Auction Theory - Lecture notes

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Abstract. These are notes for the course "Auctions, theory and practice". Some of the content is probably wrong or poorly explained.

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A brief motivation: Auctions are widely used to sell items, both in ordinary auctions for antiques, but more importantly theory on auction design is widely used when designing tenders for public projects ranging from construction to medicine prices. Additionally auctions are used as a market mechanism in the advertising market, energy market etc.

Such markets are typically characterized by having only one or few sellers (or buyers in the case of tenders) with many potential buyers interested in winning any given item for sale. In general we can think of auctions as games of incomplete information because bidders valuations or the true value of the sold item is often unknown to participants.

1.1 Bayesian Nash equilibria in the auction setting

Consider a game featuring N bidders, each of which must decide a strategy from their set of potential bids B_i. Each bidder draws a signal x_i from the stochastic variable X_i which distributed according to some density F_i. From this each bidder must choose a strategy $\beta_i : X_i \to B_i$ which gives a complete map from any possible signal to a bid in the set of possible bids B_i. Of course each bidder chooses this strategy with respect to the payoff function $\Pi_i(\boldsymbol{B}, \boldsymbol{X}) \to \mathbb{R}$.

Definition 1.1. (Pure strategy) Nash Equilibrium: A pure strategy Nash equilibrium is a vector β^* such that for all bids $b_i \in B_i$ and for all players i:

$$E[\Pi_i(\beta^*(X), X)] \geq E[\Pi_i(\beta_i(X_i), \beta^*_{-i}(X_{-i}), X)] \tag{1.1}$$

That is it is for no players a priori optimal to deviate from β.

Naturally an equilibrium like this does not have to be ex post optimal for all bidders, however if this is the case we can write

$$E[\Pi_i(\beta^*(x), x)] \geq E[\Pi_i(\beta_i(x_i), \beta^*_{-i}(x_{-i}), x)] \tag{1.2}$$

implying such a strategy will not be suboptimal to follow even when knowing the signals of all auction participants.

An important thing to note about nash equilibria is that they dont express anything about the optimality of β^* in situations where all other players are not already following it. The equilibrium concept simply states that assuming everybody follows β^* nobody would gain anything from deviating.

1.2 Simple private value auctions

Consider an auction in which a single unit of some item is to be sold. Assume there are N bidders who each draw *private* values x_i from $X_i \sim F_i$. Assume further that all bidders are risk neutral and face no liquidity or budget constraints.

There are four obvious way such an auction could be run; either one of the two open auction formats

English auctions: An open ascending auction in which the seller repeatedly raises the price until only one bidder is left. The price paid will be the price at which the second last bidder dropped out.

Dutch auction: An open decending auction in which the seller initially sets a high price and then lowers it until the first bidder raises a hand to signal willingness to buy, at which point the item is sold.

Or one of the sealed auction formats

Sealed bid first price: in which all bidders submit an envelope containing their bid. The seller then ranks bids from highest to lowest and sells the item to the highest bidder at the price this individual bid.

Sealed bid second price: which is similar to the first price model as the highest bid will still win the auction, but only pay the price of the second highest bid.

1.2.1 Strategic equivalence

It turns out these four formats pairs up nicely in terms of strategic equivalence, specifically we have

$$\begin{aligned} &\text{Dutch auction} \approx \text{Sealed first price} && \text{(Strongly equivalent)} \\ &\text{English auction} \approx \text{Sealed second price} && \text{(Weakly equivalent)} \end{aligned}$$

For the equivalence of Dutch and first price auctions, notice that in both cases the auction will be over before any meaningful information about other bidders have been revealed, and that the pricing is the same in both auctions - if you win you pay your bid. Thus either of these should entail the same bidding behaviour by auction participants.
Similarly the English auction ends at the price bid by the second highest bidder (there's no reason to bid against oneself) so the pricing mechanism is the same. This equivalence is however weak in the sense

that if signals x_i are not independent the open bidding of the English auction reveals information about the distribution of signals through the auction.

As long as we are dealing with the simple symmetic private value auction this however means we can stick to finding equilibria in one of the two equivalent formats as equilibrium strategies will transfer from one to another.

1.2.2 Symmetric equilibrium strategy in the second price sealed bid auction

In a second price auction the payoff of participating in the auction is either a) ones own private value less the second highest bid, or b) zero, depending on who wins the auction, that is

$$\Pi_i = \begin{cases} x_i - \max_{j \neq i} & \text{if } b_i > \max_{j \neq i} b_j \\ 0 & \text{otherwise} \end{cases} \tag{1.3}$$

It turns out a symmetric equilibrium in this setting is simply $\beta(x) = x$, i.e. "bid your valuation". The proof is a quite simple argument:

Proof. Symmetric eq. in second price auctions: Assume all bidders are following $\beta(x) = x$. Now for some bidder i this might result in winning the auction. In this case having bid even higher would not matter as the price is fixed at the second highest bid. Had i bid lower it could likewise either result in still winning and still paying the same price, or loosing by bidding to low, in which case i would forego a profit. Thus if i wins the auction by following $\beta(x) = x$ there is no incentive to deviate. Alternatively bidder i might loose the auction when bidding according to β. In this case i could increase the price, but either i would still loose and get 0, or i would win but at a price $p > x_i$ resulting in a negative profit. i could also consider lowering the price, but this wouldn't affect the payout which would still be 0.

In summary there is no situation in which bidders have an incentive to deviate from $\beta(x) = x$, so this is a symmetric Nash equilibrium. □

This result extends to the English auction.

Note: This is a very strong proof as it is independent of

- Other bidders strategies
- Realized valuations
- Number of other bidders
- Risk preferences
- Whether other bidders act rationally

On top of this the format is extremely simple to implement.

1.2.3 Symmetric equilibrium strategy in the first price sealed bid auction

In the first price auctions things get a little bit more complicated. This is because bidders in first price auctions directly affect the price they will pay if they win the auction, giving them an incentive to bid lower than their true valuation, if they expect other bidders to have lower valuations than their own. In the first price auction the payoff of a bidder is

$$\Pi_i = \begin{cases} x_i - b_i & \text{if } b_i > max_{j \neq i} b_j \\ 0 & \text{otherwise} \end{cases} \tag{1.4}$$

Clearly bidding above x_i is still suboptimal, but so is bidding $b_i = x_i$ as this is no better than loosing the auction.

