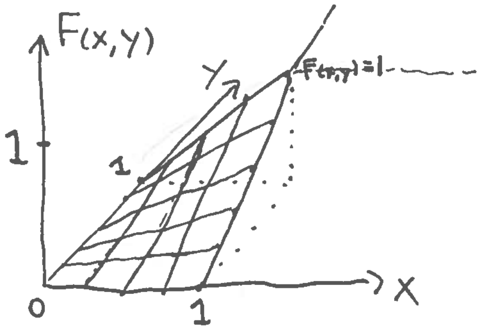
Opg. 3-1.1:

$$F(x,y) = 0 \qquad x<0,\ y<0$$

$$= xy \qquad 0 \le x,y \le 1$$

$$= 1 \qquad x>1,\ y>1$$

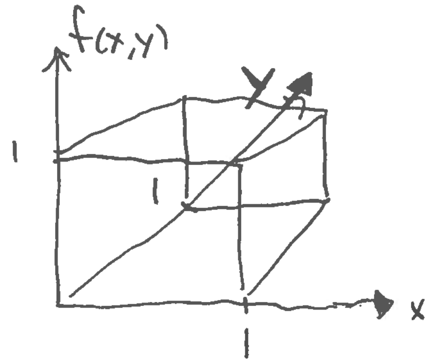
a)



b) $$f(x,y) = \frac{\partial F(x,y)}{\partial x\,\partial y} = 0 \qquad x<0,\ y<0$$

$$= 1 \qquad 0 \le x,y \le 1$$

$$= 0 \qquad x>1,\ y>1$$



Check: $\int_0^1\int_0^1 f(x,y)\,dx\,dy = 1$ ok!

(Betingelse s. 122)

c) $$\Pr\left[X \le \tfrac{3}{4},\ Y > \tfrac{1}{4}\right] = \int_{x=-\infty}^{3/4}\int_{y=\frac{1}{4}}^{1} f(x,y)\,dx\,dy$$

$$= \int_{x=0}^{3/4}\int_{y=\frac{1}{4}}^{1} 1\cdot dx\,dy$$

$$= \left(x\Big|_0^{3/4}\right)\left(y\Big|_{\frac{1}{4}}^{1}\right) = \left(\frac{3}{4}\right)\left(1-\frac{1}{4}\right) = \left(\frac{3}{4}\right)^2 = \underline{\underline{\frac{9}{16}}}$$

Opg 3-1.2:

$$f(x,y) = kxy \qquad 0 \le x,y \le 1$$

$$= 0 \qquad \text{ellers.}$$

a) $$\int_{x=0}^{1}\int_{y=0}^{1} kxy \cdot dxdy = k \int_0^1 x \left(\frac{y^2}{2}\Big|_0^1 \right) dx$$

$$= \frac{k}{2}\left(\frac{x^2}{2}\Big|_0^1 \right)$$

$$= \frac{k}{4} = 1 \qquad \text{(Betingelse 2 s. 122)}$$

$$\Rightarrow \underline{\underline{k = 4}}$$

b) $$F(x,y) = \int_{u=0}^{x}\int_{v=0}^{y} k \cdot u \cdot v \cdot du \cdot dv = \int_0^x\int_0^y 4uv \, dudv$$

$$= \int_0^x 2u \, du \cdot \int_0^y 2v \, dv$$

$$= \left(u^2\Big|_0^x\right)\left(v^2\Big|_0^y\right) = \underline{\underline{x^2y^2}} \qquad \text{for } 0 \le x,y \le 1$$

$F(x,y) = 0$ for $x,y < 0$

$F(x,y) = 1$ for $x,y > 1$

c) $$Pr\left(X \le \frac{1}{2}, Y > \frac{1}{2}\right) = \int_0^{1/2}\int_{1/2}^{1} 4xy \, dxdy = \left(x^2\Big|_0^{1/2}\right)\left(y^2\Big|_{1/2}^{1}\right) = \frac{1}{4}\cdot\frac{3}{4} = \underline{\underline{\frac{3}{16}}}$$

d) $$f_X(x) = \int_0^1 4xy \, dy = 4x \cdot \left(\frac{y^2}{2}\Big|_0^1\right) = 4x \cdot \frac{1}{2} = \underline{\underline{2x}}$$

