Definition (Definition of Derivative)

Let $f : \mathbb{R} \to \mathbb{R}$ be a real valued function $f(x)$. Then the **derivative** $f'(x)$ is given by the following formula if the limit exists:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

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$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \left[\frac{(x+h)^2 - x^2}{h} = \frac{x^2 + 2hx + h^2 - x^2}{h} \right]$$

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$$= \lim_{h \to 0} \left[\frac{2hx + h^2}{h} = 2x + h \right] = 2x$$

Powers, Constant and Sum Rules

Let $c \in \mathbb{R}$ and $f(x)$, $g(x)$ be real valued functions. Then:

- $(x^c)' = cx^{c-1}$.
- $(cf(x))' = cf'(x)$.
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If $F(x) = 10x^2 + 7x$, then

$$F'(x)$$

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If $F(x) = 10x^2 + 7x$, then

$$F'(x) = 10(2x) + 7 = 20x + 7.$$

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Product and quotient rules

$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$$

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Solution.

$$f'(x) = \frac{(1+x^2)\frac{d}{dx}(e^x) - e^x\frac{d}{dx}(1+x^2)}{(1+x^2)^2}$$

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$\square$

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Find $F'(x)$ if $F(x) = e^{x^2+1}$.

Solution:

- Let $g(x) = e^x$ and $f(x) = x^2 + 1$.

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Find $F'(x)$ if $F(x) = e^{x^2+1}$.

Solution:

- Let $g(x) = e^x$ and $f(x) = x^2 + 1$.
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$$1 = \tan'(\tan^{-1}(x)) \cdot (\tan^{-1})'(x)$$

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$$= \sec^2(\tan^{-1}(x)) \cdot (\tan^{-1})'(x) = (1 + x^2) \cdot (\tan^{-1})'(x).$$

- Solving we get, $(\tan^{-1})'(x) = \frac{1}{1+x^2}$.

Derivatives of classical functions

- $(x^c)' = cx^{c-1}$
- $(e^x)' = e^x$
- $\sin'(x) = \cos(x)$
- $\cos'(x) = -\sin(x)$
- $\tan'(x) = \sec^2(x)$
- $\ln'(x) = \frac{1}{x}$, where $\ln(x)$ is the natural log.
- $\log_a'(x) = \frac{1}{x\ln(a)}$, where $\log_a(x)$ is the log in base a.
- $\frac{d}{dx}(\sin^{-1}(x)) = \frac{1}{\sqrt{1-x^2}}$
- $\frac{d}{dx}(\tan^{-1}(x)) = \frac{1}{1+x^2}$

Definition (Antiderivatives)

A function F is called an **antiderivative** of f on an interval $\mathbf{I}$ if $F'(x) = f(x)$ for all x in $\mathbf{I}$. We use $\int f(x)\,dx$ to denote $F(x)$.

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Let $f(x) = x^2$. Then an antiderivative $F(x)$ for x^2 is $F(x) = \frac{x^3}{3}$.

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Example

Let $f(x) = x^2$. Then an antiderivative $F(x)$ for x^2 is $F(x) = \frac{x^3}{3}$.

Theorem

If F is an antiderivative of f on an interval $\mathbf{I}$, then the most general antiderivative of f on $\mathbf{I}$ is

$$F(x) + C$$

where C is an arbitrary constant.

Table of Anti-differentiation Formulas

Let $F(x)$, $G(x)$ be the antiderivative respectively for the functions $f(x)$, $g(x)$.

Function	**Particular antiderivative**
$c \cdot f(x)$	$c \cdot F(x)$
$f(x) + g(x)$	$F(x) + G(x)$
$x^n (n \neq -1)$	$\frac{x^{n+1}}{n+1}$
$\frac{1}{x}$	$\ln \lvert x \rvert$
e^x	e^x
$\cos x$	$\sin x$

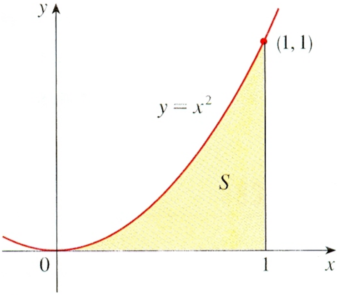
Table of Anti-differentiation Formulas

Function	Particular antiderivative
$\sin x$	$-\cos x$
$\sec^2 x$	$\tan x$
$\sec x \tan x$	$\sec x$
$\frac{1}{\sqrt{1-x^2}}$	$\sin^{-1} x$
$\frac{1}{1+x^2}$	$\tan^{-1} x$

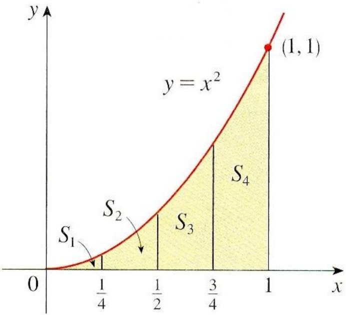
Area under $y = x^2$ from 0 to 1.

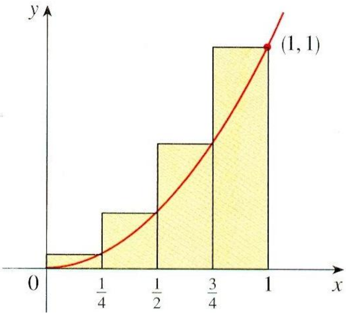
Example

Use rectangles to estimate the area under the parabola $\mathbf{y} = x^2$ from 0 to 1.



Area estimate using right end points

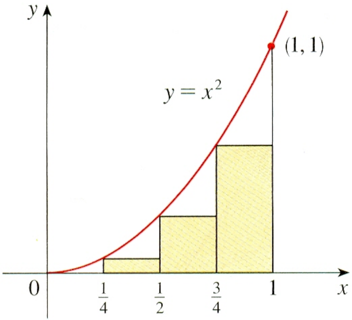




$$R_4 = \frac{1}{4}\cdot\left(\frac{1}{4}\right)^2 + \frac{1}{4}\cdot\left(\frac{1}{2}\right)^2 + \frac{1}{4}\cdot\left(\frac{3}{4}\right)^2 + \frac{1}{4}\cdot 1^2 = \frac{15}{32}$$

Note area $A < \frac{15}{32} = .46875$

Area estimate using left end points



$$L_4 = \frac{1}{4} \cdot 0^2 + \frac{1}{4} \cdot \left(\frac{1}{4}\right)^2 + \frac{1}{4} \cdot \left(\frac{1}{2}\right)^2 + \frac{1}{4} \cdot \left(\frac{3}{4}\right)^2 = \frac{7}{32} = .21875$$

Note area **A** satisfies

$$.21875 \leq \mathbf{A} \leq .46875$$

Area definition using right end points

Definition

The **area** $\mathbf{A}$ of the region S that lies under the graph of the continuous function f is the limit of the sum of the areas of approximating rectangles:

$$\begin{aligned} \mathbf{A} &= \lim_{n\to\infty} R_n \\ &= \lim_{n\to\infty} [f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x] \end{aligned}$$

Note in the above definition that if $\mathbf{I} = [a, b]$, then

$$\Delta x = \frac{b-a}{n},$$

where n is the number of rectangles or divisions.

Definition of a Definite Integral

Definition

- If f is a continuous function defined for $a \leq x \leq b$, we divide the interval $[a, b]$ into n subintervals of equal width $\Delta x = (b - a)/n$.
- We let $x_0(= a), x_1, x_2, \ldots, x_n(= b)$ be the endpoints of these subintervals and we let $x_1^*, x_2^*, \ldots, x_n^*$ be any **sample points** in these subintervals, so x_i^* lies in the i-th subinterval $[x_{i-1,x_i}]$.
- Then the **definite integral of f from a to b** is

$$\int_a^b f(x)\, dx = \lim_{n \to \infty} \sum_{i=1} f(x_i^*) \Delta x$$

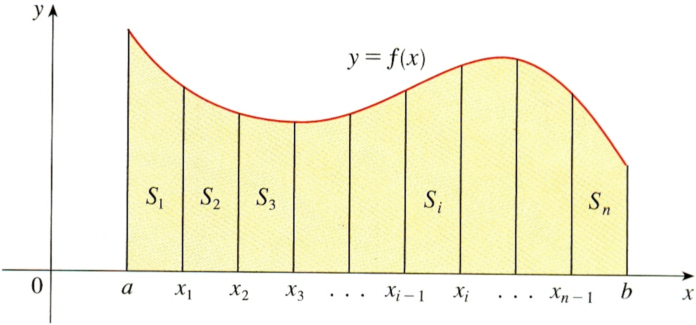


Figure: Here there are n subintervals with $b = x_n$.

Example (Trapezoidal Approximation)

The best simple estimate of **A** with n subintervals is the "trapezoidal approximation":

$$\mathbf{A} = \frac{b-a}{n}\left(f(a)/2 + f(b)/2 + \sum_{i=1}^{n-1} f(x_i)\right)$$

Properties of the Integral

- $\int_a^b [f(x) + g(x)]\, dx = \int_a^b f(x)\, dx + \int_a^b g(x)\, dx$
- $\int_a^b c \cdot f(x)\, dx = c \int_a^b f(x)\, dx$, where c is any constant
- $\int_a^b [f(x) - g(x)]\, dx = \int_a^b f(x)\, dx - \int_a^b g(x)\, dx$

Definition

A function F defined on real valued functions is said to be **linear** if given two such functions, f, g,

- $F(f+g) = F(f) + F(g)$
- For $c \in \mathbb{R}$, $F(c \cdot f) = c \cdot F(f)$.

Question

Why do you think integration is said to be ***linear****?*

Theorem (Fundamental Theorem of Calculus, Part 1)

If f is continuous on $[a, b]$, then the function g defined by

$$g(x) = \int_a^x f(t)\,dt \qquad a \leq x \leq b$$

is continuous on $[a, b]$ and differentiable on (a, b), and

$$g'(x) = f(x).$$

Theorem (Fundamental Theorem of Calculus, Part 2)

If f is continuous on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a),$$

where $F(x)$ is any antiderivative of $f(x)$, that is, a function such that

$$F'(x) = f(x).$$

Application of **FTC**

Example

Evaluate
$$\int_3^6 \frac{1}{x}\,dx.$$

Application of FTC

Example

Evaluate
$$\int_3^6 \frac{1}{x}\,dx.$$

Solution:

An antiderivative for $\frac{1}{x}$ is $F(x) = \ln x$.
So, by **FTC**,
$$\int_3^6 \frac{1}{x}\,dx = F(6) - F(3) = \ln 6 - \ln 3.$$

$\square$

Example

Evaluate

$$\int_1^3 e^x \, dx.$$

Example

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$$\int_1^3 e^x \, dx.$$

Solution:

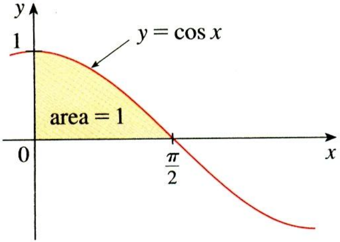
Note that an antiderivative for e^x is $F(x) = e^x$.
So, by **FTC**,

$$\int_1^3 e^x \, dx = F(3) - F(1) = e^3 - e.$$

$\square$

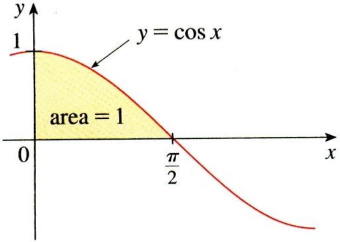
Example

Find area **A** under the cosine curve from 0 to b, where $0 \leq b \leq \frac{\pi}{2}$.



Example

Find area **A** under the cosine curve from 0 to b, where $0 \leq b \leq \frac{\pi}{2}$.



Solution:

Since an antiderivative of $f(x) = \cos(x)$ is $F(x) = \sin(x)$, we have

$$\mathbf{A} = \int_0^b \cos(x)\,dx = \sin(x)\Big|_0^b = \sin(b) - \sin(0) = \sin(b).$$

□

Theorem (Fundamental Theorem of Calculus)

Suppose f is continuous on $[a, b]$.

- If $g(x) = \int_a^x f(t)\,dt$, then $g'(x) = f(x)$.
- $\int_a^b f(x)\,dx = F(b) - F(a)$, where $F(x)$ is any antiderivative of $f(x)$, that is, $F'(x) = f(x)$.

Notation: Indefinite integral

$$\boxed{\int f(x)\,dx = F(x) \quad \textbf{means} \quad F'(x) = f(x).}$$

We use the notation $\int f(x)\,dx$ to denote an antiderivative for $f(x)$ and it is called **an indefinite integral.**

A **definite integral** has the form:

$$\int_a^b f(x)\,dx = \int f(x)\,dx\Big|_a^b = F(b) - F(a)$$

Table of Indefinite Integrals

$$\int c \cdot f(x)\,dx = c \cdot \int f(x)\,dx$$

$$\int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$$

$$\int k\,dx = kx + C$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \qquad (n \neq -1)$$

$$\int \frac{1}{x}\,dx = \ln|x| + C$$

$$\int e^x\,dx = e^x + C$$

The Substitution Rule

The **Substitution Rule** is one of the main tools used in this class for finding antiderivatives. It comes from the Chain Rule:

$$[F(g(x))]' = F'(g(x))g'(x).$$

So,

$$\int F'(g(x))g'(x)\,dx = F(g(x)).$$

Substitution Rule: If $u = g(x)$ is a differentiable function whose range is an interval $\mathbf{I}$ and f is continuous on $\mathbf{I}$, then

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du.$$

Example

Find

$$\int x^3 \cos(x^4 + 2)\,dx.$$

Substitution Rule: If $u = g(x)$ is a differentiable function whose range is an interval $\mathbf{I}$ and f is continuous on $\mathbf{I}$, then

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du.$$

Example

Find

$$\int x^3 \cos(x^4 + 2)\,dx.$$

Solution:

- Make the substitution: $u = x^4 + 2$.
- Get $du = 4x^3\,dx$.

$$\int x^3 \cos(x^4 + 2)\,dx = \int \cos u \cdot \frac{1}{4}\,du = \frac{1}{4}\int \cos u\,du$$

$$= \frac{1}{4}\sin u + C = \frac{1}{4}\sin(x^4 + 2) + C.$$

Note at the final stage we return to the original variable x. $\square$

Example

Evaluate

$$\int \sqrt{2x+1}\, dx.$$

Example

Evaluate

$$\int \sqrt{2x+1}\,dx.$$

Solution:

- Let $u = 2x + 1$.
- Then $du = 2\,dx$, so $dx = \frac{du}{2}$.
- The Substitution Rule gives

$$\int \sqrt{2x+1}\,dx = \int \sqrt{u}\,\frac{du}{2} = \frac{1}{2}\int u^{\frac{1}{2}}\,du$$

$$= \frac{1}{2} \cdot \frac{u^{\frac{3}{2}}}{\frac{3}{2}} + C = \frac{1}{3}u^{\frac{3}{2}} + C = \frac{1}{3}(2x+1)^{\frac{3}{2}} + C.$$

$\square$

Substitution Rule: If $u = g(x)$ is a differentiable function whose range is an interval **I** and f is continuous on **I**, then

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du.$$

Example

Calculate

$$\int e^{5x}\,dx.$$

Substitution Rule: If $u = g(x)$ is a differentiable function whose range is an interval $\mathbf{I}$ and f is continuous on $\mathbf{I}$, then

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du.$$

Example

Calculate

$$\int e^{5x}\,dx.$$

Solution:

- If we let $u = 5x$, then $du = 5\,dx$, so $dx = \frac{1}{5}\,du$
- Therefore

$$\int e^{5x}\,dx = \frac{1}{5}\int e^{u}\,du = \frac{1}{5}e^{u} + C = \frac{1}{5}e^{5x} + C.$$

$\square$

Substitution Rule: If $u = g(x)$ is a differentiable function whose range is an interval **I** and f is continuous on **I**, then

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du.$$

Example

Calculate $\int \tan x\,dx$.

Substitution Rule: If $u = g(x)$ is a differentiable function whose range is an interval **I** and f is continuous on **I**, then

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du.$$

Example

Calculate $\int \tan x\,dx$.

Solution:

$$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx.$$

This suggests substitution $u = \cos x$, since then $du = -\sin x\,dx$ and so, $\sin x\,dx = -du$:

$$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx = -\int \frac{du}{u}$$
$$= -\ln|u| + C = -\ln|\cos x| + C.$$

$\square$

The Substitution Rule for Definite Integrals

Theorem (Substitution Rule for Definite Integrals)

If g' is continuous on $[a, b]$ and f is continuous on the range of $u = g(x)$, then

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du.$$

Integration by parts

- Recall the method of substitution came from the chain rule.
- **Integration by parts** comes from the **product rule**:

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x).$$

- So,

$$f(x)g'(x) = (f(x)g(x))' - g(x)f'(x).$$

- Thus, after taking antiderivatives, we get

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx.$$

Integration by parts

- $$\boxed{\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx}$$
- The above is called the **formula for integration by parts**.
- If we let $u = f(x)$ and $v = g(x)$, then $du = f'(x)\,dx$ and $dv = g'(x)\,dx$.
- So the formula becomes: $$\boxed{\int u\,dv = uv - \int v\,du.}$$

Strategy for using integration by parts

- Recall the **integration by parts formula**:

$$\int u\,dv = uv - \int v\,du.$$

- To apply this formula we must choose dv so that we can integrate it!
- Frequently, we choose u so that the derivative of u is simpler than u.
- If both properties hold, then you have made the correct choice.

Examples using strategy: $\int u\,dv = uv - \int v\,du$

- $\displaystyle\int xe^x\,dx$: Choose $\mathbf{u} = \mathbf{x}$ and $\mathbf{dv} = \mathbf{e^x\,dx}$
- $\displaystyle\int t^2 e^t\,dt$: Choose $\mathbf{u} = \mathbf{t^2}$ and $\mathbf{dv} = \mathbf{e^t\,dt}$
- $\displaystyle\int \ln x\,dx$: Choose $\mathbf{u} = \ln \mathbf{x}$ and $\mathbf{dv} = \mathbf{dx}$
- $\displaystyle\int x\sin x\,dx$: $\mathbf{u} = \mathbf{x}$ and $\mathbf{dv} = \sin \mathbf{x\,dx}$
- $\displaystyle\int x^2 \sin 2x\,dx$: $\mathbf{u} = \mathbf{x^2}$ and $\mathbf{dv} = \sin \mathbf{2x\,dx}$

Example

Find
$$\int xe^x\,dx.$$

Example

Find
$$\int xe^x\,dx.$$

Solution:

- Let
$$\mathbf{u = x \quad dv = e^x\,dx.}$$

- Then
$$\mathbf{du = dx \quad v = e^x.}$$

- **Integrating by parts** gives
$$\int xe^x\,dx = xe^x - \int e^x\,dx = xe^x - e^x + C.$$

□

Example

Evaluate

$$\int \ln x \; dx.$$

Example

Evaluate
$$\int \ln x \, dx.$$

Solution:

- Let
$$\mathbf{u} = \ln \mathbf{x} \qquad \mathbf{dv} = \mathbf{dx}.$$

- Then
$$\mathbf{du} = \frac{\mathbf{1}}{\mathbf{x}}\mathbf{dx} \qquad \mathbf{v} = \mathbf{x}.$$

- **Integrating by parts**, we get
$$\int \ln x \, dx = x \ln x - \int x \frac{dx}{x}$$
$$= x \ln x - \int dx = x \ln x - x + C.$$

□

Definition

Let **n** be a positive integer. Then the **Cartesian product** of **n** copies of the real number line $\mathbb{R}$ is:

$$\mathbf{R}^{\mathbf{n}} = \mathbb{R} \times \mathbb{R} \times \ldots \times \mathbb{R} = \{(a_1, a_2, \ldots, a_{\mathbf{n}}) \mid a_j \in \mathbb{R}\},$$

which is the set of all ordered **n**-tuples of real numbers.

Example

(a) $\mathbf{R}^{\mathbf{2}} = \mathbb{R} \times \mathbb{R} = \{(a_1, a_2) \mid a_i \in \mathbb{R}\}$ is the Euclidean plane.

(b) $\mathbf{R}^{\mathbf{3}} = \mathbb{R} \times \mathbb{R} \times \mathbb{R} = \{(a_1, a_2, a_3) \mid a_i \in \mathbb{R}\}$ is Euclidian three-space.

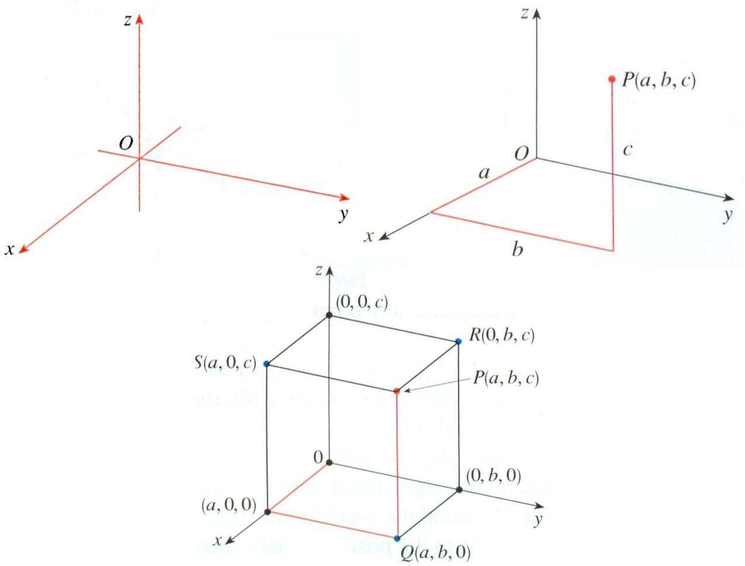
Notation and coordinates

- There are two standard notations for points in $\mathbf{R^3}$, or more generally $\mathbf{R^n}$. If $P \in \mathbf{R^3}$, then $P = (a_1, a_2, a_3)$ for some **scalars** $a_1, a_2, a_3 \in \mathbb{R}$. The book also denotes this point by writing $P(a_1, a_2, a_3)$.
- The scalar a_1 is called the ***x*-coordinate** of P, a_2 is called the ***y*-coordinate** of P and a_3 is called the ***z*-coordinate** of P.

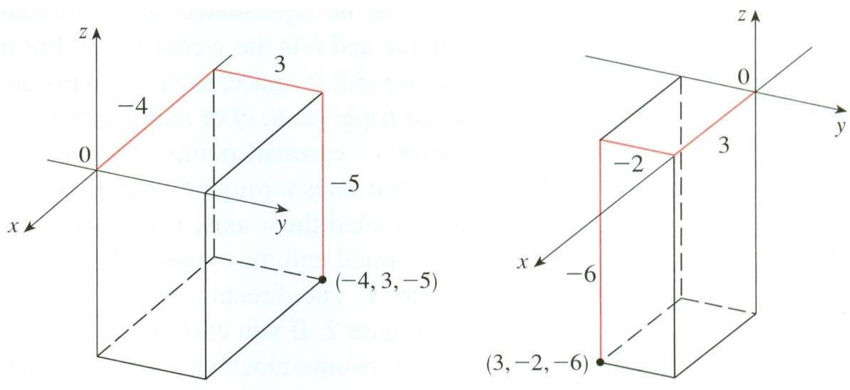
Example

The point $P = (1, 0, 7)$ in $\mathbf{R^3}$ can also be written as $P(1, 0, 7)$. Its z-coordinate is 7.

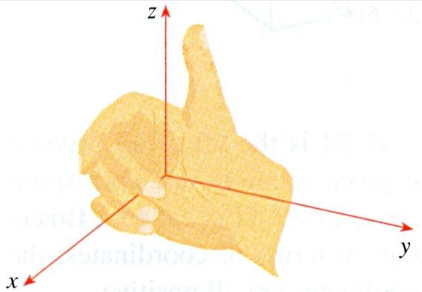
The coordinate axes and finding coordinate points

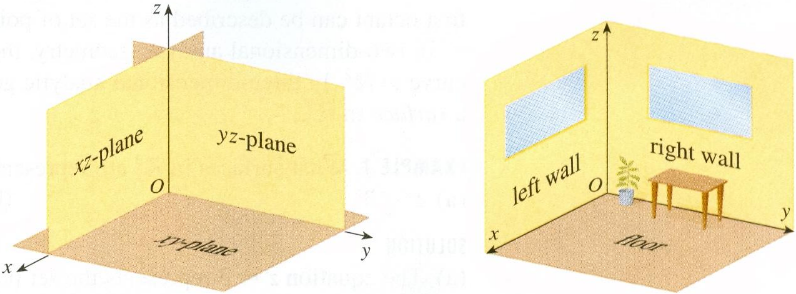


The coordinate axes and finding coordinate points



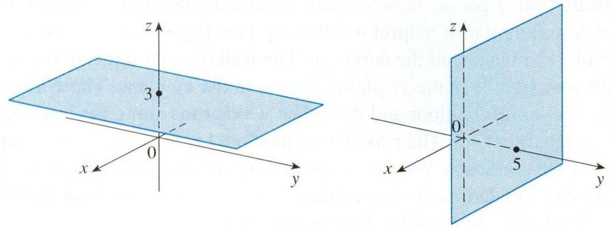
Plotting the points $(-4, 3, -5)$ and $(3, -2, -6)$





(a) Coordinate planes

(b)



(a) $z = 3$, a plane in $\mathbb{R}^3$

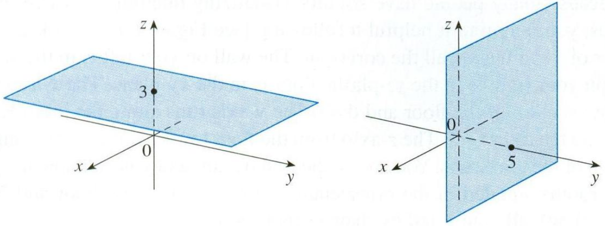
(b) $y = 5$, a plane in $\mathbb{R}^3$

Example

What surfaces in $\mathbf{R^3}$ are represented by the following equations?

(a) $z = 3$

(b) $y = 5$



(a) $z = 3$, a plane in $\mathbb{R}^3$ (b) $y = 5$, a plane in $\mathbb{R}^3$

Example

What surfaces in $\mathbf{R}^3$ are represented by the following equations?

(a) $z = 3$

(b) $y = 5$

Solution

(a) $z = 3$ is the set $\{(x, y, z) \mid z = 3\}$, which is the set of points in $\mathbf{R}^3$ whose z-coordinate is 3. This is the horizontal plane parallel to the xy-plane and 3 units above it as in Figure (a).

(b) $y = 5$ is the set of all points in $\mathbf{R}^3$ whose y-coordinate is 5. This is the vertical plane parallel to the xz-plane and 5 units to the right of it as in Figure (b).

Example

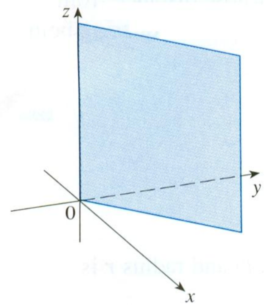
Describe and sketch the surface in $\mathbf{R}^3$ represented by the equation $y = x$.

Example

Describe and sketch the surface in $\mathbf{R}^3$ represented by the equation $y = x$.

Solution

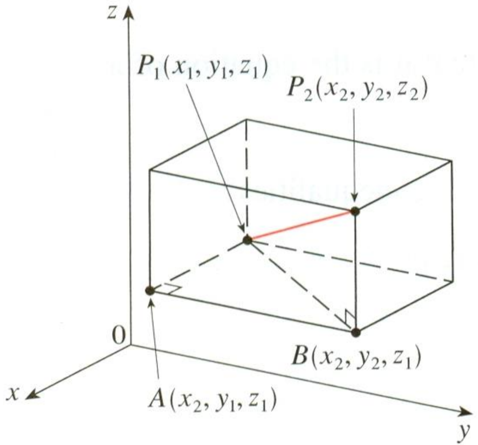
- The equation represents the set of points in $\mathbf{R}^3$ whose x- and y-coordinates are equal: $\{(x, x, z) \mid x \in \mathbb{R},\ z \in \mathbb{R}\}$.
- This is a vertical plane that intersects the xy-plane in the line $y = x$, $z = 0$.
- The portion of this plane that lies in the first octant is sketched below.



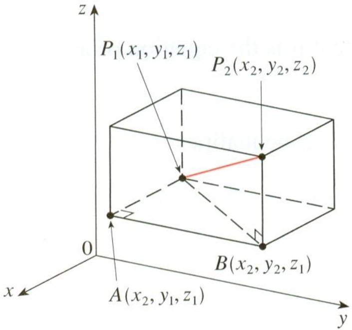
Distance Formula in Three Dimensions

The distance $|P_1P_2|$ between the points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is

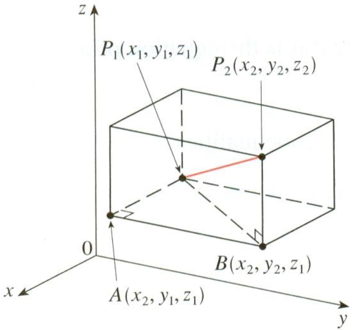
$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$



$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$



$$|P_1P_2| = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}.$$



Proof.

$|P_1P_2| = \sqrt{|P_1B|^2+|BP_2|^2}$.
$|P_1B| = \sqrt{|P_1A|^2+|AB|^2}$.
Since $|P_1A| = |x_2-x_1|$, $|AB| = |y_2-y_1|$, $|BP_2| = |z_2-z_1|$, then
$|P_1B| = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$. Hence,

$$|P_1P_2| = \sqrt{|P_1B|^2+|BP_2|^2} = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}.$$

$\square$

Example

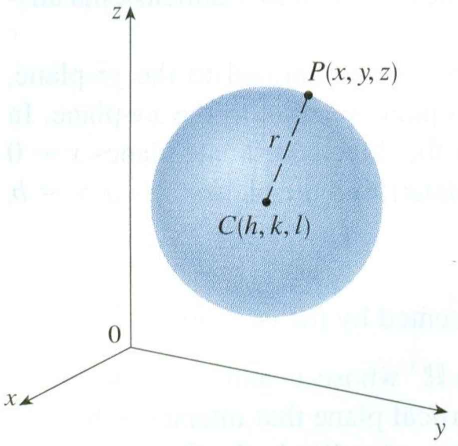
The distance from the point $P(2, -1, 7)$ to the point $Q(1, -3, 5)$ is

$$|PQ| = \sqrt{(1-2)^2 + (-3+1)^2 + (5-7)^2}$$

$$= \sqrt{1+4+4} = \sqrt{9} = 3.$$

Definition

The sphere in $\mathbf{R}^3$ with center $C(h, k, l)$ and radius $\mathbf{r}$ is the set where $(x-h)^2 + (y-k)^2 + (z-l)^2 = \mathbf{r}^2$. Note that this sphere is geometrically the set of points (x, y, z) of distance $\mathbf{r}$ from the point (h, k, l).



Example

Find the equation of the sphere $\mathbf{S}$ in $\mathbf{R}^3$ of radius 2 centered at the point $(1, 2, 3)$.

Example

Find the equation of the sphere **S** in $\mathbf{R}^3$ of radius 2 centered at the point $(1, 2, 3)$.

Solution

By definition, **S** has the equation:

$$(x-1)^2 + (y-2)^2 + (z-3)^2 = 4.$$

Recall that one can **complete the square** of $x^2 + bx$ by adding $\frac{b^2}{4}$ to get

$$x^2 + bx + \frac{b^2}{4} = \left(x + \frac{b}{2}\right)^2.$$

Example (popular exam problem)

Consider the subset $\mathbf{S}$ of $\mathbf{R^3}$ defined by

$$x^2 + y^2 + 6y + z^2 + 2z = 26.$$

By completing the square, we have

$$x^2 + (y^2 + 6y + 9) + (z^2 + 2z + 1) = 26 + 9 + 1 = 36,$$

which simplifies to be

$$x^2 + (y + 3)^2 + (z + 1)^2 = 6^2.$$

So the set $\mathbf{S}$ is the sphere centered at $(0, -3, -1)$ of radius 6.

Definition

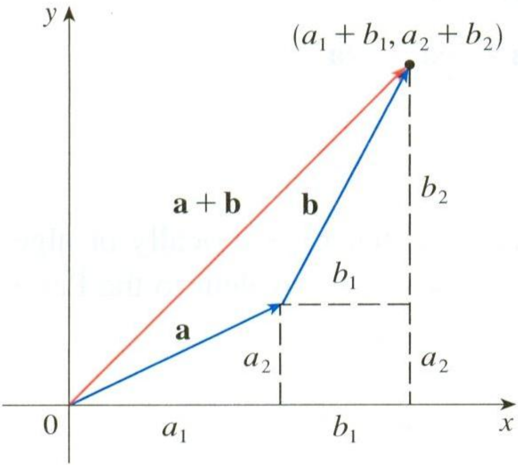
- Given points $P = (x_1, y_1, z_1)$ and $Q = (x_2, y_2, z_2)$ in $\mathbf{R}^3$, then $\overrightarrow{PQ} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$ denotes the **arrow** or **vector based at** P with **terminal point** Q.
- If $\lambda \in \mathbb{R}$ is a scalar and $\mathbf{v} = \langle a, b, c \rangle$ is a vector, then the new vector $\lambda \mathbf{v} = \langle \lambda a, \lambda b, \lambda c \rangle$.
- If $\lambda > 0$, then $\lambda \mathbf{v}$ is the vector pointed in the direction $\mathbf{v}$ and has length or **magnitude** $\lambda |\mathbf{v}|$; if $\lambda < 0$, then $\lambda \mathbf{v}$ is the vector pointed in the opposite direction of $\mathbf{v}$ with length $|\lambda||\mathbf{v}|$.
- If $\mathbf{u} = \langle x_1, y_1, z_1 \rangle$ and $\mathbf{v} = \langle x_2, y_2, z_2 \rangle$, then
  - $\mathbf{u} + \mathbf{v} = \langle x_1 + x_2, y_1 + y_2, z_1 + z_2 \rangle$. In other words, vectors add by adding their coordinates.
  - $\mathbf{u} - \mathbf{v} = \langle x_1 - x_2, y_1 - y_2, z_1 - z_2 \rangle$.

Example

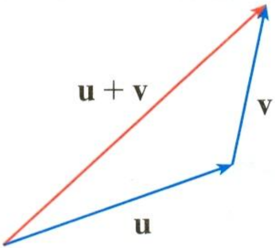
- If $P = (1, 2, 3)$ and $Q = (-2, 1, 0)$, then

$$\overrightarrow{PQ} = Q - P = \langle -2 - 1, 1 - 2, 0 - 3 \rangle = \langle -3, -1, -3 \rangle.$$

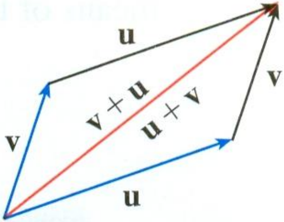
- If $\mathbf{u} = \langle 1, 2, 3 \rangle$ and $\mathbf{v} = \langle -2, 1, 0 \rangle$, then
  - $2\mathbf{u} = \langle 2, 4, 6 \rangle$
  - $\mathbf{u} + \mathbf{v} = \langle -1, 3, 3 \rangle$
  - $\mathbf{u} - \mathbf{v} = \langle 3, 1, 3 \rangle$



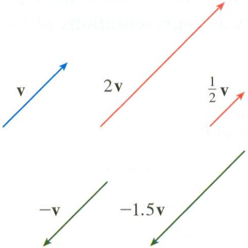
To add vectors, add their coordinates



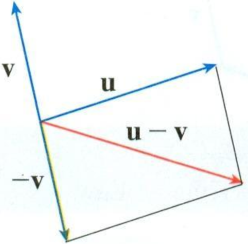
The Triangle Law



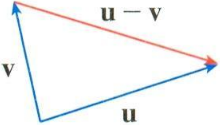
The Parallelogram Law



Scalar multiples of **v**



(a)



(b)

Example (popular exam problem)

- Three of the four vertices of a parallelogram are $P(0, -1, 1)$, $Q(0, 1, 0)$ and $R(3, 1, 1)$.
- Two of the sides are PQ and PR.
- Find the coordinates of the fourth vertex **T**.

Example (popular exam problem)

- Three of the four vertices of a parallelogram are $P(0, -1, 1)$, $Q(0, 1, 0)$ and $R(3, 1, 1)$.
- Two of the sides are PQ and PR.
- Find the coordinates of the fourth vertex **T**.

Solution:

The fourth vertex **T** is the point obtained as the vector sum

$$\overrightarrow{OQ} + \overrightarrow{PR} = \langle 0, 1, 0\rangle + \langle 3, 2, 0\rangle = (3, 3, 0),$$

where $O = (0, 0, 0)$ is the origin.

Example (popular exam problem)

- Consider the parallelogram with vertices A, B, C, D such that B and C are adjacent to A.
- If $A = (2, 5, 1)$, $B = (3, 1, 4)$, $D = (5, 2, -3)$, find the point C.

Example (popular exam problem)

- Consider the parallelogram with vertices A, B, C, D such that B and C are adjacent to A.
- If $A = (2, 5, 1)$, $B = (3, 1, 4)$, $D = (5, 2, -3)$, find the point C.

Solution:

After drawing a picture, the point C is easily seen to be:

$$\overrightarrow{OA} + \overrightarrow{BD} = \langle 2, 5, 1 \rangle + \langle 2, 1, -7 \rangle = (4, 6, -6),$$

where O is the origin.

In Class Exercises 1

1. Draw a picture and locate the points $P(1, 2, -3)$ and $Q(-2, 5, 3)$ in $\mathbf{R}^3$. Also draw $\overrightarrow{PQ}$ on your picture and label everything. Also find $|PQ|$.
2. Express the sphere **S**: $\quad x^2 + 2x + y^2 + 6y + z^2 - 4z = 11$ in standard form by completing squares. Then state what is the center and the radius of this sphere.
3. Three of the four vertices of a parallelogram are $P(2, -1, 0)$, $Q(0, 1, 0)$ and $R(0, 1, 1)$. Two of the sides are PQ and PR. Find the coordinates of the fourth vertex **T**.
4. Consider the parallelogram with vertices A, B, C, D such that B and C are adjacent to A. If $A = (2, 5, 2)$, $B = (3, 1, 4)$, $D = (5, 2, -3)$, find the point C.
5. Consider the parallelogram with vertices A, B, C, D such that B and C are adjacent to A. If $A = (2, 2)$, $B = (4, 2)$, $D = (3, 3)$, find the point C. Also draw a picture, labeling everything.

Definition

If $\mathbf{a} = \langle x_1, y_1, z_1 \rangle$ and $\mathbf{b} = \langle x_2, y_2, z_2 \rangle$, then the **dot product** of $\mathbf{a}$ and $\mathbf{b}$ is:

$$\mathbf{a} \cdot \mathbf{b} = x_1 x_2 + y_1 y_2 + z_1 z_2.$$

Definition

If $\mathbf{a} = \langle x_1, y_1, z_1 \rangle$ and $\mathbf{b} = \langle x_2, y_2, z_2 \rangle$, then the **dot product** of $\mathbf{a}$ and $\mathbf{b}$ is:

$$\mathbf{a} \cdot \mathbf{b} = x_1x_2 + y_1y_2 + z_1z_2.$$

Example

The dot product of $\langle 1, 2, 3 \rangle$ and $\langle 1, 0, 7 \rangle$ is

$$\langle 1, 2, 3 \rangle \cdot \langle 1, 0, 7 \rangle = 1 + 0 + 21 = 22.$$

Definition

If $\mathbf{a} = \langle x_1, y_1, z_1 \rangle$ and $\mathbf{b} = \langle x_2, y_2, z_2 \rangle$, then the **dot product** of $\mathbf{a}$ and $\mathbf{b}$ is:

$$\mathbf{a} \cdot \mathbf{b} = x_1 x_2 + y_1 y_2 + z_1 z_2.$$

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Theorem

Let $\mathbf{a} = \langle x_1, y_1, z_1 \rangle$ be a vector. Then:

- The **length** of $\mathbf{a}$ is $|\mathbf{a}| = \sqrt{\mathbf{a} \cdot \mathbf{a}} = \sqrt{x_1^2 + y_1^2 + z_1^2}$.

Definition

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Theorem (Basic algebraic properties of dot product)

Let $\mathbf{a} = \langle x_1, y_1, z_1 \rangle$, $\mathbf{b} = \langle x_2, y_2, z_2 \rangle$, $\mathbf{c} = \langle x_3, y_3, z_3 \rangle$ be vectors and let $\lambda \in \mathbb{R}$ be a scalar. Then:

- $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$.
- $(\mathbf{a} + \mathbf{b}) \cdot \mathbf{c} = \mathbf{a} \cdot \mathbf{c} + \mathbf{b} \cdot \mathbf{c}$.
- $(\lambda \mathbf{a}) \cdot \mathbf{b} = \lambda(\mathbf{a} \cdot \mathbf{b})$.

For convenience, it is useful to pick out the special **unit length** vectors pointed respectively along the positive x, y and z-axes, as given in the next definition.

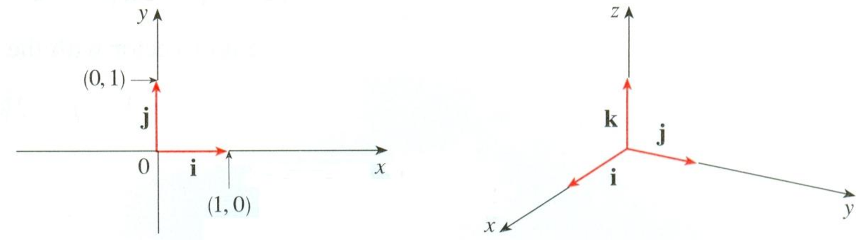
Definition

We define the **standard basis vectors** for $\mathbf{R}^3$ as follows:

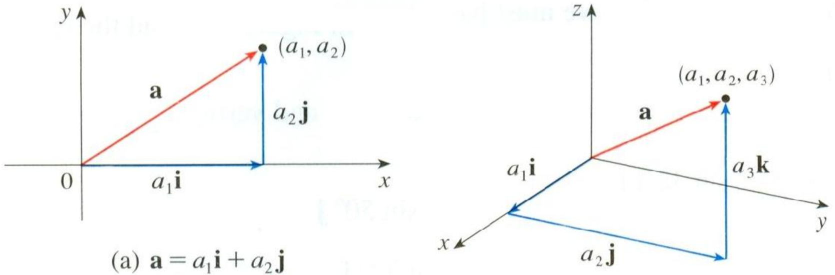
$$\mathbf{i} = \langle 1, 0, 0 \rangle, \quad \mathbf{j} = \langle 0, 1, 0 \rangle, \quad \mathbf{k} = \langle 0, 0, 1 \rangle.$$

Note that the vector $\langle a, b, c \rangle$ can be expressed by

$$\langle a, b, c \rangle = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}.$$



(a) (b)



(a) $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j}$

(b) $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$

Example (popular exam problem)

If $\mathbf{a} = \mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$ and $\mathbf{b} = 4\mathbf{i} + 7\mathbf{k}$, express the vector $2\mathbf{a} + 3\mathbf{b}$ in terms of $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ and as a coordinate vector.

Example (popular exam problem)

If $\mathbf{a} = \mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$ and $\mathbf{b} = 4\mathbf{i} + 7\mathbf{k}$, express the vector $2\mathbf{a} + 3\mathbf{b}$ in terms of $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ and as a coordinate vector.

Solution

We have
$$\begin{aligned} 2\mathbf{a} + 3\mathbf{b} &= 2(\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}) + 3(4\mathbf{i} + 7\mathbf{k}) \\ &= (2\mathbf{i} + 4\mathbf{j} - 6\mathbf{k}) + (12\mathbf{i} + 21\mathbf{k}) = 14\mathbf{i} + 4\mathbf{j} + 15\mathbf{k} \\ &= \langle 14, 4, 15 \rangle. \end{aligned}$$

Definition

- A **unit vector** is a vector whose length is 1. For instance, $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are all unit vectors.
- In general, if $\mathbf{a} \neq (0, 0, 0)$, then the unit vector that has the same direction as $\mathbf{a}$ is:
$$\mathbf{u} = \frac{1}{|\mathbf{a}|}\mathbf{a} = \frac{\mathbf{a}}{|\mathbf{a}|}.$$

Example

Find the unit vector in the direction of the vector $2\mathbf{i} - \mathbf{j} - 2\mathbf{k}$.

