

Hence,

$$\log_2(8 \times 32) = \log_2 8 + \log_2 32 = 3 + 5 = 8$$

Example 7O

Number	Power or index notation		Logarithm value
32	2^5	→	$\log_2 32 = 5$
8	2^3	→	$\log_2 8 = 3$
$(32 \div 8)$	$2^5 \div 2^2 = 2^{(5-3)} = 2^2$	→	$\log_2(32 \div 8) = 2$

Hence,

$$\log_2(32 \div 8) = \log_2 32 - \log_2 8 = 5 - 3 = 2$$

Example 7P

Number	Power or index notation		Logarithm value
8^4	$(2^3)^4$ $= 2^{(3 \times 4)} = 2^{12}$	→	$\log_2(8^4) = (3 \times 4) = 12$ $= 4\log_2 8$
64^3	$(2^6)^3$ $= 2^{(6 \times 3)} = 2^{18}$	→	$\log_2(64^3) = (6 \times 3) = 18$ $= 3\log_2 64$

Hence,

$$\log_2(8^4) = 4\log_2 8$$

and

$$\log_2(64^3) = 3\log_2 64$$

Generally,

$$\log (A \times B) = \log A + \log B$$
$$\log (A \div B) = \log A - \log B$$
$$\log A^B = B \log A$$

For all of the laws of indices there is a corresponding law of logarithms.

Result for indices	**Result for logarithms**	
$\mathbf{a^n \times a^m = a^{n+m}}$	**log A + log B = log AB**	**(i)**
$a^n \div a^m = a^{n-m}$	**log A – log B = log(A/B)**	**(ii)**
$\mathbf{(a^n)^m = a^{n\times m}}$	$\mathbf{\log A^m = m \log A}$	**(iii)**
$\mathbf{a^1 = a}$	$\mathbf{\log_a a = 1}$	**(iv)**
$\mathbf{a^0 = 1}$	**log1 = 0**	**(v)**
$a^{-1} = 1/a$	−logA = log(1/A)	**(vi)**
Where no base is specified the result applies to logarithms to any base.		

The first two laws of logarithms are the basis of doing calculations using logarithms to base 10 where multiplications are performed by adding logarithms and divisions by subtracting logarithms.

Example 7Q

Simplify the following expressions to a single logarithm:

a) log 8 + log 5 + log 3; b) log 24 – log 16 + log 4 ; c) 4log x + log x^3

Solutions: a) log 8 + log 5 + log 3 = log (8x5x3) = log 120

b) $\log 24 - \log 16 + \log 4 = \log(\frac{24\times 4}{16}) = \log 6$

c) 4log x + log x^3 = log x^4 + log x^3 = log x^{3+4} = log(x^7).

7.5.5 Using the Calculator to Find $\log_a x$

On most calculators, we can only find the buttons for evaluating the common and natural logarithms, but sometimes we also need to calculate logarithms to bases other than 10 or ***e***. For example, in computing science, you will find that logarithms to base 2 are also commonly used. To do this, we need to examine the relations between the logarithms to different base in a hope to convert the base of a logarithm to either 10 or ***e***.

Let $\log_a y = x \quad \Rightarrow \quad y = a^x$; Taking logarithms to base 10 we have

$\log_{10} y = \log_{10}(a^x) = \log_{10} y = \log_{10}(a)^x$

$\Rightarrow \log_{10} y = x \log_{10}(a)$

$\Rightarrow \dfrac{\log_{10} y}{\log_{10}(a)} = x$

Hence we have

$$\log_a y = \frac{\log_{10} y}{\log_{10}(a)}$$

If we repeat the above process by taking logarithms to base e in step 2 above we can also convert the logarithm to base e giving

$$\log_a y = \frac{\log_e y}{\log_e(a)}$$

Using these two formulas we can calculate the value of logarithms to any base, for example, to evaluate $\log_2 100$, we can convert the base to 10 and have $\log_2 100 = \dfrac{\log_{10} 100}{\log_{10} 2}$. Using your calculator to work the values of $\log_{10} 100 = 2$ and $\log_{10} 2 = 0.3010$ respectively, we have $\log_2 100 = \dfrac{2}{0.3010} = 6.64$ to two decimal places. You can also convert it to the base ***e*** first and you should get the same result.

Exercises 7.7

Simplify the following as far as possible:

1. $2^2 \times 2^3 \div 2^4$
2. $5^2 \times \left(5^2\right)^3$
3. $10^3 \times \left(10^{-2}\right)^4 \times \sqrt[3]{10^6}$
4. $\log 4 + \log 3 - \log 2$
5. $\log 4 - 2\log 3 + 3\log 5$
6. $\log 5 - 2\log 4 + \dfrac{1}{2}\log 9$

Exercises 7.8

1. Use a calculator to evaluate to 3 decimal places:
   (a) $\log 1000$; (b) $\ln 1000$; (c) $\ln 2$.

2. Express as a single logarithm:
   (a) $2\log 5 + \log 3 - \log 4$;
   (b) $3\log 4 + 2\log 5 - 3\log 7$;
   (c) $2\log x - 4\log y$;
   (d) $\frac{1}{2}\log u + 4\log v - \frac{1}{4}\log uv$.

3. Solve the following equations, giving your answers correct to 3 decimal places:
   (a) $2^3 \times 2^4 = 2^x$;
   (b) $x^3 \times x^y = x^7$;
   (c) $10^x = 1000$;
   (d) $3^x = 4$;
   (e) $\left(2^x\right)^2 = 4^3$;
   (f) $e^{2x} = 5$.

4. Solve the following logarithmic equations to 3 decimal places:
   (a) $\log 3 + \log x = \log 12$;
   (b) $\log 10 - \log(2x) = \log 8$;
   (c) $\ln x + \ln 3 - \ln 2 = 4$; (d) $\ln(2x) + \ln 4 = 5$.

Exercises 7.9

1. Evaluate to 3 dp

a) $\log_2 14$ b) $\log_2 4096$
c) $\log_2 1024$ d) $\log_2 114$
e) $\log_2 228$

2. Use the laws of logarithms to simplify
   a) $2\log 5 + 5\log 2$
   b) $\log 42 - \log 7$
3. Given that $\log_{10}3 = 0.4771$, without using a calculator, calculate
   a) $\log_{10} 9$
   b) $\log_{10} 90$

4. The temperature in a freezer which is switched off is initially -20ºC and the freezer is in a room with temperature 15ºC. The temperature difference between the inside of the freezer and the room, T, decreases with time, t, according to

$$T = T_0 e^{-kt}$$

where T_0 is the initial temperature difference
t is in hours and
k for this particular freezer is 0.05

Calculate the time when the freezer contents reach 0ºC.
The manufacturers have a new better insulated product which has a k value of 0.035, calculate the time taken for the contents of this new freezer to reach 0ºC.

5. The level of radioactivity, R, from a particular sample decreases according to

$$R = R_0 e^{-kt}$$

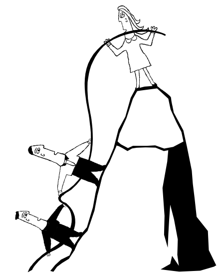
where R_0 is the initial level of activity
t is time in days and
$k = 1.2 \times 10^{-5}$

Calculate the half-life of the radioactivity in this sample i.e. the time taken for the radioactivity to reduce to half of its initial level.

There are more exercises in Foundation Mathematics 4th Edition by Croft and Davison Exercises 20.1, 20.2 20.3 question 1, 20.4, 20.5

Key points for chapter 7

Chapter 8: Calculus 1 – Differentiation



Learning objectives:

By working through this chapter, you should be able to:

- Calculate the derived function of a polynomial
- Find the gradient of a tangent to a curve at a given point
- Find the equation of a tangent to a curve at a given point
- Calculate higher derivatives
- Find the turning points on a curve
- Determine the nature of the turning points by using the second derivative
- Apply your knowledge of maxima and minima in modelling situations
- Calculate first derivatives of other commonly used functions.
- Understand the chain rule, product rule and quotient rule for differentiating more complex functions.

Calculus, or differentiation and integration, is used extensively in applications to real world problems.

You will only be introduced to the very basics of calculus in this course.

8.1 Tangents to Curves

Recap

The Gradient of a straight line is the **rate of increase** of y relative to x

$$= \frac{\text{Increase in Vertical Distance}}{\text{Increase in Horizontal Distance}} = \frac{\text{Distance Up or Down}}{\text{Distance across}}$$

We can extend this idea of rate of change (gradient) to any graph.

Example 8A
Draw a tangent to the curve $y = x^2$ at the point $x = 2$ and find the gradient of this tangent. (The tangent to a curve is a straight line which just touches the curve)

Solution

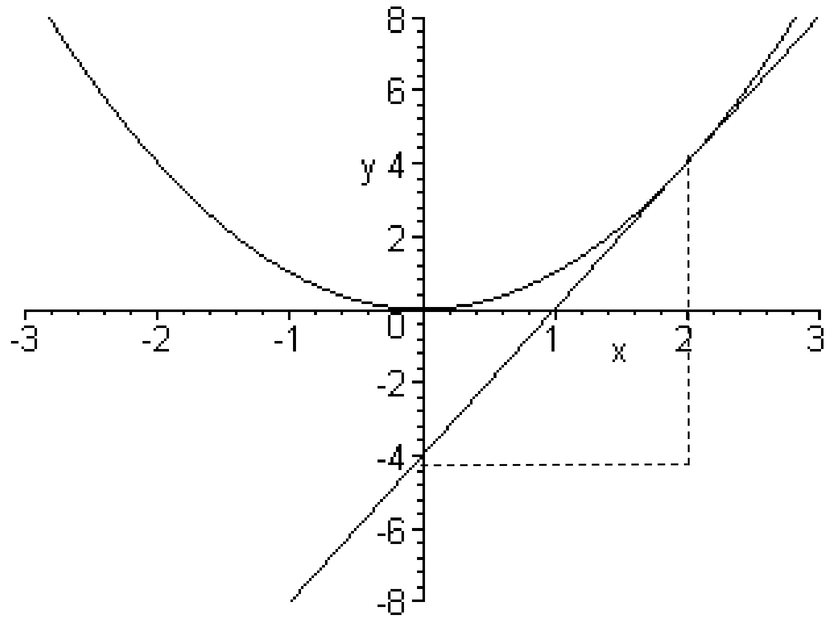


Figure 8.1 A tangent to the curve $y = x^2$ at the point (2, 4).

Gradient of the tangent at the point (2,4) = $+\dfrac{8}{2} = +4$

Example 8B

Draw a tangent to the curve $y = 4 - x^2$ at the point $x = -2$ and find the gradient of this tangent.

Solution The solution is shown in the figure below.

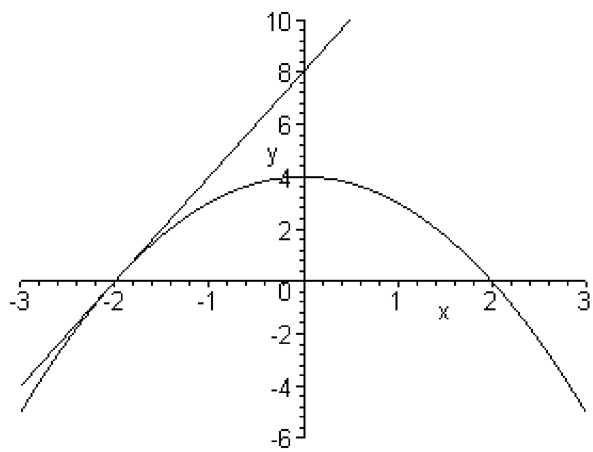


Figure 8.2 A tangent to the curve $y = 4 - x^2$ at the point $(-2, 0)$.

Gradient of the tangent at the point $(-2,0)$ =

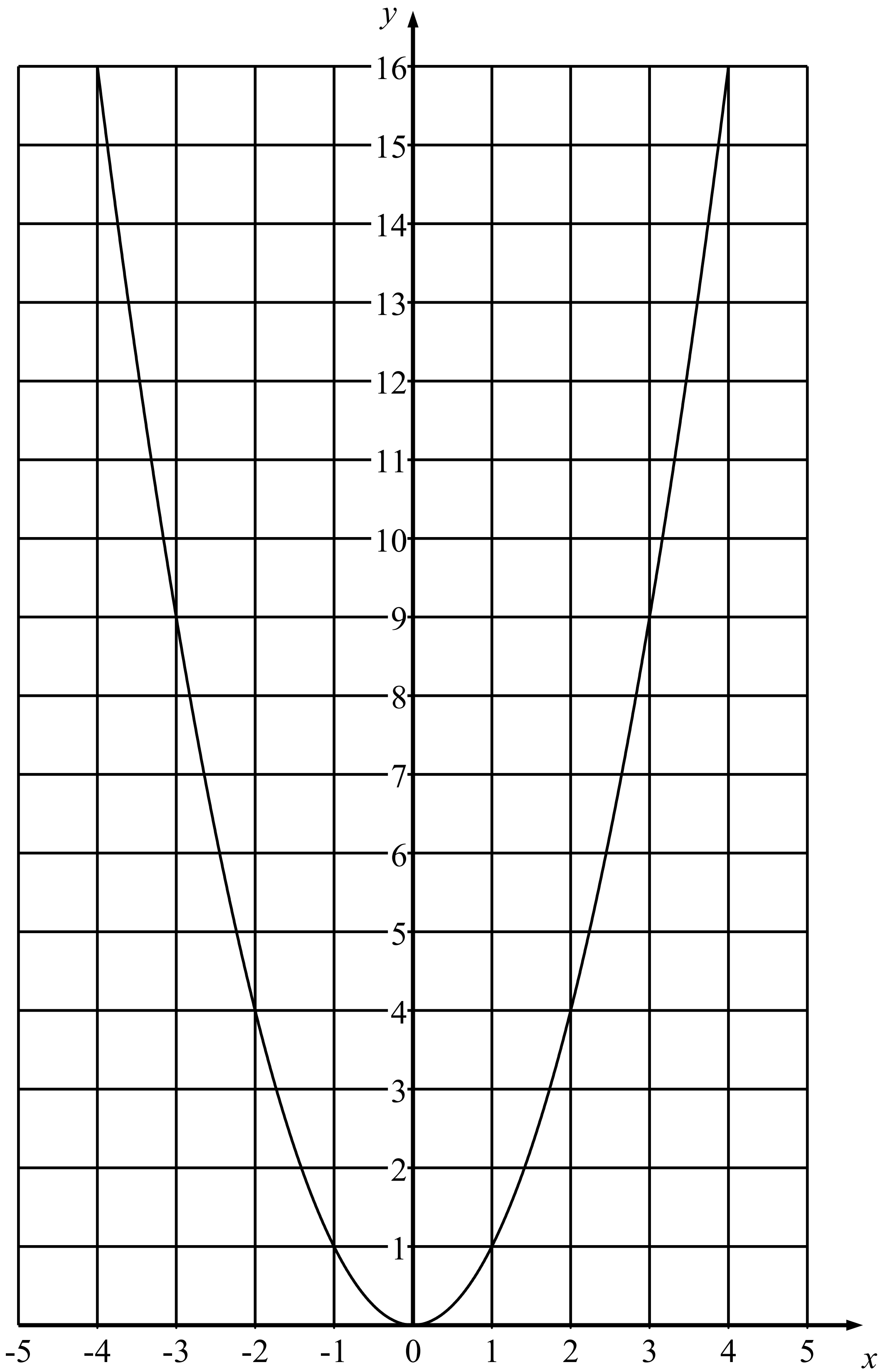
This is for you to complete

Exercise 8.1

Use the graph of $y = x^2$ on the next page to find the gradient of the tangent at

(i)	$x = 0$	Gradient =
(ii)	$x = +1$	Gradient =
(iii)	$x = -1$	Gradient =
(iv)	$x = +2$	Gradient =
(v)	$x = -2$	Gradient =
(vi)	$x = +3$	Gradient =
(vii)	$x = -3$	Gradient =

Do you notice any pattern in the values?


y
16
15
14
13
12
11
10
9
8
7
6
5
4
3
2
1
-5
-4
-3
-2
-1
0
1
2
3
4
5
x

8.2 The Gradient of a Curve and Differentiation

The **Gradient of a curve** at a particular point is defined as the gradient of the tangent at that point.

The gradient at a point on a curve tells us the rate of change of one variable with respect to another.

The gradient of the tangent at any point on a smooth curve has a single value, so we have a function which gives the slope of the tangent to a curve at any point. It is called the **derived function** or **derivative**. The process of obtaining the derived function or derivative is called **differentiation**

In exercise 8.1 you noticed that the gradient of $y = x^2$ at the point (x, x^2) is 2x, so the derivative of x^2 is $2x$.

Notation
If y is defined as a function of the variable x then the **derived function** of y is written as $\frac{dy}{dx}$ which is said 'dee y by dee x'.

Hence, if $y = x^2$ then $\frac{dy}{dx} = 2x$

8.3 Differentiation of a Polynomial

The general rule for differentiating $y = ax^n$ is:

multiply by the power and subtract one from the power

Hence, $\frac{dy}{dx} = anx^{n-1}$.