Opg. 3-1.3:

a) $E[XY] = \int_0^1 \int_0^1 xy \cdot f(x,y)\, dx\, dy$

$$= \int_0^1 \int_0^1 xy \cdot 1 \, dx\, dy = \left(\frac{x^2}{2}\Big|_0^1\right)\left(\frac{y^2}{2}\Big|_0^1\right) = \frac{1}{2}\cdot\frac{1}{2}$$

$$= \underline{\underline{\frac{1}{4}}}$$

b) $E[XY] = \int_0^1 \int_0^1 xy \cdot f(x,y)\, dx\, dy$

$$= \int_0^1 \int_0^1 xy \cdot (4xy)\, dx\, dy$$

$$= 4 \int_0^1 x^2\, dx \int_0^1 y^2\, dy$$

$$= 4 \left(\frac{x^3}{3}\Big|_0^1\right)\left(\frac{y^3}{3}\Big|_0^1\right)$$

$$= 4 \cdot \frac{1}{3} \cdot \frac{1}{3} = \underline{\underline{\frac{4}{9}}}$$

Opg. 3-2.2: $f(x,y) = 4xy$ for $0 \le x,y \le 1$
$= 0$ ellers

a)

$$f_Y(y) = \int_0^1 4xy\, dx = 4y\left(\frac{x^2}{2}\Big|_0^1\right) = 4y \cdot \frac{1}{2} = 2y$$

$$f(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{4xy}{2y} = \underline{\underline{2x}}$$

b) Ved samme fremgangsmåde som a):

$f_X(x) = 2x$

$$f(y|x) = \frac{f(x,y)}{f_X(y)} = \frac{4xy}{2x} = \underline{\underline{2y}}$$

Opg. 3-2.4: $f(x,y) = K\, e^{-(x^2+y^2+4xy)}$

a) Vi har, $f(x|Y=y) = f(x|y) = \dfrac{f(x,y)}{f_Y(y)}$.

(Formel 3-12)

Vi skal finde det bedste X, givet Y=y.

Det vil sige, vi skal finde det X, som maksimerer $f(x|y) = \frac{f(x,y)}{f_Y(y)}$. Men da $f_Y(y)$ ikke afhænger af X, skal vi bare finde det x, som maksimerer $f(x,y)$.

Dette svarer til, at $\dfrac{\partial f(x,y)}{\partial x} = 0$.

$$\frac{\partial f(x,y)}{\partial x} = K(-2x - 4y)\, e^{-(x^2+y^2+4xy)}.$$

x_{opt} skal opfylde $(-2x_{opt} - 4y) = 0$.

$$\Rightarrow \underline{\underline{x_{opt} = -2y}}$$

Opg 3-2.4: Fortsat

b) $Y = 3$

$$x_{opt} = -2y = \underline{\underline{-6}}$$

Opg. 3-4.1: $E(X) = E(Y) = 0$

$\sigma_X^2 = 16$, $\sigma_Y^2 = 36$.

$\rho = \frac{1}{2}$.

a) Formel 3-28.

$$\sigma_{X+Y}^2 = \sigma_X^2 + \sigma_Y^2 \pm 2\rho\sigma_X\sigma_Y$$

$$= 16 + 36 + 2 \cdot \frac{1}{2} \cdot 4 \cdot 6$$

$$= \underline{\underline{76}}$$

b) Formel 3-28.

$$\sigma_{X-Y}^2 = \sigma_X^2 + \sigma_Y^2 - 2\rho\sigma_X\sigma_Y$$

$$= \underline{\underline{28}}$$

c) $\sigma_{x+y}^2 = 28$ og $\sigma_{x-y}^2 = 76$.

Samme fremgangsmåde som ovenfor, men $\rho = -\frac{1}{2}$