Example

Find the unit vector in the direction of the vector $2\mathbf{i} - \mathbf{j} - 2\mathbf{k}$.

Solution

The given vector has length

$$|2\mathbf{i} - \mathbf{j} - 2\mathbf{k}| = \sqrt{2^2 + (-1)^2 + (-2)^2} = \sqrt{9} = 3.$$

The unit vector with the same direction is

$$\frac{1}{3}(2\mathbf{i} - \mathbf{j} - 2\mathbf{k}) = \frac{2}{3}\mathbf{i} - \frac{1}{3}\mathbf{j} - \frac{2}{3}\mathbf{k} = \left\langle \frac{2}{3}, -\frac{1}{3}, -\frac{2}{3} \right\rangle.$$

Theorem

For nonzero vectors $\mathbf{a}, \mathbf{b} \in \mathbf{R}^3$

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos(\theta),$$

where $\theta \in [0, \pi)$ is the angle between the vectors.

1. $\mathbf{a}$ and $\mathbf{b}$ are **orthogonal** or **perpendicular** if and only if $\mathbf{a} \cdot \mathbf{b} = 0$.
2. The angle θ between $\mathbf{a}$ and $\mathbf{b}$ is an **acute** angle if and only if $\mathbf{a} \cdot \mathbf{b} > 0$
3. The angle θ between $\mathbf{a}$ and $\mathbf{b}$ is an **obtuse** angle if and only if $\mathbf{a} \cdot \mathbf{b} < 0$.
4. $\cos(\theta) = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$.
5. $\theta = \arccos(\frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}) = \cos^{-1}(\frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|})$. In particular, if $\mathbf{a}$ and $\mathbf{b}$ are unit vectors, then $\theta = \arccos(\mathbf{a} \cdot \mathbf{b})$.

Example

The vectors $\langle 1, 2, -1 \rangle$ and $\langle 3, -1, 1 \rangle$ are **orthogonal**, since

$$\langle 1, 2, -1 \rangle \cdot \langle 3, -1, 1 \rangle = 3 - 2 - 1 = 0.$$

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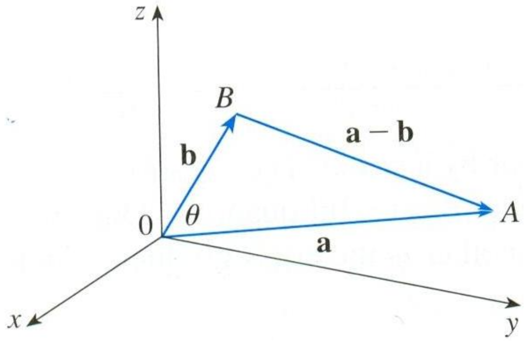
Example

The vectors $\langle 1, 2, -1\rangle$ and $\langle 0, -1, 1\rangle$ are **not** **orthogonal**, since

$$\langle 1, 2, -1\rangle \cdot \langle 0, -1, 1\rangle = 0 - 2 - 1 = -3 \neq 0.$$



Proof of $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos \theta$

- If we apply the **Law of Cosines** to triangle OAB in figure above, we get
$$|AB|^2 = |OA|^2 + |OB|^2 - 2|OA||OB| \cos \theta. \tag{1}$$
- But $|OA| = |\mathbf{a}|$, $|OB| = |\mathbf{b}|$, and $|AB| = |\mathbf{a} - \mathbf{b}|$, so Equation 1 becomes
$$|\mathbf{a} - \mathbf{b}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2|\mathbf{a}||\mathbf{b}| \cos \theta. \tag{2}$$

Continuation of the proof of $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta$.

- $$|\mathbf{a} - \mathbf{b}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2|\mathbf{a}||\mathbf{b}| \cos\theta. \tag{3}$$
- Rewrite the left side of this equation as follows:
$$|\mathbf{a} - \mathbf{b}|^2 = (\mathbf{a} - \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b})$$
$$= \mathbf{a} \cdot \mathbf{a} - \mathbf{a} \cdot \mathbf{b} - \mathbf{b} \cdot \mathbf{a} + \mathbf{b} \cdot \mathbf{b} = |\mathbf{a}|^2 - 2\mathbf{a} \cdot \mathbf{b} + |\mathbf{b}|^2.$$
- Therefore, Equation 3 gives
$$|\mathbf{a}|^2 - 2\mathbf{a} \cdot \mathbf{b} + |\mathbf{b}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2|\mathbf{a}||\mathbf{b}| \cos\theta.$$
- Thus,
$$-2\mathbf{a} \cdot \mathbf{b} = -2|\mathbf{a}||\mathbf{b}| \cos\theta.$$
or equivalently,
$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta.$$
$\square$

Definition

- The **scalar** projection (component) of $\mathbf{b}$ onto $\mathbf{a}$ is

$$\mathbf{comp_a b} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|} = \frac{\mathbf{a}}{|\mathbf{a}|} \cdot \mathbf{b} = |\mathbf{b}| \cos\theta,$$

  where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$. In particular, if $\mathbf{a}$ is a unit vector, then $\mathbf{comp_a b} = \mathbf{a} \cdot \mathbf{b}$.

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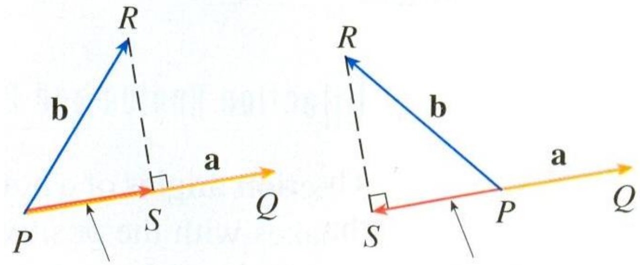
- The **vector** projection of $\mathbf{b}$ onto (in the direction of) $\mathbf{a}$ is

$$\mathbf{proj_a b} = \left(\frac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{a}\cdot\mathbf{a}}\right)\mathbf{a} = \left(\frac{\mathbf{a}}{|\mathbf{a}|}\cdot\mathbf{b}\right)\frac{\mathbf{a}}{|\mathbf{a}|} = (\mathbf{comp_a b})\frac{\mathbf{a}}{|\mathbf{a}|}.$$

  In particular, if $\mathbf{a}$ is a unit vector, then $\mathbf{proj_a b} = (\mathbf{a}\cdot\mathbf{b})\mathbf{a}$.



Example

- Consider the vectors $\mathbf{a} = \langle 1, 2, 2 \rangle$ and $\mathbf{b} = \langle 1, 1, 1 \rangle$.
- Since $\mathbf{proj_a b} = \frac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{a}\cdot\mathbf{a}}\mathbf{a}$, then

$$\mathbf{v} = \mathbf{b} - \mathbf{proj_a b} = \langle 1, 1, 1 \rangle - \frac{5}{9}\langle 1, 2, 2 \rangle = \langle \frac{4}{9}, -\frac{1}{9}, -\frac{1}{9} \rangle$$

  must be **perpendicular** to $\mathbf{a}$ and must lie in the plane containing $\mathbf{a}$ and $\mathbf{b}$.
- One can easily check this orthogonality property by taking the dot product of $\mathbf{v}$ with $\mathbf{a}$ to get zero.

Example (popular exam problem)

Find the **scalar projection** and **vector projection** of $\mathbf{b} = \langle 1, 1, 2 \rangle$ onto $\mathbf{a} = \langle -2, 3, 1 \rangle$.

Example (popular exam problem)

Find the **scalar projection** and **vector projection** of $\mathbf{b} = \langle 1, 1, 2 \rangle$ onto $\mathbf{a} = \langle -2, 3, 1 \rangle$.

Solution

- Since $|\mathbf{a}| = \sqrt{(-2)^2 + 3^2 + 1^2} = \sqrt{14}$, the **scalar projection** of $\mathbf{b}$ onto $\mathbf{a}$ is

$$\mathbf{comp_a b} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|} = \frac{(-2)(1) + 3(1) + 1(2)}{\sqrt{14}} = \frac{3}{\sqrt{14}}.$$

- The **vector projection** is this **scalar projection** times the unit vector in the direction of $\mathbf{a}$:

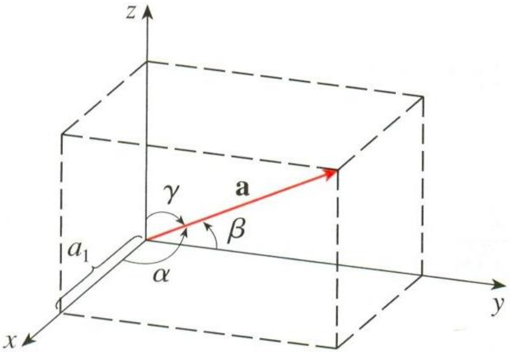
$$\mathbf{proj_a b} = \frac{3}{\sqrt{14}} \frac{\mathbf{a}}{|\mathbf{a}|} = \frac{3}{14}\mathbf{a} = \left\langle -\frac{3}{7}, \frac{9}{14}, \frac{3}{14} \right\rangle.$$

Definition

The **direction cosines** of a unit length vector $\mathbf{a}$ are:

- $\cos(\alpha) = \mathbf{a} \cdot \mathbf{i}$,
- $\cos(\beta) = \mathbf{a} \cdot \mathbf{j}$,
- $\cos(\gamma) = \mathbf{a} \cdot \mathbf{k}$,

and so, α, β, γ are the respective angles that $\mathbf{a}$ makes with the x, y and z-axes.



Definition

- Suppose $\mathbf{F}$ is a force with magnitude $\mathbf{A}$ applied in the unit direction $\frac{\mathbf{a}}{|\mathbf{a}|}$ to an object in order to move it from the point P to the point Q.
- The **work** $\mathbf{W}$ done is:

$$\mathbf{W} = \mathbf{F} \cdot \overrightarrow{PQ} = \frac{\mathbf{A}}{|\mathbf{a}|}\mathbf{a} \cdot \overrightarrow{PQ}.$$

Note that $\overrightarrow{PQ}$ is the displacement vector.

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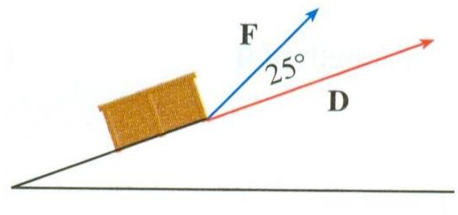
Example

- Suppose $\mathbf{F}$ is a force of $10N$ (10 Newtons) applied in the unit direction $\frac{1}{\sqrt{6}}\langle 2, 1, 1\rangle$ to an object to move it from $P = (-3, -2, 5)$ to $Q = (1, 2, 3)$.
- The work done is (length measured in meters):

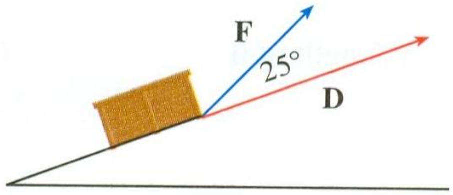
$$\mathbf{W} = \frac{10N}{\sqrt{6}}\langle 2, 1, 1\rangle \cdot \langle 4, 4, -2\rangle = \frac{100Nm}{\sqrt{6}},$$

where m is one meter.



Example

A crate is hauled $8m$ up a ramp under a constant force of 200 N applied at an angle of 25° to the ramp. Find the work done.



Example

A crate is hauled $8m$ up a ramp under a constant force of 200 N applied at an angle of 25° to the ramp. Find the work done.

Solution

If $\mathbf{F}$ and $\mathbf{D}$ are the force and displacement vectors, as pictured in figure above, then the work $\mathbf{W}$ done is

$$\mathbf{W} = \mathbf{F} \cdot \mathbf{D} = |\mathbf{F}||\mathbf{D}| \cos 25^\circ \approx 1450 N \cdot m = 1450 J.$$

Definition (Part I)

The **determinant** of the matrix $\mathbf{M}$ with rows vectors $\mathbf{v} = \langle a, b\rangle$ and $\mathbf{w} = \langle c, d\rangle$ can be calculated by:

$$|\mathbf{M}| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Fact: The absolute value $|ad - bc|$ of this determinant equals the **area of the parallelogram** with sides $\mathbf{v}$ and $\mathbf{w}$.

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Example

The area of the parallelogram with sides $\mathbf{u} = \langle 1, 2 \rangle$ and $\mathbf{v} = \langle 3, 4 \rangle$ is:

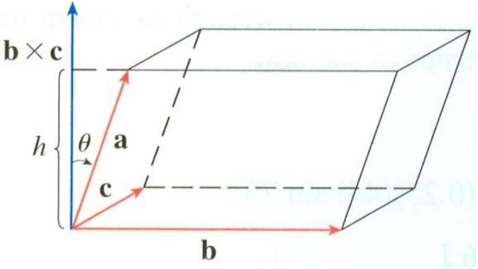
$$\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = |1 \cdot 4 - 2 \cdot 3| = |4 - 6| = 2.$$

Definition (Part II)

The **determinant** of the matrix **M** with rows vectors **a** $= \langle a_1, a_2, a_3 \rangle$, **b** $= \langle b_1, b_2, b_3 \rangle$ and **c** $= \langle c_1, c_2, c_3 \rangle$ can be calculated by:

$$|\mathbf{M}| = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

Fact: The absolute value of the determinant $|\mathbf{M}|$ equals the **volume of the parallelepiped** or box spanned by the vectors **a**, **b** and **c**.



Example

Calculate the following determinant $\begin{vmatrix} 1 & 2 & -1 \\ 3 & 0 & 1 \\ -5 & 4 & 2 \end{vmatrix}$.

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Calculate the following determinant $\begin{vmatrix} 1 & 2 & -1 \\ 3 & 0 & 1 \\ -5 & 4 & 2 \end{vmatrix}$.

Solution

$$\begin{vmatrix} 1 & 2 & -1 \\ 3 & 0 & 1 \\ -5 & 4 & 2 \end{vmatrix} = 1\begin{vmatrix} 0 & 1 \\ 4 & 2 \end{vmatrix} - 2\begin{vmatrix} 3 & 1 \\ -5 & 2 \end{vmatrix} + (-1)\begin{vmatrix} 3 & 0 \\ -5 & 4 \end{vmatrix}$$

$$= 1(0-4) - 2(6+5) + (-1)(12-0) = -38.$$

Definition (Part III)

The **cross product** $\mathbf{a} \times \mathbf{b}$ of the vectors $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$ can be calculated by:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \mathbf{k}$$

$$= \langle a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 \rangle.$$

We will show the following properties:

- The **length** of $\mathbf{a} \times \mathbf{b}$ is given by: $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin(\theta)$, where $\theta \in [0, \pi)$ is the angle between $\mathbf{a}$ and $\mathbf{b}$.
- $\mathbf{a} \times \mathbf{b}$ is **orthogonal** to both $\mathbf{a}$ and $\mathbf{b}$.
- $|\mathbf{a} \times \mathbf{b}|$ is **area of the parallelogram** with sides $\mathbf{a}$ and $\mathbf{b}$.
- The **area of the triangle** with vertices $\langle 0, 0, 0 \rangle$ and the position vectors $\mathbf{a}$ and $\mathbf{b}$ is $\frac{|\mathbf{a} \times \mathbf{b}|}{2}$.

Example

If $\mathbf{a} = \langle 1, 3, 4 \rangle$ and $\mathbf{b} = \langle 2, 7, -5 \rangle$, then

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 3 & 4 \\ 2 & 7 & -5 \end{vmatrix}$$

$$\begin{aligned} &= \begin{vmatrix} 3 & 4 \\ 7 & -5 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 4 \\ 2 & -5 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 3 \\ 2 & 7 \end{vmatrix} \mathbf{k} \\ &= (-15 - 28)\mathbf{i} - (-5 - 8)\mathbf{j} + (7 - 6)\mathbf{k} \\ &= -43\mathbf{i} + 13\mathbf{j} + \mathbf{k} = \langle -43, 13, 1 \rangle. \end{aligned}$$

Theorem

If θ is the angle between $\mathbf{a}$ and $\mathbf{b}$ $(0 \leq \theta \leq \pi)$, then $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin \theta$.

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Proof.

From the definitions of the cross product and length of a vector,

$$\mathbf{a} \times \mathbf{b} = \langle a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 \rangle.$$

$$|\mathbf{a} \times \mathbf{b}|^2 = (a_2 b_3 - a_3 b_2)^2 + (a_3 b_1 - a_1 b_3)^2 + (a_1 b_2 - a_2 b_1)^2$$

$$= (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) - (a_1 b_1 + a_2 b_2 + a_3 b_3)^2$$

$$= |\mathbf{a}|^2 |\mathbf{b}|^2 - (\mathbf{a} \cdot \mathbf{b})^2 = |\mathbf{a}|^2 |\mathbf{b}|^2 - |\mathbf{a}|^2 |\mathbf{b}|^2 \cos^2\theta$$

$$= |\mathbf{a}|^2 |\mathbf{b}|^2 (1 - \cos^2\theta) = |\mathbf{a}|^2 |\mathbf{b}|^2 \sin^2\theta.$$

Taking square roots we have $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin\theta$. $\square$

Theorem

The vector $\mathbf{a} \times \mathbf{b}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$.

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Proof.

- In order to show that $\mathbf{a} \times \mathbf{b}$ is orthogonal to $\mathbf{a}$, we compute their dot product as follows:

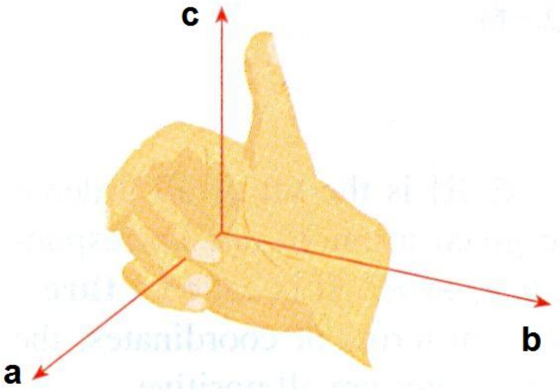
$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{a} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} a_1 - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} a_2 + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} a_3$$

$$= a_1(a_2 b_3 - a_3 b_2) - a_2(a_1 b_3 - a_3 b_1) + a_3(a_1 b_2 - a_2 b_1)$$

$$= a_1 a_2 b_3 - a_1 a_3 b_2 - a_1 a_2 b_3 + a_2 a_3 b_1 + a_1 a_3 b_2 - a_2 a_3 b_1 = 0$$

- A similar computation shows that $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{b} = 0$.
- Therefore $\mathbf{a} \times \mathbf{b}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$.

$\square$



If **a** and **b** are two nonzero vectors in "different" directions, then **c** $=$ **a** $\times$ **b** is a nonzero vector, orthogonal to both **a** and **b** in the direction given by the **righthand rule**. One sweeps fingers from **a** towards **b**.

Theorem (Basic algebraic properties of cross product)

Let $\mathbf{a} = \langle x_1, y_1, z_1 \rangle$, $\mathbf{b} = \langle x_2, y_2, z_2 \rangle$, $\mathbf{c} = \langle x_3, y_3, z_3 \rangle$ be vectors and let $\lambda \in \mathbb{R}$ be a scalar. Then:

- $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$.
- $(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}$.
- $(\lambda \mathbf{a}) \times \mathbf{b} = \lambda(\mathbf{a} \times \mathbf{b})$.
- $\mathbf{i} \times \mathbf{j} = \mathbf{k}, \quad \mathbf{j} \times \mathbf{k} = \mathbf{i}, \quad \mathbf{k} \times \mathbf{i} = \mathbf{j}$.

Theorem (Basic algebraic properties of cross product)

Let $\mathbf{a} = \langle x_1, y_1, z_1 \rangle$, $\mathbf{b} = \langle x_2, y_2, z_2 \rangle$, $\mathbf{c} = \langle x_3, y_3, z_3 \rangle$ be vectors and let $\lambda \in \mathbb{R}$ be a scalar. Then:

- $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$.
- $(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}$.
- $(\lambda \mathbf{a}) \times \mathbf{b} = \lambda(\mathbf{a} \times \mathbf{b})$.
- $\mathbf{i} \times \mathbf{j} = \mathbf{k}, \quad \mathbf{j} \times \mathbf{k} = \mathbf{i}, \quad \mathbf{k} \times \mathbf{i} = \mathbf{j}$.

Proof.

The proofs of the above properties follow immediately from the definition of the cross product. For example,

$$\mathbf{i} \times \mathbf{j} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \mathbf{k} = \mathbf{k}.$$

$\square$

Example

Find a vector perpendicular to the plane that passes through the points $P(1, 4, 6)$, $Q(-2, 5, -1)$, and $R(1, -1, 1)$.

Example

Find a vector perpendicular to the plane that passes through the points $P(1, 4, 6)$, $Q(-2, 5, -1)$, and $R(1, -1, 1)$.

Solution:

- The vector $\overrightarrow{PQ} \times \overrightarrow{PR}$ is perpendicular to both $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ and so is perpendicular to the plane through P, Q, and R.

$$\overrightarrow{PQ} = \langle -3, 1, -7 \rangle$$

$$\overrightarrow{PR} = \langle 0, -5, -5 \rangle.$$

- We compute the cross product of these vectors:

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 1 & -7 \\ 0 & -5 & -5 \end{vmatrix}$$

$$= (-5 - 35)\mathbf{i} - (15 - 0)\mathbf{j} + (15 - 0)\mathbf{k} = -40\mathbf{i} - 15\mathbf{j} + 15\mathbf{k}.$$

- So the vector $\langle -40, -15, 15 \rangle$ is perpendicular to the plane.

Theorem

The length of the cross product $\mathbf{a} \times \mathbf{b}$ is equal to the area $\mathbf{A}$ of the parallelogram determined by $\mathbf{a}$ and $\mathbf{b}$.

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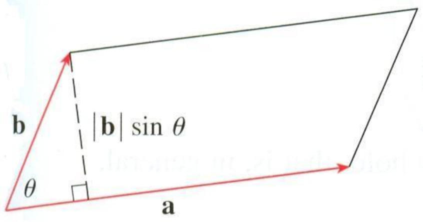
Proof.

If $\mathbf{a}$ and $\mathbf{b}$ are represented by directed line segments with the same initial point, then they determine a parallelogram with base $|\mathbf{a}|$ and altitude $|\mathbf{b}| \sin \theta$.



So, the area is

$$\mathbf{A} = |\mathbf{a}|(|\mathbf{b}| \sin \theta) = |\mathbf{a} \times \mathbf{b}|.$$

□

Example (popular exam problem)

- Consider the points $A = (1, 0, 1)$, $B = (0, 2, 3)$ and $C = (-1, -1, 0)$.
- The area of the triangle $\boldsymbol{\Delta}$ with these vertices can be found by taking the area of the parallelogram spanned by $\overrightarrow{AB}$ and $\overrightarrow{AC}$ and dividing by 2.
- Thus:

$$\textbf{Area}(\boldsymbol{\Delta}) = \frac{|\overrightarrow{AB} \times \overrightarrow{AC}|}{2} = \frac{1}{2} \left\| \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 2 & 2 \\ -2 & -1 & -1 \end{vmatrix} \right\|$$

$$= \frac{1}{2} |\langle 0, -5, 5 \rangle| = \frac{1}{2}\sqrt{0 + 25 + 25} = \frac{1}{2}\sqrt{50}.$$

In Class Exercises 2

1. Find the **scalar projection** and **vector projection** of $\mathbf{b} = \langle 1, 1, 0 \rangle$ onto $\mathbf{a} = \langle -2, 3, 1 \rangle$.

2. Calculate the following determinant $\begin{vmatrix} 1 & 2 & -1 \\ 3 & 0 & 1 \\ -1 & 4 & 2 \end{vmatrix}$.

3. If $\mathbf{a} = \langle 1, 3, 4 \rangle$ and $\mathbf{b} = \langle 2, 0, -5 \rangle$, then find the cross product $\mathbf{a} \times \mathbf{b}$.

4. Find a vector perpendicular to the plane that passes through the points $P(1, 2, 0)$, $Q(-2, 0, -1)$, and $R(1, -1, 1)$.

5. Consider the points $A = (1, 0, 1)$, $B = (0, 2, 1)$ and $C = (-1, -1, 0)$. Find the area of the triangle $\boldsymbol{\Delta}$ with these vertices.

Definition

- The product $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ is called the **scalar triple product** of the vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ (also called the **box product**).
- Notice that we can write the scalar triple product as a **determinant**:

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix},$$

where $\mathbf{a} = (a_1, a_2, a_3)$, $\mathbf{b} = (b_1, b_2, b_3)$ and $\mathbf{c} = (c_1, c_2, c_3)$.

Theorem

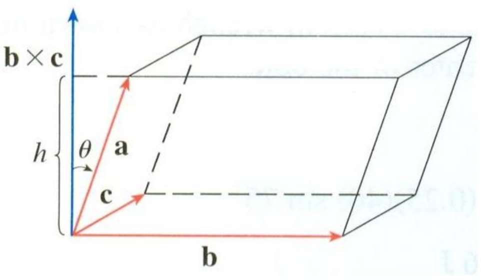
The **volume of the parallelepiped** determined by the vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ is the magnitude of their scalar triple product: $\mathbf{V} = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$.

Theorem

The **volume of the parallelepiped** determined by the vectors **a**, **b**, and **c** is the magnitude of their scalar triple product: $\mathbf{V} = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$.

Proof.

Consider the parallelepiped determined by the vectors **a**, **b**, and **c**.



- The area of the base parallelogram is $\mathbf{A} = |\mathbf{b} \times \mathbf{c}|$.
- If θ is the angle between **a** and $\mathbf{b} \times \mathbf{c}$, then the height h of the parallelepiped is $h = |\mathbf{a}||\cos\theta|$.

□

Theorem

The **volume of the parallelepiped** determined by the vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ is the magnitude of their scalar triple product: $\mathbf{V} = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$.

Continuation of proof:

- The area of the base parallelogram is $\mathbf{A} = |\mathbf{b} \times \mathbf{c}|$.
- If θ is the angle between $\mathbf{a}$ and $\mathbf{b} \times \mathbf{c}$, then the height h of the parallelepiped is $h = |\mathbf{a}||\cos\theta|$.
- Therefore, the **volume of the parallelepiped** is:

$$\mathbf{V} = \mathbf{A}h = |\mathbf{b} \times \mathbf{c}||\mathbf{a}||\cos\theta| = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|.$$

□

Example

Consider the vectors $\mathbf{a} = \langle 1, 0, 1 \rangle$, $\mathbf{b} = \langle 0, 2, 3 \rangle$ and $\mathbf{c} = \langle -1, 7, 0 \rangle$. Find the volume $\mathbf{V}$ of the parallelepiped or box spanned by these 3 vectors.

Example

Consider the vectors $\mathbf{a} = \langle 1, 0, 1 \rangle$, $\mathbf{b} = \langle 0, 2, 3 \rangle$ and $\mathbf{c} = \langle -1, 7, 0 \rangle$. Find the volume $\mathbf{V}$ of the parallelepiped or box spanned by these 3 vectors.

Solution:

$$\mathbf{V} = \left\| \begin{array}{ccc} 1 & 0 & 1 \\ 0 & 2 & 3 \\ -1 & 7 & 0 \end{array} \right\| = |-21 + 0 + 2| = |-19| = 19.$$

Example (popular exam problem)

Find the volume **V** of the **parallelepiped** such that the following four points $A = (1, 4, 2)$, $B = (3, 1, -2)$, $C = (4, 3, -3)$, $D = (1, 0, -1)$ are vertices and the vertices B, C, D are all adjacent to the vertex A.

Example (popular exam problem)

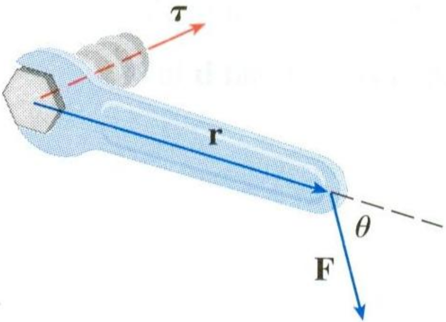
Find the volume **V** of the **parallelepiped** such that the following four points $A = (1,4,2)$, $B = (3,1,-2)$, $C = (4,3,-3)$, $D = (1,0,-1)$ are vertices and the vertices B, C, D are all adjacent to the vertex A.

Solution:

The volume **V** is equal to the absolute value of the determinant of the matrix with rows $\overrightarrow{AB} = \langle 2,-3,-4\rangle$, $\overrightarrow{AC} = \langle 3,-1,-5\rangle$, $\overrightarrow{AD} = \langle 0,-4,-3\rangle$.

$$\mathbf{V} = \left\| \begin{matrix} 2 & -3 & -4 \\ 3 & -1 & -5 \\ 0 & -4 & -3 \end{matrix} \right\|$$

$$= |2\cdot(-17) + -(-3)\cdot(-9) + (-4)\cdot(-12)| = |-13| = 13.$$

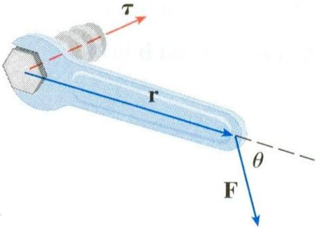


Definition

The **torque** τ on a rigid body at a point with position vector $\mathbf{r}$ induced by a force $\mathbf{F}$ is:

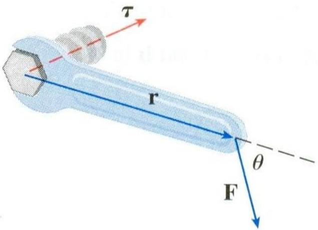
$$\tau = \mathbf{r} \times \mathbf{F};$$

it measures the tendency of the body to rotate about the origin.



Example

What is the magnitude of the **torque** τ on a rigid body with position vector $\mathbf{r} = \langle 1, -1, 3\rangle$ with a force of $10N$ in the direction of $\frac{\mathbf{a}}{|\mathbf{a}|} = \frac{1}{\sqrt{6}}\langle 2, 1, 1\rangle$ (length measured in meters m) ?



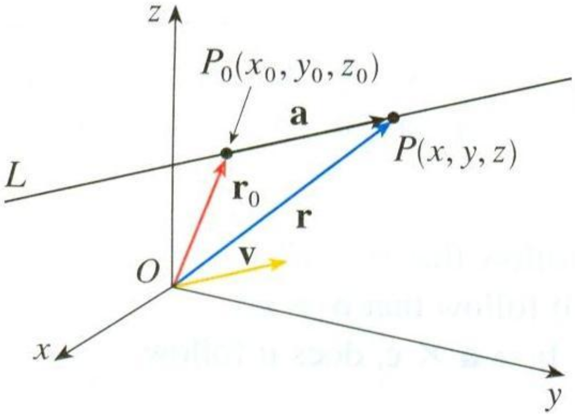
Example

What is the magnitude of the **torque** τ on a rigid body with position vector $\mathbf{r} = \langle 1, -1, 3\rangle$ with a force of $10N$ in the direction of $\frac{\mathbf{a}}{|\mathbf{a}|} = \frac{1}{\sqrt{6}}\langle 2, 1, 1\rangle$ (length measured in meters m) ?

Solution:

$$|\tau| = |\langle 1, -1, 3\rangle \times \frac{10Nm}{\sqrt{6}}\langle 2, 1, 1\rangle|$$

$$= |\frac{10Nm}{\sqrt{6}}\langle -4, 5, 3\rangle| = \frac{10Nm \cdot \sqrt{50}}{\sqrt{6}}.$$



In the above figure, $\mathbf{a}$ is a vector in the direction of $\mathbf{v}$ and $\mathbf{r}(t) = (x_0, y_0, z_0) + t\mathbf{v} = (x_0, y_0, z_0) + \mathbf{a}$, for some t. Here **L** denotes the set of points on the parameterized line.

Definition

- Given a point $\mathbf{r}_0 = (x_0, y_0, z_0)$ and a vector $\mathbf{v} = \langle a, b, c \rangle$, the **vector equation** of the line $\mathbf{L}$ passing through $\mathbf{r}_0$ in the direction of $\mathbf{v}$ is:

  $\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v} = \langle x_0, y_0, z_0 \rangle + t\langle a, b, c \rangle = \langle x_0 + at, y_0 + bt, z_0 + ct \rangle.$

- The resulting equations:

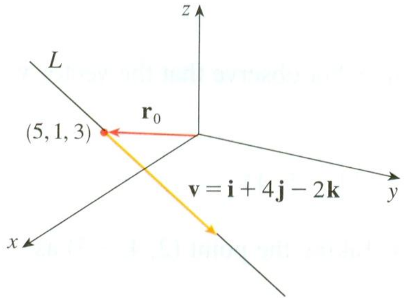
$$x = x_0 + at,$$
$$y = y_0 + bt,$$
$$z = z_0 + ct,$$

  are called the **parametric equations** for $\mathbf{L}$.

- The resulting equations (solving for t):

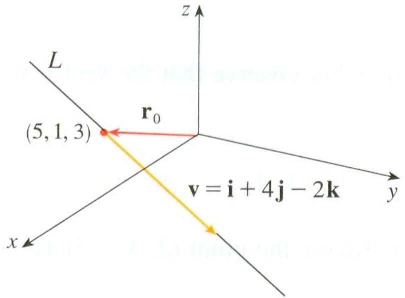
$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c},$$

  are called the **symmetric equations** for $\mathbf{L}$.



Example (popular exam problem)

Find a **vector equation** and **parametric equations** for the line that passes through the point $(5, 1, 3)$ and is parallel to the vector $\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$.



Example (popular exam problem)

Find a **vector equation** and **parametric equations** for the line that passes through the point $(5, 1, 3)$ and is parallel to the vector $\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$.

Solution:

Here $\mathbf{r}_0 = \langle 5, 1, 3 \rangle = 5\mathbf{i} + \mathbf{j} + 3\mathbf{k}$ and $\mathbf{v} = \mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$, so the **vector equation** becomes: $\mathbf{r}(t) = (5\mathbf{i} + \mathbf{j} + 3\mathbf{k}) + t(\mathbf{i} + 4\mathbf{j} - 2\mathbf{k})$ or $\mathbf{r}(t) = (5 + t)\mathbf{i} + (1 + 4t)\mathbf{j} + (3 - 2t)\mathbf{k} = \langle 5 + t, 1 + 4t, 3 - 2t \rangle$.

Parametric equations: $x = 5 + t, \quad y = 1 + 4t, \quad z = 3 - 2t.$

Example (popular exam problem)

The **vector equation** for the line **L** passing through $P = (1, 2, 3)$ and $Q = (4, 0, 7)$ is given by:

$$\mathbf{r}(t) = \vec{P} + t\overrightarrow{PQ} = \langle 1, 2, 3\rangle + t\langle 3, -2, 4\rangle$$

$$= \langle 1 + 3t, 2 - 2t, 3 + 4t\rangle.$$

Notice that $\mathbf{r}(0) = \vec{P}$ and $\mathbf{r}(1) = \vec{P} + (\vec{Q} - \vec{P}) = \vec{Q}$.

Example (popular exam problem)

Find **parametric equations** for the line which contains $A(2, 0, 1)$ and $B(-1, 1, -1)$.

Example (popular exam problem)

Find **parametric equations** for the line which contains $A(2, 0, 1)$ and $B(-1, 1, -1)$.

Solution:

Let
$$\mathbf{v} = \overrightarrow{AB} = \langle -1, 1, -1 \rangle - \langle 2, 0, 1 \rangle = \langle -3, 1, -2 \rangle.$$

Since $A(2, 0, 1)$ lies on the line, then:
$$\begin{aligned} x &= 2 - 3t, \\ y &= 0 + t = t, \\ z &= 1 - 2t. \end{aligned}$$

Example (popular exam problem)

Determine whether the lines $\mathbf{L}_1 : x = 1 + 2t,\ y = 3t,\ z = 2 - t$ and $\mathbf{L}_2 : x = -1 + s,\ y = 4 + s,\ z = 1 + 3s$ are **parallel**, **skew** or **intersecting**.

Example (popular exam problem)

Determine whether the lines $\mathbf{L}_1 : x = 1 + 2t,\ y = 3t,\ z = 2 - t$ and $\mathbf{L}_2 : x = -1 + s,\ y = 4 + s,\ z = 1 + 3s$ are **parallel**, **skew** or **intersecting**.

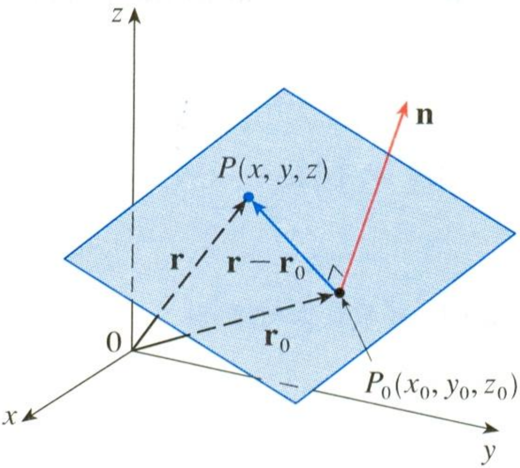
Solution:

- Vector part of line $\mathbf{L}_1$ is $\mathbf{v}_1 = \langle 2, 3, -1 \rangle$ and for line $\mathbf{L}_2$ is $\mathbf{v}_2 = \langle 1, 1, 3 \rangle$.
- Clearly, $\mathbf{v}_1$ is not a scalar multiple of $\mathbf{v}_2$ and so these lines are **not parallel**.
- If these lines intersect, then for some values of t and s:

$$x = 1 + 2t = -1 + s \implies 2t = -2 + s,$$

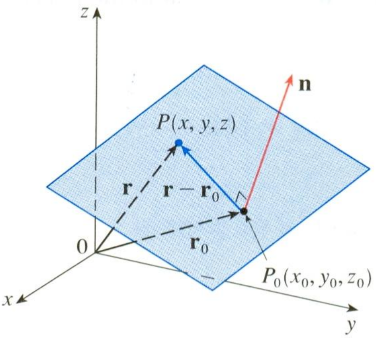
$$y = 3t = 4 + s \implies 3t = 4 + s.$$

- Solving yields: $$t = 6 \ \ \textbf{and} \ \ s = 14.$$
- Plugging these values into $z = 2 - t = 1 + 3s$ yields the inequality $-4 \neq 43$, which means there is no solution and the lines do **not intersect**.
- Thus, the lines are **skew**.



Definition

A **plane** in $\mathbf{R}^3$ is determined by a point $\mathbf{r}_0 = (x_0, y_0, z_0)$ and a vector $\mathbf{n}$ orthogonal to the plane. The vector $\mathbf{n}$ is called a **normal vector** to the plane.



Definition

The **plane** passing through the point $\mathbf{r}_0 = (x_0, y_0, z_0)$ with normal vector $\mathbf{n} = \langle a, b, c \rangle$ is given by the following equation, where (x, y, z) denotes a general point on the plane:

$$\mathbf{n} \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0.$$

Equivalently, we have:

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$

Definition

The **plane** passing through the point $\mathbf{r}_0 = (x_0, y_0, z_0)$ with normal vector $\mathbf{n} = \langle a, b, c\rangle$ is given by the following equation, where (x, y, z) denotes a general point on the plane:

$$\mathbf{n} \cdot \langle x - x_0, y - y_0, z - z_0\rangle = 0.$$

Equivalently, we have:

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$

Example

The **equation of the plane** passing through $\mathbf{r}_0 = (1, 2, 3)$ and with normal vector $\mathbf{n} = \langle -3, 4, 1\rangle$ is:

$$-3(x - 1) + 4(y - 2) + (z - 3) = 0.$$

Example (This is a very popular midterm exam problem)

Find an **equation of the plane** which contains the points $P(-1, 2, 1)$, $Q(1, -2, 1)$ and $R(1, 1, -1)$.

Example (This is a very popular midterm exam problem)

Find an **equation of the plane** which contains the points $P(-1, 2, 1)$, $Q(1, -2, 1)$ and $R(1, 1, -1)$.

Solution:

Method 1

- Consider the vectors $\overrightarrow{PQ} = \langle 2, -4, 0 \rangle$ and $\overrightarrow{PR} = \langle 2, -1, -2 \rangle$ which lie parallel to the plane.
- Then consider the normal vector:

$$\mathbf{n} = \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -4 & 0 \\ 2 & -1 & -2 \end{vmatrix} = 8\mathbf{i} + 4\mathbf{j} + 6\mathbf{k}.$$

- So the **equation of the plane** is given by:

$$\langle 8, 4, 6 \rangle \cdot \langle x + 1, y - 2, z - 1 \rangle = 8(x + 1) + 4(y - 2) + 6(z - 1) = 0.$$

Example (This is a very popular midterm exam problem)

Find an **equation of the plane** which contains the points $P(-1, 2, 1)$, $Q(1, -2, 1)$ and $R(1, 1, -1)$.

Example (This is a very popular midterm exam problem)

Find an **equation of the plane** which contains the points $P(-1,2,1)$, $Q(1,-2,1)$ and $R(1,1,-1)$.

Solution:

Method 2

- The plane consists of all the points $S(x,y,z) \in \mathbb{R}^3$, such that $\overrightarrow{PS}$, $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ are in the same plane (coplanar).
- But this happens if and only if their box product is zero.
- So the **equation of the plane** is:

$$\begin{vmatrix} x+1 & y-2 & z-1 \\ 2 & -4 & 0 \\ 2 & -1 & -2 \end{vmatrix} = 8(x+1)+4(y-2)+6(z-1)=0.$$

$\square$

$\mathbf{n} \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$

Example (popular exam problem)

Find the **equation of the plane** passing through points
$P = (1, 0, 2), \quad Q = (4, 2, 3), \quad R = (2, 0, 4).$

Example (popular exam problem)

Find the **equation of the plane** passing through points
$P = (1, 0, 2), \quad Q = (4, 2, 3), \quad R = (2, 0, 4).$

Solution:

- Since a plane is determined by its normal vector $\mathbf{n}$ and a point on it, say the point P, it suffices to find $\mathbf{n}$.
- Note that:

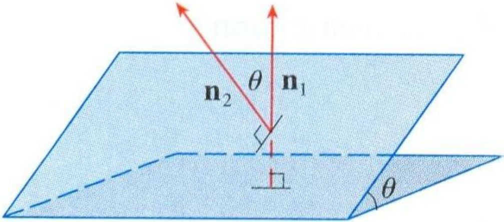
$$\mathbf{n} = \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 2 & 1 \\ 1 & 0 & 2 \end{vmatrix} = \langle 4, -5, -2 \rangle.$$

- So the **equation of the plane** is:

$$4(x - 1) - 5y - 2(z - 2) = 0.$$

In Class Exercises 3

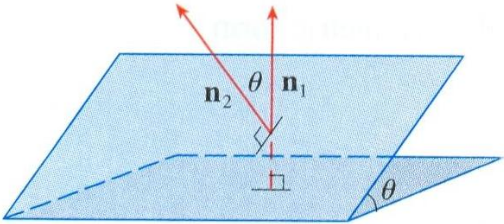
1. Consider the vectors $\mathbf{a} = \langle 1, 0, 1\rangle$, $\mathbf{b} = \langle 0, 2, 3\rangle$ and $\mathbf{c} = \langle -1, 2, 0\rangle$. Find the volume $\mathbf{V}$ of the parallelepiped or box spanned by these 3 vectors.
2. Find the volume $\mathbf{V}$ of the **parallelepiped** such that the following four points $A = (1, 0, 2)$, $B = (3, 1, -2)$, $C = (4, 3, -3)$, $D = (1, 0, -1)$ are vertices and the vertices B, C, D are all adjacent to the vertex A.
3. Find a **vector equation** and **parametric equations** for the line that passes through the point $(2, 1, 3)$ and is parallel to the vector $\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$.
4. Find **parametric equations** for the line which contains $A(2, 0, 1)$ and $B(-1, 1, -2)$.
5. Find the **equation of the plane** passing through points $P = (1, 0, 2)$, $Q = (3, 2, 3)$, $R = (2, 0, 3)$.



