Auctions - theory and practice

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University of Copenhagen - Department of Economics

Kristian Urup Olesen Larsen

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1 Lecture 1 - Simple private value auctions

A brief motivation: Auctions are widely used to sell items, both in ordinary auctions for antiques, but more importantly theory on auction design is widely used when designing tenders for public projects ranging from construction to medicine prices. Additionally auctions are used as a market mechanism in the advertising market, energy market etc.

Such markets are typically characterized by having only one or few sellers (or buyers in the case of tenders) with many potential buyers interested in winning any given item for sale. In general we can think of auctions as games of incomplete information because bidders valuations or the true value of the sold item is often unknown to participants.

1.1 Bayesian Nash equilibria in the auction setting

Consider a game featuring N bidders, each of which must decide a strategy from their set of potential bids B_i. Each bidder draws a signal x_i from the stochastic variable X_i which distributed according to some density F_i. From this each bidder must choose a strategy $\beta_i : X_i \to B_i$ which gives a complete map from any possible signal to a bid in the set of possible bids B_i. Of course each bidder chooses this strategy with respect to the payoff function $\Pi_i(\boldsymbol{B}, \boldsymbol{X}) \to \mathbb{R}$.

Definition 1.1. (Pure strategy) Nash Equilibrium: A pure strategy Nash equilibrium is a vector $\boldsymbol{\beta}^*$ such that for all bids $b_i \in B_i$ and for all players i:

$$E[\Pi_i(\boldsymbol{\beta}^*(X), X)] \geq E[\Pi_i(\beta_i(X_i), \boldsymbol{\beta}^*_{-i}(X_{-i}), X)] \tag{1}$$

That is it is for no players a priori optimal to deviate from $\boldsymbol{\beta}$.

Naturally an equilibrium like this does not have to be ex post optimal for all bidders, however if this is the case we can write

$$E[\Pi_i(\boldsymbol{\beta}^*(x), x)] \geq E[\Pi_i(\beta_i(x_i), \boldsymbol{\beta}^*_{-i}(x_{-i}), x)] \tag{2}$$

implying such a strategy will not be suboptimal to follow even when knowing the signals of all auction participants.

An important thing to note about nash equilibria is that they dont express anything about the optimality of $\boldsymbol{\beta}^*$ in situations where all other players are not already following it. The equilibrium concept simply states that assuming everybody follows $\boldsymbol{\beta}^*$ nobody would gain anything from deviating.

1.2 Simple private value auctions

Consider an auction in which a single unit of some item is to be sold. Assume there are N bidders who each draw *private* values x_i from $X_i \sim F_i$. Assume further that all bidders are risk neutral and face no liquidity or budget constraints.

There are four obvious way such an auction could be run; either one of the two open auction formats

English auctions: An open ascending auction in which the seller repeatedly raises the price until only one bidder is left. The price paid will be the price at which the second last bidder dropped out.

Dutch auction: An open decending auction in which the seller initially sets a high price and then lowers it until the first bidder raises a hand to signal willingness to buy, at which point the item is sold.

Or one of the sealed auction formats

Sealed bid first price: in which all bidders submit an envelope containing their bid. The seller then ranks bids from highest to lowest and sells the item to the highest bidder at the price this individual bid.

Sealed bid second price: which is similar to the first price model as the highest bid will still win the auction, but only pay the price of the second highest bid.

1.2.1 Strategic equivalence

It turns out these four formats pairs up nicely in terms of strategic equivalence, specifically we have

$$\begin{aligned} &\text{Dutch auction} \approx \text{Sealed first price} && \text{(Strongly equivalent)} \\ &\text{English auction} \approx \text{Sealed second price} && \text{(Weakly equivalent)} \end{aligned}$$

For the equivalence of Dutch and first price auctions, notice that in both cases the auction will be over before any meaningful information about other bidders have been revealed, and that the pricing is the same in both auctions - if you win you pay your bid. Thus either of these should entail the same bidding behaviour by auction participants.

Similarly the English auction ends at the price bid by the second highest bidder (there's no reason to bid against oneself) so the pricing mechanism is the same. This equivalence is however weak in the sense that if signals x_i are not independent the open bidding of the English auction reveals information about the distribution of signals through the auction.

As long as we are dealing with the simple symmetic private value auction this however means we can stick to finding equilibria in one of the two equivalent formats as equilibrium strategies will transfer from one to another.

