Linear Algebra Manual

Chapter 1: Linear Equations in Linear Algebra

Systems of Linear Equations

A system of linear equations has:

1. no solution, or
2. exactly one solution, or
3. infinitely many solutions

Elementary row operations:

1. Replace one row by the sum of itself and a multiple of another row.
2. Interchange two rows.
3. Multiply all entries in a row by a nonzero constant.

If the augmented matrices of two linear systems are row equivalent, then the two systems have the same solution set.

Two fundamental questions about a linear system

1. Is the system consistent; that is, does at least one solution exist?
2. If a solution exists, is it the only one; that is, is the solution unique?

Row Reduction and Echelon Forms

A rectangular matrix is in **echelon form** (or **row echelon form**) if it has the following three properties:

1. All nonzero rows are above any rows of all zeros.
2. Each leading entry of a row is in a column to the right of the leading entry of the row above it.
3. All entries in a column below a leading entry are zeros.

If a matrix in echelon from satisfies the following additional conditions, then it is in **reduced echelon form** (or **reduced row echelon form**):

1. The leading entry in each nonzero row is 1.
2. Each leading 1 is the only nonzero entry in its column.

A **pivot position** in a matrix A is a location in A that corresponds to a leading 1 in the reduced echelon form of A. A **pivot column** is a column of A that contains a pivot position.

The row Reduction Algorithm

1. Begin with the leftmost nonzero column. this is a pivot column. The pivot position is at the top.
2. Select a nonzero entry in the pivot columns as a pivot. If necessary, interchange rows to move this entry into the pivot position.
3. Use row replacement operations to create zeros in all positions below the pivot.
4. Cover (or ignore) the row containing the pivot position and cover all rows, if any, above it. Apply steps 1-3 to the sub-matrix that remains. Repeat the process until there are no more nonzero rows to modify.
5. Beginning with the rightmost pivot and working upward and to the left, create zeros above each pivot. If a pivot is not 1, make it 1 by scaling operations.

Existence and Uniqueness Theorem

A linear system is consistent if and only if the rightmost column of the augmented matrix is not a pivot column, that is, if and only if an echelon form of the augmented matrix has no row of the form.

$$\begin{bmatrix} 0 & \cdots & 0 & b \end{bmatrix} \text{ with b nonzero}$$

If a linear system is consistent, then the solution set contains either (i) a unique solution, when there are no free variables, or (ii) infinitely many solutions, when there is at least one free variable.

Using Row Reduction to Solve A Linear System

1. Write the augmented matrix of the system.
2. Use the row reduction algorithm to obtain an equivalent augmented matrix in echelon form. Decided whether the system is consistent. If there is no solution, stop; otherwise, go to the next step.
3. Continue row reduction to obtain the reduced echelon form.
4. Write the system of equations corresponding to the matrix obtains in step 3.
5. Rewrite each nonzero equations from step 4 so that its one basic variable is expressed in terms of any free variables appearing in the equation.

Vector Equations

Parallelogram Rule for Addition

If **u** and **v** are in $\mathbb{R}^2$ are represented as points in the plane, then **u** + **v** corresponds to the fourth vertex of the parallelogram whose other vertices's are **u**, **0**, and **v**.

Algebraic Properties of $\mathbb{R}^n$

For all **u**, **v**, **w** in $\mathbb{R}^n$ and all scalars *c* and *d*:

1. **u** + **v** = **v** + **u**
2. (**u** + **v**) + **w** = **u** + (**v** + **w**)
3. **u** + **0** = **0** + **u** = **u**
4. **u** + (-**u**) = -**u** + **u** = **0** where -**u** denotes (-1)**u**
5. *c*(**u** + **v**) = *c***u** + *c***v**
6. (*c* + *d*)**u** = *c***u** + *d***u**
7. *c*(*d***u**)=(*cd*)**u**

8. 1**u** = **u**

A vector equation

$$x_1a_1 + x_2a_2 + \cdots + x_na_n = b$$

has the same solution set as the linear system whose augmented matrix is

$$A = \begin{bmatrix} a_1 & a_2 & \cdots & a_n & b \end{bmatrix}$$

In particular, **b** can be generated by a linear combination of $\{a_1, \ldots, a_n\}$ if and only if there exists a solution to the linear system corresponding to the matrix A.