<u>In particular,</u>

i) if $y = x^n$, the derived function of y is $\frac{dy}{dx} = nx^{n-1}$

ii) if $y = k$, a constant, the derived function of y is $\frac{dy}{dx} = 0$

(Think of the line $y = k$, it has zero gradient for all values of x)

iii) If y is a sum of power functions then we can differentiate the power functions individually and add up the derivatives

Example 8C

Find the derived function of $y = 2x^3 + 4x^2 + 3x - 5$

Differentiating term by term we have

$$\frac{dy}{dx} = 2(3x^2) + 4(2x) + 3(1) - (0)$$

$$\frac{dy}{dx} = 6x^2 + 8x + 3$$

Exercise 8.2

1. Differentiate the following functions:

   (a) $y = x^2 + 1$;

   (b) $y = 2x^2 - x + 3$;

   (c) $y = \frac{2}{3}x^3 + \frac{1}{2}x^2 - x$;

   (d) $y = 3x^{-\frac{1}{2}}$

2. For all of the following find $\frac{dy}{dx}$

   a) $y = x^4$

   b) $y = x^{14}$

   c) $y = x$

   d) $y = c$

   e) $y = x^{-1}$

   f) $y = x^{-3}$

g) $y = x^{-2}$

h) $y = x^{1.4}$

i) $y = x^{0.2}$

j) $y = x^{-1.6}$

k) $y = x^{-0.7}$

l) $y = x^{4/3}$

m) $y = x^{-1/2}$

3. Differentiate the following

a) $s = 3t^2$

b) $a = 14b^{-0.3}$

c) $f = -2t^{-1/2}$

d) $w = 23y$

4. Rewrite the following expressions and then differentiate:

a) $y = \dfrac{7}{\sqrt{x}} + \dfrac{3}{x}$

b) $y = \dfrac{9}{\sqrt[3]{x}} - \dfrac{4}{x^2}$

c) $y = \dfrac{5}{x\sqrt{x}} - \dfrac{3}{x^7}$

<u>Note on indices:</u>

$$y = \frac{5}{\sqrt{x}} + \frac{2}{x^2}$$

$\Rightarrow$

$$y = 5x^{-\frac{1}{2}} + 2x^{-2}$$

5. Find $\frac{dy}{dx}$ for the following:

a) $y = 2x^2 - 7x + 6$

b) $y = \frac{3}{x^2}$

c) $y = 4x^3 - \frac{1}{x^2}$

d) $y = \frac{1}{x^2} + \frac{1}{x} + 1$

e) $y = \frac{2}{\sqrt{x}} - 3x^{\frac{1}{2}}$

f) $y = 2x^{1.7} - 3x^{0.2}$

g) $y = 4x^{0.9} + 7x^{1.4} - 8x^{0.1} + \frac{1}{\sqrt{x}} - 273$

Note on differentiation:

$$y = ax^n$$
$$\Rightarrow$$
$$\frac{dy}{dx} = anx^{n-1}$$

Exercise 8.3

1. Find $\frac{ds}{dt}$ for the following:

a) $s = 16t - t^2$

b) $s = t^3 - 2t^2$

c) $s = ut + \frac{1}{2}at^2$

2. For all of the following calculate the rate of change of s with respect to t at the value of t specified:

a) $s = t^2$ when t = 4

b) $s = 2t^{-2}$ when t = 3

c) $s = t^{0.5}$ when t = 9

d) $s = 20t^{-1}$ when t = 5

e) $s = t^{-0.5}$ when t =4

f) $s = 2t^2 + t$ when t = 3

8.4 Finding the Equation of a Tangent to a Curve

Note on gradients of tangents: The gradient of a tangent to a curve at a given point is the same as the value of its derivative at that point.

Example 8D

Find the gradient of the tangent to the curve $y = x^2$ at the point (2, 4).

Solution

❖ Differentiate to find the gradient,

$$\frac{dy}{dx} = 2x$$

❖ Evaluate the gradient at the point where x = 2

$$\left.\frac{dy}{dx}\right|_{x=2} = 2(2) = 4$$

Therefore the gradient of the tangent is 4

Exercise 8.4

1 Find the gradient of the curve $y = 3x^2 - 4$ at the point (2,8)

2 Find the gradient of the curve $y = 2x^3 - 5x^2 + 8$ at the point (1,5)

3 Find the gradient of the curve $y = 5x^{-1} + 3x^{-2} + 7x^{-3}$ at the point where $x=-1$

4 Find the gradient of the curve $y = \frac{1}{6x^3} + \frac{2}{x} + 6 - x^{0.4}$ at the point where $x=1$

5 An industrial plant emits sulphur dioxide into the air through a chimney. The concentration, C, of sulphur dioxide in air x miles from the chimney is given by

$$C = \frac{a}{x^2}$$

Determine the rate of change of concentration with distance from the chimney 1 mile from the chimney. Is the concentration increasing or decreasing with distance at that point?

Example 8E
To find the equation of the tangent to the curve $y = 2x^2 + 3$ at the point where $x = 4$.
Solution

- Differentiate to find the gradient of the tangent.

$$\frac{dy}{dx} = 2 \times 2x + 0 = 4x$$

- Evaluate the gradient at the point where $x = 4$,

$$\frac{dy}{dx} = 4 \times 4 = 16$$

- The equation of a straight line is $y = mx + c$

  Hence, when $m = 16$ we have

  $y = 16x + c$

- When $x = 4$, we can find y using the equation of the curve,

  so $y = 2(4)^2 + 3 = 35$

- Find c:

  As the tangent touches at this point we have $35 = 16(4) + c$

  Giving $c = -29$

Hence the equation of the tangent at the point (4,35) is

$$y = 16x - 29$$

Exercises 8.5
Find the equation of the tangent to the following curves at the points given.

1. $y = x^3 + 1$ at $x = 2$
2. $y = 2x^3 - 4x + 2$ at $x = 0$
3. $y = \frac{1}{x} + 3$ at $x = 1$
4. $y = 2\sqrt{x} + 2x$ at $x = 1$
5. $y = x^2 + x + 1$ at $x = 3$
6. $y = x^2 - \frac{1}{\sqrt{x}} + 1$ at $x = 1$

8.5 Higher Derivatives

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

and

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)$$

etc.

Note The second derivative of *y* with respect to *x* is written $\frac{d^2y}{dx^2}$ which is said 'dee two y by dee x squared', the second derivative of y with respect to x.

Example 8F

Find the first, second and third derivatives of $y = x^7 - 2x^3$;

Solution

$$\frac{dy}{dx} = 7x^6 - 6x^2;$$

$$\frac{d^2y}{dx^2} = 42x^5 - 12x;$$

$$\frac{d^3y}{dx^3} = 210x^4 - 12.$$

Exercise 8.6

1. For each of the following find $\frac{d^2y}{dx^2}$

a) $y = x^3$
b) $y = 3x^4$
c) $y = x^7$
d) $y = x^{-1}$
e) $y = 4x^{1.5}$
f) $y = 8x^{-1.5}$

2. For all of the following calculate the rate of change of $\frac{d^2y}{dx^2}$ with respect to x at the value of x that is specified.

   a) $y = x^2$ when $x = 3$

   b) $y = x^3$ when $x = 1$

   c) $y = x$ when $x = 13$

   d) $y = x^{0.5}$ when $x = 0.25$

3. Find the first, second and third derivatives of $y = 4x^{-3} + x^8$.

4. Find $\frac{d^4y}{dx^4}$ for each of the following functions:

   (a) $y = x^4 + 5$;

   (b) $y = \frac{1}{2}x^6 + 5x^3$;

   (c) $y = 2x^{\frac{1}{2}} + 4x^{-\frac{1}{2}}$;

   (d) $y = 0.2x^{0.1}$.

8.6 Finding Maximum and Minimum Points of a Curve

8.6.1 The sign of the derivative, Turning Points, Maximum and Minimum

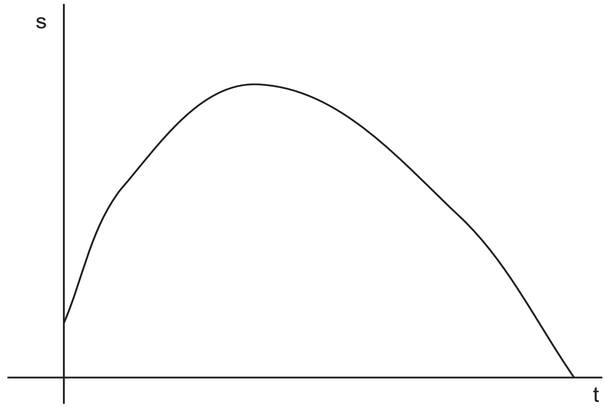


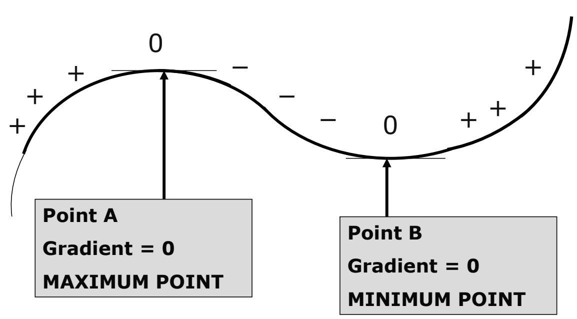
Figure 8.3

In Figure 8.3, we have plotted a graph of distance, s, against time, t. We want to know when the distance is greatest. We can do this by reading off from the graph, but how can we do it analytically. We are studying differentiation and the derived function, so does the gradient of the curve at the maximum point have any special value? Try to find the gradient at the maximum point.

The conclusion we reach is that the curve has zero gradient at the maximum. If the graph had a minimum the gradient there would also be zero

Points where the gradient is zero are called **stationary points.** Stationary points are points where the y value is neither increasing nor decreasing. There are points where the gradient is zero which are neither maximum nor minimum and the term stationary point includes these as well.

Figure 8.4



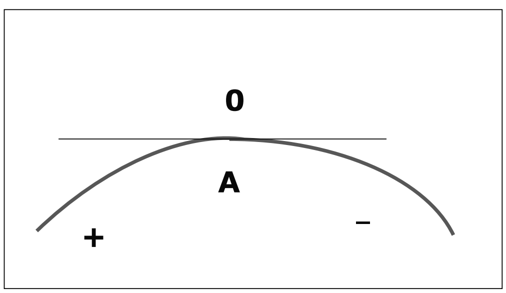
Referring to Figure 8.4, we see that at a maximum, the gradient is positive to the left of the maximum and negative to the right. As we pass through a maximum with the horizontal variable increasing the gradient goes from positive through zero to negative. As we go through a minimum in the same way, the gradient goes from negative to positive. This procedure of investigating the gradient either side of a stationary point is called the **first derivative test** for a maximum or minimum.

8.6.2 The second derivative test

As an alternative to investigating the sign of $\dfrac{dy}{dx}$ at points either side of the stationary point, we could differentiate $\dfrac{dy}{dx}$ again with respect to x, to get the **second derivative** of y with respect to x. The second derivative is the derivative of the derivative so it gives the rate of change of the first derivative. If the second derivative is positive then the first derivative is increasing and we have a minimum of y but if the second derivative is negative then the first derivative is decreasing and we have a maximum of y.

This procedure is called the **second derivative test** for a maximum or minimum.

MAXIMUM POINT



For a maximum point (Local maximum)
$\dfrac{dy}{dx} = 0$

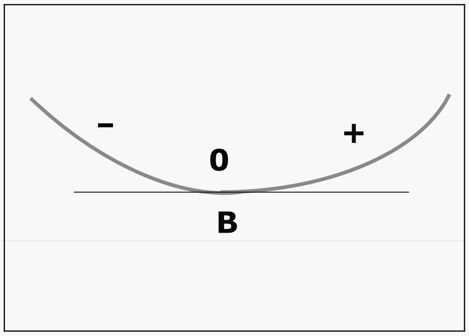
$\dfrac{dy}{dx}$ **changes from positive to negative**

$\dfrac{dy}{dx}$ **decreases**

Hence the rate of change of d*y*/d*x* with respect to *x* is NEGATIVE

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} < 0$$

MINIMUM POINT



For a minimum point (Local minimum)

$\frac{dy}{dx} = 0$

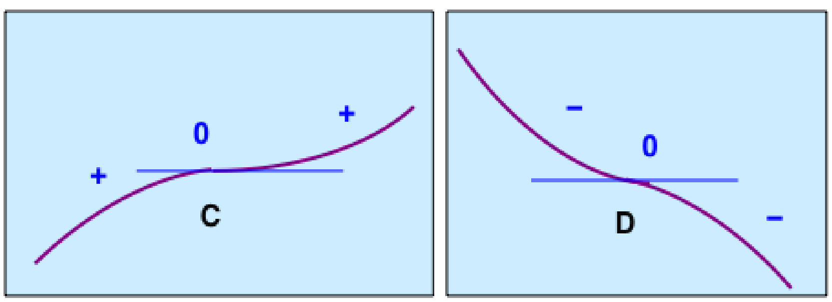
$\frac{dy}{dx}$ changes from negative to positive

$\frac{dy}{dx}$ increases

Hence the rate of change of dy/dx with respect to *x* is POSITIVE

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} > 0$$

POINT OF INFLEXION



$\frac{dy}{dx} = 0$

$\frac{dy}{dx}$ DOESN'T CHANGE

Hence the rate of change of dy/dx with respect to *x* is ZERO

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} = 0$$

Example 8G:
(Taken from Maths Extra: *Maximum and Minimum* page 4)

$$y = \frac{x^3}{3} - \frac{x^2}{2} - 2x + 1$$

Solve using the formula or by factorisation

$$\frac{dy}{dx} = x^2 - x - 2$$

$$\frac{dy}{dx} = 0 \Rightarrow x^2 - x - 2 = 0 \qquad (x-2)(x+1) = 0$$

$\Rightarrow$ $x = 2$ or $x = -1$

$$\frac{d^2y}{dx^2} = 2x - 1$$

$$\left.\frac{d^2y}{dx^2}\right|_{x=2} = 2(2) - 1 = 3 > 0 \qquad \left.\frac{d^2y}{dx^2}\right|_{x=-1} = 2(-1) - 1 = -3 < 0$$

$\Rightarrow$ Minimum at x=2 $\qquad$ $\Rightarrow$ Maximum at x=-1

Example 8H: $y=x^3$

$$\frac{dy}{dx} = 3x^2$$

$$\frac{dy}{dx} = 0 \Rightarrow 3x^2 = 0$$

$\Rightarrow$ $x = 0$

$$\frac{d^2y}{dx^2} = 6x$$

$$\left.\frac{d^2y}{dx^2}\right|_{x=0} = 6(0) = 0$$

$\Rightarrow$ $x = 0$ is a point of inflexion

Exercise 8.7
Taken from Maths Extra: Maximum and Minimum page 3 Exercise A

- Find the turning points for the following curves and determine their nature

1. $y = x^2 - 3x - 15$
2. $s = t^2 + 6t + 4$
3. $y = 1 + 5x - 2x^2$
4. $s = (t - 2)(t - 4)$
5. $y = 2x^3 - 15x^2 + 24x + 1$
6. $s = 2t^3 + 3t^2 - 36t - 1$

8.6.3 Maxima/Minima: Simple Applications

Another application of calculus is finding the maxima and minima of certain functions.

Example 8I

The height of a flower can be modelled using the equation $h = 6t - t^2$, where h is the height in centimetres and time t is measured in days. Determine the maximum height of the flower.

Solution Using calculus,

$$\frac{dh}{dt} = 6 - 2t$$

A turning point is found by solving the equation

$$\frac{dh}{dt} = 6 - 2t = 0$$

Which has one solution when $t = 3$.

To prove that this is indeed a maximum we need to consider the second derivative

$$\frac{d^2h}{dt^2} = -2$$

At the point t=3

$$\left.\frac{d^2h}{dt^2}\right|_{t=3} = -2 < 0,$$

Therefore this point gives a maximum
and $h_{\max} = 6(3) - (3)^2 = 9$ cm.

Plot the graph of this function and check the result for yourself.

Example 8J

Determine the turning points of the function $y = x^3 - x$.

Solution

$$\frac{dy}{dx} = 3x^2 - 1$$

The turning points are found by finding when this gradient is zero
So we need to solve the equation

$$\frac{dy}{dx} = 3x^2 - 1 = 0.$$

This equation has roots at $x_1 \approx 0.577$ and $x_2 \approx -0.577$.

Now consider the second derivative to determine the type of turning point.

$$\frac{d^2y}{dx^2} = 6x,$$

Evaluate the second derivative at $x_1 \approx 0.577$

$$\left.\frac{d^2y}{dx^2}\right|_{x_1} = 6x_1 \approx 3.462 > 0$$

$\Rightarrow$ x_1 is a local minimum turning point

Evaluate the second derivative at $x_2 \approx -0.577$

$$\left.\frac{d^2y}{dx^2}\right|_{x_2} = 6x_2 \approx -3.462 < 0.$$

$\Rightarrow$ x_2 is a local maximum turning point

Example 8K
The power *P* developed in a resistor is given by

$$P = -R^2 + 2R + 3,$$

where *R* is the resistance. Show that the maximum power is $P_{\max} = 4$ Watts.