1.2.2 Symmetric equilibrium strategy in the second price sealed bid auction

In a second price auction the payoff of participating in the auction is either a) ones own private value less the second highest bid, or b) zero, depending on who wins the auction, that is

$$\Pi_i = \begin{cases} x_i - \max_{j \neq i} & \text{if } b_i > \max_{j \neq i} b_j \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

It turns out a symmetric equilibrium in this setting is simply $\beta(x) = x$, i.e. "bid your valuation". The proof is a quite simple argument:

Proof. Symmetric eq. in second price auctions: Assume all bidders are following $\beta(x) = x$. Now for some bidder i this might result in winning the auction. In this case having bid even higher would not matter as the price is fixed at the second highest bid. Had i bid lower it could likewise either result in still winning and still paying the same price, or loosing by bidding to low, in which case i would forego a profit. Thus if i wins the auction by following $\beta(x) = x$ there is no incentive to deviate. Alternatively bidder i might loose the auction when bidding according to β. In this case i could increase the price, but either i would still loose and get 0, or i would win but at a price $p > x_i$ resulting in a negative profit. i could also consider lowering the price, but this wouldn't affect the pay-out which would still be 0.

In summary there is no situation in which bidders have an incentive to deviate from $\beta(x) = x$, so this is a symmetric Nash equilibrium. □

This result extends to the English auction.

Note: This is a very strong proof as it is independent of

- Other bidders strategies
- Realized valuations
- Number of other bidders

- Risk preferences
- Whether other bidders act rationally

On top of this the format is extremely simple to implement.

1.2.3 Symmetric equilibrium strategy in the first price sealed bid auction

In the first price auctions things get a little bit more complicated. This is because bidders in first price auctions directly affect the price they will pay if they win the auction, giving them an incentive to bid lower than their true valuation, if they expect other bidders to have lower valuations than their own. In the first price auction the payoff of a bidder is

$$\Pi_i = \begin{cases} x_i - b_i & \text{if } b_i > max_{j \neq i} b_j \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

Clearly bidding above x_i is still suboptimal, but so is bidding $b_i = x_i$ as this is no better than loosing the auction.

Proof. Symmetric equilibrium in the first price auction: To derive a symmetric equilibrium strategy we will introduce some structure. We do this from the perspective of bidder 1 without loss of generality. Assume all valuations are IID so that $X_i \sim U(0, \omega)$. Define further a stochastic variable $Y_1 = \max\{X_2, X_3, ..., X_N\}$ as the maximum of all the other bidders valuations. We denote the CDF of Y_1 by $G(y)$, and due to identicallity and independence we have

$$G(y) = \prod_{j \neq i} F_j(y) = (F(y))^{N-1} \tag{5}$$

Assuming all other bidders follow some equilibrium strategy $\beta(x)$, bidder 1 only wins when $b_1 > \beta(Y_1) \Rightarrow Y_1 < \beta^{-1}(b_1)$ which occurs with probability $G(\beta^{-1}(b_1))$. Because the payoff from not winning is 0 bidder 1's expected payoff is therefore

$$E[\Pi_i | x_1] = G(\beta^{-1}(b_1))(x_1 - b_1) \tag{6}$$

Now taking the derivative w.r.t b_1 will yield the following optimality condition[1]

$$\frac{g(\beta^{-1}(b_1))}{\beta'(\beta^{-1}(b_1))}(x_1 - b_1) = G(\beta^{-1}(b_1)) \tag{7}$$

[1] Using that the derivative of $f^{-1}(x)$ is $1/f'(f^{-1}(x))$

We then impose that the equilibrium must be symmetric so that $b_1 = \beta(x_1)$, whereby we get

$$\begin{aligned}
&\frac{g(x_1)}{\beta'(x_1)}(x_1 - \beta(x_1)) = G(x_1) \qquad \Leftrightarrow \\
&G(x_1)\beta'(x_1) + \beta(x_1)g(x_1) = g(x_1)x_1 \qquad \Leftrightarrow \\
&\frac{\partial}{\partial x_1}(G(x_1)\beta(x_1)) = g(x_1)x_1
\end{aligned} \tag{8}$$

Now use that $\beta(0) = 0$ by assumption to write

$$\begin{aligned}
\beta(x) &= \frac{1}{G(x)} \int_0^x yg(y)\ dy \\
&= E[Y_1 | Y_1 < x]
\end{aligned} \tag{9}$$

where i have omitted the subscript 1 to emphasise that this is a mapping for any draw of x_1. □

This result does not show us that there is a symmetric equilibrium strategy in the first price auction (we've assumed it), but that if one exists it would be to bid ones expectation of the second highest bid. The intuition is that this strategy balances the risk of loosing by bidding lower with the benefit of winning at a lower price.

