Solving the heat equation.

linear partial Differential equation

$$\frac{\partial u(x,t)}{\partial t} = K \frac{\partial^2 u(x,t)}{\partial x^2}$$

initial condition

constant

homogenous BCs.

IC: $u(x,t=0) = f(x)$ BCs: $u(0,t) = u(L,t) = 0.$

The first thing to do here is apply seperation of variables. We assume that the Solution will take the form:

$$u(x,t) = \phi(x)\, T(t).$$

Let's put this new solution into the differential equation:

$$\frac{\partial}{\partial t}\big(\phi(x)\, T(t)\big) = K \frac{\partial^2}{\partial x^2}\big(\phi(x)\, T(t)\big)$$

$$\phi(x) \frac{dT(t)}{dt} = K\, T(t) \frac{d^2\phi(x)}{dx^2}.$$

$$\div\ \phi(x).\, T(t) \Rightarrow \frac{\phi(x)}{\phi(x)\, T(t)} \frac{dT(t)}{dt} = \frac{K\, T(t)}{\phi(x)\, T(t)} \frac{d^2\phi(x)}{dx^2}$$

$$\Rightarrow \frac{1}{T(t)} \frac{dT(t)}{dt} = \frac{K}{\phi(x)} \frac{d^2\phi(x)}{dx^2}$$

$$\Rightarrow \underbrace{\frac{1}{K\, T(t)} \frac{dT(t)}{dt}}_{\text{depends on } (t)} = \underbrace{\frac{1}{\phi(x)} \frac{d^2\phi(x)}{dx^2}}_{\text{depends on } (x)} \quad \star$$

Notice that we no longer have a partial derivative left in the problem. In the time derivative we are now differentiating $T(t)$ with respect to (t) and this is now an ordinary derivative. Likewise, in the spatial derivative we are now differentiating $\phi(x)$ with respect to (x) and so we again have an ordinary derivative.

One important thing to do here is to make sure that our product solution: $u(x,t) = \phi(x)\,T(t)$ satisfies the boundary conditions so let's plug it into both of those:

$$u(0,t) = \phi(0)\,T(t) = 0$$

$$u(L,t) = \phi(L)\,T(t) = 0 .$$

For the first one: $u(0,t) = \phi(0)\,T(t) = 0$.

In this case, either $\phi(0) = 0$ or $T(t) = 0$ for every t. However, if we have $T(t) = 0$ for every t then we'll also have $u(x,t) = 0$, i.e., the trivial solution. Therefore, we will assume that in fact we must have $\phi(0) = 0$.

For the second one: $u(L,t) = \phi(L)\,T(t) = 0$

Likewise, for the second boundary condition we will get $\phi(L) = 0$ to avoid the trivial solution.

<u>Note</u> as well that we were only able to reduce the boundary conditions down like this because they were homogenous. Had they not been homogenous we <u>could not</u> have done this.

Notice that the left side of * is a function of only (t) and the right side is a function only of (x) and these two functions must be equal. Therefore, we can write the following:

$$\frac{1}{k\,T(t)} \frac{d\,T(t)}{dt} = \frac{1}{\phi(x)} \frac{d^2\phi(x)}{dx^2} = \lambda$$

where λ is the seperation constant.

Since we do not know what λ is, we need to discuss three situations: $\lambda > 0$, $\lambda = 0$, $\lambda < 0$.

The solution of the spatial equation:

$$\frac{1}{\phi(x)} \frac{d^2\phi(x)}{dx^2} = \lambda \Rightarrow \frac{d^2\phi(x)}{dx^2} = \lambda\,\phi(x).$$

$\lambda > 0$:

$$\frac{d^2\phi(x)}{dx^2} = \lambda\,\phi(x) \Rightarrow \frac{d^2\phi(x)}{dx^2} - \lambda\,\phi(x) = 0$$

The auxiliary equation for this ordinary homogenous differential equation is : $m^2 - \lambda = 0 \Rightarrow m^2 = \lambda \Rightarrow m = \pm\sqrt{\lambda}$.

The auxiliary equation has two distinct and real roots, then the general solution is given by:

$$\phi(x) = A\,e^{m_1 x} + B\,e^{m_2 x}.$$

$$\Rightarrow \phi(x) = A\,e^{\sqrt{\lambda}x} + B\,e^{-\sqrt{\lambda}x}$$

to find A and B, we apply the initial conditions:

$$\phi(0) = 0 \Rightarrow A + B = 0 \Rightarrow A = -B.$$

$$\Rightarrow \phi(x) = A\left(e^{\sqrt{\lambda}x} - e^{-\sqrt{\lambda}x}\right)$$

4

$$\phi(L) = 0 \Rightarrow A e^{\sqrt{\lambda}L} + B e^{-\sqrt{\lambda}L} = 0$$

we know from before that $A = -B \Rightarrow$

$$A \underbrace{\left(e^{\sqrt{\lambda}L} - e^{-\sqrt{\lambda}L} \right)}_{\text{can't be zero}} = 0 \Rightarrow A = 0 \text{ hence } B = 0$$

which leaves us with the trivial solution!

