

5 章 三角関数

§3 加法定理とその応用 (p.158~p.166)

問 1

$$\begin{aligned}
 \sin 75^\circ &= \sin(45^\circ + 30^\circ) \\
 &= \sin 45^\circ \cos 30^\circ + \cos 45^\circ \sin 30^\circ \\
 &= \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \cdot \frac{1}{2} \\
 &= \frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4} \\
 &= \frac{\sqrt{6} + \sqrt{2}}{4}
 \end{aligned}$$

$$\begin{aligned}
 \tan 75^\circ &= \tan(45^\circ + 30^\circ) \\
 &= \frac{\tan 45^\circ + \tan 30^\circ}{1 - \tan 45^\circ \tan 30^\circ} \\
 &= \frac{1 + \frac{1}{\sqrt{3}}}{1 - 1 \cdot \frac{1}{\sqrt{3}}} \\
 &= \frac{\left(1 + \frac{1}{\sqrt{3}}\right) \times \sqrt{3}}{\left(1 - \frac{1}{\sqrt{3}}\right) \times \sqrt{3}} \\
 &= \frac{\sqrt{3} + 1}{\sqrt{3} - 1} \\
 &= \frac{(\sqrt{3} + 1)^2}{(\sqrt{3} - 1)(\sqrt{3} + 1)} \\
 &= \frac{3 + 2\sqrt{3} + 1}{3 - 1} \\
 &= \frac{4 + 2\sqrt{3}}{2} = 2 + \sqrt{3}
 \end{aligned}$$

【別解】

$$\begin{aligned}
 \tan 75^\circ &= \frac{\sin 75^\circ}{\cos 75^\circ} = \frac{\frac{\sqrt{6} + \sqrt{2}}{4}}{\frac{\sqrt{6} - \sqrt{2}}{4}} \\
 &= \frac{\sqrt{6} + \sqrt{2}}{\sqrt{6} - \sqrt{2}} = \frac{(\sqrt{6} + \sqrt{2})^2}{(\sqrt{6} - \sqrt{2})(\sqrt{6} + \sqrt{2})} \\
 &= \frac{6 + 2 \cdot 2\sqrt{3} + 2}{6 - 2} \\
 &= \frac{8 + 4\sqrt{3}}{4} = 2 + \sqrt{3}
 \end{aligned}$$

$$\begin{aligned}
 \sin 15^\circ &= \sin(45^\circ - 30^\circ) \\
 &= \sin 45^\circ \cos 30^\circ - \cos 45^\circ \sin 30^\circ
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2} \cdot \frac{1}{2} \\
 &= \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4} \\
 &= \frac{\sqrt{6} - \sqrt{2}}{4}
 \end{aligned}$$

$$\begin{aligned}
 \cos 15^\circ &= \cos(45^\circ - 30^\circ) \\
 &= \cos 45^\circ \cos 30^\circ + \sin 45^\circ \sin 30^\circ \\
 &= \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \cdot \frac{1}{2} \\
 &= \frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4} \\
 &= \frac{\sqrt{6} + \sqrt{2}}{4}
 \end{aligned}$$

$$\begin{aligned}
 \tan 15^\circ &= \tan(45^\circ - 30^\circ) \\
 &= \frac{\tan 45^\circ - \tan 30^\circ}{1 + \tan 45^\circ \tan 30^\circ} \\
 &= \frac{1 - \frac{1}{\sqrt{3}}}{1 + 1 \cdot \frac{1}{\sqrt{3}}} \\
 &= \frac{\left(1 - \frac{1}{\sqrt{3}}\right) \times \sqrt{3}}{\left(1 + \frac{1}{\sqrt{3}}\right) \times \sqrt{3}} \\
 &= \frac{\sqrt{3} - 1}{\sqrt{3} + 1} \\
 &= \frac{(\sqrt{3} - 1)^2}{(\sqrt{3} + 1)(\sqrt{3} - 1)} \\
 &= \frac{3 - 2\sqrt{3} + 1}{3 - 1} \\
 &= \frac{4 - 2\sqrt{3}}{2} = 2 - \sqrt{3}
 \end{aligned}$$