Krishna also shows a proof which verifies that this is indeed a symmetric equilibrium in Chapter 2.

Note: this result is weaker than for the second price auctions. It is not ex-post optimal as bidders might regret their bids when they realize what others have bid. It also requires assumptions on distributions of other bidders valuations etc.

Definition 1.2. Expected k-th highest draw from a uniform distribution: Consider N independent stochastic variables $X_i \sim U(\underline{x}, \bar{x})$. The exptected value of the k-th highest draw of these N variables is

$$\underline{x} + \frac{N-k+1}{N+1}(\bar{x} - \underline{x}) \tag{10}$$

2 Lecture 2 - Private value auctions, equilibria and revenue equivalence

In lecture 1 we have shown how the following two strategies each constitute equilibrium strategies in respectively first- and second price auctions

$$\begin{aligned}\beta^{I*}(x) &= E[Y_1|Y_1 < x] \\ \beta^{II*}(x) &= x\end{aligned} \tag{11}$$

We also noted that the Dutch auction is a strong strategic equivalent of the first price auction, while the English auction is a weak equivalent of the second price auction. This implies that the derived equilibrium strategies are also equilibria in the Dutch/English auctions. So in an English price auction it is optimal to stay in the auction until the price reaches x, while in the Dutch auction one should submit a bid when the price has decended to exactly $E[Y_1|Y_1 < x]$.

2.1 Verifying the equilibrium of first price auctions

When we showed in lecture 1 that the optimal strategy in a first price auction is to bid ones expectation of the second highest value given that one self has the highest valuation, we implicitly assumed that a symmetic equilibrium existed. Instead we only proved that if such an equilibrium existed, $\beta^{I*}(x) = E[Y_1|Y_1 < x]$ would be it. To show that this is indeed an equilibrium we need to show *no incentive to deviate* when everybody is following the strategy.

To show this first note that any alternative strategy $\alpha(x)$ can be represented as "pretending" to have valuation z while playing β as all strategies are rationally bounded by $\beta(0), \beta(\omega)$ when all other players are following β. In other words it would never be optimal in the first price setting to bid higher than the highest possible bid from other bidders, nor to bid lower than the lowest possible bid from other bidders. With this information we can then proceed

Proof. No incentive to deviate under $\beta^{I*}(x)$: From bidder 1's perspective, lets consider a situation where bidder 1 draws a valuation x, but pretends to have valuation z when bidding (so $b = \beta(z)$), thus deviating from the proposed equilibrium, in this case

$$\begin{aligned}
\Pi(x,b) &= \underbrace{G(z)}_{P(win)} \cdot \overbrace{\underbrace{(x-\beta(z))}}^{x-b} \\
&= G(z)x - G(z)E[Y_1|Y_1<z] \\
&= G(z)x - \int_0^z yg(y)\ dy \\
&= G(z)x - G(z)z + \int_0^z G(y)\ dy \qquad \text{(integrate by parts)} \\
&= G(z)(x-z) + \int_0^z G(y)\ dy
\end{aligned} \tag{12}$$

Now consider the difference in profits when following either $\beta(x)$ or $\beta(z)$ for any z:

$$\begin{aligned}
\Pi(\beta(x),x) - \Pi(\beta(z),x) &= \int_0^x G(y)\ dy - G(z)(x-z) - \int_0^z G(y)\ dy \\
&= G(z)(z-x) - \int_x^z G(y)\ dy
\end{aligned} \tag{13}$$

where the integral is joined according to $\int_a^b f(x)dx - \int_a^c f(x)dx = F(b) - F(a) - F(c) + F(a)$. Now all that is left is to convince oneself that this expression is always non-negative. If $z > x$ then $G(z)(z-x)$ is larger than $\int_x^z G(y)dy$ because $G(\cdot)$ is a CDF and thus increasing on the interval $[x,z]$. Likewise for $z < x$: because $G(\cdot)$ is increasing from z to x the integral of it will be larger than $G(z)(z-x)$. □

2.2 Expected revenues in the first and second price auctions

2.2.1 Expected revenue in the second price auction

The expected revenue is naturally of great interest to the seller of the item. To derive the expected revenue we follow a cookbook approach:

1. Find expected payment $E[m(x)]$ for a bidder with a given x
2. Find the ex ante expected payment before drawing x, $E[m(X)]$ by integrating over the support of x.
3. Find expected revenue by multiplying this ex ante expected payment with the number of bidders N (if bidders are not identical this needs to be weighted.)

