Ornamenting Inductive-Recursive Definitions

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Abstract

Contents

1 Introduction

A Technical Preliminary This research development has been exclusively done formally, using the dependently–typed language Agda ([3]) as an interactive theorem–prover. As such this report is full of code snippets, following the methodology of literate programming ([1]). Theorems are presented as type declaration, proofs are implementations of such declarations and definitions are usually some kind of datastructure introduction: it definitely lies on the *program* side of the Curry–Howard correspondance. The syntax and concepts of Agda should not be too alien to a Haskell or Coq programmer but it might be interesting to start out by reading the appendix A which presents its most important features.

Motivations Although they were probably first intended as theorem provers, dependently–typed languages are currently evolving into general–purpose programming languages, leveraging their expressivity to enable correct–by–construction type–driven programming. But without the right tools this new power is unmanageable. One issues is the need to prove over and over again the same properties for similar datastructures. Ornaments (**TODO**:*ref mcbride*) tackle this problem by giving a formal syntax to describe how datastructures might be *similar*. Using these objects, we can prove generic theorems once and for all. The broad idea behind this approach is to "speak in a more intelligible way to the computer": if instead of giving a concrete declarations we gave defining properties, we would be able to systematically collect free theorems which hold by (some high level) definition.

The present work aims to generalize ornaments to the widest possible notion of datatypes: inductive–recursive families (or indexed inductive–recursive types) as recently axiomatized by Ghani et al (**??**).

Related Work

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2 Indexed Induction–Recursion

The motivation behind indexed induction–recursion is to provide a single rule that can be specialized to create most of the types that are encountered in Martin Loef's Intuitionistic Type Theory (ITT) such as inductive types (W–types), inductive families *etc.* This rule has been inspired to Dybjer (**TODO**:*ref*) by Martin Loef's definition of a universe à–la–Tarski, an inductive set of codes **data** U : Set and a recursive function el : U → Set reflecting codes into actual sets (here a simple version with only natural numbers and Π–types).

```
data U where
  `ℕ : U                                  el `ℕ       = ℕ
  `Π : (A : U) (B : el A → U) → U         el (`Π A B) = (a : el A) → el (B a)
```

We can see the most important caracteristic of inductive-recursive definitions: the simultaneous definition of an inductive type and a recursive function on it with the ability to use the recursive function in the type of the constructors, even in negative positions (left of an arrow). *Indexed* inductive-recursive definitions are a slight generalization, similar to the relationship between inductive types and inductive families. In its full generality, indexed induction recursion allows to simultaneously define an inductive predicate `U : I → Set` and an indexed recursive function `f : (i : I) → U i → X i` for any `I : Set` and `X : I → Set₁`. Using a vocabulary influenced by the *bidirectional* paradigm for typing (**TODO**:*ref*) we will call `i : I` the *input index* and `X i` the *output index*. Indeed if we think of the judgement `a : U i` as a typechecker would, the judgment requires the validity of `i : I` and suffices to demonstrate the validity of `f a : X i`. We will explore bidirectionality further in section **??**.

Induction-recursion is arguably the most powerful set former (currently known) for ITT. **TODO**:*who?* has shown that its addition gives ITT a proof-theoretic strength slightly greater than KPM, Kripke–Platek set theory together with a recursive Mahlo universe. Although its proof-theoretic strength is greater than Γ_0, ITT with induction–recursion is still considered predicative in a looser constructivist sense: it arguably has bottom–to–top construction.

2.1 Categories

Since we will use category theory as our main language we first recall the definition of a category `C`:

- a collection of objects `C : Set`
- a collection of morphisms (or arrows) `_⇒_ : (X Y : C) → Set`
- an identity `1 : (X : C) → X ⇒ X`
- a composition operation `_∘_ : ∀ {X Y Z} → Y ⇒ Z → X ⇒ Y → X ⇒ Z` that is associative and respects the identity laws `1 X ∘ F ≡ F ≡ F ∘ 1 Y`

A functor `F` between categories `C` and `D` is a mapping of objects `F : C → D` and a mapping of arrows `F[_] : ∀ {X Y} → X ⇒ F → F X ⇒ F Y`.

