# CONTROL SYSTEM DESIGN

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# I INTRODUCTION

## 1.1 What is Automatic Control?

Control is used to modify the behavior of a system so it behaves in a specific desirable way over time. For example, we may want the speed of a car on the highway to remain as close as possible to 60 miles per hour in spite of possible hills or adverse wind; or we may want an aircraft to follow a desired altitude, heading and velocity profile independently of wind gusts; or we may want the temperature and pressure in a reactor vessel in a chemical process plant to be maintained at desired levels. All these are being accomplished today by control methods and the above are examples of what automatic control systems are designed to do, without human intervention. Control is used whenever quantities such as speed, altitude, temperature or voltage must be made to behave in some desirable way over time.

This section provides an introduction to control system design methods.

To gain some insight into how an automatic control system operates we shall briefly examine the speed control mechanism in a car.

It is perhaps instructive to consider first how a typical driver may control the car speed over uneven terrain. The driver, by carefully observing the speedometer, and appropriately increasing or decreasing the fuel flow to the engine, using the gas pedal, can maintain the speed quite accurately. Higher accuracy can perhaps be achieved by looking ahead to anticipate road inclines. An automatic speed control system, also called cruise control, works by using the difference, or error, between the actual and desired speeds and knowledge of the car's response to fuel increases and decreases to calculate via some algorithm an appropriate gas pedal position, so to drive the speed error to zero. This decision process is called a *control law* and it is implemented in the *controller*. The system configuration is shown in Figure 1.1. The car dynamics of interest are captured in the *plant*. Information about the actual speed is fed back to the controller by *sensors*, and the control decisions are implemented via a device, the *actuator*, that changes the position of the gas pedal. The knowledge of the car's response to fuel increases and decreases is most often captured in a mathematical model.

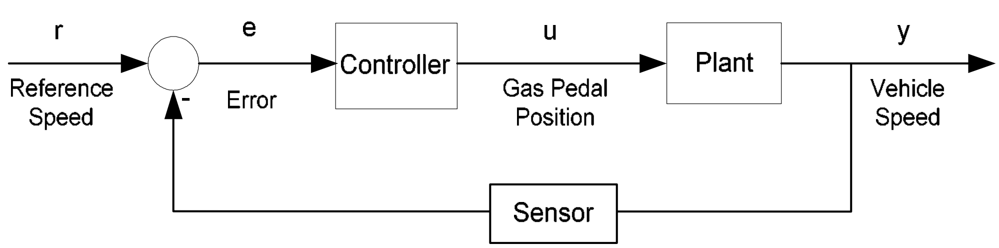


Figure 1.1 Feedback Control Configuration with Cruise Control as an Example

Certainly in an automobile today there are many more automatic control systems such as Anti-lock Brake System (ABS), emission control, tracking control. The use of feedback controller preceded control theory, outlined in the following sections, by over two thousand years. In fact, the first feedback device on record is the famous Water Clock of Ktesibios in Alexandria, Egypt from the 3rd century BC.

## 1.2 Proportional-Integral-Derivative Control

The Proportional-Integral-Derivative (PID) Controller, defined by

$$u = K_P e + K_I \int e + K_D \dot{e} \qquad (1.1)$$

is a particularly useful control approach that was invented over eighty years ago. Here $K_P$, $K_I$, *and* $K_D$ are controller parameters to be selected, often by trial and error or by the use of a look-up table in industry practice. The goal, as in the cruise control example, is to drive the error to zero in a desirable manner. All three terms in (1.1) have explicit physical meanings in that e is the current error, $\int e$ the accumulated error and $\dot{e}$ represents the trend. This, together with the basic understanding of the causal relationship between the control signal (u) and the output (y), forms the basis for engineers to "tune", or adjust, the controller parameters to meet the design specifications. This intuitive design, as it turns out, is sufficient for many control applications.

To this day, PID control is still the predominant method in industry and it is found in over 95% of industrial applications. Its success can be attributed to the simplicity, efficiency, and effectiveness of this method.

### 1.3 The Role of Control Theory

To design a controller that makes a system behave in a desirable manner we need a way to predict the behavior of the quantities of interest over time, specifically how they change in response to different inputs. Mathematical models are most oftenly used to predict future behavior and control system design methodologies are based on such models. Understanding control theory requires engineers to be well versed in basic mathematical concepts and skills, such as solving differential equations and using Laplace transform. The role of control theory is to help us gain insight on how and why feedback control systems work and how to *systematically* deal with various design and analysis issues. Specifically, the following issues are of both practical importance and theoretical interest:

1. Stability and stability margins of closed-loop systems;
2. How fast and smooth the error between the output and the setpoint is driven to zero;
3. How well the control system handles unexpected external disturbances, sensor noises, and internal dynamic changes;

In the following, modeling and analysis are first introduced, followed by an overview of the classical design methods for single-input single-output plants, design evaluation methods, and implementation issues. Alternative design methods are then briefly presented. Finally,. For the sake of simplicity and brevity, the discussion is restricted to linear, time invariant systems. Results maybe found in the literature for the cases of linear, time-varying systems, and also for nonlinear systems, systems with delays, systems described by partial differential equations and so on; these results however tend to be more restricted and case dependent.

## II. Mathematical Descriptions

Mathematical models of physical processes are the foundations of control theory. The existing analysis and synthesis tools are all based on certain types of mathematical descriptions of the systems to be controlled, also called plants. Most require that the plants are linear, causal, and time-invariant. Three different mathematical models for such plants, namely, linear ordinary differential equation, state variable or state space description , transfer function are introduced below.

### 2.1 Linear Differential Equations

In control system design the most common mathematical models of the behavior of interest are, in the time domain, linear ordinary differential equations with constant coefficients, and in the frequency or transform domain, transfer functions obtained from time domain descriptions via Laplace transforms.

Mathematical models of dynamic processes are often derived using physical laws such as Newton's and Kirchhoff's. As an example consider first a simple mechanical system, a spring/mass/damper. It consists of a weight m on a spring with spring constant k, its motion damped by friction with coefficient f.

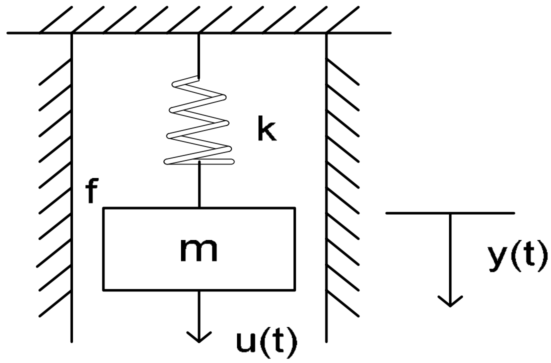

Figure 2.1 A Spring, Mass, and Damper System

If y(t) is the displacement from the resting position and u(t) the force applied, it can be shown using Newton's law that the motion is described by the following linear, ordinary differential equation with constant coefficients:

$$\ddot{y}(t)+\frac{f}{m}\dot{y}(t)+\frac{k}{m}y(t)=\frac{1}{m}u(t)$$

where $\dot{y}(t)\overset{\Delta}{=}\dfrac{dy(t)}{dt}$ with initial conditions $y(t)\big|_{t=0}=y(0)=y_0$ and $\dfrac{dy(t)}{dt}\Big|_{t=0}=\dfrac{dy(0)}{dt}=\dot{y}(0)=y_1$

Note that in the next subsection the trajectory y(t) is determined, in terms of the system parameters, the initial conditions and the applied input force u(t), using a methodology based on Laplace transform. The Laplace transform is briefly reviewed in Appendix A.

For a second example consider an electric RLC circuit with i(t) the input current of a current source, and v(t) the output voltage across a load resistance R.

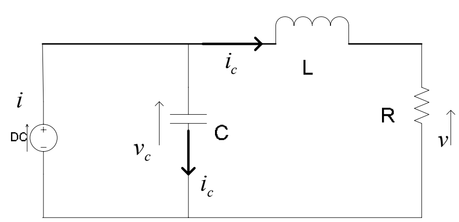

Figure 2.2 A RLC Circuit

Using Kirchhoff's laws one may derive:

$$\ddot{v}(t)+\frac{R}{L}\dot{v}(t)+\frac{1}{LC}v(t)=\frac{R}{LC}i(t)$$

which describes the dependence of the output voltage v(t) to the input current i(t). Given $i(t)$ for $t\geq 0$, the initial values $v(0)$ and $\dot{v}(0)$ must also be given to uniquely define $v$(t) for $t\geq 0$.

It is important to note the similarity between the two differential equations that describe the behavior of a mechanical and an electrical system respectively. Although the interpretation of the variables is completely different, their relations described by the same linear, second order differential equation with constant coefficients. This fact is well understood and leads to the study of mechanical, thermal, fluid systems via convenient electric circuits.

## 2.2 State Variable Descriptions

Instead of working with many different types of higher order differential equations that describe the behavior of the system, it is possible to work with an equivalent set of standardized first order vector differential equations that can be derived in a systematic way. To illustrate, consider the spring/mass/damper example. Let $x_1(t) = y(t)$ $x_2(t) = \dot{y}(t)$ be new variables, called *state variables*. Then the system is equivalently described by the equations

$\dot{x}_1(t) = x_2(t)$ and $\dot{x}_2(t) = \frac{-f}{m}x_2(t) - \frac{k}{m}x_1(t) + \frac{1}{m}u(t)$

with initial conditions $x_1(0) = y_0$ and $x_2(0) = y_1$. Since y(t) is of interest, the output equation $y(t) = x_1(t)$ is also added.

These can be written as

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k/m & -f/m \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1/m \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

which are of the general form

$$\dot{x}(t) = Ax(t) + Bu(t) \qquad y(t) = Cx(t) + Du(t)$$

Here x(t) is a 2x1 vector (a column vector) with elements the two state variables $x_1(t)$ and $x_2(t)$. It is called the *state vector*. The variable u(t) is the *input* and y(t) the *output* of the system. The first equation is a vector differential equation called the *state equation*. The second equation is an algebraic equation called the *output equation*. In the above example D=0; D is called the direct link, as it directly connects the input to the output, as opposed to through x(t) and the dynamics of the system. The above description is the *State Variable or State Space Description* of the system. The advantage is that, system descriptions can be written in a standard form (the state space form) for which many mathematical results exist. We shall present a number of them in this section.