Proof. Symmetric equilibrium in the first price auction: To derive a symmetric equilibrium strategy we will introduce some structure. We do this from the perspective of bidder 1 without loss of generality. Assume all valuations are IID so that $X_i \sim U(0, \omega)$. Define further a stochastic variable $Y_1 = \max\{X_2, X_3, ..., X_N\}$ as the maximum of all the other bidders valuations. We denote the CDF of Y_1 by $G(y)$, and due to identicallity and independence we have

$$G(y) = \prod_{j \neq i} F_j(y) = (F(y))^{N-1} \tag{1.5}$$

Assuming all other bidders follow some equilibrium strategy $\beta(x)$, bidder 1 only wins when $b_1 > \beta(Y_1) \Rightarrow Y_1 < \beta^{-1}(b_1)$ which occurs with probability $G(\beta^{-1}(b_1))$. Because the payoff from not winning is 0 bidder 1's expected payoff is therefore

$$E[\Pi_i | x_1] = G(\beta^{-1}(b_1))(x_1 - b_1) \tag{1.6}$$

Now taking the derivative w.r.t b_1 will yield the following optimality condition[1]

$$\frac{g(\beta^{-1}(b_1))}{\beta'(\beta^{-1}(b_1))}(x_1 - b_1) = G(\beta^{-1}(b_1)) \tag{1.7}$$

We then impose that the equilibrium must be symmetric so that $b_1 = \beta(x_1)$, whereby we get

$$\begin{aligned}
&\frac{g(x_1)}{\beta'(x_1)}(x_1 - \beta(x_1)) = G(x_1) \qquad \Leftrightarrow \\
&G(x_1)\beta'(x_1) + \beta(x_1)g(x_1) = g(x_1)x_1 \qquad \Leftrightarrow \\
&\frac{\partial}{\partial x_1}(G(x_1)\beta(x_1)) = g(x_1)x_1
\end{aligned} \tag{1.8}$$

Now use that $\beta(0) = 0$ by assumption to write

$$\begin{aligned}
\beta(x) &= \frac{1}{G(x)} \int_0^x yg(y)\ dy \\
&= E[Y_1 | Y_1 < x]
\end{aligned} \tag{1.9}$$

where i have omitted the subscript 1 to emphasise that this is a mapping for any draw of x_1. □

This result does not show us that there is a symmetric equilibrium strategy in the first price auction (we've assumed it), but that if one exists it would be to bid ones expectation of the second highest bid. The intuition is that this strategy balances the risk of loosing by bidding lower with the benefit of winning at a lower price.

Krishna also shows a proof which verifies that this is indeed a symmetric equilibrium in Chapter 2.

Note: this result is weaker than for the second price auctions. It is not ex-post optimal as bidders might regret their bids when they realize what others have bid. It also requires assumptions on distributions of other bidders valuations etc.

Definition 1.2. Expected k-th highest draw from a uniform distribution: Consider N independent stochastic variables $X_i \sim U(\underline{x}, \bar{x})$. The exptected value of the k-th highest draw of these N variables is

$$\underline{x} + \frac{N - k + 1}{N + 1}(\bar{x} - \underline{x}) \tag{1.10}$$

[1]Using that the derivative of $f^{-1}(x)$ is $1/f'(f^{-1}(x))$

2 Private value auctions, equilibria and revenue equivalence

In lecture 1 we have shown how the following two strategies each constitute equilibrium strategies in respectively first- and second price auctions

$$\begin{aligned} \beta^{I*}(x) &= E[Y_1 | Y_1 < x] \\ \beta^{II*}(x) &= x \end{aligned} \tag{2.1}$$

We also noted that the Dutch auction is a strong strategic equivalent of the first price auction, while the English auction is a weak equivalent of the second price auction. This implies that the derived equilibrium strategies are also equilibria in the Dutch/English auctions. So in an English price auction it is optimal to stay in the auction until the price reaches x, while in the Dutch auction one should submit a bid when the price has decended to exactly $E[Y_1 | Y_1 < x]$.

2.1 Verifying the equilibrium of first price auctions

When we showed in lecture 1 that the optimal strategy in a first price auction is to bid ones expectation of the second highest value given that one self has the highest valuation, we implicitly assumed that a symmetic equilibrium existed. Instead we only proved that if such an equilibrium existed, $\beta^{I*}(x) = E[Y_1 | Y_1 < x]$ would be it. To show that this is indeed an equilibrium we need to show *no incentive to deviate* when everybody is following the strategy.
To show this first note that any alternative strategy $\alpha(x)$ can be represented as "pretending" to have valuation z while playing β as all strategies are rationally bounded by $\beta(0), \beta(\omega)$ when all other players are following β. In other words it would never be optimal in the first price setting to bid higher than the highest possible bid from other bidders, nor to bid lower than the lowest possible bid from other bidders. With this information we can then proceed

Proof. No incentive to deviate under $\beta^{I*}(x)$: From bidder 1's perspective, lets consider a situation where bidder 1 draws a valuation x, but pretends to have valuation z when bidding (so $b = \beta(z)$), thus deviating from the proposed equilibrium, in this case

$$\begin{aligned} \Pi(x, b) &= \underbrace{G(z)}_{P(win)} \cdot \underbrace{(x - \beta(z))}_{x-b} \\ &= G(z)x - G(z)E[Y_1 | Y_1 < z] \\ &= G(z)x - \int_0^z yg(y)\ dy \\ &= G(z)x - G(z)z + \int_0^z G(y)\ dy \qquad \text{(integrate by parts)} \\ &= G(z)(x - z) + \int_0^z G(y)\ dy \end{aligned} \tag{2.2}$$

Now consider the difference in profits when following either $\beta(x)$ or $\beta(z)$ for any z:

$$\begin{aligned} \Pi(\beta(x), x) - \Pi(\beta(z), x) &= \int_0^x G(y)\ dy - G(z)(x - z) - \int_0^z G(y)\ dy \\ &= G(z)(z - x) - \int_x^z G(y)\ dy \end{aligned} \tag{2.3}$$

where the integral is joined according to $\int_a^b f(x)dx - \int_a^c f(x)dx = F(b) - F(a) - F(c) + F(a)$. Now all that is left is to convince oneself that this expression is always non-negative. If $z > x$ then $G(z)(z - x)$ is larger than $\int_x^z G(y)dy$ because $G(\cdot)$ is a CDF and thus increasing on the interval $[x, z]$. Likewise for $z < x$: because $G(\cdot)$ is increasing from z to x the integral of it will be larger than $G(z)(z - x)$. $\square$

2.2 Expected revenues in the first and second price auctions

2.2.1 Expected revenue in the second price auction

The expected revenue is naturally of great interest to the seller of the item. To derive the expected revenue we follow a cookbook approach:

1. Find expected payment $E[m(x)]$ for a bidder with a given x
2. Find the ex ante expected payment before drawing x, $E[m(X)]$ by integrating over the support of x.
3. Find expected revenue by multiplying this ex ante expected payment with the number of bidders N (if bidders are not identical this needs to be weighted.)

Proof. Revenue in the second price auction:

Step 1: The ex-post expected payment of a bidder is simply the probability of winning $G(x)$ times the expected second highest price, given that bidder 1 wins:

$$E[m^{II}(x)] = G(x)E[Y_1|Y_1 < x] \tag{2.4}$$

Step 2: Now integrating this over the support $[0, \omega]$ of x yields

$$E[m^{II}(X)] = \int_0^w m^{II}(x)f(x)\ dx \tag{2.5}$$

By a bit of algebra this can be rewritten as

$$E[m^{II}(X)] = \int_0^\omega y(1 - F(y))g(y)\ dy \tag{2.6}$$

Step 3: Now finally multiplying this by N yields

$$\begin{aligned} E[R^{II}] &= N \cdot E[m^{II}(X)] \\ &= N \cdot \int_0^\omega y(1 - F(y))g(y)\ dy \\ E[Y_2^{(N)}] \end{aligned} \tag{2.7}$$

where $Y_2^{(N)}$ is simply the second highest of the N draws (formerly written Y_1). Showing the last equality completely requires a bit of algebra (see Krishna p. 19). ☐

2.2.2 Expected revenue in the first price auction

Now in the first price auction the ex-post expected payment will simply be the equilibrium bid (pay your bid) times the probability of winning, so

$$E[m^{I}(x)] = G(x)E[Y_1|Y_1 < x] \tag{2.8}$$

This is exactly identical to the expression from the second price auction, so the next steps will be identical to the ones above, and expected revenue will also be identical between the two formats. (This of course hints at a broader concept of *revenue equivalence*).