Definition (Angle between 2 planes)

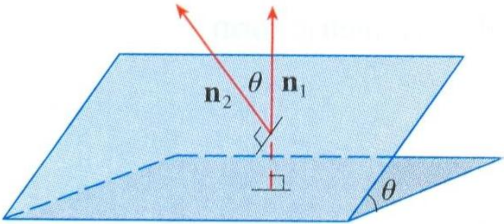
Given two planes with normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2$, respectively, then the **cosine of the angle between the planes** is the cosine of the angle θ between the lines determined by $\mathbf{n}_1$ and $\mathbf{n}_2$, which can be calculated using dot products.

$$\theta = \cos^{-1}\left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|}\right)$$



Example

Find the angle between the planes $x + y + z = 1$ and $x - 2y + 3z = 1$.



Example

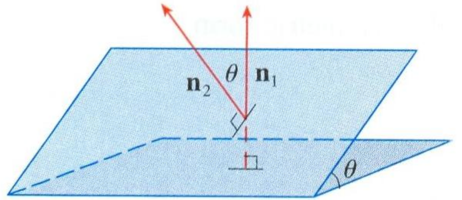
Find the angle between the planes $x + y + z = 1$ and $x - 2y + 3z = 1$.

Solution:

The normal vectors of these planes are $\mathbf{n}_1 = \langle 1, 1, 1\rangle$, $\mathbf{n}_2 = \langle 1, -2, 3\rangle$.
So, if θ is the angle between the planes, then:

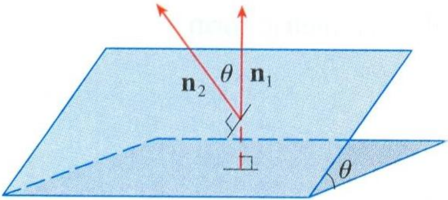
$$\cos\theta = \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|}$$

$$= \frac{1(1) + 1(-2) + 1(3)}{\sqrt{1+1+1}\sqrt{1+4+9}} = \frac{2}{\sqrt{42}}; \quad \theta = \cos^{-1}\left(\frac{2}{\sqrt{42}}\right) \approx 72^\circ.$$



Example (popular exam problem)

Find the **vector equation** of the line **L** of intersection of the planes $x + y + z = 1$ and $x - 2y + 3z = 1$.



Example (popular exam problem)

Find the **vector equation** of the line **L** of intersection of the planes $x + y + z = 1$ and $x - 2y + 3z = 1$.

Solution:

- First find a point on **L** by solving when $z = 0$.
- We have $x + y = 1$ and $x - 2y = 1$ which gives the point $(1, 0, 0) \in \mathbf{L}$.
- The vector part $\mathbf{v}$ of **L** is
$$\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 1 & -2 & 3 \end{vmatrix} = 5\mathbf{i} - 2\mathbf{j} - 3\mathbf{k} = \langle 5, -2, -3 \rangle.$$
- So $\quad \mathbf{L}(t) = (1, 0, 0) + t(5, -2, -3) = \langle 1 + 5t, -2t, -3t \rangle.$

Example (popular exam problem)

Find **parametric equations** for the line **L** of intersection of the planes $x - 2y + z = 1$ and $2x + y + z = 1$.

Example (popular exam problem)

Find **parametric equations** for the line **L** of intersection of the planes $x - 2y + z = 1$ and $2x + y + z = 1$.

Solution:

- The vector part $\mathbf{v}$ of the line **L** of intersection is orthogonal to the normal vectors $\langle 1, -2, 1\rangle$ and $\langle 2, 1, 1\rangle$. Hence $\mathbf{v}$ can be taken to be:

$$\mathbf{v} = \langle 1, -2, 1\rangle \times \langle 2, 1, 1\rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & 1 \\ 2 & 1 & 1 \end{vmatrix} = -3\mathbf{i} + \mathbf{j} + 5\mathbf{k}.$$

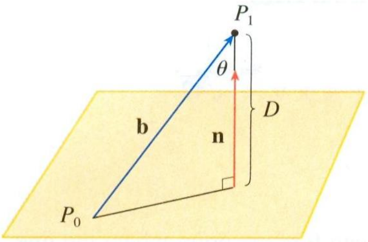
- Choose $P \in \mathbf{L}$ so the z-coordinate of P is zero. Setting $z = 0$, we obtain:

$$x - 2y = 1$$
$$2x + y = 1.$$

  Solving, we find that $x = \frac{3}{5}$ and $y = -\frac{1}{5}$. Hence, $P = \langle \frac{3}{5}, -\frac{1}{5}, 0\rangle$ lies on the line **L**.

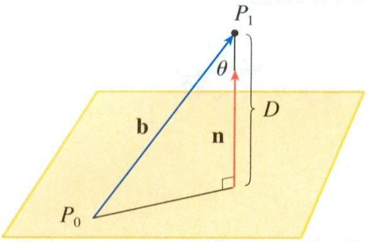
- The **parametric equations** are:

$$\begin{aligned} x &= \tfrac{3}{5} - 3t \\ y &= -\tfrac{1}{5} + t \\ z &= 0 + 5t = 5t. \end{aligned}$$



Example

Find a formula for the distance **D** from a point $P_1(x_1, y_1, z_1)$ to the plane $ax + by + cz + d = 0$.



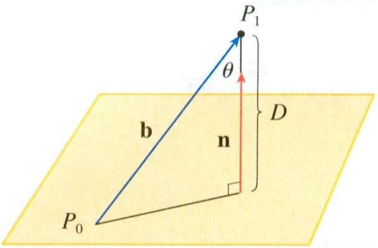
Example

Find a formula for the distance **D** from a point $P_1(x_1, y_1, z_1)$ to the plane $ax + by + cz + d = 0$.

Solution:

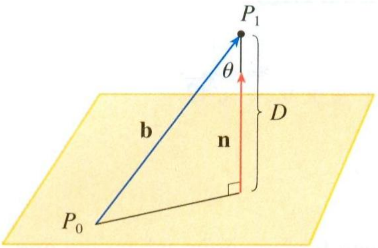
- Let $P_0(x_0, y_0, z_0)$ be any point in the given plane and let $\mathbf{b} = \overrightarrow{P_0P_1}$.
- Then $\mathbf{b} = \langle x_1 - x_0, y_1 - y_0, z_1 - z_0 \rangle$.
- The distance **D** from P_1 to the plane is equal to the absolute value of the scalar projection of **b** onto the normal vector $\mathbf{n} = \langle a, b, c \rangle$.
- Thus, $\mathbf{D} = |\mathbf{comp_n b}| = \frac{|\mathbf{n} \cdot \mathbf{b}|}{|\mathbf{n}|}$

$$= \frac{|\langle a, b, c \rangle \cdot \langle x_1 - x_0, y_1 - y_0, z_1 - z_0 \rangle|}{\sqrt{a^2 + b^2 + c^2}} = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}.$$



Example (popular midterm question)

Find the distance **D** from the point $(1, 2, -1)$ to the plane $2x + y - 2z = 1$.



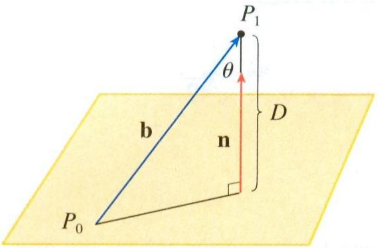
Example (popular midterm question)

Find the distance **D** from the point $(1, 2, -1)$ to the plane $2x + y - 2z = 1$.

Solution:

- Recall the distance formula $\mathbf{D} = \frac{|ax_1+by_1+cz_1+d|}{\sqrt{a^2+b^2+c^2}}$ from the previous slide.
- In order to apply the formula, rewrite the equation of the plane in standard form: $2x + y - 2z - 1 = 0$.
- So, the distance from $(1, 2, -1)$ to the plane is:

$$\mathbf{D} = \frac{|2\cdot1+1\cdot2-2\cdot(-1)-1|}{\sqrt{2^2+1^2+(-2)^2}} = \frac{5}{\sqrt{9}} = \frac{5}{3}.$$



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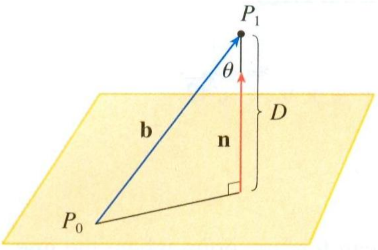
Find the distance **D** from the point $(1, 2, -1)$ to the plane $2x + y - 2z = 1$.



Example (popular midterm question)

Find the distance **D** from the point $(1, 2, -1)$ to the plane $2x + y - 2z = 1$.

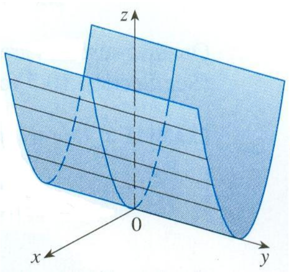
Solution:

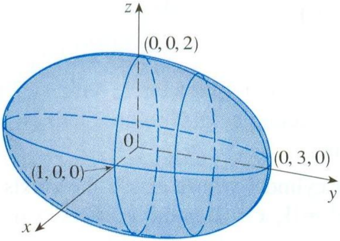
- The normal to the plane is $\mathbf{n} = \langle 2, 1, -2 \rangle$ and the point $P_0 = (0, 1, 0)$ lies on this plane.
- Consider the vector from P_0 to $(1, 2, -1)$ which is $\mathbf{b} = \langle 1, 1, -1 \rangle$.
- The distance **D** from $(1, 2, -1)$ to the plane is equal to:

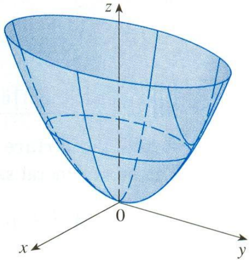
$$|\mathbf{comp_n\ b}| = \left| \mathbf{b} \cdot \frac{\mathbf{n}}{|\mathbf{n}|} \right| = |\langle 1, 1, -1 \rangle \cdot \frac{1}{3} \langle 2, 1, -2 \rangle| = \frac{5}{3}.$$

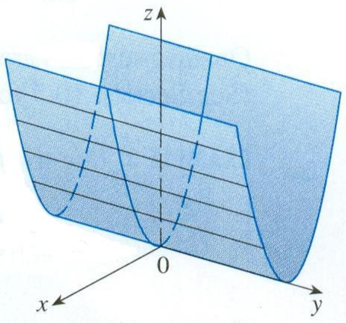
Definition (Trace Curves)

The curves of intersection of a surface **G** with planes parallel to coordinate planes are called **trace curves**.







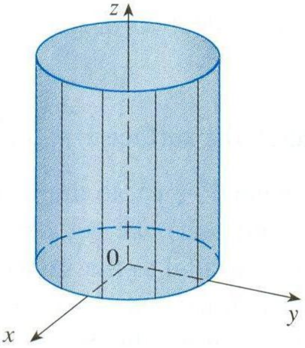


Example (Parabolic Cylinder)

Sketch the graph of the surface $z = x^2$.

Solution:

Since the graph, $z = x^2$, doesn't involve y, any vertical plane $y = k$ (parallel to the xz-plane) intersects the graph in a curve with equation $z = x^2$. So these vertical traces are **parabolas**. The figure above shows how the graph is formed by taking the **parabola** $z = x^2$ in the xz-plane and moving it in the direction of the y-axis.

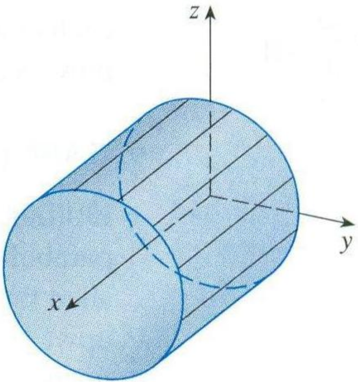


Example (Cylinder)

Identify and sketch the surface $x^2 + y^2 = 1$.

Solution:

Since z is missing and the equations $x^2 + y^2 = 1$, $z = k$ represent a circle with radius 1 in the plane $z = k$, the surface $x^2 + y^2 = 1$ is a **circular cylinder** whose axis is the z-axis. Here the rulings are vertical lines.

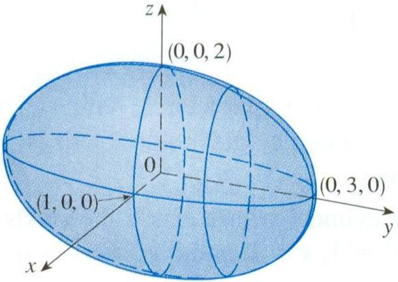


Example (Cylinder)

Identify and sketch the surface $y^2 + z^2 = 1$.

Solution:

Since x is missing and the surface is a **circular cylinder** whose axis is the x-axis. It is obtained by taking the circle $y^2 + z^2 = 1$, $x = 0$ in the yz-plane and moving it parallel to the x-axis.



Example (Ellipsoid)

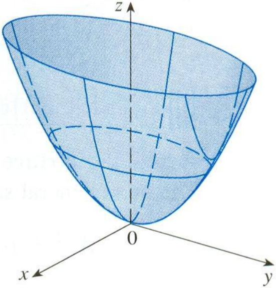
Use traces to sketch the quadric surface with equation $x^2 + \frac{y^2}{9} + \frac{z^2}{4} = 1$

Solution:

By substituting $z = 0$, we find the trace in the xy-plane is $x^2 + \frac{y^2}{9} = 1$, which an **ellipse**. The horizontal trace in the plane $z = k$ is $x^2 + \frac{y^2}{9} = 1 - \frac{k^2}{4}$ $\;z = k$ which is an **ellipse**, provided $-2 < k < 2$. The vertical traces are also **ellipses**:

$$\frac{y^2}{9} + \frac{z^2}{4} = 1 - k^2 \quad x = k \quad (\text{if} \quad -1 < k < 1)$$

$$x^2 + \frac{z^2}{4} = 1 - \frac{k^2}{9} \quad y = k \quad (\text{if} \quad -3 < k < 3).$$

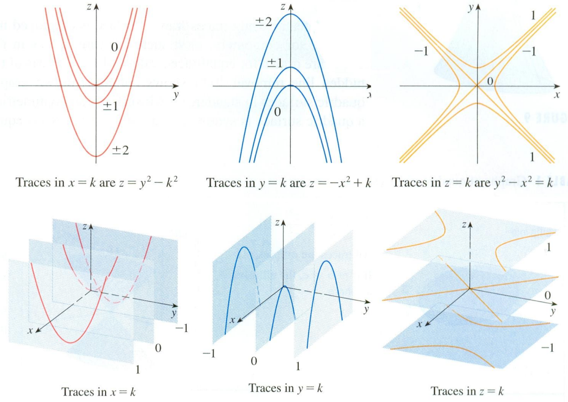


Example (Elliptic Paraboloid)

Use traces to sketch the surface $z = 4x^2 + y^2$.

Solution:

- Put $x = 0$, to get $z = y^2$.
- So the yz-plane intersects the surface in a **parabola**.
- For $x = k$ (a constant), we get $z = y^2 + 4k^2$.
- Slicing the graph with any plane parallel to the yz-plane, we obtain a **parabola** that opens upward.
- Similarly, if $y = k$, the trace is $z = 4x^2 + k^2$, which is again a **parabola** that opens upward.
- For $z = k$, the horizontal traces $4x^2 + y^2 = k$ are **ellipses**.

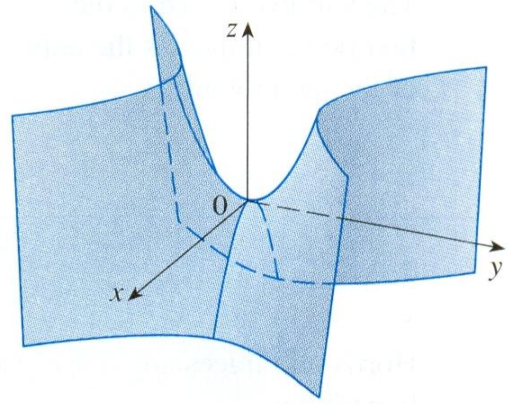


Example (Hyperbolic Paraboloid)

Sketch the surface $z = y^2 - x^2$.

Solution:

The traces in the vertical planes $x = k$ are **parabolas** $z = y^2 - k^2$, which open upward. The traces in $y = k$ are the **parabolas** $z = -x^2 + k^2$, which open downward. The horizontal traces are $y^2 - x^2 = k$, a family of **hyperbolas**.



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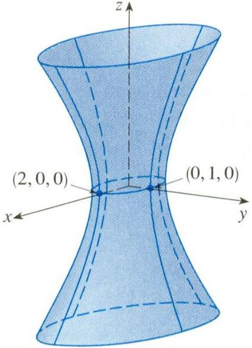
The traces in the vertical planes $x = k$ are **parabolas** $z = y^2 - k^2$, which open upward. The traces in $y = k$ are the **parabolas** $z = -x^2 + k^2$, which open downward. The horizontal traces are $y^2 - x^2 = k$, a family of **hyperbolas**.



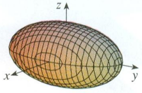
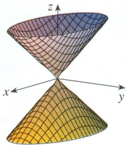
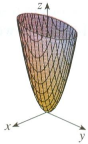
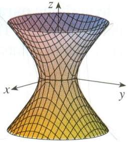
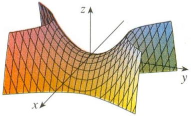
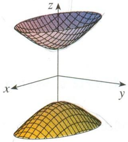
Example (Hyperboloid of One Sheet)

Sketch the surface $\frac{x^2}{4} + y^2 - \frac{z^2}{4} = 1$.

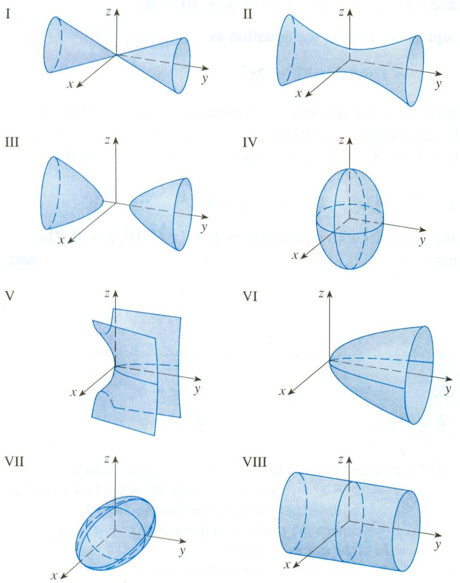
Solution:

The trace in any horizontal plane $z = k$ is the **ellipse** $\frac{x^2}{4} + y^2 = 1 + \frac{k^2}{4} \quad z = 4$. But the traces in the $xz-$ and $yz-$planes are the **hyperbolas**:

$$\frac{x^2}{4} - \frac{z^2}{4} = 1 \quad y = 0 \quad \textbf{and} \quad y^2 - \frac{z^2}{4} = 1 \quad x = 0.$$

Surface	Equation	Surface	Equation
Ellipsoid 	$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ All traces are ellipses. If $a = b = c$, the ellipsoid is a sphere.	Cone 	$\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces in the planes $x = k$ and $y = k$ are hyperbolas if $k \neq 0$ but are pairs of lines if $k = 0$.
Elliptic Paraboloid 	$\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces are parabolas. The variable raised to the first power indicates the axis of the paraboloid.	Hyperboloid of One Sheet 	$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ Horizontal traces are ellipses. Vertical traces are hyperbolas. The axis of symmetry corresponds to the variable whose coefficient is negative.
Hyperbolic Paraboloid 	$\frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2}$ Horizontal traces are hyperbolas. Vertical traces are parabolas. The case where $c < 0$ is illustrated.	Hyperboloid of Two Sheets 	$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ Horizontal traces in $z = k$ are ellipses if $k > c$ or $k < -c$. Vertical traces are hyperbolas. The two minus signs indicate two sheets.

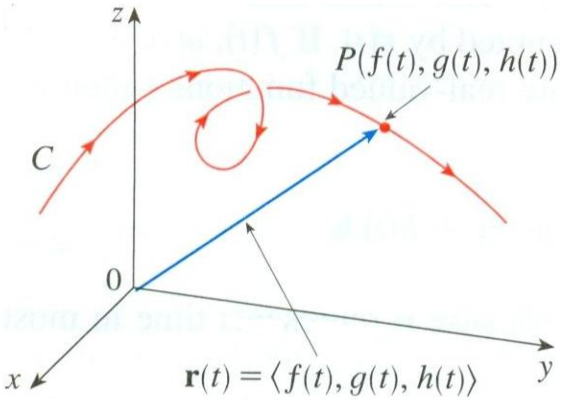
Name the surface:



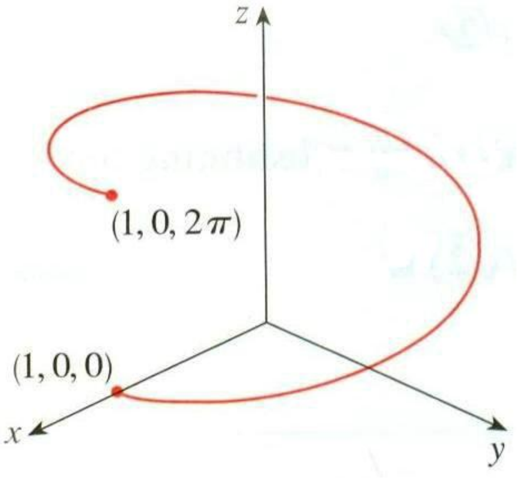
In Class Exercises 4

1. Find the **equation of the plane** passing through points $P = (1, 0, 2)$, $Q = (3, 2, 3)$, $R = (2, 0, 3)$.
2. Find the angle between the planes $x + y + z = 1$ and $x - 2y + 3z = 1$.
3. Find the **vector equation** of the line **L** of intersection of the planes $x + y + z = 1$ and $x - 2y + 3z = 1$.
4. Find the distance **D** from the point $(1, 2, -1)$ to the plane $2x + y - 2z = 1$.
5. Identify the surfaces with their equations:
    1. Parabolic Cylinder
    2. Cylinder
    3. Ellipsoid
    4. Elliptic Paraboloid
    5. Hyperbolic Paraboloid

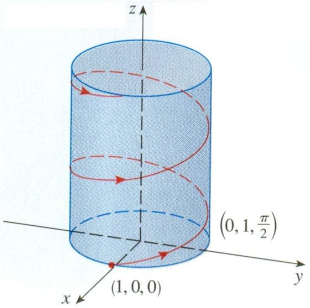
   (a) $x^2 + y^2 = 1$
   (b) $z = y^2 - x^2$
   (c) $z = x^2$
   (d) $x^2 + \frac{y^2}{9} + \frac{z^2}{4} = 1$
   (e) $z = 4x^2 + y^2$



Here is a parameterized curve $\mathbf{r}(t)$ in space with coordinate functions $x(t) = f(t)$, $y(t) = g(t)$ and $z(t) = h(t)$.



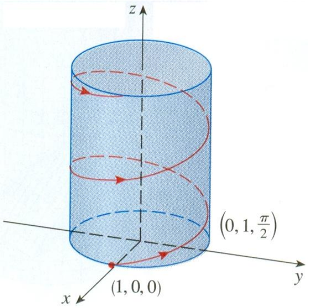
Here is a parameterized curve $\mathbf{r}(t)$ in space with coordinate functions $x(t) = \cos t$, $y(t) = \sin t$ and $z(t) = t$. This curve is called a **helix**.



Example (Helix)

Sketch the curve whose vector equation is

$$\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t\,\mathbf{k}$$



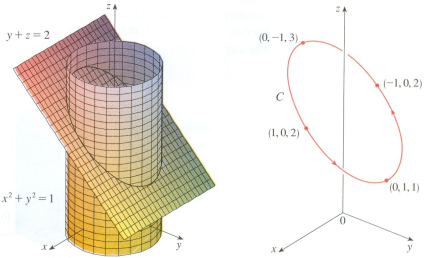
Example (Helix)

Sketch the curve whose vector equation is

$$\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}$$

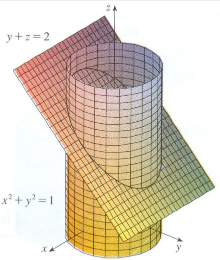
Solution:

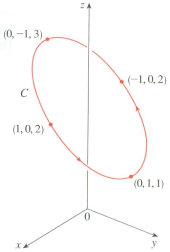
- The **parametric equations** for this curve are
$$x = \cos t\,; \quad y = \sin t, \quad z = t.$$
- Since $x^2 + y^2 = \cos^2 t + \sin^2 t = 1$, the curve lies on the cylinder $x^2 + y^2 = 1$.



Example

Find a vector function that represents the curve **C** of intersection of the cylinder $x^2 + y^2 = 1$ and the plane $y + z = 2$.





Example

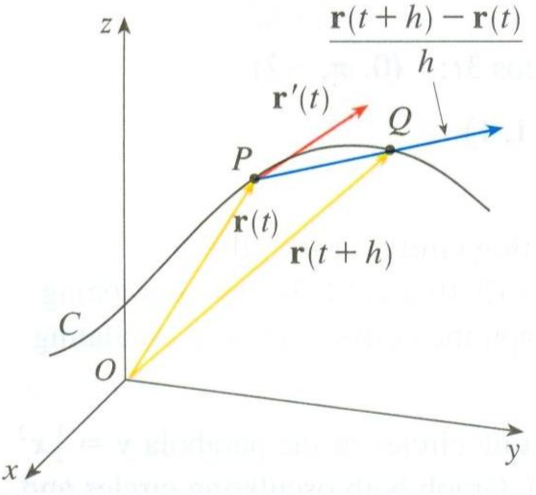
Find a vector function that represents the curve **C** of intersection of the cylinder $x^2 + y^2 = 1$ and the plane $y + z = 2$.

Solution:

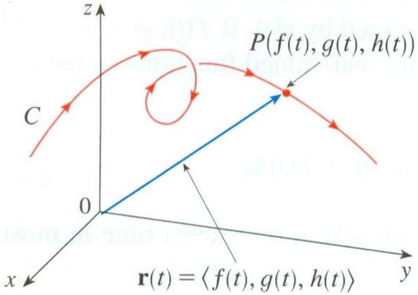
- The projection of **C** onto the *xy*-plane is the circle $x^2 + y^2 = 1$, $z = 0$.
- So we know we can write $x = \cos t \quad y = \sin t \qquad 0 \leq t \leq 2\pi$.
- From the equation of the plane, $z = 2 - y = 2 - \sin t$.
- Write **parametric equations** for **C** as
  $x = \cos t \quad y = \sin t \quad z = 2 - \sin t \qquad 0 \leq t \leq 2\pi$.
- The corresponding vector equation is

$$\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + (2 - \sin t)\mathbf{k} \qquad 0 \leq t \leq 2\pi.$$

- This equation is called a **parametrization** of the curve **C**.



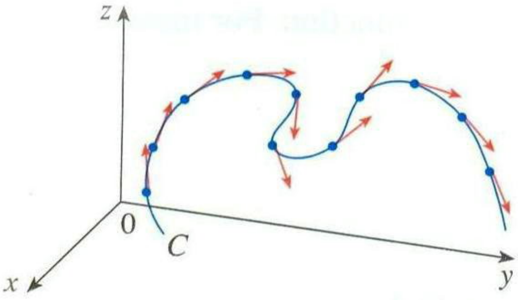
Estimating the velocity vector $\mathbf{v}(t) = \mathbf{r}'(t)$ to $\mathbf{r}(t)$.



Definition

Let $\mathbf{r}(t)$ be a vector valued curve in $\mathbf{R}^3$, where $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle$.

- Here t is called the **parameter** of $\mathbf{r}(t)$.
- If the derivative $\mathbf{r}'(t) = \lim_{h\to\infty} \frac{\mathbf{r}(t+h)-\mathbf{r}(t)}{h}$ exists for each t, then the curve $\mathbf{r}(t)$ is called **differentiable**
- $\mathbf{r}'(t)$ is called the derivative or **velocity** or **tangent** vector field $\mathbf{v}(t) = \mathbf{r}'(t)$ to the curve $\mathbf{r}(t)$.
- The length $|\mathbf{v}(t)|$ is called the **speed** of the curve $\mathbf{r}$ at the parameter value t.



Theorem

If $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$ is a differential curve in $\mathbf{R}^3$, then:

$$\mathbf{r}'(t) = \langle f'(t), g'(t), h'(t) \rangle.$$

Conversely, if $f(t)$, $g(t)$, $h(t)$ are differentiable functions, then $\mathbf{r}(t)$ is differentiable.

The **speed** function for $\mathbf{r}(t)$ is then:

$$\textbf{speed}(t) = \sqrt{(f'(t))^2 + (g'(t))^2 + (h'(t))^2}.$$

Notion of velocity vector field

- Suppose a particle moves through space so that its position vector at time t is $\mathbf{r}(t)$.
- For small values of h, the vector

$$\frac{\mathbf{r}(t+h)-\mathbf{r}(t)}{h} \tag{4}$$

  approximates the direction of the particle moving along the curve $\mathbf{r}(t)$.
- Its **magnitude** measures the size of the displacement vector per unit time.
- The vector in (4) gives the **average velocity** over a time interval of length **h** and its limit is the **velocity vector** $\mathbf{v}(t)$ at time t:

$$\mathbf{v}(t)=\lim_{h\to 0}\frac{\mathbf{r}(t+h)-\mathbf{r}(t)}{h}=\mathbf{r}'(t). \tag{5}$$

- Thus the velocity vector is also the tangent vector and **points in the direction** of the **tangent line**.

Example

- Suppose $\mathbf{r}(t) = \langle t, \sin(2t), t^2 + 1 \rangle$, then:

$$\mathbf{r}'(t) = \langle 1, 2\cos(2t), 2t \rangle$$

with $\mathbf{r}'(0) = \langle 1, 2, 0 \rangle$.

- Hence, the **tangent line** to $\mathbf{r}(t)$ at $t = 0$ is given by:

$$\mathbf{L}(t) = \mathbf{r}(0) + t\mathbf{r}'(0) = \langle 0, 0, 1 \rangle + t\langle 1, 2, 0 \rangle = \langle t, 2t, 1 \rangle.$$

- The **speed** function of $\mathbf{r}(t)$ is:

$$\textbf{speed}(t) = \sqrt{1 + 4\cos^2(2t) + 4t^2.}$$

Example (popular exam problem)

Suppose $\mathbf{r}(t) = (t, t^2, t^3)$. Find the **equation of the tangent line $\mathbf{L}$** at the point $P = (1, 1, 1)$.

Example (popular exam problem)

Suppose $\mathbf{r}(t) = (t, t^2, t^3)$. Find the **equation of the tangent line L** at the point $P = (1, 1, 1)$.

Solution:

- Note that the point $(1, 1, 1) \in \mathbf{r}$, when $t = 1$.
- So the direction of the tangent line $\mathbf{L}$ is $\mathbf{r}'(1)$.
- Since $\mathbf{r}'(t) = (1, 2t, 3t^2)$, then $\mathbf{r}'(1) = \langle 1, 2, 3 \rangle$.
- Thus the **equation of the tangent line** is:

$$\mathbf{L}(s) = \langle 1, 1, 1 \rangle + s\langle 1, 2, 3 \rangle = \langle 1 + s, 1 + 2s, 1 + 3s \rangle.$$

Definition

The length $\mathbf{L}$ of a parameterized curve $\mathbf{r}(t)$ in $\mathbf{R^3}$ on a time interval $[a, b]$ is

$$\mathbf{L} = \int_a^b |\mathbf{r}'(t)| dt.$$

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Example (popular exam problem)

- If $\mathbf{r}(t) = \langle \sin(t), \cos(t), 2t \rangle$, then $\mathbf{r}'(t) = \langle \cos(t), -\sin(t), 2 \rangle$ has constant speed $\sqrt{\cos^2(t) + \sin^2(t) + 4} = \sqrt{5}$.
- Hence, the length of $\mathbf{r}(t)$ from time $t = 1$ to time $t = 6$ is:

$$\mathbf{L} = \int_1^6 \sqrt{5}\, dt = \sqrt{5}t\Big|_1^6 = \sqrt{5}(6) - \sqrt{5}(1) = 5\sqrt{5}.$$

Theorem

Suppose $\mathbf{u}$ and $\mathbf{v}$ are differentiable vector functions, $\mathbf{c}$ is a scalar, and f is a real-valued function. Then

1. $\frac{d}{dt}[\mathbf{u}(t) + \mathbf{v}(t)] = \mathbf{u}'(t) + \mathbf{v}'(t)$
2. $\frac{d}{dt}[\mathbf{c}\mathbf{u}(t)] = \mathbf{c}\mathbf{u}'(t)$
3. $\frac{d}{dt}[f(t)\mathbf{u}(t)] = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t)$
4. $\frac{d}{dt}[\mathbf{u}(t) \cdot \mathbf{v}(t)] = \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t)$
5. $\frac{d}{dt}[\mathbf{u}(t) \times \mathbf{v}(t)] = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t)$
6. $\frac{d}{dt}[\mathbf{u}(f(t))] = f'(t)\mathbf{u}'(f(t))$ **(Chain Rule)**

Proof of Dot Product Rule.

- Let
$$\mathbf{u}(t) = \langle f_1(t), f_2(t), f_3(t)\rangle \qquad \mathbf{v}(t) = \langle g_1(t), g_2(t), g_3(t)\rangle.$$
- Then
$$\mathbf{u}(t)\cdot\mathbf{v}(t) = f_1(t)g_1(t) + f_2(t)g_2(t) + f_3(t)g_3(t) = \sum_{i=1}^{3} f_i(t)g_i(t).$$
- The ordinary Product Rule now gives
$$\begin{aligned}\frac{d}{dt}[\mathbf{u}\cdot\mathbf{v}(t)] &= \frac{d}{dt}\sum_{i=1}^{3} f_i(t)g_i(t) = \sum_{i=1}^{3}\frac{d}{dt}[f_i(t)g_i(t)] \\ &= \sum_{i=1}^{3}[f_i'(t)g_i(t) + f_i(t)g_i'(t)] \\ &= \sum_{i=1}^{3} f_i'(t)g_i(t) + \sum_{i=1}^{3} f_i(t)g_i'(t) = \mathbf{u}'(t)\cdot\mathbf{v}(t) + \mathbf{u}(t)\cdot\mathbf{v}'(t).\end{aligned}$$

□

Example

Show that if $|\mathbf{r}(t)| = \mathbf{c}$ (a constant), then $\mathbf{r}'(t)$ is **orthogonal** to $\mathbf{r}(t)$ for all t.

Example

Show that if $|\mathbf{r}(t)| = \mathbf{c}$ (a constant), then $\mathbf{r}'(t)$ is **orthogonal** to $\mathbf{r}(t)$ for all t.

Solution:

- Since $\mathbf{r}(t) \cdot \mathbf{r}(t) = |\mathbf{r}(t)|^2 = \mathbf{c}^2$ and $\mathbf{c}^2$ is a constant,

$$0 = \frac{d}{dt}[\mathbf{r}(t) \cdot \mathbf{r}(t)] = \mathbf{r}'(t) \cdot \mathbf{r}(t) + \mathbf{r}(t) \cdot \mathbf{r}'(t) = 2(\mathbf{r}'(t) \cdot \mathbf{r}(t)).$$

- Thus $\mathbf{r}'(t) \cdot \mathbf{r}(t) = 0$, which says that $\mathbf{r}'(t)$ is **orthogonal** to $\mathbf{r}(t)$.
- So **geometrically**, if a curve lies on a sphere with center the origin, then the tangent vector $\mathbf{r}'(t)$ is always perpendicular to the position vector $\mathbf{r}(t)$.

$\square$

Definition

Suppose $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ is a continuous curve in $\mathbf{R}^3$. Then

$$\int_a^b \mathbf{r}(t)\,dt = \lim_{n\to\infty} \sum_{i=1}^{n} \mathbf{r}(t_i^*)\Delta t$$

$$= \lim_{n\to\infty} \left[\left(\sum_{i=1}^{n} f(t_i^*)\Delta t \right) \mathbf{i} + \left(\sum_{i=1}^{n} g(t_i^*)\Delta t \right) \mathbf{j} + \left(\sum_{i=1}^{n} h(t_i^*)\Delta t \right) \mathbf{k} \right],$$

and so

$$\int_a^b \mathbf{r}(t)\,dt = \left(\int_a^b f(t)\,dt \right) \mathbf{i} + \left(\int_a^b g(t)\,dt \right) \mathbf{j} + \left(\int_a^b h(t)\,dt \right) \mathbf{k}.$$

Example

If $\mathbf{r}(t) = 2\cos t\mathbf{i} + \sin t\mathbf{j} + 2t\mathbf{k}$, then

$$\int \mathbf{r}(t)\,dt = \left(\int 2\cos t\,dt\right)\mathbf{i} + \left(\int \sin t\;dt\right)\mathbf{j} + \left(\int 2t\,dt\right)\mathbf{k}$$

$$= 2\sin t\mathbf{i} - \cos t\mathbf{j} + t^2\mathbf{k} + \mathbf{C}$$

where $\mathbf{C} = \langle x_0, y_0, z_0 \rangle$ is a vector constant of integration.

Definition

- The **speed** of the particle at time t is the magnitude of the velocity vector $|\mathbf{v}(t)|$.

  $\mathbf{speed}(t) = |\mathbf{v}(t)| = |\mathbf{r}'(t)| = \frac{ds}{dt}$ = rate of change of distance with respect to time.

- The **acceleration** of the particle is defined as the derivative of the velocity:

$$\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t).$$

Example (popular exam problem)

Find the **velocity**, **acceleration**, and **speed** of a particle with position vector $\mathbf{r}(t) = \langle t^2, e^t, te^t \rangle$.

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Find the **velocity**, **acceleration**, and **speed** of a particle with position vector $\mathbf{r}(t) = \langle t^2, e^t, te^t \rangle$.

Solution:

$$\mathbf{v}(t) = \mathbf{r}'(t) = \langle 2t, e^t, (1+t)e^t \rangle$$

$$\mathbf{a}(t) = \mathbf{v}'(t) = \langle 2, e^t, (2+t)e^t \rangle$$

$$\textbf{speed}(t) = |\mathbf{v}(t)| = \sqrt{4t^2 + e^{2t} + (1+t)^2 e^{2t}}.$$

Example (popular exam problem)

A moving particle starts at an initial position $\mathbf{r}(0) = \langle 1, 0, 0 \rangle$ with initial velocity $\mathbf{v}(0) = \mathbf{i} - \mathbf{j} + \mathbf{k}$. Its acceleration is $\mathbf{a}(t) = 4t\mathbf{i} + 6t\mathbf{j} + \mathbf{k}$. Find its velocity and position at time t.

Example (popular exam problem)

A moving particle starts at an initial position $\mathbf{r}(0) = \langle 1, 0, 0\rangle$ with initial velocity $\mathbf{v}(0) = \mathbf{i} - \mathbf{j} + \mathbf{k}$. Its acceleration is $\mathbf{a}(t) = 4t\mathbf{i} + 6t\mathbf{j} + \mathbf{k}$. Find its velocity and position at time t.

Solution:

- Since $\mathbf{a}(t) = \mathbf{v}'(t)$, we have
$$\mathbf{v}(t) = \int \mathbf{a}(t)\, dt = \int (4t\mathbf{i} + 6t\mathbf{j} + \mathbf{k})\, dt$$
$$= 2t^2\mathbf{i} + 3t^2\mathbf{j} + t\mathbf{k} + \mathbf{C}.$$
- To determine the constant vector $\mathbf{C}$, we use the fact that $\mathbf{v}(0) = \mathbf{i} - \mathbf{j} + \mathbf{k}$.
- The preceding equation gives $\mathbf{v}(0) = \mathbf{C}$ and
$$\mathbf{v}(t) = 2t^2\mathbf{i} + 3t^2\mathbf{j} + t\mathbf{k} + \mathbf{i} - \mathbf{j} + \mathbf{k} = (2t^2 + 1)\mathbf{i} + (3t^2 - 1)\mathbf{j} + (t + 1)\mathbf{k}.$$
- Since $\mathbf{v}(t) = \mathbf{r}'(t)$, then
$$\mathbf{r}(t) = \int \mathbf{v}(t)\, dt = \int [(2t^2 + 1)\mathbf{i} + (3t^2 - 1)\mathbf{j} + (t + 1)\mathbf{k}]\, dt =$$
$$(\frac{2}{3}t^3 + t)\mathbf{i} + (t^3 - t)\mathbf{j} + (\frac{1}{2}t^2 + t)\mathbf{k} + \mathbf{D}.$$
- Putting $t = 0$, we find that $\mathbf{D} = \mathbf{r}(0) = \mathbf{i}$, so the position at time t is:
$$\mathbf{r}(t) = (\frac{2}{3}t^3 + t + 1)\mathbf{i} + (t^3 - t)\mathbf{j} + (\frac{1}{2}t^2 + t)\mathbf{k}.$$

Example (popular exam problem)

A moving particle starts at an initial position $\mathbf{r}(0) = \langle 1, 0, 0 \rangle$ with initial velocity $\mathbf{v}(0) = \mathbf{i} - \mathbf{j} + \mathbf{k}$. Its acceleration is $\mathbf{a}(t) = 4t\mathbf{i} + 6t\mathbf{j} + \mathbf{k}$. Find its velocity and position at time t.

Solution:

- Since $\mathbf{a}(t) = \mathbf{v}'(t)$, we have

$$\mathbf{v}(t) = \int_0^t \mathbf{a}(t)\, dt + \mathbf{v}(0) = \int_0^t \langle 4t, 6t, 1 \rangle\, dt + \langle 1, -1, 1 \rangle$$

$$= \langle 2t^2, 3t^2, t \rangle \Big|_0^t + \langle 1, -1, 1 \rangle = \langle 2t^2 + 1, 3t^2 - 1, t + 1 \rangle.$$

- Since $\mathbf{v}(t) = \mathbf{r}'(t)$, then

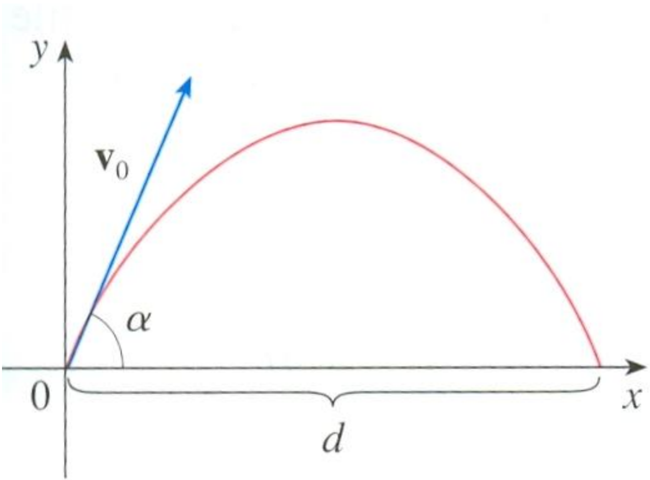
$$\mathbf{r}(t) = \int_0^t \mathbf{v}(t)\, dt + \mathbf{r}(0) = \int_0^t \langle 2t^2 + 1, 3t^2 - 1, t + 1 \rangle\, dt + \langle 1, 0, 0 \rangle$$

$$= \langle \frac{2}{3}t^3 + t), (t^3 - t), (\frac{1}{2}t^2 + t) \rangle \Big|_0^t + \langle 1, 0, 0 \rangle$$

$$= \langle \frac{2}{3}t^3 + t) + 1, (t^3 - t), (\frac{1}{2}t^2 + t) \rangle.$$

Example

A projectile is fired with angle of elevation α and initial velocity $\mathbf{v}_0$. Find the position function $\mathbf{r}(t)$ of the projectile. What value of α maximizes the horizontal distance traveled?

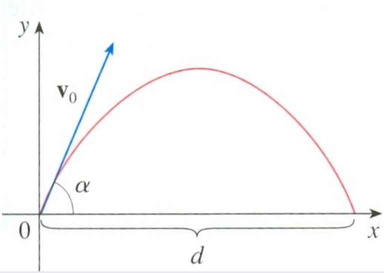


Example

A projectile is fired with angle of elevation α and initial velocity $\mathbf{v}_0$. Find the position function $\mathbf{r}(t)$ of the projectile. What value of α maximizes the horizontal distance traveled?

