Greedy algorithm and Matroid

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About this class

We will talk about:

- Greedy algorithm
- Definition of Matroid
- Optimization problem on Matroid
- Matroid intersection problem
- Matroid Union problem
- Representation of Matroid
- sth about Counting Matroid basis
- sth may interesting...

Greedy algorithm

Matroid

Matroid

A Matroid $M = (S, \mathcal{I})$ has a background set S and $\mathcal{I} \subseteq 2^S$, a family set of S, which is called the Independent set.
The Matroid satisfies the following two conditions.

- If $I \in \mathcal{I}, j \subseteq I$, $J \in \mathcal{I}$
- $\forall I, J \in \mathcal{I}$, if $|J| > |I|$, then there exists $z \in J \backslash I$ satisfies that $I + z \in \mathcal{I}$

Examples

Uniform Matroid

Matroid $U_n^k = (S, \mathcal{I})$, where $|S| = n, \mathcal{I} = \{I \subseteq S, |I| \leq k\}$

Graph Matroid

Suppose there is a undirected graph $G = (V, E)$, the Graph Matroid of G is $M = (E, \mathcal{I})$, where $\mathcal{I} = \{F \subseteq F :$ there's no cycle in $F\}$

Directed Graph "Matroid"

Suppose there is a directed graph $G = (V, E)$, the Graph Matroid of G is $M = (E, \mathcal{I})$, where $\mathcal{I} = \{F \subseteq F :$ there's no cycle in $F\}$

Matching Matroid

Suppose there is a undirected graph $G(V, E)$, the Matching Matroid of G is $M = (V, \mathcal{I})$, where $\mathcal{I} = \{S \subseteq V :$ there is a matching covering$S\}$

Basis and Cycle

Basis

For some Independent Set I in $\mathcal{I}$, if we add any element $z \in S \backslash I$ to it will lead to a Dependent Set, i.e. $I + z \notin \mathcal{I}$, we call I basis, which is also maximum Independent Set.

Cycle

For some Dependent Set I' not in $\mathcal{I}$, if we delete any element z from it will lead to a Independent Set, we call I' cycle, also the minimum Dependent Set.

Some important and useful properties.

Rank Function

Rank Function

For a matroid $M = (S, \mathcal{I})$, and U a subset of S, the rank function is defined as $r(U) = \max\limits_{I \subseteq U, I \in \mathcal{I}} |I|$.

$\forall U \subseteq S, 0 \leq r(U) \leq |U|$
$\forall A \subseteq B \subseteq S, r(A) \leq r(B)$
$\forall A, B \subseteq S, r(A \cup B) + r(A \cap B) \leq r(A) + r(B)$

Optimization problem

Given Matroid $M = (S, \mathcal{I})$ and some weight function $\omega : S \to \mathbb{R}$. For any subset I of S, define $\omega(I) = \sum_{z \in I} \omega(z)$. We need to find $\max_{I \in \mathcal{I}} \omega(I)$.

Dual Matroid

Dual Matroid

The Dual Matroid of given Matroid $M = (S, \mathcal{I})$ is defined as $M^* = (S, \mathcal{I}^*)$, where $\mathcal{I}^* = \{I \subseteq S : \text{there exists a basis of } M \text{ in } S \backslash I\}$

Proof of M^* is a Matroid?

Deletion and Contraction

Deletion

For a Matroid $M = (S, I)$, and a subset Z of S, Matroid M after deleting Z is defined as $(S \backslash Z, \mathcal{I}')$ where $\mathcal{I}' = \{I : I \subseteq S \backslash Z, I \in \mathcal{I}\}$

Contraction

For a Matroid $M = (S, I)$, and a subset Z of S, Matroid M after contracting Z is defined as $(M^* \backslash Z)^*$

Minimal of Matroid

For a given Matroid M, the Matroid after several deletions and contractions on M is called the minimal of M.

Matroid Intersection Problem

Given two Matroids $M_1 = (S, \mathcal{I}_1), M_2 = (S, \mathcal{I}_2)$ based on the same background set.
We call a subset I of S is "Independent" iff $I \in \mathcal{I}_1, I \in \mathcal{I}_2$ holds at the same time.

Min-Max Fomular

$$\max_{I\in\mathcal{I}_1\cap\mathcal{I}_2}|I| = \min_{U\subseteq S}(r_1(U) + r_2(S\backslash U))$$

Strong Basis Exchange Theorem

For a given Matroid M, suppose there are two distinct bases A, B. For any $x \in A \backslash B$, there is $y \in B \backslash A$, such that $A - x + y$ and $B - y + x$ are both bases, too.

Algorithm

More about Matroid Intersections

The problem of intersection of three or more matroids is NP-Hard, since we can use three matroids to solve the Hamilton Path problem.
If the matroid is defined with some weight function, we can also solve the intersection problem.

Applications

Matroid union problem

Given k matroids $M_i = (S_i, \mathcal{I}_i)$, the union of these matroid is defined as $M = (S, \mathcal{I})$, where $S = \bigcup_{i=1}^{k} S_i, \mathcal{I} = \{\bigcup_{i=1}^{k} I_i : I_i \in \mathcal{I}_i\}$.

Rank Function

For the union M of these matroids, the rank function is$r_M(U) = \min_{T \subseteq U}(|U \setminus T| + \sum_{i=1}^{k} r_i(T \cap S_i))$

Independence Check

The problem of checking whether a set is independent is not easy in the union of several Matroids.

Matroid Partition Algorithm

Applications

Consider the situation that we union the k same matroids $M^k = M \cup M \cup \ldots M$. We can get the following theorem.

The whole set can be covered by at most k bases iff for any $T \subseteq S$, $|T| \leq kr_M(T)$

The whole contains at least k bases iff for any $T \subseteq S$, $|S| \geq k(r_M(S) - r_M(T))$

Representation of Matroids

A Matroid M is thought as representative over the Filed $\mathbb{F}$ iff there is a linear Matroid over $\mathbb{F}$ isomorphic to M.
A Matroid can be represented over any field is called regular Matroid.
The Graph Matroid is the regular Matroid.

Counting basis

Counting spanning trees
Counting perfect matchings