Solution

Differentiate to find the turning points, thus

$$\frac{dP}{dR} = -2R + 2.$$

we want the value of R which makes this zero

$$\frac{dP}{dR} = 0 \Rightarrow R = 1.$$

Now consider the second derivative

$\frac{d^2P}{dR^2} = -2 < 0$ for all values of R and so the point $R = 1$ is a local maximum.

The maximum power is then

$$P_{\max} = -(1)^2 + 2(1) + 3 = 4 \text{ Watts}.$$

Example 8L
The charge C in a capacitor at time t is given by

$$C = t^2 - 3t + 5\,,$$

Where t is measured in seconds.
Find the minimum charge in the capacitor.

Solution

Differentiate to find the turning points, thus

$$\frac{dC}{dt} = 2t - 3\,.$$

$$\frac{dC}{dt} = 0 \qquad \text{when} \quad t = 1.5\,.$$

Now consider the second derivative

$$\frac{d^2C}{dt^2} = 2 > 0 \qquad \text{for all values of } t$$

$\Rightarrow$The point $t = 1.5$ is a local minimum.

The minimum charge is then

$$C_{\min} = (1.5)^2 - 3(1.5) + 5 = 2.75 \text{ Coulombs}\,.$$

Exercise 8.8

1. The population of locusts (measured in millions) in a certain district can be modelled using the equation

$$y = -x^2 + 2x + 8,$$

where y is the population after x weeks. Determine the maximum population of locusts.

2. The amount of drug in a human is modelled using the equation

$$y = -x^2 + 5x + 6,$$

where y is the amount of drug in millilitres after x hours. Determine the maximum amount of drug present in the body.

3. The growth of a grass can be modelled by the equation:

$$y = -x^2 + 8x,$$

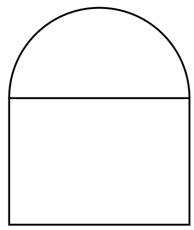
where y is the height of the grass (measured in centimetres) after x weeks. Determine the maximum height of the grass.

4. Determine the turning points of the polynomial:

$$y = x^3 - 6x^2 + 8x.$$

5. It is required to make an open box with a square base which has a volume of 1000 cm^3 and has the smallest surface area, what should the dimensions of the base and the height of the box be?

6. A window in a hospital has the shape below:



The area of a circle of radius r is πr^2. The perimeter of the window is 10 metres. Show that the dimension of the rectangular part of the window are

$$2r \text{ and } \frac{10-2r-\pi r}{2}$$

Determine the dimensions of the window which will allow the maximum amount of light to enter the hospital.

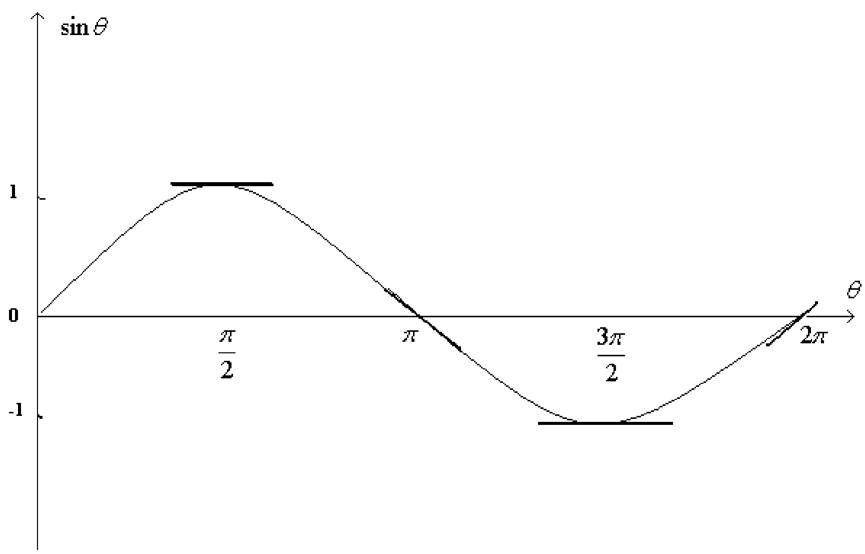
8.7 Further Differentiation

8.7.1 Derivatives for other commonly used functions

Rate of change of Trigonometric functions $\frac{d}{d\theta}(\sin\theta)$

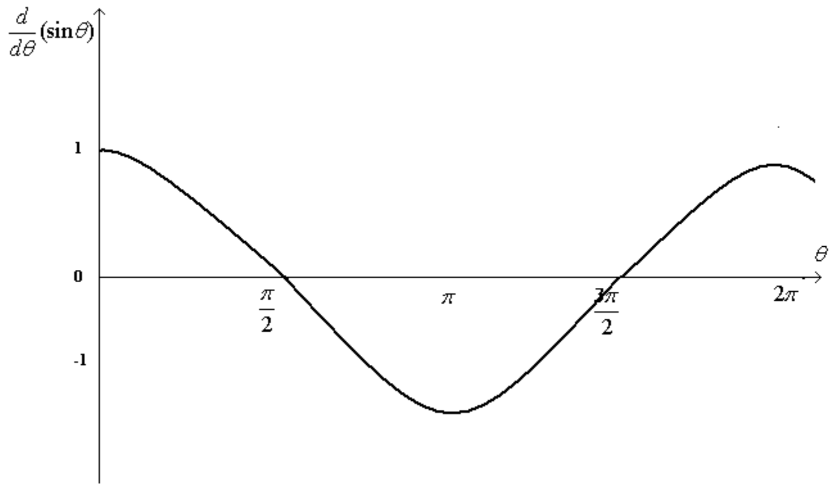
The rate of change at any point is the gradient of the tangent at that point. Consider the graph of $\sin\theta$. By drawing an accurate tangent at various points on the curve, we can calculate the gradient at these points.

Note that θ must be in **radians.**



It can be shown that the derivative is given by,

$$\frac{d}{d\theta}(\sin\theta) = \cos\theta .$$



Similarly, it can be shown that,

$$\frac{d}{d\theta}(\cos\theta) = -\sin\theta\,.$$

Derivatives of e^x and $\log_e x$

Two important results we need to consider are,

$$\frac{de^x}{dx} = e^x$$

$$\frac{d(\log_e x)}{dx} = \frac{1}{x}$$

Note: $\log_e x$ can also be written $\ln x$

8.7.2 More advanced differentiation

We saw in our study of functions that functions can be formed by putting together other functions in several ways.

Function composition
Given f(x) and u(x) then $f \circ u(x) = f(u(x))$.

Product of functions

Given u(x) and v(x) we can form a new function
y(x) = u(x).v(x) the product of u and v.

Quotient of functions

Given u(x) and v(x) we can form a new function
y(x) = u(x)/v(x) the quotient of u and v.

If we want to differentiate composite, product or quotient functions then first we have to recognise how the function is put together then apply the appropriate rule for differentiating a function of the given type.

Function composition and the chain rule

Chain Rule: Given $y = f(u(x))$, then $y = f(u)$, and $\frac{dy}{dx} = \frac{du}{dx}\frac{dy}{du}$

Example 8M

Differentiate $y = (x+3)^2$.

Solution

To do this we need to multiply out the brackets as follows.

$$y = x^2 + 6x + 9$$

then $$\frac{dy}{dx} = 2x + 6$$

Example 8N

Differentiate $y = (4x+3)^{10}$

Solution

We recognise that the y is the result of the composition of
$u = 4x + 3$ and $f(u) = u^{10}$ so we can apply the Chain Rule.

Let $u = 4x + 3$ so that, $y = u^{10}$.

Step 1 differentiate u with respect to x

$$\frac{du}{dx} = 4$$

Step 2 differentiate y with respect to u.

$$\frac{dy}{du} = 10u^9 = 10(4x+3)^9$$

Step 3 use the chain rule to combine these together.

$$\frac{dy}{dx} = \frac{du}{dx} \times \frac{dy}{du}$$

$$\frac{dy}{dx} = 4 \times 10(4x+3)^9 = 40(4x+3)^9 .$$

Exercises 8.9

Differentiate each of the following

1. $y = (-3x+1)^7$
2. $y = (4x-1)^{11}$
3. $y = (-2x+2)^8$
4. $y = (12x-3)^7$
5. $y = (-9x+5)^5$
6. $y = (2x-4)^4$
7. $y = (-4x+1)^{-2}$
8. $y = (-10x-13)^{20}$

Example 8O

Given, $y = \sin 3\theta$ find $\frac{dy}{d\theta}$.

Solution

We need to use the chain rule.

Let $u = 3\theta$ then $y = \sin u$

Differentiate u with respect to θ

$$\frac{du}{d\theta} = 3$$

Differentiate y with respect to u

$$\frac{dy}{du} = \cos u = \cos 3\theta$$

$$\frac{dy}{d\theta} = \frac{du}{d\theta} \times \frac{dy}{du} = 3\cos 3\theta$$

Exercises 8.10

For all of the following find $\frac{dy}{d\theta}$

1. $y = \cos 7\theta$
2. $y = \sin 2\theta$
3. $y = \sin 4\theta$
4. $y = -\sin 8\theta$
5. $y = \cos 3\theta$
6. $y = -\cos \frac{\theta}{3}$
7. $y = \cos 8\theta$
8. $y = -\sin \frac{\theta}{2}$

Example 8P

Given, $y = e^{4x}$ find $\frac{dy}{dx}$.

Solution

We need the chain rule.

Let $u = 4x$ then we let $y = e^u$

Differentiate u with respect to x.

$$\frac{du}{dx} = 4$$

Differentiate y with respect to u

$$\frac{dy}{du} = e^u = e^{4x}$$

$$\frac{dy}{dx} = \frac{du}{dx} \times \frac{dy}{du} = 4e^{4x}$$

Note In general, $\frac{d(e^{ax})}{dx} = ae^{ax}$

Exercises 8.11

Differentiate the following.

1. $y = e^{2x}$
2. $y = \ln 5x$
3. $y = e^{-6x}$
4. $y = 7\ln 2x$
5. $y = -3e^{8x}$
6. $y = \ln(x^2)$
7. $y = 2e^{-0.5x}$
8. $y = 8\ln\frac{x}{2}$
9. $y = 4\ln(3x+1)$
10. $y = 5e^{\frac{x}{2}} + 2\ln(6x^2)$

Exercises 8.12

Differentiate the following.

1. $y = \sin^2 3x$
2. $y = \ln(7x^4 - x)$
3. $y = \cos(5x^2 - 2x + 3)$
4. $y = 3e^{-\frac{x}{6}}$
5. $y = \ln(\frac{1}{x})$
6. $y = \ln(\cos x\ )$
7. $y = e^{\sin x}$
8. $y = 2\sin(4x^3)$
9. $y = e^{-2\cos x}$
10. $y = \cos(x^{-2})$

The following table summarises the standard first derivatives

Function	**First derivative**
$y = ax^n$	$\dfrac{dy}{dx} = anx^{n-1}$
$y = k$, a constant	$\dfrac{dy}{dx} = 0$
$y = e^{ax}$	$\dfrac{dy}{dx} = ae^{ax}$
$y = \dfrac{1}{x+a}$	$\dfrac{dy}{dx} = -\dfrac{1}{(x+a)^2}$
$y = log_e(ax)$	$\dfrac{dy}{dx} = \dfrac{1}{x}$
$y = \cos(ax)$	$\dfrac{dy}{dx} = -a\sin(ax)$
$y = \sin(ax)$	$\dfrac{dy}{dx} = a\cos(ax)$
$y = au(x) + bv(x)$ a and b constant	$\dfrac{dy}{dx} = a\dfrac{du}{dx} + b\dfrac{dv}{dx}$

Table 1 Summary of standard first derivatives

Product & Quotient Rules

Product Rule: If $y = uv$, then $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$.

Example 8Q Differentiate $y = (x^2 - 1)\sin x$.

Solution

y = uv where $u = x^2 - 1$ and $v = \sin x$,

We can differentiate u and v to get $\frac{du}{dx} = 2x$ and $\frac{dv}{dx} = \cos x$

To find $\frac{dy}{dx}$ we must put what we know together according to the product rule.

Given $y = uv$ then $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$.

$= (x^2 - 1)\cos x + \sin x \times 2x$

Quotient Rule: If $y = \frac{u}{v}$, then $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$.

Example 8R

Differentiate $y = \frac{x^2 - 1}{\sin x}$

Solution

y = u/v where u = $x^2 - 1$ and v = sin x

then $\frac{du}{dx} = 2x$ and $\frac{dv}{dx} = \cos x$

To find $\frac{dy}{dx}$ we must put what we know together according to the quotient rule.

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} = \frac{\sin x \times 2x - (x^2 - 1)\cos x}{\sin^2 x}.$$

Exercises 8.13

Differentiate the following:

1. $y = 2x(x^3 - 4x)$
2. $y = x^2 \sin(x)$
3. $y = e^x \cos x$
4. $y = e^{2x}\left(x^2 - 5\right)$
5. $y = \dfrac{x^3}{x+1}$
6. $y = \dfrac{e^x}{x}$
7. $y = \dfrac{\sin x}{x}$
8. $y = \dfrac{xe^x}{x^2 - 1}$

Key points from Chapter 8

Chapter 9: Curve Sketching

Learning objectives:

By working through this chapter you should be able to:

- Apply your knowledge to curve sketching, including Quadratic, Cubic and Exponential.

9.1 Introduction

A sketch of a graph shows the important features of the graph and general shape of a graph without accurate plotting of lots of points.

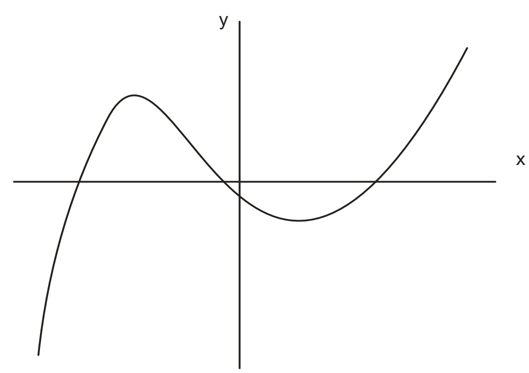


Figure 9.1

The most important features of the graph in Figure 10.1 are:

1. Where the graph crosses the x axis
2. Where the graph crosses the y axis
3. What happens when x gets very big and positive and when x gets very big and negative?
4. Whether the graph has any points where it reverses direction.

9.2 Important Features of a Graph

1. Crossing the y axis

All points on the y-axis have $x = 0$ so all we have to do is to put $x = 0$ into the function.

2. Crossing the x-axis

At all points on the x-axis $y = 0$. We have to find the values of x that make $y = 0$. We can do this for a quadratic function using the formula in chapter 9. If a polynomial is given in factored form then we need to find where each of the factors is zero.

3. What happens when x is large?

Consider the graph of $y = ax^n + bx^{n-1} + cx^{n-2} + \ldots + \text{const}$

If we look at the size of the leading term ax^n relative to the next term we get

$$\frac{ax^n}{bx^{n-1}} = \frac{a}{b}x$$

Now, whatever the size of a and b, ax/b will have an absolute value bigger than 1 when the absolute value of x becomes greater than the absolute value of b/a. Thus, each term in a polynomial is bigger than the next provided that x is big enough. x may be either positive or negative.
The implication of this is that when x gets big enough, the behaviour of

$$y = ax^n + bx^{n-1} + cx^{n-2} + \ldots + \text{const}$$

is dominated by the behaviour of $y = ax^n$.
When x is <u>big and positive</u>, x^n is always big and positive
but When x is <u>big and negative</u>, the behaviour of x^n depends on whether n is odd or even. If n is even then x^n is positive and if n is odd x^n is negative.

4. Where the graph reverses direction (Stationary points and points of inflexion)

Points where the graph changes direction are those where the graph is horizontal. Put another way, it is where the graph has gradient equal to zero.

We can establish where the graph has zero gradient by finding the points where $\frac{dy}{dx} = 0$. We first have to differentiate y then solve $\frac{dy}{dx} = 0$

9.3 Sketching Polynomials

Example 9A. Sketch the graph of $y = -x^2 + 9x - 20$

Step 1 Graph crosses the y axis at where $x = 0$
$\Rightarrow$Graph crosses the y axis at the point (0, -20)

Step 2 Graph crosses the x-axis where $y = 0$
$\Rightarrow$Graph crosses the y axis at where $-x^2 + 9x - 20 = 0$
We need to use the quadratic formula to solve this equation

$$\boxed{a = -1 \qquad b = +9 \qquad\qquad c = -20}$$

$$x = \frac{-9 \pm \sqrt{9^2 - 4 \times -1 \times -20}}{2 \times -1}$$
$$x = \frac{-9+1}{-2} \quad \text{or} \quad x = \frac{-9-1}{-2}$$
$$x = 4 \quad \text{or} \quad x = 5$$

Step 3 For big x (positive and negative)
$-x^2$ has an even power but is multiplied by -1 so $-x^2$ is big and negative.