2.2 Data types

The different notions of data types, by which we mean inductive types, inductive–recursive types and their indexed variants, share their semantics: initial algebras of endfunctors. In a first approximation, we can think of an "initial algebra" as the categorical notion for the "least closed set" (just not only for sets). As such, we will study a certain class of functors with initial algebras that give rise to our indexed inductive–recursive types.

We shall determine the category our data types live in. The most simple data types, inductive types, live in the category `Set`. On the other hand, as we have seen, inductive–recursive data types are formed by couples in `(U : Set) × (U → X)`. Categorically, this an `X`-indexed set and it is an object of the slice category of `Set/X`. We will be representing these objects by the record type `Fam γ X`[1].

```
record Fam (α : Level) (X : Set β) : Set (lsuc α ⊔ β) where
  constructor _,_
  field
```

[1]See section **TODO**:*ref* for some explainations of `Level`, but for most part it can be safely ignored, together with its artifacts `Lift`, `lift` and the greek letters `α`, `β`, `γ` and `δ`.

```
    Code : Set α
    decode : Code → X

_⇝_ : Fam α₀ X → Fam α₁ X → Set _
F ⇝ G = (i : Code F) → Σ (Code G) λ j → decode G j ≡ decode F i
```

This definition already gives us enough to express our first example of inductive–recursive definition.

```
ΠΣ-univ : Fam lzero Set
ΠΣ-univ = U , el
```

Now we can get to indexed inductive–recursive data types which essentially are functions from an input index `i : I` to `(X i)`-indexed sets. We will use couples `(I , X)` a lot as they define the input and output indexing sets so we call their type `ISet`.

```
ISet : (α β : Level) → Set _
ISet α β = Fam α (Set β)

𝔽 : (γ : Level) → ISet α β → Set _
𝔽 γ (I , X) = (i : I) → Fam γ (X i)

_⟹_ : 𝔽 γ₀ X → 𝔽 γ₁ X → Set _
F ⟹ G = (i : _) → F i ⇝ G i
```

Again we might consider our universe example as a trivially indexed type.

```
ΠΣ-univᵢ : 𝔽 lzero (⊤ , λ _ → Set)
ΠΣ-univᵢ _ = U , el
```

TODO:*mention* F*Σ and* F*Π*

2.3 A Universe of Strictly Positive Functors

Dybjer and Setzer have first presented codes for (indexed) inductive-recursive definitions (**TODO:***ref*) by constructing a universe of functors. However, as conjectured by [2], this universe lacks closure under composition, *eg* if given the codes of two functors, we don't know how to construct a code for the composition of the functors. I will thus use an alternative universe construction devised by McBride which we call *Irish* induction–recursion[2].

In this section we fix a given pair of input/output indexes `X Y : ISet α β` and i will define codes `ρ : IIR δ X Y : Set` for some functors `⟦ ρ ⟧ : 𝔽 γ X → 𝔽 (γ ⊔ δ) Y`.

First we give a datatype of codes that will describe the first component inductive–recursive functors. This definition is itself inductive–recursive: we define a type `poly γ X : Set` representing the shape of the constructor[3] and a recursive predicate `info : poly γ X → Set` representing the information contained in the final datatype, underapproximating the information contained in a subnode by the output index `X i` it delivers.

Lets give some intuition for these constructors.

[2] It has also been called *polynomial* induction–recursion because it draws similarities to polynomial functors, yet they are different notions and should not be confused.

[3] It is easy to show that in a dependent theory, restricting every type to a single constructor does not lose generality.

- ι i codes an inductive position with input index i, *eg* the indexed identity functor. Its info is decode X i *eg* the output index that we will obtain from the later constructed recursive function.
- κ A codes the constant functor, with straighforward information content A.
- σ A B codes the dependent sum of a functor A and a functor family B depending on A's information.
- π A B codes the dependent product, but strict positivity rules out inductive positions in the domain. As such the functor A must be a constant functor and we can (and must) make it range over Set, not poly.

