Handin 4

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Problem 1

Which of the following sets are well-ordered? (Why/why not?)

a $S = \{x \in \mathbb{Q} : x \geq -10\}$

S is not well-ordered, since there is a subset which has no least number. A subset could be written as $x = (-10 : \infty)$

b $S = \{-2, -1, 0, 1, 2\}$

S is well-ordered, since it is finite and and only with natural numbers which can be written as $S = [-2 : 2]$.

c $S = \{x \in \mathbb{Q} : -1 \leq x \leq 1\}$

S is not well-ordered, since it is in $\mathbb{Q}$ and a subset can be written as $U = (-1, 1]$ which means we have no least value when we can divide by a larger number to get an even smaller number.

d $S = \{p : p \text{ is prime}\} = \{2, 3, 5, 7, 9, 11, 13, \dots\}$

S is well-ordered, since there is a lower limit for any subset.

Problem 2

Use mathematical induction to prove that $1 + 5 + 9 + \ldots + (4n - 3) = 2n^2 - n$ **for every positive integer** n.

Base case
$P(1)$ should be true:

$$4n - 3 = 2n^2 - n \tag{1}$$
$$4 \cdot 1 - 3 = 2 \cdot 1^2 - 1 \tag{2}$$
$$1 = 1 \tag{3}$$

From (**??**) we have shown the base case is in fact true.

Induction step
The following should apply: $\forall k \in \mathbb{N}, P(k) \Rightarrow P(k+1)$

Proof.

$$
\begin{aligned}
1+5+9+\ldots+(4k-3) &= 2k^2-k && P(k) && (4)\\
1+5+9+\ldots+(4(k+1)-3) &= 2(k+1)^2-k+1 && P(k+1) && (5)\\
&= \underbrace{1+5+9+\ldots+(4k-3)}_{\text{Lefthand side of P(k)}}+(4(k+1)-3) && && (6)\\
&= 2k^2-k+4k+1 && && (7)\\
&= 2k^2+3k+1 && && (8)\\
&= 2(k+1)^2-(k+1) && && (9)
\end{aligned}
$$

The result the same as (**??**) true and therefor follows the Principle of Mathematical Induction. □

Problem 3

Prove that $2^n \geq n^3$ for every integer $n \geq 10$.

Proof. The inequality holds for $n = 10$ since $2^{10} \geq 10^3$. Assume that $2^k \geq k^3$, where k is a nonnegative integer. We show that $2^{k+1} \geq (k+1)^3$. When $k = 10$, we have $2^{10} = 1024 \geq 1000 = 10^3$. We therefor assume that $k \geq 10$. Then

$$
\begin{aligned}
2^{k+1} = 2 \cdot 2^k &\geq k^3+3k^2+3k+1 && && (10)\\
2 \cdot 2^k &\geq 2 \cdot k^3 && 2 \cdot 10^2 = 2048, 2 \cdot 10^3 = 2000 && (11)\\
&\geq k^3+k^3 && \text{Isolate the first } k^3 && (12)\\
&\geq k^3+10k^2 && k^3 \geq 10k^2 && (13)\\
&= k^3+3k^2+7k^2 && && (14)\\
&\geq k^3+3k^2+3k+1 && 7k^2 \geq 3k+1 && (15)
\end{aligned}
$$

By the Principle of Mathematical Induction, $2^n \geq n^3$ for every integer $n \geq 10$, since (**??**) is equal to (**??**). □

Problem 4

Use the method of minimum counterexample to prove that $3|\left(2^{2n}-1\right)$ for every positive integer n.

Proof. Assume, to the contraty, that there are positive integers n such, that $6 \not|(2^{2n}-1$. Then there is a a smallest positive integer n such that $6 \not|(2^{2n}-1$. Let m be this integer. If $n = 1$, then $2^{2n}-1 = 3$. Since $3|3$, it follows that $3|(2^{2n}-1)$ for $n = 1$. Therefor, $m \geq 2$. So we can write $m = k+1$ where $1 \leq k < m$.
So $3|(2^{2k}-1)$, hence $3x = 2^{2k}-1 \Leftrightarrow 3x+1 = 2^{2k}$. Observe that

$$
\begin{aligned}
m^{2m}-1 &= 2^{2(k+1)}-1 && (16)\\
&= 2^2 \cdot 2^{2k}-1 && (17)\\
&= 4 \cdot (2^{2k}-1) && (18)\\
&= 4 \cdot (3x+1)-1 && (19)\\
&= 3(4x+1) && (20)
\end{aligned}
$$

This show us that $3|(2^{2m}-1)$ is true, since $4x+1 = \mathbb{Z}$. This is a contradiction, so the original statement is proved. □

Problem 5

Use the Strong Principle of Mathematical to prove the following.
Let $S = \{i \in \mathbb{Z} : i \geq 2\}$
and the P be a subset of S with the properties that $2, 3 \in P$
and if $n \in S$, then either $n \in P$ or $n = ab$, where $a, b \in S$.
Then every element of S either belongs to P or it can be expressed as a product of elements of P.

Proof. The set, S, consist of only integers greater than 2 and P is a subset of S. Should n be in the set S, it can be expressed as a product of two numbers in the subset, or it is in the subset it self.
Case 1: n is not a prime – let's say 6. Here n can be expressed as $2 \cdot 3 = 6$ and n is therefor not in the subset, P.
Case 2: n is not a prime – let's say 12. Here n can be expressed as $3 \cdot 4 = 12$ and n is not in the subset, but 4 is.
Case 3: n is a prime – let's say 5. Here n can not be expressed by the product of the elements in P and is only the the set, S.
With the cases we can see, that all elements of S can be in the subset, P. Some elements of S cannot be the product of elements in P and must therefor be in the subset.

It can also be stated a number in the subset is $k \geq 2$, which means that $2 \leq a \leq k$ and $2 \leq b \leq k$ which then follow, that $k + 1 = a \cdot b$. □

Problem 6

Use the Strong Principle of Mathematical Induction to prove, that for each integer $n \geq 12$, there are non-negative integers a and b such the $n = 3a + 7b$.

Proof. Prove the base case:

$$12 = 3 \cdot 4 + 7 \cdot 0 \tag{21}$$

Next we assume, that for an integer $k \geq 12$ that for every integer i within $12 \leq i \leq k$, there exist non-negative integer a and b such that $i = 3a + 7b$.
We show that there exists non-negative integers x and y such that $k+1 = 3x+7y$. Since $13 = 3 \cdot 2 + 7 \cdot 1$ and $14 = 3 \cdot 0 + 7 \cdot 2$ we may assume that $k \geq 14$.
Since $k - 2 \geq 12$, there exist non-negative integers c and d such that $k - 2 = 3x + 7d$. Hence $k + 1 = 3(c + 1) + 7d$.
By the Strong Principle of of Mathematical Induction, for each integer $n \geq 12$, there are non-negative integers a and b such that $n = 3a + 7b$ is true. □