Step 4 Stationary points or turning points
We want to know where $\frac{dy}{dx} = 0$

$$\frac{dy}{dx} = -2x + 9$$
$$\frac{dy}{dx} = 0 \Rightarrow -2x + 9 = 0 \quad \Rightarrow \boldsymbol{x = 4.5}$$

Step 5 Put all of this information onto a set of axes, then join them up in the simplest way to complete the sketch

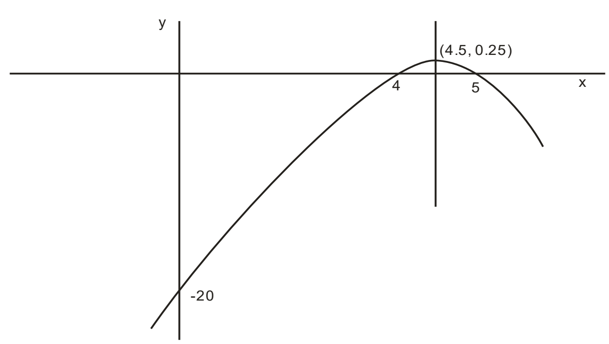


Figure 9.2

Exercises 9.1

1. Sketch the following graphs:

a) $y = 2x^2 - x - 10$ b) $y = 2x^2 - x + 1$

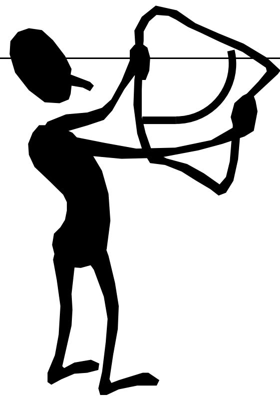
c) $y = -x^2 + 4x + 20$ d) $y = -5x^2 - 30x - 40$

2. Sketch the following graphs:

a) $y = (x + 3)(x - 2)(x - 1) = x^3 - 7x + 6$

b) $y = (-x + 3)(x - 2)(x - 1) = -x^3 + 6x^2 - 11x + 6$

9.4 Sketching Exponential graphs



Exponential Graphs
$y = e^{ax}$

Where a is positive

Note:
$\boldsymbol{x} = 0 \Rightarrow \boldsymbol{y} = 1$

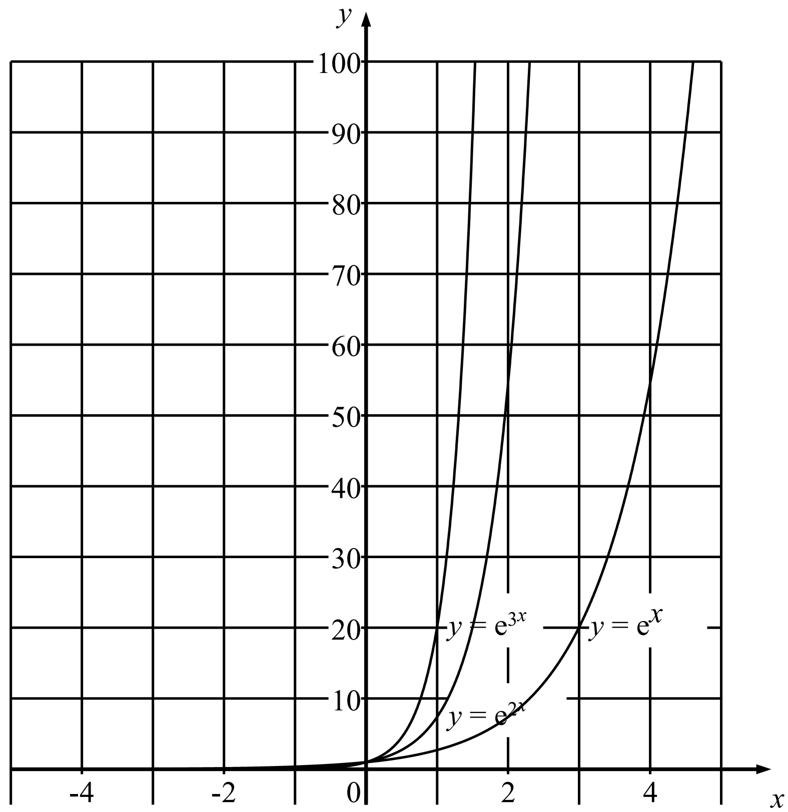


Figure 9.3

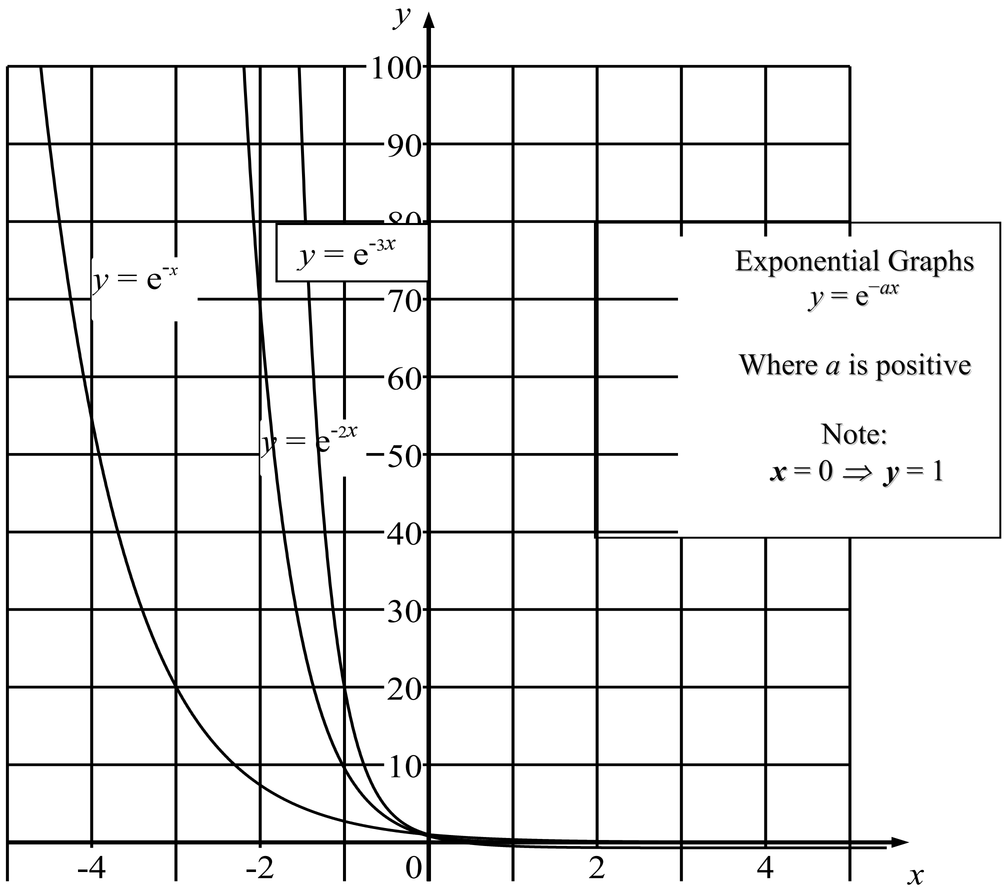

y
100
90
80
70
60
50
40
30
20
10
-4
-2
0
2
4
x
y = e^{-x}
y = e^{-2x}
y = e^{-3x}
Exponential Graphs
y = e^{−ax}
Where a is positive
Note:
x = 0 ⇒ y = 1

Figure 9.4

Example 9B

Sketch the graph $y = 2 + e^{2x}$

Solution

Shape	**Exponential**
Step 1	**Graph crosses the y axis where $x = 0$**
$\Rightarrow$	**Graph crosses the y axis at the point (0, +3)**
Step 2	**Graph crosses the x-axis where $y = 0$**
$\Rightarrow$	**Graph crosses the x-axis where $2 + e^{2x} = 0$**
Step 3	**For big positive x, e^{2x} is big and positive**
$\Rightarrow$	**$2 + e^{2x}$ is big and positive**
	For big negative x, e^{2x} is small and positive and $e^{2x} \to 0$
$\Rightarrow$	**$2 + e^{2x} \to 2$**
Step 4	**NO Stationary points or turning points**

This equation has NO solution so graph doesn't cut the x-axis

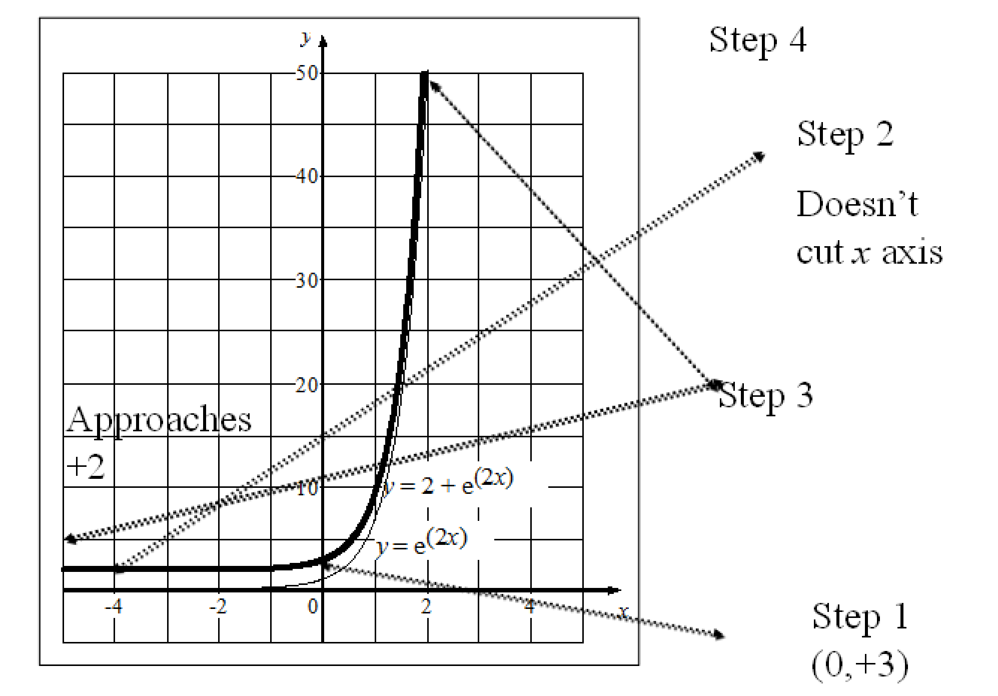


Figure 9.5

Example 10B (Alternative method)

Sketch the graph $y = 2 + e^{2x}$

<u>Solution</u>

Shape Exponential

Consider the graph of $y = e^{2x}$

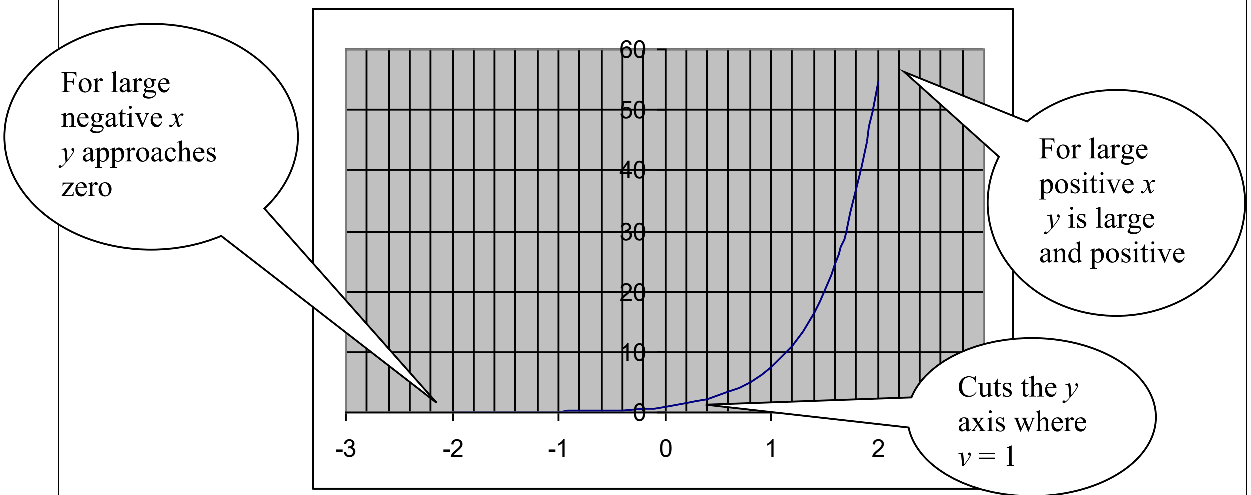


Figure 9.6

Now move the graph up 2 units

The graph of $y = 2 + e^{2x}$

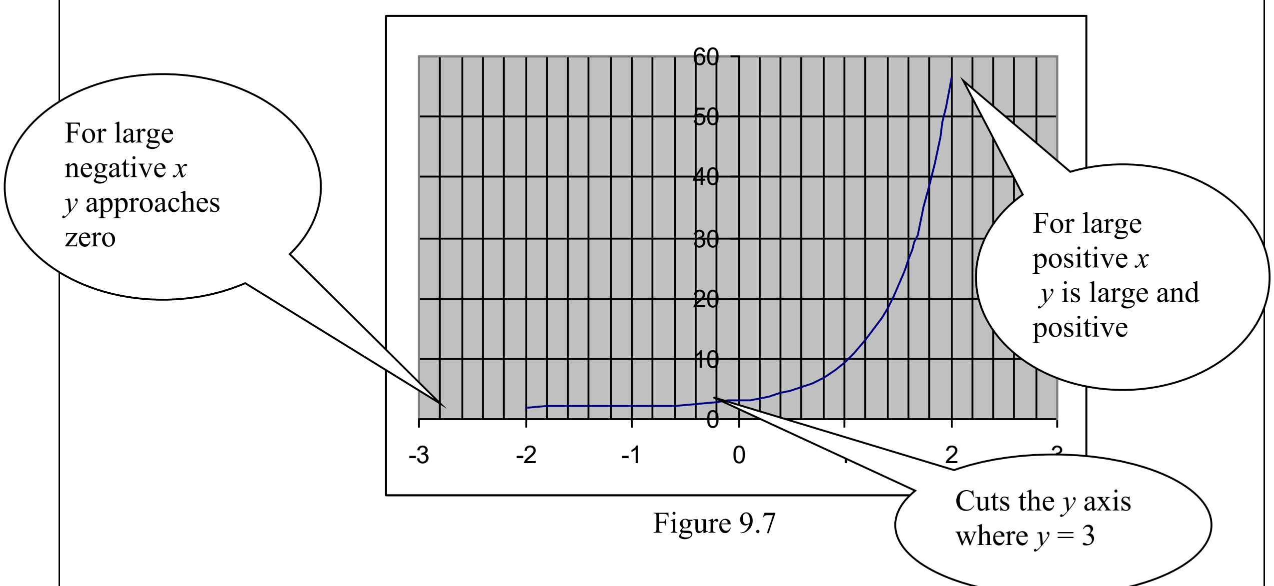


Figure 9.7

The standard graphs for the exponential function are presented in figure 9.8.

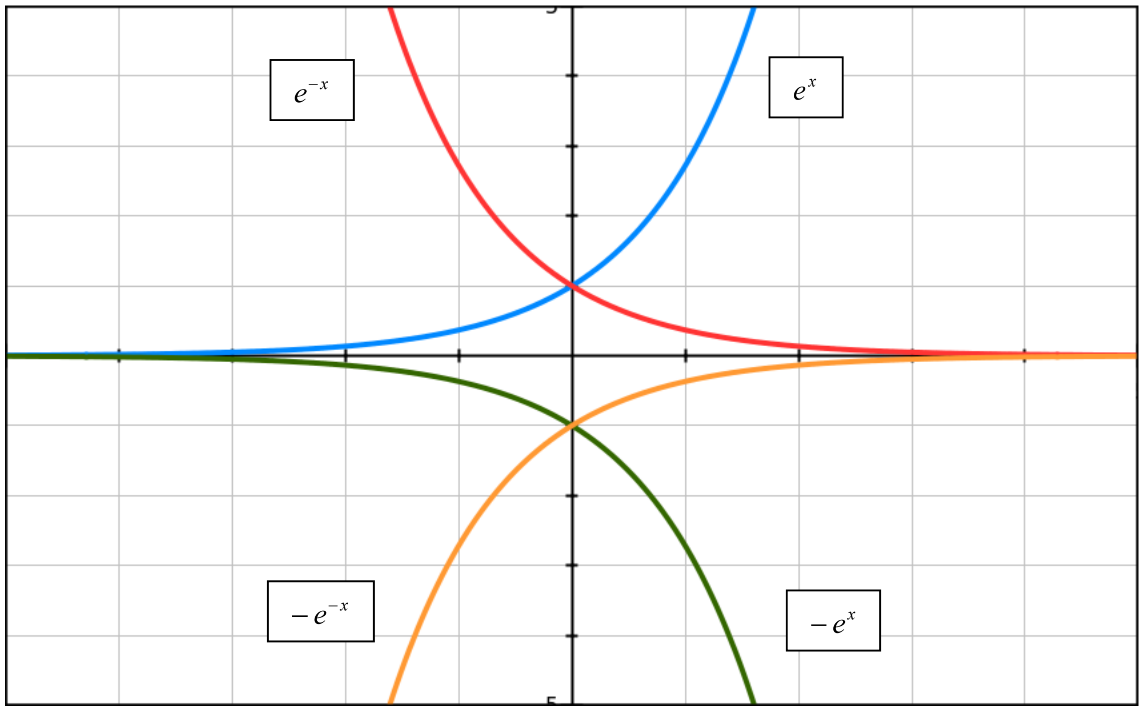


Figure 9.8

Exercise 9.2

Sketch the following functions

1. (a) $y = e^{5x}$ (b) $y = e^{5x} - 2$ (c) $y = e^{5x} + 4$

2. (a) $y = e^{-4x}$ (b) $y = e^{-4x} - 2$ (c) $y = e^{-4x} + 2$

3. (a) $y = -e^{2x}$ (b) $y = -e^{2x} - 1$ (c) $y = -e^{2x} + 1$

4. (a) $y = -e^{-3x}$ (b) $y = -e^{-3x} - 1$ (c) $y = -e^{-3x} + 1$

Key points from Chapter 9

Chapter 10: Calculus 2 – Integration

Learning objectives:

By working through this chapter you should be able to:

- Understand the definition of indefinite and definite integration and the rules for integrating commonly used functions
- Perform integration by parts
- Apply integration to find the area under a curve

10.1 Introduction

The topic **integration** can be approached in a number of different ways. Maybe the simplest way of introducing it is to think of it as **differentiation in reverse**.