A state variable description of a system can sometimes be derived directly, and not through a higher order differential equation. To illustrate, consider the circuit example presented above:

Using Kirchhoff's current law $i_c = C\frac{dv_c}{dt} = i - i_L$ and from the voltage law $L\frac{di_L}{dt} = -Ri_L + v_c$. If the state variables are selected to be $x_1 = v_c$, $x_2 = i_L$ then the equations may be written as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & -1/C \\ 1/L & -R/L \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1/C \\ 0 \end{bmatrix} i$$

$$v = \begin{bmatrix} 0 & R \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

where $v = Ri_L = Rx_2$ is the output of interest. Note that the choice of state variables is not unique. In fact if we start from the 2nd order differential equation and set $\bar{x}_1 = v$ and $\bar{x}_2 = \dot{v}$, we derive an equivalent state variable description, namely

$$\begin{bmatrix} \dot{\bar{x}}_1 \\ \dot{\bar{x}}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1/LC & -R/L \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ R/LC \end{bmatrix} i$$

$$v = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix}$$

Equivalent state variable descriptions are obtained by a change in the basis (coordinate system) of the vector state space. Any two equivalent representations

$\dot{x} = Ax + Bu \quad y = Cx + Du$ and $\dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}u \qquad y = \bar{C}\bar{x} + \bar{D}u$

are related by $\bar{A} = PAP^{-1}$, $\bar{B} = PB$, $\bar{C} = CP^{-1}$, $\bar{D} = D$

and $\bar{x} = Px$ where P is a square and nonsingular matrix. Note that state variables can represent physical quantities that may be measured, for instance $x_1 = v_c$ voltage, $x_2 = i_L$ current in the above example. Or they can be mathematical quantities, which may not have direct physical interpretation.

**Linearization**

The linear models studied here are very useful not only because they describe linear dynamical processes, but also because they can be approximations of nonlinear dynamical processes in the neighborhood of an operating point. The idea in linear approximations of nonlinear dynamics is analogous to using Taylor series approximations of functions to extract a linear approximation. A simple example is that of a simple pendulum $\dot{x}_1 = x_2$, $\dot{x}_2 = -k \sin x_1$ where for small excursions from the equilibrium at zero, $\sin x_1$ is approximately equal to $x_1$ and the equations become linear, namely $\dot{x}_1 = x_2$ $\dot{x}_2 = -kx_1$.

## 2.3 Transfer Functions

The *Transfer Function* of a linear, time-invariant system is the ratio of the Laplace transform of the output Y(s) to the Laplace transform of the corresponding input U(s) with all initial conditions assumed to be zero.

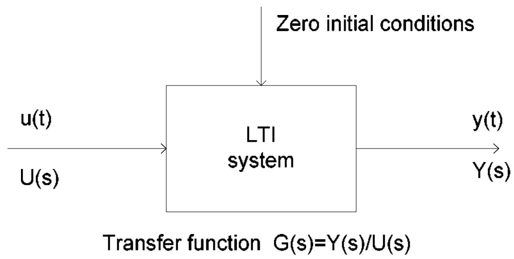


Figure 2.3 The Transfer Function Model

### 2.3.1 From Differential Equations to Transfer Functions

Let the equation

$$\frac{d^2 y(t)}{dt^2} + a_1 \frac{dy(t)}{dt} + a_0 y(t) = b_0 u(t)$$

with some initial conditions $y(t)\big|_{t=0} = y(0)$ and $\frac{dy(t)}{dt}\Big|_{t=0} = \frac{dy(0)}{dt} = \dot{y}(0)$ describe a process of interest, for example a spring/mass/damper system; see previous subsection.

Taking Laplace transform of both sides we obtain

$$\left[s^2Y(s) - sy(0) - \dot{y}(0)\right] + a_1[sY(s) - y(0)] + a_0Y(s) = b_0U(s)$$

where $Y(s) = L\{y(t)\}$ and $U(s) = L\{u(t)\}$. Combining terms and solving with respect to Y(s) we obtain:

$$Y(s) = \frac{b_0}{s^2 + a_1 s + a_0}U(s) + \frac{(s + a_1)y(0) + \dot{y}(0)}{s^2 + a_1 s + a_0}$$

Assuming the initial conditions are zero,

Y(s)/U(s) $= G(s) = \frac{b_0}{s^2 + a_1 s + a_0}$

where G(s) *is the transfer function of the system* defined above.

We are concerned with transfer functions G(s) that are rational functions, that is ratios of polynomials in s $G(s) = \frac{n(s)}{d(s)}$. We are interested in *proper* G(s) where $\lim_{s\to\infty} G(s) < \infty$. Proper G(s) have degree n(s) ≤ degree d(s). In most cases degree n(s) < degree d(s) in which case G(s) is called *strictly proper*. Consider the transfer function

$$G(s) = \frac{b_m s^m + b_{m-1}s^{m-1} + \ldots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0} \quad \text{with } m \le n$$

Note that the system described by this G(s) (Y(s)=G(s)U(s)) is described in the time domain by the following differential equation:

$$y^{(n)}(t) + a_{n-1}y^{(n-1)}(t) + \ldots + a_1 y^{(1)}(t) + a_0 y(t) = b_m u^{(m)}(t) + \ldots + b_1 u^{(1)}(t) + b_0 u(t)$$

where $y^{(n)}(t)$ denotes the nth derivative of y(t) with respect to time t. Taking Laplace transform of both sides of this differential equation, assuming that all initial conditions are zero, one obtains the above transfer function G(s).

### 2.3.2 From State Space Descriptions to Transfer Functions

Consider $\dot{x}(t) = Ax(t) + Bu(t)$, $y(t) = Cx(t) + Du(t)$ with x(0) initial conditions; x(t) is in general an n-tuple, that is a (column) vector with n elements. Taking Laplace transform of both sides of the state equation: $sX(s) - x(0) = AX(s) + BU(s)$ or $(sI_n - A)X(s) = BU(s) + x(0)$

where $I_n$ is the nxn *identity* matrix; it has 1 on all diagonal elements and 0 everywhere else e.g. $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Then

$$X(s) = (sI_n - A)^{-1}BU(s) + (sI_n - A)^{-1}x(0)$$

Taking now Laplace transform on both sides of the output equation we obtain $Y(s) = CX(s) + DU(s)$. Substituting we obtain,

$$Y(s) = [C(sI_n - A)^{-1}B + D]U(s) + C(sI - A)^{-1}x(0)$$

The response y(t) is the inverse Laplace of Y(s). Note that the second term on the right-hand side of the expression depends on x(0) and it is zero when the initial conditions are zero, i.e. when x(0)=0. The first term describes the dependence of Y on U and it is not difficult to see that the transfer function G(s) of the systems is

$$G(s) = C(sI_n - A)^{-1}B + D$$

**Example**

Consider the spring/mass/damper example discussed previously with state variable description

$\dot{x} = Ax + Bu, y = Cx$. If m=1, f=3, k=2 then

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

and its transfer function G(s) (Y(s)=G(s)U(s)) is

$$G(s) = C(sI_2 - A)^{-1}B = \begin{bmatrix} 1 & 0 \end{bmatrix} \left[ \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \right]^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \end{bmatrix} \frac{1}{s^2 + 3s + 2} \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \frac{1}{s^2 + 3s + 2} \quad \text{as before.}$$

Using the state space description and properties of Laplace transform an explicit expression for y(t) in terms of u(t) and x(0) may be derived. To illustrate, consider the *scalar case* $\dot{z} = az + bu$ with z(0) initial condition. Using Laplace transform:

$$Z(s) = \frac{1}{s-a} z(0) + \frac{b}{s-a} U(s) \quad \text{from which}$$

$$z(t) = L^{-1}\{Z(s)\} = e^{at} z(0) + \int_0^t e^{a(t-\tau)} bu(\tau) d\tau$$

Note that the second term is a convolution integral. Similarly in the *vector case*, given

$\dot{x}(t) = Ax(t) + Bu(t); y(t) = Cx(t) + B(t)u(t)$ it can be shown that

$$x(t) = e^{At} x(0) + \int_0^t e^{A(t-\tau)} Bu(\tau) d\tau \quad \text{and}$$

$$y(t) = Ce^{At} x(0) + \int_0^t Ce^{A(t-\tau)} Bu(\tau) d\tau + Du(t)$$

Notice that $e^{At} = L^{-1}\{(sI - A)^{-1}\}$. The *matrix exponential* $e^{At}$ is defined by the (convergent) series

$$e^{At} = I + e^{At} + \frac{A^2 t^2}{2!} + \ldots + \frac{A^k t^k}{k!} + \ldots = I + \sum_{k=1}^{\infty} \frac{t^k}{k!} A^k$$

### 2.3.3 Poles and Zeros

The n roots of the denominator polynomial d(s) of G(s) are the *poles of G(s)*. The m roots of the numerator polynomial n(s) of G(s) are (finite) *zeros of G(s)*.

**Example**

$$G(s)=\frac{s+2}{s^2+2s+2}=\frac{s+2}{(s+1)^2+1}=\frac{s+2}{(s+1-j)(s+1+j)}$$

G(s) has one (finite) zero at –2 and two complex conjugate poles at $-1\pm j$

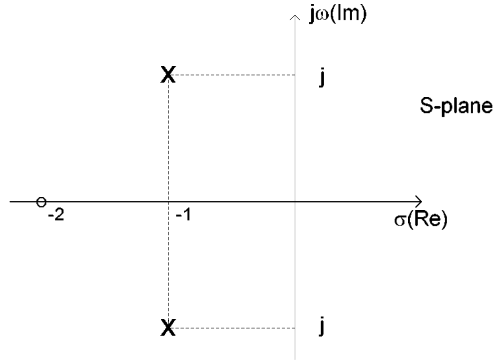


Figure 2.2 Complex Conjugate Poles of G(s)

In general, a transfer function with m zeros and n poles can be written as

$G(s)=k\frac{(s-z_1)...(s-z_m)}{(s-p_1)...(s-p_n)}$ where k is the *gain*.

## 2.4 Frequency Response

The *frequency response of a system* is given by its transfer function G(s) evaluated at s=jω, that is, G(jω). The frequency response is a very useful means of characterizing a system, since typically it can be determined experimentally, and since control system specifications are frequently expressed in terms of the frequency response. When the poles of G(s) have negative real parts, the system turns out to be bounded-input/bounded-output (BIBO) stable-see subsection 3.1. Under these conditions the frequency response G(jω) has a clear physical meaning, and this fact can be used to determine G(jω) experimentally. In particular, it can be shown that if the input u(t) = k sin($\omega_o$t) is applied to a system with a stable transfer function G(s) (Y(s) = G(s)U(s)), then the output y(t) at steady-state (after all transients have died out) is given by

$y_{ss}(t)=k\,|G(\omega_o)|\,\sin(\omega_o t+\theta(\omega_o))$

where $|G(\omega_o)|$ denotes the magnitude of $G(j\omega_o)$ and $\theta(\omega_o)=\arg G(j\omega_o)$ is the argument or phase of the complex quantity $G(j\omega_o)$. Applying sinusoidal inputs with different frequencies $\omega_o$ and measuring the magnitude and phase of the output at steady state it is possible to determine the full frequency response of the system $G(j\omega_o) = |G(\omega_o)|\,e^{(j\theta(\omega o))}$.

