問 1

$$\sin 75^\circ = \sin(45^\circ + 30^\circ)$$
$$= \sin 45^\circ \cos 30^\circ + \cos 45^\circ \sin 30^\circ$$
$$= \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2}\cdot\frac{1}{2}$$
$$= \frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4}$$
$$= \boldsymbol{\frac{\sqrt{6}+\sqrt{2}}{4}}$$

$$\tan 75^\circ = \tan(45^\circ + 30^\circ)$$
$$= \frac{\tan 45^\circ + \tan 30^\circ}{1 - \tan 45^\circ \tan 30^\circ}$$
$$= \frac{1 + \frac{1}{\sqrt{3}}}{1 - 1\cdot\frac{1}{\sqrt{3}}}$$
$$= \frac{\left(1 + \frac{1}{\sqrt{3}}\right)\times\sqrt{3}}{\left(1 - \frac{1}{\sqrt{3}}\right)\times\sqrt{3}}$$
$$= \frac{\sqrt{3}+1}{\sqrt{3}-1}$$
$$= \frac{\left(\sqrt{3}+1\right)^2}{\left(\sqrt{3}-1\right)\left(\sqrt{3}+1\right)}$$
$$= \frac{3+2\sqrt{3}+1}{3-1}$$
$$= \frac{4+2\sqrt{3}}{2} = \boldsymbol{2+\sqrt{3}}$$

【別解】

$$\tan 75^\circ = \frac{\sin 75^\circ}{\cos 75^\circ} = \frac{\frac{\sqrt{6}+\sqrt{2}}{4}}{\frac{\sqrt{6}-\sqrt{2}}{4}}$$
$$= \frac{\sqrt{6}+\sqrt{2}}{\sqrt{6}-\sqrt{2}} = \frac{\left(\sqrt{6}+\sqrt{2}\right)^2}{\left(\sqrt{6}-\sqrt{2}\right)\left(\sqrt{6}+\sqrt{2}\right)}$$
$$= \frac{6+2\cdot 2\sqrt{3}+2}{6-2}$$
$$= \frac{8+4\sqrt{3}}{4} = \boldsymbol{2+\sqrt{3}}$$

$$\sin 15^\circ = \sin(45^\circ - 30^\circ)$$
$$= \sin 45^\circ \cos 30^\circ - \cos 45^\circ \sin 30^\circ$$
$$= \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}\cdot\frac{1}{2}$$
$$= \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4}$$
$$= \boldsymbol{\frac{\sqrt{6}-\sqrt{2}}{4}}$$

$$\cos 15^\circ = \cos(45^\circ - 30^\circ)$$
$$= \cos 45^\circ \cos 30^\circ + \sin 45^\circ \sin 30^\circ$$
$$= \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2}\cdot\frac{1}{2}$$
$$= \frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4}$$
$$= \boldsymbol{\frac{\sqrt{6}+\sqrt{2}}{4}}$$

$$\tan 15^\circ = \tan(45^\circ - 30^\circ)$$
$$= \frac{\tan 45^\circ - \tan 30^\circ}{1 + \tan 45^\circ \tan 30^\circ}$$
$$= \frac{1 - \frac{1}{\sqrt{3}}}{1 + 1\cdot\frac{1}{\sqrt{3}}}$$
$$= \frac{\left(1 - \frac{1}{\sqrt{3}}\right)\times\sqrt{3}}{\left(1 + \frac{1}{\sqrt{3}}\right)\times\sqrt{3}}$$
$$= \frac{\sqrt{3}-1}{\sqrt{3}+1}$$
$$= \frac{\left(\sqrt{3}-1\right)^2}{\left(\sqrt{3}+1\right)\left(\sqrt{3}-1\right)}$$
$$= \frac{3-2\sqrt{3}+1}{3-1}$$
$$= \frac{4-2\sqrt{3}}{2} = \boldsymbol{2-\sqrt{3}}$$

【別解】

$$\tan 15^\circ = \frac{\sin 15^\circ}{\cos 15^\circ} = \frac{\frac{\sqrt{6}-\sqrt{2}}{4}}{\frac{\sqrt{6}+\sqrt{2}}{4}}$$
$$= \frac{\sqrt{6}-\sqrt{2}}{\sqrt{6}+\sqrt{2}} = \frac{\left(\sqrt{6}-\sqrt{2}\right)^2}{\left(\sqrt{6}+\sqrt{2}\right)\left(\sqrt{6}-\sqrt{2}\right)}$$
$$= \frac{6-2\cdot 2\sqrt{3}+2}{6-2}$$