If $v_1, \ldots, v_p$ are in $\mathbb{R}^n$, then the set of all linear combinations of $v_1, \ldots, v_p$ is denoted by Span$\{v_1, \ldots, v_p\}$ and is called the **subset of** $\mathbb{R}^n$ **spanned** (or **generated**) **by** $v_1, \ldots, v_p$. That is, Span$\{v_1, \ldots, v_p\}$ is the collection of all vectors that can be written in the form

$$c_1v_1 + c_2v_2 + \cdots + c_pv_p$$

with $c_1, \ldots, c_p$ scalars.

The Matrix Equation Ax = b

If A is an *m x n* matrix, with columns $a_1, \ldots, a_n$ and if **x** is in $\mathbb{R}^n$, then the **product of** *A* **and x**, denoted by A**x**, is **the linear combination of the columns of** *A* using the corresponding entries in x as weights; that is,

$$Ax = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1a_1 + x_2a_2 + \cdots + x_na_n$$

If *A* is an *m x n* matrix, with columns $a_1, \ldots, a_n$, and if **b** is in $\mathbb{R}^m$, the matrix equation

$$Ax = b$$

has the same solution set as the vector equation

$$x_1a_1 + x_2a_2 + \cdots + x_na_n = b$$

which, in turn, has the same solution set as the system of linear equations whose augmented matrix is

$$\begin{bmatrix} a_1 & a_2 & \cdots & a_n & b \end{bmatrix}$$

The equation *A***x** = **b** has a solution if and only if **b** is a linear combination of the columns of *A*.

Let *A* be an *m x n* matrix. Then the following statements are logically equivalent. That is, for a particular A, either they are all true statements or they are all false.

1. For each **b** in $\mathbb{R}^m$, the equation A**x** = **b** has a solution
2. Each **b** in $\mathbb{R}^m$ is a linear combination of the columns of *A*.
3. The columns of *A* span $\mathbb{R}^m$.
4. *A* has a pivot position in every row.

Row-Vector Rule for Computing Ax

If the product $A\mathbf{x}$ is defined, then the ith entry of $A\mathbf{x}$ is the sum of the products of corresponding entries from row i of A and from the vector $\mathbf{x}$.

Properties of the Matrix-Vector Product Ax

If A is an m x n matrix, **u** and **v** are vectors in $\mathbb{R}^n$, and c is a scalar, then:

1. $A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v}$
2. $A(c\mathbf{u}) = c(A\mathbf{u})$

Solution Sets of Linear Systems

Homogeneous Linear Systems

The homogeneous equation $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution if and only if the equation has at least one free variable.

Suppose the equation $A\mathbf{x} = \mathbf{b}$ is consistent for some given **b**, and let **p** be a solution. Then the solution set of $A\mathbf{x} = \mathbf{b}$ is the set of all vectors of the form $\mathbf{w} = \mathbf{p} + \mathbf{v_h}$, where $\mathbf{v_h}$ is any solution of the homogeneous equation $A\mathbf{x} = \mathbf{0}$.

Writing a Solution Set In Parametric Vector Form

1. Row reduce the augmented matrix to reduced echelon form.
2. Express each basic variable in terms of any free variables appearing in an equation.
3. Write a typical solution **x** as a vector whose entries depend on the free variables, if any.
4. Decompose **x** into a linear combination of vectors (with numeric entries) using the free variables as parameters.

Linear Independence

An indexed set of vectors $\{v_1, \ldots, v_p\}$ in $\mathbb{R}^n$ is said to be **linearly independent** if the vector equation

$$x_1 v_1 + x_2 v_2 + \cdots + x_p v_p = 0$$

has only the trivial solution. The set $\{v_1, \ldots, v_p\}$ is said to be **linearly dependent** if there exist weights $c_1, \ldots, c_p$, not all zero, such that

$$c_1 v_1 + c_2 v_2 + \cdots + c_p v_p = 0$$

Linear Independence of matrix Columns

The columns of matrix A are linearly independent if and only if the equation $A\mathbf{x} = \mathbf{0}$ has *only* the trivial solution.

Sets of One or Two Vectors

A set of two vectors $\{v_1, v_2\}$ is linearly dependent if at least one of the vectors is a multiple of the other. The set is linearly independent if and only if neither of the vectors is a multiple of the other.