Solution:

- Since the force due to gravity acts downward, we have
$$\mathbf{F} = m\mathbf{a} = -mg\mathbf{j},$$
where $g = |a| \approx 9.8 \ \mathbf{m}/\mathbf{s}^2$. Thus $\mathbf{a} = -g\mathbf{j}$.
- Since $\mathbf{v}'(t) = \mathbf{a}$, we have
$$\mathbf{v}(t) = -gt\mathbf{j} + \mathbf{C},$$
where $\mathbf{C} = \mathbf{v}(0) = \mathbf{v}_0$.
- Therefore,
$$\mathbf{r}'(t) = \mathbf{v}(t) = -gt\mathbf{j} + \mathbf{v}_0$$
$$\mathbf{r}(t) = -\frac{1}{2}gt^2\mathbf{j} + t\mathbf{v}_0 + \mathbf{D}.$$
- Since $\mathbf{D} = \mathbf{r}(0) = \langle 0, 0\rangle$, $\mathbf{r}(t) = -\frac{1}{2}gt^2\mathbf{j} + t\mathbf{v}_0$.
- If $|\mathbf{v}_0| = v_0$, then $\mathbf{v}_0 = v_0 \cos\alpha\mathbf{i} + v_0 \sin\alpha\mathbf{j}$, and so,
$$\mathbf{r}(t) = (v_0 \cos\alpha)t\mathbf{i} + [(v_0 \sin\alpha)t - \frac{1}{2}gt^2]\mathbf{j}.$$



$$\mathbf{r}(t) = (v_0 \cos\alpha) t\mathbf{i} + [(v_0 \sin\alpha)t - \frac{1}{2}gt^2]\mathbf{j}.$$

- The **parametric equations** of the trajectory are therefore:
$$x = (v_0 \cos\alpha)t \quad y = (v_0 \sin\alpha)t - \frac{1}{2}gt^2. \tag{6}$$
- The horizontal distance **d** is the value of x when $y = 0$.
- Setting $y = 0$, we obtain $t = 0$ or $t = \frac{(2v_0 \sin\alpha)}{g}$.
- This second value of t then gives
$$\mathbf{d} = x = (v_0 \cos\alpha)\frac{2v_0 \sin\alpha}{g} = \frac{v_0^2(2\sin\alpha\cos\alpha)}{g} = \frac{v_0^2 \sin 2\alpha}{g}.$$
- **d** has its maximum value when $\sin 2\alpha = 1$.
- That is, $\alpha = \frac{\pi}{4}$.

Recall the following definitions.

Definition

- $\mathbf{i} = (1, 0, 0)$, $\mathbf{j} = (0, 1, 0)$, $\mathbf{k} = (0, 0, 1)$ are the standard basis vectors in $\mathbf{R}^3$.
- $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$, where $\mathbf{r}\colon [a, b] \to \mathbf{R}^3$.
- The **speed** of $\mathbf{r}(t)$ is $|\mathbf{r}'(t)| = \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}$.
- The **length function** is $\mathbf{s}(t) = \int_a^t |\mathbf{r}'(t)| \, dt$.

Definition

- The speed of $\mathbf{r}(t)$ is $\frac{\mathbf{ds}}{\mathbf{dt}} = \mathbf{s}'(t) = |\mathbf{r}'(t)|$.
- The **unit tangent vector** of $\mathbf{r}(t)$ is $\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$.
- If $\mathbf{r}(t)$ is parameterized by arc length, then $\frac{\mathbf{ds}}{\mathbf{dt}} = |\mathbf{r}'(t)| = 1$.
- The **curvature** of a curve $\mathbf{r}(t)$ parameterized by arc length is $\kappa(t) = |\mathbf{T}'(t)|$.
- The **curvature** of a curve $\mathbf{r}(t)$ **not** parameterized by arc length is $\kappa(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|}$ (this follows by the chain rule).

Theorem

The curvature of the curve given by the vector function $\mathbf{r}$ is

$$\kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}.$$

Theorem

The curvature of the curve given by the vector function $\mathbf{r}$ is

$$\kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}.$$

Proof.

- Since $\mathbf{T} = \frac{\mathbf{r}}{|\mathbf{r}'|}$ and $|\mathbf{r}'| = \frac{\mathbf{ds}}{\mathbf{dt}}$, we have $\mathbf{r}' = |\mathbf{r}'|\mathbf{T} = \frac{\mathbf{ds}}{\mathbf{dt}}\mathbf{T}$.
- So the Product Rule gives $\mathbf{r}'' = \frac{\mathbf{d^2s}}{\mathbf{dt^2}}\mathbf{T} + \frac{\mathbf{ds}}{\mathbf{dt}}\mathbf{T}'$.
- Using the fact that $\mathbf{T} \times \mathbf{T} = (0, 0, 0)$, we have $\mathbf{r}' \times \mathbf{r}'' = \left(\frac{\mathbf{ds}}{\mathbf{dt}}\right)^{\mathbf{2}} (\mathbf{T} \times \mathbf{T}')$.
- Now $|\mathbf{T}(t)| = 1$ for all t, so $\mathbf{T}$ and $\mathbf{T}'$ are orthogonal.
- Therefore, $|\mathbf{r}' \times \mathbf{r}''| = \left(\frac{\mathbf{ds}}{\mathbf{dt}}\right)^{\mathbf{2}} |\mathbf{T} \times \mathbf{T}'| = \left(\frac{\mathbf{ds}}{\mathbf{dt}}\right)^{\mathbf{2}} |\mathbf{T}||\mathbf{T}'| = \left(\frac{\mathbf{ds}}{\mathbf{dt}}\right)^{\mathbf{2}} |\mathbf{T}'|$.
- Thus,
$$|\mathbf{T}'| = \frac{|\mathbf{r} \times \mathbf{r}''|}{(\frac{\mathbf{ds}}{\mathbf{dt}})^{\mathbf{2}}} = \frac{|\mathbf{r}' \times \mathbf{r}''|}{|\mathbf{r}'|^{\mathbf{2}}}$$
and
$$\kappa = \frac{|\mathbf{T}'|}{|\mathbf{r}'|} = \frac{|\mathbf{r}' \times \mathbf{r}''|}{|\mathbf{r}'|^3}.$$

$\square$

Example

Consider the helix $\mathbf{r}(t) = \langle \cos t, \sin t, t \rangle$. What is its curvature function $\kappa(t)$?

Example

Consider the helix $\mathbf{r}(t) = \langle \cos t, \sin t, t \rangle$. What is its curvature function $\kappa(t)$?

Solution:

- We first calculate the relevant information.

$$\mathbf{r}'(t) = \langle -\sin t, \cos t, 1 \rangle,$$

and so
$$\mathbf{r}''(t) = \langle -\cos t, -\sin t, 0 \rangle,$$
$$|\mathbf{r}'(t)| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2},$$
$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \langle \sin t, -\cos t, 1 \rangle,$$
$$|\mathbf{r}'(t) \times \mathbf{r}''(t)| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}.$$

- Therefore, by the previous theorem,

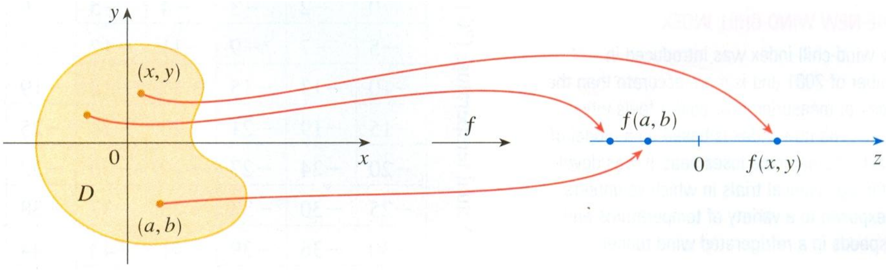
$$\kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} = \frac{\sqrt{2}}{(\sqrt{2})^3} = \frac{1}{2}.$$

In Class Exercises 5

1. Suppose $\mathbf{r}(t) = (2t, t^2, t^3)$. Find the **equation of the tangent line L** at the point $P = (2, 1, 1)$.
2. If $\mathbf{r}(t) = \langle \sin(t), \cos(t), 3t \rangle$, then what is the length of $\mathbf{r}(t)$ from time $t = 1$ to time $t = 6$?
3. A moving particle starts at an initial position $\mathbf{r}(0) = \langle 1, 0, 0 \rangle$ with initial velocity $\mathbf{v}(0) = \langle 1, 1, -1 \rangle$. Its acceleration is $\mathbf{a}(t) = 4t\mathbf{i} + 6t\mathbf{j} + \mathbf{k}$. Find its velocity vector at time t.
4. A moving particle starts at an initial position $\mathbf{r}(0) = \langle 1, 0, 0 \rangle$. Its velocity is $\mathbf{v}(t) = 4\mathbf{i} + 6t\mathbf{j} + \mathbf{k}$. Find its position at time t.
5. Use the formula
$$\kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$$
to calculate the curvature of $\mathbf{r}(t) = (2t, t^2, t^3)$ at the point $P = (2, 1, 1)$.

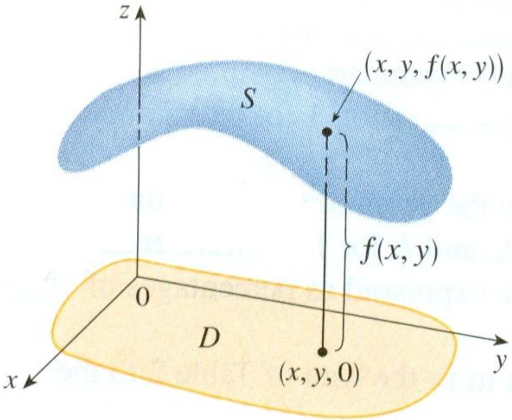
Definition

- A function f of two variables is a rule that assigns to each ordered pair of real numbers (x, y) in a set $\mathbf{D} \subset \mathbf{R}^2$, a real number denoted by $f(x, y)$.
- The set $\mathbf{D}$ is called the **domain** of f and its **range** is the set of values that f takes.
- We then write $f : \mathbf{D} \to \mathbb{R}$.



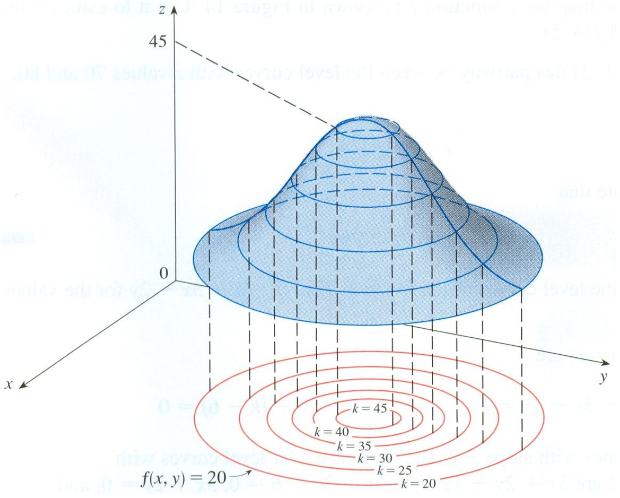
Definition

The **graph** of a function $f : \mathbf{D} \subset \mathbf{R}^2 \to \mathbb{R}$ is the set of points $\mathbf{G} = \{(x, y, f(x, y)) \in \mathbf{R}^3 \mid (x, y) \in \mathbf{D}\}$.

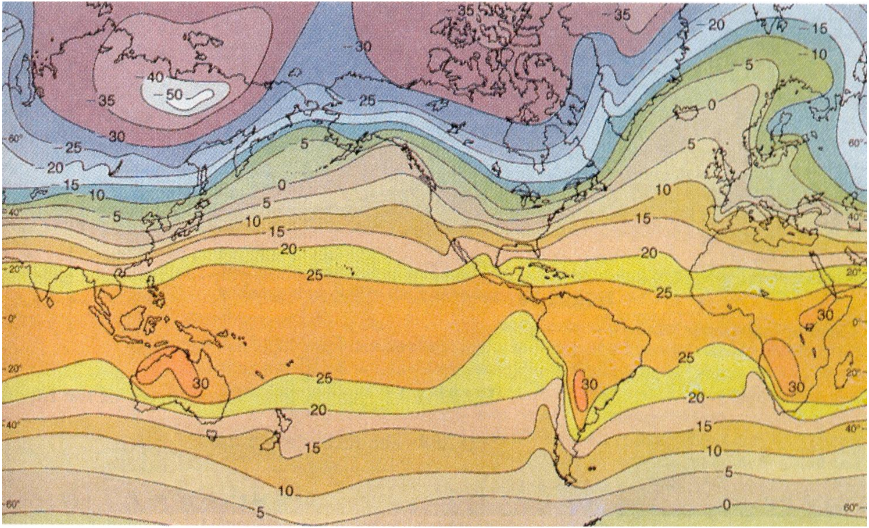


Definition

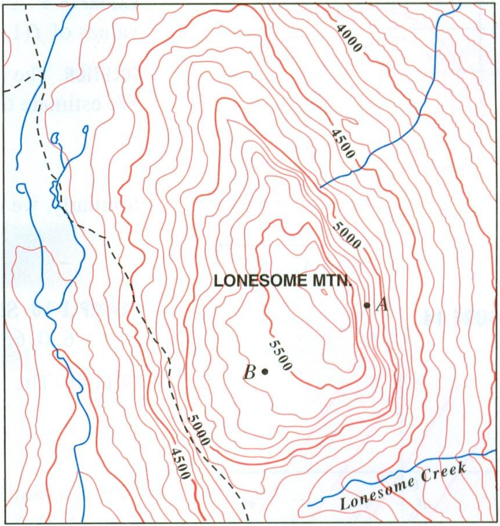
The **level curves** of a function f of two variables are the curves with equations $f(x, y) = k$, where k is a constant.



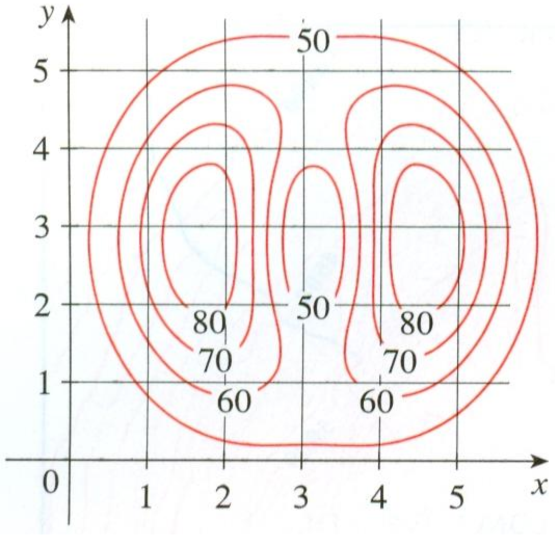
The level set curves of the temperature function on a world map.

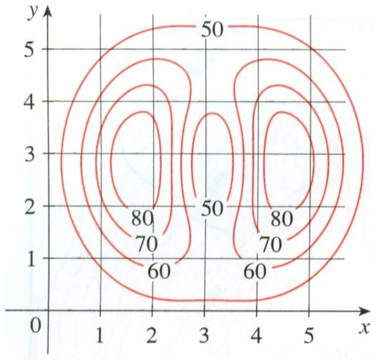


The level set curves of the altitude function on a map of the Lonesome Mountain Park region.



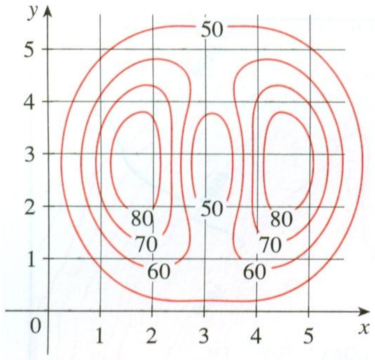
The contour map of a function f.





Example

A contour map for a function f is shown in the figure. Use it to estimate the values of $f(1,3)$ and $f(4,5)$.



Example

A contour map for a function f is shown in the figure. Use it to estimate the values of $f(1,3)$ and $f(4,5)$.

Solution:

The point $(1,3)$ lies partway between the level curves with z-values 70 and 80.
We estimate that $f(1,3) \approx 73$.
Similarly, we estimate that $f(4,5) \approx 56$.

Definition

- Let f be a function of two variables whose domain **D** includes points arbitrary close to (a, b).
- We say that the **limit of $\mathbf{f(x, y)}$ as $\mathbf{(x, y)}$ approaches $\mathbf{(a, b)}$** is **L** and we write

$$\lim_{(x,y)\to(a,b)} f(x, y) = \mathbf{L}$$

if for every number $\varepsilon > 0$ there is a corresponding number $\delta > 0$ such that $|f(x, y) - \mathbf{L}| \leq \varepsilon$, whenever $(x, y) \in \mathbf{D}$ and $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$.

Example (popular exam problem)

Explain why the **limit** of $f(x, y) = (3x^2y^2)/(2x^4 + y^4)$ does not exist as (x, y) approaches $(0, 0)$.

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Solution:

- Along the line $\langle t, t\rangle$, $t \neq 0$, $f(x, y)$ has the value $\frac{3}{3} = 1$.
- Along the line $\langle 0, t\rangle$, $t \neq 0$, $f(x, y)$ has the value $\frac{0}{1} = 0$.
- Since $f(x, y)$ has **2 different limiting values** along lines at $(0, 0)$, it does **not** have a **limit** at $(0, 0)$.

Definition

A function f of two variables is called **continuous at** (a, b) if

$$\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b).$$

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We say f is **continuous on D** if f is continuous at every point (a, b) in **D**.

Definition (Partial Derivatives)

If f is a function of two variables, its **partial derivatives** are the functions $f_x = \frac{\partial f}{\partial x}$ and $f_y = \frac{\partial f}{\partial y}$ defined by

$$f_x(x, y) = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

$$f_y(x, y) = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h}.$$

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Definition (Notation for Partial Derivatives)

If $z = f(x, y)$, we write

$$f_x(x, y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x, y) = \frac{\partial z}{\partial x} = f_1 = D_1 f = D_x f$$

$$f_y(x, y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x, y) = \frac{\partial z}{\partial y} = f_2 = D_2 f = D_y f.$$

Calculating partial derivatives

Rule

We have the following rules for calculating partial derivatives.

- To find f_x, regard y as a constant and differentiate $f(x, y)$ with respect to x.
- To find f_y, regard x as a constant and differentiate $f(x, y)$ with respect to y.

Example

If $f(x, y) = x^3 + x^2y^3 - 2y^2$, find $f_x(2, 1)$ and $f_y(2, 1)$.

Example

If $f(x, y) = x^3 + x^2y^3 - 2y^2$, find $f_x(2, 1)$ and $f_y(2, 1)$.

Solution:

- Holding y constant and differentiating with respect to x, we get

$$f_x(x, y) = 3x^2 + 2xy^3$$

and so $f_x(2, 1) = 3 \cdot 2^2 + 2 \cdot 2 \cdot 1^3 = 16$.

- Holding x constant and differentiating with respect to y, we get

$$f_y(x, y) = 3x^2y^2 - 4y$$

$$f_y(2, 1) = 3 \cdot 2^2 \cdot 1^2 - 4 \cdot 1 = 8.$$

Example

Calculate f_x, f_y for $f(x, y) = x^2 e^{xy} + y^2$.

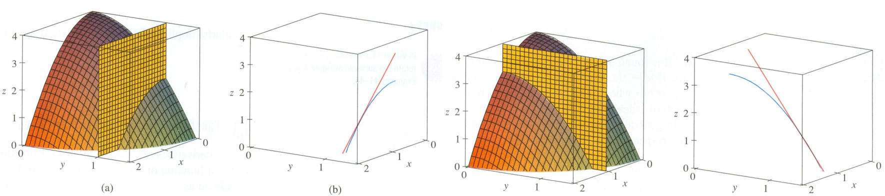
Example

Calculate f_x, f_y for $f(x, y) = x^2 e^{xy} + y^2$.

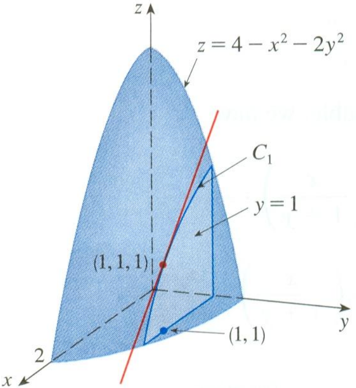
Solution:

$$f_x(x, y) = 2xe^{xy} + x^2 e^{xy} y = 2xe^{xy} + x^2 y e^{xy}$$

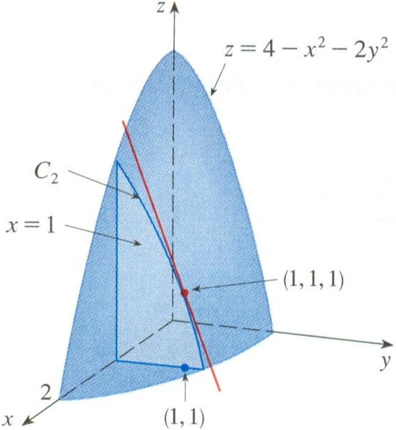
$$f_y(x, y) = x^2 e^{xy} x + 2y = x^3 e^{xy} + 2y.$$



(a)
(b)



$z = 4 - x^2 - 2y^2$
C_1
$y = 1$
$(1, 1, 1)$
$(1, 1)$



$z = 4 - x^2 - 2y^2$
C_2
$x = 1$
$(1, 1, 1)$
$(1, 1)$

Definition (Second Partial Derivatives)

For $z = f(x, y)$, we use the following notation:

$$(f_x)_x = f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 z}{\partial x^2}$$

$$(f_x)_y = f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 z}{\partial y \partial x}$$

$$(f_y)_x = f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 z}{\partial x \partial y}$$

$$(f_y)_y = f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 z}{\partial y^2}$$

Example

Find the second partial derivatives of

$$f(x, y) = x^3 + x^2y^3 - 2y^2.$$

Example

Find the second partial derivatives of

$$f(x,y) = x^3 + x^2y^3 - 2y^2.$$

Solution:

- Note:

$$f_x(x,y) = 3x^2 + 2xy^3 \qquad f_y(x,y) = 3x^2y^2 - 4y.$$

- Therefore,

$$f_{xx} = 6x + 2y^3 \qquad f_{xy} = 6xy^2$$
$$f_{yx} = 6xy^2 \qquad f_{yy} = 6x^2y - 4.$$

- Note that $f_{xy} = f_{yx}$!!

Theorem (Clairaut's Theorem)

Suppose f is defined on a disk **D** that contains the point (a, b). If the functions f_{xy} and f_{yx} are both continuous on **D**, then

$$f_{xy}(a, b) = f_{yx}(a, b).$$

Definition (Laplace's Equation)

A function $u(x, y)$ of 2 variables is said to be **harmonic** if

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

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Example

Show that the function $u(x, y) = e^x \sin y$ is harmonic.

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Example

Show that the function $u(x, y) = e^x \sin y$ is harmonic.

Solution:

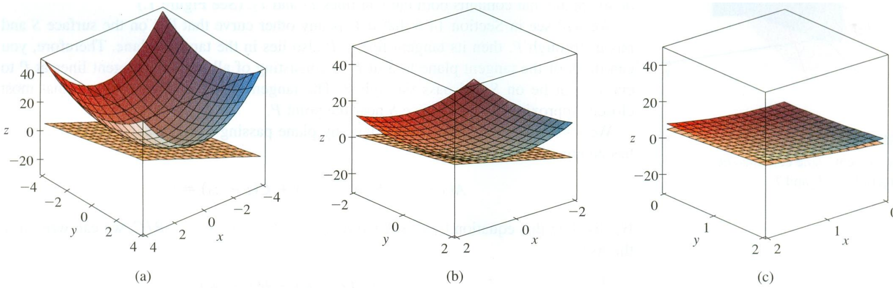
$$u_x = e^x \sin y \quad u_y = e^x \cos y$$

$$u_{xx} = e^x \sin y \quad u_{yy} = -e^x \sin y$$

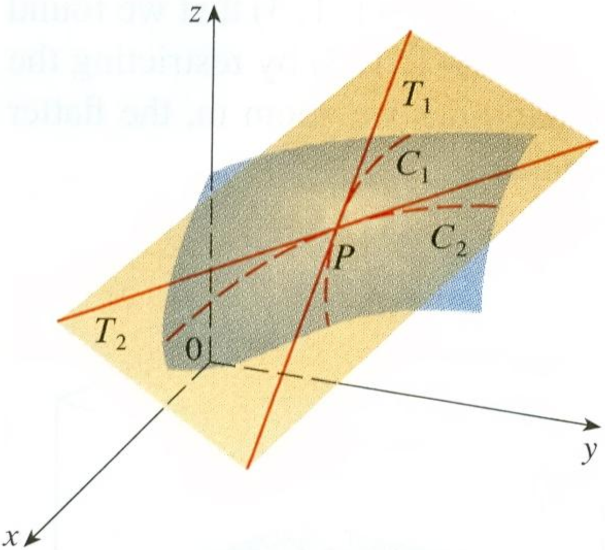
$$u_{xx} + u_{yy} = e^x \sin y - e^x \sin y = 0$$

Therefore, u is harmonic.

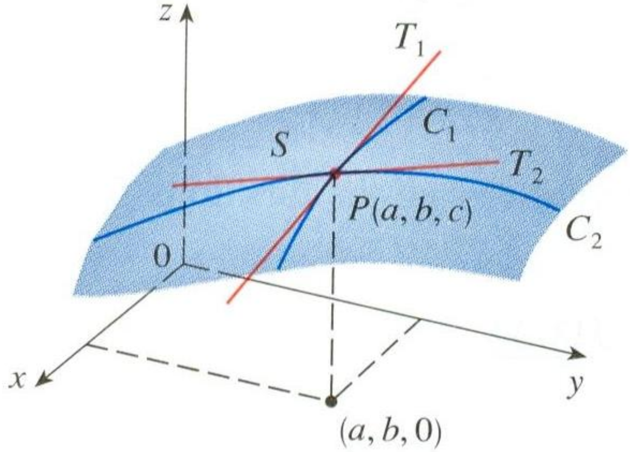
Zooming in on the graph of $f = 2x^2 + y^2$ at a point $(1, 1, 3)$.



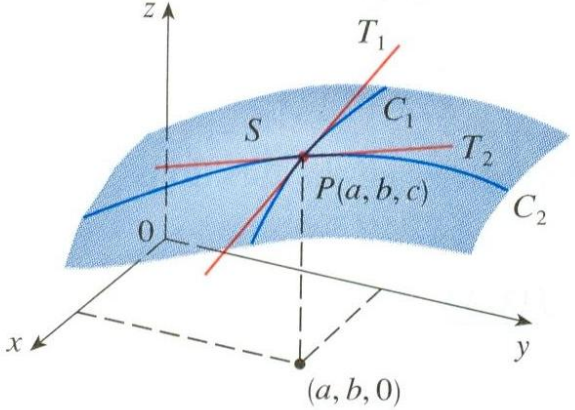
The **tangent plane** to the graph of $f = 2x^2 + y^2$ at a point $(1, 1, 3)$ is the plane which most closely approximates the graph.



The tangent plane contains the tangent lines $\mathbf{T}_1$ and $\mathbf{T}_2$ to the curves $\mathbf{C}_1$ and $\mathbf{C}_2$ that lie on it.

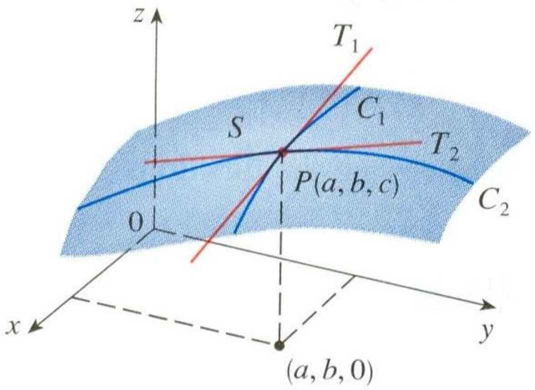


The partial derivatives f_x, f_y of f at (a, b) are the **slopes of tangent lines** to the curves $\mathbf{C}_1$ and $\mathbf{C}_2$ at the point $P(a, b, c)$, where $c = f(a, b)$. Here $\mathbf{C}_1$ and $\mathbf{C}_2$ are the intersection curves of the graph of f with the planes $P_1 = \{y = b\}$, $P_2 = \{x = a\}$, respectively.



At $t = 0$ the tangent vector to the curve $\mathbf{C}_1(t) = (a + t, b, f(a + t, b))$ is $(\mathbf{1}, \mathbf{0}, \mathbf{f_x(a, b)})$ and for $\mathbf{C}_2(t) = (a, b + t, f(a, b + t))$ the tangent vector is $(\mathbf{0}, \mathbf{1}, \mathbf{f_y(a, b)})$. Hence, the **normal vector** to the tangent plane at $P(a, b, c = f(a, b))$ is their cross product:

$$(\mathbf{1}, \mathbf{0}, \mathbf{f_x(a, b)}) \times (\mathbf{0}, \mathbf{1}, \mathbf{f_y(a, b)}) = (-\mathbf{f_x(a, b)}, -\mathbf{f_y(a, b)}, \mathbf{1}).$$



Definition (Tangent Plane)

Suppose f has continuous partial derivatives. An equation of the **tangent plane** to the surface $z = f(x, y)$ at the point $P = (x_0, y_0, z_0)$ is

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

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Example (popular exam problem)

Find the **tangent plane** to the elliptic paraboloid $z = 2x^2 + y^2$ at the point $(1, 1, 3)$.

$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$

Example (popular exam problem)

Find the **tangent plane** to the elliptic paraboloid $z = 2x^2 + y^2$ at the point $(1, 1, 3)$.

Solution:

- Let $f(x, y) = 2x^2 + y^2$.
- Then
$$f_x(x, y) = 4x \qquad f_y(x, y) = 2y$$
$$f_x(1, 1) = 4 \qquad f_y(1, 1) = 2.$$
- So the equation of the **tangent plane** at $(1, 1, 3)$ is
$$z - 3 = 4(x - 1) + 2(y - 1)$$
or
$$z = 4x + 2y - 3.$$

Definition (Linear Approximation)

The **linear approximation** of $f(x, y)$ at (a, b) is

$$\mathbf{L}(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b).$$

Definition (Linear Approximation)

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$$\mathbf{L}(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b).$$

Example (Very popular exam problem)

Use the linear approximation $\mathbf{L}(x, y)$ to $f(x, y) = xe^{xy}$ at $(1, 2)$ to estimate $f(1.1, 1.8)$.

Definition (Linear Approximation)

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Example (Very popular exam problem)

Use the linear approximation $\mathbf{L}(x, y)$ to $f(x, y) = xe^{xy}$ at $(1, 2)$ to estimate $f(1.1, 1.8)$.

Solution:

$$f_x(x, y) = e^{xy} + xye^{xy} \qquad f_y(x, y) = x^2 e^{xy}$$

$$f_x(1, 2) = e^2 + 2e^2 \qquad f_y(1, 2) = e^2.$$

$$\mathbf{L}(x, y) = f(1, 2) + f_x(1, 2)(x - 1) + f_y(1, 2)(y - 2).$$

Hence,

$$\mathbf{L}(1.1, 1.8) = e^2 + (e^2 + 2e^2)(.1) + e^2(-.2).$$

In Class Exercises 6

1. Explain why the **limit** of $f(x,y) = (x^2y^2)/(x^4 + y^4)$ does not exist as (x,y) approaches $(0,0)$.
2. For the function $f(x,y) = 2x^2 + xy^2$, calculate f_x, f_y, f_{xy}, f_{xx}.
3. Find the **tangent plane** to the elliptic paraboloid $z = 2x^2 + y^2$ at the point $(1,1,3)$.
4. Find the **linear approximation** $\mathbf{L}(x,y)$ to the function $f(x,y) = 2x^2 + y$ at the point $(1,2)$ and use it to estimate $f(1.1, 1.8)$.
5. Find the **linear approximation** $\mathbf{L}(x,y)$ to $f(x,y) = xe^y$ at $(1,2)$.

Definition of differentiable function

Definition

If $z = f(x, y)$, then f is **differentiable** at (a, b) if the change Δz in z can be expressed in the form

$$\Delta z = f_x(a, b)\Delta x + f_y(a, b)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y,$$

where ε_1 and $\varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$.

The next theorem gives a simple condition for $f(x, y)$ to satisfy in order to be differentiable.

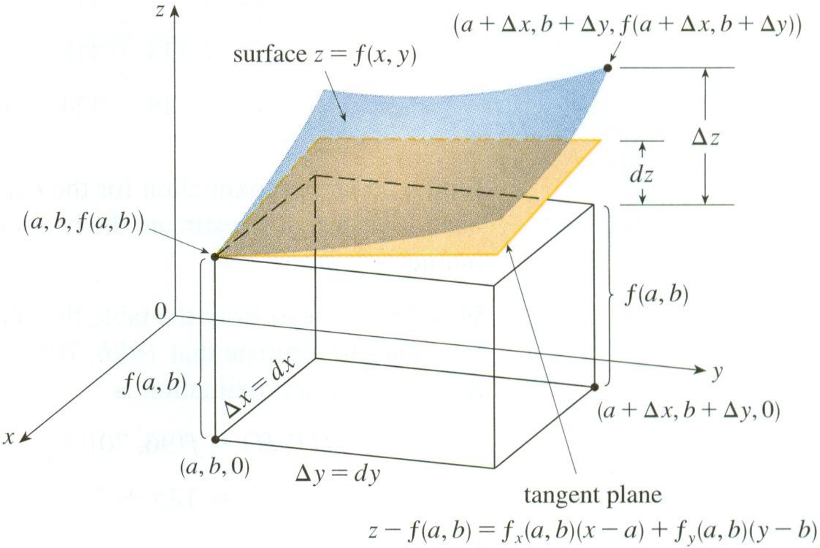
Theorem

If the partial derivatives f_x and f_y exist near (a, b) and are continuous at (a, b), then f is differentiable at (a, b).

Definition (Total Differential dz)

For $z = f(x, y)$,

$$dz = f_x(x, y)\,dx + f_y(x, y)\,dy = \frac{\partial z}{\partial x}\,dx + \frac{\partial z}{\partial y}\,dy.$$



A picture depicting the **total differential** dz of a function $z = f(x, y)$.

Example

If $z = f(x, y) = x^2 + 3xy - y^2$, find the differential dz.

Example

If $z = f(x, y) = x^2 + 3xy - y^2$, find the differential dz.

Solution:

By definition,

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = (2x + 3y)dx + (3x - 2y)dy.$$

Example

The base radius and height of a right circular cone are measured as 10 cm and 25 cm, respectively, with a possible error in measurement of as much as 0.1 cm. in each. Use differentials to estimate the maximum error in the calculated volume of the cone.

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The base radius and height of a right circular cone are measured as 10 cm and 25 cm, respectively, with a possible error in measurement of as much as 0.1 cm. in each. Use differentials to estimate the maximum error in the calculated volume of the cone.

Solution:

- The volume $\mathbf{V}$ of a cone with base radius r and height h is $\mathbf{V} = \pi r^2 \frac{h}{3}$.
- So the differential of $\mathbf{V}$ is
$$\mathbf{dV} = \frac{\partial \mathbf{V}}{\partial r}\, dr + \frac{\partial \mathbf{V}}{\partial h}\, dh = \frac{2\pi rh}{3}\, dr + \frac{\pi r^2}{3}\, dh.$$
- Since each error is at most 0.1 cm, then $|\mathbf{\Delta} r| \leq 0.1$, $|\mathbf{\Delta} h| \leq 0.1$. In this case, $dr = 0.1$, $dh = 0.1$, $r = 10$ and $h = 25$.
- This gives $\mathbf{dV} = \frac{500\pi}{3}(0.1) + \frac{100\pi}{3}(0.1) = 20\pi.$
- Thus, the maximum error in the calculated volume is about $20\pi\ \mathrm{cm}^3 \approx 63\ \mathrm{cm}^3$.

Example

The dimensions of a rectangular box are measured to be 75 cm, 60 cm, and 40 cm, and each measurement is correct to within 0.2 cm. Use differentials to estimate the largest possible error when the volume of the box is calculated form these measurements.

Example

The dimensions of a rectangular box are measured to be 75 cm, 60 cm, and 40 cm, and each measurement is correct to within 0.2 cm. Use differentials to estimate the largest possible error when the volume of the box is calculated form these measurements.

Solution:

- If the dimensions of the box are x, y, and z, its volume is $\mathbf{V} = xyz$.
- So
$$\mathbf{dV} = \frac{\partial \mathbf{V}}{\partial x}\, dx + \frac{\partial \mathbf{V}}{\partial y}\, dy + \frac{\partial \mathbf{V}}{\partial z}\, dz = yz\, dx + xz\, dy + xy\, dz.$$
- We have $|\boldsymbol{\Delta} x| \leq 0.2$, $|\boldsymbol{\Delta} y| \leq 0.2$, and $|\boldsymbol{\Delta} z| \leq 0.2$. Using $dx = 0.2$, $dy = 0.2$, and $dz = 0.2$ together with $x = 75$, $y = 60$, and $z = 40$:
$$\boldsymbol{\Delta}\mathbf{V} \approx \mathbf{dV} = (60)(40)(0.2) + (75)(40)(0.2) + (75)(60)(0.2) = 1980.$$
- Thus, an error of only 0.2 cm in measuring each dimension could lead to an error of as much as 1980 cm^3.

Theorem (Chain Rule Case 1)

Suppose that $z = f(x, y)$ is a differentiable function of x and y, where $x = g(t)$ and $y = h(t)$ are both differentiable functions of t. Then z is a differentiable function of t and

$$\frac{\mathbf{dz}}{\mathbf{dt}} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$

Proof of the Chain Rule.

$$\frac{\mathbf{dz}}{\mathbf{dt}} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$

- A change of Δt in t produces changes of Δx in x and Δy in y.
- These, in turn, produce a change of Δz in z.
- From the definition of differentiable function,

$$\Delta z = \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y,$$

  where $\varepsilon_1 \to 0$ and $\varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0,0)$.
- Dividing both sides by Δt:

$$\frac{\Delta z}{\Delta t} = \frac{\partial f}{\partial x}\frac{\Delta x}{\Delta t} + \frac{\partial f}{\partial y}\frac{\Delta y}{\Delta t} + \varepsilon_1 \frac{\Delta x}{\Delta t} + \varepsilon_2 \frac{\Delta y}{\Delta t}.$$

- Letting $\Delta t \to 0$:

$$\frac{\mathbf{dz}}{\mathbf{dt}} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + 0 \cdot \frac{dx}{dt} + 0 \cdot \frac{dy}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$

Theorem (Chain Rule Case 2)

Suppose that $z = f(x, y)$ is a differentiable function of x and y, where $x = g(s, t)$ and $y = h(s, t)$ are both differentiable functions of s and t. Then z is a differentiable function of t and s with

$$\frac{\partial \mathbf{z}}{\partial \mathbf{s}} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \qquad \frac{\partial \mathbf{z}}{\partial \mathbf{t}} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}.$$

Theorem (Chain Rule General Version)

Suppose that u is a differentiable function of n variables $x_1,\ x_2,\ldots,x_n$ and each x_j is a differentiable function of the m variables $t_1,\ t_2,\ldots,t_m$. Then u is a function of $t_1,\ t_2,\ldots,t_m$ and

$$\frac{\partial \mathbf{u}}{\partial \mathbf{t_i}} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_i} + \ldots + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t_i}$$

for each $i = 1,\ 2,\ldots,m$.

Example (popular exam problem)

If $z = x^2y + 3xy^4$, where $x = \sin 2t$ and $y = \cos t$, find $\frac{dz}{dt}$ when $t = 0$.

Example (popular exam problem)

If $z = x^2 y + 3xy^4$, where $x = \sin 2t$ and $y = \cos t$, find $\frac{\mathbf{dz}}{\mathbf{dt}}$ when $t = 0$.

Solution:

- Case 1 of the **Chain Rule** gives

$$\frac{\mathbf{dz}}{\mathbf{dt}} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

$$= (2xy + 3y^4)(2\cos 2t) + (x^2 + 12xy^3)(-\sin t).$$

- When $t = 0$, we have $x = \sin 0 = 0$ and $y = \cos 0 = 1$.
- Therefore,

$$\frac{\mathbf{dz}}{\mathbf{dt}}(0) = (0+3)(2\cos 0) + (0+0)(-\sin 0) = 6.$$

Example

The pressure $\mathbf{P}$ (in kilopascals), volume V (in liters), and temperature T (in kelvins) of a mole of an ideal gas are related by the equation $\mathbf{P}V = 8.31T$. Find the rate at which the pressure is changing when the temperature is 300 K and increasing at a rate of 0.1 K/s and volume is 100 L and increasing at a rate of 0.2 L/s.

Example

The pressure $\mathbf{P}$ (in kilopascals), volume V (in liters), and temperature T (in kelvins) of a mole of an ideal gas are related by the equation $\mathbf{P}V = 8.31T$. Find the rate at which the pressure is changing when the temperature is 300 K and increasing at a rate of 0.1 K/s and volume is 100 L and increasing at a rate of 0.2 L/s.

Solution:

- If t represents the time elapsed in seconds, then $T = 300,\ dT/dt = 0.1,\ V = 100,\ dV/dt = 0.2.$
- Since
$$\mathbf{P} = 8.31\frac{T}{V},$$
the **Chain Rule** gives
$$\frac{\mathbf{dP}}{\mathbf{dt}} = \frac{\partial \mathbf{P}}{\partial T}\frac{dT}{dt} + \frac{\partial \mathbf{P}}{\partial V}\frac{dV}{dt} = \frac{8.31}{V}\frac{dT}{dt} - \frac{8.31\,T}{V^2}\frac{dV}{dt}$$
$$= \frac{8.31}{100}(0.1) - \frac{8.31(300)}{100^2}(0.2) = -0.04155.$$
- So the pressure is decreasing at a rate of about 0.042 kPa/s.

Example (popular exam problem)

If $z = e^x \sin y$, where $x = st^2$ and $y = s^2 t$, find $\frac{\partial \mathbf{z}}{\partial \mathbf{s}}$ and $\frac{\partial \mathbf{z}}{\partial \mathbf{t}}$.

Example (popular exam problem)

If $z = e^x \sin y$, where $x = st^2$ and $y = s^2 t$, find $\frac{\partial \mathbf{z}}{\partial \mathbf{s}}$ and $\frac{\partial \mathbf{z}}{\partial \mathbf{t}}$.

Solution:

Applying Case 2 of the **Chain Rule**, we get

- $$\begin{aligned} \frac{\partial \mathbf{z}}{\partial \mathbf{s}} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = (e^x \sin y)(t^2) + (e^x \cos y)(2st) \\ &= t^2 e^{st^2} \sin(s^2 t) + 2ste^{st^2} \cos(s^2 t). \end{aligned}$$

- $$\begin{aligned} \frac{\partial \mathbf{z}}{\partial \mathbf{t}} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} = (e^x \sin y)(2st) + (e^x \cos y)(s^2) \\ &= 2ste^{st^2} \sin(s^2 t) + s^2 e^{st^2} \cos(s^2 t). \end{aligned}$$

Example (popular exam problem)

If $u = x^4 y + y^2 z^3$, where $x = rse^t$, $y = rs^2 e^{-t}$, and $z = r^2 s \sin t$, find the value of $\partial \mathbf{u} / \partial \mathbf{s}$ when $r = 2$, $s = 1$, $t = 0$.

Example (popular exam problem)

If $u = x^4y + y^2z^3$, where $x = rse^t$, $y = rs^2e^{-t}$, and $z = r^2 s \sin t$, find the value of $\partial \mathbf{u}/\partial \mathbf{s}$ when $r = 2$, $s = 1$, $t = 0$.