The encoding of our ΠℕN–universe goes as follows:

```
data ΠℕN-tag : Set where `ℕ `Π : ΠℕN-tag

ΠℕN₀ : poly lzero (⊤ , λ _ → Set)
ΠℕN₀ = σ (κ ΠℕN-tag) λ {   -- select a constructor
    -- no argument for `ℕ
  (lift `ℕ) → κ ⊤;
    -- first argument, an inductive position whose output index we bind to A
    -- second argument, a (non-dependent) Π type from A to an inductive position
  (lift `Π) → σ (ι *) λ {(lift A) → π A λ _ → ι *}}
```

We can now give the interpretation of a code ρ : poly δ X into a functor ⟦ ρ ⟧₀.

```
⟦_⟧₀ : (ρ : poly γ X) → 𝔽 δ X → Fam (γ ⊔ δ) (info ρ)
⟦_⟧₀ (ι i) F = lift ◁ lft γ F i
⟦_⟧₀ (κ A) F = Lift δ A , lift ∘ lower
⟦ σ A B ⟧₀ F = FΣ (⟦ A ⟧₀ F) λ a → ⟦ B a ⟧₀ F
⟦ π A B ⟧₀ F = FΠ A λ a → ⟦ B a ⟧₀ F

⟦_⟧[_]₀ : (ρ : poly γ X) → F ⟹ G → ⟦ ρ ⟧₀ F ⇝ ⟦ ρ ⟧₀ G
⟦ ι i   ⟧[ φ ]₀ = λ x → let j , p = φ i $ lower x in lift j , cong lift p
⟦ κ A   ⟧[ φ ]₀ = λ a → lift $ lower a , refl
⟦ σ A B ⟧[ φ ]₀ = FΣ⇝ ⟦ A ⟧[ φ ]₀ λ a → ⟦ B $ decode (⟦ A ⟧₀ _) a ⟧[ φ ]₀
⟦ π A B ⟧[ φ ]₀ = FΠ⇝ λ a → ⟦ B a ⟧[ φ ]₀
```

It would be time to check if this interpretation does the right thing on our example, alas even simple examples of induction–recursion are somewhat complicated, as such I don't think it would be informative to display here the normalized expression of ⟦ ΠℕNc ⟧₀ F. The reader is still encouraged to normalize it by hand to familiarize with the interpretation.

While taking as parameter a indexed family 𝔽 γ X, our intepreted functors only output a family Fam (γ ⊔ δ) (info ρ). In other words, ρ : poly γ X only gives the structure of the definition for a given input index i : Code Y. To account for that, the full description of the first component of inductive–recursive functors has to be a function node : Code Y → poly γ X. We are left to describe the recursive function, which can be done with a direct emit : (j : Code Y) → info (node j) → decode Y j computing the output index from the full information.

```
record IIR (γ : Level) (X Y : ISet α β) : Set (lsuc α ⊔ β ⊔ lsuc γ) where
  constructor _,_
  field
    node : (j : Code Y) → poly γ X
    emit : (j : Code Y) → info (node j) → decode Y j
```

We can now explain the index emitting function el, completing our encoding of the ΠIN–universe.

```
ΠINc : IIR lzero (⊤ , λ _ → Set) (⊤ , λ _ → Set)
node ΠINc _ = ΠIN₀
emit ΠINc _ (lift `IN , lift ∗) = IN
emit ΠINc _ (lift `Π , A , B) = (a : lower A) → lower $ B a

⟦_⟧ : IIR γ X Y → F δ X → F (γ ⊔ δ) Y
⟦ ρ ⟧ F = λ j → emit ρ j ◁ ⟦ node ρ j ⟧₀ F

⟦_⟧[_] : (ρ : IIR γ X Y) → F ⇒ G → ⟦ ρ ⟧ F ⇒ ⟦ ρ ⟧ G
⟦ ρ ⟧[ φ ] j = emit ρ j ◂ ⟦ node ρ j ⟧[ φ ]₀
```

We have use the post–composition functor defined as follows:

```
_◁_ : (f : X → Y) → Fam α X → Fam α Y
f ◁ F = _ , f ∘ decode F

_◂_ : (f : X → Y) → A ↛ B → f ◁ A ↛ f ◁ B
(f ◂ m) i = let (j , p) = m i in j , cong f p
```