# III. Analysis of Dynamical Behavior

## 3.1 System Response, Modes and Stability

It was shown above how the response of a system to an input and under some given initial conditions can be calculated from its differential equation description using Laplace transforms. Specifically, $y(t)=L^{-1}\{Y(s)\}$ where $Y(s)=\frac{n(s)}{d(s)}U(s)+\frac{m(s)}{d(s)}$ with n(s)/d(s) = G(s) the system transfer function; the numerator m(s) of the second term depends on the initial conditions and it is zero when all initial conditions are zero, i.e. when the system is initially at rest.

In view now of the partial fraction expansion rules, see Appendix A, Y(s) can be written as follows:

$$Y(s) = \frac{c_1}{s-p_1} + \ldots + \frac{c_{i1}}{s-p_i} + \frac{c_{i2}}{(s-p_i)^2} + \ldots + \frac{b_1 s + b_0}{s^2 + a_1 s + a_0} + \ldots + I(s)$$

This expression shows real poles of G(s), namely $p_1$, $p_2$ etc. and it allows for multiple poles $p_i$; it also shows complex conjugate poles a±jb written as second order terms. I(s) denotes the terms due to the input U(s); they are fractions with poles the poles of U(s). Note that if G(s) and U(s) have common poles they are combined to form multiple-pole terms.
Taking now the inverse Laplace transform of Y(s) :

$$y(t) = L^{-1}\{Y(s)\} = c_1 e^{p_1 t} + \ldots + c_{i1} e^{p_i t} + (\cdot) t e^{p_i t} + \ldots + e^{at}((\cdot)\sin bt + (\cdot)\cos bt) + \ldots + i(t)$$

where i(t) depends on the input. Note that the terms of the form $ct^k e^{p_i t}$ are the *modes of the system*. The system behavior is the aggregate of the behaviors of the modes. Each mode depends primarily on the location of the pole $p_i$ ; the location of the zeros affect the size of its coefficient c.

If the input u(t) is a *bounded signal*, i.e. $|u(t)| < \infty$ for all t, then all the poles of I(s) have real parts that are negative or zero, and this implies that I(t) is also bounded for all t. In that case, the response y(t) of the system will also be bounded for any bounded u(t) if and only if all the poles of G(s) have strictly negative real parts. Note that poles of G(s) with real parts equal to zero are not allowed, since if U(s) also has poles at the same locations, y(t) will be unbounded. Take for example G(s)=1/s and consider the bounded step input U(s)=1/s; the response y(t)=t which is not bounded.

Having all the poles of G(s) located in the open left half of the s-plane is very desirable and it corresponds to the system being stable. In fact, *a system is bounded-input, bounded-output (BIBO) stable if and only if all poles of its transfer function have negative real parts*. If at least one of the poles has positive real part then the system is *unstable*. If a pole has zero real part then the term *marginally stable* is sometimes used.

Note that in a BIBO stable system if there is no forcing input, but only initial conditions are allowed to excite the system, then y(t) will go to zero as t goes to infinity. This is a very desirable property for a system to have, because nonzero initial conditions always exist in most real systems. For example, disturbances such as interference may add charge to a capacitor in an electric circuit, or a sudden brief gust of wind may change the heading of an aircraft. In a stable system the effect of the disturbances will diminish and the system will return to its previous desirable operating condition. For these reasons a control system should first and foremost be guaranteed to be stable, that is it should always have poles with negative real parts. There are a many design methods to stabilize a system or if it is initially stable to preserve its stability, and several are discussed later in this section.

### 3.2 Response of First and Second Order Systems

Consider a system described by a first order differential equation, namely

$\dot{y}(t) + a_0 y(t) = a_0 u(t)$ and let y(0)=0. In view of the previous subsection, the transfer function of the system is

$$G(s) = \frac{a_0}{s + a_0}$$

and the response to a *unit step input* q(t) (q(t)=1 for t≥0, q(t)=0 for t<0) may be found as follows:

$$y(t) = L^{-1}\{Y(s)\} = L^{-1}\{G(s)U(s)\} = L^{-1}\left\{\frac{a_0}{s+a_0}\frac{1}{s}\right\}$$

$$= L^{-1}\left\{\frac{1}{s}+\frac{-1}{s+a_0}\right\} = \left[1-e^{-a_0 t}\right]q(t)$$

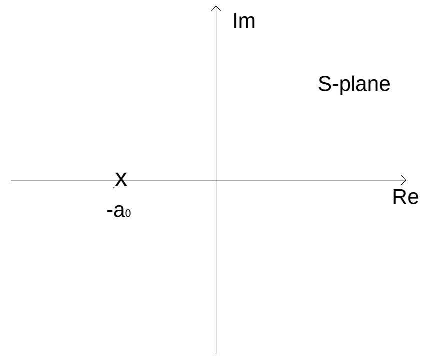


Figure 3.1 Pole location of a First Order System

Note that the pole of the system is $p=-a_0$ (in Figure 3.1 we have assumed that $a_0>0$). As that pole moves to the left on the real axis, i.e. as $a_0$ becomes larger, the system becomes faster. This can be seen from the fact that the steady state value of the system response $y_{ss} = \lim_{t\to\infty} y(t) = 1$ is approached by the trajectory of y(t) faster, as $a_0$ becomes larger. To see this, note that the value $1-e^{-1}$ is attained at time $\tau=1/a_0$, which is smaller as $a_0$ becomes larger. The time $\tau$ is *the time constant* of this first order system; see below for further discussion of the time constant of a system.
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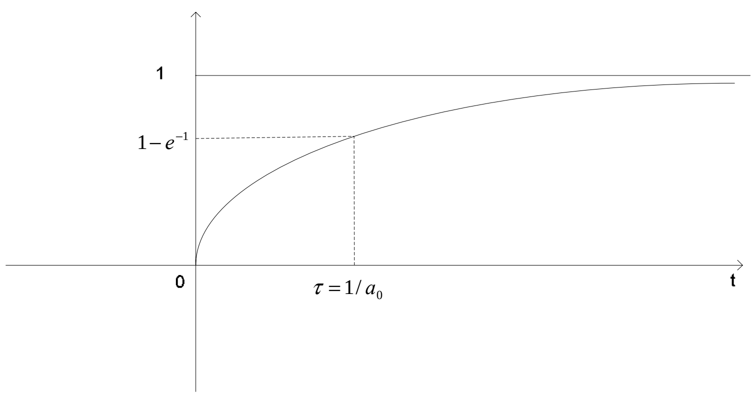


Figure 3.2 Step Response of A First Order Plant

We now derive the response of a 2nd order system to a unit step input. Consider a system described be $\ddot{y}(t)+a_1\dot{y}(t)+a_0 y(t) = a_0 u(t)$ which gives rise to the transfer function:

$$G(s)=\frac{a_0}{s^2+a_1 s+a_0}$$

Notice that the steady state value of the response to a unit step is $y_{ss} = \lim_{s\to 0} sG(s)\frac{1}{s} = 1$; note that this normalization or scaling to 1 is in fact the reason for selecting the constant numerator to be $a_0$. G(s) above does not have any finite zeros-only poles- as we want to study first the effect of the poles on the system behavior. We shall discuss the effect of adding a zero or an extra pole later.

It is customary, and useful as we will see, to write the above transfer function as

$$G(s)=\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

where $\zeta$ is the *damping ratio* of the system and $\omega_n$ is the *(undamped) natural frequency* of the system, i.e. the frequency of oscillations when the damping is zero.
The poles of the system are

$$p_{1,2}=-\zeta\omega_n\pm\omega_n\sqrt{\zeta^2-1}$$

When $\zeta$ >1 the poles are real and distinct and the unit step response approaches its steady state value of 1 without overshoot. In this case the system is *overdamped*. The system is called critically damped when $\zeta$ =1 in which case the poles are real, repeated and located at $-\zeta\omega_n$.
The more interesting case is when the system is *underdamped* ($\zeta$ <1). In this case the poles are complex conjugate and are given by

$$p_{1,2}=-\zeta\omega_n\pm j\omega_n\sqrt{1-\zeta^2}=\sigma+j\omega_d$$

The response to a unit step input in this case is

$$y(t)=\left[1-\frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin(\omega_d t+\theta)\right]q(t)$$

where $\theta=\cos^{-1}\zeta=\tan^{-1}(\sqrt{1-\zeta^2}/\zeta)$ , $\omega_d=\omega_n\sqrt{1-\zeta^2}$ and q(t) is the step function.

The response to an *impulse input* $(u(t)=\delta(t))$ also called the *impulse response* h(t) of the system is given in this case by

$$h(t)=\left[\omega_n\frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin(\omega_n\sqrt{1-\zeta^2}\,t)\right]\text{q(t)}$$

The second order system is parameterized by the two parameters $\zeta$ and $\omega_n$. Different choices for $\zeta$ and $\omega_n$ lead to different pole locations and to different behavior of the (modes of) the system. The graph below shows the relation between the parameters and the pole location.

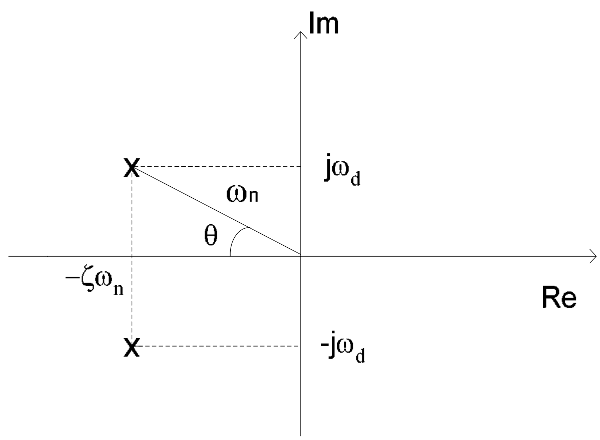


Figure 3.3 Relation between pole location and parameters

<u>Time constant of a mode and of a system</u>
The time constant of a mode $ce^{pt}$ of a system is the time value that makes $|pt|$=1, i.e. $\tau=1/|p|$. For example, in the above fist order system we have seen that $\tau=1/a_0=RC$. A pair of complex conjugate poles

$p_{1,2}=\sigma \pm j\omega$ give rise to the term of the form $Ce^{\sigma t}sin(\omega t+\theta)$. In this case, $\tau=1/|\sigma|$. i.e. τ is again the inverse of the distance of the pole from the imaginary axis. The time constant of a system is the time constant of its dominant modes.

### 3.3 Transient Response Performance Specifications for a Second Order Underdamped System

For the system $G(s)=\dfrac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$ and a unit step input, explicit formulas for important measures of performance of its transient response can be derived. Note that the steady state is $y_{ss}=\lim_{s\to 0} sG(s)\dfrac{1}{s}=1$.