2.2.3 Variation in realized revenue

Altough expected revenue is identical between the first and second price auctions, the realizations will not follow the same distributions (theyre only mean-identical). One way of seeing this is by noticing that in the second price auctions $\beta^{II}(x) = x$ so $\beta^{II} : [0, \omega] \to [0, \omega]$, while the shading in the first price auction implies bidding below ω even when drawing $x = \omega$. In other words **the spread of revenues is larger in second price auctions than in first price auctions.**

2.3 The revenue equivalence theorem

From the realization that expected revenues are identical in first and second price sealed bid auctions we imediately also learn that this must also be true for English and Dutch auctions, as these are strategic equivalent to one of the two sealed formats.

Proposition 2.1. Revenue equivalence: Consider any standard[2] auction in which 1) values are identically distributed and 2) bidders are risk neutral. Then any symmetric and increasing equilibrium where the expected payment of a bidder with value 0 is 0 yield the same expected revenue.

Proof. Expected payment in "nice" auctions does not depend on the auction format: Consider a standard auction A and the expected payoff for a bidder with valuation x who bids as if his valuation were z

$$\Pi(z,x) = \underbrace{G(z)x}_{P(win)\cdot value} - \underbrace{m^A(z)}_{\text{expected payment}} \tag{2.9}$$

The bidder will want to maximise his payoff w.r.t the type he plays, so taking

$$\frac{\partial \Pi(z,x)}{\partial z} = g(z)x - \frac{\partial m^A(z)}{\partial z} \tag{2.10}$$

stating that in optimum the bidder will seek to equate the marginal costs of changing the bid $\partial m^A(z)/\partial z$ to the marginal gains in expected value of winning $g(z)x$. Taking the integral on $[0,x]$ we get

$$\begin{aligned} m^A(z) &= \int_0^x yg(y)\ dy \\ &= G(x)E[Y_1|Y_1 < x] \end{aligned} \tag{2.11}$$

where we have used the fact that $m^A(0) = 0$. This shows that expected payment is the same in any auction satisfying the above assumptions. □

The intuition in this result is that ...

2.4 Reserve prices

Reserve prices are a way to ensure a minimal selling price of items. Implementing a reserve price produces some additional mathematical notation, but the intuition in bidding strategies remain unchanged. In the second price auction it is still optimal to bid x, and in the first price auction it is still optimal to shade to the expected value of the second highest bid. The only complication is that if ones valuation is below r one should not bid, and in the first price auction there will for some bidders be a binding lower limit to their shading, forcing them to bid $\max\{r, E[Y_1|Y_1 < r]\}$.

From the sellers perspective the expected profit from the auction when setting a reserve price r is

$$\Pi_0 = N \cdot E[m^A(X,r)] + F(r)^N x_0 \tag{2.12}$$

where m^A is a modified expected payment function, that takes into account that some bidders will be constrained in their bidding by the reserve price and $F(r)^N$ is the probability that all N *independent* bidders draw valuations below r. x_0 is the value of the item to the seller if unsold. To show that it is optimal to set a positive reserve price we will study the sign of the derivative of this expession when $r = x_0$. Krishna shows (this is just a bunch of algebra) that

$$\frac{\partial \Pi_0}{\partial r} = N\left[1 - (r - x_0)\frac{f(r)}{1-F(r)}\right](1 - F(r))G(r) \tag{2.13}$$

[2] Standard auctions are roughly auctions in which it is guaranteed that the highest bidder wins.

Now when $r = x_0$ this collapses to

$$\left.\frac{\partial \Pi_0}{\partial r}\right|_{r=x_0} = N(1 - F(r))G(r) > 0 \tag{2.14}$$

showing that it is optimal to set a reserve price larger than 0. We can further deduce that the optimal reserve price reached when

$$1 = (r^* - x_0)\frac{f(r^*)}{1 - F(r^*)} \tag{2.15}$$

This principle that it is optimal to set a positive reserve price in almost all auctions is known as the *exclusion principle*.

The intuition to take away is that a) reserve prices only matter for the seller when 1 or 0 individuals draws above r, if more than 2 individuals draw valuations above r, the usual auction mechanism kicks in. Thus the seller needs to weight the risk of nobody drawing above r (and thus being stuck with a value of x_0) against the chance that only one individual draws above r, in which case the reserve price becomes a binding minimum payment, increasing the salesprice. Since it is more likely that one individuals draws above r, than that nobody does, it is benefitial to set the reserve price above x_0.

Note: the exclusion principle does not hold with private affiliated values.

Note: Challenges for reserve prices - it requires that sellers are credibly commited to not reauctioning the auction if nobody bids above r. Bidders behavior might (in the real world) be affected by the reserve price, as it could be perceived as a signal that the item for sale is valuable.

Note: in the derivation we use m^A as the expected payment in an arbitrary auction. For this proof to hold we need the auction we consider to be revenue equivalent with first and second price auctions with reserve prices. Look at Krishna p. 22 for math.

2.5 Key action tradeoffs

Auctioneers might care for more than earning a high profit, especially when there is some element of repetition in the auction setting. Auctioneers might care for maximizing revenue, efficent allocation, simlicity of auction format, long run competition (if auctions are repeated), fairness, public perception etc. In this list are some common tradeoffs.

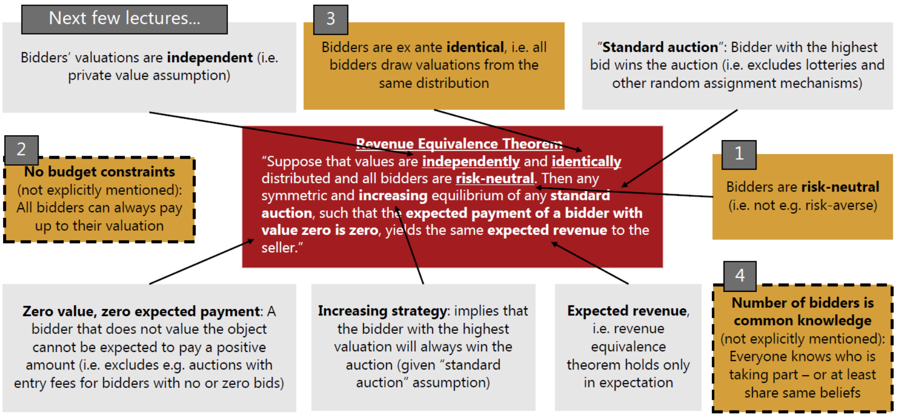
The reserve price prevents efficient allocation, as some bidders are excluded even though their valuation is higher than x_0, but increases revenue.

Efficency is often sought for, but in more complex markets auctions might easily become to complicated for bidders to fully understand, making it difficult to arrive at the efficient equilibrium.

2.6 Key takeaways

- The equilibirum in second price auctions is very simple, and therefore most likely a good prediction of actual auction outcomes. The equilibrium in first price auctions relies on assumptions about valuations being independent to actually predict anything we also need to assume known form of the value-distributions (and more?).
- in first price auctions the degree of shading is decreasing in the number of bidders N.
- Expected revenue is the same in all *standard auctions*

Figure 1: Assumptions required for revenue equivalence



- The variance on realized revenue is higher in second price auctions than in first price auctions (as shading implies never bidding above $E[Y_1|Y_1 < \omega] < \omega$).
- More bidders implies a higher expected revenue (less shading, more probability of someone drawing a high x).
- Reserve prices can increase the expected revenue (at the cost of potentially not being efficient, c.f. the two-state model).

