Assignment 4

Course name: PROBABILITY AND STOCHASTIC PROCESSES (AI5030)

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Question 1

A simple random sample of size n will be drawn from a class of 125 students, and the mean mathematics score of the sample will be computed. If the standard error of the sample mean for "**with sampling**" is twice as much as the standard error of the sample mean for "**without replacement sampling**", the value of n is-

1. 32
2. 63
3. 79
4. 94

Answer:
For SRSWR,

$$Var_{SRSWR}(\bar{y}) = (\sigma_Y^2)_{SRSWR} = \frac{\sigma^2}{n} \tag{1}$$

For SRSWOR,

$$Var_{SRSWOR}(\bar{y}) = (\sigma_Y^2)_{SRSWOR} = \frac{\sigma^2}{n}(\frac{N-n}{N-1}) \tag{2}$$

Given in the question,
Population Size= N=125
Sample Size=n
Population Variance=σ^2
Sample Standard Error for SRSWR= $(\sigma_Y)_{SRSWR}$
Sample Standard Error for SRSWR= $(\sigma_Y)_{SRSWOR}$
and, $(\sigma_Y)_{SRSWR} = 2 \times ((\sigma_Y)_{SRSWOR})$

So,

$$
\begin{aligned}
&\therefore (\sigma_Y)_{SRSWR} = 2 \times ((\sigma_Y)_{SRSWOR}) \\
&\implies \frac{\sigma}{\sqrt{n}} = 2 \times \frac{\sigma}{\sqrt{n}} \sqrt{\frac{N-n}{N-1}} \\
&\implies 1 = 2 \times \sqrt{\frac{125-n}{125-1}} \\
&\implies n = 94
\end{aligned}
\tag{3}
$$

Proof. Let's prove the formulas used above.

- N : Population Size
- Population Mean:
$$\bar{Y} = \frac{1}{N} \sum_{i=1}^{N} Y_i \tag{4}$$
- Population Variance:
$$\sigma^2 = \frac{1}{N} \sum_{i=1}^{N} (Y_i - \bar{Y})^2 = \frac{1}{N} \sum_{i=1}^{N} Y_i^2 - \bar{Y}^2 \tag{5}$$
- Population Mean Square:
$$S^2 = \frac{1}{N-1} \sum_{i=1}^{N} (Y_i - \bar{Y})^2 \tag{6}$$
- $$S^2 = (\frac{N}{N-1})\sigma^2 \tag{7}$$
- n: Sample Size
- $\bar{y} = \frac{1}{n} \sum_{i=1}^{n} y_i$: Sample Mean
- $s^2 = \frac{1}{n-1} \sum_{i=1}^{n} (y_i - \bar{y})^2$: Sample Mean Square

Theorem 1. *In SRSWOR,*

a) *P(Selecting a specified unit at r^{th} draw)=P(Selecting the unit at first draw)=$\frac{1}{N}$*

b) *P(Inclusion of specified sample of size n)=$\frac{n}{N}$*

Theorem 2. *In SRSWR, number of ways upto n units can be selected, one-by-one, is N^n ways.*

Theorem 3. *For both SRSWOR and SRSWR, sample mean $\bar{y}$ is an unbiased estimator of population mean $\bar{Y}$.*

$$E[\bar{y}] = \bar{Y}$$

Theorem 4. *In SRSWOR, the sample mean square s^2 is an unbiased estimator of population mean square S^2.*

$$E[s^2] = S^2$$

Remarks.

1) *From equation (5) and (7), we get-*

$$\sum_{i=1}^{N} Y_i^2 = (N-1)S^2 + N\bar{Y}^2 \tag{8}$$

2) *Generalized algebraic expansion for square of summation of variables can be used to write $\sum_{i \neq j=1}^{N} Y_i Y_j$ in terms of known quantities.*

$$(\sum_{i=1}^{N} Y_i)^2 = \sum_{i=1}^{N} Y_i^2 + \sum_{i \neq j=1}^{N} Y_i Y_j$$

