Test of Distributed Systems

Lecture 11: Linear Temporal Logic

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Should these statements be true?

- $\Box p \leftrightarrow \neg\Diamond\neg p$ (duality)
- $\Box(p \to q) \to (\Box p \to \Box q)$ (distributivity)
- $\Box p \to p$ (reflexivity)
- $p \to \Diamond p$ (reflexivity)
- $\Box p \to \Box\Box p$ (transitivity)
- $\Diamond p \to \Diamond\Diamond p$ (transitivity)
- $\Box p \to \bigcirc p$ (step)
- $\bigcirc p \to \Diamond p$ (step)
- $\bigcirc p \leftrightarrow \neg \bigcirc \neg p$ (linearity)

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Axioms of LTL

1. Predicate Calculus
2. $\Box(p \to q) \to (\Box p \to \Box q)$ (distributivity of $\Box$ over $\to$)
3. $\bigcirc(p \to q) \to (\bigcirc p \to \bigcirc q)$ (distributivity of $\bigcirc$ over $\to$)
4. $\Box p \to (p \land \bigcirc p \land \bigcirc\Box p)$ (expansion of $\Box$)
5. $\Box(p \to \bigcirc p) \to (p \to \Box p)$ (induction)
6. $\bigcirc p \leftrightarrow \neg \bigcirc \neg p$ (linearity)

Inference rules:

$$\frac{p,\ p \to q}{q} \qquad \frac{p}{\Box p} \quad \frac{p \to q}{\Box p \to \Box q} \quad \frac{p \to q}{\bigcirc p \to \bigcirc q} \qquad \frac{p \to \bigcirc p}{p \to \Box p}$$

modus ponens generalization induction

Define:

- $\Diamond p \leftrightarrow \neg\Box\neg p$ (duality)

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$\Box$ is transitive and reflexive

Theorem: $\Box p \leftrightarrow \Box\Box p$.
Proof:

$\Box p \leftrightarrow \Box\Box p$

{PC}

- $\Box p \rightarrow \Box\Box p$

  {induction}

  $\Box p \rightarrow \bigcirc\Box p$

  {expansion}

- $\Box\Box p \rightarrow \Box p$

  {expansion}

$\Box$ distributes over $\wedge$

Theorem: $\Box(p \wedge q) \to (\Box p \wedge \Box q)$.
Proof:

$$\Box(p \wedge q) \to (\Box p \wedge \Box q)$$

$\{$PC$\}$

- $\Box(p \wedge q) \to \Box p$

  $\{$generalization$\}$

  $p \wedge q \to p$

  $\{$PC$\}$

- $\Box(p \wedge q) \to \Box q$

  $\{$generalization$\}$

  $p \wedge q \to q$

  $\{$PC$\}$

$\Box$ contraction

Theorem: $p \wedge \bigcirc\Box p \to \Box p$.

Proof:

$p \wedge \bigcirc\Box p \to \Box p$

$\{$PC$\}$

$p \wedge \bigcirc\Box p \to \Box p \wedge \Box\bigcirc\Box p$

$\{$distribution over $\wedge\}$

$p \wedge \bigcirc\Box p \to \Box(p \wedge \bigcirc\Box p)$

$\{$induction$\}$

$p \wedge \bigcirc\Box p \to \bigcirc(p \wedge \bigcirc\Box p)$

$\{$PC$\}$

$\bigcirc\Box p \to \bigcirc(p \wedge \bigcirc\Box p)$

$\{$generalization$\}$

$\Box p \to p \wedge \bigcirc\Box p$

$\{$expansion$\}$

$\lozenge$ current

Theorem: $p \to \lozenge p$.
Proof:

$$p \to \lozenge p$$

$\{$definition$\}$

$$p \to \neg\Box\neg p$$

$\{$PC$\}$

$$\neg\neg p \to \neg\Box\neg p$$

$\{$PC$\}$

$$\Box\neg p \to \neg p$$

$\{$expansion$\}$

$\Diamond$ next

Theorem: $\bigcirc p \to \Diamond p$.
Proof:

$$\bigcirc p \to \Diamond p$$

{definition}

$$\bigcirc p \to \neg\Box\neg p$$

{linearity}

$$\neg\bigcirc\neg p \to \neg\Box\neg p$$

{PC}

$$\Box\neg p \to \bigcirc\neg p$$

{expansion}

$\Box$ implies $\Diamond$

Theorem: $\Box p \to \Diamond p$.
Proof:

$$\Box p \to \Diamond p$$

$\{$PC$\}$

- $\Box p \to \bigcirc p$

  $\{$expansion$\}$

- $\bigcirc p \to \Diamond p$

  $\{\Diamond$ next$\}$

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Distribution

Law: $\Box(p \wedge q) \leftrightarrow (\Box p \wedge \Box q)$.
Theorem: $\Diamond(p \vee q) \leftrightarrow (\Diamond p \vee \Diamond q)$.
Proof:

$$\Diamond(p \vee q) \leftrightarrow (\Diamond p \vee \Diamond q)$$

{definition}

$$\neg\Box\neg(p \vee q) \leftrightarrow (\neg\Box\neg p \vee \neg\Box\neg q)$$

{PC}

$$\neg\Box(\neg p \wedge \neg q) \leftrightarrow \neg(\Box\neg p \wedge \Box\neg q)$$

{PC}

$$\Box(\neg p \wedge \neg q) \leftrightarrow (\Box\neg p \wedge \Box\neg q)$$

{the law above}

More laws

- $\Diamond p \leftrightarrow p \vee \bigcirc \Diamond p$
- $\Box(p \to q) \to (\Diamond p \to \Diamond q)$ { generalization }
- $\Diamond(p \to q) \to (\Diamond p \to \Diamond q)$
- $\Box \bigcirc p \leftrightarrow \bigcirc \Box p$ { exchange }
- $\Diamond \bigcirc p \leftrightarrow \bigcirc \Diamond p$ { exchange }
- $\Diamond \Diamond p \leftrightarrow \Diamond p$

Contraction

Theorem: $\Box\Diamond\Box p \leftrightarrow \Diamond\Box p$.
Proof:

$\Box\Diamond\Box p \leftrightarrow \Diamond\Box p$

{PC}

- $\Box\Diamond\Box p \rightarrow \Diamond\Box p$

  {expansion}

- $\Diamond\Box p \rightarrow \Box\Diamond\Box p$

  {induction}

  $\Diamond\Box p \rightarrow \bigcirc\Diamond\Box p$

  {exchange}

  $\Diamond\Box p \rightarrow \Diamond\bigcirc\Box p$

  {generalization}

  $\Box p \rightarrow \bigcirc\Box p$

  {expansion}

More laws

Theorem (Contraction): $\Diamond\Box\Diamond p \leftrightarrow \Box\Diamond p$.
Proof: exercise.

Theorem: $\Diamond\Box p \rightarrow \Box\Diamond p$.
Proof: exercise.

Theorem: $\Box\Diamond p \rightarrow \Diamond\Box p$.
Proof?

Counterexample!