【別解】

$$\begin{aligned}
 \tan 15^\circ &= \frac{\sin 15^\circ}{\cos 15^\circ} = \frac{\frac{\sqrt{6} - \sqrt{2}}{4}}{\frac{\sqrt{6} + \sqrt{2}}{4}} \\
 &= \frac{\sqrt{6} - \sqrt{2}}{\sqrt{6} + \sqrt{2}} = \frac{(\sqrt{6} - \sqrt{2})^2}{(\sqrt{6} + \sqrt{2})(\sqrt{6} - \sqrt{2})} \\
 &= \frac{6 - 2 \cdot 2\sqrt{3} + 2}{6 - 2}
 \end{aligned}$$

$$= \frac{8 - 4\sqrt{3}}{4} = 2 - \sqrt{3}$$

問2

$$\begin{aligned} \text{与式} &= \frac{\tan \theta + \tan \frac{\pi}{4}}{1 - \tan \theta \tan \frac{\pi}{4}} \\ &= \frac{\tan \theta + 1}{1 - \tan \theta \cdot 1} \\ &= \frac{1 + \tan \theta}{1 - \tan \theta} \end{aligned}$$

問3

α は第2象限の角だから, $\cos \alpha < 0$

よって

$$\begin{aligned} \cos \alpha &= -\sqrt{1 - \sin^2 \alpha} \\ &= -\sqrt{1 - \left(\frac{1}{3}\right)^2} \\ &= -\sqrt{1 - \frac{1}{9}} \\ &= -\sqrt{\frac{8}{9}} = -\frac{2\sqrt{2}}{3} \end{aligned}$$

β は第2象限の角だから, $\cos \beta > 0$

よって

$$\begin{aligned} \cos \beta &= \sqrt{1 - \sin^2 \beta} \\ &= \sqrt{1 - \left(-\frac{2}{5}\right)^2} \\ &= \sqrt{1 - \frac{4}{25}} \\ &= \sqrt{\frac{21}{25}} = \frac{\sqrt{21}}{5} \end{aligned}$$

したがって

$$\begin{aligned} \sin(\alpha + \beta) &= \sin \alpha \cos \beta + \cos \alpha \sin \beta \\ &= \frac{1}{3} \cdot \frac{\sqrt{21}}{5} + \left(-\frac{2\sqrt{2}}{3}\right) \cdot \left(-\frac{2}{5}\right) \\ &= \frac{\sqrt{21}}{15} + \frac{4\sqrt{2}}{15} \\ &= \frac{\sqrt{21} + 4\sqrt{2}}{15} \end{aligned}$$

$$\begin{aligned} \cos(\alpha + \beta) &= \cos \alpha \cos \beta - \sin \alpha \sin \beta \\ &= \left(-\frac{2\sqrt{2}}{3}\right) \cdot \left(\frac{\sqrt{21}}{5}\right) - \frac{1}{3} \cdot \left(-\frac{2}{5}\right) \\ &= -\frac{2\sqrt{42}}{15} + \frac{2}{15} \end{aligned}$$

$$= \frac{2 - 2\sqrt{42}}{15}$$

問4

$0 < \alpha < \frac{\pi}{2}$, $0 < \beta < \frac{\pi}{2}$ の辺々を加えると

$$0 < \alpha + \beta < \frac{\pi}{2} + \frac{\pi}{2}$$

すなわち, $0 < \alpha + \beta < \pi \cdots \textcircled{1}$

$$\tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}$$

$$= \frac{\frac{1}{2} + \frac{1}{3}}{1 - \frac{1}{2} \cdot \frac{1}{3}}$$

$$= \frac{\frac{5}{6}}{1 - \frac{1}{6}} = \frac{\frac{5}{6}}{\frac{5}{6}} = 1$$

また, ①より, $\alpha + \beta = \frac{\pi}{4}$

問5

α は第4象限の角だから, $\sin \alpha < 0$

よって

$$\begin{aligned} \sin \alpha &= -\sqrt{1 - \cos^2 \alpha} \\ &= -\sqrt{1 - \left(\frac{3}{5}\right)^2} \\ &= -\sqrt{1 - \frac{9}{25}} \\ &= -\sqrt{\frac{16}{25}} = -\frac{4}{5} \end{aligned}$$