Sets of Two or More Vectors

An index set $S = \{v_1, \ldots, v_p\}$ of two ore more vectors is linearly dependent if and only if at least one of the vectors in *S* is a linear combination of the others. In fact, if *S* is linearly dependent and $v_1 \neq 0$, then some v_j (with *j* > 1) is a linear combination of the preceding vectors, $v_1, \ldots, v_{j-1}$.

If a set contains more vectors than there are entries in each vector, then the set is linearly dependent. That is, any set $\{v_1, \ldots, v_p\}$ in $\mathbb{R}^n$ is linearly dependent if *p* > *n*.

If a set $S = \{v_1, \ldots, v_p\}$ in $\mathbb{R}^n$ contains the zero vector, then the set is linearly dependent.

Introduction to Linear Transformations

A transformation (or mapping) *T* is **linear** if:

1. *T*(**u** + **v**) = *T*(**u**) + *T*(**v**) for all **u**, **v** in the domain of *T*
2. *T*(*c***u**) = *cT*(**u**) for all scalars *c* and all **u** in the domain of *T*

If *T* is a linear transformation, then

$$T(0) = 0$$

and

$$T(cu + dv) = cT(u) + dT(v)$$

for all vectors **u**, **v** in the domain of *T* and all scalars *c*, *d*.

The Matrix of a Linear Transformation

Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear transformation. Then there exists a unique matrix *A* such that

$$T(x) = Ax \text{ for all x in } \mathbb{R}^n$$

In fact, *A* is the *m x n* matrix whose *j*th column is the vector *T*($\mathbf{e}_\mathrm{j}$) is the *j*th column of the identity matrix in $\mathbb{R}^n$:

$$A = \begin{bmatrix} T(e_1) & \cdots & T(e_n) \end{bmatrix}$$

A mapping $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is said to be **onto** $\mathbb{R}^m$ if each **b** in $\mathbb{R}^m$ is the image of *at least one* **x** in $\mathbb{R}^n$.

A mapping $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is said to be **one-to-one** if each **b** in $\mathbb{R}^m$ is the image of *at most one* **x** in $\mathbb{R}^n$.

Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear transformation. Then *T* is one-to-one if and only if the equation *T*(**x**) = **0** has only the trivial solution.

Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear transformation, and let *A* be the standard matrix for *T*. Then:

1. *T* maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ if and only if the columns of *A* span $\mathbb{R}^m$
2. *T* is one-to-one if and only if the columns of *A* are linearly independent.

Chapter 2: Matrix Algebra

Matrix Operations

Let A, B, and C be matrices of the same size, and let r and s be scalars.

1. $A + B = B + A$
2. $(A + B) + C = A + (B + C)$
3. $A + 0 = A$
4. $r(A + B) = rA + rB$
5. $(r + s)A = rA + sA$
6. $r(sA) = (rs)A$

If A is an $m \times n$ matrix, and if B is an $n \times p$ matrix with columns $b_1, \ldots, b_p$, then the product AB is the $m \times p$ matrix whose columns are $Ab_1, \ldots, Ab_p$. That is,

$$AB = A \begin{bmatrix} b_1 & b_2 & \cdots & b_p \end{bmatrix} = \begin{bmatrix} Ab_1 & Ab_2 & \cdots & Ab_p \end{bmatrix}$$

Each column AB is a linear combination of the columns of A using weights from the corresponding column of B.

Row-Column Rule for Computing AB

If the product AB is defined, then the entry in row i and column j of AB is the sum of the products of corresponding entries from row i for A and column j of B. If $(AB)_{ij}$ denotes the (i, j)-entry in AB, and if A is an $m \times n$ matrix, then

$$(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$$

Properties of Matrix Multiplication

Let A be an $m \times n$ matrix, and let B and C have sizes for which the indicated sums and products are defined.

1. $A(BC) = (AB)C$
2. $A(B + C) = AB + AC$
3. $(B + C)A = BA + CA$
4. $r(AB) = (rA)B = A(rB)$
5. $I_mA = A = $ *AI_m

Warnings:

1. In general, $AB \neq BA$
2. The cancellation laws do *not* hold for matrix multiplication. That is, if $AB = AC$, then it is *not* true in general that $B = C$.
3. If a product AB is the zero matrix, you *cannot* conclude in general that either $A = 0$ or $B = 0$.

The Transpose of a Matrix

Let A and B denote matrices whose sizes are appropriate for the following sums and products.