$\lambda = 0$

$$\frac{d^2\phi(x)}{dx^2} = 0$$

The auxiliary equation here is: $m^2 = 0 \Rightarrow m_{1,2} = 0$

The equation has two repeated roots, therefore, the general solution is given by:

$$\phi(x) = (A + Bx) e^{mx}$$

$$\Rightarrow \quad \phi(x) = A + Bx$$

Let's find A and B from the boundary conditions:

$\phi(0) = 0 \Rightarrow A = 0.$

$\phi(L) = 0 \Rightarrow A + BL = 0$

Since $A = 0 \Rightarrow BL = 0 \Rightarrow B = 0$

we have trivial solution again!

$\lambda < 0$

$$\frac{d^2\phi(x)}{dx^2} = -\lambda \phi(x) \Rightarrow \frac{d^2\phi(x)}{dx^2} + \lambda \phi(x) = 0$$

The auxiliary equation in this case is given by:

$$m^2 + \lambda = 0 \Rightarrow m^2 = -\lambda \Rightarrow m^2 = i^2 \lambda$$

$$\Rightarrow m = \pm i \sqrt{\lambda}.$$

The roots for the auxiliary equations are Complex, therefore the general Solution is given by:

$$\phi(x) = e^{\alpha x}\left(A\cos(\beta x) + B\sin(\beta x)\right)$$

where $\underline{\underline{\alpha}}$ is the real part of the Complex root

$\underline{\underline{\beta}}$ is the imaginary part of the Complex root

$\Rightarrow$

$$\phi(x) = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x)$$

To find A and B, and as before, we use the boundary Conditions:

$\phi(0) = 0 \quad \Rightarrow \quad A = 0 \quad \Rightarrow \quad \phi(x) = B\sin(\sqrt{\lambda}x)$

$\phi(L) = 0 \quad \Rightarrow \quad B\sin(\sqrt{\lambda}L) = 0$

Since we are after the non_trivial Solution, this Can only work if: $\sqrt{\lambda_n}\,L = n\pi$ where $n = 1, 2, 3, \ldots$

$$\Rightarrow \sqrt{\lambda_n} = \frac{n\pi}{L} \quad \text{and} \quad \lambda_n = \left(\frac{\pi n}{L}\right)^2.$$

The positive eigenvalues and their Corresponding eigenfunctions of this boundary value Problem are then,

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad \phi_n(x) = B_n\sin\left(\frac{n\pi x}{L}\right) \qquad n = 1, 2, 3, \ldots$$

The solution of the time equation:

$$\frac{dT(t)}{dt} = -K\lambda\, T(t)$$

As we have seen before, λ must be negative to have a non-trivial solution.

Similarly, the auxiliary equation is:

$$m + K\lambda = 0 \implies m = -k\lambda.$$

The root of this equation is real, therefore, the solution is:

$$T(t) = A\, e^{mt} \implies T(t) = A\, e^{-k\lambda t}$$

Let's now combine the solutions of both parts:

$$U_n(x,t) = \phi_n(x) . T(t)$$

$$= B_n \sin\left(\frac{n\pi x}{L}\right) . A_n\, e^{-K\lambda t} \quad , \lambda = \left(\frac{n\pi}{L}\right)^2.$$

B_n and A_n are constants and can be combined into a new one:

$$U_n(x,t) = C_n \sin\left(\frac{n\pi x}{L}\right) e^{-k\left(\frac{n\pi}{L}\right)^2 . t}$$

where: $C_n = B_n . A_n$.

We know from the principle of superposition: if $U_1(x,t)$ and $U_2(x,t)$ are two solutions to the PDE and BCs, then $C_1U_1 + C_2U_2$ is also a solution, for any constants C_1, C_2. This relies on the linearity of the PDE and BCs. Therefore, the principle of superposition says that the solution can be written as an infinite sum of all solutions $U_n(x,t)$:

$$U(x,t) = \sum_{n=1}^{\infty} U_n(x,t) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{L}\right) e^{-k\left(\frac{n\pi}{L}\right)^2 . t}$$

$n = 1, 2, 3, \dots$

Each function $u_n(x,t)$ is a solution to the PDE and the BCs.

But in general they will not individually satisfy the IC.

To solve the IC Condition, we need all the Solutions at $t=0$ as following:

$$t=0 \Rightarrow f(x) = u(x,0) = \sum_{n=1}^{\infty} u_n(x,0) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi}{L}x\right)$$

This is a Sine Fourier Series of $f(x)$. To solve for the C_n's, we use the orthogonality property of the eigenfunctions $\sin\left(\frac{n\pi x}{L}\right)$ and we get:

$$C_n = \frac{2}{L}\int_0^L f(x).\sin\left(\frac{n\pi x}{L}\right)dx.$$

In Conclusion, to derive the solution of the heat equation and corresponding BCs and IC, we use properties of linear operators and infinite series.