The *rise time* $t_r$ shows how long it takes for the system's output to rise from 0 to 66% of its final value (equal to 1 here) and it can be shown to be $t_r=\dfrac{\pi-\theta}{\omega_n}$ where $\theta=\cos^{-1}\zeta$ and $\omega_d=\omega_n\sqrt{1-\zeta^2}$.

The *settling time* $t_s$ is the time required for the output to settle within some percentage, typically 2% or 5%, of its final value.

$t_s \cong \dfrac{4}{\zeta\omega_n}$ is the 2% settling time ( $t_s \cong \dfrac{3}{\zeta\omega_n}$ is the 5%settling time)

Before the underdamped system settles, it will overshoot its final value. The *peak time* $t_p$ measures the time it takes for the output to reach its first (and highest) peak value. $M_p$ measures the actual *overshoot* that occurs in percentage terms of the final value. $M_p$ occurs at time $t_p$, which is the time of the first and largest overshoot.

$$t_p=\frac{\pi}{\omega_d}\ ,\quad M_p=100e^{-\zeta\pi/\sqrt{1-\zeta^2}}\ \%$$

It is important to notice that the overshoot depends only on $\zeta$. Typically, tolerable overshoot values are between 2.5% and 25%, which correspond to damping ratio $\zeta$ between .8 and .4.

### 3.4 Effect of Additional Poles and Zeros

The addition of an extra pole in the left-half s-plane (LHP) tends to slow the system down-the rise time of the system for example will become larger. When the pole is far to the left of the imaginary axis, its effect tends to be small. The effect becomes more pronounced as the pole moves towards the imaginary axis.

The addition of a zero in the LHP has the opposite effect, as it tends to speed the system up. Again the effect of a zero far away to the left of the imaginary axis tends to be small. It becomes more pronounced as the zero moves closer to the imaginary axis.

The addition of a zero in the right-half s-plane (RHP) has a delaying effect much more severe than the addition of a LHP pole. In fact a RHP zero causes the response (say, to a step input) to start towards the wrong direction. It will move down first and become negative for example before it becomes positive again and starts towards its steady state value. Systems with RHP zeros are called *non-minimum phase systems* (for reasons that will become clearer after the discussion of the frequency design methods) and are typically difficult to control. Systems with only LHP poles (stable) and LHP zeros are called *minimum phase systems*.

## IV. Classical Control Design Methods

In this section, we focus on the problem of controlling a single-input and single-output (SISO) LTI plant. It is understood from the above sections that such a plant can be represented by a transfer function, $G_p(s)$. The closed-loop system is shown in Figure 4.1 below:

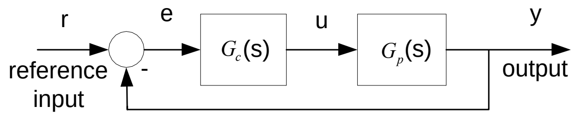


Figure 4.1 Feedback Control Configuration

The goal of feedback control is to make the output of the plant, y, follow the reference input, r, as closely as possible. Classical design methods are those used to determine the controller transfer function, $G_c(s)$, so that the closed-loop system, represented by the transfer function:

$$G_{CL}(s) = \frac{G_c(s)G_p(s)}{1+G_c(s)G_p(s)} \tag{4.1}$$

has desired characteristics.

### 4.1 Design Specifications and Constraints

The design specifications are typically described in terms of step response, i.e. r is the setpoint described as a step like function. These specifications are given in terms of transient response and steady state error, assuming the feedback control system is stable. The transient response is characterized by the rise time, i.e. the time it takes for the output to reach 66% of its final value, the settling time, i.e. the time it takes for the output to settle within 2% of its final value, and the percent overshoot, which is how much the output exceeds the setpoint r percentage wise during the period that y converges to r. The steady state error refers to the difference, if any, between y and r as y reaches its steady state value. See subsection 3.3.

There are many constraints a control designer has to deal with in practice, as shown in Figure 4.2. They can be described as follows:

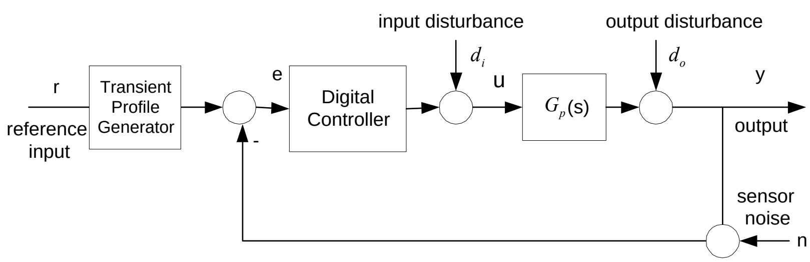


Figure 4.2 Closed-Loop Simulator Setup

1. Actuator Saturation: the input, u, to the plant is physically limited to a certain range, beyond which it "saturates", i.e. becomes a constant.
2. Disturbance Rejection and Sensor Noise Reduction: There are always disturbances and sensor noises in the plant to be dealt with.
3. Dynamic Changes in the Plant: physical systems are almost never truly linear nor time invariant.
4. Transient Profile: In practice, it is often not enough to just move y from one operating point to another. How it gets there is sometimes just as important. Transient profile is a mechanism to define the desired trajectory of y in transition, which is of great practical concerns. The smoothness of y and its derivatives, the energy consumed, the maximum value and the rate of change required of the control action are all influenced by the choice of transient profile.

5. Digital Control: Most controllers are implemented today in digital forms, which makes the sampling rate and quantization errors limiting factors in the controller performance.

### 4.2 Control Design Strategy Overview

The control strategies are summarized here in ascending order of complexity and, hopefully, performance.

1. **Open-Loop Control:**
   If the plant transfer function is known and there is very little disturbance, a simple open loop controller, as shown in Figure 4.3, would satisfy most design requirements.

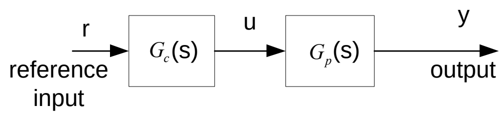


Figure 4.3 Open-loop Control Configuration

   where $G_c(s)$ is an approximate inverse of $G_p(s)$. Such control strategy has been used as an economic means in controlling stepper motors, for example.

2. **Feedback Control with a Constant Gain**
   With significant disturbance and dynamic variations in the plant, feedback control, as shown in Figure 4.1, is the only choice; see also Appendix B. Its simplest form is $G_c(s)=k$, or $u= ke$, where k is a constant. Such proportional controller is very appealing because of its simplicity. The common problems with this controller are significant steady state error and overshoot.

3. **Proportional-Integral-Derivative Controller**
   To correct the above problems with the constant gain controller, two additional terms are added: $u = k_p e + k_i \int e + k_d \dot{e}$ or $G_c(s) = k_p + k_i / s + k_d s$. This is the well-known PID controller, which is used by most engineers in industry today. The design can be quite intuitive: the proportional term usually plays the key role, with the integral term added to reduce/eliminate the steady state error and the derivative term the overshoot. The primary drawbacks of PID is that the integrator introduces phase lag that could lead to stability problems and the differentiator makes the controller sensitive to noise.

4. **Root Locus Method**
   A significant portion of most current control textbooks is devoted to the question of how to place the poles of the closed-loop system in Figure 4.1 at desired locations, assuming we know where they are. Root Locus is a graphical technique to manipulate the closed-loop poles given the open-loop transfer function. This technique is most effective if disturbance rejection, plant dynamical variations, and sensor noise are not to be considered. This is because these properties can not be easily linked to closed loop pole locations.

5. **Loop-Shaping Method**
   Loop-shaping [5] refers to the manipulation of the *loop gain* frequency response, $L(j\omega)=G_p(j\omega)G_c(j\omega)$, as a control design tool. It is the only existing design method that can bring most of design specifications and constraints, as discussed above, under one umbrella and systematically find a solution. This makes it a very useful tool in understanding, diagnosing and solving practical control problems. The loop-shaping process consists of two steps:
   1) convert all design specifications to loop gain constraints, as shown in Figure 4.4;
   2) find a controller $G_c(s)$ to meet the specifications.

Loop-shaping as a concept and a design tool helped the practicing engineers greatly in improving the PID loop performance and stability margins. For example, a PID implemented as a lead-lag compensator is commonly seen in industry today. This is where the classical control theory provides the mathematical and design insights on why and how feedback control works. It has also laid the foundation for modern control theory.

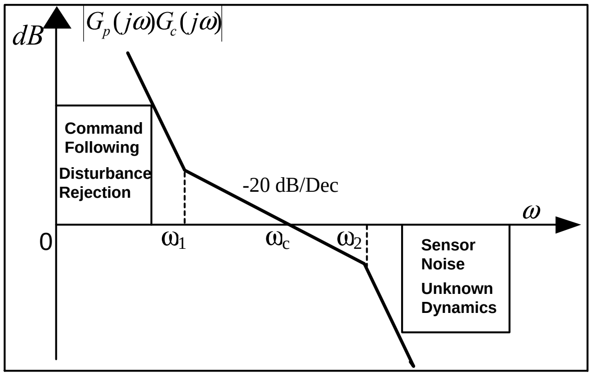


Figure 4.4 Loop-shaping

**Example 4.1** Consider a motion control system as shown in Figure 4.5 below. It consists of a digital controller, a DC motor drive (motor and power amplifier), and a load of 235 lbs that is to be moved linearly by 12 inches in 0.3 second with an accuracy of 1% or better. A belt and pulley mechanism is used to convert the motor ration to a linear motion. Here a servo motor is used to drives the load to perform a linear motion. The motor is coupled with the load through a pulley.

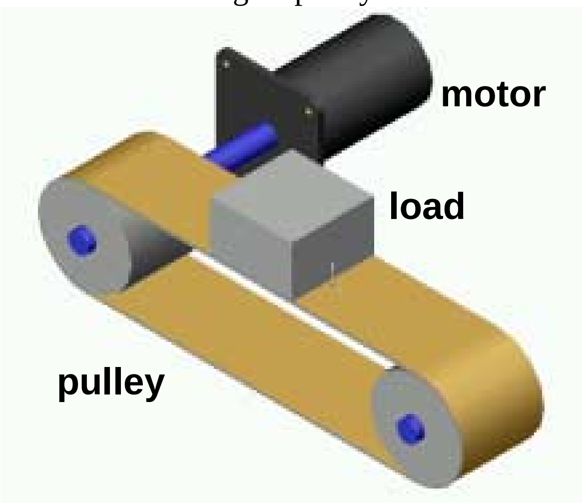


Figure 4.5 A Digital Servo Control Design Example

The design process involves:

1. Selection of components including: motor, power amplifier, the belt-and-pulley, the feedback devices (position sensor and/or speed sensor)
2. Modeling of the plant
3. Control design and simulation
4. Implementation and tuning

The first step results in a system with the following parameters:

*1) Electrical*:

- Winding resistance and inductance: $R_a$=.4 om $L_a$= 8 mH (the transfer function of armature voltage to current is $(1/R_a)/((L_a/R_a)s+1)$ ;
- back emf constant:$K_E$=1.49 v/(rad/sec),
- power amplifier gain:$K_{pa}$=80,
- current feedback gain: $K_{cf}$ =.075 v/amp;

*2) Mechanical*:

- Torque constant: $K_t$=13.2 in-lb/amp;
- motor inertia $J_m$=.05 lb-in-sec$^2$;
- Pulley radius $R_p$=1.25 in;
- load weight: W=235 lbs (including the assembly);
- total inertia $J_t=J_m+J_l=.05+(W/g)R_p^2=1.0$ lb-in-sec$^2$.