Proof. Revenue in the second price auction:

Step 1: The ex-post expected payment of a bidder is simply the probability of winning $G(x)$ times the expected second highest price, given that bidder 1 wins:

$$E[m^{II}(x)] = G(x)E[Y_1|Y_1 < x] \tag{14}$$

Step 2: Now integrating this over the support $[0, \omega]$ of x yields

$$E[m^{II}(X)] = \int_0^w m^{II}(x)f(x)\ dx \tag{15}$$

By a bit of algebra this can be rewritten as

$$E[m^{II}(X)] = \int_0^\omega y(1 - F(y))g(y)\ dy \tag{16}$$

Step 3: Now finally multiplying this by N yields

$$\begin{aligned} E[R^{II}] &= N \cdot E[m^{II}(X)] \\ &= N \cdot \int_0^\omega y(1 - F(y))g(y)\ dy \\ E[Y_2^{(N)}] \end{aligned} \tag{17}$$

where $Y_2^{(N)}$ is simply the second highest of the N draws (formerly written Y_1). Showing the last equality completely requires a bit of algebra (see Krishna p. 19). □

2.2.2 Expected revenue in the first price auction

Now in the first price auction the ex-post expected payment will simply be the equilibrium bid (pay your bid) times the probability of winning, so

$$E[m^I(x)] = G(x)E[Y_1|Y_1 < x] \tag{18}$$

This is exactly identical to the expression from the second price auction, so the next steps will be identical to the ones above, and expected revenue will also be identical between the two formats. (This of course hints at a broader concept of *revenue equivalence*).

2.2.3 Variation in realized revenue

Altough expected revenue is identical between the first and second price auctions, the realizations will not follow the same distributions (theyre only mean-identical). One way of seeing this is by noticing that in the second price auctions

$\beta^{II}(x) = x$ so $\beta^{II} : [0, \omega] \to [0, \omega]$, while the shading in the first price auction implies bidding below ω even when drawing $x = \omega$. In other words **the spread of revenues is larger in second price auctions than in first price auctions.**

2.3 The revenue equivalence theorem

From the realization that expected revenues are identical in first and second price sealed bid auctions we imediately also learn that this must also be true for English and Dutch auctions, as these are strategic equivalent to one of the two sealed formats.

Proposition 2.1. Revenue equivalence: Consider any standard[2] auction in which 1) values are identically distributed and 2) bidders are risk neutral. Then any symmetric and increasing equilibrium where the expected payment of a bidder with value 0 is 0 yield the same expected revenue.

Proof. Expected payment in "nice" auctions does not depend on the auction format: Consider a standard auction A and the expected payoff for a bidder with valuation x who bids as if his valuation were z

$$\Pi(z, x) = \underbrace{G(z)x}_{P(win)\cdot value} - \underbrace{m^A(z)}_{\text{expected payment}} \tag{19}$$

The bidder will want to maximise his payoff w.r.t the type he plays, so taking

$$\frac{\partial \Pi(z, x)}{\partial z} = g(z)x - \frac{\partial m^A(z)}{\partial z} \tag{20}$$

stating that in optimum the bidder will seek to equate the marginal costs of changing the bid $\partial m^A(z)/\partial z$ to the marginal gains in expected value of winning $g(z)x$. Taking the integral on $[0, x]$ we get

$$\begin{aligned} m^A(z) &= \int_0^x yg(y)\ dy \\ &= G(x)E[Y_1|Y_1 < x] \end{aligned} \tag{21}$$

where we have used the fact that $m^A(0) = 0$. This shows that expected payment is the same in any auction satisfying the above assumptions. □

The intuition in this result is that ...

[2]Standard auctions are roughly auctions in which it is guaranteed that the highest bidder wins.