1. $(A^T)^T = A$
2. $(A + B)^T = A^T + B^T$
3. For any scalar r, $(rA)^T = rA^T$
4. $(AB)^T = B^T A^T$

The transpose of a product of matrices equals the product of their transposes in the reverse order.

The Inverse of a Matrix

$$A^{-1}A = I \text{ and } AA^{-1} = I$$

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad - bc \neq 0$, then A is invertible and

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

if $ad - bc = 0$, then A is not invertible.

If A is an invertible n x n matrix, then for each **b** in $\mathbb{R}^n$, the equation A**x** = **b** has the unique solution **x** = A^{-1}**b**.

1. If A is an invertible matrix, then A^{-1} is invertible and $(A^{-1})^{-1} = A$
2. If A and B are n x n invertible matrices, then so is AB, and the inverse of AB is the product of the inverse of A and B in the reverse order. That is, $(AB)^{-1} = B^{-1}A^{-1}$
3. If A is an invertible matrix, then so is A^T, and the inverse of A^T is the transpose of A^{-1}. That is, $(A^T)^{-1} = (A^{-1})^T$

The product of n x n invertible matrices is invertible, and the inverse is the product of their inverses in the reverse order.

If an elementary row operation is performed on an m x n matrix A, the resulting matrix can be written as EA, where the m x m matrix E is created by performing the same row operations on I_m.

Each elementary matrix E is invertible. The inverse of E is the elementary matrix of the same type that transforms E back into I.

An n x n matrix A is invertible if and only if A is row equivalent to I_n, and in this case, any sequence of elementary row operations that reduces A to I_n also transforms I_n into A^{-1}.

Algorithm for Finding A^{-1}

Row reduce the augmented matrix $[A \quad I]$. If A is row equivalent to I, then $[A \quad I]$ is row equivalent to $[I \quad A^{-1}]$. Otherwise, A does not have an inverse.

Characterizations of Invertible Matrices

The Invertible Matrix Theorem

Let A be a square n x n matrix. Then the following statements are equivalent. That is, for a given A, the statements are either all true or all false.

1. A is an invertible matrix.
2. A is row equivalent to the n x n identity matrix.
3. A has n pivot positions.
4. The equation $Ax = 0$ has only the trivial solution.
5. The columns of A form a linearly independent set.
6. The linear transformation $x \to Ax$ is one-to-one.
7. The equation $Ax = b$ has at least one solution for **b** in $\mathbb{R}^n$.
8. The columns of A span $\mathbb{R}^n$.

9. The linear transformation $x \to Ax$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.
10. There is an *n x n* matrix *C* such that $CA = I$.
11. There is an *n x n* matrix D such that $AD = I$.
12. A^T is an invertible matrix.

Let *A* and *B* be square matrices. If $AB = I$, then *A* and *B* are both invertible, with $B = A^{-1}$ and $A = B^{-1}$.

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation and let *A* be the standard matrix for *T*. Then *T* is invertible if and only if *A* is an invertible matrix. In that case, the linear transformation *S* given by $S(x) = A^{-1}x$ is the unique function satisfying equations $S(T(x)) = x$ and $T(S(x)) = x$.

Partitioned Matrices

Column-Row Expansion of *AB*

If *A* is *m x n* and *B* is *n x p*, then

$$\begin{aligned} AB &= \begin{bmatrix} col_1(A) & col_2(A) & \cdots & col_n(A) \end{bmatrix} \begin{bmatrix} row_1(B) \\ row_2(B) \\ \vdots \\ row_n(B) \end{bmatrix} \\ &= col_1(A)row_1(B) + \cdots + col_n(A)row_n(B) \end{aligned}$$

Matrix Factorizations

Algorithm for an LU Factorization

1. Reduce *A* to an echelon form *U* by a sequence of row replacement operations, if possible.
2. Place entries in *L* such that the *same sequence of row operations* reduces *L* to *I*.

Subspaces of $\mathbb{R}^n$

A **subspace** of $\mathbb{R}^n$ in any set *H* in $\mathbb{R}^n$ that has three properties:

1. the zero vector is in *H*.
2. For each **u** and **v** in *H*, the sum **u** + **v** is in *H*.
3. For each **u** in *H* and each scalar *c*, the vector *c***u** is in *H*.

The **column space** of a matrix *A* is the set Col *A* of all linear combinations of the columns of *A*.

The **null space** of a matrix *A* is the set Nul *A* of all solutions of the homogeneous equation *A***x** = **0**.