We know that if $y = x^2$, then $\frac{dy}{dx} = 2x$. Now suppose that we are given $\frac{dy}{dx} = 2x$ and asked to find *y* in terms of *x*. This process is the **reverse of differentiation** and is called **integration**.

We know that $y = x^2$ will satisfy $\frac{dy}{dx} = 2x$, but so will $y = x^2 + 1$ and $y = x^2 + 2$. In fact any expression of the form, $y = x^2 + c$, where c is a constant, will also satisfy $\frac{dy}{dx} = 2x$.

In other words, we do not know whether the original function contained a constant term or not. For this reason, we write $y = x^2 + c$, where c is called the **constant of integration**.

Notation: $y = x^2 + c$ is called the **integral** of 2*x* with respect to *x*.

$\int 2x$	dx	$= x^2 + c$
Integral Sign	Indicating that the integration is with respect to ***x***	

Function	Indefinite Integral
$y = ax^n$	$\int y\,dx = \frac{ax^{n+1}}{n+1} + C,\ n \neq -1$
$y = k$, a constant	$\int y\,dx = kx + C$
$y = e^{ax}$	$\int y\,dx = \frac{e^{ax}}{a} + C$
$y = \frac{1}{x+a}$	$\int y dx = \ln\lvert x+a\rvert + C$
$y = \cos(ax)$	$\int y\,dx = \frac{\sin(ax)}{a} + C$
$y = \sin(ax)$	$\int y\,dx = -\frac{\cos(ax)}{a} + C$
$y = au(x) + bv(x)$ a and b constant	$\int y\,dx = a\int u dx + b\int v dx + C$

Table 1: The Rules of Integration

Note: All the integrations above which need a constant C are called **indefinite integrations.**

Example 10.A

Find the following integrals

a) $\int 3x^2 dx$

$$\int 3x^2 dx = 3\frac{x^{2+1}}{2+1} + c$$
$$\int 3x^2 dx = x^3 + c$$

b) $\int \frac{1}{x^2} dx$

$$\int \frac{1}{x^2} dx = \frac{x^{-2+1}}{-2+1} + c = \frac{x^{-1}}{-1} + c$$
$$\int \frac{1}{x^2} dx = -\frac{1}{x} + c$$

c) $\int 6\sqrt{x}\, dx$

$$\int 6\sqrt{x}\,dx = \frac{6x^{\frac{1}{2}+1}}{(\frac{1}{2}+1)} + c = \frac{6x^{\frac{3}{2}}}{(\frac{3}{2})} + c$$

$$\int 6\sqrt{x}\,dx = 4x^{\frac{3}{2}} + c = 4\sqrt{x^3} + c$$

d) $\int (x^2 + 6x - 3)dx$

$$\int (x^2 + 6x - 3)dx = \frac{x^3}{3} + \frac{6x^2}{2} - 3x + c$$

$$\int (x^2 + 6x - 3)dx = \frac{x^3}{3} + 3x^2 - 3x + c$$

Exercises 10.1

Integrate the following

1.	$\int (7x^4 - 5x^3 - x)dx$	6.	$\int (\frac{1}{2}x^6 + 2x^3 - x^{\frac{3}{2}})dx$
2.	$\int (2x^5 + 3x^3 - 3x^{\frac{1}{2}})dx$	7.	$\int (\frac{4}{5}x - \frac{2}{3}x^7 + x^{\frac{5}{2}})dx$
3.	$\int (-x^6 - 5x^4 + 3x)dx$	8.	$\int (1.4x^4 + 3.2x^{-3} + 0.5x)dx$
4.	$\int (x^5 - 9x^{-7} - x^{\frac{3}{4}})dx$	9.	$\int (5x^{-6} + \frac{5}{4}x^{\frac{3}{4}} + 3x^{-\frac{5}{2}})dx$
5.	$\int (x^{1/5} + 5x^{1/2} + 0.2x^{-\frac{1}{3}})dx$	10.	$\int (\frac{1}{3}x^{-\frac{1}{6}} + 8x^{-9} - 5x^{\frac{3}{2}})dx$

Example 10.B

Find **a)** $\int \cos 5x \, dx$ **b)** $\int e^{-3x} dx$ **c)** $\int \frac{7}{(2x-5)} dx$

Solution

a) $\int \cos 5x \, dx = \frac{\sin 5x}{5} + c$

b) $\int e^{-3x} dx = \frac{e^{-3x}}{-3} + c = -\frac{1}{3} e^{-3x} + c$

c) $\int \frac{7}{(2x-5)} dx = \frac{7}{2} \ln(2x-5) + c$

Exercises 10.2

Integrate the following

1. $\int 7 \cos 4x \, dx$
2. $\int 4e^{3x} \, dx$
3. $\int \frac{8}{2x+3} \, dx$
4. $\int -5e^{-2x} \, dx$
5. $\int 6 \sin \frac{1}{2} x \, dx$
6. $\int \frac{-2}{x} \, dx$
7. $\int 8 \cos \frac{1}{4} x \, dx$
8. $\int -1.2e^{0.5x} \, dx$
9. $\int \frac{1}{x^5} \, dx$

10.2 Calculating the constant of integration C

Before we move to a different type of integration where there is no constant of integration C. We need to know how to calculate C.

Example 10.C

If the gradient of a curve, which passes through the point (3, 1) is given by $2x^2 + 5x$, find the equation of the curve.

Solution

$$\frac{dy}{dx} = 2x^2 + 5x,$$

so $$y = \int (2x^2 + 5x)dx = \tfrac{2}{3}x^3 + \tfrac{5}{2}x^2 + C$$

From the point (3, 1) the constant can be found from

$$y = \tfrac{2}{3}x^3 + \tfrac{5}{2}x^2 + C$$

$$1 = \tfrac{2}{3}(3)^3 + \tfrac{5}{2}(3)^2 + C$$

$$C = 1 - 18 - \frac{45}{2} = \frac{-79}{2}$$

The equation of the curve is therefore given by,

$$y = \tfrac{2}{3}x^3 + \tfrac{5}{2}x^2 - \tfrac{79}{2}$$

Exercises 10.3

1. If the gradient of a curve, which passes through the point (1, 3) is given by $5x^3 + 2x$, find the equation of the curve.

2. If the gradient of a curve, which passes through the point (0, -1) is given by $\tfrac{1}{4}x^3 + \tfrac{2}{3}x^2 - \tfrac{1}{2}x$, find the equation of the curve.

3. If the gradient of a curve, which passes through the point ($\tfrac{\pi}{2}$, -1) is given by $2\cos 3x$, find the equation of the curve.

10.3 Definite integrals

So far we deal with indefinite integrals where the constant C can be seen in the result. In this section we introduce definite integrals, so called because the result will be a definite answer, usually a number, with no constant of integration. Definite integrals have many applications, for example in finding areas bounded by curves and finding volumes of solids.

Definite integrals can be recognised by numbers written to the upper and lower right of the integral sign.

$$\int_a^b f(x)\,dx$$

It is called the **definite integral of f(x) from a to b**. The numbers a and b are known as the **lower and upper limits of the integral**.

Example 10.D

Find **a)** $\int_1^4 x^2\,dx$

$$\int_1^4 x^2\,dx = \left[\frac{x^3}{3}\right]_1^4$$

Step 1 Here we have integrated x^2 and we get $x^3/3$
The term $x^3/3$ is written in square brackets to indicate that we have to evaluate it.

Step 2 We evaluate $x^3/3$ at x = 4 (the upper limit of integration) and subtract the the value of $x^3/3$ at x = 1 (the lower limit of integration)

$$= \left(\frac{4^3}{3}\right) - \left(\frac{1^3}{3}\right) = \frac{64}{3} - \frac{1}{3} = \frac{63}{3} = 21$$

b) $\int_0^{\frac{\pi}{2}} \sin 2x\,dx$

Step 1 $$\int_0^{\frac{\pi}{2}} \sin 2x\,dx = \left[-\frac{1}{2}\cos 2x\right]_0^{\frac{\pi}{2}}$$

Step 2 $$= \left(-\frac{1}{2}\cos\pi\right) - \left(-\frac{1}{2}\cos 0\right) = 1$$

c) $\int_2^3 e^{2x}\,dx$

Step 1 $$\int_2^3 e^{2x}\,dx = \left[\frac{1}{2}e^{2x}\right]_2^3 = \left(\frac{1}{2}e^{2*3}\right) - \left(\frac{1}{2}e^{2*2}\right)$$

Step 2 $= 0.5e^6 - 0.5e^4 = 174.4$ to one decimal place.

Exercises 10.4

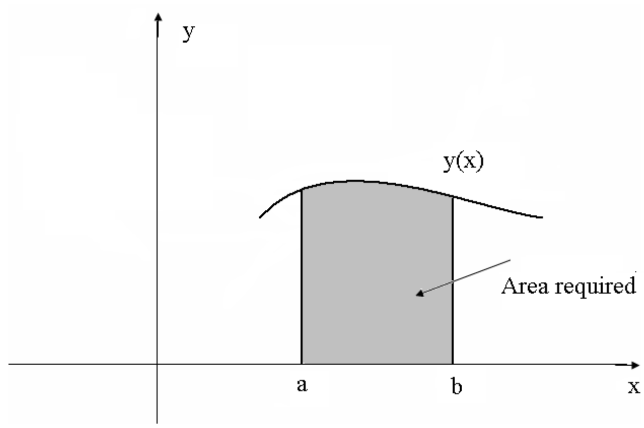
Evaluate the following definite integrals, giving the answers correct to two decimal places where appropriate.

1. $\int_0^{\frac{\pi}{4}} -7\sin 2x\,dx$
2. $\int_1^3 \frac{5}{x}\,dx$
3. $\int_2^3 x^{-2}\,dx$
4. $\int_1^3 12e^{-3x}\,dx$
5. $\int_0^2 15e^{5x}\,dx$
6. $\int_1^8 x^{\frac{1}{3}}\,dx$
7. $\int_0^{\frac{\pi}{6}} -2\cos 3x\,dx$
8. $\int_1^3 (5x^4 - 6x^2)\,dx$

10.4 Application of integration (area under a curve)

One of the important applications of integration is to find the area bounded by a curve. Consider the graph of the function $y(x)$ shown below. We are interesting in calculating the area underneath the graph and above the x-axis, between the points where $x = a$ and $x = b$. This area (area entirely above the x-axis) is given by the definite integral.

$$A = \int_a^b y(x)\,dx$$

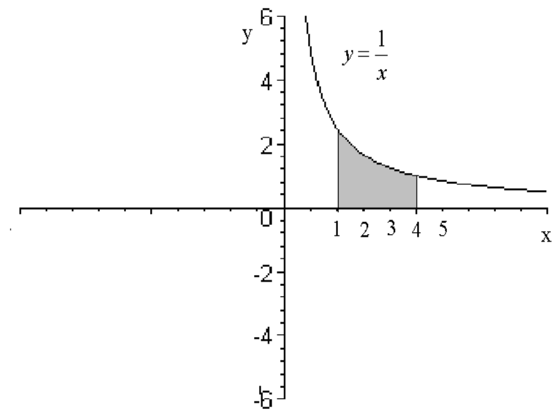


Example 10.E

Calculate the area between the curve $y = \frac{1}{x}$ and the x axis between the values $x = 1$ and $x = 4$.

Solution

$$A = \int_1^4 \frac{1}{x}\,dx$$
$$= \left[\ln(x)\right]_1^4$$
$$= \ln 4 - \ln 1$$
$$= \ln 4 \quad \textit{where}\ \ln 1 = 0$$
$$= 1.386\ (3\,d.p.)$$
square units



Exercises 10.5

Answer correct to two decimal places, where appropriate.

1. Calculate the area between the curve $y = -5 + 6x - x^2$ and the x-axis between the values $x = 1$ and $x = 5$.

2. Calculate the area between the curve $y = 8x - 6x^2 + x^3$ and the x-axis between the values $x = 0$ and $x = 2$.

3. Calculate the area between the curve $y = 5\cos 6x$ and the x-axis between the values $x = 0$ and $x = \frac{\pi}{12}$.

4. Calculate the area between the curve $y = \frac{4}{x}$ and the x-axis between the values $x = 1$ and $x = 4$.

5. Calculate the area between the curve $y = 0.8\,e^{2x}$ and the x-axis between the values $x = 0.2$ and $x = 0.5$.

6. Calculate the area between the curve $y = 7\sin 3x$ and the x-axis between the values $x = \frac{\pi}{6}$ and $x = \frac{5\pi}{6}$.

7. Calculate the area between the curve $y = 4\sqrt{x}$ and the x-axis between the values $x = 1$ and $x = 9$.

8. Calculate the area between the curve $y = 1 + 6x^{-2}$ and the x-axis between the values $x = 1$ and $x = 4$.

Key points from Chapter 10

Answers to Exercises

Answers to exercises

Exercise 1.1

1.

a) $x + 4 = 7$

b) $2x = 5$

c) $\frac{y}{2} = 8$

d) $y - 6 = 4$

e) $\frac{x}{4} = 2$

2.

a) Let C be the cost of the rope (in £) and l be the length of the rope (in m)

$$C = 0.9 * l \longrightarrow C = 0.9l$$

b) Let D be the distance travelled (in miles), v be the average velocity and t be the time taken (in hrs)

$$D = v * t$$

c) Let c be the concentration (in g/ml), m be the mass (in g) and v be the volume (in ml)

$$c = \frac{m}{v}$$

d) Let A be the area (in m^2), b be the length of the base (in m) and h be the height (in m)

$$A = \frac{1}{2}bh$$

e) Let C be the cost (in pence), s be the standing charge (in pence) and u be the number of units used

$$C = s + 0.9u$$

3.

a) 15 b) 5 c) 50 d) 2

4.

$$7x + 4y = 100$$

5.

$$2x + 2 = T$$

6.

$$y = 3x$$

Exercise 1.2

a) 50 b) 13 c) 7 d) 10 e) 8 f) 30

g) 5 h) 16 i) 4 j) 4 k) $\frac{8}{15}$

Exercise 1.3

1. a) 27 b) 81 c) 12 d) 25 e) 12 f) 36

2.

a) $(6u)^3$ b) $\frac{v^2}{2}$ c) $(p+2q)^2$ d) $b^2 - a^3$

Exercise 1.4

a) $6p+3$
b) $6q$
c) $5x+7y$
d) $4z^2+z$, $4z^2$ and z are not like terms.
e) $5a-2b+2c$
f) $5s^2+5r^2+4$
g) $18u+15v+1$
h) $xy+xz+yz$, xy, xz, yz are not like terms.

Exercise 1.5

1. $4x+4y$
2. $30a-50b$
3. $6u+6v-120$
4. $11a+15b$
5. $6s+7t$
6. $p+30q+5r$
7. $2x+8y-5$
8. $3x^2+9y^2+8$
9. x^4-2x^3
10. $x^4-5x^3+2x^2$
11. $2x^2+x$
12. $-7x^2-x$
13. $7x^3+4x^2-5x$

Exercise 1.6

1. $7(u+3v)$
2. $8(2x+3y)$
3. $z(z+3)$
4. $4(2p-5q+3r)$
5. $a(2a-11b)$
6. $yz(9x+8)$

Exercise 1.7

1. $x^2+7x+10$
2. x^2-x-2
3. $x^2-11x+10$
4. y^2-16
5. $z^2+10z+25$
6. s^2-2s+1

Exercise 1.8

1. $(x+3)(x+1)$
2. $(x+5)(x+2)$
3. $(x+6)(x+2)$
4. $(x-6)(x-1)$
5. $(x+2)(x-3)$

Exercise 1.9

1. a) a^5 b) a^2
   c) a^{60} d) a^2
   e) a^7 f) a^{12}
   g) a^5 h) a^7
   i) a^{-2} j) a^8

2. (a) 2^5; (b) 3; (c) x^2;

   (d) 2^{12}; (e) x^6; (f) u.