Solution:

- We have
$$\frac{\partial \mathbf{u}}{\partial \mathbf{s}} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial s}$$
$$= (4x^3y)(re^t) + (x^4 + 2yz^3)(2rse^{-t}) + (3y^2z^2)(r^2 \sin t).$$

- When $r = 2$, $s = 1$, and $t = 0$, we have $x = 2$, $y = 2$, and $z = 0$, so
$$\frac{\partial \mathbf{u}}{\partial \mathbf{s}} = (64)(2) + (16)(4) + (0)(0) = 192.$$

Theorem (Implicit Differentiation)

Suppose that z is given implicitly as a function $z = f(x, y)$ by an equation $\mathbf{F}(x, y, z) = 0$, i.e., $\mathbf{F}(x, y, f(x, y)) = 0$ for all (x, y) in the domain of $f(x, y)$. Then:

$$\frac{\partial \mathbf{z}}{\partial \mathbf{x}} = -\frac{\frac{\partial \mathbf{F}}{\partial x}}{\frac{\partial \mathbf{F}}{\partial z}} \qquad \frac{\partial \mathbf{z}}{\partial \mathbf{y}} = -\frac{\frac{\partial \mathbf{F}}{\partial y}}{\frac{\partial \mathbf{F}}{\partial z}}.$$

Proof of the implicit differentiation formulas.

- We compute the first formula

$$\frac{\partial \mathbf{z}}{\partial \mathbf{x}} = -\frac{\frac{\partial \mathbf{F}}{\partial x}}{\frac{\partial \mathbf{F}}{\partial z}}$$

  in the theorem.

- Consider $z = f(x, y)$ and so $\mathbf{F}(x, y, z) = 0$ means $\mathbf{F}(x, y, f(x, y)) = 0$ for all (x, y) in the domain of $\mathbf{F}$.

- Apply the **Chain Rule** to differentiate the equation $\mathbf{F}(x, y, z) = 0$:

$$\frac{\partial \mathbf{F}}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial \mathbf{F}}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial \mathbf{F}}{\partial z}\frac{\partial \mathbf{z}}{\partial \mathbf{x}} = 0,$$

  with $\frac{\partial}{\partial x}(x) = 1$ and $\frac{\partial}{\partial x}(y) = 0$.

- Thus,

$$\frac{\partial \mathbf{F}}{\partial x} + \frac{\partial \mathbf{F}}{\partial z}\frac{\partial \mathbf{z}}{\partial \mathbf{x}} = 0.$$

- Solving for $\frac{\partial \mathbf{z}}{\partial \mathbf{x}}$, proves the first formula in the theorem.

$\square$

Example (popular exam problem)

Find $\frac{\partial \mathbf{z}}{\partial \mathbf{x}}$ and $\frac{\partial \mathbf{z}}{\partial \mathbf{y}}$ if $x^3 + y^3 + z^3 + 6xyz = 1$.

Example (popular exam problem)

Find $\frac{\partial \mathbf{z}}{\partial \mathbf{x}}$ and $\frac{\partial \mathbf{z}}{\partial \mathbf{y}}$ if $x^3 + y^3 + z^3 + 6xyz = 1$.

Solution:

- Let $\mathbf{F}(x, y, z) = x^3 + y^3 + z^3 + 6xyz - 1$.
- Then, from the previous theorem, we have

$$\frac{\partial \mathbf{z}}{\partial \mathbf{x}} = -\frac{\mathbf{F}_x}{\mathbf{F}_z} = -\frac{3x^2 + 6yz}{3z^2 + 6xy} = -\frac{x^2 + 2yz}{z^2 + 2xy}$$

$$\frac{\partial \mathbf{z}}{\partial \mathbf{y}} = -\frac{\mathbf{F}_y}{\mathbf{F}_z} = -\frac{3y^2 + 6xz}{3z^2 + 6xy} = -\frac{y^2 + 2xz}{z^2 + 2xy}.$$

In Class Exercises 7

1. If $z = x^2 y + y^4$, where $x = \sin 2t$ and $y = \cos t$, find $\frac{\mathbf{dz}}{\mathbf{dt}}$ when $t = 0$.
2. If $z = e^x \sin y$, where $x = s$ and $y = s^2 t$, find $\frac{\partial \mathbf{z}}{\partial \mathbf{s}}$.
3. If $z = e^x \sin y$, where $x = s$ and $y = s^2 t$, find $\frac{\partial \mathbf{z}}{\partial \mathbf{t}}$.
4. Find $\frac{\partial \mathbf{z}}{\partial \mathbf{x}}$ if $y^2 + z^3 + xyz = 1$.
5. Find $\frac{\partial \mathbf{z}}{\partial \mathbf{y}}$ if $y^2 + z^3 + xyz = 1$.

Definition (Directional Derivative)

The **directional derivative** of $f(x, y)$ at (x_0, y_0) in the direction of a unit vector $\mathbf{u} = \langle a, b \rangle$ is

$$\mathbf{D}_{\mathbf{u}} f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

if this limit exists.

Definition (Directional Derivative)

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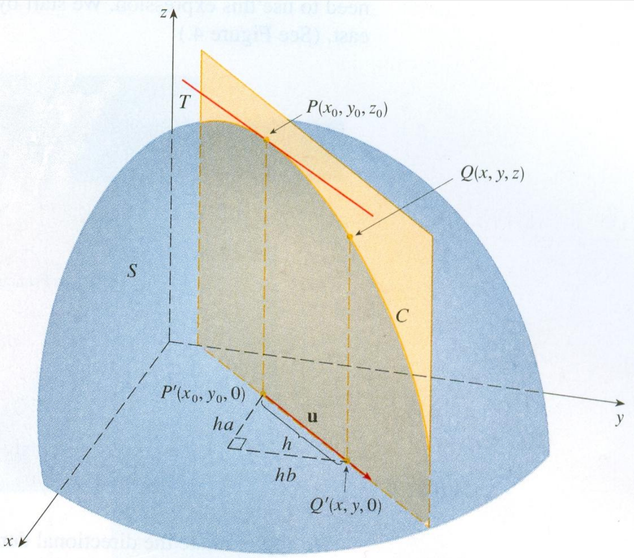
if this limit exists.

Definition (Directional Derivative)

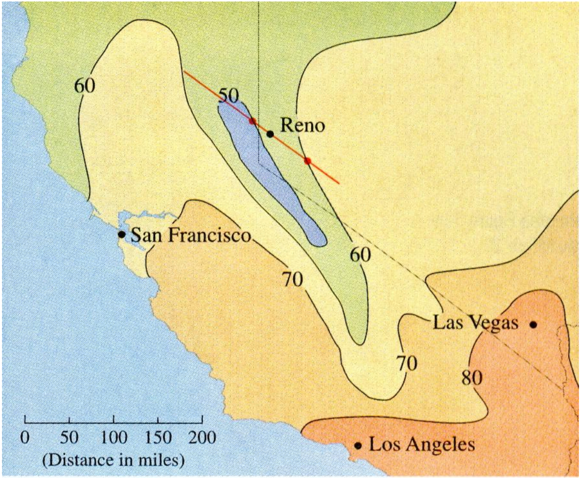
The **directional derivative** of $f(x, y, z)$ at (x_0, y_0, z_0) in the direction of a unit vector $\mathbf{u} = \langle a, b, c\rangle$ is

$$\mathbf{D_u} f(x_0, y_0, z_0) = \lim_{h\to 0} \frac{f(x_0 + ha, y_0 + hb, z_0 + hc) - f(x_0, y_0, z_0)}{h}$$

if this limit exists.



The **interpretation of directional derivative** as being the **slope** of the curve **C** contained in the intersection of the vertical plane P containing **u** and the graph of f.



An estimation of the directional derivative of the temperature function $\mathbf{T}(x, y)$ in the northwest direction $\mathbf{u} = \frac{(-1,1)}{\sqrt{2}}$. Since the change of temperature is -10 degrees over a distance of about 100 miles, then the estimation of the rate of change is $-\frac{1}{10}$ deg/mi.

Definition (Gradient)

If f is a function of two variables x and y, then the **gradient** of f is the vector function ∇f defined by

$$\nabla f(x,y) = \langle f_x(x,y), f_y(x,y)\rangle = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}.$$

Definition (Gradient)

For $f(x,y,z)$, a function of three variables, then the **gradient** of f is the vector function

$$\nabla f = \langle f_x, f_y, f_z\rangle = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}.$$

Example (popular exam problem)

If $f(x, y) = \sin x + e^{xy}$, then

$$\nabla f(x, y) = \langle f_x, f_y \rangle = \langle \cos x + ye^{xy}, xe^{xy} \rangle$$

and

$$\nabla f(0, 1) = \langle 2, 0 \rangle.$$

Example (popular exam problem)

If $f(x, y) = \sin x + e^{xy}$, then

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Example (popular exam problem)

If $f(x, y, z) = x \sin yz$, find the gradient of f.

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If $f(x,y) = \sin x + e^{xy}$, then

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and

$$\nabla f(0,1) = \langle 2, 0 \rangle.$$

Example (popular exam problem)

If $f(x,y,z) = x \sin yz$, find the gradient of f.

Solution:

The gradient of f is

$$\nabla f(x,y,z) = \langle f_x(x,y,z), f_y(x,y,z), f_z(x,y,z) \rangle$$

$$= \langle \sin yz, xz \cos yz, xy \cos yz \rangle.$$

The next two theorems give a simple rule for calculating the directional derivative of a function in 2 or 3 variables in terms of the gradient of the function.

The next two theorems give a simple rule for calculating the directional derivative of a function in 2 or 3 variables in terms of the gradient of the function.

Theorem

If f is a differentiable function of x and y, then f has a directional derivative in the direction of any unit vector $\mathbf{u} = \langle a, b \rangle$ and

$$\mathbf{D_u} f(x, y) = \nabla f(x, y) \cdot \mathbf{u} = f_x(x, y)a + f_y(x, y)b.$$

Theorem

If f is a differentiable function of x, y, and z, then f has a directional derivative in the direction of any unit vector $\mathbf{u} = \langle a, b, c \rangle$ and

$$\mathbf{D_u} f(x, y, z) = \nabla f(x, y, z) \cdot \mathbf{u}.$$

Proof.

For $\mathbf{u} = \langle a, b \rangle$, we will show $\mathbf{D_u} f(x, y) = \nabla f(x, y) \cdot \mathbf{u}$.

- Define a function g of the single variable h by
$$g(h) = f(x_0 + ha, y_0 + hb).$$
- Then, we have: $$g'(0) = \lim_{h \to 0} \frac{g(h) - g(0)}{h}$$
$$= \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h} = \mathbf{D_u} f(x_0, y_0).$$
- Write $g(h) = f(x, y)$, where $x = x_0 + ha,\ y = y_0 + hb$.
- The **Chain Rule** gives:
$$g'(h) = \frac{\partial f}{\partial x} \frac{dx}{dh} + \frac{\partial f}{\partial y} \frac{dy}{dh} = f_x(x, y)a + f_y(x, y)b.$$
- Put $h = 0$, then $x = x_0$, $y = y_0$ and
$$g'(0) = f_x(x_0, y_0)a + f_y(x_0, y_0)b.$$
- Comparing equations, we find
$$\mathbf{D_u} f(x_0, y_0) = f_x(x_0, y_0)a + f_y(x_0, y_0)b.$$

$\square$

By previous theorems, for any unit vector $\mathbf{u}$,

$$\mathbf{D}_{\mathbf{u}} f = \nabla f \cdot \mathbf{u} = |\nabla f||\mathbf{u}| \cos(\theta) = |\nabla f| \cos(\theta).$$

Thus, the next theorem holds.

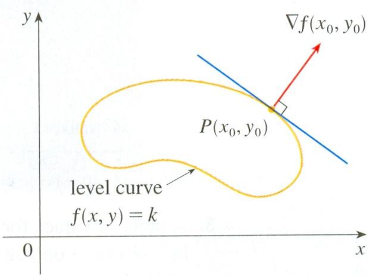
By previous theorems, for any unit vector $\mathbf{u}$,

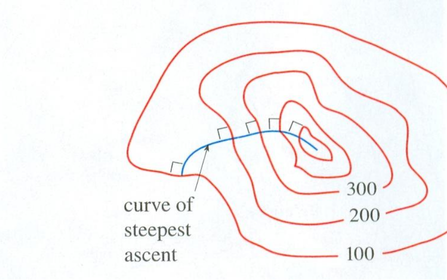
$$\mathbf{D_u}f = \nabla f \cdot \mathbf{u} = |\nabla f||\mathbf{u}|\cos(\theta) = |\nabla f|\cos(\theta).$$

Thus, the next theorem holds.

Theorem

- Suppose f is a differentiable function of two or three variables.
- The **maximum value** of the **directional derivative** $\mathbf{D_u}f(P)$ at a point P is $|\nabla f(P)|$ and it occurs when $\mathbf{u}$ has the same direction as the gradient vector $\nabla f(P)$.
- The number $|\nabla f(P)|$ is called the **maximum rate of change** of f at P.





The interpretation of the **gradient** as being the **direction of largest increase or accent** and as being **orthogonal** to the level set curves.

Example (popular exam problem)

Find the directional derivative of the function $f(x, y) = x^2y^3 - 4y$ at the point $(2, -1)$ in the direction of the vector $\mathbf{v} = \langle 2, 5 \rangle$.

Example (popular exam problem)

Find the directional derivative of the function $f(x,y) = x^2y^3 - 4y$ at the point $(2,-1)$ in the direction of the vector $\mathbf{v} = \langle 2,5\rangle$.

Solution:

- First compute the gradient vector at $(2,-1)$:

$$\nabla f(x,y) = \langle 2xy^3, 3x^2y^2 - 4\rangle \qquad \nabla f(2,-1) = \langle -4, 8\rangle.$$

- Since $|\mathbf{v}| = \sqrt{29}$, the unit vector in the direction of $\mathbf{v}$ is

$$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \langle \frac{2}{\sqrt{29}}, \frac{5}{\sqrt{29}}\rangle.$$

- By the previous theorem,

$$\mathbf{D_u} f(2,-1) = \nabla f(2,-1)\cdot \mathbf{u} = \langle -4, 8\rangle \cdot \langle \frac{2}{\sqrt{29}}, \frac{5}{\sqrt{29}}\rangle$$

$$= \frac{-4\cdot 2 + 8\cdot 5}{\sqrt{29}} = \frac{32}{\sqrt{29}}.$$

In Class Exercises 8

1. For the function $f(x, y) = 2x^2 + xy^2$, calculate f_x, f_y, f_{xy}, f_{xx}:
2. What is the **gradient** $\nabla f(x, y)$ of f at the point $(1, 2)$? Also, what is the **maximum rate of change** of f at $(1, 2)$.
3. Assuming that $\nabla f(1, 2) = \langle 8, 4 \rangle$, calculate the **directional derivative** of f at the point $(1, 2)$ in the direction of the vector $\mathbf{v} = \langle 3, 4 \rangle$?
4. Show the **linearization** $\mathbf{L}(x, y)$ of $f(x, y)$ at $(1, 2)$ is $\mathbf{L}(x, y) = 6 + 8(x - 1) + 4(y - 2)$.
5. Use the **linearization** $\mathbf{L}(x, y) = 6 + 8(x - 1) + 4(y - 2)$ in the previous part to estimate $f(0.9, 2.1)$.

Theorem

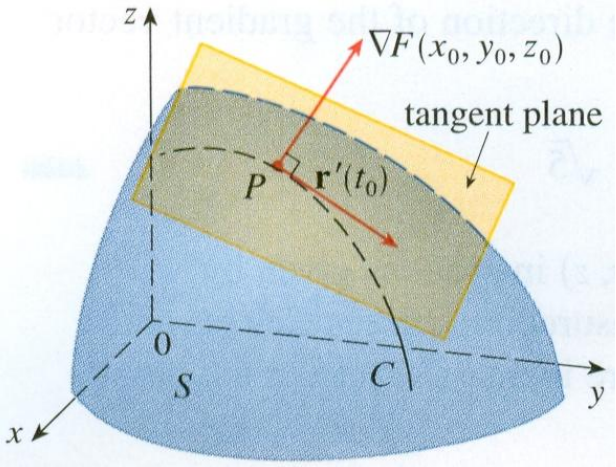
- Suppose **S** is a surface determined as $\mathbf{F}(x, y, z) = k = \textbf{constant}$.
- Then $\nabla\mathbf{F}$ is everywhere normal or orthogonal to **S**.
- In particular, if $P = (x_0, y_0, z_0) \in \mathbf{S}$, then the equation of the tangent plane to **S** at p is:

$$\nabla\mathbf{F} \cdot (x - x_0, y - y_0, z - z_0) = 0$$

.

Note that this dot product is equal to:

$$\mathbf{F}_x(x_0, y_0, z_0)(x - x_0) + \mathbf{F}_y(x_0, y_0, z_0)(y - y_0) + \mathbf{F}_z(x_0, y_0, z_0)(z - z_0) = 0.$$



Since the function $\mathbf{F}$ has constant value along any curve $\mathbf{r}(t)$ on $\mathbf{F}(x, y, z) = k$, the gradient $\nabla\mathbf{F}$ is orthogonal to $\mathbf{r}'(t_0)$ at $P = \mathbf{r}(t_0)$.

Example (Very popular exam problem)

Find the equations of the **tangent plane** and **normal line** at the point $(-2, 1, -3)$ to the ellipsoid $\frac{x^2}{4} + y^2 + \frac{z^2}{9} = 3$.

Example (Very popular exam problem)

Find the equations of the **tangent plane** and **normal line** at the point $(-2, 1, -3)$ to the ellipsoid $\frac{\mathbf{x}^2}{\mathbf{4}} + \mathbf{y}^2 + \frac{\mathbf{z}^2}{\mathbf{9}} = \mathbf{3}$.

Solution:

- The ellipsoid is the level surface

$$\mathbf{F}(x, y, z) = \frac{x^2}{4} + y^2 + \frac{z^2}{9} = 3.$$

- Therefore,

$$\mathbf{F}_x(x, y, z) = \frac{x}{2} \quad \mathbf{F}_y(x, y, z) = 2y \quad \mathbf{F}_z(x, y, z) = \frac{2z}{9}$$

$$\mathbf{F}_x(-2, 1, -3) = -1 \quad \mathbf{F}_y(-2, 1, -3) = 2 \quad \mathbf{F}_z(-2, 1, -3) = -\frac{2}{3}.$$

- Then the equation of the **tangent plane** at $(-2, 1, -3)$ is

$$-1(x + 2) + 2(y - 1) - \frac{2}{3}(z + 3) = 0,$$

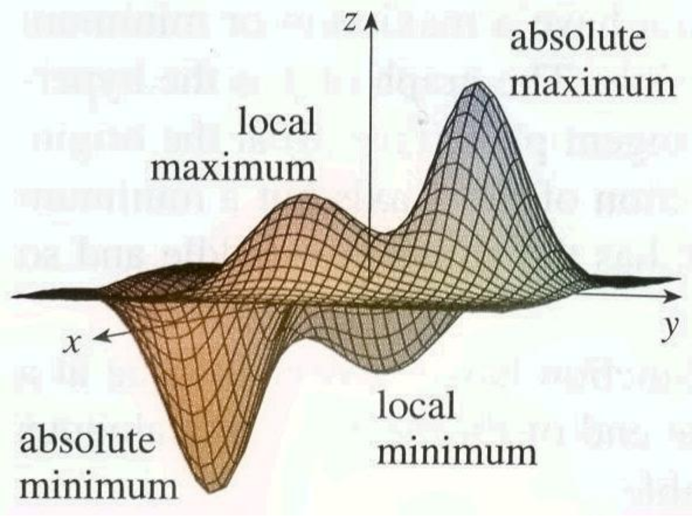
  which simplifies to $3x - 6y + 2z + 18 = 0$.

- Since $\nabla\mathbf{F}(-2, 1, -3) = \langle -1, 2, -\frac{2}{3} \rangle$, the vector equation of the **normal line** is: $\mathbf{L}(t) = \langle -2, 1, -3 \rangle + t\langle -1, 2, -\frac{2}{3} \rangle$.

Maxima and minima of functions

Definition

- A function of two variables has a **local maximum** at (a, b) if $f(x, y) \leq f(a, b)$ when (x, y) is near (a, b). (This means that $f(x, y) \leq f(a, b)$ for all points (x, y) in some disk with center (a, b).)
- The number $f(a, b)$ is called a **local maximum value**.
- If $f(x, y) \leq f(a, b)$ for all $f(x, y)$ in the domain of f, then f has an **absolute maximum** at (a, b).
- If $f(x, y) \geq f(a, b)$ when (x, y) is near (a, b), then $f(a, b)$ is a **local minimum value**.
- If $f(x, y) \geq f(a, b)$ for all (x, y) in the domain of f, then f has an **absolute minimum** at (a, b).



The above picture of the graph of a function $z = f(x, y)$ shows the local and absolute **maxima** and **minima** of the function.

The next theorem explains how to find local maxima and local minima for a function in two variables.

Theorem

If f has a local **maximum** or **minimum** at (a, b) and the first-order partial derivatives of f exist there, then $f_x(a, b) = 0$ and $f_y(a, b) = 0$.

The next theorem explains how to find local maxima and local minima for a function in two variables.

Theorem

If f has a local **maximum** or **minimum** at (a, b) and the first-order partial derivatives of f exist there, then $f_x(a, b) = 0$ and $f_y(a, b) = 0$.

Proof.

- At a local maximum or minimum (a, b) of the function f, the tangent plane to the graph of f is horizontal.
- Hence, $f_x(a, b) = 0$ and $f_y(a, b) = 0$.

□

Definition

A point (a, b) is called a **critical point** of $f(x, y)$ if $f_x(a, b) = f_y(a, b) = 0$.

Definition

A point (a, b) is called a **critical point** of $f(x, y)$ if $f_x(a, b) = f_y(a, b) = 0$.

- The next theorem gives a method for testing critical points of a function $f(x, y)$ to see if they represent local minima, local maxima or saddle points.
- A critical point (a, b) is a **saddle** point if the **Hessian D** defined in the next theorem is negative.

Theorem (Second Derivative Test)

Suppose the second partial derivatives of f are continuous on a disk with center (a, b), and suppose that $f_x(a, b) = 0$ and $f_y(a, b) = 0$ (that is, (a, b) is a **critical point** of f). Let

$$\mathbf{D} = \mathbf{D}(a, b) = f_{xx}(a, b)f_{yy}(a, b) - [f_{xy}(a, b)]^2.$$

- If $\mathbf{D} > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a **local minimum**.
- If $\mathbf{D} > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a **local maximum**.
- If $\mathbf{D} < 0$, then (a, b) is a **saddle point**.

To remember the formula for $\mathbf{D}$ it's helpful to write it as a determinant:

$$\mathbf{D} = \begin{vmatrix} \mathbf{f_{xx}} & \mathbf{f_{xy}} \\ \mathbf{f_{yx}} & \mathbf{f_{yy}} \end{vmatrix} = \mathbf{f_{xx}f_{yy}} - (\mathbf{f_{xy}})^2.$$

Example (Very popular exam problem)

Find the local **maximum** and **minimum** values and **saddle points** of $f(x, y) = x^4 + y^4 - 4xy + 1$.

Example (Very popular exam problem)

Find the local **maximum** and **minimum** values and **saddle points** of $f(x, y) = x^4 + y^4 - 4xy + 1$.

Solution:

- First locate the **critical points** by setting $\nabla f = \langle 0, 0 \rangle$:

$$\nabla f = \langle 4x^3 - 4y, 4y^3 - 4x \rangle = \langle 0, 0 \rangle$$
$$\Longrightarrow x^3 - y = 0 \qquad y^3 - x = 0.$$

- Substituting $y = x^3$ from the first equation into the second one gives

$$0 = x^9 - x = x(x^8 - 1) = x(x^4 - 1)(x^4 + 1)$$
$$= x(x^2 - 1)(x^2 + 1)(x^4 + 1)$$

  with three real roots: $x = 0, 1, -1$.

- So the **critical points** are $(0, 0)$, $(1, 1)$, and $(-1, -1)$.

Example (Very popular exam problem)

Find the local **maximum** and **minimum** values and **saddle points** of $f(x,y) = x^4 + y^4 - 4xy + 1$.

Example (Very popular exam problem)

Find the local **maximum** and **minimum** values and **saddle points** of $f(x, y) = x^4 + y^4 - 4xy + 1$.

Solution:

Recall the **critical points** are $(0, 0)$, $(1, 1)$, and $(-1, -1)$.

- Next calculate the second partial derivatives and $\mathbf{D}(x, y)$:

$$f_{xx} = 12x^2 \qquad f_{xy} = -4 \qquad f_{yy} = 12y^2$$

$$\mathbf{D}(x, y) = f_{xx} f_{yy} - (f_{xy})^2 = 144x^2y^2 - 16.$$

- Since $\mathbf{D}(0, 0) = -16 < 0$, the origin is a **saddle point**.
- Since $\mathbf{D}(1, 1) = 128 > 0$ and $f_{xx}(1, 1) = 12 > 0$, $f(1, 1) = -1$ is a **local minimum**.
- Similarly, $\mathbf{D}(-1, -1) = 128 > 0$ and $f_{xx}(-1, -1) = 12 > 0$, so $f(-1, -1) = -1$ is also a **local minimum**.

In Class Exercises 9

1. Find the equation of the **tangent plane** at the point $(-2, 1, -3)$ to the ellipsoid $\frac{x^2}{4} + y^2 + \frac{z^2}{9} = 3$.
2. Find the equation of the **normal line** at the point $(-2, 1, -3)$ to the ellipsoid $\frac{x^2}{4} + y^2 + \frac{z^2}{9} = 3$.
3. Find the equation of the **tangent plane** at the point $(0, 1, 3)$ to the ellipsoid $\frac{x^2}{4} + y^2 + \frac{z^2}{9} = 2$.
4. (True or False) Given a function $f(x, y)$, let (a, b) be a critical point. Recall the **Hessian**:

$$\mathbf{D} = \begin{vmatrix} \mathbf{f_{xx}} & \mathbf{f_{xy}} \\ \mathbf{f_{yx}} & \mathbf{f_{yy}} \end{vmatrix} = \mathbf{f_{xx}f_{yy}} - (\mathbf{f_{xy}})^2.$$

   (a) If $\mathbf{D} < 0$, then (a, b) is a **saddle point**.
   (b) If $\mathbf{D} > 0$, then $f(a, b)$ is a **local minimum**.
   (c) If $\mathbf{D} < 0$, then $f(a, b)$ is a **local minimum**.

5. Find the critical points of $f(x, y) = x^2 + 2y^2$. What is the **Hessian** $\mathbf{D}$ at this critical point?

Definition

A subset $\mathbf{D} \subset \mathbf{R}^2$ is **closed** if it contains all of its boundary points.

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Example

The closed unit disk $\mathbf{D} = \{x^2 + y^2 \leq 1\}$ is a closed set in $\mathbf{R^2}$.

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Definition

A subset $\mathbf{D} \subset \mathbf{R}^2$ is **bounded** if it is contained within some disk in the plane.

Example

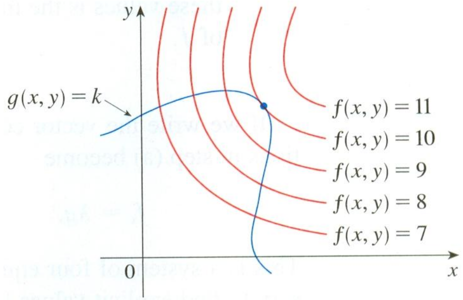
The solid ellipse $\{2x^2 + 3y^2 \leq 1\}$ is a bounded region of $\mathbf{R}^3$. However, the halfspace $\{x > 0\}$ is **not** a bounded region.

Theorem (Extreme Value Theorem for Functions of Two Variables)

If f is continuous on a closed, bounded set **D** in $\mathbf{R}^2$, then f attains an absolute **maximum** value $f(x_1, y_1)$ and an absolute **minimum** value $f(x_2, y_2)$ at some points (x_1, y_1) and (x_2, y_2) in **D**.

To find the absolute **maximum** and **minimum** values of a continuous function f on a closed, bounded set **D**:

1. Find the values of f at the **critical points** of f in **D**.
2. Find the **extreme values** of f on the boundary of **D**.
3. The largest of the values from steps 1 and 2 is the absolute **maximum** value; the smallest of these values is the absolute **minimum** value.



- Consider a function $f(x, y)$ of two variable and suppose that we want to find its **maximum** value subject to the **constraint** $g(x, y) = k$ as described in the figure above.
- At the **maximum**, the level set of f is tangent to the level set $g(x, y) = k$.
- In particular, for some number λ, $\underline{\nabla f = \lambda \nabla g}$ at the **blue** point in $\{g(x, y) = k\} \cap \{f(x, y) = 10\}$.
- Thus, one way to the find the absolute **maximum** is to look for solutions of the above equation of gradients.

The next theorem is stated for a function f of three variables but there is a similar theorem for a function of two variables.

Theorem (Method of Lagrange Multipliers)

To find the **maximum** and **minimum** values of $f(x, y, z)$ subject to the constraint $g(x, y, z) = k$ (assuming that these extreme values exist and $\nabla g \neq \langle 0, 0, 0 \rangle$ on the surface $g(x, y, z) = k$):

1. Find all values of x, y, z, and λ such that
$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$$
and
$$g(x, y, z) = k.$$
2. Evaluate f at all points (x, y, z) that result from step 1.
The largest of these values is the **maximum value** of f; the smallest is the **minimum value** of f.

Example (Very popular exam problem)

Find the extreme values of $f(x, y) = x^2 + 2y^2$ on the circle $x^2 + y^2 = 1$.

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Find the extreme values of $f(x, y) = x^2 + 2y^2$ on the circle $x^2 + y^2 = 1$.

Solution:

First find the extreme values of f subject to the constraint $g(x, y) = x^2 + y^2 = 1$. Using Lagrange multipliers, we solve the equations $\nabla f = \lambda \nabla g$, $g(x, y) = 1$, which can be written as

$$f_x = \lambda g_x \qquad f_y = \lambda g_y \qquad g(x, y) = 1$$

or as

$$2x = 2x\lambda \tag{7}$$

$$4y = 2y\lambda \tag{8}$$

$$x^2 + y^2 = 1. \tag{9}$$

From (7) we have $x = 0$ or $\lambda = 1$. If $x = 0$, then (9) gives $y = \pm 1$. If $\lambda = 1$, then $y = 0$ from (8), so then (9) gives $x = \pm 1$. So f might have extreme values at the points $(0, 1)$, $(0, -1)$, $(1, 0)$ and $(-1, 0)$. Evaluating f at these four points, we find that

$$f(0, 1) = 2 \qquad f(0, -1) = 2 \qquad f(1, 0) = 1 \qquad f(-1, 0) = 1.$$

Therefore, the **maximum value** of f on the circle $x^2 + y^2 = 1$ is $f(0, \pm 1) = 2$ and the **minimum value** is $f(\pm 1, 0) = 1$.

Example (Very popular exam problem)

Find the extreme values of $f(x, y) = x^2 + 2y^2$ on the disk $x^2 + y^2 \leq 1$.

Example (Very popular exam problem)

Find the extreme values of $f(x, y) = x^2 + 2y^2$ on the disk $x^2 + y^2 \leq 1$.

Solution:

- Compare the values of f at the critical points with values at the points on the boundary.
- Since $f_x = 2x$ and $f_y = 4y$, the only **critical point** is $(0, 0)$.
- Compare the value of f at that point with the extreme values on the boundary from the previous example:

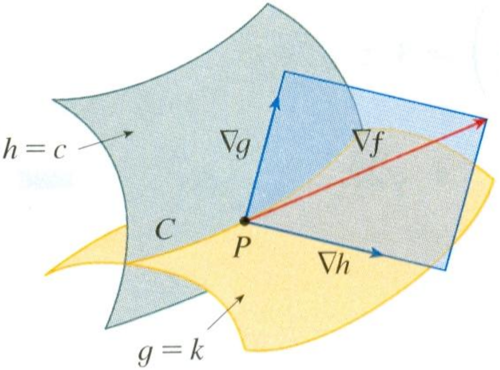
$$f(0,0) = 0 \qquad f(\pm 1, 0) = 1 \qquad f(0, \pm 1) = 2.$$

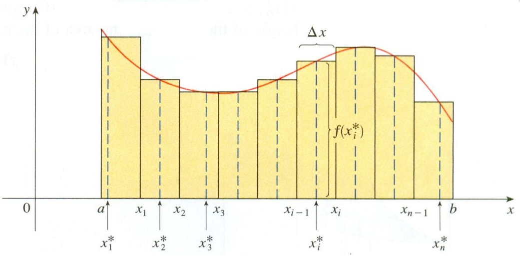
- Therefore, the **maximum** value of f on the disk $x^2 + y^2 \leq 1$ is $f(0, \pm 1) = 2$ and the **minimum** value is $f(0, 0) = 0$.

- The method of Lagrange multipliers can also be applied to the case of finding a **minimum** or **maximum** value for a function $f(x, y, z)$ subject to two constraints $g(x, y, z) = k$ and $h(x, y, z) = c$.
- The equation to solve in this case is:

$$\nabla f = \lambda \nabla g + \mu \nabla h,$$

for some numbers λ and μ. See the figure below.

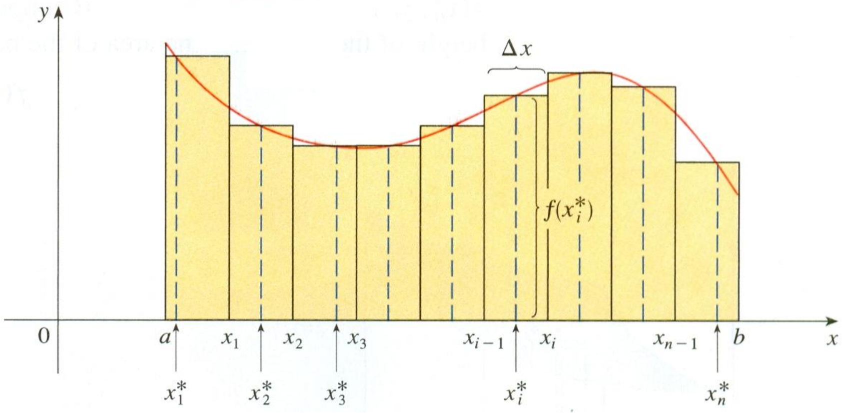


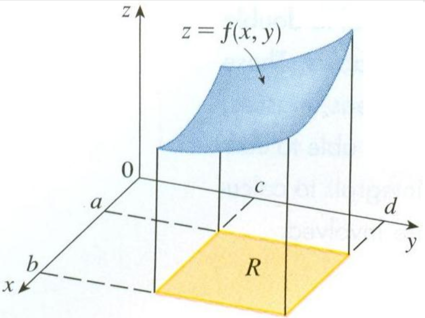


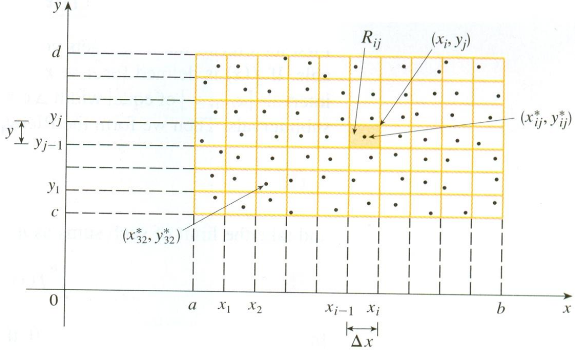
Definition

- For a positive, continuous function $f(x, y)$ defined on a closed and bounded domain $\mathbf{D} \subset \mathbf{R}^2$, we denote by
$$\int\int_{\mathbf{D}} f(x, y)\, dA,$$
the **volume** under the graph of $f(x, y)$ over $\mathbf{D}$.
- The volume of the rectangle $\mathbf{R} = \{(x, y) \mid a \leq x \leq b,\ c \leq y \leq d\} = [a, b] \times [b, c] \subset \mathbf{R}^2$ can be **estimated** by using the one used to estimate the area under the graph of a function $\mathbf{f}(x)$ of a single variable.
- We also use this method for defining the double integral when $f(x, y)$ is not necessarily positive.

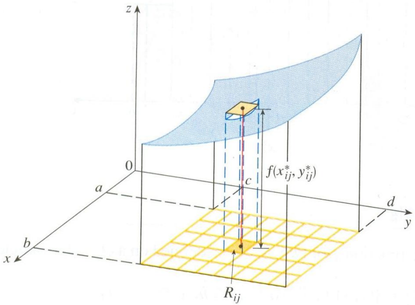
Estimating the area $\int_a^b \mathbf{f}(x)\,dx$ under the graph of $\mathbf{f(x)}$

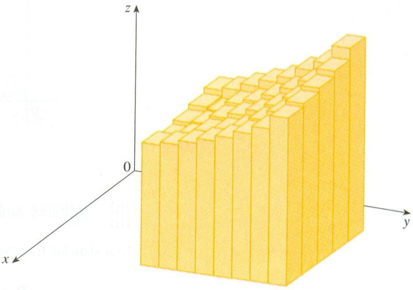



z
$z = f(x, y)$
0
a
c
d
b
x
y
R


y
d
y_j
y
y_{j-1}
y_1
c
R_{ij}
(x_i, y_j)
(x_{ij}^*, y_{ij}^*)
(x_{32}^*, y_{32}^*)
0
a
x_1
x_2
x_{i-1}
x_i
b
x
Δx

Estimating the volume under the graph of f(x,y) over the rectangle $\mathbf{R} = [a, b] \times [c, d]$, as a sum of volumes of solid rectangles.





Definition

The **double integral** of $f(x, y)$ over a rectangle $\mathbf{R}$ is

$$\int\int_{\mathbf{R}} f(x,y)\,dA = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*) \boldsymbol{\Delta} A$$

if this limit exists.
Fact: This limit exists if $f(x, y)$ is a continuous function.

From now on all functions considered will be **continuous**.

Theorem (Midpoint Rule for Double Integrals)

Let m, n be positive integers and
$\mathbf{R} = \{(x, y) \mid a \leq x \leq b,\ c \leq y \leq d\} = [a, b] \times [b, c] \subset \mathbf{R}^2$.

- Let $x_0 = a < x_1 < x_2 < \ldots < x_m = b$ be a division of $[a, b]$ into n intervals $[x_i, x_i + 1]$ of equal width $\mathbf{\Delta} x = \frac{b-a}{m}$.
- Similarly, let $y_0 = c < y_1 < y_2 < \ldots < y_n = d$ be a division of $[c, d]$ into m intervals $[y_j, y_{j+1}]$ of equal widths $\mathbf{\Delta} y = \frac{d-c}{n}$.

Then:

$$\int \int_{\mathbf{R}} f(x, y)\, dA \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(\overline{x_i}, \overline{y_j})\ \mathbf{\Delta} A,$$

where $\overline{x_i}$ is the midpoint of $[x_{i-1}, x_i]$ and $\overline{y_j}$ is the midpoint of $[y_{j-1}, y_j]$.

Example

Use the Midpoint Rule with $m = n = 2$ to estimate the integral $\int\int_{\mathbf{R}} (x - 3y^2)\, dA$, where $\mathbf{R} = \{(x, y) \mid 0 \le x \le 2,\ 1 \le y \le 2\}$.

Example

Use the Midpoint Rule with $m = n = 2$ to estimate the integral $\int\int_{\mathbf{R}}(x - 3y^2)\,dA$, where $\mathbf{R} = \{(x, y) \mid 0 \leq x \leq 2,\ 1 \leq y \leq 2\}$.

Solution:

In using the Midpoint Rule with $m = n = 2$, we evaluate $f(x, y) = x - 3y^2$ at the centers of the four subrectangles. So $\overline{x}_1 = \frac{1}{2}, \overline{x}_2 = \frac{3}{2}, \overline{y}_1 = \frac{5}{4}$, and $\overline{y}_2 = \frac{7}{4}$. The area of each subrectangle is $\boldsymbol{\Delta A} = \frac{1}{2}$. Thus

$$\int\int_{\mathbf{R}}(x - 3y^2)\,dA \approx \sum_{i=1}^{2}\sum_{j=1}^{2} f(\overline{x}_i, \overline{y}_j)\boldsymbol{\Delta A}$$

$$= f(\overline{x}_1, \overline{y}_1)\boldsymbol{\Delta A} + f(\overline{x}_1, \overline{y}_2)\boldsymbol{\Delta A} + f(\overline{x}_2, \overline{y}_1)\boldsymbol{\Delta A} + f(\overline{x}_2, \overline{y}_2)\boldsymbol{\Delta A}$$

$$= f\left(\frac{1}{2}, \frac{5}{4}\right)\boldsymbol{\Delta A} + f\left(\frac{1}{2}, \frac{7}{4}\right)\boldsymbol{\Delta A} + f\left(\frac{3}{2}, \frac{5}{4}\right)\boldsymbol{\Delta A} + f\left(\frac{3}{2}, \frac{7}{4}\right)\boldsymbol{\Delta A}$$

$$= \left(-\frac{67}{16}\right)\frac{1}{2} + \left(-\frac{139}{16}\right)\frac{1}{2} + \left(-\frac{51}{16}\right)\frac{1}{2} + \left(-\frac{123}{16}\right)\frac{1}{2}$$

$$= -\frac{95}{8} = -11.875.$$

Definition

If f is a continuous function of two variables, then its **average value** on a domain $\mathbf{D} \subset \mathbf{R}^2$ is:

$$\frac{\int\int_{\mathbf{D}} f(x,y)\, dA}{\mathbf{Area}(\mathbf{D}) = \int\int_{\mathbf{D}} dA}.$$

Definition

The **iterated integral** of $f(x, y)$ on a rectangle $\mathbf{R} = [a, b] \times [c, d]$ is

$$\int_a^b \int_c^d f(x, y)\, dy\ dx \qquad \textbf{or} \qquad \int_c^d \int_a^b f(x, y)\, dx\, dy.$$

One calculates the integral $\int_a^b \int_c^d f(x, y)\, dy\, dx$ by:

- First calculating $A(x) = \int_c^d f(x, y)\, dy$, holding x constant.
- Next calculating $\int_a^b A(x)\, dx$.
- Similarly, one calculates the other integral

$$\int_c^d \int_a^b f(x, y)\, dx\, dy.$$

.

Example

Evaluate the iterated integral

$$\int_0^3 \int_1^2 x^2 y \; dy \; dx.$$

Example

Evaluate the iterated integral

$$\int_0^3 \int_1^2 x^2 y \, dy \, dx.$$

Solution:

- Regarding x as a constant, we obtain

$$\int_1^2 x^2 y \, dy = \left[x^2 \frac{y^2}{2} \right]_{y=1}^{y=2} = x^2 \left(\frac{2^2}{2} \right) - x^2 \left(\frac{1^2}{2} \right) = \frac{3}{2} x^2.$$

- Thus, $A(x) = \frac{3}{2} x^2$ in this example.

- We now integrate $A(x)$ from 0 to 3:

$$\int_0^3 \int_1^2 x^2 y \, dy \, dx = \int_0^3 \left[\int_1^2 x^2 y \, dy \right] dx$$

$$= \int_0^3 \frac{3}{2} x^2 \, dx = \frac{x^3}{2} \Big|_0^3 = \frac{27}{2}.$$

Example

Evaluate the iterated integral

$$\int_1^2 \int_0^3 x^2 y \, dx \, dy.$$

Example

Evaluate the iterated integral

$$\int_1^2 \int_0^3 x^2 y \, dx \, dy.$$

Solution:

Here we first integrate with respect to x:

$$\int_1^2 \int_0^3 x^2 y \, dx \, dy = \int_1^2 \left[\int_0^3 x^2 y \, dx \right] \, dy$$

$$= \int_1^2 \left[\frac{x^3}{3} y \right]_{x=0}^{x=3} dy = \int_1^2 9y \, dy = 9 \frac{y^2}{2} \bigg|_1^2 = 9 \cdot 2 - \frac{9}{2} = \frac{27}{2}.$$

Changing the order of integration in the previous 2 examples gives the same result. This fact also follows from the next theorem.

