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Some Useful Mathematical Relations

Functional Relations

- $a^x = e^{x \ln a}$
- $2^{\log n} = n$
- Stirling's Theorem: $n! \approx \sqrt{2\pi n}\left(\dfrac{n}{e}\right)^n$ for large n
- $\log n! = c_1 + c_2 \log n + n \log n - c_3 n = O(n \log n)$
- $\sum_{i=1}^{n} \log i = \log n! = O(n \log n)$
- $\sum_{i=1}^{n} i \log i = \sum_{i=1}^{n} \log i^i \le n \log n^n = O(n^2 \log n)$

Finite Series

- **Arithmetic Series:** $\sum_{i=0}^{n-1} (a + b\,i) = (n/2)\,\{2a + (n-1)b\}$

  $\sum_{i=0}^{n-1} (n-i) = \sum_{i=1}^{n} i = n(n+1)/2$

- **Geometric Series:** $\sum_{i=0}^{n} x^i = (x^{n+1} - 1)/(x-1)$ for $x \neq 1$

  $\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$ $\sum_{i=1}^{n} 2^i = 2^{n+1} - 2$ $\sum_{j=1}^{k} j\,2^j = (k-1)\,2^{k+1} + 2$

  $\sum_{i=0}^{n-1} i\,2^{n-i} = \sum_{i=1}^{n} (n-i)2^i = 2^{n+1} - 2(n+1)$

Sums of Powers of Natural Numbers

- $\sum_{i=1}^{n} i = n(n+1)/2$
- $\sum_{i=1}^{n} i^2 = n(n+1)(2n+1)/6$
- $\sum_{i=1}^{n} i^3 = [n(n+1)/2]^2$

- $\sum_{i=1}^{n} i^4 = n(n+1)(2n+1)(3n^2 + 3n - 1) / 30$
- $\sum_{i=1}^{n} i^5 = n^2 (n+1)^2 (2 n^2 + 2n - 1) / 12$

Sums of Odd Numbers and their Powers

- $\sum_{i=1}^{n} (2 i - 1) = n^2$
- $\sum_{i=1}^{n} (2 i - 1)^2 = n(4n^2 - 1)/3$
- $\sum_{i=1}^{n} (2 i - 1)^3 = n^2 (2n^2 - 1)$

Other Finite Series

- $\sum_{i=1}^{n} i (i + 1)^2 = (1/12) n(n+1)(n+2)(3n + 5)$
- $\sum_{i=1}^{n} i . i! = (n+1)! - 1$
- $\sum_{i=2}^{n} 1 /(i^2 - 1) = (3/4) - (2n+1)/[2n(n+1)]$
- $\sum_{i=1}^{n} 1/ i \sim \gamma + \ln n + 1/(2n)$ for large n, $\gamma = 0.577$ = Euler's constant
- $\sum_{i=1}^{n} \frac{1}{i\,(i+1)} = \frac{n}{n+1}$

Infinite Series and Products

- $\sum_{k=0}^{\infty} a^{kx} = 1 / (1 - a^x)$ for ($a > 1$ and $x < 0$) or ($0 < a < 1$ and $x > 0$)
- $\sum_{k=0}^{\infty} x^k / k! = e^x$
- $\sum_{k=0}^{\infty} x^k = 1 / (1-x)$ for $|x| < 1$

- $\sum_{k=0}^{\infty}$ (a + bk) x^k = a /(1-x) + b x /(1 – x^2) for |x| < 1
- $\sum_{k=1}^{\infty}$ $(-1)^{k+1}$ / k = ln 2
- $\sum_{k=1}^{\infty}$ $(-1)^{k+1}$ / k^2 = π^2 /12
- $\sum_{k=1}^{\infty}$ 1 / $(2k – 1)^2$ = π^2 / 8
- $\sum_{k=1}^{\infty}$ 1 / (k 2^k) = ln 2
- $\sum_{k=1}^{\infty}$ 1 / (k^2 2^k) = π^2 /12 - $(\ln 2)^2$ /2
- $\sum_{k=0}^{\infty} \frac{1}{k!} = e$ = 2.71828
- $\sum_{k=0}^{\infty} \frac{(-1)^k}{k!} = 1/e$ = 0.36787
- $\sum_{k=1}^{\infty}$ k /(k+1)! = 1
- $\prod_{k=2}^{\infty} \left(1 - \frac{1}{k^2}\right) = \frac{1}{2}$
- $\prod_{k=0}^{\infty} (1 + x^{2^k}) = \frac{1}{1-x}$ for |x| < 1

General Solution of 1st Order Linear Recurrences

T(n) = a_n T(n-1) + b_n , given T(0) or T(1)

Solution:

Successive substitution gives for the cases of T(0) given and T(1) given:

$$T(n) = T(0)\prod_{i=1}^{n} a_i + \sum_{i=1}^{n-1} b_i \prod_{j=i+1}^{n} a_j + b_n$$

$$T(n) = T(1)\prod_{i=2}^{n} a_i + \sum_{i=2}^{n-1} b_i \prod_{j=i+1}^{n} a_j + b_n$$

Common D & Q Recurrence :

$$T(n) = aT(n/c) + bn^x$$

The solutions become :

$$T(n) = a^m T(1) + bn^x \sum_{i=0}^{m-1} (\frac{a}{c^x})^i \quad \textbf{for T(1) given.}$$

$$T(n) = a^{m-1} T(c) + bn^x \sum_{i=0}^{m-2} (\frac{a}{c^x})^i \quad \textbf{for T(c) given.}$$

Special Case $a = c^x$ **has the solutions :**

$$T(n) = a^m T(1) + bmn^x = a^m T(1) + bn^x \log_c n \quad \textbf{For T(1) given}$$

$$T(n) = a^{m-1} T(c) + bn^x (\log_c n - 1) \quad \textbf{For T(c) given}$$

In the above equations, $\boldsymbol{m = log_c\, n}$