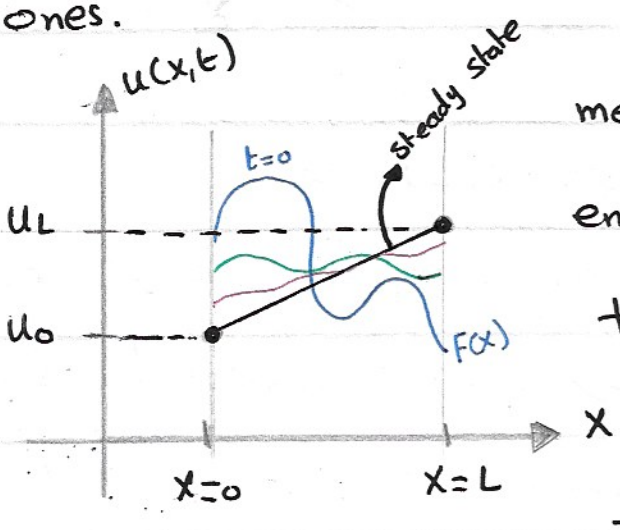
Solving the heat equation with nonhomogenous BCs.

$$\frac{\partial u(x,t)}{\partial t} = k\,\frac{\partial^2 u(x,t)}{\partial x^2}$$

$$\left.\begin{aligned} u(0,t) &= u_0 \\ u(L,t) &= u_L \end{aligned}\right\} \text{BCs.} \qquad\qquad u(x,t=0) = F(x)\Big\}\ \text{IC.}$$

Since seperation of variables method can only be used when our boundary conditions are homogenous. So, clearly in this situation here we cannot use this method. However, you will see that we will end up using seperation of variables eventually after converting our boundary conditions to homogenous ones.




At the beginning, we need to mention that as time goes by, the ends of the function F(x) are going to drift so they can obey the boundary conditions. Eventually, the part remaining in the middle will follow suit in order to get rid of whatever local of concentration gradients are present. So once the system reaches a steady state, we'll get a linear behaviour.

Because I Know that my steady state will be linear, I can make the hypothesis that my solution for $u(x,t)$ will be the sum of a steady state solution and a transient solution. The idea is that initially this transient solution has an effect on what $u(x,t)$ looks like but as time goes on, its effect becomes smaller and smaller until we're only left with the steady state solution. $\Rightarrow$

$$u(x,t) = \underbrace{U_{ss}(x)}_{\text{steady}} + \underbrace{U_{tr}(x,t)}_{\text{transient}} \quad \bigstar$$

Solving for the steady state solution:

Since it's a steady state then $\frac{\partial U_{ss}(x)}{\partial t} = 0 \Rightarrow \frac{d^2 U_{ss}(x)}{dx^2} = 0$

The boundary conditions:

$U_{ss}(x=0) = U_0 \qquad U_{ss}(x=L) = U_L.$

To solve the steady state differential equation, let's write the auxiliary equation: $m^2=0 \Rightarrow m_{1,2}=0$ repeated roots.

The general solution for the equation is:

$$U_{ss}(x) = (A+Bx)\,e^{mx} \Rightarrow U_{ss}(x) = A + Bx$$

Notice that the solution of the steady state equation is a line which explains the linear behaviour.

To find A and B, we apply the boundary Conditions: ③

$U_{ss}(x=0) = u_0 \quad \Rightarrow \quad A = u_0.$

$U_{ss}(x=L) = U_L \quad \Rightarrow \quad U_L = U_0 + BL \Rightarrow BL = U_L - U_0$

$$\Rightarrow \quad B = \frac{U_L - U_0}{L}$$

Therefore, we can write:

$$U_{ss}(x) = U_0 + \left(\frac{U_L - U_0}{L}\right) x.$$

Solving for the transient solution:

At this stage, we don't quite know what equation and what boundary conditions that transient solution has to satisfy so we will have to do some work. To find the "transient sub problem" we need to put ★ back into the differential equation:
$$\frac{\partial u(x,t)}{\partial t} = K \frac{\partial^2 u(x,t)}{\partial x^2}$$

$$\Rightarrow$$

$$\frac{\partial}{\partial t}\Big[U_{ss}(x) + U_{tr}(x,t)\Big] = K \frac{\partial^2}{\partial x^2}\Big[U_{ss}(x) + U_{tr}(x,t)\Big]$$

$$\underbrace{\frac{\partial U_{ss}(x)}{\partial t}}_{=0} + \frac{\partial U_{tr}(x,t)}{\partial t} = K\left(\underbrace{\frac{\partial^2 U_{ss}(x)}{\partial x^2}}_{=0} + \frac{\partial^2 U_{tr}(x,t)}{\partial x^2}\right)$$

$$\Rightarrow \quad \frac{\partial U_{tr}(x,t)}{\partial t} = K \frac{\partial^2 U_{tr}(x,t)}{\partial x^2}$$

which is the exact PDE we had for $u(x,t)$!

Let's find the boundary conditions for this equation / problem:

Originally we know that:

$$U(0,t) = u_0 \quad \underline{\underline{\text{and}}} \quad U(L,t) = U_L$$

and since $U(x,t) = U_{ss}(x) + U_{tr}(x,t)$ we can write:

For the first BC:

$U_{ss}(0) + U_{tr}(0,t) = U_0$ but we already know that

$$U_{ss}(0) = U_0 \Rightarrow U_{tr}(0,t) = 0$$

For the second BC:

$U_{ss}(L) + U_{tr}(L,t) = U_L$ but we already know that

$$U_{ss}(L) = U_L \Rightarrow U_{tr}(L,t) = 0$$

Notice that boundary conditions now are homogenous and our problem can be solved using seperation of variables method.

Let's check the initial condition of the problem:

$$U_{ss}(x)\Big|_{t=0} + U_{tr}(x, t=0) = f(x)$$

$$\Rightarrow \quad U_{tr}(x, t=0) = f(x) - U_{ss}(x)\Big|_{t=0}$$

$$= \underbrace{f(x) - \left[\left(\frac{U_L - U_0}{L}\right)x + U_0\right]}_{\psi(x)}$$

$$\Rightarrow$$

$$U_{tr}(x, t=0) = \psi(x).$$

Applying seperation of variables method to solve the transient problem leads to :

$$U_{tr}(x,t) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{kn^2\pi^2}{L^2}\right)t}$$

$n = 1, 2, 3, \ldots$

Therefore, the solution for the whole problem can be written as following :

$$U(x,t) = U_{ss}(x) + U_{tr}(x,t)$$

$$\Rightarrow$$

$$U(x,t) = \left(\frac{U_L - U_0}{L}\right)x + U_0 + \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{kn^2\pi^2}{L^2}\right)t}$$

We can see here that as $t \to \infty$ the transient solution vanishes away leaving us with the steady state solution.