$$= \frac{8 - 4\sqrt{3}}{4} = \mathbf{2 - \sqrt{3}}$$

$\boxed{問 2}$

$$与式 = \frac{\tan\theta + \tan\frac{\pi}{4}}{1 - \tan\theta\tan\frac{\pi}{4}}$$

$$= \frac{\tan\theta + 1}{1 - \tan\theta \cdot 1}$$

$$= \boldsymbol{\frac{1 + \tan\theta}{1 - \tan\theta}}$$

$\boxed{問 3}$

αは第 2 象限の角だから，$\cos\alpha < 0$

よって

$$\cos\alpha = -\sqrt{1 - \sin^2\alpha}$$

$$= -\sqrt{1 - \left(\frac{1}{3}\right)^1}$$

$$= -\sqrt{1 - \frac{1}{9}}$$

$$= -\sqrt{\frac{8}{9}} = -\frac{2\sqrt{2}}{3}$$

βは第 2 象限の角だから，$\cos\beta > 0$

よって

$$\cos\beta = \sqrt{1 - \sin^2\beta}$$

$$= \sqrt{1 - \left(-\frac{2}{5}\right)^2}$$

$$= \sqrt{1 - \frac{4}{25}}$$

$$= \sqrt{\frac{21}{25}} = \frac{\sqrt{21}}{5}$$

したがって

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

$$= \frac{1}{3} \cdot \frac{\sqrt{21}}{5} + \left(-\frac{2\sqrt{2}}{3}\right) \cdot \left(-\frac{2}{5}\right)$$

$$= \frac{\sqrt{21}}{15} + \frac{4\sqrt{2}}{15}$$

$$= \boldsymbol{\frac{\sqrt{21} + 4\sqrt{2}}{15}}$$

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$$= \left(-\frac{2\sqrt{2}}{3}\right) \cdot \left(\frac{\sqrt{21}}{5}\right) - \frac{1}{3} \cdot \left(-\frac{2}{5}\right)$$

$$= -\frac{2\sqrt{42}}{15} + \frac{2}{15}$$

$$= \boldsymbol{\frac{2 - 2\sqrt{42}}{15}}$$

$\boxed{問 4}$

$0 < \alpha < \frac{\pi}{2}$，$0 < \beta < \frac{\pi}{2}$の辺々を加えると

$$0 < \alpha + \beta < \frac{\pi}{2} + \frac{\pi}{2}$$

すなわち，$0 < \alpha + \beta < \pi$・・・①

$$\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

$$= \frac{\frac{1}{2} + \frac{1}{3}}{1 - \frac{1}{2} \cdot \frac{1}{3}}$$

$$= \frac{\frac{5}{6}}{1 - \frac{1}{6}} = \frac{\frac{5}{6}}{\frac{5}{6}} = \mathbf{1}$$

また，①より，$\alpha + \beta = \boldsymbol{\frac{\pi}{4}}$

$\boxed{問 5}$

αは第 4 象限の角だから，$\sin\alpha < 0$

よって

$$\sin\alpha = -\sqrt{1 - \cos^2\alpha}$$

$$= -\sqrt{1 - \left(\frac{3}{5}\right)^2}$$

$$= -\sqrt{1 - \frac{9}{25}}$$

$$= -\sqrt{\frac{16}{25}} = -\frac{4}{5}$$

したがって

$$\sin 2\alpha = 2\sin\alpha\cos\alpha$$

$$= 2 \cdot \left(-\frac{4}{5}\right) \cdot \frac{3}{5}$$

$$= -\boldsymbol{\frac{24}{25}}$$

$$\cos 2\alpha = 2\cos^2\alpha - 1$$

$$= 2 \cdot \left(\frac{3}{5}\right)^2 - 1$$

$$= \frac{18}{25} - 1 = -\boldsymbol{\frac{7}{25}}$$

$$\tan 2\alpha = \frac{\sin 2\alpha}{\cos 2\alpha}$$

$$= \frac{-\frac{24}{25}}{-\frac{7}{25}} = \mathbf{\frac{24}{7}}$$

問６

$$\cos^2\frac{\pi}{8} = \cos^2\frac{\frac{\pi}{4}}{2}$$

$$= \frac{1+\cos\frac{\pi}{4}}{2}$$

$$= \frac{1+\frac{1}{\sqrt{2}}}{2}$$

$$= \frac{\left(1+\frac{\sqrt{2}}{2}\right)\times 2}{2\times 2}$$

$$= \frac{2+\sqrt{2}}{4}$$

$\cos\frac{\pi}{8} > 0$ であるから

$$\cos\frac{\pi}{8} = \sqrt{\frac{2+\sqrt{2}}{4}} = \mathbf{\frac{\sqrt{2+\sqrt{2}}}{2}}$$