3 Extensions and exceptions to the simple auctions

This lecture studies some of the key assumptions we have previously imposed on the auction to derive various result, and what happens if these assumptions are not upheld.

3.1 Risk neutrality

So far we have assumed bidders to be risk neutral, so that their decision does not put any additional weight to winning the auction appart from the pure value of the item. When this assumption holds, agents can maximize expected profits (instead of expected utility) simplifying their decisions. Mathematically we would say risk-neutral bidders have quasi-linear preferences such that $u(\cdot) : \mathbb{R}_+ \to \mathbb{R}$ has $u' > 0$, $u'' = 0$. Risk averse bidders on the other hand have concave utility.

3.1.1 Risk aversion in the second price auction

In the second price auction risk preferences does *not* play any role in deriving the symmetric equilibrium. This is in essence because the equilibrium is ex-post optimal in the seond price auction, meaning there is no potential for regretting ones bid.

3.1.2 Risk aversion in the first price auction

In the second price auction risk preferences play a role, in particular we solve for the equilibrium by assuming the bidder wants to maximize expected profits. In the case of risk aversion the bidder would want to maximize expected profits while balancing the risk of not winning the auction. every time the risk averse bidder lowers his bid he not only reduces the probability of winning, but also incurs a risk-cost from the reduced change of winning.

oppositely risk loving bidders will shade even more than risk neutral ones, because they assign a positive value to the risk of not winning the auction.

$$\begin{array}{llll} \text{Gain from winning} & \underset{\textit{balance}}{-} & \text{P(win)} & \text{Risk neutral} \\ \text{Gain from winning} & \underset{\textit{balance}}{-} & \text{P(win)} + \text{price of loosing} & \text{non-neutral} \end{array}$$

In conclusion risk averse bidders shade less than risk-neutral bidders in the first price format.

3.1.3 Revenues and efficiency under risk aversion

Risk averse bidders also affect the revenue equivalence of first and second price auctions.

Proposition 3.1. Revenue equivalence under risk aversion: Risk averse bidders with the same utility function and symmetric independent private values give a higher expected revenue in first price auctions than in second price auctions.

Proof. Risk profiles does not affect the second price auction, but cause all bidders to shade less in the first price format. Thus revenues cannot be equivalent, and must be higher in the first price auction. □

Note that as long as all bidders are equally risk averse, both auction formats are still efficient.

3.2 Budget constraints

So far we've assumed that bidders can always bid their intended bid, but often this is not the case (think the case with entrepreneurs in Copenhagen).

3.2.1 Budget constraints in the second price auction

In second price auctions the optimal strategy under a budget constraint becomes to bid ones valuation if possible, and otherwise the full budget, i.e. $\min\{x, w\}$ where w is the budget limit. The proof of this is completely parallel to the one without a budget constraint, execpt now we need to account for the cases where $x > w$ where one could bid below w and risk loosing out on a win, or bid above w and risk winning at a price higher than affordable (which we assume gives a profit of 0).

3.2.2 Budget constraints in the first price auction

In first price auctions once again the bidding strategy is only affects when the budget constraints become binding, so $\beta^I(x, w) = \min\{\beta^I(x), w\}$. The reasoning is that when $\beta(x) < w$ the logic without budget constraints applies, while at $\beta(x) > w$ it is a weakly dominated strategy to bid above w as winning in this case gives 0 profit.

3.2.3 Revenues and efficiency under budget constraints

With budget constraints expected revenues are higher in first price auctions than in second price auction. The reason is that in first price auctions the bidders shade, meaning less of them will be budget constrained. Since revenues are identical without budget constraints this implies a larger decrease in revenue in second price auctions.

Clearly budget constraints can lead to inefficiency - as the bidder with the highest valuation might be budget constrained at a very small budget. If all bidders face the same budget, auctions can still be inefficient as the auctioneer will have no way to determine the right winner if bids are at the constraints limit w.

3.3 Asymmetric distributions

So far our baseline assumption has been that all individuals have identical valuation distributions, i.e. bidders are ex ante identical

3.3.1 Asymmetric bidders in the second price auction

In the second price auction bidders valuation distributions are not important for deriving the equilibrium strategies.

3.3.2 Asymmetric bidders in the first price auction

In the first price auction we use the assumption of symmetric bidders in out definition of $G(\cdot)$, we will consider a special case of asymmetry, namely one with two bidders with different distributions of x, so

$$X_1 \sim F_1 : [0, \omega_1] \qquad X_2 \sim F_2 : [0, \omega_2] \tag{3.1}$$

where bidders follow β_1, β_2 and we impose that $\beta_1(0) = \beta_2(0) = 0$ while $\beta_1(\omega_1) = \beta_2(\omega_2) = \bar{(}b)$ i.e. that both bidders submit bids on the same range. We then consider a special case of asymmetry where F_2 stochastically dominates F_1 so $\forall x \in (0, \omega_2) : F_1(x) \leq F_2(x)$ where $\omega_2 \leq \omega_1$ meaning the bidder with the largest range of valuations (bidder 1)'s probability of drawing at least x is always larger than the probability that the weak bidder draws at least x.
In this case the weak bidder (bidder 2) will bid more aggressively than the strong bidder (bidder 1), that is $\beta_2(x) > \beta_1(x)$ for all x in $[0, \omega_2]$. This is intuitively because the weak bidder realizes he is very unlikely to win, making it worth shading less, to increase the probability of winning.

3.3.3 Revenue equivalence and efficiency under asymmetry

In general we do not have revenue equivalence under the assymetric model, but which auction performs better depends on the concrete distributions of bidders valuations, meaning no general ranking of auction formats can be made. When distributions are uniform however, the expected revenue is higher in a first price auction than in the second price auction (see example 4.3, 4.4 in krishna).

With asymmetry the first price format can be inefficient, while the second price format will still be efficient. In the first price format the aggresiveness of the weak bidder implies there will be some cases where the weak bidder wins even though the strong bidder had drawn a higher valuation.

3.4 Uncertain number of bidders

So far we've assumed that the number of bidders is common knowledge but in many cases especially whn there are few potential bidders this might not be the case.

3.4.1 Uncertain number of bidders in the second price auction

The second price auction strategies is independent of the number of bidders. As a single opposing bidder is enough for the "bid your valuation" argument to hold.

3.4.2 Uncertain number of bidders in the first price auction

In the first price auction the number of bidders directly influence ones expectation of Y_1. In this case uncertainty on N requires bidders to form beliefs about the true N by guessing that $N = n$ with some probability. The bidding strategy then becomes an average of optimal bids under each possible N weighted by the probability that it is the true N. This is of course assuming all bidders have the same beliefs over possible values of N, which might not be the case. In many cases it is likely that this density is asymmetric between bidders - imagine for example an oil drilling rights auction. In this case there might be 2-3 certain bidders, who have to guess if a new company will enter, while the new company can for certain know if they enter or not.

3.5 Resale and efficiency

One key concern in auction design is efficiency - i.e. selling to the bidder who wants the item the most. However one might argue that resale markets will always lead to efficient outcomes so auctioneers might as well maximize revenue regardless of the concerns for efficiency.