Exercise 2.1

1. $x = 15$ 2. $x = 17$ 3. $x = 9$ 4. $x = 66$ 5. $x = 5$

Exercise 2.2

1. $x = \dfrac{3}{2}$ 2. $y = 6$ 3. $z = 9$ 4. $u = 19$

5. $t = 3$ 6. $s = 2$ 7. $r = 1$ 8. $q = \dfrac{1}{6}$

Exercise 2.3

1. a) $x = 8$ b) $x = 0$ c) $x = \dfrac{9}{2}$

Exercise 3.1

1. $x = dt$ 2. $x = a - b$ 3. $x = \dfrac{r+u}{t}$ 4. $x = st - r$

5. $x = \dfrac{1}{p}$ 6. $x = vy + 1$ 7 $x = \dfrac{ad}{b}$

Exercise 3.2

1. a) $a = \dfrac{v-u}{t}$ b) 3

2. a) $r = \dfrac{v}{w}$ b) 3.2

3. a) $F = 1.8C + 32$ b) 104

4. a) $V = \dfrac{CT}{P}$ b) 6

5. a) $R = \dfrac{E-v}{I}$ b) 2.8

6. $\mathrm{v} = \dfrac{\mathrm{r-q}}{\mathrm{t}}$

7. a) $u = \dfrac{mv - Ft}{m}$ b) 8

8. a) $v = \dfrac{2S - ut}{t}$ b) 6

9. a) 0.9 b) $x = \dfrac{5(T+\lambda)}{\lambda}$ c) 30

10. a) 105cm^2 b) $\mathrm{h} = \dfrac{\mathrm{A} - 2\pi \mathrm{r}^2}{2\pi \mathrm{r}}$ c) 2.24cm

11. a) 12cm b) $b = \sqrt{c^2 - a^2}$ c) 40cm

12. a) 4ms^{-1} b) $m = \dfrac{2E}{v^2}$ c) 0.04kg

13. a) $g = \dfrac{4\pi^2 I}{T^2}$ b) 9.81 ms^{-2}

14. a) $h = \dfrac{V}{\pi r^2}$ b) 5m

Exercise 3.3

1. a) $x = \dfrac{y - 6a}{a}$ b) $x = \dfrac{9 - z}{2}$ c) $x = \dfrac{6v - 2u}{3}$

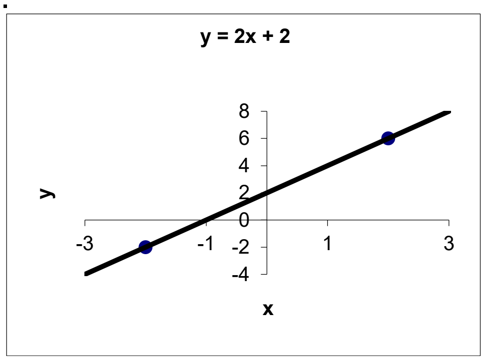
   d) $x = \dfrac{a}{a - 2}$ e) $x = \dfrac{3 - r}{p^2 + q^2}$ f) $x = \dfrac{2u - st}{2 - s}$

Exercise 4.1

1. $y = x$
2. $y = x + 2$
3. $y = x - 1$
4. $y = 2x$
5. $y = 2x + 1$

Exercise 4.2

1.



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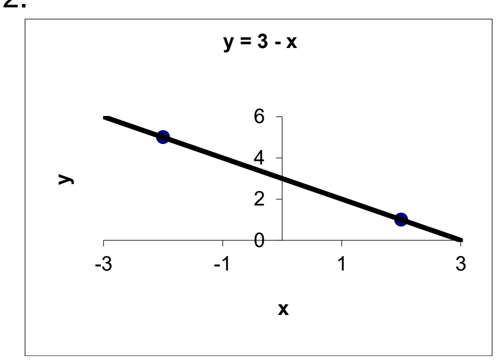

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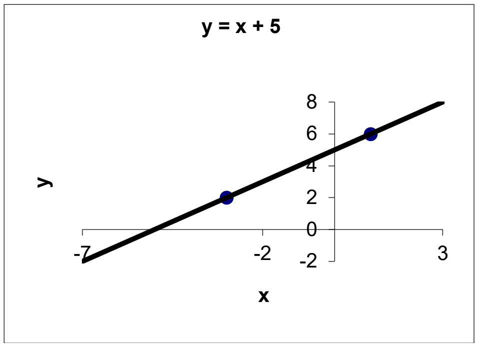

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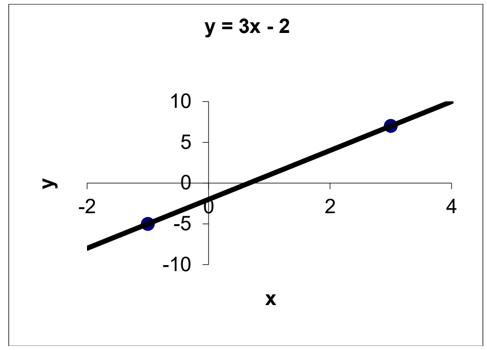

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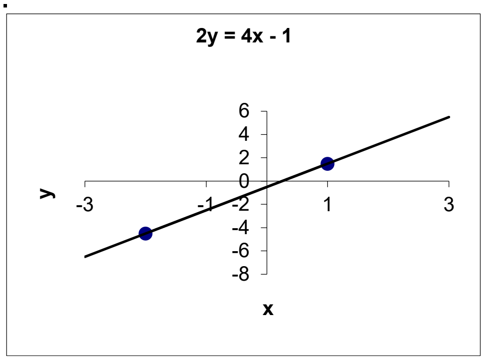


Exercise 4.3

AB = $\sqrt{4^2+2^2} = \sqrt{20} = 4.472$

AC = $\sqrt{5^2+6^2} = \sqrt{61} = 7.810$

AD = $\sqrt{1^2+7^2} = \sqrt{50} = 7.071$

BC = $\sqrt{1^2+4^2} = \sqrt{17} = 4.123$

BD = $\sqrt{3^2+5^2} = \sqrt{34} = 5.831$

CD = $\sqrt{1^2+4^2} = \sqrt{17} = 4.123$

Exercise 4.4

1. gradient = 2, y-intercept = 3
2. gradient = –1, y-intercept = –2
3. gradient = ½, x-intercept= 2, y-intercept = –1

4.

a) gradient = 5 x-intercept = $-2/5$ and y-intercept = 2

b) gradient = –3 x-intercept = $-1/3$ and y-intercept = -1

c) gradient = –1 x-intercept = 3 and y-intercept = 3

d) gradient = 1/2 x-intercept = 4 and y-intercept = 2

e) gradient = 1/2 x-intercept = -18 and y-intercept = 9

f) gradient = 3/5 x-intercept = $-4/3$ and y-intercept = $4/5$

g) gradient = $-1/2$ x-intercept = $-5/2$ and y-intercept = $-5/4$

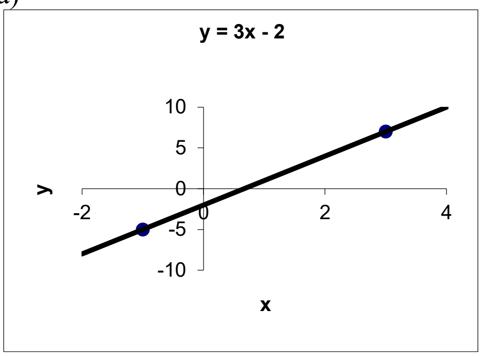
h) gradient = 7 x-intercept = 0.5 and y-intercept = -3.5

i) gradient = –3 x-intercept = 0.1333 and y-intercept = 0.4

j) gradient = 6 x-intercept = $2/3$ and y-intercept = -4

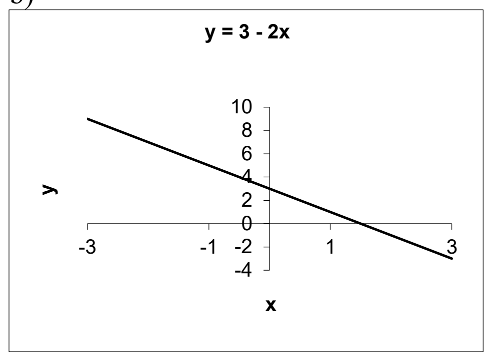
5.

a)



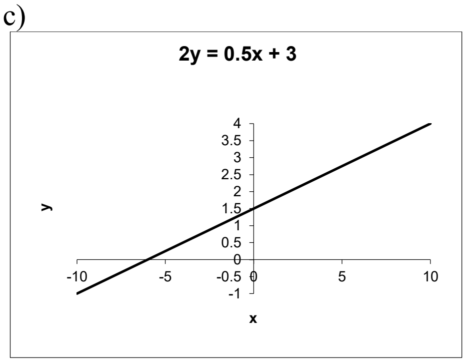
gradient = 3, intercept = –2

b)



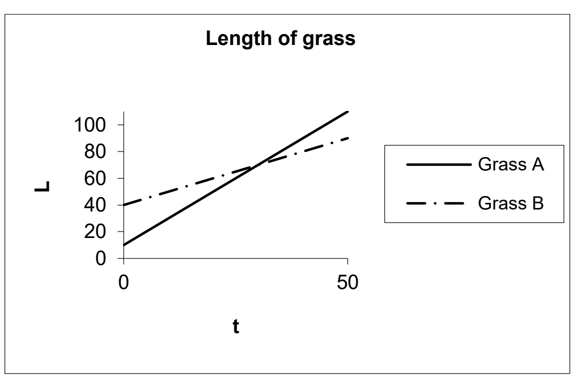
gradient = -2, intercept = 3

c)

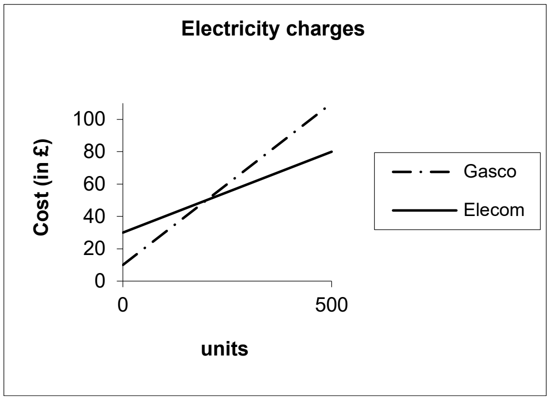


gradient = ¼, intercept = 3/2

6. Lengths of grasses are equal when $t = 30$ days.



7. Companies charge the same when 200 units consumed. Gasco is cheaper when less than 200 units are used.



Exercise 4.5

1. $C = 12 + 0.11u$ (everything is in £)

2. gradient = - 1 cm/h (candle is getting shorter) intercept = 10cm
   Equation of line $L = -t + 10 \rightarrow L = 10 - t$.

3. x = distance from home (m) and t = time taken (s)
   $x = 500 - 2t$ valid for $0 \le t \le 250$

Exercise 4.6

1. a) $y = 2x + 4$
   b) $y = 4x - 1$
   c) $y = \frac{4}{3}x - \frac{10}{3} \quad \rightarrow \quad 3y = 4x - 10$
   d) $m_1 = \frac{1}{2}$, $m_1 = m_2$ (parallel), $y = \frac{1}{2}x + \frac{5}{2}$
   e) $m_1 = -2$, $m_2 = -\frac{1}{m_1}$ (perpendicular), $y = -2x$

Exercise 4.7

1. $x = 3$, $y = 4$
2. $x = 2$, $y = 5$
3. $x = 1$, $y = 7$
4. $x = -1$, $y = 3$
5. $x = -4$, $y = -2$
6. $x = 0$, $y = 5$
7. $x = 6$, $y = -2$
8. $x = -1$, $y = 2$
9. $x = -1$, $y = -1$
10. $x = 2$, $y = -1$

Exercise 4.8

1. $c_s = 12 + 0.11u$

   $c_m = 15 + 0.098u$

   If two companies charge the same $c_s = c_m$ i.e. when u = 250

   $12 + 0.11u = 15 + 0.098u$

   $0.012u = 3$

   $u = 250$

   $c_s = 12 + 0.11 \times 250 = 39.5 = c_m$

2. Let c be the number of chickens and d be the number of sheep

   $c + d = 19$

   $2c + 4d = 46$ (chickens have two legs and sheep have four legs)

   Multiply first equation by 2

   $2c + 2d = 38$

   $2c + 4d = 46$

   $38 - 2d = 46 - 4d$

   $4d - 2d = 46 - 38$

   $d = 4, c = 15$

3. Let E be the charge from Eurohammer

   Let J be the charge from Jackhurt

   Let t be the length of the hire

   Then

   $E = 4t + 10$

   $J = 5.5t$

   Equal cost when $J = E$

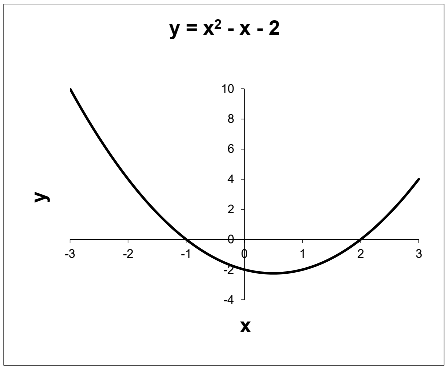
   i.e $4t + 10 = 5.5t$

   $1.5t = 10$

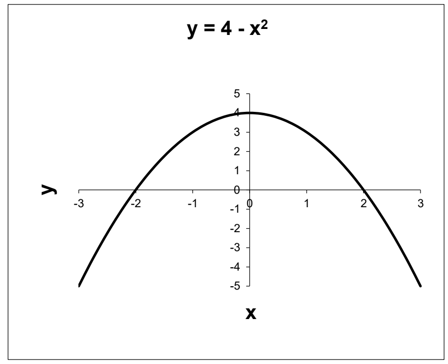
   $t = 6\frac{2}{3}$ days

Exercise 5.1

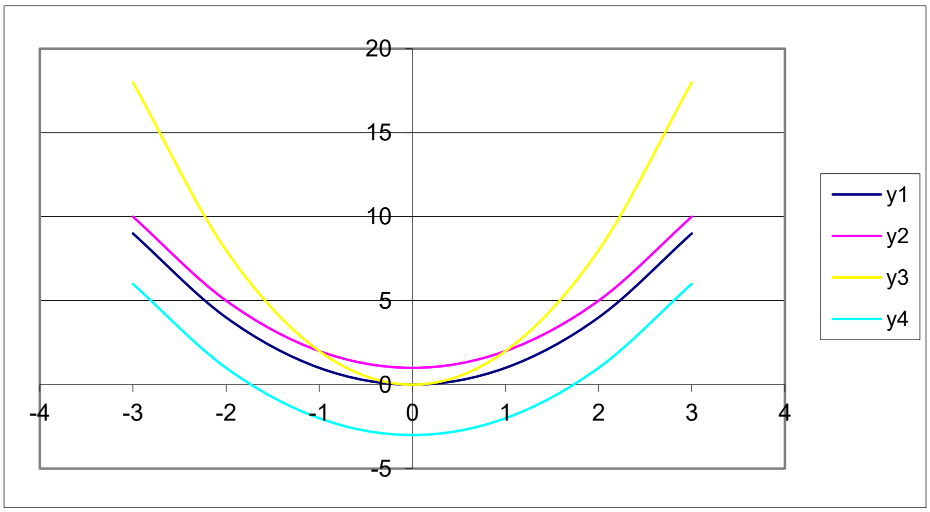
a)

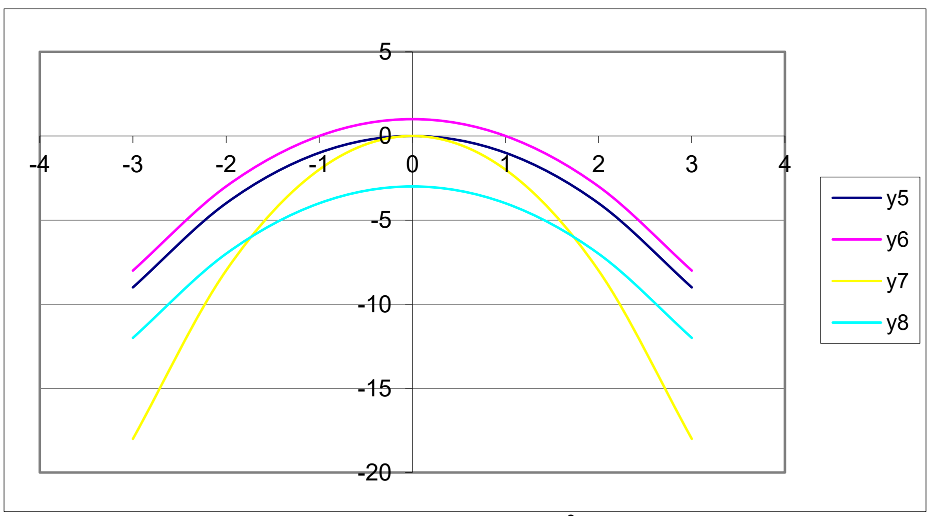


b)



Exercise 5.2





a) 1 to 4 are all cup-shaped – coefficient of x^2 is +ve
5 to 8 are all cap-shaped – coefficient of x^2 is –ve

b) 1, 2 and 4 are al the same shape but are moved vertically relative to each other

c) 3 is a tighter curve – coefficient of x^2 is 2

Exercise 5.3

1. x = 3/2 or x = 1
2. x = ½ or x = –7
3. x = 2 or x = –5/2
4. x = –1/3 or x = –1
5. x = 4/3 or x = –1
6. x = 2/3 or x = 3
7. x = 5/2 or x = –3/2
8. x = 3/5 or x = –3
9. x = 2 or x = –9/2
10. x = 2/3 or x = 3/2

Exercise 5.4

1. x = 1 or x = 4
2. x = 2 or x = 4
3. x = –1 or x = 8
4. x = 3 or x = 4
5. x = 9 or x = -9
6. x =3/2 or x = -3/2
7. x = 4 or x = ½
8. x = -1/3 or x = –3/2
9. x = –7/3 or x = 2
10. x = 1/4 or x = 3/2

Exercise 5.5

1. $x(x+3)=75$

$$x^2+3x=75$$

$$x=\frac{-3+\sqrt{309}}{2}=7.289\,,\ x=\frac{-3-\sqrt{309}}{2}=-10.289$$

2. Two consecutive even integer numbers are: x and $x+2$

$$x^2=3(x+2)+4$$
$$x^2=3x+6+4$$
$$x^2-3x-10=0$$
$$(x-5)(x+2)=0$$
$$x=5 \quad or \quad x=-2$$

since 5 is not even integer, the solution is $x=-2$ and $x+2=-2+2=0$.The consecutive even integers are -2 and 0.