The null space of an *m x n* matrix *A* is a subspace of $\mathbb{R}^n$. Equivalently, the set of all solutions of a system *A***x** = **0** of *m* homogeneous linear equations *n* unknowns is a subspace of $\mathbb{R}^n$.

A **basis** for a subspace *H* of $\mathbb{R}^n$ is a linearly independent set of *H* that spans *H*.

The pivot columns of a matrix *A* from the basis for the column space of *A*.

Dimension and Rank

Coordinate Systems

Suppose the set $\beta = \{b_1, \ldots, b_p\}$ is a basis for a subspace *H*. For each **x** in *H*, the **coordinates of x relative to the basis** β are the weights $c_1, \ldots, c_p$ such that $x = c_1 b_1 + \cdots + c_p b_p$, and the vector in $\mathbb{R}^p$

$$[x]_\beta = \begin{bmatrix} c_1 \\ \vdots \\ c_p \end{bmatrix}$$

is called the **coordinate vector of x (relative to β) or the β-coordinate vector of x.**

The **dimension** of a nonzero subspace *H*, denoted by dim *H*, is the number of vectors in any basis for *H*. The dimension of the zero subspace $\{0\}$ is defined to be zero.

The **rank** of a matrix *A*, denoted by rank *A*, is the dimension of the columns pace of *A*.

The Rank Theorem

If a matrix *A* has *n* columns, then rank *A* + dim Nul *A* = *n*.

The Basis Theorum

Let *H* be a *p*-dimensional subspace of $\mathbb{R}^n$. Any linearly independent set of exactly *p* elements in *H* is automatically a basis for *H*. Also, any set of *p* elements of *H* that spans *H* is automatically a basis for *H*.

The Invertible Matrix Theorem (continued)

1. The columns of *A* from a basis of $\mathbb{R}^n$.
2. Col *A* = $\mathbb{R}^n$
3. dim Col *A* = *n*
4. rank *A* = *n*
5. Nul *A* = $\{0\}$
6. dim Nul *A* = $\{0\}$

Chapter 3: Determinants

Introduction to Determinants

For $n \geq 2$, the **determinant** of an *n x n* matrix $A = [a_{ij}]$ is the sum of *n* terms of the form $\pm a_{ij}\ detA_{1j}$, with plus and minus signs alternating, where the entries $a_{11}, a_{12}, \ldots, a_{1n}$ are from the first row of *A*. In symbols,

$$
\begin{aligned}
detA &= a_{11}detA_{11} - a_{12}detA_{12} + \cdots + (-1)^{1+n}a_{1n}detA_{1n} \\
&= \sum_{j=1}^{n}(-1)^{1+j}a_{1j}detA_{1j}
\end{aligned}
$$

The determinant of an *n x n* matrix *A* can be computed by a cofactor expansion across any row or down any column. The expansion across the *i*th row using the cofactors in $C_{ij} = (-1)^{i+j}detA_{ij}$ is

$$detA = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$

The cofactor expansion down the *j*th column is

$$detA = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$$

If *A* is a triangular matrix, then det *A* is the product of the entires on the main diagonal of *A*.

Properties of Determinants

Row Operations

Let *A* be a square matrix.

1. If a multiple of one row of *A* is added to another row to produce matrix *B*, then $detB = detA$
2. If two rows of *A* are interchanged to produce *B*, then $B = -detA$
3. If own row of *A* is multiplied by *k* to produce *B*, then $B = k \cdot detA$

A square matrix *A* is invertible if and only if $detA \neq 0$.

If *A* is an *n x m* matrix, then $detA^T = detA$.

Multiplicative Property

If *A* and *B* are *n x n* matrices, then $detAB = (detA)(detB)$.

Cramer's Rule, Volume, and Linear Transforamtions

Cramer's Rule

Let *A* be an invertible *n x n* matrix. For any **b** in $\mathbb{R}^n$, the unique solution **x** of *A***x** = **b** has entries given by

$$x_i = \frac{detA_i(B)}{detA}, i = 1, 2, \ldots, n$$

An Inverse Formula

Let *A* be an invertible *n x n* matrix. Then

$$A^{-1} = \frac{1}{detA} adjA$$

Determinants as Area of Volume

If *A* is a *2 x 2* matrix, the area of the parallelogram determined by the columns of *A* is $|detA|$. If *A* is a *3 x 3* matrix, the volume of the parallelepiped determined by the columns of *A* is $|detA|$.