したがって

$$\begin{aligned} \sin 2\alpha &= 2 \sin \alpha \cos \alpha \\ &= 2 \cdot \left(-\frac{4}{5}\right) \cdot \frac{3}{5} \\ &= -\frac{24}{25} \end{aligned}$$

$$\begin{aligned} \cos 2\alpha &= 2 \cos^2 \alpha - 1 \\ &= 2 \cdot \left(\frac{3}{5}\right)^2 - 1 \end{aligned}$$

$$= \frac{18}{25} - 1 = -\frac{7}{25}$$

$$\tan 2\alpha = \frac{\sin 2\alpha}{\cos 2\alpha}$$

$$= \frac{-\frac{24}{25}}{-\frac{7}{25}} = \frac{24}{7}$$

問 6

$$\begin{aligned}\cos^2 \frac{\pi}{8} &= \cos^2 \frac{\frac{\pi}{4}}{2} \\ &= \frac{1 + \cos \frac{\pi}{4}}{2} \\ &= \frac{1 + \frac{1}{\sqrt{2}}}{2} \\ &= \frac{\left(1 + \frac{\sqrt{2}}{2}\right) \times 2}{2 \times 2} \\ &= \frac{2 + \sqrt{2}}{4}\end{aligned}$$

$\cos \frac{\pi}{8} > 0$ であるから

$$\cos \frac{\pi}{8} = \sqrt{\frac{2 + \sqrt{2}}{4}} = \frac{\sqrt{2 + \sqrt{2}}}{2}$$

問 7

$$\frac{\pi}{2} < \alpha < \pi \text{ より, } \frac{\pi}{4} < \frac{\alpha}{2} < \frac{\pi}{2} \cdots \textcircled{1}$$

$$\begin{aligned}\sin^2 \frac{\alpha}{2} &= \frac{1 - \cos \alpha}{2} \\ &= \frac{1 - \left(-\frac{7}{9}\right)}{2} \\ &= \frac{\frac{16}{9}}{2} = \frac{8}{9}\end{aligned}$$

①より, $\sin \frac{\alpha}{2} > 0$ であるから

$$\sin \frac{\alpha}{2} = \sqrt{\frac{8}{9}} = \frac{2\sqrt{2}}{3}$$

また

$$\begin{aligned}\cos^2 \frac{\alpha}{2} &= \frac{1 + \cos \alpha}{2} \\ &= \frac{1 + \left(-\frac{7}{9}\right)}{2} \\ &= \frac{\frac{2}{9}}{2} = \frac{1}{9}\end{aligned}$$

①より, $\cos \frac{\alpha}{2} > 0$ であるから

$$\begin{aligned}\cos \frac{\alpha}{2} &= \sqrt{\frac{1}{9}} = \frac{1}{3} \\ \tan \frac{\alpha}{2} &= \frac{\sin \frac{\alpha}{2}}{\cos \frac{\alpha}{2}} \\ &= \frac{\frac{2\sqrt{2}}{3}}{\frac{1}{3}} = 2\sqrt{2}\end{aligned}$$