3. $N(T)=20T^2-20T-120$, when the number of bacteria equal zero, we have

$$20T^2-20T-120=0$$
$$T^2-T-6=0$$
$$(T-3)(T+2)=0$$
$$T=3 \quad or \quad T=-2$$

Exercise 6.1

1. H² = 4² + 2² = 20 ... <u>H = 4.47</u> $H^2 = 4^2 + 2^2 = 20 \quad \to H = 4.47$

2. $\tan(F^\circ) = \dfrac{2}{4} \quad \to \quad F^\circ = \tan^{-1}(0.5) = 26.57^\circ$

3. $\sin(G^\circ) = \dfrac{4}{4.47} \quad \to \quad G^\circ = \sin^{-1}(0.89485) = 63.49^\circ$

4. 90° + 63.49° + 26.57° = **<u>180</u>.06** (close enough!)

5. $\cos(F^\circ) = \dfrac{4}{4.7} \quad \to \quad F^\circ = \cos^{-1}(0.89485) = 26.51^\circ$ (close enough!)

Exercise 6.2

1. **(a)** 22.84 cm **(b)** 24.57 cm **(c)** 7.41 cm

2. **(a)** 36° **(b)** 43° **(c)** 38°

Exercise 6.3

1.

a) π b) $\dfrac{\pi}{2}$ c) $\dfrac{\pi}{4}$ d) $\dfrac{137\pi}{180}$

2.

a) 60° b) 30° c) 15° d) 135°

Exercise 6.4

1.

(a) 1; (b) 0; (c) error; (d) 0.7071; (e) 0.7071;

(f) 30°; (g) 11.31°; (h) 0.5; (i) 95.74°; (j) 0.

2.

(a) 1; (b) -1; (c) 2.5722; (d) -0.2955; (e) -0.98999;

(f) 0.6442; (g) 0.1974; (h) 1.5708; (i) 1.0472; (j) 0.

Exercise 6.5

1. **(a)** 18.1 miles **(b)** 8.5 miles **(c)** 4.9 miles **(d)** 23.0 miles

2. **(a)** 131 m **(b)** 55° **(c)** 128 m **(d)** 44°

3. **(a)** 14 cm **(b)** 14 cm **(c)** 49 cm **(d)** 63 cm

Exercise 7.1

1. 1, 17, 23
2. -9, 3, 19
3. 6, 3, -6
4. 46, 4, 28
5. 8, 32, 113

Exercise 7.2

(a) $y = f^{-1}(x) = x - 1$; **(b)** $y = f^{-1}(x) = \frac{x}{2} - \frac{3}{2}$; **(c)** $y = f^{-1}(x) = \frac{x-1}{7}$

(d) $y = f^{-1}(x) = -\frac{x}{3} + \frac{5}{3}$; (e) $y = f^{-1}(x) = 3x + 3$; (f) $y = f^{-1}(x) = 2 + x^2$

Exercise 7.3

1. a) 19.953 b) 3.981
2. a) 5042.768 b) 9.190 c) 0.871
3. a) 0.459 b) 87.847
4. a) 2.000 b) 0.125

Exercise 7.4

1. £474.16
2. 24 °C
3. i) 10.87615 ii) 23.60816 c) 37.92723
   Greatest ever speed of typing is 60 words per minute.

Exercise 7.5

1. a) 0.60206 b) 1.60206 c) 2.60206
2. a) 1.146128 b) 2.146128 c) 3.146128
3. a) 0.778151 b) -0.22185 c) -1.22185 d) -2.2218

Exercise 7.6

1. a) 1.60944 b) 1.50408 c) 2.30259
2. a) 0.40547 b) 0.69315 c) -0.51083

Exercise 7.7

1. $2^2 \times 2^3 \div 2^4 = 2^{2+3-4} = 2^1 = 2$
2. $5^2 \times \left(5^2\right)^3 = 5^{2+2\times 3} = 5^8$
3. $10^3 \times \left(10^{-2}\right)^4 \times \sqrt[3]{10^6} = 10^{3+(-8)+2} = 10^{-3} = 0.001$
4. $\log 4 + \log 3 - \log 2 = \log\left(4 \times 3 \div 2\right) = \log 6$

5. $\log 4 - 2\log 3 + 3\log 5 = \log 4 - \log 3^2 + \log 5^3 = \log\left(\frac{4\times 5^3}{3^2}\right) = \log 55.56$

6. $\log 5 - 2\log 4 + \frac{1}{2}\log 9 = \log 5 - \log 4^2 + \log 9^{\frac{1}{2}} = \log\left(\frac{5\times 3}{16}\right) = \log 0.94$

Exercise 7.8

1. (a) 3; (b) 6.908; (c) 0.693.
2. (a) $\log\frac{75}{4}$; (b) $\log\frac{1600}{343}$; (c) $\log\frac{x^2}{y^4}$; (d) $\log u^{\frac{1}{4}} v^{\frac{15}{4}}$.
3. (a) $x = 7$; (b) $y = 4$; (c) $x = 3$;
   (d) $x = 1.262$; (e) $x = 3$; (f) 0.805.
4. (a) $x = 4$; (b) $x = \frac{5}{8}$; (c) $x = 36.399$; (d) 18.552.

Exercise 7.9

1. a) 3.807 b) 12 c) 10 d) 6.833 e) 7.833

2. a) $2\log 5 + 5\log 2 = \log(5^2 * 2^5) = \log(800)$

   b) $\log 42 - \log 7 = \log(42/7) = \log 6$

3. a) $\log_{10} 9 = \log_{10} 3^2 = 2\log_{10} 3 = 0.9542$

   b) $\log_{10} 90 = \log_{10}(9*10) = \log_{10} 9 + \log_{10} 10 = 0.9542 + 1 = 1.9542$

4. $T_0 = 15 - -20 = 35$ and finally $T = 15 - 0 = 15$

$$15 = 35 * e^{0.05t}$$

$$\frac{35}{15} = e^{0.05t}$$

$$\ln(\frac{7}{3}) = 0.05t$$

$$t = 20 * \ln(\frac{7}{3}) = 16.94\,hr$$

new fridge $t = 24.2\,hr$ t = 24.2 h

5.

$$R_0 = R_o e^{-1.2\times 10^{-5}t}$$

$$1/2 = e^{-1.2\times 10^{-5}t} \qquad t = \ln 2/(1.2\times 10^{-5})$$

t = 0.577 x 10^5 days = 158.25 years

Exercise 8.1

i) 0 ii) 2 iii) -2 iv) 4
v) -4 vi) 6 vii) -6

Exercise 8.2

1. a) $2x$
   b) $4x-1$
   c) $2x^2+x-1$
   d) $\frac{3}{2}x^{-\frac{1}{2}}$
2. a) $\frac{dy}{dx}=4x^3$ b) $\frac{dy}{dx}=14x^{13}$ c) $\frac{dy}{dx}=1$ d) $\frac{dy}{dx}=0$
   e) $\frac{dy}{dx}=-1x^{-2}$ f) $\frac{dy}{dx}=-3x^{-4}$ g) $\frac{dy}{dx}=-2x^{-3}$ h) $\frac{dy}{dx}=1.4x^{0.4}$
   i) $\frac{dy}{dx}=0.2x^{-0.8}$ j) $\frac{dy}{dx}=-1.6x^{-2.6}$ k) $\frac{dy}{dx}=-0.7x^{-1.7}$ l) $\frac{dy}{dx}=\frac{4}{3}x^{\frac{1}{3}}$
   m) $\frac{dy}{dx}=-\frac{1}{2}x^{-\frac{3}{2}}$

3. a) $\frac{ds}{dt}=6t$ b) $\frac{da}{db}=-4.2b^{-1.3}$
   c) $\frac{df}{dt}=t^{-\frac{3}{2}}$ d) $\frac{dw}{dy}=23$

4. a) $y=7x^{-\frac{1}{2}}+3x^{-1}$ $\frac{dy}{dx}=-\frac{7}{2}x^{-\frac{3}{2}}-3x^{-2}$
   b) $y=9x^{-\frac{1}{3}}-4x^{-2}$ $\frac{dy}{dx}=-\frac{9}{3}x^{-\frac{4}{3}}+8x^{-3}$
   c) $y=5x^{-\frac{3}{2}}-3x^{-7}$ $\frac{dy}{dx}=-\frac{15}{2}x^{-\frac{5}{2}}+21x^{-8}$

5. a) $\frac{dy}{dx}=4x-7$ b) $\frac{dy}{dx}=-\frac{6}{x^3}$
   c) $\frac{dy}{dx}=12x^2+\frac{2}{x^3}$ d) $\frac{dy}{dx}=-\frac{2}{x^3}-\frac{1}{x^2}$
   e) $\frac{dy}{dx}=-\frac{1}{x^{\frac{3}{2}}}-\frac{3}{2x^{\frac{1}{2}}}$ f) $\frac{dy}{dx}=3.4x^{0.7}-0.6x^{-0.8}$
   g) $\frac{dy}{dx}=3.6x^{-0.1}+9.8x^{0.4}-0.8x^{-0.9}-\frac{1}{2\sqrt{x^3}}$

Exercise 8.3

1. a) $\frac{ds}{dt} = 16 - 2t$ b) $\frac{ds}{dt} = 3t^2 - 4t$ c) $\frac{ds}{dt} = u + at$

2. a) $\frac{ds}{dt} = 2t, \quad t = 4 \rightarrow \frac{ds}{dt} = 8$ b) $\boldsymbol{\frac{ds}{dt} = -4t^{-3}, t = 3 \rightarrow \frac{ds}{dt} = -\frac{4}{27}}$

c) $\frac{ds}{dt} = 0.5t^{-0.5}, \quad t = 9 \rightarrow \frac{ds}{dt} = \frac{1}{6}$ d) $\frac{ds}{dt} = -20t^{-2}, \quad t = 5 \rightarrow \frac{ds}{dt} = \frac{-4}{5}$

e) $\frac{ds}{dt} = -0.5t^{-1.5}, t = 4 \rightarrow \frac{ds}{dt} = -\frac{1}{16}$ f) $\frac{ds}{dt} = 4t + 1, \quad t = 3 \rightarrow \frac{ds}{dt} = 13$

Exercise 8.4

1. $\frac{dy}{dx} = 6x$ $\left.\frac{dy}{dx}\right|_{x=2} = 12$
2. $\frac{dy}{dx} = 6x^2 - 10x$ $\left.\frac{dy}{dx}\right|_{x=1} = 6 - 10 = -4$
3. $\frac{dy}{dx} = -5x^{-2} - 6x^{-3} - 21x^{-4}$ $\left.\frac{dy}{dx}\right|_{x=-1} = -20$
4. $\frac{dy}{dx} = -\frac{1}{2}x^{-4} - 2x^{-2} - 0.4x^{-0.6}$ $\left.\frac{dy}{dx}\right|_{x=1} = -2.9$
5. $\frac{dC}{dx} = -2ax^{-3}$ $\left.\frac{dC}{dx}\right|_{x=1} = -2a$

   $\frac{dC}{dx}$ is always negative for positive x so C is decreasing as x increases.

Exercise 8.5

1. $y = 12x - 15$ 2. $y = -4x + 2$ 3. $y = -x + 5$
4. $y = 3x + 1$ 5. $y = 7x - 8$ 6. $y = 2.5x - 1.5$

Exercise 8.6

1. a) $\frac{d^2y}{dx^2} = 6x$ b) $\frac{d^2y}{dx^2} = 36x^2$ c) $\frac{d^2y}{dx^2} = 42x^5$ d) $\frac{d^2y}{dx^2} = 2x^{-3}$

   e) $\frac{d^2y}{dx^2} = 3x^{-0.5}$ f) $\frac{d^2y}{dx^2} = 30x^{-3.5}$

2.

a) $\frac{d^2y}{dx^2}=2, \quad x=3 \rightarrow \frac{d^2y}{dx^2}=2$ b) $\frac{d^2y}{dx^2}=6x, \quad x=1 \rightarrow \frac{d^2y}{dx^2}=6$

c) $\frac{d^2y}{dx^2}=0, \quad x=13 \rightarrow \frac{d^2y}{dx^2}=0$ d) $\frac{d^2y}{dx^2}=-0.25x^{-1.5}, \quad x=0.25 \rightarrow \frac{d^2y}{dx^2}=-2$

3. $\frac{dy}{dx}=-12x^{-4}+8x^7$; $\frac{d^2y}{dx^2}=48x^{-5}+56x^6$; $\frac{d^3y}{dx^3}=-240x^{-6}+336x^5$.

4. a) $\frac{d^4y}{dx^4}=24$; b) $\frac{d^4y}{dx^4}=180x^2$;

c) $\frac{d^4y}{dx^4}=-\frac{15}{8}x^{-\frac{7}{2}}+\frac{105}{4}x^{-\frac{9}{2}}$; d) $\frac{d^4y}{dx^4}=-0.09918x^{-3.9}$.