2.4 Initial Algebra

2.4.1 Least Fixed–Point

Now that we have a universe of functors, we need to translate that into actual indexed inductive–recursive types. This amounts to taking its least fixed–point μ ρ.

```
μ : (ρ : IIR γ X X) → F γ X
Code (μ ρ i) = μ–c ρ i
decode (μ ρ i) = μ–d ρ i
```

It consists of two parts, the inductive family `μ–c ρ : Code X → Set` and the recursive function `μ–d ρ : (i : Code X) → μ–c ρ i → decode X i`. By chance Agda has a primitive for constructing these kinds of sets: the **data** keyword. Applying the interpreted functor to the least fixed–point with ⟦ ρ ⟧ (μ ρ) and the two components of the indexed family basically gives us the implementation of respectively μ–c ρ and μ–d ρ.

```
data μ–c (ρ : IIR γ X X) (i : Code X) : Set γ where
  ⟨_⟩ : Code (⟦ ρ ⟧ (μ ρ) i) → μ–c ρ i

μ–d : (ρ : IIR γ X X) (i : Code X) → μ–c ρ i → decode X i
μ–d ρ i ⟨ x ⟩ = decode (⟦ ρ ⟧ (μ ρ) i) x
```

We have now completed the encoding of ΠIN and we can write pretty versions the constructors!

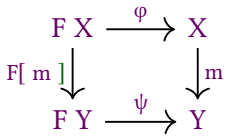
TODO:*minipage*

```
U₁ : Set
U₁ = μ–c ΠINc ∗

el₁ : U₁ → Set
el₁ = μ–d ΠINc ∗

`IN₁ : U₁
```

```
`ℕ₁ = ⟨ lift `ℕ , lift ∗ ⟩
`Π₁ : (A : U₁) (B : el₁ A → U₁) → U₁
`Π₁ A B = ⟨ lift `Π , lift A , lift ∘ B ⟩
```

2.4.2 Catamorphism and Paramorphism

I previously said that this least–fixed point has in category theory the semantic of an initial algebra. Let's break it down. Given an endofunctor `F : C → C`, an F-algebras is a carrier `X : C` together with an arrow `F X ⇒ X`. An arrow between two F-algebras `(X , φ)` and `(Y , ψ)` is an arrow `m : X ⇒ Y` subject to the commutativity of the usual square diagram `ψ ∘ F[ m ] ≡ m ∘ φ`.



Additionaly, an object `X : C` is initial if for any `Y : C` we can give an arrow `X ⇒ Y`.

We almost already have constructed an `⟦ ρ ⟧`-algebra with carrier `μ ρ` and the constructor `⟨_⟩` mapping the object part of `⟦ ρ ⟧ (μ ρ)` to `μ ρ`. What is left is to add a trivial proof.

```
roll : ⟦ ρ ⟧ (μ ρ) ⇒ μ ρ
roll _ x = ⟨ x ⟩ , refl
```

TODO:*interlude: intro example distinct elt list*

To prove the fact that our algebra is initial we have first have to formally write the type of algebras.

```
record alg (δ : Level) (ρ : IIR γ X X) : Set (α ⊔ β ⊔ lsuc δ ⊔ γ) where
  constructor _,_
  field
    {obj} : 𝔽 δ X
    mor : ⟦ ρ ⟧ obj ⇒ obj
open alg public
```

We can now give for every `φ : alg δ ρ` the initiality arrow `μ ρ ⇒ obj φ`.

```
fold : (φ : alg δ ρ) → μ ρ ⇒ obj φ
fold φ = mor φ ⊙ foldm φ
```

With the helper `foldm ρ` is defined as:

```
foldm : (φ : alg δ ρ) → μ ρ ⇒ ⟦ ρ ⟧ (obj φ)
foldm {ρ = ρ} φ i ⟨ x ⟩ = ⟦ ρ ⟧[ fold φ ] i x
```

