

Solving the heat equation.

①

Initial condition

Linear Partial Differential equation

$$\frac{\partial u(x,t)}{\partial t} = k \frac{\partial^2 u(x,t)}{\partial x^2}$$

Constant

homogenous BCs.

$$IC: u(x, t=0) = f(x)$$

$$BCs: u(0, t) = u(L, t) = 0.$$

The first thing to do here is apply separation of variables. We assume that the solution will take the form:

$$u(x, t) = \phi(x) T(t).$$

Let's put this new solution into the differential equation:

$$\frac{\partial}{\partial t} (\phi(x) T(t)) = k \frac{\partial^2}{\partial x^2} (\phi(x) T(t))$$

$$\phi(x) \frac{dT(t)}{dt} = k T(t) \frac{d^2 \phi(x)}{dx^2}.$$

$$\div \phi(x) \cdot T(t) \Rightarrow \frac{\phi(x)}{\phi(x) T(t)} \frac{dT(t)}{dt} = \frac{k T(t)}{\phi(x) T(t)} \frac{d^2 \phi(x)}{dx^2}$$

$$\Rightarrow \frac{1}{T(t)} \frac{dT(t)}{dt} = \frac{k}{\phi(x)} \frac{d^2 \phi(x)}{dx^2}$$

$$\Rightarrow \underbrace{\frac{1}{k T(t)} \frac{dT(t)}{dt}}_{\text{depends on } (t)} = \underbrace{\frac{1}{\phi(x)} \frac{d^2 \phi(x)}{dx^2}}_{\text{depends on } (x)} \quad \star$$

depends on (t)

depends on (x)

Notice that we no longer have a partial derivative left in the problem. In the time derivative we are now differentiating  $T(t)$  with respect to  $(t)$  and this is now an ordinary derivative.

Likewise, in the spatial derivative we are now differentiating  $\phi(x)$  with respect to  $(x)$  and so we again have an ordinary derivative.

②

One important thing to do here is to make sure that our product solution:  $u(x,t) = \phi(x) T(t)$  satisfies the boundary conditions so let's plug it into both of those:

$$u(0,t) = \phi(0) T(t) = 0$$

$$u(L,t) = \phi(L) T(t) = 0.$$

For the first one:  $u(0,t) = \phi(0) T(t) = 0.$

In this case, either  $\phi(0) = 0$  or  $T(t) = 0$  for every  $t$ .

However, if we have  $T(t) = 0$  for every  $t$  then we'll also have  $u(x,t) = 0$ , i.e., the trivial solution. Therefore, we will assume that in fact we must have  $\phi(0) = 0$ .

For the second one:  $u(L,t) = \phi(L) T(t) = 0$

Likewise, for the second boundary condition we will get  $\phi(L) = 0$  to avoid the trivial solution.

Note as well that we were only able to reduce the boundary conditions down like this because they were homogenous. Had they not been homogenous we could not have done this.



Notice that the left side <sup>of \*</sup> is a function of only  $(t)$  and the <sup>③</sup> right side is a function only of  $(x)$  and these two functions must be equal. Therefore, we can write the following:

$$\frac{1}{kT(t)} \frac{dT(t)}{dt} = \frac{1}{\phi(x)} \frac{d^2\phi(x)}{dx^2} = \lambda$$

where  $\lambda$  is the Separation Constant.

Since we do not know what  $\lambda$  is, we need to discuss three situations:  $\lambda > 0$ ,  $\lambda = 0$ ,  $\lambda < 0$ .

The solution of the spatial equation:

$$\frac{1}{\phi(x)} \frac{d^2\phi(x)}{dx^2} = \lambda \Rightarrow \frac{d^2\phi(x)}{dx^2} = \lambda \phi(x).$$

$\lambda > 0$ :  $\frac{d^2\phi(x)}{dx^2} = \lambda \phi(x) \Rightarrow \frac{d^2\phi(x)}{dx^2} - \lambda \phi(x) = 0$

The auxiliary equation for this ordinary homogenous differential equation is:  $m^2 - \lambda = 0 \Rightarrow m^2 = \lambda \Rightarrow m = \pm \sqrt{\lambda}$ .