Let a_1 and a_2 be nonzero vectors. Then for any scalar *c*, the area of the parallelogram determined by a_1 and a_2 equals the area of the parallelogram determined by a_1 and $a_2 + ca_1$.

Linear Transforamtions

let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation determined by a *2 x 2* matrix *A*. If *S* is a parallelogram in $\mathbb{R}^2$, then

$$\{\text{area of } T(S)\} = |detA| \cdot \{\text{area of } S\}$$

If T is determined by a *3 x 3* matrix *A*, and if *S* is a parallelepiped in $\mathbb{R}^3$, then

$$\{\text{volume of } T(S)\} = |detA| \cdot \{\text{volume of } S\}$$

Chapter 4: Vector Spaces

Vector Spaces and Subspaces

A **vector space** is a nonempty set *V* of objects, called *vectors*, on which are defined two operations, called *addition* and *multiplication by scalars* (real numbers), subject to the ten axioms (or rules) listed below. The axioms must hold for all vectors **u**, **v**, and **w** in *V* and for all scalars *c* and *d*.

1. The sum of **u** and **v**, denoted by **u** + **v**, is in *V*
2. **u** + **v** = **v** + **u**
3. (**u** + **v**) + **w** = **u** + (**v** + **w**)
4. There is a **zero** vector **0** in *V* such that **u** + **0** = **u**
5. For each **u** in *V*, there is a vector **-u** in *V* such that **u** + (**-u**) = **0**
6. The scalar multiple of **u** by *c*, donated by *c***u**, is in *V*
7. *c*(**u** + **v**) = *c***u** + *c***v**
8. (*c* + *d*)**u** = *c***u** + *d***u**
9. *c*(*d***u**) = (*cd*)**u**
10. 1**u** = **u**

Subspaces

A **subspace** of a vector space *V* is a subspace *H* of *V* that has three properties:

1. the zero vector of *V* is in *H*
2. *H* is closed under vector addition. That is, for each **u** and **v** in *H*, the sum **u** + **v** is in *H*
3. *H* is closed under multiplication by scalars. That is, for each **u** in *H* and each scalar *c*, the vector *c***u** is in *H*

If $v_1, \ldots, v_p$ are in a vector space *V*, then Span$\{v_1, \ldots, v_p\}$ is a subspace of *V*.

Null Spaces, Column Spaces, and Linear Transformations

The **null space** of an *m x n* matrix *A*, written as Nul *A*, is the set of all solutions of the homogeneous equation *A***x** = **0**. In set notation,

$$NulA = \{x : x \text{ is in } \mathbb{R}^n \text{ and } Ax = 0\}$$

the null space of an *m x n* matrix *A* is a subspace of $\mathbb{R}^n$. Equivalently, the set of all solutions to a system *A***x** = **0** of *m* homogeneous linear equations *n* unknowns is a subspace of $\mathbb{R}^n$.

The Column Space of a Matrix

The **column space** of an *m x n* matrix *A*, written as Col *A*, is the set of all linear combinations of the columns of *A*. If $A = [\, a_1, \dots, a_n \,]$, then

$$ColA = Span\{a_1, \dots, a_n\}$$

The column space of *m x n* matrix *A* is a subspace of $\mathbb{R}^m$.

The column space of an *m x n* matrix *A* is all of $\mathbb{R}^m$ if and only if the equation *A***x** = **b** has a solution for each **b** in $\mathbb{R}^m$.

A **linear transformation** *T* from a vector space *V* into a vector space *W* is a rule that assigns to each vector **x** in *V* a unique vector *T*(**x**) in *W*, such that

1. *T*(**u** + **v**) = *T*(**u**) + *T*(**v**) for all **u**, **v**, in *V*
2. *T*(*c***u**) = *cT*(**u**) for all **u** in *V* and all scalars *c*

Linearly Independent Sets; Bases

An indexed set $\{v_1, \dots, v_p\}$ of two or more vectors, with $v_1 \neq 0$, is linearly dependent if and only if some v_j (with j > 1) is a linear combination of the preceding vectors, $v_1, \dots, v_{j-1}$.

Let *H* be a subspace of a vector space *V*. An indexed set of vectors $\beta = \{b_1, \dots, b_p\}$ in *V* is a **basis** for *H* if

1. β is a linearly independent set
2. The subspace spanned by β coincides with *H*; that is, $H = Span\{b_1, \dots, b_p\}$

The Spanning Set Theorem

Let $S = \{v_1, \dots, v_p\}$ be a set in *V*, and let $H = Span\{v_1, \dots, v_p\}$.