Exercise 8.7

1. $\frac{dy}{dx}=2x-3$; minimum at x = 1.5
2. $\frac{ds}{dt}=2t+6$; minimum at t = -3
3. $\frac{dy}{dx}=5-4x$; maximum at x = 1.25
4. $\frac{ds}{dt}=2t-6$; minimum at t = 3
5. $\frac{dy}{dx}=6x^2-30x+24$; maximum at x = 1; minimum at x = 4
6. $\frac{ds}{dt}=6t^2+6t-36$; maximum at t = -3; minimum at t = 2

Exercise 8.8

1. Maximum population is 9 million locusts after 1 week
2. Maximum amount of drug is 12.25 ml after 2.5 hours
3. The maximum height of the grass is 16 cm after 4 weeks
4. Two turning points at x=3.155 (local min) and x=0.845 (local max)

5. Let x be the side of the base of the box, and h be the height
   - The volume is

     $v=x^2h$

     $1000=x^2h$
   - The surface area A is

$$A = base + 4\ sides$$
$$A = x^2 + 4xh$$
$$A = x^2 + 4x(\frac{1000}{x^2})$$
$$A = x^2 + \frac{4000}{x}$$

- The derivative is

$$\frac{dA}{dx} = 2x - 4000x^{-2}$$
$$2x - 4000x^{-2} = 0$$
$$x^3 = 2000$$
$$x = 12.6\ cm\ h = 6.3\ cm$$

- The dimensions of the box is

$$x = 12.6\ cm\ h = 6.3\ cm$$

6. The dimensions of the rectangular part are 2r and h

- The total area of the window is
$$A = \frac{1}{2}\pi r^2 + 2hr$$
- The perimeter is
$$P = 2h + 2r + \pi r$$
$$10 = 2h + 2r + \pi r$$
$$h = \frac{10 - 2r - \pi r}{2}$$
- $$A = \frac{1}{2}\pi r^2 + r(10 - 2r - \pi r)$$
$$A = \frac{1}{2}\pi r^2 + 10r - 2r^2 - \pi r^2$$
$$\frac{dA}{dr} = \pi r \ + 10 - 4r \ - 2\pi r$$
- $$10 = 4r + \pi r$$
$$r = \frac{10}{4 + \pi}$$
$$r = 1.4\ metres$$

Exercise 8.9

1. $\frac{dy}{dx} = -21(-3x+1)^6$
2. $\frac{dy}{dx} = 44(4x-1)^{10}$
3. $\frac{dy}{dx} = -16(-2x+2)^7$
4. $\frac{dy}{dx} = 84(12x-3)^6$
5. $\frac{dy}{dx} = -45(-9x+5)^4$
6. $\frac{dy}{dx} = 8(2x-4)^3$
7. $\frac{dy}{dx} = 8(-4x+1)^{-3}$
8. $\frac{dy}{dx} = -200(-10x-13)^{19}$

Exercise 8.10

1. $\frac{dy}{d\theta} = -7\sin 7\theta$
2. $\frac{dy}{d\theta} = 2\cos 2\theta$
3. $\frac{dy}{d\theta} = 4\cos 4\theta$
4. $\frac{dy}{d\theta} = -8\cos 8\theta$
5. $\frac{dy}{d\theta} = -3\sin 3\theta$
6. $\frac{dy}{d\theta} = \frac{1}{3}\sin\frac{\theta}{3}$
7. $\frac{dy}{d\theta} = -8\sin 8\theta$
8. $\frac{dy}{d\theta} = -\frac{1}{2}\cos\frac{\theta}{2}$

Exercise 8.11

1. $\frac{dy}{dx} = 2e^{2x}$
2. $\frac{dy}{dx} = \frac{1}{x}$
3. $\frac{dy}{dx} = -6e^{-6x}$
4. $\frac{dy}{dx} = \frac{7}{x}$
5. $\frac{dy}{dx} = -24e^{8x}$
6. $\frac{dy}{dx} = \frac{2}{x}$
7. $\frac{dy}{dx} = -e^{-0.5x}$
8. $\frac{dy}{dx} = \frac{8}{x}$
9. $\frac{dy}{dx} = \frac{12}{(3x+1)}$
10. $\frac{dy}{dx} = \frac{5}{2}e^{x/2} + \frac{4}{x}$

Exercise 8.12

1. $\frac{dy}{dx} = 6\sin 3x\cos 3x$
2. $\frac{dy}{dx} = \frac{28x^3-1}{7x^4-x}$
3. $\frac{dy}{dx} = -(10x-2)\sin(5x^2-2x+3)$
6. $\frac{dy}{dx} = \frac{-\sin x}{\cos x}\ or -\tan x$
7. $\frac{dy}{dx} = co\,s\,x(e^{\sin x})$
8. $\frac{dy}{dx} = 24x^2\cos(4x^3)$

4. $\dfrac{dy}{dx} = -\dfrac{1}{2}e^{-x/6}$

5. $\dfrac{dy}{dx} = \dfrac{-1}{x}$

9. $\dfrac{dy}{dx} = 2\sin x(e^{-2\cos x})$

10. $\dfrac{dy}{dx} = 2x^{-3}\sin(x^{-2})$

Exercise 8.13

1. $\dfrac{dy}{dx} = 2x(3x^2 - 4) + 2(x^3 - 4x)$

2. $\dfrac{dy}{dx} = x^2 \cos x + 2x\sin x$

3. $\dfrac{dy}{dx} = -e^x \sin x + e^x \cos x$

4. $\dfrac{dy}{dx} = 2xe^{2x} + 2e^{2x}(x^2 - 5) = 2e^{2x}(x + x^2 - 5)$

5. $\dfrac{dy}{dx} = \dfrac{3x^2(x+1) - x^3(1)}{(x+1)^2} = \dfrac{x^2(2x+3)}{(x+1)^2}$

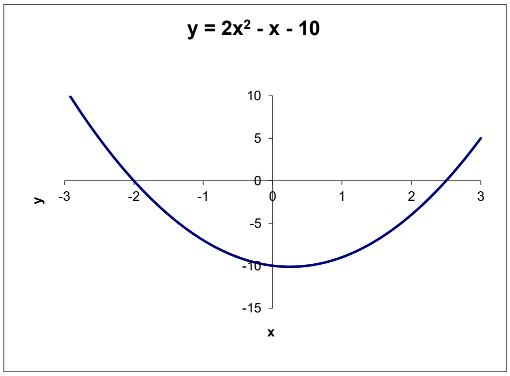
6. $\dfrac{dy}{dx} = \dfrac{xe^x - e^x}{x^2} = \dfrac{e^x(x-1)}{x^2}$

7. $\dfrac{dy}{dx} = \dfrac{x\cos x - \sin x(1)}{x^2} = \dfrac{x\cos x - \sin x}{x^2}$

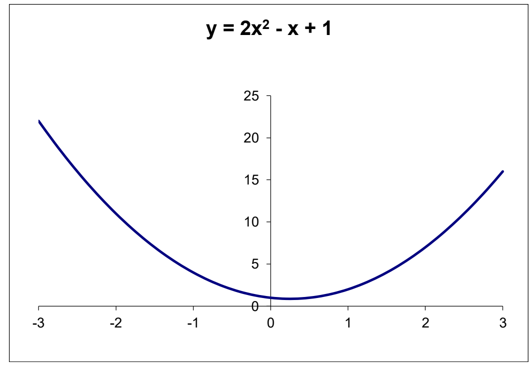
8. $\dfrac{dy}{dx} = \dfrac{(x^2-1)(xe^x + e^x) - xe^x(2x)}{(x^2-1)^2} = \dfrac{e^x[(x^2-1)(x+1) - 2x^2]}{(x^2-1)^2}$

Exercise 9.1

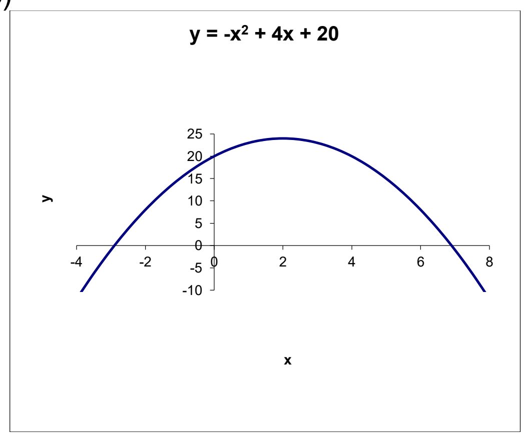
1. a)



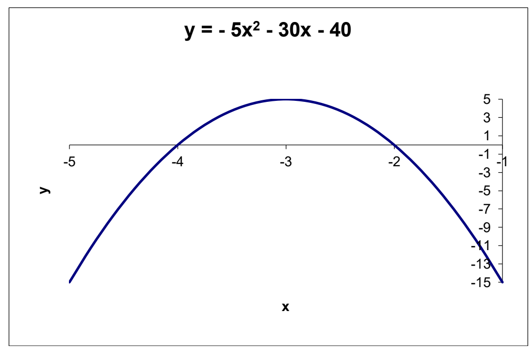
b)



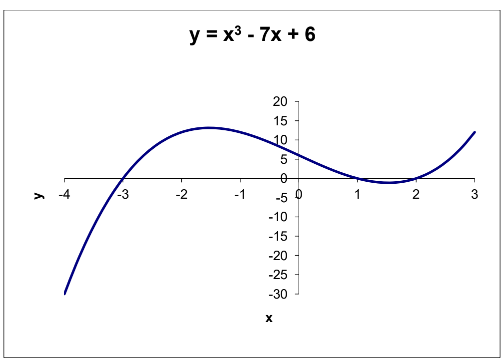
c)

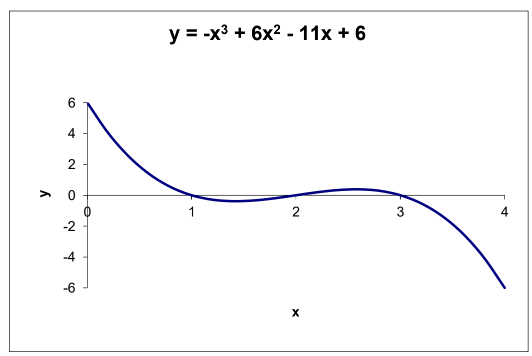


d)



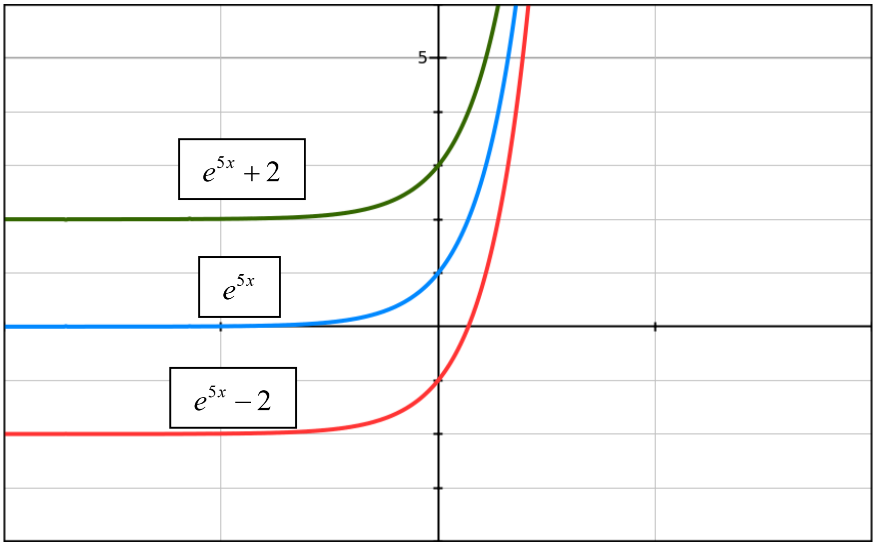
2. a)



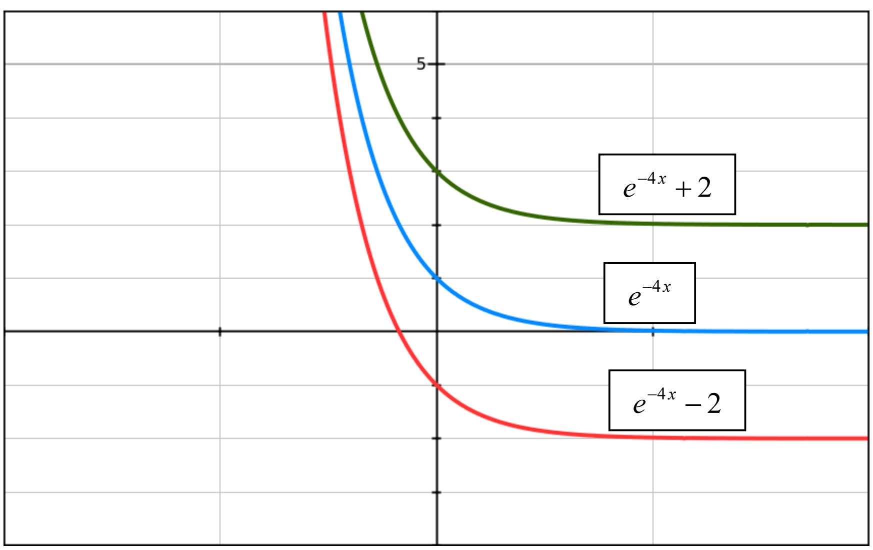


Exercise 9.2

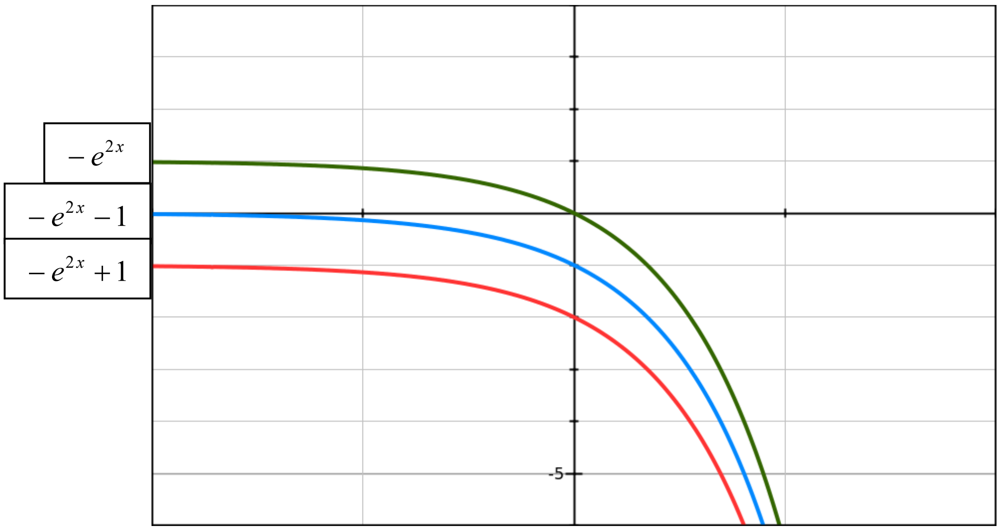
1. (a), (b), (c)



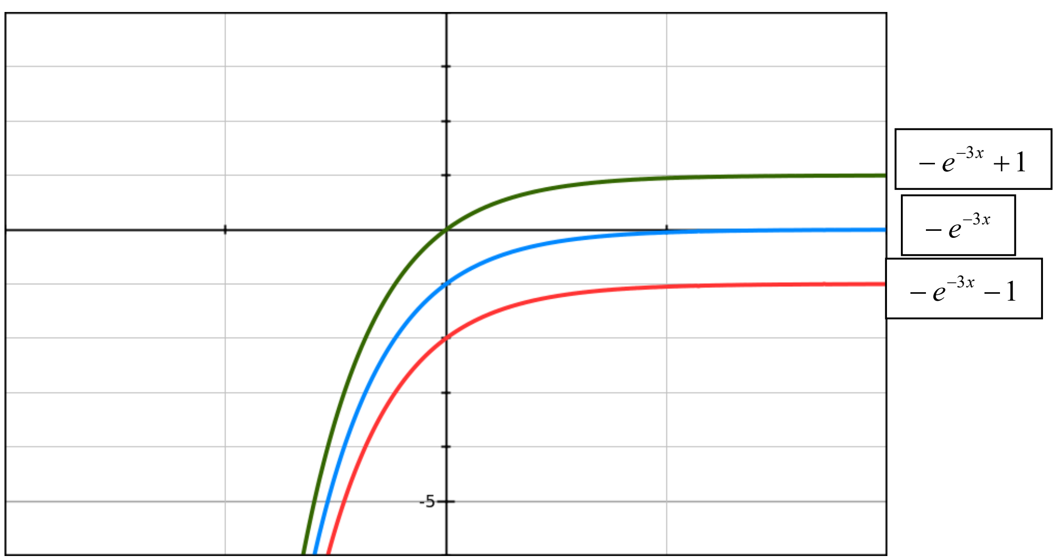
2. (a), (b), (c)



3. (a), (b), (c)



4. (a), (b), (c)



Exercise 10.1

1. $\frac{7}{5}x^5 - \frac{5}{4}x^4 - \frac{1}{2}x^2 + c$
2. $\frac{1}{3}x^6 + \frac{3}{4}x^4 - 2x^{\frac{3}{2}} + c$
3. $-\frac{1}{7}x^7 - x^5 + \frac{3}{2}x^2 + c$
4. $\frac{1}{6}x^6 + \frac{3}{2}x^{-6} - \frac{4}{7}x^{\frac{7}{4}} + c$
5. $\frac{5}{6}x^{\frac{6}{5}} + \frac{10}{3}x^{\frac{3}{2}} + 0.3x^{\frac{2}{3}} + c$
6. $\frac{1}{14}x^7 + \frac{1}{2}x^4 - \frac{2}{5}x^{5/2} + C$
7. $\frac{2}{5}x^2 - \frac{1}{12}x^8 + \frac{2}{7}x^{\frac{7}{2}} + c$
8. $0.28x^5 - 1.6x^{-2} + 0.25x^2 + c$
9. $-x^{-5} + \frac{5}{7}x^{\frac{7}{4}} - 2x^{-\frac{3}{2}} + c$
10. $\frac{2}{5}x^{\frac{5}{6}} - x^{-8} - 2x^{\frac{5}{2}} + c$

Exercise 10.2

1. $\frac{7}{4}\sin 4x + c$
2. $\frac{4}{3}e^{3x} + c$
3. $4\ln\,(2x+3) + c$
4. $\frac{5}{2}e^{-2x} + c$
5. $-12\cos\frac{1}{2}x + c$
6. $-2\ln x + c$
7. $32\sin\frac{1}{4}x + c$
8. $-2.4e^{0.5x} + c$
9. $\frac{-1}{4}x^{-4} + c$

Exercise 10.3

1. $y = \frac{5}{4}x^4 + x^2 + \frac{3}{4}$
2. $y = \frac{1}{16}x^4 + \frac{2}{9}x^3 - \frac{1}{4}x^2 - 1$
3. $y = \frac{2}{3}\sin 3x - \frac{1}{3}$

Exercise 10.4

1. $-\frac{7}{2}$ *or* -3.5
2. 5.49
3. $\frac{1}{6}$ *or* 0.17
4. 0.1987
5. 66076.40
6. $\frac{45}{4}$ *or* 11.25
7. $-\frac{2}{3}$ *or* -0.67
8. 190

Exercise 10.5

1. *10.67* sq. units
2. *4* sq. units
3. *0.83* sq. units
4. *5.55* sq. units
5. *0.49* sq. units
6. *0* sq. units
7. *69.33* sq. units
8. *7.5* sq. units