Theorem (Fubini's Theorem)

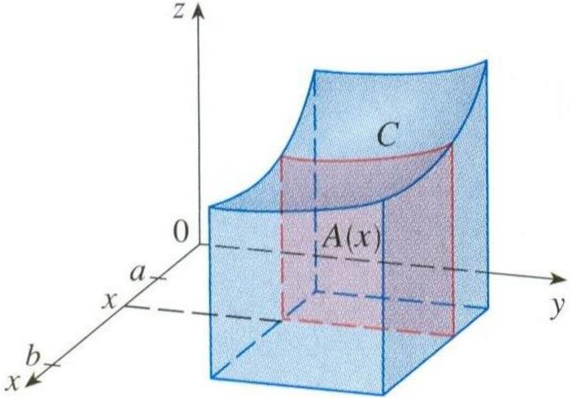
If f is continuous on the rectangle $\mathbf{R} = \{(x, y) \mid a \leq x \leq b,\ c \leq y \leq d\}$, then

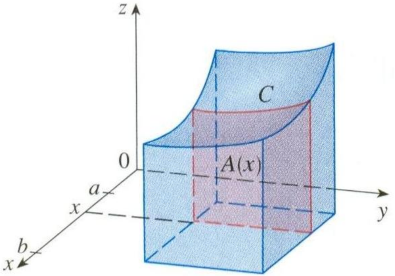
$$\int\int_{\mathbf{R}} f(x, y)\, dA = \int_a^b \int_c^d f(x, y)\, dy\, dx =$$

$$\int_c^d \int_a^b f(x, y)\, dx\, dy.$$

More generally, this is true if we assume that f is bounded on $\mathbf{R}$, f is discontinuous only on a finite number of smooth curves, and the iterated integrals exist.

Remember: Volume $=$ Area $\times$ Distance





Proof of Fubini's Theorem.

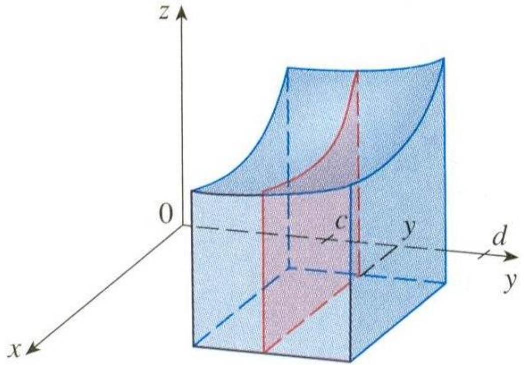
- Assume $f(x, y) \geq 0$. Interpret the double integral $\int\int_R f(x, y)\, dA$ as the volume $\mathbf{V}$ of the solid $\mathbf{S}$ that lies above $\mathbf{R}$ and under the surface $z = f(x, y)$.
- Let $A(x)$ be the area of a cross-section of $\mathbf{S}$ in the plane through x perpendicular to the x-axis. $A(x)$ is the area under the curve C where $z = f(x, y)$, and x is held constant and $c \leq y \leq d$.
- Therefore

$$A(x) = \int_c^d f(x, y)\, dy$$

and we have

$$\int\int_{\mathbf{R}} f(x, y)\, dA = \mathbf{V} = \int_a^b A(x)\, dx = \int_a^b \int_c^d f(x, y) dy\, dx.$$

□



A similar argument, using cross-sections perpendicular to the y-axis as in the above figure, shows that

$$\int\int_{\mathbf{R}} f(x, y)\, dA = \int_c^d A(y)\ dy = \int_c^d \int_a^b f(x, y)\, dx\, dy.$$

Example (popular exam problem)

Evaluate the double integral $\int\int_{\mathbf{R}}(x - 3y^2)\, dA$, where $\mathbf{R} = \{(x, y) \mid 0 \leq x \leq 2,\ 1 \leq y \leq 2\}$.

Example (popular exam problem)

Evaluate the double integral $\int\int_{\mathbf{R}}(x-3y^2)\,dA$, where $\mathbf{R}=\{(x,y)\mid 0\le x\le 2,\ 1\le y\le 2\}$.

Solution:

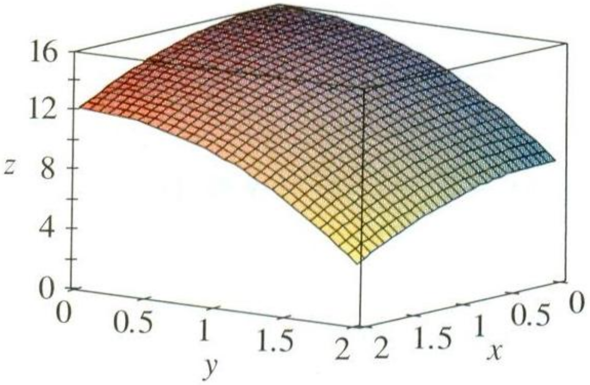
Fubini's Theorem gives

$$\int\int_{\mathbf{R}}(x-3y^2)\,dA=\int_0^2\int_1^2(x-3y^2)\,dy\,dx=\int_0^2\left[xy-y^3\right]_{y=1}^{y=2}\,dx$$

$$=\int_0^2(x-7)\,dx=\frac{x^2}{2}-7x\Big|_0^2=2-14=-12.$$

Example (popular exam problem)

Find the volume of the solid **S** that is bounded by the elliptic paraboloid $x^2 + 2y^2 + z = 16$, the planes $x = 2$ and $y = 2$, and the three coordinate planes.



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Example (popular exam problem)

Find the volume of the solid **S** that is bounded by the elliptic paraboloid $x^2 + 2y^2 + z = 16$, the planes $x = 2$ and $y = 2$, and the three coordinate planes.

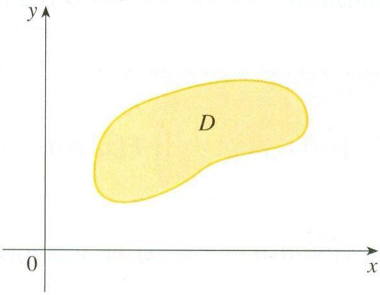
Solution:

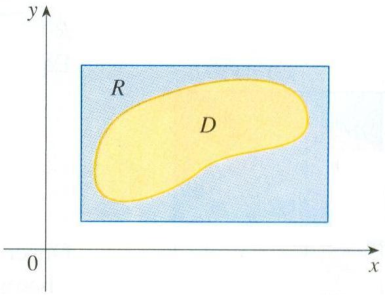
- First observe that **S** is the solid that lies under the surface $z = 16 - x^2 - y^2$ and above the square $\mathbf{R} = [0, 2] \times [0, 2]$.
- Now evaluate the double integral using Fubini's Theorem.

$$\mathbf{V} = \int\int_{\mathbf{R}} (16 - x^2 - 2y^2)\, dA = \int_0^2 \int_0^2 (16 - x^2 - 2y^2)\, dx\, dy$$

$$= \int_0^2 \left[16x - \frac{1}{3}x^3 - 2y^2 x\right]_{x=0}^{x=2} dy$$

$$= \int_0^2 (\frac{88}{3} - 4y^2)\, dy = \frac{88}{3}y - \frac{4}{3}y^3 \Big|_0^2 = 48.$$





Double Integral for arbitrary domains **D**

Define a new function **F** with domain **R** by

$$\mathbf{F}(x,y) = \begin{cases} f(x,y) & \text{if}(x,y) \text{ is in } \mathbf{D} \\ 0 & \text{if } (x,y) \text{ is in } \mathbf{R} \text{ but not in } \mathbf{D} \end{cases}.$$

If the double integral of **F** exists over **R**, then define the **double integral of f over D** by

$$\int\int_{\mathbf{D}} f(x,y)\,dA = \int\int_{\mathbf{R}} \mathbf{F}(x,y)\,dA.$$

Type I and Type II domains

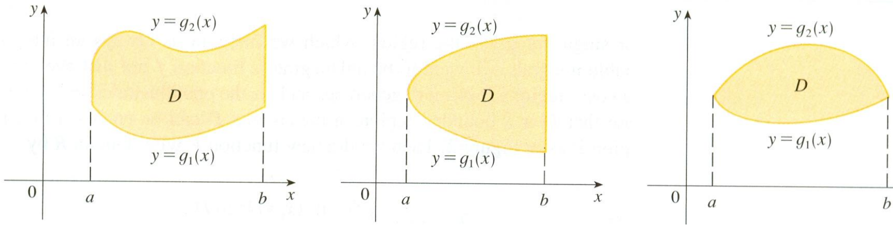
- For any continuous function $f(x, y)$ on a closed and bounded domain $\mathbf{D} \subset \mathbf{R}^2$, the integral $\int\int_{\mathbf{D}} f(x, y)\, dA$ is defined and it is equal to the volume under the graph of $f(x, y)$ on $\mathbf{D}$ when the function is positive.
- **Type I and Type II domains.**
  There are two cases for $\mathbf{D}$, called **type I** and **types II**, where the integral
  $$\int\int_{\mathbf{D}} f(x, y)\, dA$$
  can be calculated in a straightforward manner.
- These regions are described in the next 2 slides.

Definition

A plane region **D** is said to be of **type I**, if it can be expressed as

$$\mathbf{D} = \{(x, y) \mid a \le x \le b, \;\; g_1(x) \le y \le g_2(x)\},$$

where $g_1(x)$ and $g_2(x)$ are continuous.

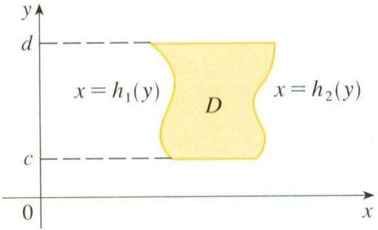


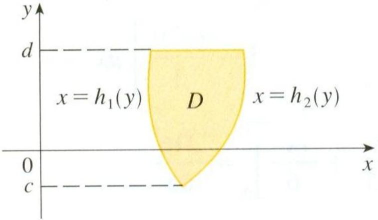
Definition

A plane region **D** is said to be of **type II**, if it can be expressed as

$$\mathbf{D} = \{(x, y) \mid c \le y \le d,\ h_1(y) \le x \le h_2(y)\},$$

where $h_1(y)$ and $h_2(y)$ are continuous.





Theorem

If f is continuous on a **type I** region **D** such that

$$\mathbf{D} = \{(x, y) \mid a \le x \le b, \;\; g_1(x) \le y \le g_2(x)\},$$

then

$$\int\int_{\mathbf{D}} f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx.$$

Theorem

If f is continuous on a **type I** region $\mathbf{D}$ such that

$$\mathbf{D} = \{(x, y) \mid a \leq x \leq b, \;\; g_1(x) \leq y \leq g_2(x)\},$$

then

$$\int\int_{\mathbf{D}} f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx.$$

Proof.

The proof of the above formula follows from the proof of Fubini's Theorem. $\square$

Theorem

If f is continuous on a **type II** region **D** such that

$$\mathbf{D} = \{(x, y) \mid c \leq y \leq d, \;\; h_1(y) \leq x \leq h_2(y)\},$$

then

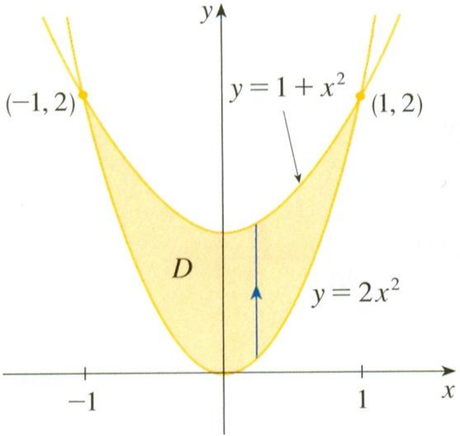
$$\int\int_{\mathbf{D}} f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy.$$

Proof.

The proof of the above formula follows from the proof of Fubini's Theorem. □

Example (popular exam problem)

Evaluate $\int\int_{\mathbf{D}}(x + 2y)\, dA$, where **D** is the region bounded by the parabolas $y = 2x^2$ and $y = 1 + x^2$.



Example (popular exam problem)

Evaluate $\int\int_{\mathbf{D}} (x + 2y)\, dA$, where **D** is the region bounded by the parabolas $y = 2x^2$ and $y = 1 + x^2$.

Example (popular exam problem)

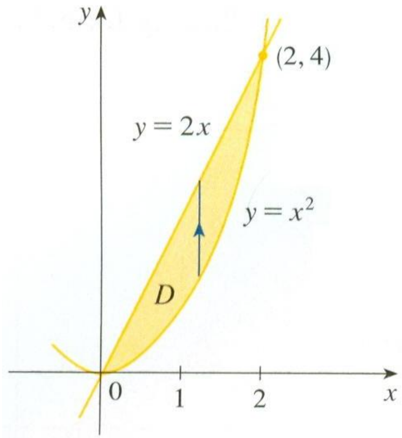
Evaluate $\int\int_{\mathbf{D}}(x+2y)\,dA$, where **D** is the region bounded by the parabolas $y = 2x^2$ and $y = 1 + x^2$.

Solution

- The parabolas intersect when $2x^2 = 1 + x^2$, that is $x^2 = 1$, so $x = \pm 1$.
- **D** is a type I region:
$$\mathbf{D} = \{(x, y) - 1 \le x \le 1,\ \ 2x^2 \le y \le 1 + x^2\}.$$
- Since the lower boundary is $y = 2x^2$ and the upper boundary is $y = 1 + x^2$,
$$\int\int_{\mathbf{D}}(x+2y)\,dA = \int_{-1}^{1}\int_{2x^2}^{1+x^2}(x+2y)\,dy\,dx$$
$$= \int_{-1}^{1}\left[xy + y^2\right]_{y=2x^2}^{y=1+x^2}\,dx = \int_{-1}^{1}(-3x^4 - x^3 + 2x^2 + x + 1)\,dx$$
$$= -3\,\frac{x^5}{5} - \frac{x^4}{4} + 2\frac{x^3}{3} + \frac{x^2}{2} + x\Big|_{-1}^{1} = \frac{32}{15}.$$

Example (popular exam problem)

Find the volume **V** of the solid that lies under the paraboloid $z = x^2 + y^2$ and above the region **D** in the xy-plane bounded by the line $y = 2x$ and the parabola $y = x^2$.



Example (popular exam problem)

Find the volume **V** of the solid that lies under the paraboloid $z = x^2 + y^2$ and above the region **D** in the xy-plane bounded by the line $y = 2x$ and the parabola $y = x^2$.

Solution 1:

We see that **D** is a type I region and

$$\mathbf{D} = \{(x, y) \mid 0 \leq x \leq 2, \;\; x^2 \leq y \leq 2x\}.$$

Therefore, the volume **V** under $z = x^2 + y^2$ and above **D** is

$$\mathbf{V} = \int\int_{\mathbf{D}} (x^2 + y^2)\, dA = \int_0^2 \int_{x^2}^{2x} (x^2 + y^2)\, dy\, dx$$

$$= \int_0^2 \left[x^2 y + \frac{y^3}{3} \right]_{y=x^2}^{y=2x} dx = \int_0^2 \left[x^2(2x) + \frac{(2x)^3}{3} - x^2 x^2 - \frac{(x^2)^3}{3} \right] dx$$

$$= \int_0^2 \left(-\frac{x^6}{3} - x^4 + \frac{14x^3}{3} \right) dx = -\frac{x^7}{21} - \frac{x^5}{5} + \frac{7x^4}{6} \bigg|_0^2 = \frac{216}{35}.$$

In Class Exercises 10

1. Evaluate the iterated integral

$$\int_0^3 \int_0^1 x^2 y \, dy \, dx.$$

2. Evaluate the double integral $\int\int_{\mathbf{R}} (x - 3y^2) \, dA$, where $\mathbf{R} = \{(x, y) \mid 0 \le x \le 2, \; 1 \le y \le 2\}$.

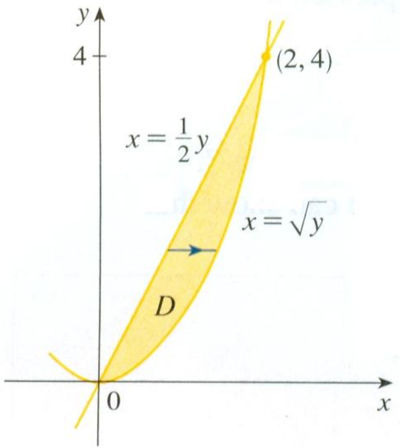
3. Find the volume of the solid **S** that is bounded by the elliptic paraboloid $x^2 + 2y^2 + z = 16$, the planes $x = 2$ and $y = 2$, and the three coordinate planes.

4. Evaluate $\int\int_{\mathbf{D}} (x + 2y) \, dA$, where **D** is the region bounded by the parabolas $y = 2x^2$ and $y = 1 + x^2$.

5. Find the volume **V** of the solid that lies under the paraboloid $z = x^2 + y^2$ and above the region **D** in the xy-plane bounded by the line $y = 1$ and the parabola $y = x^2$.

Example (popular exam problem)

Find the volume **V** of the solid that lies under the paraboloid $z = x^2 + y^2$ and above the region **D** in the xy-plane bounded by the line $y = 2x$ and the parabola $y = x^2$.



Example (popular exam problem)

Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$ and above the region **D** in the xy-plane bounded by the line $y = 2x$ and the parabola $y = x^2$.

Solution 2:

- We see that **D** can also be written as a type II region:

$$\mathbf{D} = \{(x, y) \mid 0 \le y \le 4, \ \ \frac{1}{2}y \le x \le \sqrt{y}\}.$$

- Therefore, another expression for **V** is

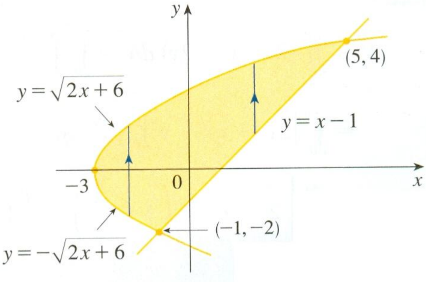
$$\mathbf{V} = \int\int_{\mathbf{D}} (x^2 + y^2)\, dA = \int_0^4 \int_{\frac{1}{2}y}^{\sqrt{y}} (x^2 + y^2)\, dx\, dy$$

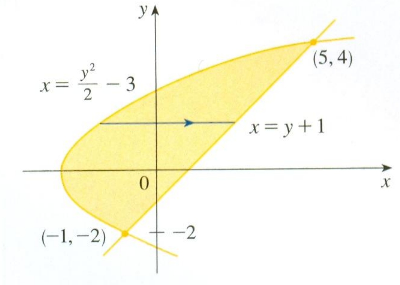
$$= \int_0^4 \left[\frac{x^3}{3} + y^2 x\right]_{x=\frac{1}{2}y}^{x=\sqrt{y}} dy = \int_0^4 \left(\frac{y^{\frac{3}{2}}}{3} + y^{\frac{5}{2}} - \frac{y^3}{24} - \frac{y^3}{2}\right) dy$$

$$= \frac{2}{15}y^{\frac{5}{2}} + \frac{2}{7}y^{\frac{7}{2}} - \frac{13}{96}y^4\Big|_0^4 = \frac{216}{35}.$$

Example (popular exam problem)

Evaluate $\int\int_{\mathbf{D}} xy\, dA$, where **D** is the region bounded by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$.





Example (popular exam problem)

Evaluate $\int\int_{\mathbf{D}} xy\, dA$, where **D** is the region bounded by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$.

Solution:

$$\mathbf{D} = \{(x, y) \mid -2 \le y \le 4, \quad \frac{1}{2}y^2 - 3 \le x \le y + 1\}.$$

Thus,

$$\int\int_{\mathbf{D}} xy\, dA = \int_{-2}^{4}\int_{\frac{1}{2}y^2-3}^{y+1} xy\, dx\, dy = \int_{-2}^{4}\left[\frac{x^2}{2}y\right]_{x=\frac{1}{2}y^2-3}^{x=y+1} dy$$

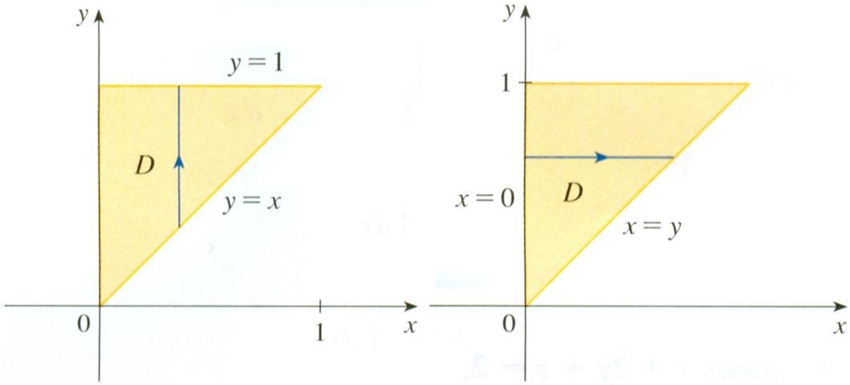
$$= \frac{1}{2}\int_{-2}^{4} y\left[(y+1)^2 - (\frac{1}{2}y^2 - 3)^2\right] dy$$

$$= \frac{1}{2}\int_{-2}^{4}\left(-\frac{y^5}{4} + 4y^3 + 2y^2 - 8y\right) dy$$

$$= \frac{1}{2}\left[-\frac{y^6}{24} + y^4 + 2\frac{y^3}{3} - 4y^2\right]_{-2}^{4} = 36.$$

Example (popular exam problem)

Evaluate the iterated integral $\int_0^1 \int_x^1 \sin(y^2)\, dy\, dx$.



Example (popular exam problem)

Evaluate the iterated integral $\int_0^1 \int_x^1 \sin(y^2)\, dy\, dx$.

Example (popular exam problem)

Evaluate the iterated integral $\int_0^1 \int_x^1 \sin(y^2)\; dy\; dx$.

Solution:

We must **change the order of integration**. First express the given iterated integral as a double integral.

$$\int_0^1 \int_x^1 \sin(y^2)\; dy\; dx = \int\int_{\mathbf{D}} \sin(y^2)\; dA,$$

where
$$\mathbf{D} = \{(x, 0 \mid 0 \le x \le 1,\;\; x \le y \le 1)\}.$$

An alternative description of $\mathbf{D}$ is
$$\mathbf{D} = \{(x, y) \mid 0 \le y \le 1,\;\; 0 \le x \le y\}.$$

Express the double integral as an iterated integral in reverse order:

$$\int_0^1 \int_x^1 \sin(y^2)\; dy\; dx = \int\int_{\mathbf{D}} \sin(y^2)\; dA = \int_0^1 \int_0^y \sin(y^2)\; dx\; dy$$

$$= \int_0^1 \left[x \sin(y^2)\right]_{x=0}^{x=y}\; dy = \int_0^1 y \sin(y^2)\; dy = -\frac{1}{2}\cos(y^2)\Big|_0^1 = \frac{1}{2} - \frac{\cos 1}{2}.$$

In Class Exercises 11

1. Find the volume **V** of the solid that lies under the paraboloid $z = x^2 + y^2$ and above the region **D** in the xy-plane bounded by the line $y = x$ and the parabola $y = x^2$.

2. Consider the iterated integral $\mathbf{I} = \int_0^1 \int_{2x}^2 \sin(y^2)\, dy\, dx$.

   (a) Sketch the region of integration.
   (b) Write the integral **I** with the order of integration reversed.

3. Consider the double integral $\mathbf{I} = \int\int_{\mathbf{R}} x^2 y - x \, dA$ where **R** is the first quadrant region enclosed by the curves $y = 0$, $y = x^2$ and $y = 2 - x$.

   (a) Sketch the region of integration.
   (b) Express the integral **I** as an iterated integral.

4. Find an equivalent iterated integral expression for the double integral $\int_0^1 \int_x^1 \sin(y^2)\, dy\, dx$, where the order of integration is reversed.

Polar Coordinates $(\mathbf{r}, \theta)$ in the plane are described by

$$\mathbf{r} = \quad \textbf{distance} \quad \textbf{from} \quad \textbf{the} \quad \textbf{origin}$$

and

$$\boldsymbol{\theta} \in [\mathbf{0}, \mathbf{2}\boldsymbol{\pi}) \quad \textbf{is} \quad \textbf{the} \quad \textbf{counter-clockwise} \quad \textbf{angle}.$$

We make the convention

$$(-\mathbf{r}, \theta) = (\mathbf{r}, \theta + \pi).$$

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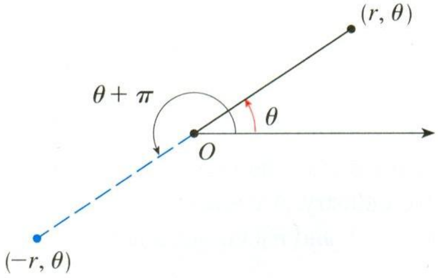
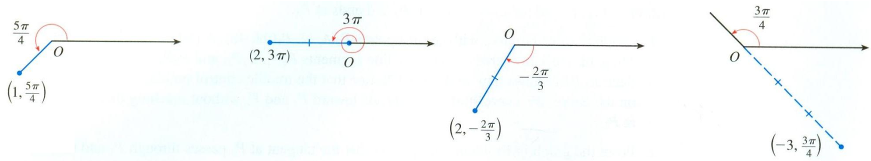


Example

Plot the points whose **polar** coordinates are given.

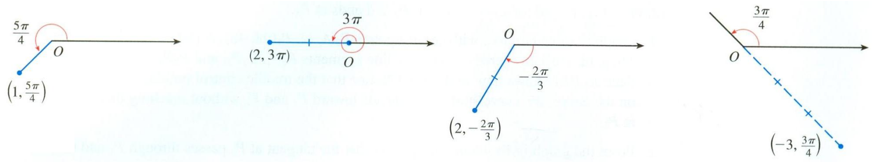
$$\text{(a) } \left(1, 5\frac{\pi}{4}\right) \quad \text{(b) } (2, 3\pi) \quad \text{(c) } \left(2, -2\frac{\pi}{3}\right) \quad \text{(d) } \left(-3, 3\frac{\pi}{4}\right).$$



Example

Plot the points whose **polar** coordinates are given.

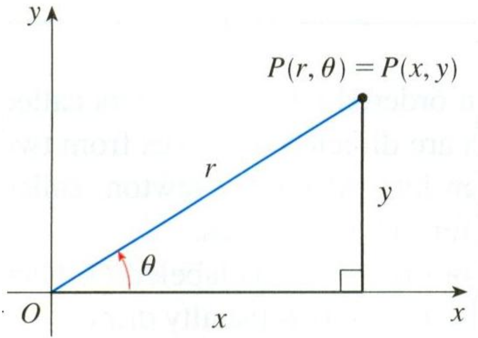
$$\text{(a)}\ \left(1, 5\frac{\pi}{4}\right) \quad \text{(b)}\ (2, 3\pi) \quad \text{(c)}\ \left(2, -2\frac{\pi}{3}\right) \quad \text{(d)}\ \left(-3, 3\frac{\pi}{4}\right).$$



Solution:

In part (d) the point $\left(-3, 3\frac{\pi}{4}\right)$ is located three units from the pole in the fourth quadrant because the angle $3\frac{\pi}{4}$ is in the second quadrant and $r = -3$ is negative.

Coordinate conversion - Polar/Cartesian



$$\mathbf{x} = \mathbf{r}\cos\theta \qquad \mathbf{y} = \mathbf{r}\sin\theta$$

$$\mathbf{r}^2 = \mathbf{x}^2 + \mathbf{y}^2 \qquad \tan\theta = \frac{\mathbf{y}}{\mathbf{x}}$$

Example

Convert the point $\left(2, \frac{\pi}{3}\right)$ from **polar** to **Cartesian** coordinates.

Example

Convert the point $\left(2, \frac{\pi}{3}\right)$ from **polar** to **Cartesian** coordinates.

Solution:

Since $r = 2$ and $\theta = \frac{\pi}{3}$,

$$x = r\cos\theta = 2\cos\frac{\pi}{3} = 2 \cdot \frac{1}{2} = 1$$

$$y = r\sin\theta = 2\sin\frac{\pi}{3} = 2 \cdot \frac{\sqrt{3}}{2} = \sqrt{3}$$

Therefore, the point is $(1, \sqrt{3})$ in **Cartesian** coordinates.

Example

Represent the point with **Cartesian** coordinates $(1, -1)$ in terms of **polar** coordinates.

Example

Represent the point with **Cartesian** coordinates $(1, -1)$ in terms of **polar** coordinates.

Solution:

- If we choose r to be positive, then

$$r = \sqrt{x^2 + y^2} = \sqrt{1^2 + (-1)^2} = \sqrt{2}$$
$$\tan\theta = \frac{y}{x} = -1.$$

- Since the point $(1, -1)$ lies in the fourth quadrant, we choose $\theta = -\frac{\pi}{4}$ or $\theta = 7\,\frac{\pi}{4}$.
- Thus, one possible answer is $\left(\sqrt{2}, -\frac{\pi}{4}\right)$; another is $\left(\sqrt{2}, 7\,\frac{\pi}{4}\right)$.

Polar Coordinates

- The coordinates of a point $(x, y) \in \mathbf{R}^{3}$ can be described by the equations:
$$\mathbf{x} = \mathbf{r}\cos(\theta) \qquad \mathbf{y} = \mathbf{r}\sin(\theta), \tag{10}$$
where $r = \sqrt{x^2 + y^2}$ is the distance from the origin and $(\frac{x}{r}, \frac{y}{r})$ is $(\cos(\theta), \sin(\theta))$ on the unit circle.
- Note that $r \geq 0$ and θ can be taken to lie in the interval $[0, 2\pi)$.
- To find r and θ when x and y are known, we use the equations:
$$\mathbf{r}^2 = \mathbf{x}^2 + \mathbf{y}^2 \qquad \tan(\theta) = \frac{\mathbf{y}}{\mathbf{x}}. \tag{11}$$

Graph of a polar equation

Definition

The **graph of a polar equation** $r = f(\theta)$, or more generally $\mathbf{F}(r, \theta) = 0$, consists of all points P that have at least one polar representation (r, θ) whose coordinates satisfy the equation.

Example

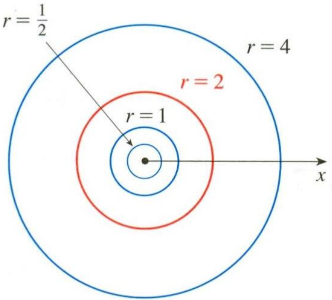
What curve is represented by the **polar equation** $\mathbf{r = 2}$?

Example

What curve is represented by the **polar equation $\mathbf{r = 2}$**?

Solution:

The curve consists of all points $(\mathbf{r}, \theta)$ with $r = 2$. Since r represents the distance from the point to the origin O, the curve $r = 2$ represents the circle with center O and radius 2. In general, the equation $r = a$ represents a circle with center O and radius $|a|$.



Example

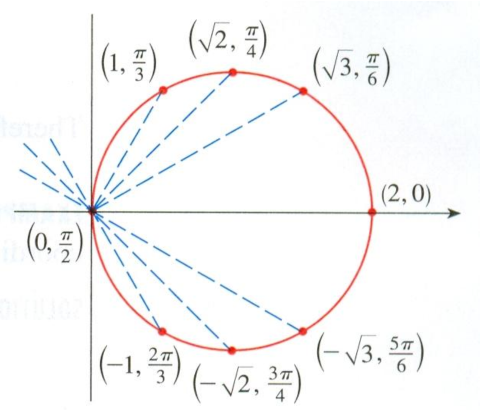
Sketch the curve with **polar equation** $\mathbf{r = 2}\cos\theta$.

Example

Sketch the curve with **polar equation** $\mathbf{r = 2}\cos\theta$.

Solution:

Plotting points we find what seems to be a circle:



Example

Find the **Cartesian** coordinates for $r = 2\cos\theta$.

Example

Find the **Cartesian** coordinates for $r = 2\cos\theta$.

Solution

Since $x = r\cos\theta$, the equation $r = 2\cos\theta$ becomes $r = \frac{2x}{r}$ or

$$2x = r^2 = x^2 + y^2$$

or

$$x^2 - 2x + y^2 = 0$$

or

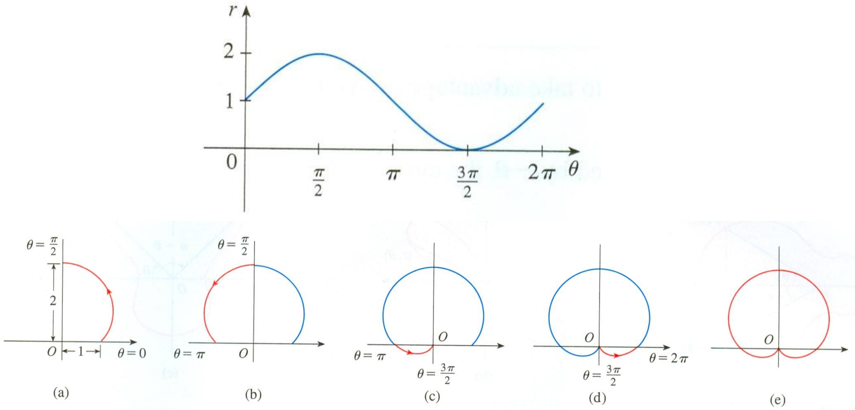
$$(x-1)^2 + y^2 = 1.$$

This is the equation of a **circle** of radius 1 centered at $(1, 0)$.

Example

Sketch the curve $r = 1 + \sin\theta$. This curve is called a **cardioid**.

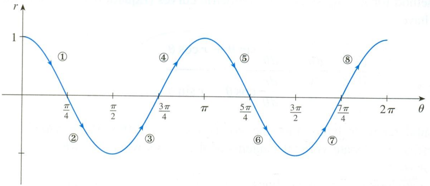
Solution

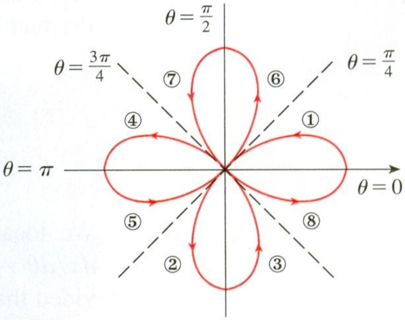


Example

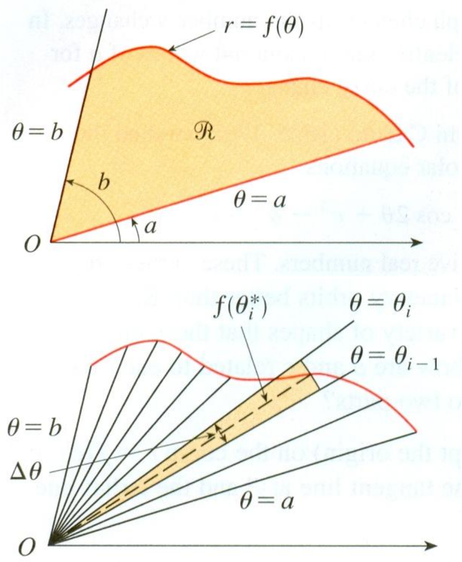
Sketch the curve $r = \cos 2\theta$. This curve is called a **four-leaved rose**.

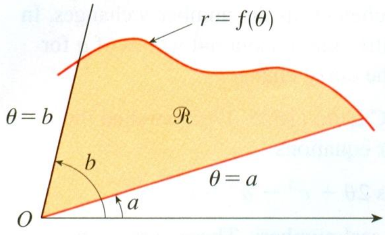
Solution

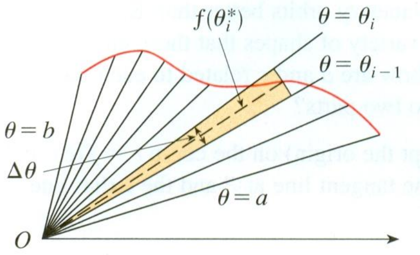




Area under a polar graph $\mathbf{r} = \mathbf{f}(\theta)$







The **area** of a region "under" a polar function $\mathbf{r} = \mathbf{f}(\theta)$ is described by either of the following formulas. These formulas arise from the fact that the area of a $\theta_1 \leq \theta \leq \theta_2$ portion of a circle of radius r is given by $\frac{1}{2}(\theta_2 - \theta_1)r^2$.

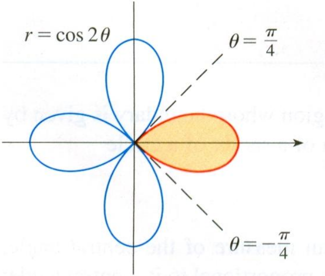
$$\boxed{\mathbf{A} = \int_{\mathbf{a}}^{\mathbf{b}} \frac{1}{2}[\mathbf{f}(\theta)]^2 \, \mathbf{d}\theta,}$$

$$\boxed{\mathbf{A} = \int_{\mathbf{a}}^{\mathbf{b}} \frac{1}{2}\mathbf{r}^2 \, \mathbf{d}\theta,}$$

Example

Find the **area** enclosed by one loop of the four-leaved rose $r = \cos 2\theta$.

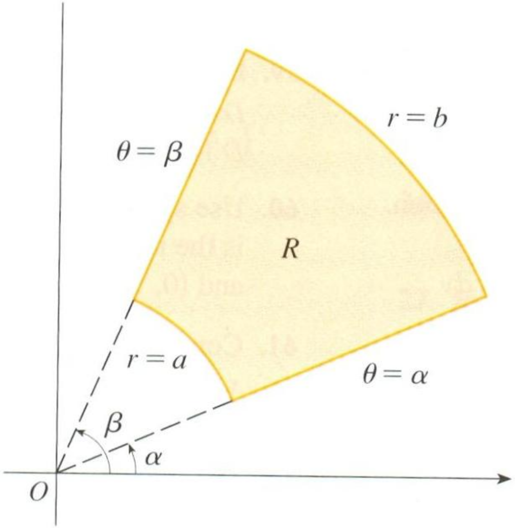
Solution First recall the picture of this curve:



By our **area formulas**,

$$\textbf{Area} = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{2} r^2 \, d\theta = \frac{1}{2} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos^2 2\theta \, d\theta = \frac{1}{2} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos^2 2\theta \, d\theta$$

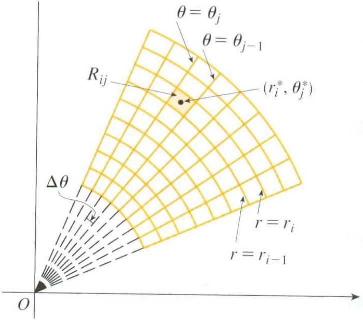
$$= \frac{1}{2} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{2}(1 + \cos 4\theta) \, d\theta = \frac{1}{4}(\theta + \frac{1}{4} \sin 4\theta)\Big|_{-\frac{\pi}{4}}^{\frac{\pi}{4}} = \frac{\pi}{8}.$$

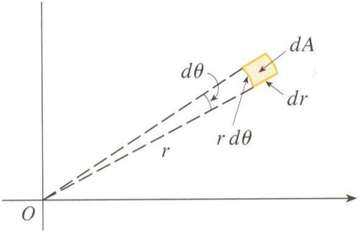


A picture of a polar rectangle $\mathbf{R} = [a, b] \times [\alpha, \beta]$

The next theorem describes how to calculate the integral of a function $f(x, y)$ over a polar rectangle $\mathbf{R} = [a, b] \times [\alpha, \beta]$. Note that

$$dA = r\,dr\,d\theta.$$





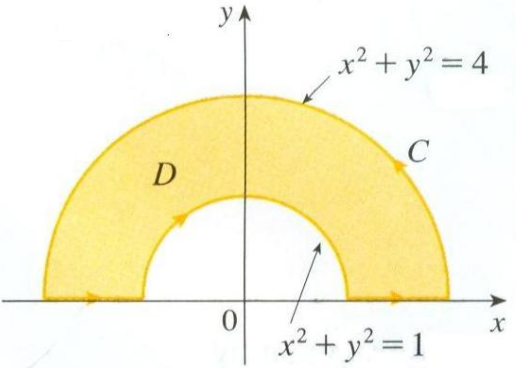
Theorem (Change to Polar Coordinates in a Double Integral)

If f is continuous on a polar rectangle $\mathbf{R}$ given by $0 \leq a \leq r \leq b$, $\alpha \leq \theta \leq \beta$, where $0 \leq \beta - \alpha \leq 2\pi$, then

$$\int\int_{\mathbf{R}} f(x, y)dA = \int_{\alpha}^{\beta}\int_{a}^{b} f(r\cos(\theta), r\sin(\theta))r\,dr\,d\theta.$$

Example (popular exam problem)

Evaluate $\int\int_{\mathbf{R}}(3x + 4y^2)\, dA$, where **R** is the region in the upper half-plane bounded by the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.



Example (popular exam problem)

Evaluate $\int\int_{\mathbf{R}}(3x+4y^2)\,dA$, where **R** is the region in the upper half-plane bounded by the circles $x^2+y^2=1$ and $x^2+y^2=4$.

Example (popular exam problem)

Evaluate $\int\int_{\mathbf{R}}(3x+4y^2)\,dA$, where **R** is the region in the upper half-plane bounded by the circles $x^2+y^2=1$ and $x^2+y^2=4$.

Solution:

The region **R** can be described as

$$\mathbf{R}=\{(x,y)\mid y\geq 0,\ 1\leq x^2+y^2\leq 4\}.$$

It is the half-ring given by $1\leq r\leq 2$, $0\leq\theta\leq\pi$. Therefore,

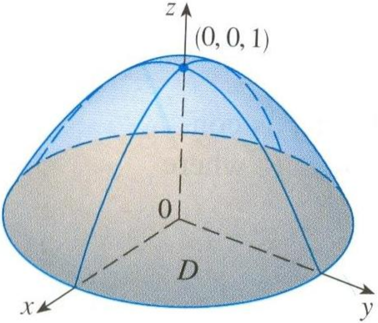
$$\int\int_{\mathbf{R}}(3x+4y^2)dA=\int_0^{\pi}\int_1^2(3r\cos(\theta)+4r^2\sin^2(\theta))r\;dr\;d\theta$$

$$=\int_0^{\pi}\int_1^2(3r^2\cos(\theta)+4r^3\sin^2(\theta))\;dr\;d\theta$$

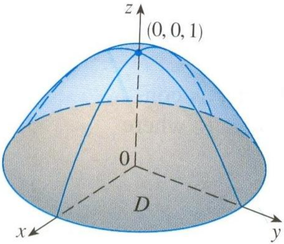
$$=\int_0^{\pi}[r^3\cos(\theta)+r^4 sin^2(\theta)]_{r=1}^{r=2}\;d\theta=\int_0^{\pi}(7\cos(\theta)+15\sin^2(\theta))\;d\theta$$

$$=\int_0^{\pi}[7\cos(\theta)+\frac{15}{2}(1-\cos(2\theta))]\;d\theta$$

$$=7\sin(\theta)+\frac{15\theta}{2}-\frac{15}{4}\sin(2\theta)\Big|_0^{\pi}=\frac{15\pi}{2}.$$

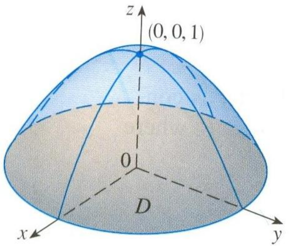


The solid bounded by the plane $z = 0$ and the paraboloid $z = 1 - x^2 - y^2$.



Example (popular exam problem)

Find the volume of the solid bounded by the plane $z = 0$ and the paraboloid $z = 1 - x^2 - y^2$.

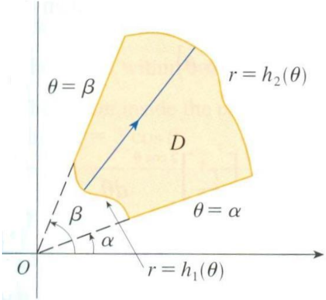


Solution:

- For $z = 0$, we get $x^2 + y^2 = 1$. This means that the plane intersects the paraboloid in the circle $x^2 + y^2 = 1$.
- So the solid lies under the paraboloid and above the circular disk **D** given by $x^2 + y^2 \le 1$.
- In polar coordinates **D** is given by $0 \le r \le 1,\ 0 \le \theta \le 2\pi$. Since $1 - x^2 - y^2 = 1 - r^2$, the volume is:

$$\mathbf{V} = \int\int_{\mathbf{D}} (1 - x^2 - y^2)dA = \int_0^{2\pi} \int_0^1 (1 - r^2)r\ dr\ d\theta$$

$$= \int_0^{2\pi} \int_0^1 (r - r^3)\ dr\ d\theta = 2\pi \left[\frac{r^2}{2} - \frac{r^4}{4}\right]_0^1 = \frac{\pi}{2}.$$



The next theorem extends our previous application of Fubini's theorem from Chapter 15.3 for type II regions.