With the maximum armature current set at 100 amp, the Maximum Torque = $K_t I_{a,max}$=13.2x100 = 1320 in-lbs; the maximum angular acceleration = 1320/$J_t$=1320 rad/sec$^2$, and the maximum linear acceleration = 1320*$R_p$=1650 in/sec$^2$=4.27 g's(1650/386). As it turned out, they are sufficient for this application.

The second step produces a simulation model as:

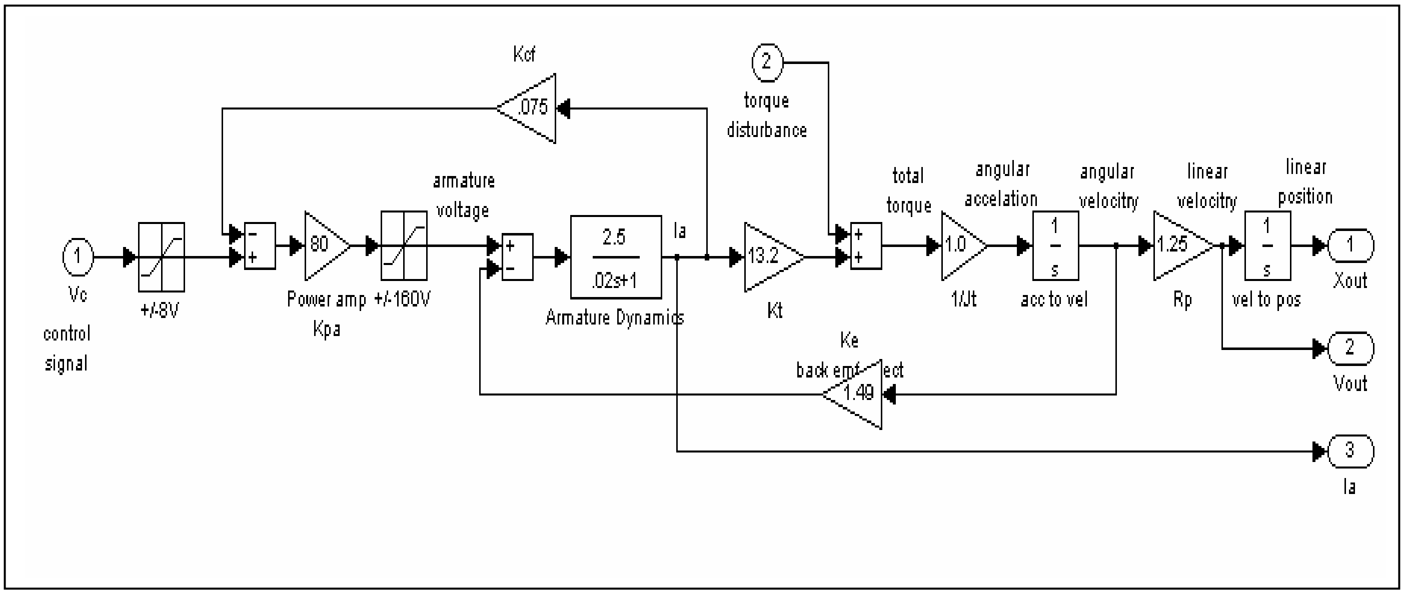


Figure 4.6 Simulation model of the motion control system

A simplified transfer function of the plant, from the control input, $v_c$ (in volts), to the linear position output, $x_{out}$ (in inches), is

$$G_p(s) = \frac{206}{s(s+3)} \tag{4.1}$$

An open loop controller is not suitable here because it cannot handle the torque disturbances and the inertia change in the load. Now consider the feedback control scheme in Figure 4.1 with a constant controller, u=ke. The root locus plot in Figure 4.7 indicates that, even at a high gain, the real part of the closed-loop poles does not exceed -1.5, which corresponds to a settling time of about 2.7 sec. This is far slower than desired.

In order to make the system respond faster, the closed-loop poles must be moved further away from the jω axis. In particular, a settling time of .3 sec or less corresponds to the closed-loop poles with real parts smaller than -13.3 This is achieved by using a PD controller of the form

$$G_c(s)= K(s+3)\ ,\ K\geq 13.3/206 \tag{4.2}$$

will result in a settling time of less than .3 sec.

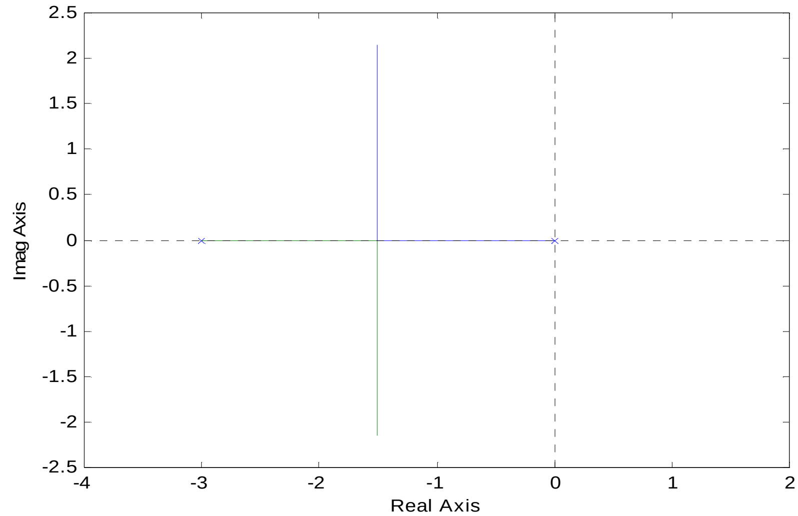


Figure 4.7 Root Locus plot of the servo design problem

The above PD design is a simple solution in servo design that is commonly used. There are several issues, however, that can not be completely resolved in this framework:

1) Low frequency torque disturbance induces steady state error that affect the accuracy;
2) The presence of a resonant mode within or close to the bandwidth of the servo loop may create undesirable vibrations;
3) Sensor noise may cause the control signal to be very noisy;
4) The change in the dynamics of the plant, for example, the inertia of the load, may require frequency tweaking of the controller parameters;
5) The step like setpoint change results in an initial surge in the control signal and could shorten the life span of the motor and other mechanical parts.

These are problems that most control textbook do not adequately address, but they are of significant importance in practice. The first three problems can be tackled using the loop-shaping design technique introduced above. The tuning problem is an industry wide design issue and the focus of various research and development efforts. The last problem is addressed by employing a smooth transient as the set point, instead of a step like setpoint. This is known as the “motion profile” in industry.

## 4.3 Evaluation of Control Systems

Analysis of control system provides crucial insights to control practitioners on why and how feedback control works. Although the use of PID precedes the birth of classical control theory of the 1950s by at least two decades, it is the latter that established the control engineering discipline. The core of classical control theory are the frequency response based analysis techniques, namely, Bode and Nyquist plots, stability margins, etc.

In particular, by examining the loop gain frequency response of the system in Figure 4.1, that is $L(j\omega)=G_c(j\omega)G_p(j\omega)$, and the sensitivity function $1/(1+L(j\omega))$, one can determine the following:

1) How fast the control system responds to the command or disturbance input (i.e. the bandwidth);
2) Whether the closed-loop system is stable (Nyquist Stability Theorem); If it is stable, how much dynamic variation it takes to make the system unstable (in terms of the gain and phase change in the plant). It leads to the definition of gain and phase margins. More broadly, it defines how robust the control system is.

3) How sensitive the performance (or closed-loop transfer function) is to the changes in the parameters of the plant transfer function (described by the sensitivity function);
4) The frequency range and the amount of attenuation for the input and output disturbances shown in Figure 4.2 (again described by the sensitivity function).

Evidently, these characteristics obtained via frequency response analysis are invaluable to control engineers. The efforts to improve these characteristics led to the lead-lag compensator design and, eventually, loop-shaping technique described above.

**Example 4.2**: The PD controller in (4.2) is known to be sensitive to sensor noises. A practical cure to this problem is add a low pass filter to the controller to attenuate high frequency noises, that is

$$G_c(s) = \frac{13.3(s+3)}{206(\frac{s}{133}+1)^2} \tag{4.3}$$

The loop gain transfer function is now

$$\mathrm{L(s)=G_p}(s)G_c(s) = \frac{13.3}{s(\frac{s}{133}+1)^2} \tag{4.4}$$

The bandwidth of the low pass filter is chosen to be one decade higher the loop gain bandwidth to maintain proper gain and phase margins. The Bode plot of the new loop gain, as shown in Figure 4.8, indicates that a) the feedback system has a bandwidth 13.2 rad/sec, which corresponds to a .3 sec settling time as specified, b) This design has adequate stability margins (gain margin is 26 dB and phase margin is 79 degrees).

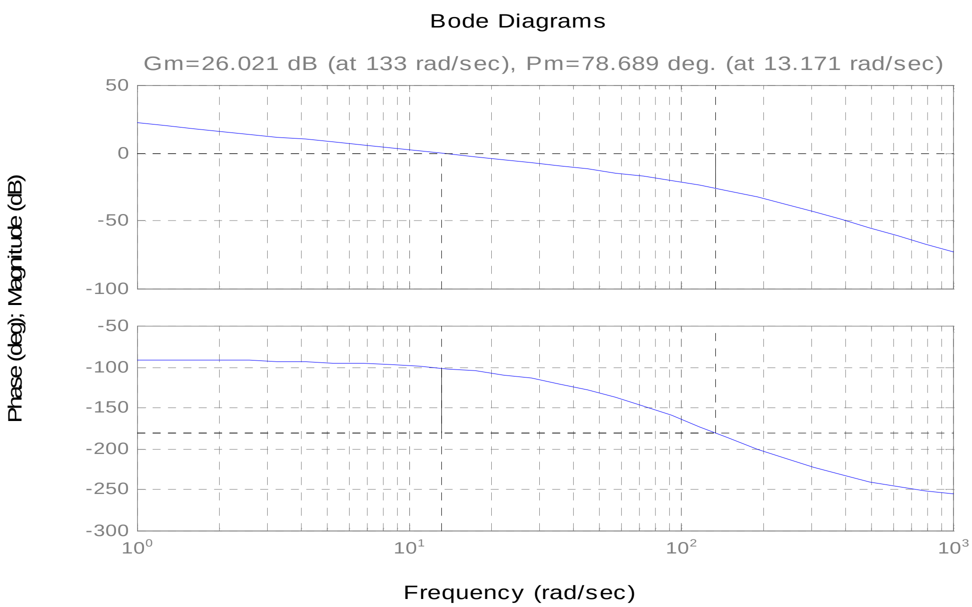


Figure 4.8 Bode plot evaluation of the control design

## 4.4 Digital Implementation

Once the controller is designed and simulated successfully, the next step is to digitize it so that it can be programmed into the processor in the digital control hardware. To do this:

1) Determine the sampling period, $T_s$, and the number of bits used in Analog-to-Digital Converter (ADC) and Digital-to-Analog Converter (DAC);

2) Convert the continuous time transfer function $G_c(s)$ to its corresponding form in discrete time transfer function, $G_{cd}(z)$, using, for example, the Tustin's method, $s=(1/T)(z-1)/(z+1)$;
3) From $G_{cd}(z)$, derive the difference equation, $u(k)=g(u(k-1), u(k-2), \ldots y(k), y(k-1), \ldots)$, where g is a linear algebraic function;

After the conversion, the sampled data system, with the plant running in continuous time and the controller in discrete time, should be verified in simulation first before the actual implementation. The quantization error and sensor noise should also be included to make it realistic.