1. if one of the vectors in *S*, say v_k is a linear combination of the remaining vectors in *S*, then the set formed from *S* by removing v_k still spans *H*.
2. If $H \neq \{0\}$, some subset of *S* is a basis for *H*.

Bases for Nul A and Col A

The pivot columns of a matrix *A* from a basis for Col *A*

Coordinate Systems

The Unique Representation Theorem

Let $\beta = \{b_1, \ldots, b_n\}$ be a basis for a vector space *V*. Then for each **x** in *V*, there exists a unique set of scalars $c_1, \ldots, c_n$ such that

$$x = c_1 b_1 + \cdots + c_n b_n$$

Suppose $\beta = \{b_1, \cdots, b_n\}$ is a basis for *V* and **x** is in *V*. the **coordinates of x relative to the basis** β **(or the** β**-coordinates of x)** are the weights $c1, \ldots, c_n$ such that $x = c_1 b_1 + \cdots + c_n b_n$

Coordinates in $\mathbb{R}^n$

$P_\beta = [\, b_1 \quad b_2 \quad \cdots \quad b_n \,]$

The Coordinate Mapping

Let $\beta = \{b_1, \ldots, b_n\}$ be a basis for a vector space *V*. Then the coordinate mapping $x \to [x]_\beta$ is a one-to-one linear transformation from *V* onto $\mathbb{R}^n$.

The Dimension of a Vector Space

If a vector space *V* has a basis $\beta = \{b_1, \ldots, b_n\}$, then any set in *V* containing more than *n* vectors must be linearly dependent.

If a vector space *V* has a basis of *n* vectors, then every basis of *V* must consist of exactly *n* vectors.

If *V* is spanned by a finite set, then *V* is said to be **finite-dimensional**, and the **dimension** of *V*, written as dim *V*, is the number of vectors in a basis for *V*. The dimension of the zero vector space $\{0\}$ is defined to be zero. If *V* is not spanned by a fine set, then *V* is said to be **infinite-dimensional**.

Subspaces of a Finite-Dimensional Space

Let *H* be a subspace of a finite-dimensional vector space *V*. Any linearly independent set in *H* can be expanded to a basis for *H*. Also, *H* is finite-dimensional and $dimH \leq dimV$.

The Basis Theorem

Let *V* be a *p*-dimensional vector space, $p \geq 1$. Any linearly independent set of exactly *p* elements in *V* is automatically a basis for *V*. Any set of exactly *p* elements that spans *V* is automatically a basis for *V*.

The Dimensions of Nul A and Col A

The dimension of Nul A is the number of free variables in the equation $Ax = 0$, and the dimension of Col A is the number of pivot columns in *A*.

Rank

The Row Space

If two matrices *A* and *B* are row equivalent, then their, row spaces are the same. If *B* is in echelon form, the nonzero rows of *B* form a basis for the row space of *A* as well as for that of *B*.

The Rank Theorem

The dimensions of the column space and the row space of an *m x n* matrix *A* are equal. This common dimension, the rank of *A*, also equals the number of pivot positions in *A* and satisfies the equation $rankA + dimNulA = n$

The **rank** of *A* is the dimension of the column space of *A*.

Rank and the Invertible Matrix Theorem

Let *A* be an *n x n* matrix. Then the following statements are each equivalent to the statement that *A* is an invertible matrix.

1. The columns of *A* form a basis for $\mathbb{R}^n$
2. $ColA = \mathbb{R}^n$
3. $dimColA = n$
4. $rankA = n$
5. $NulA = \{0\}$
6. $dimNulA = 0$

Change of Basis

Let $B = \{b_1, \dots, b_n\}$ and $C = \{c_1, \dots, c_n\}$ be bases of a vector space *V*. then there is a unique *n x n* matrix $P_{C \leftarrow B}$ such that

$$[x]_C = P_{C \leftarrow B}[x]_B$$

The columns of $P_{C \leftarrow B}$ are the *C*-coordinate vectors of the vectors in the basis *B*. That is,

$$P_{C \leftarrow B} = \left[\, [b_1]_c \quad [b_2]_c \quad \cdots \quad [b_n]_c \,\right]$$