Complying to the proof obligation for the equality condition, we get:

```
foldm-⊙ : (φ : alg δ ρ) → foldm φ ⊙ roll ≡ ⟦ ρ ⟧[ fold φ ]
foldm-⊙ φ = funext λ i → funext λ x → cong-Σ refl (uoip _ _)
fold-⊙ : (φ : alg δ ρ) → fold φ ⊙ roll ≡ mor φ ⊙ ⟦ ρ ⟧[ fold φ ]
fold-⊙ φ = trans ⊙-assoc $ cong (_⊙_ $ mor φ) (foldm-⊙ φ)
```

Note that we make use of `uoip` the unicity of identity proofs, together with the associativity lemma `⊙-assoc`.

As hinted by its name, the initiality arrow fold ρ is in fact a generic fold or with fancier wording an elimination rule, precisely the catamorphism (also called recursor). An elimination scheme is the semantic of recursive functions with pattern matching. Diggressing a little on elimination rules, we can notice that this is not the only one.

TODO:*introduce paramorphism, factorial on nat* **TODO**:*para is the most generic (non-dependent) eliminator, ref meeertens*

```
record alg≈ (δ : Level) (Y : Code X → Set β₁) (ρ : IIR γ X X) : Set (α ⊔ β₀ ⊔ β₁ ⊔ lsuc δ ⊔ γ) where
  constructor _,_
  field
    {obj} : 𝔽 δ (Code X , Y)
    down : (i : Code X) → decode X i → Y i
    mor : (down ◃ ⟦ ρ ⟧ (μ ρ & obj)) ⇒ obj
open alg≈ public

para₀ : (Y : Code X → Set β′) (φ : alg≈ δ Y ρ) → μ ρ ⇒ μ ρ & obj φ
π₀ (para₀ Y φ i ⟨ x ⟩) = ⟨ x ⟩ , π₀ $ mor φ i (π₀ $ ⟦ ρ ⟧[ para₀ Y φ ] i x)
π₁ (para₀ Y φ i ⟨ x ⟩) = refl

para : (Y : Code X → Set β′) (φ : alg≈ δ Y ρ) → (down φ ◃ μ ρ) ⇒ obj φ
π₀ (para Y φ i ⟨ x ⟩) = π₀ $ mor φ i (π₀ $ ⟦ ρ ⟧[ para₀ Y φ ] i x)
π₁ (para Y φ i ⟨ x ⟩) = trans (π₁ $ mor φ i _) (cong (down φ i) (π₁ $ ⟦ ρ ⟧[ _ ] i x))
```

2.5 Induction Principle

We have given several elimination rules, but dependent languages are used to do mathematics and the only elimination rule a mathematican would want on an inductive type is the most powerful one: an induction principle. In substance the induction principle states that, for any predicate `P : (i : Code X) (x : Code (μ ρ i)) → Set`, if given that the predicate holds for every subnode we can show it hold for the node itself, then we can show the predicate to hold for every possible node.

Let's formalize that a bit. I define a predicate all stating that a property hold for all subnodes. It looks a lot like ⟦ ρ ⟧ but does something slightly more powerful at inductive positions.

```
all : (ρ : poly γ X) (P : ∀ i → Code (F i) → Set δ) →
      Code (⟦ ρ ⟧₀ F) → Set (α ⊔ γ ⊔ δ)
all (ι i)     P (lift x) = Lift (α ⊔ γ) (P i x)
all (κ A)     P x        = ⊤
all (σ A B) P (a , b) = Σ (all A P a) λ _ → all (B (decode (⟦ A ⟧₀ _) a)) P b
all (π A B) P f       = (a : A) → all (B a) P (f a)
```

Given that I can state the induction principle.

```
induction : (ρ : IIR γ X X) (P : ∀ i → Code (μ ρ i) → Set δ)
  (p : ∀ i (xs : Code (⟦ ρ ⟧ (μ ρ) i)) → all (node ρ i) P xs → P i (⟨_⟩ xs)) →
  (i : Code X) (x : Code (μ ρ i)) → P i x
induction ρ P p i ⟨ x ⟩ = p i x $ every (node ρ _) P (induction ρ P p) x
```