The minimum sampling frequency required for a given control system design has not been established analytically. The rule of thumb given in control textbooks is that $f_s=1/T_s$ should be chosen approximately 30-60 times the bandwidth of the closed-loop system. Lower sampling frequency is possible after careful tuning but the aliasing, or signal distortion, will occur when the data to be sampled has significant energy above the Nyquist frequency. For this reason, an anti-aliasing filter is often placed in front of the ADC to filter out the high frequency contents in the signal.

Typical ADC and DAC chips have 8, 12, and 16 bits of resolution. It is the length of the binary number used to approximate an analog one. The selection of the resolution depends on the noise level in the sensor signal and the accuracy specification. For example, the sensor noise level, say .1%, must be below the accuracy specification, say .5%. Allowing one bit for the sign, an 8-bit ADC with a resolution of $1/2^7$, or, .8%, is not good enough; similarly, a 16-bit ADC with a resolution .003%. is unnecessary because several bits are "lost" in the sensor noise. Therefore, a 12-bit ADC, which has a resolution of .04%, is appropriate for this case. This is an example of "error budget", as it is known among designers, where components are selected economically so that the sources of inaccuracies are distributed evenly.

Converting $G_c(s)$ to $G_{cd}(z)$ is a matter of numerical integration. There have been many methods suggested, some are too simple and inaccurate (such as the Euler's forward and backward methods), others are too complex. The Tustin's method suggested above, also known as trapezoidal method or bilinear transformation, is a good compromise. Once the discrete transfer function, $G_{cd}(z)$, is obtained, finding the corresponding difference equation that can be easily programmed in C is straightforward. For example, given a controller with input e(k) and output u(k), and the transfer function $G_{cd}(z)=\frac{z+2}{z+1}=\frac{1+2z^{-1}}{1+z^{-1}}$ the corresponding input-output relationship is $u(k)=\frac{1+2z^{-1}}{1+z^{-1}}e(k)$, or equivalently, $(1+z^{-1})u(k)= (1+2z^{-1})e(k)$. That is, $u(k)=-u(k-1)+e(k)+2e(k-1)$.

Finally, the presence of the sensor noise usually requires that an anti-aliasing filter be used in front of the ADC to avoid distortion of the signal in ADC. The phase lag from the such as filter must not occur at the crossover frequency (bandwidth) or it will reduce the stability margin or even destabilize the system. This puts yet another constraint on the controller design.

## V. Alternative Design Methods:

### 5.1 Nonlinear PID

Using Nonlinear PID (NPID) is an alternative to PID for better performance. It maintains the simplicity and intuition of PID, but empowers it with nonlinear gains. An example of NPID is shown in Figure 5.1. The need for the integral control is reduced, by making the proportional gain larger, when the error is small. The limited authority integral control has its gain zeroed outside a small interval around the origin to reduce the phase lag. Finally the differential gain is reduced for small errors to reduce sensitivities to sensor noise. More details can be found in [8].

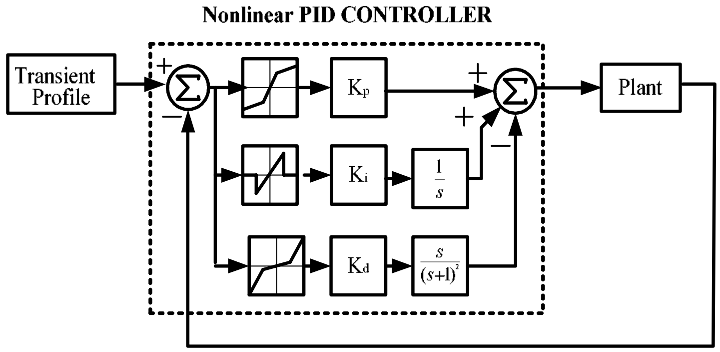


Figure 5.1 Nonlinear PID for a Power Converter Control Problem

### 5.2 State Feedback and Observer Based Design

If the state space model of the plant

$$\begin{aligned}\dot{x} &= Ax + Bu \\ y &= Cx + Du\end{aligned} \tag{5.1}$$

is available, the pole-placement design can be achieve via state feedback

$$u=r+Kx \tag{5.2}$$

where K is the gain vector to be determined so that the eigenvalues of the closed-loops system

$$\begin{aligned}\dot{x} &= (A + BK)x + Br \\ y &= Cx + Du\end{aligned} \tag{5.3}$$

are at the desired locations, assuming they are known. Usually the state vector is not available through measurements and the state observer of the form

$$\begin{aligned}\dot{\hat{x}} &= A\hat{x} + Bu + L(y - \hat{y}) \\ \hat{y} &= C\hat{x} + Du\end{aligned} \tag{5.4}$$

where $\hat{x}$ is the estimate of x and L is the observer gain vector to be determined.

The state feedback design approach has the same drawbacks as those of Root Locus approach, but the use of the state observer does provide a means to extract the information about the plant that is otherwise unavailable in the previous control schemes, which are based the input-output descriptions of the plant. This proves to be valuable in many applications. In addition, the state space methodologies are also applicable to systems with many inputs and outputs.

#### 5.2.1 Controllability and Observability

Controllability and observability are useful system properties and are defined as follows. Consider an $n^{th}$ order system described by

$\dot{x} = Ax + Bu \;\; y = Cx + Du$

where A is an nxn matrix.

The system is *controllable* if it is possible to transfer the state to any other state in finite time. This property is important as it measures, for example, the ability of a satellite system to re-orient itself to face another part of the earth’s surface using the available thrusters; or to shift the temperature in an industrial oven to a specified temperature. Two equivalent tests for controllability are:

The system (or the pair (A,B)) is *controllable* if and only if the controllability matrix

$\mathcal{C} = \left[B, AB, \ldots A^{n-1}B\right]$

has full (row) rank n. Equivalently if and only if

$\left[s_i I - A, B\right]$
has full (row) rank n for all eigenvalues $s_i$ of A.

The system is *observable* if by observing the output and the input over a finite period of time it is possible to deduce the value of the state vector of the system. If for example a circuit is observable it may be possible to determine all the voltages across the capacitors and all currents through the inductances by observing the input and output voltages.

The system (or the pair (A,C)) is *observable* if and only if the observability matrix

$$\mathcal{O} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

has full (column) rank n. Equivalently if and only if

$$\begin{bmatrix} s_i I - A \\ C \end{bmatrix}$$

has full (column) rank n for all eigenvalues $s_i$ of A.

Consider now the transfer function
$G(s) = C(sI - A)^{-1} B + D$
Note that, by definition, in a transfer function all possible cancellations between numerator and denominator polynomials are assumed to have already taken place. In general, therefore, the poles of G(s) are some (or all) of the eigenvalues of A. It can be shown that when the system is both controllable and observable no cancellations take place and so in this case the poles of G(s) are exactly the eigenvalues of A.

### 5.2.2 Eigenvalue Assignment Design

Consider (5.1) and (5.2). It is known that when the system is controllable, K can be selected to assign the closed loop eigenvalues to any desired locations (real or complex conjugate) and so to significantly modify the behavior of the open loop system. Many algorithms exist to determine such K. In the case of a single input, there is a convenient formula called Ackermann's formula
$K = -[0, \ldots 0,1]\ \mathcal{C}^{-1}\alpha_d(A)$
where $\mathcal{C} = \left[B, ..., A^{n-1}B\right]$ is the nxn controllability matrix and the roots of $\alpha_d(s)$ are the desired closed loop eigenvalues.

**Example 5.1**
Let $A = \begin{bmatrix} 1/2 & 1 \\ 1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and the desired eigenvalues be $-1 \pm j$
Here

$$\mathcal{C} = \left[B, AB\right] = \begin{bmatrix} 1 & 3/2 \\ 1 & 3 \end{bmatrix}$$

Note that A has eigenvalues at 0 and 5/2. We wish to determine K so that the eigenvalues of A+BK are at $-1 \pm j$, which are the roots of $\alpha_d(s) = s^2 + 2s + 2$.

Here $\mathcal{C} = [B, AB] = \begin{bmatrix} 1 & 3/2 \\ 1 & 3 \end{bmatrix}$ and

$$\alpha_d(A) = A^2 + 2A + 2I = \left( \begin{bmatrix} 1/2 & 1 \\ 1 & 2 \end{bmatrix}^2 + 2\begin{bmatrix} 1/2 & 1 \\ 1 & 2 \end{bmatrix} + 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) = \begin{bmatrix} 17/4 & 9/2 \\ 9/2 & 11 \end{bmatrix}$$

Then

$$K = -[0 \quad 1] C^{-1} \alpha_d(A) = [-1/6 \quad -13/3]$$

Here $A + BK = \begin{bmatrix} 1/3 & -10/3 \\ 5/6 & -7/3 \end{bmatrix}$ which has the desired eigenvalues.