問７

$\frac{\pi}{2} < \alpha < \pi$より，$\frac{\pi}{4} < \frac{\alpha}{2} < \frac{\pi}{2}$ ・・・①

$$\sin^2\frac{\alpha}{2} = \frac{1-\cos\alpha}{2}$$

$$= \frac{1-\left(-\frac{7}{9}\right)}{2}$$

$$= \frac{\frac{16}{9}}{2} = \frac{8}{9}$$

①より，$\sin\frac{\alpha}{2} > 0$ であるから

$$\sin\frac{\alpha}{2} = \sqrt{\frac{8}{9}} = \mathbf{\frac{2\sqrt{2}}{3}}$$

また

$$\cos^2\frac{\alpha}{2} = \frac{1+\cos\alpha}{2}$$

$$= \frac{1+\left(-\frac{7}{9}\right)}{2}$$

$$= \frac{\frac{2}{9}}{2} = \frac{1}{9}$$

①より，$\cos\frac{\alpha}{2} > 0$ であるから

$$\cos\frac{\alpha}{2} = \sqrt{\frac{1}{9}} = \mathbf{\frac{1}{3}}$$

$$\tan\frac{\alpha}{2} = \frac{\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}}$$

$$= \frac{\frac{2\sqrt{2}}{3}}{\frac{1}{3}} = \mathbf{2\sqrt{2}}$$

問８

（１）与式 $= \frac{1}{2}\{\cos(2\theta+5\theta)+\cos(2\theta-5\theta)\}$

$$= \frac{1}{2}\{\cos 7\theta + \cos(-3\theta)\}$$

$$= \boldsymbol{\frac{1}{2}(\cos 7\theta + \cos 3\theta)}$$

（２）与式 $= -\frac{1}{2}\{\cos(4\theta+3\theta)-\cos(4\theta-3\theta)\}$

$$= -\frac{1}{2}(\cos 7\theta - \cos\theta)$$

$$= \boldsymbol{\frac{1}{2}(\cos\theta - \cos 7\theta)}$$

（３）与式 $= \frac{1}{2}\{\sin(5\theta+3\theta)-\sin(5\theta-3\theta)\}$

$$= \boldsymbol{\frac{1}{2}(\sin 8\theta - \sin 2\theta)}$$

問９

（１）与式 $= 2\sin\frac{5\theta+3\theta}{2}\cos\frac{5\theta-3\theta}{2}$

$$= 2\sin\frac{8\theta}{2}\cos\frac{2\theta}{2}$$

$$= \boldsymbol{2\sin 4\theta\cos\theta}$$

（２）与式 $= 2\cos\frac{2\theta+4\theta}{2}\cos\frac{2\theta-4\theta}{2}$

$$= 2\cos\frac{6\theta}{2}\cos\frac{-2\theta}{2}$$

$$= 2\cos 3\theta\cos(-\theta)$$

$$= \boldsymbol{2\cos 3\theta\cos\theta}$$

（３）与式 $= 2\cos\frac{6\theta+2\theta}{2}\sin\frac{6\theta-2\theta}{2}$

$$= 2\cos\frac{8\theta}{2}\sin\frac{4\theta}{2}$$

$$= \boldsymbol{2\cos 4\theta\sin 2\theta}$$

$\boxed{問 10}$

（１）$y = \sqrt{1^2 + 1^2}\sin(x + \alpha)$

$= \sqrt{2}\sin(x + \alpha)$

ここで,

$\cos\alpha = \dfrac{1}{\sqrt{2}}$，$\sin\alpha = \dfrac{1}{\sqrt{2}}$ より，$\alpha = \dfrac{\pi}{4}$

よって，$\boldsymbol{y = \sqrt{2}\sin\left(x + \dfrac{\pi}{4}\right)}$

（２）$y = \sqrt{1^2 + \left(\sqrt{3}\right)^2}\sin(x + \alpha)$

$= 2\sin(x + \alpha)$

ここで,

$\cos\alpha = \dfrac{1}{2}$，$\sin\alpha = -\dfrac{\sqrt{3}}{2}$ より，$\alpha = -\dfrac{\pi}{3}$

よって，$\boldsymbol{y = 2\sin\left(x - \dfrac{\pi}{3}\right)}$

$\boxed{問 11}$

$y = \sqrt{1^2 + (-1)^2}\sin(x + \alpha)$

$= \sqrt{2}\sin(x + \alpha)$

また，$\cos\alpha = \dfrac{1}{\sqrt{2}}$，$\sin\alpha = -\dfrac{1}{\sqrt{2}}$ より，$\alpha = -\dfrac{\pi}{4}$

$y = \sqrt{2}\sin\left(x - \dfrac{\pi}{4}\right)$

ここで，$0 \leqq x \leqq 2\pi$であるから

$-\dfrac{\pi}{4} \leqq x - \dfrac{\pi}{4} \leqq 2\pi - \dfrac{\pi}{4}$

$x - \dfrac{\pi}{4} = \dfrac{\pi}{2}$ すなわち，$x = \dfrac{3}{4}\pi$のとき

最大値　$\sqrt{2}\sin\dfrac{\pi}{2} = \sqrt{2}$

$x - \dfrac{\pi}{4} = \dfrac{3}{2}\pi$ すなわち，$x = \dfrac{7}{4}\pi$のとき

最小値　$\sqrt{2}\sin\dfrac{3}{2}\pi = -\sqrt{2}$

したがって,

最大値　$\boldsymbol{\sqrt{2}}$　$\boldsymbol{\left(x = \dfrac{3}{4}\pi\right)}$

最大値　$\boldsymbol{-\sqrt{2}}$　$\boldsymbol{\left(x = \dfrac{7}{4}\pi\right)}$