In the second price format we are almost always guaranteed efficiency, but under asymmetry the first price format is sometimes inefficient. It can be shown that this inefficiency is not easily cured by resale markets, as bidders will try to hide their true types as to not reveal it before the resale market. In this case the buyer who ends up with the item has no information about who actually has the highest valuation, and thus is no better of than in the first auction, and gives the possibility that no buyers really want the item on the resale market.

4 Interdependent common value auctions

So far we've assumed signals to be private and independent of what signals other bidders have drawn. There are two ways in which we will relax this assumption, 1) we can allow for only partial information about the value of an item and 2) we can allow for interdependence between bidders values.
In this setting it is important to distinguish between the *value* of an item, which is the actual "true" value of the item for sale, and the *valuation* which is the bidders best estimate of the value.
In this setup we have that bidders draw an unknown value v_i from V_i, and a known signal x_i from X_i. The ex post value of an item to bidder i, is the expecation of V_i conditional on all of the signals, so

$$v_i(x_1, ..., x_N) = E[V_i | X_1 = x_1, ..., X_N = x_N] \tag{4.1}$$

Which we can interpret as the bidders estimate of v_i given all common knowledge and any knowledge gained when the other bidders signals are revealed. Bidders payoffs are $v_i - p_i$ where p_i is whatever price is paid. In this framework we can get back to the private value setting by assuming $v_i(X) = X_i$ and as another extreme have *common value* auction in which $v = v(X_1, ..., X_N)$ where the value only depends on the signals of all other bidders.

4.1 Interdependent common value with independent signals

Let us consider a simple model where the signals are independent, but the value is common. Let $X_1, ...X_N$ be independently distributed signals, their distribution known to all bidders. Assume that each X_i provides an unbiased estimate of V, so $E[X_i | V = v] = v$. Turning this definition around we should be able to estimate v by

$$v = \frac{1}{N} \sum_{i=1}^{N} x_i \tag{4.2}$$

Now of course bidders signals are initially private but if bids are revealed throughout the auction they can update their estimate of v continously. If the auction is sealed-bid other bidders bids (and implicitly signals) are revealed when the auction is over, leading to the *winners curse*.

4.1.1 Winners curse

Consider a sealed bid auction with common value and independent signals. If bidder i bids $E[V | X_i = x_i]$ and wins the auction this implies $x_i > x_j$ for any $j \neq i$, and consequently bidder i's estimate of V should drop. Essentially winning reveals to bidder i that his estimate of V was to high. Mathematically winning implies

$$E[V|X_i = x_i, Y_1 < x] < E[V|X_i = x_i] \tag{4.3}$$

so bidder i regrets winning. Note **this is purely an out-of-equilibirum effect** as bidders will take into account this effect when forming their bid, to bid according to $E[V|X_i = x_i, Y_1 < x]$ not $E[V|X_i = x_i]$.

Notice that this is also true in the second price auction, as the private-value equilibrium was derived using that bidders know exactly what value they assign to the item. Now it is optimal to shade both in first- and second price auctions. Furthermore the strategic equivalence between the second price sealed bid auction and the English auction is lost, because the gradual revealing of when bidders drop out in the English auction carries information about v.

4.2 Affiliated signals

Above we made some simplifying assumptions. For example we assumed that signal distributions were common knowledge. This in turn would imply that drawing the minimum value in the case of uniform signals would very clearly signal something about the relative signals of other bidders. We also assumed intdependence of signals, so in an English auction a bidder exiting the auction should only affect the remaining bidders estimate of the true value, but not their idea about which valuations other bidders have drawn. Often we would think that the valuations bidders draw are also in some way related.

Affiliation implements exactly this. Bidders will not know if their bid is high or low, as their signals are no longer independent. What we allow for now is that signals are not independent, so

$$f(X) \neq \prod_{i=1}^{N} f_i(x_i) \tag{4.4}$$

The way we allow for this is through *affiliation*, this is a form of strong relation where if a subset of X_i's turn out large, this implies a high probability of the remaining X_j's also being large. We wont show a formal definition of affiliation but note some implications of affiliated signals:

- If $X_1, ..., X_N$ are affiliated, then $X_1, Y_1, ..., Y_{N-1}$ are also affiliated. (Intuitively: the winning signal is affiliated with loosing bidders signals)
- if $x > x'$ then $G(\cdot|x)$ dominates $G(\cdot|x')$ in terms of the inverse hazard rate, that is

  $$\forall y : \quad \frac{g(y|x)}{G(y|x)} \geq \frac{g(y|x')}{G(y|x')} \tag{4.5}$$

  (intuitively: for any given y a higher signal $x > x'$ implies it is less likely that y is the second highest signal.)
- For any increasing function(strategy) γ and $x > x'$ we have

  $$E[\gamma(Y_1)|X_1 = x] \geq E[\gamma(Y_1)|X_1 = x'] \tag{4.6}$$

  (Intuitively: Drawing a higher $x > x'$ implies expecting higher bids.)

4.3 Symmetric model with affiliated signals

Let us consider the case with symmetric valuations, so all signals are drawn on $[0, \omega]$ and for all bidders the valuations are symmetric in the signals of all other bidders except for themselves:

$$v_i(X) = u(X_i, X-i) \tag{4.7}$$

We also assume that the joint distribution of signals $f(\cdot)$ is symmetric in its arguments, and that signals are affiliated.

4.3.1 Second price auction with affiliated signals

In the second price auction with affiliated signals the symmetric equilibrium is given by $\beta^{II}(x) = v(x, x)$ where v is defined as the expected value of the ex ante true value V_1, conditional on $X_i = x$ and $Y_1 = y$, that is from the perspective of bidder 1

$$v(x, y) = E[V_1 | X_1 = x, Y_1 = y] \tag{4.8}$$

The equilibrium is then to bid as if the second highest bid $y = x$, i.e. you are just tied with the second highest bidder. The intuition in this result is that in the second price auction the worst off a bidder will be, before dropping out of the auction, is to be tied with the second highest bid (this yields a payoff of 0). Bidding higher than this does not influence the price (but can lead to "over-winning"), bidding lower risks not winning the auction.

One technque for solving these, often complicated models, is to assume a diffuse prior on the signals. This technique involves assuming that $v \sim U(-\infty, \infty)$ and then bounding the signals around the actual v. In particular we typically assume that $X_i \sim U(0.75 \cdot v, 1.25 \cdot v)$ giving some bound on the distance from x to v.

Notice this format is not ex-post optimal (as is the case in private value auctions), as the bidders do not know the signals of all the non-winning bidders, which influence the value v.

4.3.2 English auction with affiliated signals

In the english auction with affiliated signals bidders should remain in the auction until the prices reaches the bidders valuation just like in the private value examples, but with the added twist that the valuation changes whenever another bidder exits the auction. This is because the price at which other bidders drop out provide information about their valuation, and thus the density of estimates of v. This also means the English and second price auctions are no longer strategically equivalent.

English auction using a diffuse prior Consider the auction for skolekridt as an English auction. We will assume a diffuse prior on the signals, namely that

$$X_i \in [0.75 \cdot v, 1.25 \cdot v] \tag{4.9}$$

imagine now a case where all bidders share their signals, in this case the expected value of v will be $E[v|X] = \frac{1}{2}(X_{(1)} + X_{(n)})$ where $X_{(n)}$ is the lowest signal and $X_{(1)}$ the highest signal. (That is v is expected to be right in the middle of the draws). At the beginning of the auction all bidders are active and the price p has some low value. The only way a bidder would win at this price is if all other bidders decided to drop out immediately, implying $X_{(n)} = X_{(1)} = v$. Thus as p increases the first bidder will drop out when the price is equal to his own signal.