### 5.2.3 Linear Quadratic Regulator (LQR) Problem

Consider

$\dot{x} = Ax + Bu \quad z = Mx$

We wish to determine u(t) $t \ge 0$ which minimizes the quadratic cost

$$J(u) = \int_0^\infty \left[ x^T(t)(M^T Q M)x + u^T(t) R u(t) \right] dt$$

for any initial state x(0). The weighting matrices Q and R are real, symmetric $(Q = Q^T, R = R^T)$, Q and R are positive definite $(R > 0, Q > 0)$ and $M^T Q M$ is positive semi-definite ($M^T Q M \ge 0$). Since R>0,the term $u^T R u$ is always positive for any $u \ne 0$, by definition. Minimizing its integral forces u(t) to remain small. $M^T Q M \ge 0$ implies that $x^T M^T Q M x$ is positive but it can also be zero for some $x \ne 0$ ; this allows some of the states to be treated as “do not care states.” Minimizing the integral of $x^T M^T Q M x$ forces the states to become smaller as time progresses. It is convenient to take Q (and R in the multi-input case) to be diagonal with positive entries on the diagonal. The above performance index is designed so that the minimizing control input drives the states to the zero state-or as close as possible- without using excessive control action, in fact minimizing the control energy. When $(A, B, Q^{1/2} M)$ is controllable and observable the solution $u^*(t)$ of this optimal control problem is a state feedback control law , namely

$u^*(t) = K^* x(t) = -R^{-1} B^T P_c^* \; x(t)$

where $P_c^*$ is the unique symmetric positive definite solution of the *algebraic Riccati equation*:

$$A^T P_c + P_c A - P_c B R^{-1} B^T P_c + M^T Q M = 0$$

**Example 5.2**

Consider $\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$, $y = [1 \quad 0] x$

And let

$$J = \int_0^\infty \left( y^2(t) + 4u^2(t) \right) dt$$

Here $M = C$, $Q = 1$, $M^T Q M = C^T C = \begin{bmatrix} 1 \\ 0 \end{bmatrix} [1 \quad 0] = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $R = 4$

Solving the Riccati equation we obtain

$$P_c^* = \begin{bmatrix} 2 & 2 \\ 2 & 2\sqrt{2} \end{bmatrix}$$

and

$$u^*(t) = K^* x(t) = -\frac{1}{4}\begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}2 & 2\\ 2 & 2\sqrt{2}\end{bmatrix} x(t) = -\frac{1}{2}\begin{bmatrix}1, \sqrt{2}\end{bmatrix} x(t)$$

### 5.2.4 Linear State Observers

Since the states of a system contain a great deal of useful information, knowledge of the state vector is desirable. Frequently, however, it may be either impossible or impractical to obtain measurements of all states. Therefore it is important to be able to estimate the states from available measurements namely of inputs and outputs.

Let the system be

$$\dot{x} = Ax + Bu\,,\ \ y = Cx + Du$$

An asymptotic state estimator of the full state, also called *Luenberger observer* is given by

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - \hat{y})$$

where L is selected so that all eigenvalues of A-LC are in the LHP (have negative real parts). Note that a L that arbitrarily assigns the eigenvalues of A-LC exists if and only if the system is observable. The observer may be written as

$$\dot{\hat{x}} = (A - LC)\hat{x} + \left[B - LD, K\right]\begin{bmatrix}u\\ y\end{bmatrix}$$

which clearly shows the role of u and y-they are the inputs to the observer. If the error $e(t) = x(t) - \hat{x}(t)$ then

$$e(t) = e^{[(A-LC)t]}e(0)$$

which shows that $e(t) \to 0$ or $\hat{x}(t) \to x(t)$ as $t \to \infty$

To determine appropriate L, note that $(A - LC)^T = A^T + C^T(-L) = \overline{A} + \overline{B}\overline{K}$ , which is the problem addressed above in the state feedback case. One could also use the following observable version of Ackermann's formula, namely

$L = \alpha_d(A)\ \mathcal{O}^{-1}[0, \ldots 0,1]^T$

where $\mathcal{O} = \begin{bmatrix} C\\ CA\\ \vdots\\ CA^{n-1}\end{bmatrix}$.

The gain L in the above estimator may be determined so that it is optimal in an appropriate sense. In the following, some of the key equations of such an optimal estimator (*Linear Quadratic Gaussian* (LQG)), also known as the *Kalman-Bucy filter* are briefly outlined.

Consider

$$\dot{x} = Ax + Bu + \Gamma w\,,\ \ y = Cx + v$$

where w and v represent process and measurement noise terms. Both w and v are assumed to be white, zero-mean Gaussian stochastic processes, i.e. they are uncorrelated in time and have expected values $E[w] = 0$ and $E[v] = 0$. Let $E\left[ww^T\right] = W$, $E\left[vv^T\right] = V$ denote the covariances where W, V are real, symmetric and positive definite matrices. Assume also that the noise processes w and v are independent, i.e. $E\left[wv^T\right] = 0$ . Also assume that the initial state x(0) is a Gaussian random variable of known mean, $E[x(0)] = x_0$ and known covariance $E\left[(x(0) - x_0)(x(0) - x_0)^T\right] = P_{e0}$.

Assume also that x(0) is independent of w and v.

Consider now the estimator

$$\dot{\hat{x}} = (A - LC)\hat{x} + Bu + Ly$$

and let $(A, \Gamma W^{1/2}, C)$ be controllable and observable. Then the error covariance $E\left[(x-\hat{x})(x-\hat{x})^T\right]$ is minimized when the filter gain

$$L^* = P_e^* C^T V^{-1}$$

where $P_e^*$ denotes the symmetric, positive definite solution of the (dual to control) *algebraic Riccati equation*

$$P_e A^T + A P_e - P_e C^T V^{-1} C P_e + \Gamma W \Gamma^T = 0$$

The above Riccati is the *dual* to the Riccati equation for optimal control and can be obtained from the optimal control equation by making use of the substitutions :

$A \to A^T$, $B \to C^T$, $M \to \Gamma^T$, $R \to V$, $Q \to W$

In the state feedback control law u=Kx+r, when state measurements are not available, it is common to use the estimate of the state $\hat{x}$ from a Luenberger observer. That is, given

$\dot{x} = Ax + Bu \quad y = Cx + Du$

the control law is $u = K\hat{x} + r$ where $\hat{x}$ is the state estimate from the observer

$$\dot{\hat{x}} = (A - LC)\hat{x} + \left[B - KD, K\right]\begin{bmatrix} u \\ y \end{bmatrix}$$

The closed loop system is then of order 2n since the plant and the observer are each of order n. It can be shown that in this case, of linear output feedback control design, the design of the control law and of the gain K (using for example LQR) can be carried out independently of the design of the estimator and the filter gain L (using for example LQG). This is known as the *separation property*.

It is remarkable to notice that the overall transfer function of the compensated system that includes the state feedback and the observer is

$$T(s) = (C + DK)\left[sI - (A + BK)\right]^{-1} B + D$$

which is exactly the transfer function one would obtain if the state x were measured directly and the state observer were not present. This is of course assuming zero initial conditions (to obtain the transfer function); if nonzero initial conditions are present then there is some deterioration of performance due to observer dynamics, and the fact that at least initially the state estimate typically contains significant error.

## VI. Advanced Analysis and Design Techniques

This section covered so far some fundamental analysis and design methods in classical control theory, the development of which was primarily driven by engineering practice and needs. Over the last few decades, vast efforts in control research have led to the creation of modern mathematical control theory, or advanced control, or control science. This development started with optimal control theory in the 50s and 60s to study the optimality of control design; a brief glimpse of optimal control was given above, in subsection 5. In optimal control theory, a cost function is to be minimized, and analytical or computational methods are used to derive optimal controllers. Examples include minimum fuel problem, time-optimal control (Bang-Bang) problem, LQ, $H_2$, and $H_\infty$, each corresponding to a different cost function. Other major branches in modern control theory include Multi-input Multi-output (MIMO) control systems methodologies, which attempt to extend well known SISO design methods and concepts to MIMO problems; Adaptive Control, designed to extend the operating range of a controller by adjusting automatically the controller parameters based on the estimated dynamic changes in the plants; Analysis and design of nonlinear control systems, etc.

A key problem is the robustness of the control system. The analysis and design methods in control theory are all based on the mathematical model of the plants, which is an approximate description of physical processes. Whether a control system can tolerate the uncertainties in the dynamics of the plants, or how much uncertainty it takes to make a system unstable, is studied in robust control theory, where $H_2$, $H_\infty$, and other analysis and design methods were originated. Even with recent progress, open problems remain when dealing with real world applications. Some recent approaches such as in [8] attempt to address some of these difficulties in a realistic way.

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**Appendix A**

**A Brief Review of Laplace Transform**

Given a function f(t) its *Laplace transform* is given by

$$L\{f(t)\} = F(s) = \int_0^{\infty} f(t)e^{-st}dt$$

The Laplace variable s can be seen as a generalized frequency $s = \sigma + j\omega$.
This is the one-sided Laplace transform, as the lower integration limit starts at 0 and not at $-\infty$. Note that the lower integration limit is in fact $0^-$, that is the integration starts just before 0. This is so to include any possible discontinuities in f(t) that may occur exactly at 0; this is the case for example with the impulse or delta function $\delta(t)$. In control, it is common for the signals of interest to be assumed to be zero for t<0 and so no signal information is lost when moving into the transform domain. This one-sided Laplace transform is particularly convenient when solving linear ordinary differential equations using Laplace transforms.

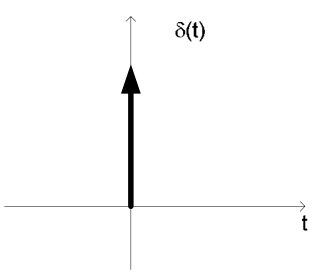
There is one-to-one correspondence between f(t) and F(s). To recover f(t) from F(s) one can use the inverse Laplace transform given by

$$L^{-1}\{F(s)\} = f(t) = \frac{1}{2\pi j}\int_{\sigma - j\infty}^{\sigma + j\infty} F(s)e^{st}ds$$

However, in control it is common to obtain f(t) from F(s) using partial fraction expansion, properties and tables of common Laplace transforms.

It is not difficult to derive the Laplace transforms of simple functions. To demonstrate, we select some functions common in control:

*Impulse (delta or Dirac) function* $\delta(t)$



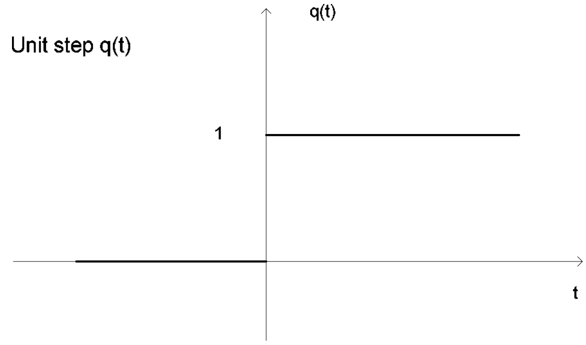

An important property of the delta function is given by

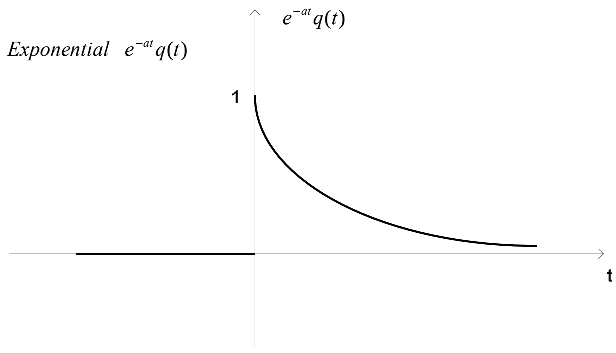
$$f(t) = \int_{-\infty}^{+\infty} \delta(t-\tau)f(\tau)d\tau$$

where in fact the integration range needs to cover only the point where the argument in $\delta(.)$ becomes zero. Then

$$L\{\delta(t)\} = \int_{0^-}^{\infty} \delta(t)e^{-st}dt = e^{-st}\Big|_{t=0} = 1$$

*Unit Step*




It is not difficult to show that $L\{q(t)\}=1/s$ where q(t) is the *unit step* (zero for t<0 and 1 for $t \le 0$). Also that $L\{e^{-at}\cdot q(t)\}=1/(s+a)$ ; note that $e^{-at}q(t)$ is zero for t<0 and $e^{-at}$ for $t \le 0$.