I used the helper every which explains how to construct a proof of all for ⟦ ρ ⟧ F if we can prove the predicate for F.

```
every : (ρ : poly γ X) (P : ∀ i → Code (D i) → Set δ)
```

```
          (p : ∀ i (x : Code (D i)) → P i x) (xs : Code (⟦ ρ ⟧₀ D)) →
          all ρ P xs
every (ι i)     _ p (lift x) = lift $ p i x
every (κ A)     P _ _        = *
every (σ A B) P p (a , b)    = every A P p a , every (B (decode (⟦ A ⟧₀ _) a)) P p b
every (π A B) P p f          = λ a → every (B a) P p (f a)
```

Note that I could have derived the other elimination rules from this induction principle, but cata– and paramorphisms are very useful non–dependent special cases that diserve to be treated separately and possibly optimized. Non-dependent functions still have a place of choice in dependent languages: just because we can replace every implication by universal quantification doesn't mean we should.

3 Ornaments

3.1 Fancy Data

A major use for indexes in type families is to refine a type to contain computational relevant information about objects of that type. Suppose we have a type of lists.

```
data vec (X : Set) : Set where
  nil : vec X
  cons : X → vec X → vec X
```

We may want to define a function `zip : vec X → vec Y → vec (X × Y)` pairing up the items of two arguments.

```
zip : vec X → vec Y → vec (X × Y)
zip nil         nil         = nil
zip (cons x xs) (cons y ys) = cons (x , y) (zip xs ys)
zip (cons x xs) nil         = ?
zip nil         (cons y ys) = ?
```

It is clear that there is nothing really sensible to do for the two last cases. We should signal some incompatibility by throwing an exception or we may just return an empty list. But this is not very principled. What we would like is to enforce on the type level that the two arguments have the same length and that we additionally will return a list of that exact length. This type is called vec.

```
data vec (X : Set) : ℕ → Set where
  nil : vec X zero
  cons : ∀ {n} → X → vec X n → vec X (suc n)
```

I wrote the constructors such that they maintain the invariant that `vec X n` is only inhabited by sequences of length `n`. I may now write the stronger version of `zip` which explicitely states what is possible to zip.

```
zip : {X Y : Set}{n : ℕ} → vec X n → vec Y n → vec (X × Y) n
zip nil         nil         = nil
zip (cons x xs) (cons y ys) = cons (x , y) (zip xs ys)
```

This is made possible because of the power dependent pattern matching has: knowing a value is of a particular constructor may add constraints to the type of the expression we have to produce

and to the type of other arguments. As such when we pattern match with cons on the first argument, the implicit index n gets unified with suc m, which implies that the second argument has no choice but to be a cons too.

Several comments can be made about vec and vec. The first one is that they are almost same. More precisely, they have the same shape, the only added argument is the natural number n in cons for vec[4]. Because only a sprinkle of information has been added to something of the same shape, we should be able to derive a function from vec X n to vec X. The second comment is that there is an straightforward isomorphism between vec X and Σ ℕ (vec X). As such we should be able to come up with the reverse function (x : vec X) → vec X (length x).

The rest of this section will be dedicated to formalizing prose definitions such as "vectors are lists indexed by their length" and generically deriving the properties that they imply.

3.2 Reindexing

Another take on the previous example of lists and vectors is that vectors have a more informative index (natural numbers) than lists (trivial indexation by the unit type). This can be expressed by the fact that there is a function ℕ → ⊤ giving a non-fancy index given a fancy one. Because we work with inductive–recursive types and not just inductive ones, we have two indexes—the input index I : Set and the output index X : I → Set—and we have to translate this notion. For this we introduce the datatype PRef (index refinement using powersets).

```
record PRef (α₁ β₁ : Level) (X : ISet α₀ β₀) : Set (α₀ ⊔ β₀ ⊔ lsuc α₁ ⊔ lsuc β₁) where
  field
    Code : Set α₁
    down : Code → Fam.Code X
    decode : (j : Code) → decode X (down j) → Set β₁
open PRef public
```

Let X : ISet α₀ β₀ and R : PRef α₁ β₁ X. Code R represents the new input index, together with the striping function down R taking new input indexes to old ones. Additionaly we have to define a new output index Y : Code R → Set such that we can derive a stripping function (j : Code R) → Y j → X (down j). Directly defining Y together with this second striping function would not have been practical[5]. Thus instead of the stripping function, we ask for its fibers (called its graph), given by decode R. This reversal is the classical choice between families (A : Set) × A → X and powersets X → Set to represent indexation.