The auxiliary equation has two distinct and real roots, then the general solution is given by:

$$\begin{aligned} \phi(x) &= A e^{m_1 x} + B e^{m_2 x} \\ \Rightarrow \phi(x) &= A e^{\sqrt{\lambda} x} + B e^{-\sqrt{\lambda} x} \end{aligned}$$

to find  $A$  and  $B$ , we apply the initial conditions:

$$\begin{aligned} \phi(0) = 0 &\Rightarrow A + B = 0 \Rightarrow A = -B. \\ \Rightarrow \phi(x) &= A \left( e^{\sqrt{\lambda} x} - e^{-\sqrt{\lambda} x} \right) \end{aligned}$$

$$\phi(L) = 0 \Rightarrow A e^{\sqrt{\lambda}L} + B e^{-\sqrt{\lambda}L} = 0 \quad (4)$$

we know from before that  $A = -B \Rightarrow$

$$A \left( e^{\sqrt{\lambda}L} - e^{-\sqrt{\lambda}L} \right) = 0$$

can't be zero  $\Rightarrow A=0$  hence  $B=0$

which leaves us with the trivial solution!

$$\lambda = 0$$

$$\frac{d^2\phi(x)}{dx^2} = 0$$

The auxiliary equation here is:  $m^2 = 0 \Rightarrow m_{1,2} = 0$

The equation has two repeated roots, therefore, the general solution is given by:

$$\phi(x) = (A + Bx) e^{mx}$$

$$\Rightarrow \phi(x) = A + Bx$$

Let's find  $A$  and  $B$  from the boundary conditions:

$$\phi(0) = 0 \Rightarrow A = 0.$$

$$\phi(L) = 0 \Rightarrow A + BL = 0$$

$$\text{Since } A = 0 \Rightarrow BL = 0 \Rightarrow B = 0$$

we have trivial solution again!

$$\lambda < 0$$

$$\frac{d^2\phi(x)}{dx^2} = -\lambda \phi(x) \Rightarrow \frac{d^2\phi(x)}{dx^2} + \lambda \phi(x) = 0$$

The auxiliary equation in this case is given by:

$$m^2 + \lambda = 0 \Rightarrow m^2 = -\lambda \Rightarrow m^2 = i^2 \lambda$$

$$\Rightarrow m = \pm i \sqrt{\lambda}.$$



The roots for the auxiliary equations are complex, therefore <sup>⑤</sup>  
the general solution is given by:

$$\phi(x) = e^{\alpha x} (A \cos(\beta x) + B \sin(\beta x))$$

where  $\alpha$  is the real part of the complex root

$\beta$  is the imaginary part of the complex root

$\Rightarrow$

$$\phi(x) = A \cos(\sqrt{\lambda} x) + B \sin(\sqrt{\lambda} x)$$

To find A and B, and as before, we use the boundary

Conditions:

$$\phi(0) = 0 \Rightarrow A = 0 \Rightarrow \phi(x) = B \sin(\sqrt{\lambda} x)$$

$$\phi(L) = 0 \Rightarrow B \sin(\sqrt{\lambda} L) = 0$$

Since we are after the non-trivial solution, this can only work if:

$$\sqrt{\lambda_n} L = n\pi \quad \text{where } n=1, 2, 3, \dots$$

$$\Rightarrow \sqrt{\lambda_n} = \frac{n\pi}{L} \quad \text{and} \quad \lambda_n = \left(\frac{n\pi}{L}\right)^2.$$

The positive eigenvalues and their corresponding eigenfunctions of this boundary value problem are then,

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad \phi_n(x) = B_n \sin\left(\frac{n\pi x}{L}\right) \quad n=1, 2, 3, \dots$$

The solution of the time equation:

$$\frac{dT(t)}{dt} = -K\lambda T(t)$$

⑥ As we have seen before  
 $\lambda$  must be negative  
to have a non-trivial  
solution

Similarly, the auxiliary equation is:

$$m + K\lambda = 0 \Rightarrow m = -K\lambda$$

The root of this equation is a real root, therefore the

solution is given by:

$$T(t) = A e^{mt} \Rightarrow T(t) = A e^{-K\lambda t}$$

Let's now combine the solutions of both parts:

$$\begin{aligned} u(x,t) &= \phi(x) T(t) \\ &= B_n \sin\left(\frac{n\pi x}{L}\right) \cdot A_n e^{-\left(\frac{Kn^2\pi^2}{L^2}\right)t} \end{aligned}$$

$A_n$  and  $B_n$  are constants and can be combined into a new

constant:

$$u(x,t) = C_n \sin\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{Kn^2\pi^2}{L^2}\right)t}$$

This solution is true for any value of  $n$ . Therefore,

the general solution for the 1D heat equation with

homogenous boundary conditions can be written as:

$$u(x,t) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{Kn^2\pi^2}{L^2}\right)t}$$