Assignment 5

Course name: PROBABILITY AND STOCHASTIC PROCESSES (AI5030)

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Question 1

Which of the following are correct?

1. If X and Y are independent $\mathcal{N}(0,1)$, then $\frac{X+Y}{\sqrt{2}}$ is $\mathcal{N}(0,1)$.
2. If X and Y are independent $\mathcal{N}(0,1)$, then $\frac{X}{Y}$ has t-distribution.
3. If X and Y are independent $\mathcal{U}niform(0,1)$, then $(\frac{X+Y}{2})$ is $\mathcal{U}niform(0,1)$.
4. If X is $\mathcal{B}inomial(n,p)$, then $n-X$ is $\mathcal{B}inomial(n,1-p)$.

Answer:

Theorem 1 (Uniqueness Theorem)**.** *If two random variables have same moment generating functions, then they have the same distribution.*

Theorem 2. *Product of normally distributed random variables with a constant gives a normally distributed random variable.*

i.e., $X \sim \mathcal{N}(\mu,\sigma^2)$ and, $Z = kX$, where k is a constant, then-

$$Z \sim \mathcal{N}(k\mu,(k\sigma)^2)$$

Proof:
Moment Generating Function of a Normal Random Variable $X \sim \mathcal{N}(\mu,\sigma^2)$ is given as-

$$M_X(t) = E[e^{tX}] = e^{\left(\mu_X t + \frac{\sigma_X^2 t^2}{2}\right)}$$

$$\therefore M_Z(t) = E[e^{tZ}]$$
$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tz} e^{\frac{-1}{2}\left(\frac{(x-\mu)^2}{\sigma^2}\right)} dx$$
$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ktx} e^{\frac{-1}{2}\left(\frac{(x-\mu)^2}{\sigma^2}\right)} dx$$
$$\text{Let,} y = \frac{x-\mu}{\sigma}$$
$$\implies dy = \frac{dx}{\sigma}$$
$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{kt(\mu+\sigma y)} e^{\left(\frac{-y^2}{2}\right)} dy$$
$$= \frac{e^{k\mu t}}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{-1}{2}\left((y-k\sigma t)^2-(k\sigma t)^2\right)} dy$$
$$= \left(\frac{e^{\left(k\mu t+\frac{(k\sigma)^2 t^2}{2}\right)}}{\sqrt{2\pi}}\right) \int_{-\infty}^{\infty} e^{\frac{-1}{2}(y-k\sigma t)^2} dy$$

Since,the integral has an Even function.

$$= \left(\frac{2e^{\left(k\mu t+\frac{(k\sigma)^2 t^2}{2}\right)}}{\sqrt{2\pi}}\right) \int_{0}^{\infty} e^{\frac{-1}{2}(y-k\sigma t)^2} dy \tag{1}$$
$$\text{Let,} r = \frac{1}{2}(y-k\sigma t)^2$$
$$\implies dy = \frac{dr}{\sqrt{2r}}$$
$$= \left(\frac{2e^{\left(k\mu t+\frac{(k\sigma)^2 t^2}{2}\right)}}{\sqrt{2\pi}}\right) \int_{0}^{\infty} \frac{e^{-r}}{\sqrt{2r}} dr$$
$$= \left(\frac{e^{\left(k\mu t+\frac{(k\sigma)^2 t^2}{2}\right)}}{\sqrt{\pi}}\right) \int_{0}^{\infty} r^{\frac{-1}{2}} e^{-r} dr$$

Evaluating the integral using the gamma function,we get-

$$= \left(\frac{e^{\left(k\mu t+\frac{(k\sigma)^2 t^2}{2}\right)}}{\sqrt{\pi}}\right) \times \Gamma\left(\frac{1}{2}\right)$$
$$= \left(\frac{e^{\left(k\mu t+\frac{(k\sigma)^2 t^2}{2}\right)}}{\sqrt{\pi}}\right) \times (\sqrt{\pi})$$
$$= e^{\left((k\mu)t+\frac{(k\sigma)^2 t^2}{2}\right)}$$