問 8

$$(1) \text{ 与式 } = \frac{1}{2} \{ \cos(2\theta + 5\theta) + \cos(2\theta - 5\theta) \}$$

$$= \frac{1}{2} \{ \cos 7\theta + \cos(-3\theta) \}$$

$$= \frac{1}{2} (\cos 7\theta + \cos 3\theta)$$

$$(2) \text{ 与式 } = -\frac{1}{2} \{ \cos(4\theta + 3\theta) - \cos(4\theta - 3\theta) \}$$

$$= -\frac{1}{2} (\cos 7\theta - \cos \theta)$$

$$= \frac{1}{2} (\cos \theta - \cos 7\theta)$$

$$(3) \text{ 与式 } = \frac{1}{2} \{ \sin(5\theta + 3\theta) - \sin(5\theta - 3\theta) \}$$

$$= \frac{1}{2} (\sin 8\theta - \sin 2\theta)$$

問 9

$$(1) \text{ 与式 } = 2 \sin \frac{5\theta + 3\theta}{2} \cos \frac{5\theta - 3\theta}{2}$$

$$= 2 \sin \frac{8\theta}{2} \cos \frac{2\theta}{2}$$

$$= 2 \sin 4\theta \cos \theta$$

$$(2) \text{ 与式 } = 2 \cos \frac{2\theta + 4\theta}{2} \cos \frac{2\theta - 4\theta}{2}$$

$$= 2 \cos \frac{6\theta}{2} \cos \frac{-2\theta}{2}$$

$$= 2 \cos 3\theta \cos(-\theta)$$

$$= 2 \cos 3\theta \cos \theta$$

$$(3) \text{ 与式 } = 2 \cos \frac{6\theta + 2\theta}{2} \sin \frac{6\theta - 2\theta}{2}$$

$$= 2 \cos \frac{8\theta}{2} \sin \frac{4\theta}{2}$$

$$= 2 \cos 4\theta \sin 2\theta$$

問 10

$$(1) \ y = \sqrt{1^2 + 1^2} \sin(x + \alpha)$$

$$= \sqrt{2} \sin(x + \alpha)$$

ここで,

$$\cos \alpha = \frac{1}{\sqrt{2}}, \quad \sin \alpha = \frac{1}{\sqrt{2}} \text{ より, } \alpha = \frac{\pi}{4}$$

$$\text{よって, } y = \sqrt{2} \sin\left(x + \frac{\pi}{4}\right)$$

$$(2) \ y = \sqrt{1^2 + (\sqrt{3})^2} \sin(x + \alpha)$$

$$= 2 \sin(x + \alpha)$$

ここで,

$$\cos \alpha = \frac{1}{2}, \quad \sin \alpha = -\frac{\sqrt{3}}{2} \text{ より, } \alpha = -\frac{\pi}{3}$$

$$\text{よって, } y = 2 \sin\left(x - \frac{\pi}{3}\right)$$

問 11

$$y = \sqrt{1^2 + (-1)^2} \sin(x + \alpha)$$

$$= \sqrt{2} \sin(x + \alpha)$$

$$\text{また, } \cos \alpha = \frac{1}{\sqrt{2}}, \quad \sin \alpha = -\frac{1}{\sqrt{2}} \text{ より, } \alpha = -\frac{\pi}{4}$$

$$y = \sqrt{2} \sin\left(x - \frac{\pi}{4}\right)$$

ここで, $0 \leq x \leq 2\pi$ であるから

$$-\frac{\pi}{4} \leq x - \frac{\pi}{4} \leq 2\pi - \frac{\pi}{4}$$

$$x - \frac{\pi}{4} = \frac{\pi}{2} \text{ すなわち, } x = \frac{3}{4}\pi \text{ のとき}$$

$$\text{最大値 } \sqrt{2} \sin \frac{\pi}{2} = \sqrt{2}$$

$$x - \frac{\pi}{4} = \frac{3}{2}\pi \text{ すなわち, } x = \frac{7}{4}\pi \text{ のとき}$$

$$\text{最小値 } \sqrt{2} \sin \frac{3}{2}\pi = -\sqrt{2}$$

したがって,

$$\text{最大値 } \sqrt{2} \quad \left(x = \frac{3}{4}\pi\right)$$

$$\text{最大値 } -\sqrt{2} \quad \left(x = \frac{7}{4}\pi\right)$$