The remaining bidders update their belief as they now know $X_{(n)}$. Conditional on winning their estimate of v is then $\frac{1}{2}(X_i + X_{(n)})$, and they drop out when p is equal to this value. This continues until the second last bidder drops out at $p = \frac{1}{2}(X_{(2)} + X_{(n)})$ which is the price at which the bidder with the highest signal wins the item.

This reasoning leads us to a statement of the general equilibrium in an common value English auction with affiliated signals.

Proposition 4.1. Equilibrium in the English auction The symmetric equilibrium strategies in an common value english auction β are given by

$$\beta^N(x) = u(x, ..., x) \tag{4.10}$$

when all bidders are active and

$$\beta^k(x) = u(x, p_{k+1}, p_{k+2}, ..., p_N) = u(x, ..., x, x_{k+1}, ..., x_N) \tag{4.11}$$

when there are $k < N$ bidders active.

The reasoning behind this proposition is outlined above. Essentially the loosing bidders drop out when the price reaches a level where winning would be a break-even situation, and remaining in the auction would imply a cost if winning. The winning bidder cannot affect the price himself, and is guaranteed a positive payoff by winning. This equilibrium is also ex-post optimal as no bidders risk regretting their bid.

4.3.3 First price auctions with affiliated signals

The equilibrium strategy in a first price auction with common value and affiliated signals can be shown to be

$$\beta^I(x) = \int_0^x v(y,y)\ dL(y|x), \quad L(y|x) = \exp\left(-\int_y^x \frac{g(t|t)}{G(t|t)}\ dt\right) \tag{4.12}$$

Which essentially states to bid ones expectation of the second highest bidders expected value conditional on winning. If values are private $v(y,y) = y$ and $L(y|x) = G(x)/G(y)$. Using this information it is straight forward to return to the private value expression of $E[Y_1|Y_1 < x]$.

4.4 Revenue comparison

With affiliated signals revenue equivalence does *not* hold. Instead it can be shown that

$$E[R^{Eng}] \geq E[R^{II}] \geq E[R^I] \tag{4.13}$$

Thus the English auction yields the highest payoffs. It is also a very strong equilibrium as it is ex-post optimal while the first and second price equilibria are bayesian nash equilibria.

4.5 Efficiency

All three auction formats have increasing equilibrium functions, but as these are formulated in terms of the *signals*, not the *value* this does not imply efficiency. In general neither of the three formats are thus efficient.

It can be shown that if the valuation functions satisfy the *single crossing property* all three formats will be efficient. The single crossing property is simply that

$$\forall x, i, j \neq i: \qquad \frac{\partial v_i}{\partial x_i}(x) \geq \frac{\partial v_j}{\partial x_i}(x) \tag{4.14}$$

That is every bidders signal matters most to the bidder himself.

5 Common value auction (cont.) and the linkage principle

This lecture is quite short as the second half was spent with a guest lecturer from the Danish energy company Ørsted. I wont cover the guest lecturers talk.

Is there shading in equilibrium bidding strategies?

	2nd-price	**1st-price**
Private value	No shade	Shade
Common value	Shade	Larger shade

Figure 2: Auctions overview Quick overview of the shading behavior of bidders in the auction formats covered so far.

5.1 Revenue linkage principle

In lecture 4 we saw that it was possible to rank the expected revenues of the three auction formats. This observation leads to a more general formulation about revenues in common value auctions.
Consider a standard auction in which the highest bidder wins and bidders follow the symmetric equilibrium β^A. Then from the perspective of bidder 1, let $W^A(z,x)$ be the expected price paid by bidder 1 if he wins by pretending to have signal z, while he actually has signal x. Examples of W^A are

$$W^I(z,x) = \beta^I(x) \tag{5.1}$$
$$W^{II}(z,x) = E[\beta^{II}(Y_1)|X_1 = x, Y_1 < z] \tag{5.2}$$

The linkage principle states that if A and B are two standard format auctions where the highest bidder wins and *only* he pays a positive amount, and it holds that $W^A(0,0) = W^B(0,0) = 0$ and[3]

$$\forall x: \quad \frac{\partial_2 W^A(x,x)}{\partial x} \geq \frac{\partial_2 W^A(x,x)}{\partial x} \tag{5.3}$$

Then the expected revenue in auction A is at least as large as it is in B. Intuitively that is if an increase in signal x increases the expected price paid more when playing truthfully, then expected revenue will be higher. We can use this principle to reason about the revenues in different types of common value auctions.

6 Extensions to the interdependent value auction

This lecture focuses on extensions similar the ones studies in lectures 2 and 3, but in the setting of common value auctions.

6.1 The role of public information

A central question we have not yet touched upon is what the seller should do with any private information about the item being sold. In tenders for oil fields or large infrastructure projects the seller (often a state) might have private information about the future plans for legislation governing the area or have specialty knowledge about the ground on which to build. To the seller it might be tempting to keep this information private in some cases and release it in others. We will assume that the seller must commit to either strategy across all auctions, which seems reasonable when considering the seller needs credibility for the (lack of) information to be trusted.

6.1.1 Public information in the symmetric model

Let S denote private information available to the seller. The information affects bidders valuations such that

$$V_i = v_i(S, X_1, ..., X_N) \tag{6.1}$$

we still have that $v_i(0) = 0$ and v_i is symmetric so $v_i(S,X) = u(S, X_i, X_{-i})$. We assume that $(S, X_1, ...X_N)$ are affiliated and distributed according to the joint density f.
We augment our model with two versions of valuations. If no public information is available we define

$$v(x,y) = E[V_i|X_i = x, Y_1 = y] \tag{6.2}$$

If public information is made available this changes to

$$\hat{v}(s,x,y) = E[V_i|S = s, X_i = x, Y_1 = y] \tag{6.3}$$

Now to study the consequence of information on revenues define a function $W^A(z,x)$ such that

$$W^A(z,x) = E[p(x)|X_i = x, Y_1 < z] \tag{6.4}$$

which is the expected price paid when winning.

[3]where ∂_2 indicates we take the derivative w.r.t. the second argument.

Public information in the first price model Notice in the first price model we have that $W^I(z,x) = \beta(z)$ without any public information (the winner pays his bid). However if public information is available $W^I(z,x) = E[\hat{\beta}(S,z)|X_i = x]$. Clearly the derivative of the first expression w.r.t. x is 0, while it is positive for the second expression because of affiliation between X_i and S. Thus by the linkage principle expected revenue is higher when releasing the information.

6.2 Reserve prices with affiliated signals

In the private value case we have seen that reserve prices are optimal under very weak assumptions (the exclusion principle) and we learned that they are only effective in the case where a single bidder draws a signal above the limit.
If signals are affiliated the exclusion principle does not hold. This is because affiliation implies signals are less spread out than in the independent case, why the chance of a reserve price becoming effective are lower. The higher the degree of affiliation the lower the likelihood of only one bidder bidding above r. At the same time it becomes more likely that all bids fall below r in which case the reserve price causes a loss for the seller. So with strong affiliation setting $r = 0$ will be better than $r > 0$.
This does not mean reserve prices are never useful. In cases with few bidders, or if the seller assigns some value to the item sold (or incurs a "shitstorm cost" at low prices), the reserve price might still be benefitial. (discuss the ambulance tender case).