2.4 Reserve prices

Reserve prices are a way to ensure a minimal selling price of items. Implementing a reserve price produces some additional mathematical notation, but the intuition in bidding strategies remain unchanged. In the second price auction it is still optimal to bid x, and in the first price auction it is still optimal to shade to the expected value of the second highest bid. The only complication is that if ones valuation is below r one should not bid, and in the first price auction there will for some bidders be a binding lower limit to their shading, forcing them to bid $\max\{r, E[Y_1|Y_1 < r]\}$.

From the sellers perspective the expected profit from the auction when setting a reserve price r is

$$\Pi_0 = N \cdot E[m^A(X, r)] + F(r)^N x_0 \tag{22}$$

where m^A is a modified expected payment function, that takes into account that some bidders will be constrained in their bidding by the reserve price and $F(r)^N$ is the probability that all N *independent* bidders draw valuations below r. x_0 is the value of the item to the seller if unsold. To show that it is optimal to set a positive reserve price we will study the sign of the derivative of this expession when $r = x_0$. Krishna shows (this is just a bunch of algebra) that

$$\frac{\partial \Pi_0}{\partial r} = N\left[1 - (r - x_0)\frac{f(r)}{1 - F(r)}\right](1 - F(r))G(r) \tag{23}$$

Now when $r = x_0$ this collapses to

$$\left.\frac{\partial \Pi_0}{\partial r}\right|_{r=x_0} = N(1 - F(r))G(r) > 0 \tag{24}$$

showing that it is optimal to set a reserve price larger than 0. We can further deduce that the optimal reserve price reached when

$$1 = (r^* - x_0)\frac{f(r^*)}{1 - F(r^*)} \tag{25}$$

This principle that it is optimal to set a positive reserve price in almost all auctions is known as the *exclusion principle*.

The intuition to take away is that a) reserve prices only matter for the seller when 1 or 0 individuals draws above r, if more than 2 individuals draw valuations above r, the usual auction mechanism kicks in. Thus the seller needs to weight the risk of nobody drawing above r (and thus being stuck with a value of x_0) against the chance that only one individual draws above r, in which case the reserve price becomes a binding minimum payment, increasing the salesprice. Since it is more likely that one individuals draws above r, than that nobody does, it is benefitial to set the reserve price above x_0.

Note: the exclusion principle does not hold with private affiliated values.

Note: Challenges for reserve prices - it requires that sellers are credibly commited to not reauctioning the auction if nobody bids above r. Bidders behavior might (in the real world) be affected by the reserve price, as it could be perceived as a signal that the item for sale is valuable.

Note: in the derivation we use m^A as the expected payment in an arbitrary auction. For this proof to hold we need the auction we consider to be revenue equivalent with first and second price auctions with reserve prices. Look at Krishna p. 22 for math.

2.5 Key action tradeoffs

Auctioneers might care for more than earning a high profit, especially when there is some element of repetition in the auction setting. Auctioneers might care for maximizing revenue, efficent allocation, simlicity of auction format, long run competition (if auctions are repeated), fairness, public perception etc. In this list are some common tradeoffs.

The reserve price prevents efficient allocation, as some bidders are excluded even though their valuation is higher than x_0, but increases revenue.

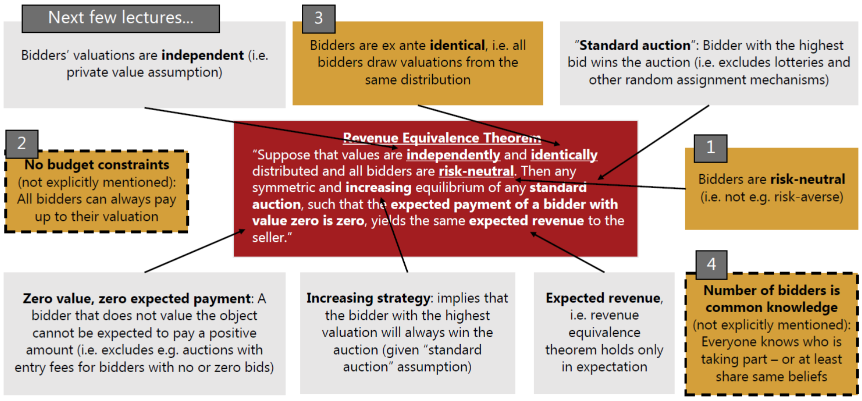
Efficency is often sought for, but in more complex markets auctions might easily become to complicated for bidders to fully understand, making it difficult to arrive at the efficient equilibrium.