Theorem 3. *Sum of two independent normally distributed random variables is a normally distributed random variable.*

$$X \sim \mathcal{N}(\mu_X, \sigma_X^2)$$

$$Y \sim \mathcal{N}(\mu_Y, \sigma_Y^2)$$

and, $Z = X + Y$*, then-*

$$Z \sim \mathcal{N}(\mu_X + \mu_Y, \sigma_X^2 + \sigma_Y^2)$$

Proof:

$$\begin{aligned}
\therefore M_Z(t) &= E[e^{tZ}] \\
&= E[e^{t(X+Y)}] \\
&= E[e^{tX}]E[e^{tY}] \\
&= e^{\left(\mu_X t + \frac{\sigma_X^2 t^2}{2}\right)} \times e^{\left(\mu_Y t + \frac{\sigma_Y^2 t^2}{2}\right)} \\
&= e^{\left((\mu_X + \mu_Y)t + \frac{(\sigma_X^2 + \sigma_Y^2)t^2}{2}\right)}
\end{aligned} \tag{2}$$

Definition 4 (Ratio Distribution)**.** *A Ratio Distribution is a probability distribution constructed as the distribution of the ratio of random variables having two other known distributions.*
Given two (usually independent) random variables X and Y, the distribution of the random variables $X = (\frac{U}{V})$ *is a ratio distribution.*

Theorem 5. *Ratio distribution of two standard normally distributed variables is a standard cauchy distribution.*

Proof: PDF of U and V is given as,

$$f_U(u) = \frac{1}{\sqrt{2\pi}}.e^{-\frac{u^2}{2}}, f_V(v) = \frac{1}{\sqrt{2\pi}}.e^{-\frac{v^2}{2}} \tag{3}$$

where $-\infty <$u,v $< \infty$
Since U and V are independent, their joint PDF is given by,

$$f_{UV}(u,v) = \frac{1}{2\pi}.e^{-\frac{u^2+v^2}{2}} \tag{4}$$

$$F_X(x) = P(X \leq x) = P\left(\frac{U}{V} \leq x\right) \tag{5}$$

$$= P(U \leq V.x, V \geq 0) + P(U \geq V.x, V < 0) \tag{6}$$

$$= \int_0^{\infty} \int_{-\infty}^{v.x} f_{uv}(u,v)\,du\,dv + \int_{-\infty}^{0} \int_{-\infty}^{v.x} f_{uv}(u,v)\,du\,dv \tag{7}$$

Differentiating $F_X(x)$ to obtain PDF of X, $f_X(x)$
(Using Leibniz's Integral Rule)

$$f_X(x) = \int_0^{\infty} v.f_{uv}(vx,v)\,dv + \int_{-\infty}^{0} (-v).f_{uv}(vx,v)\,dv \tag{8}$$

$$f_X(x) = \int_0^{\infty} v.\frac{1}{2\pi}.e^{-\frac{(1+x^2)v^2}{2}}\,dv + \int_{-\infty}^{0} (-v).\frac{1}{2\pi}.e^{-\frac{(1+x^2)v^2}{2}}\,dv \tag{9}$$

Let,

$$u = \frac{(1+x^2)v^2}{2} \tag{10}$$

$$\frac{du}{dv} = (1+x^2)v \Rightarrow dv = \frac{du}{(1+x^2)v} \tag{11}$$

$$f_X(x) = \int_0^{\infty} \frac{1}{2\pi(1+x^2)}.e^{-u}\,du - \int_{-\infty}^{0} \frac{1}{2\pi(1+x^2)}.e^{-u}\,du \tag{12}$$

$$f_X(x) = \frac{1}{\pi(1+x^2)} \tag{13}$$

Hence, X~ Cauchy(0,1)

Theorem 6. *Product of Uniform Random Variable with a constant results in a Uniform Random Variable.*
i.e., $X \sim \mathcal{U}niform(a,b)$ and $Z = kX$, where k is a constant, then-

$$Z \sim \mathcal{U}niform(ka,kb)$$

Proof:
Moment Generating Function of Uniform Distribution $X \sim \mathcal{U}niform(a,b)$ is given as-

$$M_X(t) = E[e^{tX}] = \frac{e^{tb} - e^{ta}}{t(b-a)}$$

$$\begin{aligned} \therefore M_Z(t) &= E[e^{tZ}] \\ &= \int_{-\infty}^{\infty} e^{t(kx)} f_X(x) dx \\ &= \int_a^b e^{t(kx)} \left(\frac{1}{b-a}\right) dx \\ &= \left(\frac{1}{b-a}\right)\left[\frac{e^{t(kx)}}{kt}\right]_a^b \\ &= \frac{e^{t(kb)} - e^{t(ka)}}{t\,((kb)-(ka))} \end{aligned} \tag{14}$$

Theorem 7. *Sum of two independent, uniform random variables is not uniform.*

$$X \sim \mathcal{U}(a,b)$$

$$Y \sim \mathcal{U}(a,b)$$

and, $Z = X + Y$, then Z is not Uniformly distributed.