6.3 Asymmetries in auctions with affiliated signals

We have seen three key features of symmetric equilibria in common value auctions namely

- They can be ranked in revenue, so $E[R^{Eng}] \geq E[R^{II}] \geq E[R^I]$.
- They are all efficient under "reasonable" assumptions.
- Releasing public information increases revenue because of affiliation.

If bidders are asymmetric the revenue ranking no longer holds, and releasing public information might actually decrease revenue. One way to understand this in the case of ascending auctions is that if one bidder has a slight edge, this bidder will bid slightly more aggressively. This means all other bidders risk of the winners curse increases (beating the strong bidder is really bad news about your own signals accuracy). Because other bidders correct for this increased risk of winners curse, the strong bidder in turn will bid even more aggressively (if other bidders shade their bids, seeing them drop out is not that bad for the expected value of v). This shows how even a small advantage can lead to relatively large differences in outcome.

Asymmetries in information In many real worlds tenders an incumbent participant probably have more information about the true value of winning the tender than other bidders. In these cases information is asymmetrically distributed. In the case where some bidders are completely uninformed Krishna shows it is optimal to follow an mixing strategy bidding some random number between 0 and the expected value of the informed bidders signals.

7 Bidder collusion

In order to study collusion in auctions we return to the private value auctions. In particular we study a second price auction with N bidders the set of which we denote $\mathcal{N}$. Bidders valuations are distributed on $[0, \omega_i]$ and follow a distribution F_i which can vary between bidders.
We consider a case where the subset $\mathcal{I} \in \mathcal{N}$ of all bidders have joined a bidding ring. That is bidders $1, 2, ..., I$ are in the ring while $I + 1, ..., N$ are not.

Define also a variable $Y_1^{\mathcal{S}}$ to be the highest valuation in the subset of bidders $\mathcal{S} \in \mathcal{N}$. $Y_1^{\mathcal{I}}$ is thus the highest signal among the bidding ring members.

Bidders in the ring needs to coordinate to identify one representative for their ring, who will bid to win the item, while all others submit shill bids. In this way the ring can earn profit from exploiting the difference in perception about the number of bidders in the auction. The way the ring does this is by identifying the within-ring bidder with the highest value Y_1^I who will represent the ring, and bid according to $\beta(x) = x$. At the same time all other ring-members bid low enough to be certain not to win the auction. To bidders outside the ring seemingly nothing changes so for them it is still optimal to bid according to $\beta(x) = x$.

Clearly the expected payment $m_i(x_i)$ for members of the ring is lower than it would have been in absence of the ring $\hat{m}_i(x_i)$ (the average is reduced by all those who shade their bids). We can thus define the gain from parttaking in the ring to be

$$t_i(x_i) = m_i(x_i) - \hat{m}_i(x_i) \geq 0 \tag{7.1}$$

and we can compute the expected total profit to the ring as

$$t_I = \sum_{i \in I} E[t_i(x_i)] \tag{7.2}$$

To begin with we can notice that given the existence of the ring no members has any incentive to deviate from it, as the ring representative bids with the regular strategy $\beta(x) = x$ as he would have done anyways, while all other members could have participated in the auction but certainly lost. However from this consideration it is not clear why the ring should form in the first place, as there is currently no mechanism for identifying Y_1^I (which could be costly) nor any mechanism for distributing the rings gain from the representative to other members.

To properly function the ring needs a ring-centre which is responsible for identifying the member with the highest value and ensure financing to pay members for participating in the ring.

The PAKT To identify the highest valuing member the ring center arranges a PAKT (Pre Auction Knockout) auction where the members bid to become the ring representative. The PAKT is a second price auction as well, revealing the valuation of all ring members to the centre. With this information as well as observations from the actual auction, the ring centre can calculate the price the representative would have paid in absence of the ring $\hat{p}_i$, and "tax" the representative with $\hat{p}_i - p_i$ to compensate the other ring members. Note that this implies the ring only realizes a surplus if the representative wins the auction, as there is otherwise nothing to "tax".
Like the real auction the PAKT will be a second price auction. This serves both to identify the representing member and later to calculate reimbursements for the remaining member on the basis of their expected gain from participating in the ring. The auction is a second price format ensuring all participants reveal their true preferences. Clearly this ring is incentive compatible, as nobody would be better off by leaving the ring, but only in expectation. (since it is only in the case where the ring representative wins that the ring has anything to pay back to members).

This in-expectation budget balance makes it unlikely to observe these kinds of bidding rings in the world. Only all inclusive bidding rings can guarantee a profit for all of its members, meaning if we should expect to find collusion in single round auctions it would most likely be of this kind. Typically however bidding collusion is seen in settings with multiple rounds or items. An example of single-ish round collusion is tried in State vs. Pool.

Expanding the ring increases the per bidder expected profit. This is because it doesn't affect the probability of winning the auction in the end, but the expected price if winning decreases because one less bidder is outside the ring and thus the expected highest out-of-ring bid is decreased.
Unlike the kind of positive cartel-externalities that sometimes arise in regular markets, there is no benefit for out-of-ring members from a ring, as the distribution if signals is independent and only one item is sold. This gives bidders outside the ring an increased incentive to join the ring.

7.1 Collusion from the auctioneers perspective

First of, the auction will still yield an efficient outcome when the ring exists, since draws are still the same and the bidder with the highest valuation wins. The expected revenue will be lower. The argument for this is that bidders outside the ring have the same expected payments, while in-ring bidders have lower than normal expected payments so overall the revenue will expectedly be lower. The revenue equivalence theorem does not hold, because the equilibria is no longer symmetric with a ring involved.

Reserve prices and rings To counter the bidding ring the auctioneer can implement a reserve price (or if there is proof of collusion go to the courts). The ring implies there are only $N - I + 1$ de facto bidders, where the ring representatives valuations are distributed differently from the rest. Say valuations are $Y_1^I, X_{I+1}, ..., X_N$. Now define $Z^{\mathcal{I}}$ to be the second highest value of $Y_1^I, X_{I+1}, ..., X_N$ and notice that because of the second price structure the price is

$$\hat{P} = \max\{Z^{\mathcal{I}}, r\} \tag{7.3}$$

where r is the reserve price. Letting $H^{\mathcal{I}}$ be the distribution of $Z^{\mathcal{I}}$ (with density $h^{\mathcal{I}}$) we can write the expected selling price as

$$r \underbrace{\left(H^{\mathcal{I}}(r) - G(r)\right)}_{P(Y_1^N > r,\ Z^{\mathcal{I}} < 0)} + \int_r^{\omega} z h^{\mathcal{I}}(z)\ dz \tag{7.4}$$

Taking the derivative of this it can be seen that in optimum it must be that

$$H(r^*) - G(r^*) - r^* g(r^*) = 0 \tag{7.5}$$

Now consider what happens if the ring grows from I members to $I + 1$ members. This can affect H in two ways. Either the new member had the second highest value before, which can lower the second highest value if the bidder does not get to represent the ring. Or the bidder had the previous highest value in which case he will represent the ring and reduce the second highest bid if this was submitted by the ring. The implication is that adding more members can only shift the distribution of second highest values lower, so $H^{\mathcal{I}+1}(r^*) > H^{\mathcal{I}}(r^*)$. This implies the optimality condition is not satisfied because the derivative is positive, and (assuming this is a single peaked condition) the optimality condition must therefore require an $r^{**} > r^*$ when considering $H^{\mathcal{I}+1}$.