Adding extra N^2 at LHS and substituting value from equation (8), we get-

$$\implies N^2(\frac{1}{N}\sum_{i=1}^{N} Y_i)^2 = [(N-1)S^2 + N\bar{Y}^2] + \sum_{i \neq j=1}^{N} Y_i Y_j$$

$$\implies N^2(\bar{Y})^2 = [(N-1)S^2 + N\bar{Y}^2] + \sum_{i \neq j=1}^{N} Y_i Y_j$$

$$\implies \sum_{i \neq j=1}^{N} Y_i Y_j = N(N-1)\bar{Y}^2 - (N-1)S^2 \tag{9}$$

3) *Let an indicator function a_i for the case of SRSWOR, where-*

$$a_i = \begin{cases} 0; & i^{th} \text{ unit is included in sample} \\ 1; & Otherwise \end{cases} \tag{10}$$

So, from theorem 1.b,

$$P(a_i = 0) = P(a_i = 1) = \frac{n}{N}$$

Expectation for First-Order Inclusion,

$$\begin{aligned} E[a_i] &= 0 \times P(a_i = 0) + 1 \times P(a_i = 1) \\ &= 0 \times \frac{n}{N} + 1 \times \frac{n}{N} \\ &= \frac{n}{N} \end{aligned} \tag{11}$$

Expectation for Second-Order Inclusion,

$$\begin{aligned} E[a_i a_j] &= 0 \times P(a_i a_j = 0) + 1 \times P(a_i a_j = 1) \\ &= 0 + 1 \times P(a_i = 1 \cap a_j = 1) \\ &= P(a_i = 1) \cdot P(a_i = 1 | a_j = 1) \\ &= (\frac{n}{N})(\frac{n-1}{N-1}) \end{aligned} \tag{12}$$

4) *In SRSWOR,*

- *Sample Mean is given as-*

$$\bar{y} = \frac{1}{n} \sum_{i=1}^{N} a_i Y_i \tag{13}$$

- *Expectation of Sample Mean is given as-*

$$E[\bar{y}] = \frac{1}{n} \sum_{i=1}^{N} E[a_i] Y_i \tag{14}$$

where, Y_i is the i^{th} unit of the population.

- *Square of Sample Mean is given as-*

$$\begin{aligned} \bar{y}^2 &= (\frac{1}{n} \sum_{i=1}^{N} a_i Y_i)^2 \\ &= (\frac{1}{n})^2 (\sum_{i=1}^{N} a_i Y_i^2 + \sum_{i \neq j = 1}^{N} a_i a_j Y_i Y_j) \end{aligned} \tag{15}$$

- *Expectation of Square of Sample Mean is given as-*

$$
\begin{aligned}
E[\bar{y}^2] &= E[(\frac{1}{n})^2(\sum_{i=1}^{N} a_i Y_i^2 + \sum_{i\neq j=1}^{N} a_i a_j Y_i Y_j)] \\
&= (\frac{1}{n})^2(\sum_{i=1}^{N} E[a_i] Y_i^2 + \sum_{i\neq j=1}^{N} E[a_i a_j] Y_i Y_j) \\
&= (\frac{1}{n})^2[\frac{n}{N}\sum_{i=1}^{N} Y_i^2 + \frac{n(n-1)}{N(N-1)}\sum_{i\neq j=1}^{N} Y_i Y_j)] \\
&\textit{Substituting values from equ. 8 and 9,} \\
&= \frac{1}{nN}[(N-1)S^2 + N\bar{Y}^2] + \frac{n-1}{nN(N-1)}[N(N-1)\bar{Y}^2 - (N-1)S^2] \\
&= (\frac{N-n}{nN})S^2 + \bar{Y}^2
\end{aligned}
\tag{16}
$$

In case of SRSWR, Variance of Sample Mean is given as-

$$
\begin{aligned}
Var(\bar{y}) &= Var(\frac{1}{n}\sum_{i=1}^{n} y_i) \\
&= \frac{1}{n^2}(\sum_{i=1}^{n} Var(y_i)) \\
&= \frac{1}{n^2}(n \times \sigma^2) \\
&= \frac{\sigma^2}{n}
\end{aligned}
\tag{17}
$$

In case of SRSWOR, Variance of Sample Mean is given as-

$$
\begin{aligned}
Var(\bar{y}) &= E[\bar{y} - E(\bar{y})]^2 \\
&= E(\bar{y}^2) - \bar{Y}^2 \\
&\text{Substituting value from equation (16),} \\
&= (\frac{N-n}{nN})S^2 + \bar{Y}^2 - \bar{Y}^2 \\
&= (\frac{N-n}{nN})S^2 \\
&\text{Substituting value from equ. (7) and rearranging,} \\
&= (\frac{\sigma^2}{n})(\frac{N-n}{N-1})
\end{aligned}
\tag{18}
$$

□