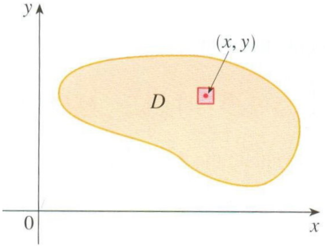
Theorem

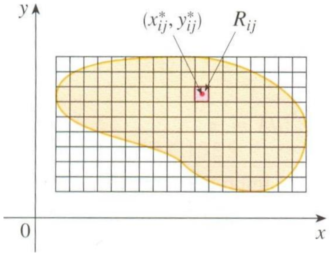
If f continuous on a polar region of the form

$$\mathbf{D} = \{(r, \theta) \mid \alpha \leq \theta \leq \beta, \ h_1(\theta) \leq r \leq h_2(\theta)\},$$

then

$$\int\int_{\mathbf{D}} f(x, y) dA = \int_{\alpha}^{\beta} \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos(\theta), r\sin(\theta)) r \, dr \, d\theta.$$



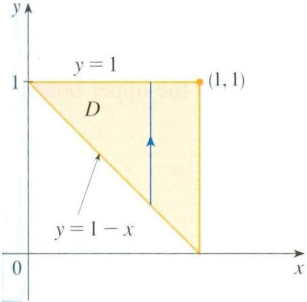


Definition

- Suppose that a lamina occupies a region **D** of the $xy-$plane and its **density** (in units of mass per unit area) is given by a density function $\rho(x, y)$.
- Then the **total mass m** is:

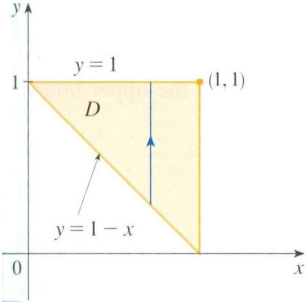
$$\mathbf{m} = \lim_{k,l\to\infty} \sum_{i=1}^{k} \sum_{j=1}^{l} \rho(x_{ij}^*, y_{ij}^*) \boldsymbol{\Delta} A = \int\int_{\mathbf{D}} \rho(x, y)\, dA.$$

- One has a similar formula for the total charge **Q** if $\rho(x, y)$ represents the charge density instead of the mass density.



Example

Charge is distributed over the triangular region **D** in the above figure with charge density at (x, y) is $\sigma(x, y) = xy$, measured in coulombs per square meter $(\frac{C}{m^2})$. Find the total charge **Q**.



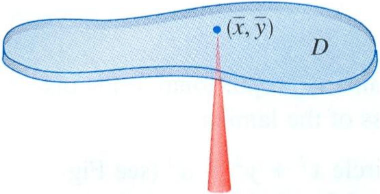
Example

Charge is distributed over the triangular region **D** in the above figure with charge density at (x, y) is $\sigma(x, y) = xy$, measured in coulombs per square meter $(\frac{C}{m^2})$. Find the total charge **Q**.

Solution:

$$\mathbf{Q} = \int\int_{\mathbf{D}} \sigma(x, y) dA = \int_0^1 \int_{1-x}^1 xy \, dy \, dx = \int_0^1 \left[x \frac{y^2}{2} \right]_{y=1-x}^{y=1} dx$$

$$= \int_0^1 \frac{x}{2}[1^2 - (1-x)^2] \, dx = \frac{1}{2} \int_0^1 (2x^2 - x^3) \, dx = \frac{1}{2} \left[\frac{2x^3}{3} - \frac{x^4}{4} \right]_0^1 = \frac{5}{24}.$$



Definition

Suppose that a lamina occupies a region **D** and has density function $\rho(x, y)$.

- The **moment** of the lamina **about the** x**-axis** is:

$$M_x = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} y_{ij}^{*} \rho(x_{ij}^{*}, y_{ij}^{*}) \boldsymbol{\Delta} A = \int\int_{\mathbf{D}} y\rho(x, y)\, dA$$

- The **moment about the** y**-axis** is:

$$M_y = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij}^{*} \rho(x_{ij}^{*}, y_{ij}^{*}) \boldsymbol{\Delta} A = \int\int_{\mathbf{D}} x\rho(x, y)\, dA.$$

Definition

The coordinates $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$ of the **center of mass** of a lamina occupying the region $\mathbf{D}$ and having density function $\rho(x, y)$ are

$$\bar{\mathbf{x}} = \frac{M_y}{\mathbf{m}} = \frac{1}{\mathbf{m}} \int\int_{\mathbf{D}} x\rho(x, y)\, dA \qquad \bar{\mathbf{y}} = \frac{M_x}{\mathbf{m}} = \frac{1}{\mathbf{m}} \int\int_{\mathbf{D}} y\, \rho(x, y)\, dA$$

where the mass $\mathbf{m}$ is given by

$$\mathbf{m} = \int\int_{\mathbf{D}} \rho(x, y)\, dA$$

The next definition describes the notion of a **vector field**.

Definition

Let $\mathbf{D}$ be a set in $\mathbf{R}^2$ (a plane region). A **vector field on $\mathbf{R}^2$** is a function $\mathbf{F}$ that assigns to each point (x, y) in $\mathbf{D}$ a two-dimensional vector $\mathbf{F}(x, y)$.

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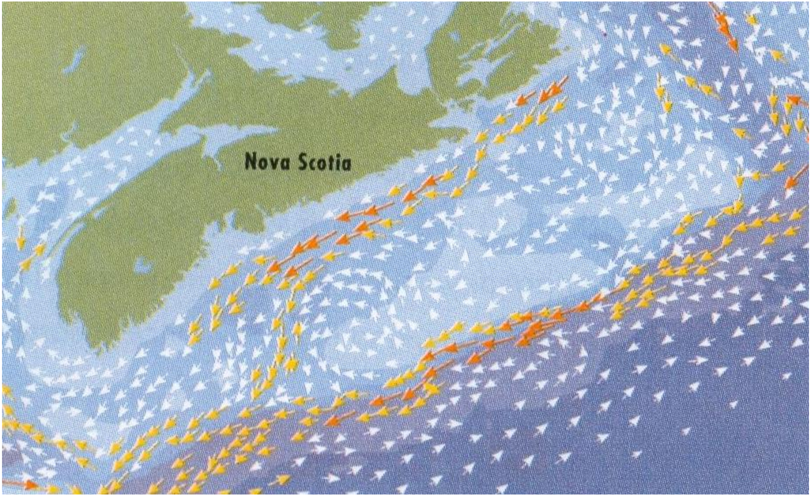
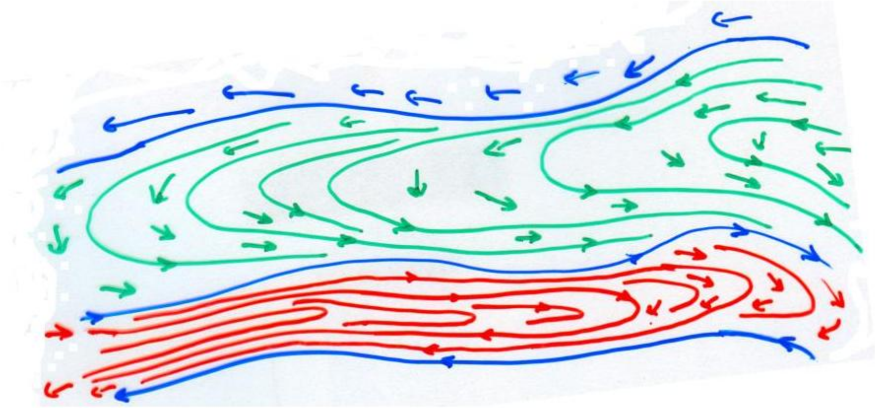
Example

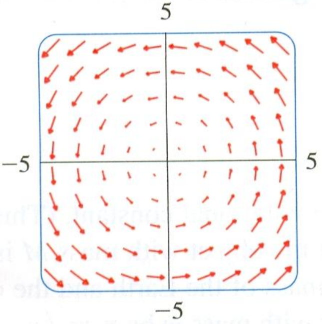
- We have already seen an example of a vector field associated to a function $f(x, y)$ defined on a domain $\mathbf{D} \subset \mathbf{R}^2$, namely the gradient vector field $\nabla f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle$.
- In nature and in physics, we have the familiar examples of the velocity vector field in weather and force vector fields that arise in gravitational fields, electric and magnetic fields.



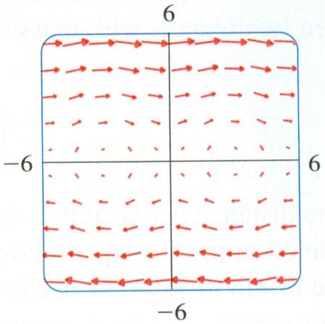
Ocean currents off Nova Scotia

A vector field in the plane together with its integral curves.

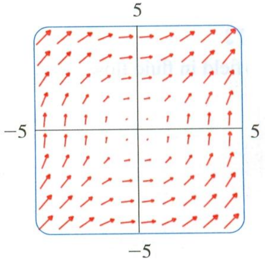




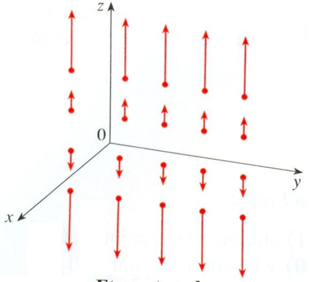
$\mathbf{F}(x, y) = \langle -y, x \rangle$



$\mathbf{F}(x, y) = \langle y, \sin x \rangle$



$\mathbf{F}(x, y) = \langle \ln(1 + y^2), \ln(1 + x^2) \rangle$



$\mathbf{F}(x, y, z) = z\,\mathbf{k}$

Definition

- Let $\mathbf{E}$ be a subset of $\mathbf{R^3}$.
- A **vector field on $\mathbf{R^3}$** is a function $\mathbf{F}$ that assigns to each point (x, y, z) in $\mathbf{E}$ a three-dimensional vector $\mathbf{F}(x, y, z)$.

Definition

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- A **vector field on $\mathbf{R}^3$** is a function $\mathbf{F}$ that assigns to each point (x, y, z) in $\mathbf{E}$ a three-dimensional vector $\mathbf{F}(x, y, z)$.

- Note that a vector field $\mathbf{F}$ on $\mathbf{R}^3$ can be expressed by its component functions.
- So if $\mathbf{F} = \langle P, Q, R \rangle$, then:

$$\mathbf{F}(x, y, z) = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}.$$

- We now describe our first kind of "line integral".
- These type of integrals arise from integrating a function along a curve $\mathbf{C}$ in the plane or in $\mathbf{R}^3$.
- The types of line integral described in the next definition is called a **line integral with respect to arc length**.

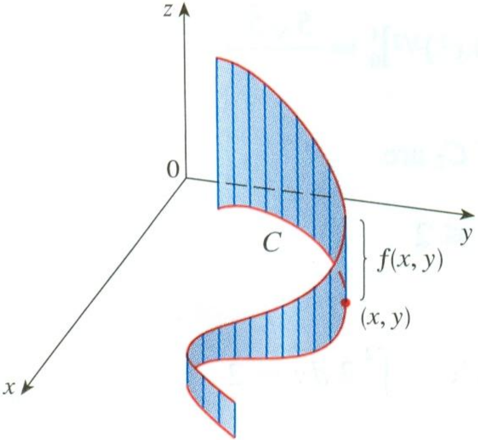
- We now describe our first kind of "line integral".
- These type of integrals arise from integrating a function along a curve **C** in the plane or in $\mathbf{R}^3$.
- The types of line integral described in the next definition is called a **line integral with respect to arc length**.

Definition

Let C be a smooth curve in $\mathbf{R}^2$. Given n, consider n equal subdivisions of lengths $\boldsymbol{\Delta}_i s$; let (x_i^*, y_i^*) denote the midpoints of the i-th subdivision. If f is a real valued function defined on **C**, then the **line integral of** f **along C** is

$$\int_{\mathbf{C}} f(x, y)\, ds = \lim_{n \to \infty} \sum_{j=1}^{n} f(x_i^*, y_i^*) \boldsymbol{\Delta} s_i$$

if this limit exists.



Interpretation of the line integral $\int_{\mathbf{C}} f(x, y)\, ds$ as being the area under the graph of f over **C**

The following formula can be used to evaluate a line integral.

Theorem

Suppose $f(x, y)$ is a continuous function on a differentiable curve $\mathbf{C}(t)$, $\mathbf{C}\colon [a, b] \to \mathbf{R}^2$. Then

$$\int_{\mathbf{C}} f(x, y)\, ds = \int_a^b f(x(t), y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

Note that in the above formula,

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2},$$

is the speed of $\mathbf{C}(t)$ at time t.

Example

Evaluate $\int_{\mathbf{C}} 2x \, ds$, where **C** consists of the arc **C** of the parabola $y = x^2$ from $(0, 0)$ to $(1, 1)$.

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Solution:

- We can choose x as the parameter and the equations for **C** become

$$x = x \qquad y = x^2 \qquad 0 \le x \le 1.$$

- Therefore

$$\int_{\mathbf{C}} 2x\,ds = \int_0^1 2x\sqrt{\left(\frac{dx}{dx}\right)^2 + \left(\frac{dy}{dx}\right)^2}\,dx$$

$$= \int_0^1 2x\sqrt{1+4x^2}\,dx = \frac{1}{4}\cdot\frac{2}{3}(1+4x^2)^{\frac{3}{2}}\Big|_0^1 = \frac{5\sqrt{5}-1}{6}.$$

One can also define in a similar manner the line integral of a function f along a curve $\mathbf{C}$ in $\mathbf{R^3}$.

Theorem

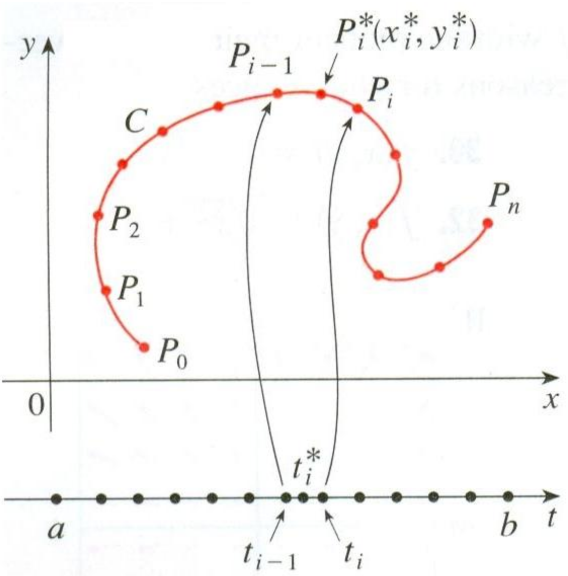
$$\int_{\mathbf{C}} f(x, y, z)\, ds$$

$$= \int_a^b f(x(t), y(t), z(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt.$$

- For what we will study next, another important type of line integral.
- These line integrals are called **line integrals of f along C with respect to x and y.**
- They are defined respectively for x and y by the following limits:

$$\int_C f(x,y)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta x_i$$

$$\int_C f(x,y)\,dy = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta y_i.$$



$P_i^*(x_i^*, y_i^*)$
y
P_{i-1}
P_i
C
P_n
P_2
P_1
P_0
0
x
t_i^*
a
b
t
t_{i-1}
t_i

- Note that if $(x(t), y(t))$ is a parametrization of $\mathbf{C}$, then $f(x_i^*, y_i^*)\mathbf{\Delta} x_i$ is approximately equal to

$$f(x(t_i), y(t_i)) \cdot x'(t_i)\mathbf{\Delta} t,$$

where $(x_i^*, y_i^*) = (x(t_i), y(t_i))$.

- So the following formulas show how to calculate these new type line integrals.
- Note that these integrals depend on the **orientation** of the curve $\mathbf{C}$, i.e., the initial and terminal points.

Theorem

$$\int_{\mathbf{C}} f(x, y)\, dx = \int_a^b f(x(t), y(t))x'(t)\, dt$$

$$\int_{\mathbf{C}} f(x, y)\, dy = \int_a^b f(x(t), y(t))y'(t)\, dt.$$

Example (popular exam problem)

Evaluate $\int_{\mathbf{C}} y^2\,dx + x\,dy$, where **C** is the line segment from $(-5,-3)$ to $(0,2)$.

Example (popular exam problem)

Evaluate $\int_{\mathbf{C}} y^2\, dx + x\, dy$, where $\mathbf{C}$ is the line segment from $(-5,-3)$ to $(0,2)$.

Solution:

- A parametric representation for the line segment is

$$x = 5t - 5, \qquad y = 5t - 3, \qquad 0 \leq t \leq 1.$$

- Then $dx = 5\, dt,\ dy = 5\, dt.$
- Also,

$$\int_{\mathbf{C}} y^2\, dx + x\, dy = \int_0^1 (5t-3)^2 (5dt) + (5t-5)(5\, dt)$$

$$= 5\int_0^1 (25t^2 - 25t + 4)\, dt = 5\left[\frac{25t^3}{3} - \frac{25t^2}{2} + 4t\right]_0^1 = -\frac{5}{6}.$$

Example (popular exam problem)

Evaluate $\int_{\mathbf{C}} y\,dx + x^2\,dy$, where $\mathbf{C}$ is the arc of the parabola $y = 4 - x^2$ from $(-3, -5)$ to $(2, 0)$.

Example (popular exam problem)

Evaluate $\int_{\mathbf{C}} y\,dx + x^2\,dy$, where **C** is the arc of the parabola $y = 4 - x^2$ from $(-3,-5)$ to $(2,0)$.

Solution:

- Since the parabola is given as a function of x, take x as the parameter.
- Write **C** as
$$y = 4 - x^2 \quad x = x, \qquad -3 \leq x \leq 2.$$
- Then $dy = -2x\,dx$ and we have:

$$\int_{\mathbf{C}} y\,dx + x^2 dy = \int_{-3}^{2} (4 - x^2)\,dx + x^2(-2x)\,dx$$

$$= \int_{-3}^{2} (-2x^3 - x^2 + 4)\,dx = -\frac{x^4}{2} - \frac{x^3}{3} + 4x\Big|_{-3}^{2} = 40\frac{5}{6}.$$

Example (popular exam problem)

Evaluate $\int_{\mathbf{C}} y\, dx + z\, dy + x\, dz$, where $\mathbf{C}$ consists of the line segment $\mathbf{C}_1$ from $(2,0,0)$ to $(3,4,5)$ followed by the vertical line segment $\mathbf{C}_2$ from $(3,4,5)$ to $(3,4,0)$.

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Solution:

- Write $\mathbf{C}_1$ as $\mathbf{r}(t) = \langle 2,0,0\rangle + t\langle 1,4,5\rangle = \langle 2+t, 4t, 5t\rangle$
  or, in parametric form, as
  $$x = 2+t \qquad y = 4t \qquad z = 5t \qquad 0 \le t \le 1.$$
- Thus,
  $$\int_{\mathbf{C}_1} y\,dx + z\,dy + x\,dz = \int_0^1 (4t)\,dt + (5t)4\,dt + (2+t)5\,dt$$
  $$= \int_0^1 (10+29t)\,dt = 10t + 29\frac{t^2}{2}\bigg|_0^1 = 24.5.$$
- For $\mathbf{C}_2$: $\mathbf{r}(t) = \langle 3,4,5\rangle + t\langle 0,0,-5\rangle = \langle 3,4,5-5t\rangle$ or
  $x = 3 \quad y = 4 \quad z = 5-5t \quad 0 \le t \le 1.$
- Then $dx = 0 = dy$, so $\int_{\mathbf{C}_2} y\,dx + z\,dy + x\,dz = \int_0^1 2(-5)\,dt = -15.$
- Adding the values, we obtain $\int_{\mathbf{C}} y\,dx + z\,dy + x\,dz = 24.5 - 15 = 9.5.$

In Class Exercises 12

1. Express the volume **V** of the solid that lies under the paraboloid $z = 4 - x^2 - y^2$ and above the xy-plane as an iterated integral in polar coordinates.
2. Consider the integral $\int\int_{\mathbf{R}} y\sqrt{x^2+y^2}\, dA$ with **R** the region $\{(x,y)\colon 1 \le x^2+y^2 \le 2,\ \ 0 \le y \le x\}$
   (a) First describe the domain **R** in polar coordinates.
   (b) Write the integral as an iterated integral in polar coordinates.
3. Consider the integral $\int_{\mathbf{C}} 2x\, ds$, where **C** consists of the arc **C** of the parabola $y = x^2$ from $(0,0)$ to $(1,1)$.
   (a) Parameterize **C** by a curve $\mathbf{r}(t)$, $0 \le t \le 1$.
   (b) Express this integral as an integral of the form $\int_0^1 \mathbf{F}(t)\, dt$, i.e., find $\mathbf{F}(t)$.
4. Evaluate $\int_{\mathbf{C}} y^2\, dx + x\, dy$, where **C** is the line segment from $(-5,-3)$ to $(0,2)$, by first parameterizing **C**.
5. Evaluate $\int_{\mathbf{C}} y\, dx + z\ dy + x\, dz$, where **C** consists of the line segment from $(4,0,0)$ to $(3,4,5)$, by first parameterizing **C**.

- We now get to our final type of line integral which can be considered to be a line integral of a vector field.
- This type of integral is used to calculate the work $\mathbf{W}$ done by a force field $\mathbf{F}$ in moving a particle along a smooth curve $\mathbf{C}$.

Theorem

If $\mathbf{C}$ is given by the vector equation $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ on the interval $[a, b]$, then the work $\mathbf{W}$ can be calculated by

$$\mathbf{W} = \int_a^b \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt,$$

where $\cdot$ is the dot product.

Example (popular exam problem)

Find the work done by the force field $\mathbf{F}(x, y) = x^2\mathbf{i} - xy\mathbf{j}$ in moving a particle along the quarter-circle $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j}$, $0 \leq t \leq \frac{\pi}{2}$.

Example (popular exam problem)

Find the work done by the force field $\mathbf{F}(x, y) = x^2\mathbf{i} - xy\mathbf{j}$ in moving a particle along the quarter-circle $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j},\ 0 \leq t \leq \frac{\pi}{2}$.

Solution:

- Since $x = \cos t$ and $y = \sin t$, we have

$$\mathbf{F}(\mathbf{r}(t)) = \cos^2 t\mathbf{i} - \cos t \sin t\mathbf{j}$$

and

$$\mathbf{r}'(t) = -\sin t\mathbf{i} + \cos t\mathbf{j}.$$

- Therefore, the work done is:

$$\int_{\mathbf{C}} \mathbf{F} \cdot \mathbf{dr} = \int_0^{\frac{\pi}{2}} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)dt = \int_0^{\frac{\pi}{2}} (-2\cos^2 t \sin t)dt$$

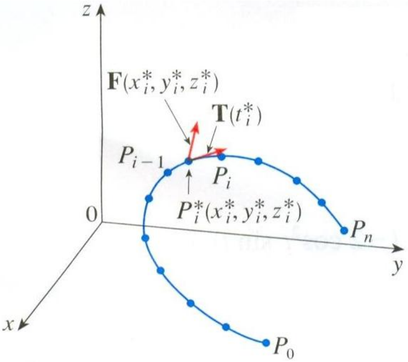
$$= 2\frac{\cos^3 t}{3}\Big|_0^{\frac{\pi}{2}} = -\frac{2}{3}.$$

Definition

- Let $\mathbf{F}$ be a continuous vector field defined on a smooth curve $\mathbf{C}$ given by a vector function $\mathbf{r}(t)$, $a \leq t \leq b$.
- Then the **line integral of F along C** is

$$\int_{\mathbf{C}} \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt = \int_{\mathbf{C}} \mathbf{F} \cdot \mathbf{T}\, ds.$$

- Here, $\mathbf{T}(t)$ denotes the unit tangent vector $\frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$.



Example (popular exam problem)

Evaluate $\int_{\mathbf{C}} \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y, z) = xy\mathbf{i} + yz\mathbf{j} + zx\mathbf{k}$ and $\mathbf{C}$ is the twisted cubic given by

$$x = t \qquad y = t^2 \qquad z = t^3, \qquad 0 \leq t \leq 1.$$

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$$x = t \qquad y = t^2 \qquad z = t^3, \qquad 0 \le t \le 1.$$

Solution:

- We have

$$\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$$

$$\mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j} + 3t^2\mathbf{k}$$

$$\mathbf{F}(\mathbf{r}(t)) = t^3\mathbf{i} + t^5\mathbf{j} + t^4\mathbf{k}.$$

- Thus,

$$\int_{\mathbf{C}} \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)dt$$

$$= \int_0^1 (t^3 + 5t^6)dt = \frac{t^4}{4} + \frac{5t^7}{7}\Big|_0^1 = \frac{27}{28}.$$

Theorem

If $\mathbf{C}$ in $\mathbf{R}^3$ is parameterized by $\mathbf{r}(t)$ and $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$, then

$$\int_{\mathbf{C}} \mathbf{F} \cdot \mathbf{dr} = \int_{\mathbf{C}} P\,dx + Q\,dy + R\,dz.$$

- We now apply the material covered so far on line integrals to obtain several versions of the fundamental theorem of calculus in the multivariable setting.
- Recall that the fundamental theorem calculus can be written as

$$\int_a^b \mathbf{F}'(x)dx = \mathbf{F}(b) - \mathbf{F}(a),$$

when the derivative $\mathbf{F}'(x)$ is continuous on $[a, b]$.

Theorem (Fundamental Theorem of Calculus for line integrals)

- Let $\mathbf{C}$ be a smooth curve given by the vector function $\mathbf{r}(t),\ a \leq t \leq b$.
- Let f be a differentiable function of two or three variables whose gradient vector ∇f is continuous on $\mathbf{C}$.
- Then

$$\int_{\mathbf{C}} \nabla f \cdot \mathbf{dr} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)).$$

Proof.

We now prove the formula in the previous theorem.

$$\int_{\mathbf{C}} \nabla f \cdot d\mathbf{r} = \int_a^b \nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt = \int_a^b \left(\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} \right) dt$$

$$= \int_a^b \frac{d}{dt} f(\mathbf{r}(t))\, dt \qquad (\text{ ChainRule })$$

$$= f(\mathbf{r}(b)) - f(\mathbf{r}(a)).$$

The last step follows from the Fundamental Theorem of Calculus. □

For further discussion, we make the following definitions.

Definition

A curve $\mathbf{r} \colon [a, b] \to \mathbf{R^3}$ (or $\mathbf{R^3}$) **closed** if $\mathbf{r}(a) = \mathbf{r}(b)$.

Definition

A domain $\mathbf{D} \subset \mathbf{R^3}$ (or $\mathbf{R^2}$) is **open** if for any point p in $\mathbf{D}$, a small ball (or disk) centered at p in $\mathbf{R^3}$ (in $\mathbf{R^2}$) is contained in $\mathbf{D}$.

Definition

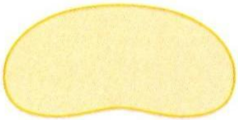
A curve $\mathbf{r} \colon [a, b] \to \mathbf{R^3}$ (or $\mathbf{R^2}$) is a **simple curve** if it doesn't intersect itself anywhere between its end points ($\mathbf{r}(t_1) \neq \mathbf{r}(t_2)$ when $a < t_1 < t_2 < b$).

Definition

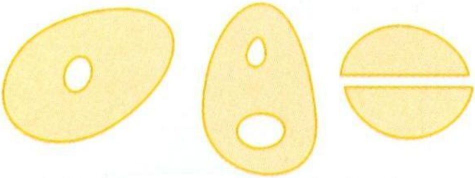
A domain $\mathbf{D} \subset \mathbf{R^3}$ (or $\mathbf{R^2}$) is **connected** if any two points in $\mathbf{D}$ can be joined by a path contained inside $\mathbf{D}$.

Definition

An open, connected region $\mathbf{D} \subset \mathbf{R}^2$ is a **simply-connected** region if any simple closed curve in $\mathbf{D}$ encloses only points that are in $\mathbf{D}$.



simply-connected region



regions that are not simply-connected

Definition

- A vector field **F** is called a **conservative vector field** if it is the gradient of some scalar function $f(x, y)$; the function $f(x, y)$ is called a **potential function** for **F**.
- For example,

$$f(x, y) = xy + y^2, \qquad \nabla f = \langle y, x + 2y \rangle$$

and so, $\mathbf{F}(x, y) = y\mathbf{i} + (x + 2y)\mathbf{j}$ is a **conservative** vector field.

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Definition

If **F** is a continuous vector field with domain **D**, we say that the line integral $\int_{\mathbf{C}} \mathbf{F} \cdot d\mathbf{r}$ is **independent of path** if

$$\int_{\mathbf{C}_1} \mathbf{F} \cdot d\mathbf{r} = \int_{\mathbf{C}_2} \mathbf{F} \cdot d\mathbf{r}$$

for any two paths $\mathbf{C}_1$ and $\mathbf{C}_2$ in **D** with the same initial and the same terminal points.

Theorem

$$\int_{\mathbf{C}} \mathbf{F} \cdot \mathbf{dr}$$

is **independent of path** in **D** if and only if

$$\oint_{\mathbf{C}} \mathbf{F} \cdot \mathbf{dr} = 0$$

for every closed path **C** in **D**.

Theorem

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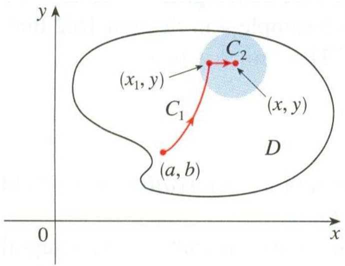
is **independent of path** in **D** if and only if

$$\oint_{C} \mathbf{F} \cdot d\mathbf{r} = 0$$

for every closed path **C** in **D**.

Theorem

- Suppose **F** is a vector field that is continuous on an open connected region **D**.
- If $\int_C \mathbf{F} \cdot d\mathbf{r}$ is **independent of path** in **D**, then **F** is a **conservative vector field** on **D**; that is, there exists a function f such that $\nabla f = \mathbf{F}$.



Path Independence $\Longrightarrow$ Conservative

- Let (a, b) be a fixed point in **D**.
- Construct the desired **potential function** f by defining $f(x, y) = \int_{(a,b)}^{(x,y)} \mathbf{F} \cdot d\mathbf{r}$ for any point (x, y) in **D**.
- Since **D** is open, there exists a disk contained in **D** with center (x, y). Choose any point (x_1, y) in the disk with $x_1 < x$ and let **C** consist of any path $\mathbf{C}_1$ from (a, b) to (x_1, y) followed by the horizontal line segment $\mathbf{C}_2$ from (x_1, y) to (x, y). Then
- $f(x, y) = \int_{\mathbf{C}_1} \mathbf{F} \cdot d\mathbf{r} + \int_{\mathbf{C}_2} \mathbf{F} \cdot d\mathbf{r} = \int_{(a,b)}^{(x_1,y)} \mathbf{F} \cdot d\mathbf{r} + \int_{\mathbf{C}_2} \mathbf{F} \cdot d\mathbf{r}$.

Continuation of the proof.

- $f(x, y) = \int_{\mathbf{C}_1} \mathbf{F} \cdot \mathbf{dr} + \int_{\mathbf{C}_2} \mathbf{F} \cdot \mathbf{dr} = \int_{(a,b)}^{(x_1,y)} \mathbf{F} \cdot \mathbf{dr} + \int_{\mathbf{C}_2} \mathbf{F} \cdot \mathbf{dr}$.
- The first of these integrals does not depend on x, so

$$\frac{\partial}{\partial x} f(x, y) = 0 + \frac{\partial}{\partial x} \int_{\mathbf{C}_2} \mathbf{F} \cdot \mathbf{dr}.$$

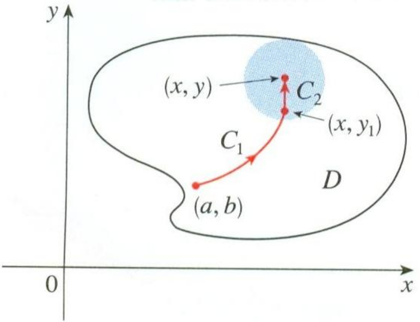
- If we write $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$, then

$$\int_{\mathbf{C}_2} \mathbf{F} \cdot \mathbf{dr} = \int_{\mathbf{C}_2} P\,dx + Q\,dy.$$

- On $\mathbf{C}_2$, y is constant, so $dy = 0$.
- Using t as the parameter, where $x_1 \leq t \leq x$, we have:

$$\frac{\partial}{\partial x} f(x, y) = \frac{\partial}{\partial x} \int_{\mathbf{C}_2} P\ dx + Q\,dy = \frac{\partial}{\partial x} \int_{x_1}^{x} P(t, y)\,dt = P(x, y)$$

  by the Fundamental Theorem of Calculus.



Continuation of the proof.

A similar argument, using a vertical line segment, shows that

$$\frac{\partial}{\partial y} f(x, y) = \frac{\partial}{\partial y} \int_{C_2} P\,dx + Q\,dy = \frac{\partial}{\partial y} \int_{y_1}^{y} Q(x, t)\,dt = Q(x, y).$$

Thus, $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} = \nabla f$ and so $\mathbf{F}$ is **conservative**. $\square$

Theorem

If $\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$ is a **conservative** vector field, where P and Q have continuous first-order partial derivatives on a domain **D**, then throughout **D** we have

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

Theorem

- Let $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ be a vector field on an open simply-connected region **D**.
- Suppose that P and Q have continuous first-order derivatives and

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \qquad \textbf{throughout} \qquad \mathbf{D}.$$

- Then **F** is **conservative**.

Example (popular exam problem)

Determine whether or not the vector field $\mathbf{F}(x, y) = (x - y)\mathbf{i} + (x - 2)\mathbf{j}$ is **conservative**.

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Solution:

- Let $P(x, y) = x - y$ and $Q(x, y) = x - 2$.
- Then:
$$\frac{\partial P}{\partial y} = -1 \qquad \frac{\partial Q}{\partial x} = 1.$$
- Since $\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x}$, then $\mathbf{F}$ is **not** **conservative**.

Example (popular exam problem)

Determine whether or not the vector field $\mathbf{F}(x, y) = (3 + 2xy)\mathbf{i} + (x^2 - 3y^2)\mathbf{j}$ is **conservative**.

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Determine whether or not the vector field $\mathbf{F}(x, y) = (3 + 2xy)\mathbf{i} + (x^2 - 3y^2)\mathbf{j}$ is **conservative**.

Solution:

- Let $P(x, y) = 3 + 2xy$ and $Q(x, y) = x^2 - 3y^2$.
- Then
$$\frac{\partial P}{\partial y} = 2x = \frac{\partial Q}{\partial x}.$$
- Also, the domain of $\mathbf{F}$ is the entire plane ($\mathbf{D} = \mathbf{R}^2$), which is open and simply-connected.
- We conclude that $\mathbf{F}$ is **conservative**.

Example (Problem on final exam!!)

If $\mathbf{F}(x, y) = (3 + 2xy)\mathbf{i} + (x^2 - 3y^2)\mathbf{j}$, find a function f such that $\mathbf{F} = \nabla f$.

Example (Problem on final exam!!)

If $\mathbf{F}(x, y) = (3 + 2xy)\mathbf{i} + (x^2 - 3y^2)\mathbf{j}$, find a function f such that $\mathbf{F} = \nabla f$.

Solution:

By the previous example, $\mathbf{F}$ is **conservative**. So there exists a function f with $\nabla f = \mathbf{F}$:

$$f_x(x, y) = 3 + 2xy \tag{12}$$

$$f_y(x, y) = x^2 - 3y^2. \tag{13}$$

Integrating (12) with respect to x,

$$f(x, y) = 3x + x^2y + g(y). \tag{14}$$

The constant of integration $g(y)$ is a constant with respect to x.
Next differentiate both sides of (14) with respect to y:

$$f_y(x, y) = x^2 + g'(y). \tag{15}$$

Comparing (13) and (15), we find $g'(y) = -3y^2$.
Integrating with respect to y,

$$g(y) = -y^3 + \mathbf{K},$$

where $\mathbf{K}$ is a constant. Putting this in (14), the desired **potential function** is $f(x, y) = 3x + x^2y - y^3 + \mathbf{K}$.

Example (Very popular exam problem)

Evaluate the line integral $\int_C \mathbf{F} \cdot \mathbf{dr}$, where $\mathbf{F} = \nabla(f(x,y) = 3x + x^2y - y^3 + \mathbf{K})$ and $\mathbf{C}$ is the curve given by $\mathbf{r}(t) = e^t \sin t\,\mathbf{i} + e^t \cos t\,\mathbf{j},\ 0 \le t \le \pi$.

Example (Very popular exam problem)

Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F} = \nabla(f(x,y) = 3x + x^2y - y^3 + \mathbf{K})$ and $\mathbf{C}$ is the curve given by $\mathbf{r}(t) = e^t \sin t\,\mathbf{i} + e^t \cos t\,\mathbf{j}$, $0 \leq t \leq \pi$.

Solution:

- The initial and terminal points of $\mathbf{C}$ are $\mathbf{r}(0) = (0,1)$ and $\mathbf{r}(\pi) = (0, -e^{\pi})$.
- In the expression for $f(x,y)$, any value of the constant $\mathbf{K}$ works, so choose $\mathbf{K} = 0$.
- Then by the Fundamental Theorem of Calculus for Line Integrals,

$$\int_{\mathbf{C}} \mathbf{F} \cdot d\mathbf{r} = \int_{\mathbf{C}} \nabla f \cdot d\mathbf{r} = f(0, -e^{\pi}) - f(0,1)$$

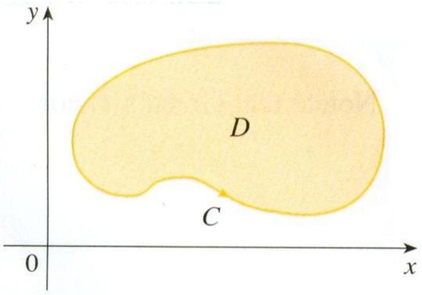
$$= e^{3\pi} - (-1) = e^{3\pi} + 1.$$

- This method is shorter than the straightforward method for evaluating line integrals we learned in Chapter 16.2.

Definition

A simple closed parameterized curve **C** in $\mathbf{R}^2$ always bounds a bounded simply-connected domain **D**.

- We say that **C** is **positively oriented** if for the parametrization $\mathbf{r}(t)$ of **C**, the region **D** is always on the left at $\mathbf{r}(t)$ traverses **C**.
- Note that this parametrization is the **counterclockwise** one on the boundary of unit disk $\mathbf{D} = \{(x, y) \mid x^2 + y^2 \leq 1\}$.



The next theorem is a version of the Fundamental Theorem of Calculus. It allows one to carry out a two dimensional integral on a domain **D** by calculating a related integral on the boundary of **D**. **There will be at least one final exam problem related to the following theorem.**

Theorem (Green's Theorem)

- Let **C** be a **positively oriented**, piecewise-smooth, simple closed curve in the plane and let **D** be the region bounded by **C**.
- If P and Q have continuous partial derivatives on an open region that contains **D**, then

$$\oint_{\mathbf{C}} P\,dx + Q\,dy = \int\int_{\mathbf{D}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right)\,dA.$$

Proof of Green's Theorem for when D is a Simple Region

- Green's Theorem follows if

$$\oint_{C} P\,dx = -\int\int_{D} \frac{\partial P}{\partial y}\,dA \tag{16}$$

and

$$\oint_{C} Q\,dy = \int\int_{D} \frac{\partial Q}{\partial x}\,dA. \tag{17}$$

- We prove Equation 16 by expressing **D** as a type I region:

$$\mathbf{D} = \{(x,y) \mid a \le x \le b, g_1(x) \le y \le g_2(x)\}$$

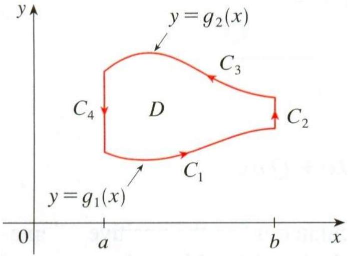
where g_1 and g_2 are continuous functions.

- This enables us to compute the double integral on the right side of Equation 16

$$\int\int_{D} \frac{\partial P}{\partial y}\,dA$$

$$= \int_a^b \int_{g_1(x)}^{g_2} \frac{\partial P}{\partial y}(x,y)\,dy\,dx = \int_a^b [P(x,g_2(x)) - P(x,g_1(x))]\,dx. \tag{18}$$

- Last step follows from the Fundamental Theorem of Calculus.



Continuation of the proof.

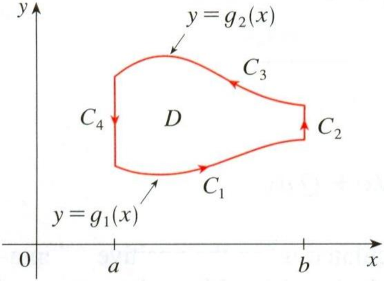
Now compute the left side of Equation 16 by breaking up $\mathbf{C}$ as the union of the four curves $\mathbf{C}_1, \mathbf{C}_2, \mathbf{C}_3$ and $\mathbf{C}_4$. On $\mathbf{C}_1$ we take x as the parameter and write the **parametric equations** as $x = x, y = g_1(x), a \leq x \leq b$. Thus,

$$\int_{\mathbf{C}_1} P(x, y)\, dx = \int_a^b P(x, g_1(x))\, dx$$

$\mathbf{C}_3$ goes from right to left but $-\mathbf{C}_3$ goes from left to right. Write the **parametric equations** of $-\mathbf{C}_3$ as $x = x,\ y = g_2(x),\ a \leq x \leq b$. Therefore,

$$\int_{\mathbf{C}_3} P(x, y)\, dx = -\int_{-\mathbf{C}_3} P(x, y)\, dx = -\int_a^b P(x, g_2(x))\, dx.$$

□



Continuation of the proof.

On $\mathbf{C}_2$ or $\mathbf{C}_4$, x is constant, so $dx = 0$ and $\int_{\mathbf{C}_2} P(x, y)\,dx = 0 = \int_{\mathbf{C}_4} P(x, y)\,dx$. Hence

$$\oint_{\mathbf{C}} P(x,y)\,dx = \int_{\mathbf{C}_1} P(x,y)\,dx + \int_{\mathbf{C}_2} P(x,y)\,dx + \int_{\mathbf{C}_3} P(x,y)\,dx + \int_{\mathbf{C}_4} P(x,y)\,dx$$

$$= \int_a^b P(x, g_1(x))\,dx - \int_a^b P(x, g_2(x))\,dx.$$

Comparing this expression with the one in Equation 18, we see that

$$\oint_{\mathbf{C}} P(x,y)\,dx = -\int\int_{\mathbf{D}} \frac{\partial P}{\partial y}\,dA.$$

□

An immediate consequence of Green's Theorem is the existence of area formulas described in the next theorem.

Theorem (Area Formulas)

- Let **D** be a simply-connected domain in the plane with simple closed oriented boundary curve **C**.
- Let **A** be the area of **D**.
- Then:

$$\mathbf{A} = \oint_{\mathbf{C}} x\,dy = -\oint_{\mathbf{C}} y\,dx = \frac{1}{2}\oint_{\mathbf{C}} x\,dy - y\,dx.$$

Example (popular exam problem)

Find the area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Example (popular exam problem)

Find the area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Solution:

- The ellipse has **parametric equations** $x = a\cos t$ and $y = b\sin t$, where $0 \leq t \leq 2\pi$.
- Using the formula in the previous theorem, we have:

$$\begin{aligned}
\mathbf{A} &= \frac{1}{2}\oint_{\mathbf{C}} x\,dy - y\,dx \\
&= \frac{1}{2}\int_0^{2\pi} (a\cos t)(b\cos t)\,dt - (b\sin t)(-a\sin t)\,dt \\
&= \frac{ab}{2}\int_0^{2\pi} (\cos^2 t + \sin^2 t)\,dt = \pi ab.
\end{aligned}$$

Example (popular exam problem)

Evaluate
$$\oint_{\mathbf{C}} (3y - e^{\sin x})\,dx + (7x + \sqrt{y^4+1})\,dy,$$
where **C** is the circle $x^2 + y^2 = 9$.