2.6 Key takeaways

- The equilibirum in second price auctions is very simple, and therefore most likely a good prediction of actual auction outcomes. The equilibrium in first price auctions relies on assumptions about valuations being independent to actually predict anything we also need to assume known form of the value-distributions (and more?).

- in first price auctions the degree of shading is decreasing in the number of bidders N.

Figure 1: Assumptions required for revenue equivalence



- Expected revenue is the same in all *standard auctions*
- The variance on realized revenue is higher in second price auctions than in first price auctions (as shading implies never bidding above $E[Y_1|Y_1 < \omega] < \omega$).
- More bidders implies a higher expected revenue (less shading, more probability of someone drawing a high x).
- Reserve prices can increase the expected revenue (at the cost of potentially not being efficient, c.f. the two-state model).

3 Lecture 3 - Extensions and exceptions to the simple auctions

This lecture studies some of the key assumptions we have previously imposed on the auction to derive various result, and what happens if these assumptions are not upheld.

3.1 Risk neutrality

So far we have assumed bidders to be risk neutral, so that their decision does not put any additional weight to winning the auction appart from the pure value of the item. When this assumption holds, agents can maximize expected profits (instead of expected utility) simplifying their decisions. Mathematically we would say risk-neutral bidders have quasi-linear preferences such that $u(\cdot) : \mathbb{R}_+ \to \mathbb{R}$ has $u' > 0, u'' = 0$.

Risk averse bidders on the other hand have concave utility.

3.1.1 Risk aversion in the second price auction

In the second price auction risk preferences does *not* play any role in deriving the symmetric equilibrium. This is in essence because the equilibrium is ex-post optimal in the seond price auction, meaning there is no potential for regretting ones bid.

3.1.2 Risk aversion in the first price auction

In the second price auction risk preferences play a role, in particular we solve for the equilibrium by assuming the bidder wants to maximize expected profits. In the case of risk aversion the bidder would want to maximize expected profits while balancing the risk of not winning the auction. every time the risk averse bidder lowers his bid he not only reduces the probability of winning, but also incurs a risk-cost from the reduced change of winning.

oppositely risk loving bidders will shade even more than risk neutral ones, because they assign a positive value to the risk of not winning the auction.

$$\begin{array}{lcll} \text{Gain from winning} & \underset{balance}{-} & \text{P(win)} & \text{Risk neutral} \\ \text{Gain from winning} & \underset{balance}{-} & \text{P(win)} + \text{price of loosing} & \text{non-neutral} \end{array}$$

In conclusion risk averse bidders shade less than risk-neutral bidders in the first price format.

3.1.3 Revenues and efficiency under risk aversion

Risk averse bidders also affect the revenue equivalence of first and second price auctions.

Proposition 3.1. Revenue equivalence under risk aversion: Risk averse bidders with the same utility function and symmetric independent private values give a higher expected revenue in first price auctions than in second price auctions.

Proof. Risk profiles does not affect the second price auction, but cause all bidders to shade less in the first price format. Thus revenues cannot be equivalent, and must be higher in the first price auction. □

Note that as long as all bidders are equally risk averse, both auction formats are still efficient.

3.2 Budget constraints

So far we've assumed that bidders can always bid their intended bid, but often this is not the case (think the case with entrepreneurs in Copenhagen).

3.2.1 Budget constraints in the second price auction

In second price auctions the optimal strategy under a budget constraint becomes to bid ones valuation if possible, and otherwise the full budget, i.e. $\min\{x, w\}$ where w is the budget limit. The proof of this is completely parallel to the one without a budget constraint, execpt now we need to account for the cases where $x > w$ where one could bid below w and risk loosing out on a win, or bid above w and risk winning at a price higher than affordable (which we assume gives a profit of 0).

3.2.2 Budget constraints in the first price auction

In first price auctions once again the bidding strategy is only affects when the budget constraints become binding, so $\beta^I(x, w) = \min\{\beta^I(x), w\}$. The reasoning is that when $\beta(x) < w$ the logic without budget constraints applies, while at $\beta(x) > w$ it is a weakly dominated strategy to bid above w as winning in this case gives 0 profit.