Because of the small fiber twist we operated, we have a bit of work to get the new indexing pair (in ISet) from a PRef.

```
PFam : PRef α₁ β₁ X → ISet α₁ (β₀ ⊔ β₁)
Code (PFam P) = Code P
decode (PFam P) j = Σ _ (decode P j)
```

In substance, the new output index is simply the old one to which we add some information that can depend on it. The stripping function is thus simply the projection π_0.

[4]Actually this n does not contain any information as it can be derived from the type index. As such there is ongoing research to optimize away these kind of arguments and we will see that because of our index–first formalism of indexed datatypes it will not even be added in the first hand.

[5]Later we would have needed to define preimages which necessarily embed some notion of equality. As explained in ?? we want to avoid any mention of equality when formalizing the unrelated matters of data types.

3.3 A Universe of Ornaments

Step by step, following the construction of induction–recursion I will start by describing ornaments of poly, the inductive part of the definition. For R : PRef α_1 β_1 X and ρ : poly γ_0 X I define a universe of decriptions orn_0 γ R ρ : Set _. Simultaneously I define an interpretation ⌊ o ⌋$_0$: poly (γ_0 ⊔ γ_1) (PFam R) taking the description of the "delta" to the actual fancy description it represents, and a stripping function info↓ : info ⌊ o ⌋$_0$ → info ρ taking new node informations to old ones.

```
data orn₀ (γ₁ : Level) (R : PRef α₁ β₁ X) : poly γ₀ X → Set
⌊_⌋₀ : (o : orn₀ γ₁ R ρ) → poly (γ₀ ⊔ γ₁) (PFam R)
info↓ : info ⌊ o ⌋₀ → info ρ

data orn₀ γ₁ R where
  ι    : (j : Code R)                                  → orn₀ γ₁ R (ι (down R j))
  κ    : {A : Set γ₀}                                  → orn₀ γ₁ R (κ A)
  σ    : (A : orn₀ γ₁ R U)
         (B : (a : info ⌊ A ⌋₀) → orn₀ γ₁ R (V (info↓ a)))
           → orn₀ γ₁ R (σ U V)
  π    : (B : (a : A) → orn₀ γ₁ R (V a)) → orn₀ γ₁ R (π A V)
  add₀ : (A : poly (γ₀ ⊔ γ₁) (PFam R))
         (B : info A → orn₀ γ₁ R U)                    → orn₀ γ₁ R U
  add₁ : (A : orn₀ γ₁ R U)
         (B : info ⌊ A ⌋₀ → poly (γ₀ ⊔ γ₁) (PFam R)) → orn₀ γ₁ R U
  del-κ : (a : A) → orn₀ γ₁ R (κ A)

⌊ ι j       ⌋₀ = ι j
⌊_⌋₀ (κ {A}) = κ (Lift γ₁ A)
⌊ σ A B     ⌋₀ = σ ⌊ A ⌋₀ λ a → ⌊ B a ⌋₀
⌊_⌋₀ (π {A} B) = π (Lift γ₁ A) λ {(lift a) → ⌊ B a ⌋₀}
⌊ add₀ A B ⌋₀ = σ A λ a → ⌊ B a ⌋₀
⌊ add₁ A B ⌋₀ = σ ⌊ A ⌋₀ B
⌊ del-κ _  ⌋₀ = κ ⊤

info↓ {o = ι i}      (lift (x , _)) = lift x
info↓ {o = κ}        (lift a)       = lift $ lower a
info↓ {o = σ A B}    (a , b)        = info↓ a , info↓ b
info↓ {o = π B}      f              = λ a → info↓ (f $ lift a)
info↓ {o = add₀ A B} (a , b)        = info↓ b
info↓ {o = add₁ A B} (a , _)        = info↓ a
info↓ {o = del-κ a}  _              = lift a
```