Example (popular exam problem)

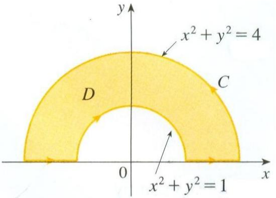
Evaluate

$$\oint_{\mathbf{C}} (3y - e^{\sin x})\, dx + (7x + \sqrt{y^4 + 1})\, dy,$$

where **C** is the circle $x^2 + y^2 = 9$.

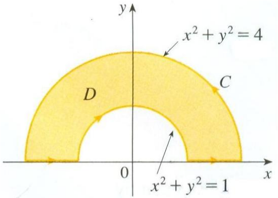
Solution:

- The region **D** bounded by **C** is the disk $x^2 + y^2 \leq 9$. Changing to polar coordinates after applying Green's Theorem, we obtain:
- $$\begin{aligned} &\oint_{\mathbf{C}} (3y - e^{\sin x})\, dx + (7x + \sqrt{y^4 + 1})\, dy \\ &= \int\int_{\mathbf{D}} \left[\frac{\partial}{\partial x}(7x + \sqrt{y^4 + 1}) - \frac{\partial}{\partial y}(3y - e^{\sin x}) \right] dA \\ &= \int_0^{2\pi} \int_0^3 (7 - 3)\, r\, dr\, d\theta = 4 \int_0^{2\pi} d\theta \int_0^3 r\, dr = 36\pi. \end{aligned}$$



Example (popular exam problem)

Evaluate $\oint_C y^2\,dx + 3xy\,dy$, where **C** is the boundary of the semiannular region **D** in the upper half-plane between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.



Example (popular exam problem)

Evaluate $\oint_{\mathbf{C}} y^2\,dx + 3xy\,dy$, where **C** is the boundary of the semiannular region **D** in the upper half-plane between the circles $x^2+y^2=1$ and $x^2+y^2=4$.

Solution:

- Notice that although **D** is not simple, the y-axis divides it into two simple regions.
- In polar coordinates we can write

$$\mathbf{D} = \{(r,\theta) \mid 1 \le r \le 2, 0 \le \theta \le \pi\}.$$

Therefore, Green's Theorem gives

$$\oint_{\mathbf{C}} y^2\,dx + 3xy\,dy = \int\int_{\mathbf{D}} \left[\frac{\partial}{\partial x}(3xy) - \frac{\partial}{\partial y}(y^2)\right] dA = \int\int_{\mathbf{D}} y\,dA$$

$$= \int_0^{\pi}\int_1^2 (r\sin\theta) r\,dr\,d\theta = \int_0^{\pi} \sin\theta\,d\theta \int_1^2 r^2\,dr = \ -\cos\theta\Big|_0^{\pi} \cdot \left[\frac{1}{3}r^3\right]_1^2 = \frac{14}{3}.$$

In Class Exercises 13

1. Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F} = \nabla(f(x, y) = 3x + x^2y - y^3)$ and $\mathbf{C}$ is the curve given by $\mathbf{r}(t) = e^t \sin t\, \mathbf{i} + e^t \cos t\, \mathbf{j}$, $0 \le t \le \pi$.

2. Determine whether the following vector fields are conservative or not.

   (a) $\mathbf{F}(x, y) = (x^2 + xy)\mathbf{i} + (xy - y^2)\mathbf{j}$.
   (b) $\mathbf{F}(x, y) = (3x^2y + y^2)\mathbf{i} + (x^3 + 2xy + 3y^2)\mathbf{j}$.

3. Find a **potential function** for the vector field $\mathbf{F} = \langle 2y^2 + y^2 \cos(xy^2), 4xy + 2xy \cos(xy^2)\rangle$.

4. Use Green's Theorem to show that if $\mathbf{D} \subset \mathbf{R}^2$ is the bounded region with boundary a positively oriented simple closed curve $\mathbf{C}$, then the area of $\mathbf{D}$ can be calculated by the formula:
   $\textbf{Area}(\mathbf{D}) = \frac{1}{2} \oint_{\mathbf{C}} -y\, dx + x\, dy$

5. Consider the ellipse $4x^2 + y^2 = 1$. Use the above area formula to calculate the area of the region $\mathbf{D} \subset \mathbf{R}^2$ with boundary this ellipse. (Hint: This ellipse can be parameterized by $\mathbf{r}(t) = \langle \frac{1}{2} \cos(t), \sin(t)\rangle$ for $0 \le t \le 2\pi$.)

Definition

If $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is a vector field on $\mathbf{R}^3$ and the partial derivatives of P, Q, and R all exist, then the curl of $\mathbf{F}$ is the vector field on $\mathbf{R}^3$ defined by

$$\text{curl } \mathbf{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}$$

- We introduce the vector differential operator ∇ ("del") as

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}.$$

- It has meaning when it operates on a scalar function to produce the gradient of f:

$$\nabla f = \mathbf{i}\frac{\partial f}{\partial x} + \mathbf{j}\frac{\partial f}{\partial y} + \mathbf{k}\frac{\partial f}{\partial z} = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

- Think of ∇ as a vector with components $\partial/\partial x$, $\partial/\partial y$, and $\partial/\partial z$ and consider the formal cross product of ∇ with the vector field $\mathbf{F}$ as follows:

$$\operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

$$= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}.$$

Example

If $\mathbf{F}(x, y, z) = xz\mathbf{i} + xyz\mathbf{j} - y^2\mathbf{k}$, find curl $\mathbf{F}$.

Example

If $\mathbf{F}(x, y, z) = xz\mathbf{i} + xyz\mathbf{j} - y^2\mathbf{k}$, find curl $\mathbf{F}$.

Solution

Computing, we have

$$\text{curl } \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz & xyz & -y^2 \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}(-y^2) - \frac{\partial}{\partial z}(xyz)\right]\mathbf{i} - \left[\frac{\partial}{\partial x}(-y^2) - \frac{\partial}{\partial z}(xz)\right]\mathbf{j} + \left[\frac{\partial}{\partial x}(xyz) - \frac{\partial}{\partial y}(xz)\right]\mathbf{k}$$

$$= (-2y - xy)\mathbf{i} - (0 - x)\mathbf{j} + (yz - 0)\mathbf{k} = -y(2 + x)\mathbf{i} + x\mathbf{j} + yz\mathbf{k}$$

Theorem

If $f : \mathbf{R^3} \to \mathbb{R}$ is a function of three variables that has continuous second-order partial derivatives, then

$$\operatorname{curl}(\nabla f) = 0$$

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Proof

We have

$$\operatorname{curl}(\nabla f) = \nabla \times (\nabla f) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix}$$

$$= \left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}\right)\mathbf{i} - \left(\frac{\partial^2 f}{\partial z \partial x} - \frac{\partial^2 f}{\partial x \partial z}\right)\mathbf{j} + \left(\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x}\right)\mathbf{k}$$

$$= 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = 0$$

by Clairaut's Theorem.

Theorem

If $\mathbf{F}$ is a vector field defined on all of $\mathbf{R}^3$ whose component functions have continuous partial derivatives and curl $\mathbf{F} = 0$, then $\mathbf{F}$ is a conservative vector field. In this case, curl $\mathbf{F} = 0 \Leftrightarrow \mathbf{F} = \nabla f$ for some f.

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Example

Show that $\mathbf{F}(x, y, z) = y^2z^3\mathbf{i} + 2xyz^3\mathbf{j} + 3xy^2z^2\mathbf{k}$ is a conservative vector field.

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Example

Show that $\mathbf{F}(x, y, z) = y^2z^3\mathbf{i} + 2xyz^3\mathbf{j} + 3xy^2z^2\mathbf{k}$ is a conservative vector field.

Solution

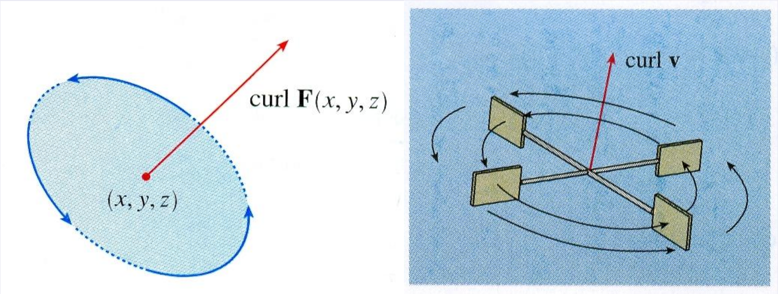
- We compute the curl of $\mathbf{F}$:

$$\text{curl } \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2z^3 & 2xyz^3 & 3xy^2z^2 \end{vmatrix}$$

$$= (6xyz^2 - 6xyz^2)\mathbf{i} - (3y^2z^2 - 3y^2z^2)\mathbf{j} + (2yz^3 - 2yz^3)\mathbf{k} = 0.$$

- Since curl $\mathbf{F} = 0$ and the domain of $\mathbf{F}$ is $\mathbf{R}^3$, $\mathbf{F}$ is a conservative vector field.

- The reason for the name *curl* is that the curl vector is associated with rotations.
- Suppose $\mathbf{F}$ represents the velocity field in fluid flow.
- Particles near (x, y, z) in the fluid tend to rotate about the axis that points in the direction of curl $\mathbf{F}(x, y, z)$ and the length of this curl vector is a measure of how quickly the particles move around the axis.
- If curl $\mathbf{F} = 0$ at a point P, then the fluid is free from rotations at P and $\mathbf{F}$ is called **irrotational** at P. In other words, there is no whirlpool or eddy at P.
- If curl $\mathbf{F} = 0$, then a tiny paddle wheel moves with the fluid but doesn't rotate about its axis.
- If curl $\mathbf{F} \neq 0$, the paddle wheel rotates about its axis.



- Imagine a tiny paddle wheel placed in a fluid at a point P.
- The paddle wheel rotates the fastest when its axis is parallel to curl $\mathbf{v}$.

Definition

If $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is a vector field on $\mathbf{R}^3$ and $\partial P/\partial x$, $\partial Q/\partial y$, and $\partial R/\partial z$ exist, then the **divergence of** $\mathbf{F}$ is the function of three variables defined by

$$\text{div } \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

- Observe that curl $\mathbf{F}$ is a vector field but div $\mathbf{F}$ is a scalar field.
- In terms of the gradient operator $\nabla = (\partial/\partial x)\mathbf{i} + (\partial/\partial y)\mathbf{j} + (\partial/\partial z)\mathbf{k}$, the divergence of $\mathbf{F}$ can be written symbolically as the dot product of ∇ and $\mathbf{F}$:

$$\text{div } \mathbf{F} = \nabla \cdot \mathbf{F}$$

Example

If $\mathbf{F}(x, y, z) = xz\mathbf{i} + xyz\mathbf{j} - y^2\mathbf{k}$, find div $\mathbf{F}$.

Example

If $\mathbf{F}(x, y, z) = xz\mathbf{i} + xyz\mathbf{j} - y^2\mathbf{k}$, find div $\mathbf{F}$.

Solution

By the definition of divergence, we have

$$\text{div } \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}(xz) + \frac{\partial}{\partial y}(xyz) + \frac{\partial}{\partial z}(-y^2)$$

$$= z + xz$$

Theorem

If $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is a vector field on $\mathbf{R}^3$ and P, Q, and R have continuous second-order partial derivatives, then

$$\text{div curl } \mathbf{F} = 0$$

Theorem

If $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is a vector field on $\mathbf{R}^3$ and P, Q, and R have continuous second-order partial derivatives, then

$$\text{div curl } \mathbf{F} = 0$$

Solution

Using the definitions of divergence and curl, we have

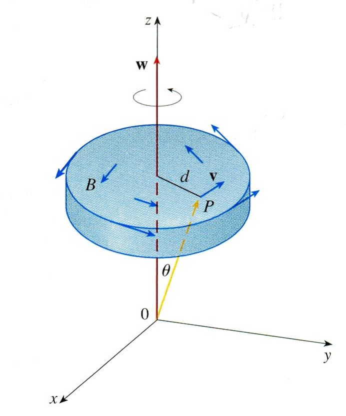
$$\text{div curl } \mathbf{F} = \nabla \cdot (\nabla \times \mathbf{F})$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$$

$$= \frac{\partial^2 R}{\partial x \partial y} - \frac{\partial^2 Q}{\partial x \partial z} + \frac{\partial^2 P}{\partial y \partial z} - \frac{\partial^2 R}{\partial y \partial x} + \frac{\partial^2 R}{\partial y \partial x} + \frac{\partial^2 Q}{\partial z \partial x} - \frac{\partial^2 P}{\partial z \partial y} = 0$$

because the terms cancel in pairs by Clairaut's Theorem.

- The reason for the name **divergence** can be understood in the context of fluid flow.
- If $\mathbf{F}(x, y, z)$ is the velocity of a fluid (or gas), then div $\mathbf{F}(x, y, z)$ represents the net rate of chance (with respect to time) of the mass of fluid (or gas) flowing from the point (x, y, z) per unit volume.
- In other words, div $\mathbf{F}(x, y, z)$ measures the tendency of the fluid to diverge from the point (x, y, z).
- If div $\mathbf{F} = 0$, then $\mathbf{F}$ is said to be **incompressible**.



Maxwell's equations related the electric field **E** and the magnetic field **H** as they vary with time in a region containing no charge and no current. They are:

- $\operatorname{div} \mathbf{E} = 0, \qquad \operatorname{div} \mathbf{H} = 0,$
- $\operatorname{curl} \mathbf{E} = -\dfrac{1}{c}\dfrac{\partial \mathbf{H}}{\partial t}, \qquad \operatorname{curl} \mathbf{H} = \dfrac{1}{c}\dfrac{\partial \mathbf{E}}{\partial t},$ where c is the speed of light.

- If f is a function of three variables, we have

$$\text{div}\,(\nabla f) = \nabla \cdot (\nabla f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

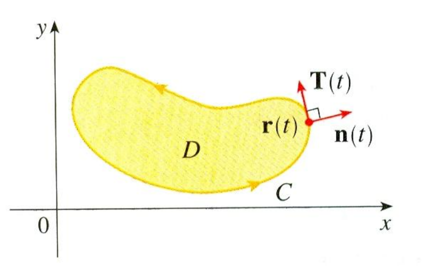
  and this expression occurs so often that we abbreviate it as $\nabla^2 f$.

- The operator is called the **Laplace operator** because of its relation to **Laplace's equation**: A function $f : \mathbf{R}^3 \to \mathbb{R}$ is **harmonic** if

$$\boldsymbol{\Delta} f = \nabla^2 f = \frac{\partial^2}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0,$$

Theorem

$$\oint_{\mathbf{C}} \mathbf{F} \cdot \mathbf{n}\, ds = \int\int_{\mathbf{D}} \operatorname{div} \mathbf{F}(x, y)\, dA$$



In this figure:

- $\mathbf{r}(t)$ is a parametrization of the boundary curve $\mathbf{C}$ of $\mathbf{D}$.
- $\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{\mathbf{r}'(t)}$ is the unit tangent field to $\mathbf{C}$.
- $\mathbf{n}(t)$ is the outward pointing unit normal vector to $\mathbf{D}$ along $\mathbf{r}(t)$.

Theorem

$$\oint_{\mathbf{C}} \mathbf{F} \cdot \mathbf{n} \, ds = \int \int_{\mathbf{D}} \text{div } \mathbf{F}(x, y) \, dA$$

Theorem

$$\oint_{\mathbf{C}} \mathbf{F} \cdot \mathbf{n}\, ds = \int\int_{\mathbf{D}} \text{div } \mathbf{F}(x, y)\, dA$$

Solution

- $$\oint_{\mathbf{C}} \mathbf{F} \cdot \mathbf{n}\, ds = \int_a^b (\mathbf{F} \cdot \mathbf{n})(t)|\mathbf{r}'(t)|\, dt$$
  $$= \int_a^b \left[\frac{P(x(t), y(t))y'(t)}{|\mathbf{r}'(t)|} - \frac{Q(x(t), y(t))x'(t)}{|\mathbf{r}'(t)|} \right] |\mathbf{r}'(t)|\, dt$$
  $$= \int_a^b P(x(t), y(t))y'(t)\, dt - Qx((t), y(t))x'(t)\, dt$$
  $$= \int_{\mathbf{C}} P\, dy - Q\, dx = \int_{\mathbf{D}} \int \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right)\, dA$$
  by Green's Theorem.
- The integrand in this double integral is just the divergence of **F**. So we have a second vector form of Green's Theorem.

Parameter Surfaces

- We suppose that

$$\mathbf{r}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k}$$

  is a vector-valued function defined on a region $\mathbf{D}$ in the uv-plane.
- So x, y, and z, the component functions of $\mathbf{r}$, are functions of the two variables u and v with domain $\mathbf{D}$.
- The set of all points (x, y, z) in $\mathbf{R}^3$ such that

$$x = x(u, v) \quad y = y(u, v) \quad z = z(u, v)$$

  and (u, v) varies throughout $\mathbf{D}$, is called a **parametric surface $\mathbf{S}$**.
- The above equations are called **parametric equations** of $\mathbf{S}$.

Tangent Planes

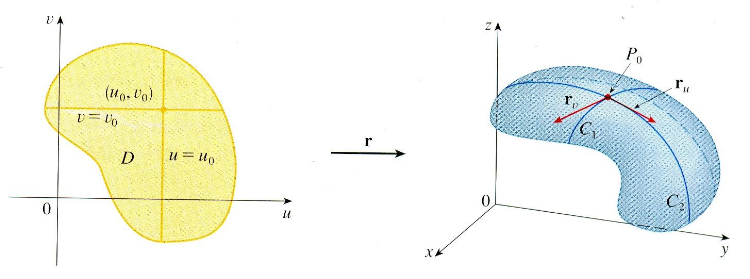
- We now find the tangent plane to a parametric surface **S** traced out by a vector function $\mathbf{r}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k}$ at a point P_0 with position vector $\mathbf{r}(u_0, v_0)$.
- If we keep u constant by putting $u = u_0$, then $\mathbf{r}(u_0, v)$ becomes a vector function of the single parameter v and defines a grid curve C_1 lying on **S**.
- The tangent vector to C_1 at P_0 is obtained by taking the partial derivative of $\mathbf{r}$ with respect to v:

$$\mathbf{r}_v = \frac{\partial x}{\partial v}(u_0, v_0)\mathbf{i} + \frac{\partial y}{\partial v}(u_0, v_0)\mathbf{j} + \frac{\partial z}{\partial v}(u_0, v_0)\mathbf{k}.$$

- Similarly, if we keep v constant by putting $v = v_0$, we get a grid curve C_2 given by $\mathbf{r}(u, v_0)$ that lies on **S**, and its tangent vector at P_0 is

$$\mathbf{r}_u = \frac{\partial x}{\partial u}(u_0, v_0)\mathbf{i} + \frac{\partial y}{\partial u}(u_0, v_0)\mathbf{j} + \frac{\partial z}{\partial u}(u_0, v_0)\mathbf{k}.$$

- If $\mathbf{r}_u \times \mathbf{r}_v$ is not 0, then the surface **S** is called **smooth** (it has no "corners").
- For a smooth surface, the **tangent plane** is the plane that contains the tangent vectors $\mathbf{r}_u$ and $\mathbf{r}_v$.
- The vector $\mathbf{r}_u \times \mathbf{r}_v$ is a normal vector to the tangent plane.



- In this figure the parametric surface **S** is traced out by a vector function

$$\mathbf{r}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k}.$$

- This figure also shows 2 grid curves $\mathbf{C}_1, \mathbf{C}_2$ that arise as the images of lines in the parameter domain **D**.

- Here $\mathbf{r}_v = \frac{\partial x}{\partial v}\mathbf{i} + \frac{\partial y}{\partial v}\mathbf{j} + \frac{\partial z}{\partial v}\mathbf{k}$ and $\mathbf{r}_u = \frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j} + \frac{\partial z}{\partial u}\mathbf{k}$.

Definition

- Suppose $\mathbf{S}$ is a smooth parametric surface given by the equation

$$\mathbf{r}(u,v) = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k}, \quad (u,v) \in \mathbf{D}$$

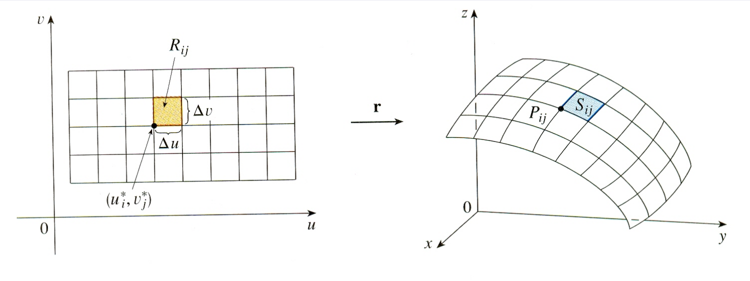
  and $\mathbf{S}$ is covered just once as (u,v) ranges throughout the parameter domain $\mathbf{D}$.

- Then the **surface area** of $\mathbf{S}$ is

$$A(\mathbf{S}) = \int\int_{\mathbf{D}} |\mathbf{r}_u \times \mathbf{r}_v| \, dA,$$

  where

$$\mathbf{r}_u = \frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j} + \frac{\partial z}{\partial u}\mathbf{k}, \quad \mathbf{r}_v = \frac{\partial x}{\partial v}\mathbf{i} + \frac{\partial y}{\partial v}\mathbf{j} + \frac{\partial z}{\partial v}\mathbf{k}.$$



The image of the subrectangle R_{ij} is the patch $\mathbf{S}_{ij}$.

Surface Area of the Graph of a Function

- For the special case of a surface $\mathbf{S}$ with equation $z = f(x, y)$, where (x, y) lies in $\mathbf{D}$ and f has continuous partial derivatives, we take x and y as parameters.
- The parametric equations are $x = x, \quad y = y, \quad z = f(x, y)$.
- $$\mathbf{r}_x = \mathbf{i} + \left(\frac{\partial f}{\partial x}\right)\mathbf{k} \quad \mathbf{r}_y = \mathbf{j} + \left(\frac{\partial f}{\partial y}\right)\mathbf{k}$$
  and
  $$\mathbf{r}_x \times \mathbf{r}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & \frac{\partial f}{\partial x} \\ 0 & 1 & \frac{\partial f}{\partial y} \end{vmatrix} = -\frac{\partial f}{\partial x}\mathbf{i} - \frac{\partial f}{\partial y}\mathbf{j} + \mathbf{k}.$$
- Thus, we have
  $$|\mathbf{r}_x \times \mathbf{r}_y| = \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1} = \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}.$$
- The surface area formula becomes
  $$A(\mathbf{S}) = \int\int_{\mathbf{D}} \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA.$$

Surface Integral Formula

Suppose $f : \mathbf{S} \to \mathbb{R}$ is a function on a surface $\mathbf{S}$ parameterized by a domain $\mathbf{D}$ in the (u, v)-plane. Then

$$\int\int_{\mathbf{S}} f(x, y, z)\, d\mathbf{S} = \int\int_{\mathbf{D}} f(\mathbf{r}(u, v))|\mathbf{r}_u \times \mathbf{r}_v|\, dA$$

Example

Compute the surface integral $\int\int_{\mathbf{S}} x^2 \, d\mathbf{S}$, where $\mathbf{S}$ is the unit sphere $x^2 + y^2 + z^2 = 1$.

Example

Compute the surface integral $\int\int_{\mathbf{S}} x^2 \, d\mathbf{S}$, where $\mathbf{S}$ is the unit sphere $x^2 + y^2 + z^2 = 1$.

Solution

- We use the parametric representation

$$x = \sin\phi\cos\phi \quad y = \sin\phi\sin\theta \quad z = \cos\phi \quad 0 \le \phi \le \pi \quad 0 \le \theta \le 2\pi,$$

  that is, $\mathbf{r}(\phi, \theta) = \sin\phi\cos\theta\mathbf{i} + \sin\phi\sin\theta\mathbf{j} + \cos\phi\mathbf{k}$.

- We can compute that $|\mathbf{r}_\phi \times \mathbf{r}_\theta| = \sin\phi$.
- Therefore,

$$\int\int_{\mathbf{S}} x^2 \, d\mathbf{S} = \int\int_{\mathbf{D}} (\sin\phi\cos\theta)^2 |\mathbf{r}_\phi \times \mathbf{r}_\theta| \, dA$$

$$\int_0^{2\pi}\int_0^{\pi} \sin^2\phi\cos^2\theta\sin\phi \, d\phi \, d\theta = \int_0^{2\pi} \cos^2\theta \, d\theta \int_0^{\pi} \sin^3\phi \, d\phi$$

$$= \int_0^{2\pi} \frac{1}{2}(1 + \cos 2\theta) \, d\theta \int_0^{\pi} (\sin\phi - \sin\phi\cos^2\phi) \, d\phi$$

$$= \frac{1}{2}[\theta + \frac{1}{2}\sin 2\theta]_0^{2\phi} [-\cos\phi + \frac{1}{3}\cos^3\phi]_0^{\pi} = \frac{4\pi}{3}.$$

Definition

- If $\mathbf{F}$ is a continuous vector field defined on an oriented surface $\mathbf{S}$ with unit normal vector $\mathbf{n}$, then the **surface integral of F over S** is

$$\int\int_{\mathbf{S}} \mathbf{F} \cdot d\mathbf{S} = \int\int_{\mathbf{S}} \mathbf{F} \cdot \mathbf{n}\, d\mathbf{S}.$$

- This integral is also called the **flux** of $\mathbf{F}$ across $\mathbf{S}$.

Parametric Surface

- If $\mathbf{S}$ is given by a vector function $\mathbf{r}(u, v)$, then we have

$$\int\int_{\mathbf{S}} \mathbf{F} \cdot d\mathbf{S} = \int\int_{\mathbf{S}} \mathbf{F} \cdot \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}\, d\mathbf{S}$$

$$= \int\int_{\mathbf{D}} \left[\mathbf{F}(\mathbf{r}(u, v)) \cdot \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}\right] |\mathbf{r}_u \times \mathbf{r}_v|\, dA,$$

  where $\mathbf{D}$ is the parameter domain.

- Thus, we have

$$\int\int_{\mathbf{S}} \mathbf{F} \cdot d\mathbf{S} = \int\int_{\mathbf{D}} \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v)\, dA$$

Stokes' Theorem

- Let $\mathbf{S}$ be an oriented piecewise-smooth surface that is bounded by a simple, closed piecewise-smooth boundary curve $\mathbf{C}$ with positive orientation.
- Let $\mathbf{F}$ be a vector field whose components have continuous partial derivatives on an open region in $\mathbf{R}^3$ that contains $\mathbf{S}$.
- Then

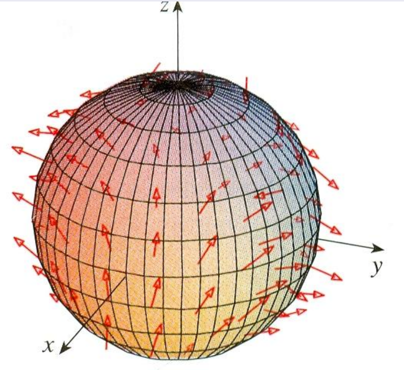
$$\int_{\mathbf{C}} \mathbf{F} \cdot d\mathbf{r} = \int\int_{\mathbf{S}} \text{curl } \mathbf{F} \cdot d\mathbf{S}$$

Divergence Theorem

- Let $\mathbf{E}$ be a simple solid region and let $\mathbf{S}$ be the boundary surface of $\mathbf{E}$, given with positive (outward) orientation.
- Let $\mathbf{F}$ be a vector field whose component functions have continuous partial derivatives on an open region that contains $\mathbf{E}$.
- Then
$$\int\int_{\mathbf{S}} \mathbf{F} \cdot d\mathbf{S} = \int\int\int_{\mathbf{E}} \operatorname{div} \mathbf{F} \, dV.$$

Example

Find the flux of the vector field $\mathbf{F}(x, y, z) = z\mathbf{i} + y\mathbf{j} + x\mathbf{k}$ over the unit sphere $x^2 + y^2 + z^2 = 1$.



This figure shows the vector field $\mathbf{F}(x, y, z) = z\mathbf{i} + y\mathbf{j} + x + \mathbf{k}$.

Example

Find the flux of the vector field $\mathbf{F}(x, y, z) = z\mathbf{i} + y\mathbf{j} + x\mathbf{k}$ over the unit sphere $x^2 + y^2 + z^2 = 1$.

Example

Find the flux of the vector field $\mathbf{F}(x, y, z) = z\mathbf{i} + y\mathbf{j} + x\mathbf{k}$ over the unit sphere $x^2 + y^2 + z^2 = 1$.

Solution

- First we compute the divergence of $\mathbf{F}$:

$$\text{div } \mathbf{F} = \frac{\partial}{\partial x}(z) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(x) = 1.$$

- The unit sphere $\mathbf{S}$ is the boundary of the unit ball $\mathbf{B}$ given by $x^2 + y^2 + z^2 \leq 1$.
- Thus, the Divergence Theorem gives the flux as

$$\int\int_{\mathbf{S}} \mathbf{F} \cdot d\mathbf{S} = \int\int\int_{\mathbf{B}} \text{div } \mathbf{F} dV = \int\int\int_{\mathbf{B}} 1 \ dV = V(\mathbf{B}) = \frac{4}{3}\pi(1)^3 = \frac{4\pi}{3}.$$

In Class Exercises 1

1. Draw a picture and locate the points $P(1,2,-3)$ and $Q(-2,5,3)$ in $\mathbf{R}^3$. Also draw $\overrightarrow{PQ}$ on your picture and label everything. Also find $|PQ|$.
2. Express the sphere $\mathbf{S}$: $\quad x^2+2x+y^2+6y+z^2-4z=11$ in standard form by completing squares. Then state what is the center and the radius of this sphere.
3. Three of the four vertices of a parallelogram are $P(2,-1,0)$, $Q(0,1,0)$ and $R(0,1,1)$. Two of the sides are PQ and PR. Find the coordinates of the fourth vertex $\mathbf{T}$.
4. Consider the parallelogram with vertices A, B, C, D such that B and C are adjacent to A. If $A=(2,5,2)$, $B=(3,1,4)$, $D=(5,2,-3)$, find the point C.
5. Consider the parallelogram with vertices A, B, C, D such that B and C are adjacent to A. If $A=(2,2)$, $B=(4,2)$, $D=(3,3)$, find the point C. Also draw a picture, labeling everything.

In Class Exercises 2

1. Find the **scalar projection** and **vector projection** of $\mathbf{b} = \langle 1, 1, 0 \rangle$ onto $\mathbf{a} = \langle -2, 3, 1 \rangle$.
2. Calculate the following determinant $\begin{vmatrix} 1 & 2 & -1 \\ 3 & 0 & 1 \\ -1 & 4 & 2 \end{vmatrix}$.
3. If $\mathbf{a} = \langle 1, 3, 4 \rangle$ and $\mathbf{b} = \langle 2, 0, -5 \rangle$, then find the cross product $\mathbf{a} \times \mathbf{b}$.
4. Find a vector perpendicular to the plane that passes through the points $P(1, 2, 0)$, $Q(-2, 0, -1)$, and $R(1, -1, 1)$.
5. Consider the points $A = (1, 0, 1)$, $B = (0, 2, 1)$ and $C = (-1, -1, 0)$. Find the area of the triangle $\boldsymbol{\Delta}$ with these vertices.

In Class Exercises 3

1. Consider the vectors $\mathbf{a} = \langle 1, 0, 1\rangle$, $\mathbf{b} = \langle 0, 2, 3\rangle$ and $\mathbf{c} = \langle -1, 2, 0\rangle$. Find the volume **V** of the parallelepiped or box spanned by these 3 vectors.
2. Find the volume **V** of the **parallelepiped** such that the following four points $A = (1, 0, 2)$, $B = (3, 1, -2)$, $C = (4, 3, -3)$, $D = (1, 0, -1)$ are vertices and the vertices B, C, D are all adjacent to the vertex A.
3. Find a **vector equation** and **parametric equations** for the line that passes through the point $(2, 1, 3)$ and is parallel to the vector $\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$.
4. Find **parametric equations** for the line which contains $A(2, 0, 1)$ and $B(-1, 1, -2)$.
5. Find the **equation of the plane** passing through points $P = (1, 0, 2)$, $Q = (3, 2, 3)$, $R = (2, 0, 3)$.

In Class Exercises 4

1. Find the **equation of the plane** passing through points $P = (1, 0, 2), \quad Q = (3, 2, 3), \quad R = (2, 0, 3)$.
2. Find the angle between the planes $x + y + z = 1$ and $x - 2y + 3z = 1$.
3. Find the **vector equation** of the line **L** of intersection of the planes $x + y + z = 1$ and $x - 2y + 3z = 1$.
4. Find the distance **D** from the point $(1, 2, -1)$ to the plane $2x + y - 2z = 1$.
5. Identify the surfaces with their equations:
   1. Parabolic Cylinder
   2. Cylinder
   3. Ellipsoid
   4. Elliptic Paraboloid
   5. Hyperbolic Paraboloid

   (a) $x^2 + y^2 = 1$
   (b) $z = y^2 - x^2$
   (c) $z = x^2$
   (d) $x^2 + \frac{y^2}{9} + \frac{z^2}{4} = 1$
   (e) $z = 4x^2 + y^2$

In Class Exercises 5

1. Suppose $\mathbf{r}(t) = (2t, t^2, t^3)$. Find the **equation of the tangent line** $\mathbf{L}$ at the point $P = (2, 1, 1)$.
2. If $\mathbf{r}(t) = \langle \sin(t), \cos(t), 3t \rangle$, then what is the length of $\mathbf{r}(t)$ from time $t = 1$ to time $t = 6$?
3. A moving particle starts at an initial position $\mathbf{r}(0) = \langle 1, 0, 0 \rangle$ with initial velocity $\mathbf{v}(0) = \langle 1, 1, -1 \rangle$. Its acceleration is $\mathbf{a}(t) = 4t\mathbf{i} + 6t\mathbf{j} + \mathbf{k}$. Find its velocity vector at time t.
4. A moving particle starts at an initial position $\mathbf{r}(0) = \langle 1, 0, 0 \rangle$. Its velocity is $\mathbf{v}(t) = 4\mathbf{i} + 6t\mathbf{j} + \mathbf{k}$. Find its position at time t.
5. Use the formula
$$\kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$$
to calculate the curvature of $\mathbf{r}(t) = (2t, t^2, t^3)$ at the point $P = (2, 1, 1)$.

In Class Exercises 6

1. Explain why the **limit** of $f(x, y) = (x^2y^2)/(x^4 + y^4)$ does not exist as (x, y) approaches $(0, 0)$.
2. For the function $f(x, y) = 2x^2 + xy^2$, calculate f_x, f_y, f_{xy}, f_{xx}.
3. Find the **tangent plane** to the elliptic paraboloid $z = 2x^2 + y^2$ at the point $(1, 1, 3)$.
4. Find the **linear approximation** $\mathbf{L}(x, y)$ to the function $f(x, y) = 2x^2 + y$ at the point $(1, 2)$ and use it to estimate $f(1.1, 1.8)$.
5. Find the **linear approximation** $\mathbf{L}(x, y)$ to $f(x, y) = xe^y$ at $(1, 2)$.

In Class Exercises 7

1. If $z = x^2y + y^4$, where $x = \sin 2t$ and $y = \cos t$, find $\frac{\mathbf{dz}}{\mathbf{dt}}$ when $t = 0$.
2. If $z = e^x \sin y$, where $x = s$ and $y = s^2t$, find $\frac{\partial \mathbf{z}}{\partial \mathbf{s}}$.
3. If $z = e^x \sin y$, where $x = s$ and $y = s^2t$, find $\frac{\partial \mathbf{z}}{\partial \mathbf{t}}$.
4. Find $\frac{\partial \mathbf{z}}{\partial \mathbf{x}}$ if $y^2 + z^3 + xyz = 1$.
5. Find $\frac{\partial \mathbf{z}}{\partial \mathbf{y}}$ if $y^2 + z^3 + xyz = 1$.

In Class Exercises 8

1. For the function $f(x,y) = 2x^2 + xy^2$, calculate f_x, f_y, f_{xy}, f_{xx}:
2. What is the **gradient** $\nabla f(x,y)$ of f at the point $(1,2)$? Also, what is the **maximum rate of change** of f at $(1,2)$.
3. Assuming that $\nabla f(1,2) = \langle 8,4 \rangle$, calculate the **directional derivative** of f at the point $(1,2)$ in the direction of the vector $\mathbf{v} = \langle 3,4 \rangle$?
4. Show the **linearization** $\mathbf{L}(x,y)$ of $f(x,y)$ at $(1,2)$ is $\mathbf{L}(x,y) = 6 + 8(x-1) + 4(x-2)$.
5. Use the **linearization** $\mathbf{L}(x,y) = 6 + 8(x-1) + 4(x-2)$ in the previous part to estimate $f(0.9, 2.1)$.

In Class Exercises 9

1. Find the equation of the **tangent plane** at the point $(-2, 1, -3)$ to the ellipsoid $\frac{\mathbf{x^2}}{\mathbf{4}} + \mathbf{y^2} + \frac{\mathbf{z^2}}{\mathbf{9}} = \mathbf{3}$.
2. Find the equation of the **normal line** at the point $(-2, 1, -3)$ to the ellipsoid $\frac{\mathbf{x^2}}{\mathbf{4}} + \mathbf{y^2} + \frac{\mathbf{z^2}}{\mathbf{9}} = \mathbf{3}$.
3. Find the equation of the **tangent plane** at the point $(0, 1, 3)$ to the ellipsoid $\frac{\mathbf{x^2}}{\mathbf{4}} + \mathbf{y^2} + \frac{\mathbf{z^2}}{\mathbf{9}} = \mathbf{2}$.
4. (True or False) Recall the **Hessian**:

$$\mathbf{D} = \begin{vmatrix} \mathbf{f_{xx}} & \mathbf{f_{xy}} \\ \mathbf{f_{yx}} & \mathbf{f_{yy}} \end{vmatrix} = \mathbf{f_{xx}f_{yy}} - (\mathbf{f_{xy}})^2.$$

   (a) Given a function $f(x, y)$, if (a, b) is a critical point and $\mathbf{D} < 0$, then (a, b) is a **saddle point**.
   (b) If $\mathbf{D} > 0$, then $f(a, b)$ is a **local minimum**.
   (c) If $\mathbf{D} < 0$, then $f(a, b)$ is a **local minimum**.
5. Find the critical points of $f(x, y) = x^2 + 2y^2$. What is the **Hessian** $\mathbf{D}$ at this critical point?

In Class Exercises 10

1. Evaluate the iterated integral

$$\int_0^3 \int_0^1 x^2 y \, dy \, dx.$$

2. Evaluate the double integral $\int\int_{\mathbf{R}} (x - 3y^2) \, dA$, where $\mathbf{R} = \{(x, y) \mid 0 \leq x \leq 2, \ 1 \leq y \leq 2\}$.

3. Find the volume of the solid **S** that is bounded by the elliptic paraboloid $x^2 + 2y^2 + z = 16$, the planes $x = 2$ and $y = 2$, and the three coordinate planes.

4. Evaluate $\int\int_{\mathbf{D}} (x + 2y) \, dA$, where **D** is the region bounded by the parabolas $y = 2x^2$ and $y = 1 + x^2$.

5. Find the volume **V** of the solid that lies under the paraboloid $z = x^2 + y^2$ and above the region **D** in the xy-plane bounded by the line $y = 1$ and the parabola $y = x^2$.

In Class Exercises 11

1. Find the volume **V** of the solid that lies under the paraboloid $z = x^2 + y^2$ and above the region **D** in the xy-plane bounded by the line $y = x$ and the parabola $y = x^2$.

2. Consider the iterated integral $\mathbf{I} = \int_0^1 \int_{2x}^2 \sin(y^2)\, dy\, dx$.

   (a) Sketch the region of integration.
   (b) Write the integral **I** with the order of integration reversed.

3. Consider the double integral $\mathbf{I} = \int\int_{\mathbf{R}} x^2 y - x \; dA$ where **R** is the first quadrant region enclosed by the curves $y = 0$, $y = x^2$ and $y = 2 - x$.

   (a) Sketch the region of integration.
   (b) Express the integral **I** as an iterated integral.

4. Find an equivalent iterated integral expression for the double integral $\int_0^1 \int_x^1 \sin(y^2)\, dy\, dx$, where the order of integration is reversed.

In Class Exercises 12

1. Express the volume **V** of the solid that lies under the paraboloid $z = 4 - x^2 - y^2$ and above the xy-plane as an iterated integral in polar coordinates.
2. Consider the integral $\int\int_{\mathbf{R}} y\sqrt{x^2 + y^2}\, dA$ with **R** the region $\{(x, y)\colon 1 \leq x^2 + y^2 \leq 2,\ \ 0 \leq y \leq x\}$
   - (a) First describe the domain **R** in polar coordinates.
   - (b) Write the integral as an iterated integral in polar coordinates.
3. Consider the integral $\int_{\mathbf{C}} 2x\, ds$, where **C** consists of the arc **C** of the parabola $y = x^2$ from $(0, 0)$ to $(1, 1)$.
   - (a) Parameterize **C** by a curve $\mathbf{r}(t)$, $0 \leq t \leq 1$.
   - (b) Express this integral as an integral of the form $\int_0^1 \mathbf{F}(t)\, dt$, i.e., find $\mathbf{F}(t)$.
4. Evaluate $\int_{\mathbf{C}} y^2\, dx + x\, dy$, where **C** is the line segment from $(-5, -3)$ to $(0, 2)$, by first parameterizing **C**.
5. Evaluate $\int_{\mathbf{C}} y\, dx + z\ dy + x\, dz$, where **C** consists of the line segment from $(4, 0, 0)$ to $(3, 4, 5)$, by first parameterizing **C**.

In Class Exercises 13

1. Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F} = \nabla(f(x, y) = 3x + x^2y - y^3)$ and $\mathbf{C}$ is the curve given by $\mathbf{r}(t) = e^t \sin t\,\mathbf{i} + e^t \cos t\,\mathbf{j}$, $0 \leq t \leq \pi$.

2. Determine whether the following vector fields are conservative or not.

   (a) $\mathbf{F}(x, y) = (x^2 + xy)\mathbf{i} + (xy - y^2)\mathbf{j}$.
   (b) $\mathbf{F}(x, y) = (3x^2y + y^2)\mathbf{i} + (x^3 + 2xy + 3y^2)\mathbf{j}$.

3. Find a **potential function** for the vector field $\mathbf{F} = \langle 2y^2 + y^2\cos(xy^2), 4xy + 2xy\cos(xy^2)\rangle$.

4. Use Green's Theorem to show that if $\mathbf{D} \subset \mathbf{R}^2$ is the bounded region with boundary a positively oriented simple closed curve $\mathbf{C}$, then the area of $\mathbf{D}$ can be calculated by the formula:
   Area$(\mathbf{D}) = \frac{1}{2}\oint_{\mathbf{C}} -y\,dx + x\,dy$

5. Consider the ellipse $4x^2 + y^2 = 1$. Use the above area formula to calculate the area of the region $\mathbf{D} \subset \mathbf{R}^2$ with boundary this ellipse. (Hint: This ellipse can be parameterized by $\mathbf{r}(t) = \langle \frac{1}{2}\cos(t), \sin(t)\rangle$ for $0 \leq t \leq 2\pi$.)

In Class Exercises 14

1. Consider the vector field $\mathbf{F} = \langle y^2, -xy^2, 2xyz \rangle$.
    1. Calculate curl $\mathbf{F}$.
    2. Is F conservative? Why?
    3. Calculate $\operatorname{div} \mathbf{F}$.
    4. Is F incompressible? Why?
2. Consider the parameterized surface $\mathbf{S}$ which is given by $\mathbf{r}(u, v) = \langle u, 2\sin(v), \cos(v) \rangle : [0, 1] \times [0, \pi] \to \mathbf{R^3}$. Write down an integral formula for the area of $\mathbf{S}$ and calculate it.
3. Use the Divergence Theorem to calculate the flux of the vector field $\mathbf{F}(x, y, z) = z\mathbf{i} + y\mathbf{j} + 2x\mathbf{k}$ over the unit sphere $x^2 + y^2 + z^2 = 1$.