Proof:

$$\begin{aligned}
\therefore M_Z(t) &= E[e^{tZ}] \\
&= E[e^{t(X+Y)}] \\
&= E[e^{tX}]E[e^{tY}] \\
&= \left(\frac{e^{tb}-e^{ta}}{t(b-a)}\right) \times \left(\frac{e^{tb}-e^{ta}}{t(b-a)}\right) \\
&= \left(\frac{e^{tb}-e^{ta}}{t(b-a)}\right)^2
\end{aligned} \tag{15}$$

From theorem 1, we can conclude that- since, this moment generating function is not similar to that of uniform random variable's moment, this distribution is not uniform.

Theorem 8. *Binomial Random Variable X counts the number of successes in 'n' independent Bernoulli trials.*
i.e., $X = X_1 + X_2 + \cdots + X_n$; where, $X_i \sim Bernoulli(p) \sim \mathscr{B}(p)$
then, $X \sim Binomial(n,p) \sim b(n,p)$.

Proof:
Moment Generating Function of $X_i \sim \mathscr{B}(p)$ is given as-

$$\therefore M_{X_i}(t) = E[e^{tX_i}]$$

$$= e^{t\times 0} \times q + e^{t\times 1} \times p$$

$$= q + pe^t$$

For summation of independent random variables, Moment generating function is written as-

$$\therefore M_X(t) = \Pi_{i=1}^{i=n} M_{X_i}(t)$$

$$= \Pi_{i=1}^{i=n} E[e^{tX_i}]$$

$$= (q + pe^t)^n$$

Using Binomial formula, we get-
$(a+b)^n = \sum_{x=0}^{n} \binom{n}{k} a^x b^{n-x}$

$$= \sum_{x=0}^{n} \binom{n}{k} (pe^t)^x (q)^{n-x}$$

Theorem 9. *If X is Bernoulli random variable with success probability p ,i.e., $X \sim \mathscr{B}(p)$, then, $Z = (1 - X) \sim \mathscr{B}(p)$.*

Proof:

$$\therefore M_Z(t) = E[e^{tZ}]$$
$$= \sum_{x=0,1} e^{t(1-x)} f_X(x)$$
$$= e^{t\times(1-0)} \times q + e^{t\times(1-1)} \times p$$
$$= qe^t + p$$

Hence, $Z \sim \mathscr{B}(q) \sim \mathscr{B}(1-p)$.

Analysis of Option 1 :

From theorem 2 and 3, we can say that option 1 is correct.

Analysis of Option 2 :

From theorem 5, we can conclude that option 2 is incorrect.

Analysis of Option 3 :

From theorem 7, we can conclude that option 3 is incorrect.

Analysis of Option 4 :

From theorem 8, we know that-
if $X_i \sim \mathscr{B}(p)$, then $X = \sum_{i=1}^{i=n} X_i \sim b(n, p)$.

From theorem 9, we know that-
if $X_i \sim \mathscr{B}(p)$, then $Z_i = (1 - X_i) \sim \mathscr{B}(q)$.

Let, $Z = \sum_{i=1}^{i=n} Z_i$
$= \sum_{i=1}^{i=n}(1 - X_i)$
$= \sum_{i=1}^{i=n}(1) - \sum_{i=1}^{i=n}(X_i)$
$= n - X$.

$$\therefore M_Z(t) = \Pi_{i=1}^{i=n} M_{Z_i}(t)$$
$$= \Pi_{i=1}^{i=n}(p + qe^t)$$
$$= (p + qe^t)^n$$

This shows that-

$$Z \sim b(n, q) \sim b(n, 1-p)$$

So, we can conclude that option 4 is correct.

Hence, the correct options are : 1 and 4.