Lets break down the constructors. First we have the constructors that look like poly: ι, κ, σ and π. They essentially say that nothing is changed. ι j ornaments poly of the form ι i where down R j ≡ i *ie* we replace inductive positions by a fancy index such that the stripping matches. σ A B has to use the interpretation ⌊_⌋$_0$ and info↓ to express how the family B depends on the info of A. κ and σ B change nothing and as such some of their arguments are implicit because there is no choice possible.

The next 3 constructors allow to change things. add_0 allows to delay the ornamenting, it interprets into a σ where the first component has no counterpart in the initial data. In other

words we add a new argument to the constructor and then give an ornament which might depend on it. add_1 is the other way around, it gives an ornament and then adds new arguments which might depend on it. And finally del–κ allows you to erase a constant argument given that you can provide an element of it. It is restricted to delete only constants because for an inductive position it is not really clear what the notion of "element of it" is.

$\lfloor \_ \rfloor_0$ and info↓ are straightforward, the first 4 constructors are unsurprising, the additions interpret into sigmas where info↓ ignores the new component and the deletion interprets into a trivial constant, info↓ giving back the element we have provided in the definition.

As for inductive–recursive types in this part of the construction we are not yet taking input indexes into account so we can't give the ornament of lists into vectors yet. But we can give the ornament of natural numbers into lists: we identify zero with nil and suc with cons where cons demands an additional constant argument in addition to the recursive position.

```
data ℕ–tag : Set where `ze `su : ℕ–tag

nat–c : poly lzero (⊤ , λ _ → ⊤)
nat–c = σ (κ ℕ–tag) λ {
  (lift `ze) → κ ⊤;
  (lift `su) → ι * }

list–R : PRef lzero lzero (⊤ , λ _ → ⊤)
Code list–R = ⊤
down list–R _ = *
decode list–R _ _ = ⊤

list–o : (X : Set) → orn₀ lzero list–R nat–c
list–o X = σ κ λ {
  (lift (lift `ze)) → κ                    ;
  (lift (lift `su)) → add₀ (κ X) λ _ → ι *}
```

I define the type orn γ_1 R S ρ : Set ornamenting ρ : IIR γ_0 X Y.

```
record orn (γ₁ : Level) (R : PRef α₁ β₁ X) (S : PRef α₁ β₁ Y) (ρ : IIR γ₀ X Y) : Set where
  field
    node : (j : Code S) → orn₀ γ₁ R (node ρ (down S j))
    emit : (j : Code S) → (x : info ⌊ node j ⌋₀) → decode S j (emit ρ (down S j) (info↓ x))
```

node is not surprising, for every fancy input index we give an ornament of the description with the corresponding old index. The emit function starts off like the one for IIR, taking an input index and the info, here of the interpretation of the ornament. Having that, we can already compute the old decoding using info↓ and emit ρ (down R j). We thus require to generate an output index compatible with the old output index we have derived.

We eventually reach the full interpretation ⌊_⌋ taking an ornament to a fancy IIR.

```
⌊_⌋ : (o : orn γ₁ R S ρ) → IIR (γ₀ ⊔ γ₁) (PFam R) (PFam S)
node ⌊ o ⌋ j = ⌊ node o j ⌋₀
emit ⌊ o ⌋ j = λ x → _ , emit o j x
```

TODO:*list to vec here?*

3.4 Ornamental Algebra

3.5

4 Case Study: Bidirectional Simply-Typed Lambda Calculus

??

5 Discussion

5.1 Index-First Datatypes and a Principled Treatment of Equality

?? **TODO:***bidirectional flow discipline in formalizations* **TODO:***no choice about equality, explicit proof obligation instead of weird pattern matching conditions*

5.2 Further Work

TODO:*extend to fibred IR*
TODO:*precise the paramorphism thing*
TODO:*study datastructure reorganizations (eg optimizations)*

A Introduction to Agda

B Bibliography

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