7.2 Bidding rings in first price auctions

In principle a mechanism similar to the PAKT exists in first price auctions, but here bidders outside the ring will also change their strategy in response to the ring. The reason we are less likely to see collusion is first price auctions is that ring members have an incentive to cheat and bid just above the agreed upon price and thus winning at a low price. Theoretically this is unfixable, so in the world collusion in first price auctions are most likely also associated with some kind of coercion by the ring leader.

8 Repeated auctions and tenders

So far the auctions we have considered have had only one item for sale, with no possibility of acquiring the item when the auction ends. In the real world however many auctions are repeated, either to sell the same item or one identical to the first one, and bidders anticipate this repeatedness.

Consider a case where there is to be held K first price auctions. There are N bidders who only want to buy one item each. Their valuations are distributed according to $F : [0, \omega] \to [0, 1[$ if a bidder wins he drops out of the following rounds.
From the perspective of bidder 1: let Y_r be the rth highest bid among the $n-1$ other bidders, with distribution F_r. We will look for a set of symmetric equilibrium strategies $(\beta_1, \beta_2, ..., \beta_K)$ where in each stage the bidder has both his own value and the prices at which items sold in previous rounds as information $\beta_k(x, p_1, p_2, ..., p_{k-1})$.

Consider the case where $K = 2$, since β_k's are increasing functions we can see that items will be sold in order of descending values. Furthermore because we assume a symmetric equilibrium all bidders can infer $y_1 = \beta^{-1}(p_1)$ where y_1 is the value of the first winner.
Working backwards through the problem begin by considering a bidder in the second round. He knows he shouldn't bid above $\beta(y_1, y_1)$ because the descending order means no bidders have a value above y_1 in the second round. The expected payoff from bidding with some $z \leq y_1$ is

$$\Pi_i(z, x|y_1) = \underbrace{F_2(z|Y_1 = y_1)}_{P(\text{winning round 2})}(x - \beta_2(z, y_1)) \tag{8.1}$$

maximizing w.r.t z gives

$$\beta_2'(z, y_1) = \frac{f_2(z|Y_1 = y_1)}{F_2(z|Y_1 = y_1)}(x - \beta_2(z, y_1)) \tag{8.2}$$

Now use that draws are independent so the only information about Y_1 contained in y_1 is that $Y_2 < y_1$, so $F_2(x|Y_1 = y_1) = \frac{F(x)^{N-2}}{F(y_1)^{N-2}}$. Inserting this above, along with the assumption that $x = z$ in the equilibrium, gives

$$\beta_2'(x, y_1) = \frac{(N-2)f(x)}{F(x)}(x - \beta_2(x, y_1)) \tag{8.3}$$

Krishna p.215 shows that solving this differential equation yields

$$\beta_2(x) = E[Y_2|Y_2 < x < Y_1] \tag{8.4}$$

i.e. bid your expectation of the highest remaining bidders value (apart from yourself), conditional on you having the actual highest value, both of which are surely smaller than y_1.

This solves the second round game, and we can plug in this solution to derive the first round stategies. In the first round we need to consider the two cases $z \geq x$ and $z < x$ separately, because the outcomes in the second round will differ between the two. The expected payoff from bidding $\beta_1(z)$ with $z \geq x$ is

$$\Pi_i(z, x) = F_1(z)(x - \beta_1(z)) + (N-1)(1 - F(z))F(x)^{N-2}(x - \beta_2(x)) \tag{8.5}$$

which is simply the probability of winning in round 1 times the payoff, plus the probability of winning in round 2 times the payoff here. Winning in round two requires $Y_2 \leq x \leq z \leq Y_1$, giving rise to the probability.
Similarly if $z < x$ we have

$$\Pi_i(z, x) = F_1(z)(x - \beta_1(z)) + (F_2(x) - F_1(x))(x - \beta_2(x)) + \int_z^x [x - \beta_2(y_1)]f_1(y_1)\, dy_1 \tag{8.6}$$

Which is the probability of winning in the first round times the payoff from this, plus the payoff from loosing the first auction but winning the second, and the third term arises from the probability that the bidder pretends to have value z in the first round and only in the second round realize that his true valuation x is larger than Y_1. Solving this (Krishna p.216) it can be shown that

$$\beta_1(x) = E[Y_2 | Y_1 < x] \tag{8.7}$$

In summary the equilibrium strategy when $K = 2$ is always to bid your expectation of the second highest value among all the bidders. Generally it can be shown that for K round first price auctions with single-unit demand the symmetric equilibrium strategy is

$$\beta_k(x) = E[Y_K | Y_k < x < Y_{K-1}] \tag{8.8}$$

i.e. your expectation on the K'th highest value conditional on your knowledge of the already revealed values. This also implies that bidders bid more aggressively in later rounds until the final round which functions like a regular first price auction with $N - K + 1$ bidders. The expected price however does not trend upwards, because the higher bids are relative to their expectations of other the remaining bidders values.

9 Repeated auctions and tenders II

In lecture 8 we saw that the sequential first price auctions with unit demand had the equilibrium

$$\beta_k(x) = E[Y_K | Y_K < x < Y_{K-1}] \tag{9.1}$$

Krishna shows this is equivalent to

$$\beta_k(x) = E[\beta_{k+1}(Y_K) | Y_K < x < Y_{K-1}] \tag{9.2}$$

that is the optimal bid in round K is equal to the expectation of what the second highest bidder would do in the following round. This balances the tradeoff between not wanting to bid to high in the current round with the knowledge that shading of all the other bidders decreases as the auction moves through the rounds.

9.1 Repeated second price auctions

The single round result in second price auctions is that bidders will bid their valuation. In the last round of a sequential auction this is clearly still the case, as the auction is then just a regular second price auction. To investigate the mechanism in the auctions $k < K$ we can use the revenue equivalence theorem to link the revenue from K sequential first price auctions to the revenue in the sequential second price auction (notice we haven't shown this formally, but RE extends to multi-unit and sequential auctions as well). Thus the sum of expected payments across all K auctions is identical in the two auctions

$$m^I(x) = \sum_{k=1}^{K} m_k^I(x) = m^{II}(x) = \sum_{k=1}^{K} m_k^{II}(x) \tag{9.3}$$

This implies that revenue equivalence also holds for any local k such that $m_k^I(x) = m_k^{II}(x)$. (See slide 9.12). Using this it can be shown that

$$\beta_k^{II}(x) = \beta_{k+1}^I(x) \tag{9.4}$$

meaning the second price auction is in principle like the first price counterpart except bids are shifted one round back. This shows it is optimal to shade in a multiround second price auction, but not as much as in the first price auction.

9.2 Overall intuition of sequential auctions

Overall we see that across the two investigated formats, the bidders shade in the first rounds of the auction as they can "gamble" on the chance that other bidders have low valuations. Towards the end the auction approaches the regular one-shot auctions from earlier. In the real world we often have sequential auctions without a predetermined number of rounds, i.e. where $K = "\infty"$. Here we should expect bidders to shade heavily. Oppositely bidders might also have "infinite" demand meaning they value winning every round, which intuitively will reduce shading.

10 Policy considerations & multi unit auctions

References