Below the Laplace transforms of useful in control functions are listed.

| f(t) | F(s) |
|---|---|
| $\delta(t)$ | 1 |
| q(t) | $1/s$ |
| $e^{-at}\cdot q(t)$ | $1/(s+a)$ |
| $\sin\omega t\cdot q(t)$ | $\omega/(s^2+\omega^2)$ |
| $\cos\omega t\cdot q(t)$ | $s/(s^2+\omega^2)$ |
| $t^n\cdot q(t)$ | $n!/s^{n+1}$ |
| $t^n e^{-at}\cdot q(t)$ | $n!/(s+a)^{n+1}$ |

**Some Properties of Laplace transform**

When solving linear ordinary differential equations with constant coefficients using Laplace transform properties that involve derivatives of the time functions are useful. These include

$$L\left\{\frac{df(t)}{dt}\right\}=sF(s)-f(0)$$

$$L\left\{\frac{d^2f(t)}{dt^2}\right\}=s^2F(s)-sf(0)-\frac{df(0)}{dt}$$

where $F(s) = L\{f(t)\}$. These and other properties are included in the list below:

| f(t) | F(s) |
|---|---|
| $\frac{d^k f^{(k)}}{dt^k}\left(= f^{(k)}(t)\right)$ | $s^k F(s) - s^{k-1} f(0) - s^{k-2} f^{(1)}(0) - \ldots - f^{(k-1)}(0)$ |
| $e^{-at} f(t) \cdot q(t)$ | $F(s+a)$ |
| Delay $f(t-a) \cdot q(t-a)$ | $e^{-as} F(s)$ |
| Convolution $\int f(t-\tau) g(\tau) d\tau$ | $F(s) \cdot G(s)$ |
| Final value $\lim\limits_{t \to \infty} f(t)$ | $\lim\limits_{s \to 0} sF(s)$ |
| Initial value $\lim\limits_{t \to 0^+} f(t)$ | $\lim\limits_{s \to \infty} sF(s)$ |

**Partial Fraction Expansion**

Write

$$G(s) = \frac{b_m s^m + \ldots + b_1 s \ + b_0}{s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0} \qquad m \le n$$

as the sum of simple terms, namely

$$G(s) = \frac{c_1}{s - p_1} + \frac{c_2}{s - p_2} + \ldots + \frac{c_n}{s - p_n}$$

where $p_j$ j=1,…,n are the n roots of the denominator polynomial (poles of G(s))

1.When all n $p_j$ are distinct then $c_j = \lim\limits_{s \to p_j} \left((s - p_j) G(s)\right)$

2.When 2 poles are complex conjugate, that is $p_1 = a + jb$ and $p_2 = a - jb$
(also written as $p_2 = p^*_1$) then

$$F(s) = \frac{c_1}{s - p_1} + \frac{c_1^*}{s - p_1^*}$$

from which

$$f(t) = 2|c_1| e^{at} \cos\left[bt + \mathrm{a}\,rg(c_1)\right]$$

Note that it is also common in the case of complex conjugate poles to allow a second order term in the partial fraction expansion of the form

$$\frac{b_1 s + b_0}{s^2 + a_1 s + a_0}$$

which could then be found directly from the tables in some cases.

3.When a $p_j$ is repeated r times then the expansion must include r terms that correspond to the pole $p_j$ These r terms are of the form

$$\frac{c_{j1}}{s - p_j} + \frac{c_{j2}}{(s - p_j)^2} + \ldots + \frac{c_{jr}}{(s - p_j)^r}$$

where

$$c_{jr} = \lim_{s \to p_j} \left((s - p_j)^r G(s)\right)$$

…
…

$$c_{j1} = \lim_{s \to p_j} \left( \frac{1}{(r-1)!} \frac{d^{r-1}}{ds^{r-1}} \left\{ (s - p_j)^r G(s) \right\} \right)$$

**Examples**

1. $F(s) = \dfrac{2s^2 + 3s + 3}{(s+1)(s+3)^3} = \dfrac{1/4}{s+1} + \dfrac{-1/8}{(s+3)} + \dfrac{2/3}{(s+3)^2} + \dfrac{-6}{(s+3)^3}$

Then

$$f(t) = L^{-1}\{F(s)\} = \left( \frac{1}{4} e^{-t} - \frac{1}{8} e^{-3t} + \frac{2}{3} t e^{-3t} - 3t^2 e^{-3t} \right) q(t)$$

Note that if F(s) represents the transfer function of a system then $f(t) = L^{-1}\{F(s)\}$ is the impulse response of the system.

2. $F(s) = \dfrac{2s+1}{s^2+1} = \dfrac{2s}{s^2+1} + \dfrac{1}{s^2+1}$

From the tables,

$$f(t) = (2\cos t + \sin t) q(t)$$

Note that the poles of F(s) are at $\pm j$. Alternatively, using the formulas for poles $p_1$ =j and $p_2$ =-j

$$\frac{1}{s^2+1} = \frac{-1/2j}{s-j} + \frac{+1/2j}{s-(-j)}$$

and

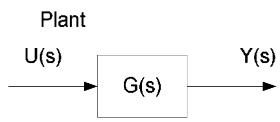
$$L^{-1}\left\{ \frac{1}{s^2+1} \right\} = 2 \cdot 1/2 \cdot e^{at} \cos(t + \pi/2) = \sin t \cdot q(t)$$

Similarly for the other term.

**Appendix B**
**Open and Closed Loop Stabilization**

In this appendix, it is shown that it is impossible to stabilize an unstable system using open loop control, due to system uncertainties. In general, closed loop or feedback control is necessary to control a system-stabilize if unstable and improve performance-because of uncertainties that are always present. Feedback provides current information about the system and so the controller does not have to rely solely on incomplete system information contained in a nominal plant model. These uncertainties are system parameter uncertainties and also uncertainties induced on the system by its environment, including uncertainties in the initial condition of the system, and uncertainties due to disturbances and noise.

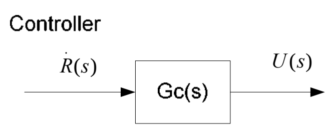
Consider the plant with transfer function $G(s) = \dfrac{1}{s-(1+\varepsilon)}$ where the pole location at 1 is inaccurately known.




The corresponding description in the time domain using differential equations is $\dot{y}(t) - (1+\varepsilon)y(t) = u(t)$ .
Solving using Laplace transform, we obtain $sY(s) - y(0) - (1+\varepsilon)Y(s) = U(s)$ from which

$$Y(s) = \frac{y(0)}{s-(1+\varepsilon)} + \frac{1}{s-(1+\varepsilon)}U(s)$$

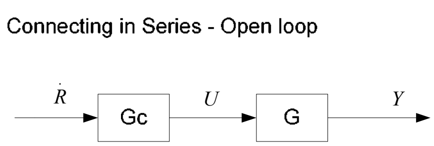
Consider now the controller with transfer function $G_c(s) = \dfrac{s-1}{s+2}$.




The corresponding description in the time domain using differential equations is $\dot{u}(t) + 2u(t) = \dot{r}(t) - r(t)$. Solving, using Laplace transform we obtain $sU(s) - u(0) + 2U(s) = sR(s) - R(s)$ from which

$$U(s) = \frac{u(0)}{s+2} + \frac{s-1}{s+2}R(s)$$

Connect now the plant and the controller in series (open loop control)




The overall transfer function is $T = GG_c = \dfrac{s-1}{(s-(1+\varepsilon))(s+2)}$

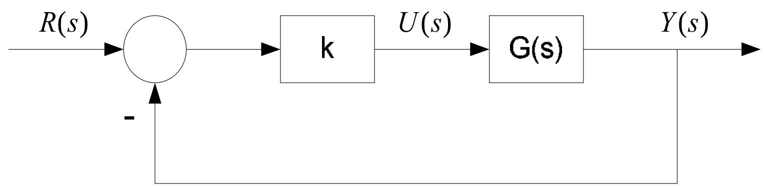
Including the initial conditions

$$Y(s) = \frac{(s+2)y(0) + u(0)}{(s-(1+\varepsilon))(s+2)} + \frac{s-1}{(s-(1+\varepsilon))(s+2)}R(s)$$

It is now clear that open loop control cannot be used to stabilize the plant:

1. First because of the uncertainties in the plant parameters. Note that the plant pole is not exactly at +1 but at $1+\varepsilon$ and so the controller zero cannot cancel the plant pole exactly.
2. Secondly, even if we had knowledge of the exact pole location, that is $\varepsilon = 0$, and $Y(s) = \dfrac{(s+2)y(0)+r(0)}{(s-1)(s+2)} + \dfrac{1}{s+2}R(s)$, still we cannot stabilize the system because of the uncertainty in the initial conditions. We cannot for example select r(0) so to cancel the unstable pole at +1 because y(0) may not be known exactly.

We shall now stabilize the above plant using a simple feedback controller.




Consider a unity feedback control system with the controller being just a gain k to be determined. The closed loop transfer function is $T(s) = \dfrac{kG(s)}{1+kG(s)} = \dfrac{k}{s-(1+\varepsilon)+k}$. Working in the time domain, $\dot{y} - (1+\varepsilon)y = u = k(r-y)$ from which $\dot{y} + [k-(1+\varepsilon)]y = kr$. Using Laplace transform we obtain

$sY(s) - y(0) + (k-(1+\varepsilon))Y(s) = kR(s)$ and

$$Y(s) = \frac{y(0)}{s+k-(1+\varepsilon)} + \frac{k}{s+k-(1+\varepsilon)}R(s)$$

It is now clear that if the controller gain is selected so that $k > 1+\varepsilon$ then the system is stable. In fact we could have worked with the nominal system to derive $k > 1$ for stability. For stability robustness, we take k somewhat larger than 1 so to have some safety margin and satisfy $k > 1+\varepsilon$ for some unknown small $\varepsilon$.