3.2.3 Revenues and efficiency under budget constraints

With budget constraints expected revenues are higher in first price auctions than in second price auction. The reason is that in first price auctions the bidders shade, meaning less of them will be budget constrained. Since revenues are identical without budget constraints this implies a larger decrease in revenue in second price auctions.

Clearly budget constraints can lead to inefficiency - as the bidder with the highest valuation might be budget constrained at a very small budget. If all bidders face the same budget, auctions can still be inefficient as the auctioneer will have no way to determine the right winner if bids are at the constraints limit w.

3.3 Asymmetric distributions

So far our baseline assumption has been that all individuals have identical valuation distributions, i.e. bidders are ex ante identical

3.3.1 Asymmetric bidders in the second price auction

In the second price auction bidders valuation distributions are not important for deriving the equilibrium strategies.

3.3.2 Asymmetric bidders in the first price auction

In the first price auction we use the assumption of symmetric bidders in out definition of $G(\cdot)$, we will consider a special case of asymmetry, namely one with two bidders with different distributions of x, so

$$X_1 \sim F_1 : [0, \omega_1] \qquad X_2 \sim F_2 : [0, \omega_2] \tag{26}$$

where bidders follow β_1, β_2 and we impose that $\beta_1(0) = \beta_2(0) = 0$ while $\beta_1(\omega_1) = \beta_2(\omega_2) = \bar(b)$ i.e. that both bidders submit bids on the same range. We then consider a special case of asymmetry where F_2 stochastically dominates F_1 so $\forall x \in (0, \omega_2) : F_1(x) \leq F_2(x)$ where $\omega_2 \leq \omega_1$ meaning the bidder with the largest range of valuations (bidder 1)'s probability of drawing at least x is always larger than the probability that the weak bidder draws at least x.

In this case the weak bidder (bidder 2) will bid more aggressively than the strong bidder (bidder 1), that is $\beta_2(x) > \beta_1(x)$ for all x in $[0, \omega_2]$. This is intuitively because the weak bidder realizes he is very unlikely to win, making it worth shading less, to increase the probability of winning.

3.3.3 Revenue equivalence and efficiency under asymmetry

In general we do not have revenue equivalence under the assymetric model, but which auction performs better depends on the concrete distributions of bidders valuations, meaning no general ranking of auction formats can be made. When distributions are uniform however, the expected revenue is higher in a first price auction than in the second price auction (see example 4.3, 4.4 in krishna).

With asymmetry the first price format can be inefficient, while the second price format will still be efficient. In the first price format the aggresiveness of the weak bidder implies there will be some cases where the weak bidder wins even though the strong bidder had drawn a higher valuation.

3.4 Uncertain number of bidders

So far we've assumed that the number of bidders is common knowledge but in many cases especially whn there are few potential bidders this might not be the case.

3.4.1 Uncertain number of bidders in the second price auction

The second price auction strategies is independent of the number of bidders. As a single opposing bidder is enough for the "bid your valuation" argument to hold.

3.4.2 Uncertain number of bidders in the first price auction

In the first price auction the number of bidders directly influence ones expectation of Y_1. In this case uncertainty on N requires bidders to form beliefs about the true N by guessing that $N = n$ with some probability. The bidding strategy then becomes an average of optimal bids under each possible N weighted by the probability that it is the true N. This is of course assuming all bidders have the same beliefs over possible values of N, which might not be the case. In many cases it is likely that this density is asymmetric between bidders - imagine for example an oil drilling rights auction. In this case there might be 2-3 certain bidders, who have to guess if a new company will enter, while the new company can for certain know if they enter or not.

3.5 Resale and efficiency

One key concern in auction design is efficiency - i.e. selling to the bidder who wants the item the most. However one might argue that resale markets will always lead to efficient outcomes so auctioneers might as well maximize revenue regardless of the concerns for efficiency.

In the second price format we are almost always guaranteed efficiency, but under asymmetry the first price format is sometimes inefficient. It can be shown that this inefficiency is not easily cured by resale markets, as bidders will try to hide their true types as to not reveal it before the resale market. In this case the buyer who ends up with the item has no information about who actually has the highest valuation, and thus is no better of than in the first auction, and gives the possibility that no buyers really want the item on the resale market.

4 Lecture 4 - Interdependent common value auctions