Convolutional Codes

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Introduction

- Convolutional coding is a second technique in ECC.
- Convolutional codes differ from block codes in that encoder contains memory
  - The encoder outputs at any given time instant depends not only on the inputs at that time instant but also on some previous inputs.
- To simplify decoding, linear convolutional codes are preferred.
- A rate $R = k/n$ convolutional encoder with memory order K can be realized as a k-input and n-output linear sequential circuit with input memory K.
  - It is denoted as $C_{conv}(n, k, K)$.
- In CC, large minimum distances and high error correction capability are achieved by increasing the memory order K, which is different from block codes.
  - i.e. the higher the level of memory, the higher the complexity of the convolutional decoder and the stronger the error correction capability.

Linear Sequential Circuits

- Linear sequential circuits are constructed by using basic memory units (or delays), combined with adder or scalar multiplies that operate over $GF(q)$.
- Linear sequential circuits are known as finite state sequential machines(FSSMs).
- FSSM analysis is usually performed by means of a rational transfer function $G(D)$ of polynomial in the D domain, called the delay domain.
    - where message sequence and code sequence use the polynomial form $M(D)$ and $C(D)$, respectively.
- A convolutional encoder is a structure created using FSSMs which generates an output sequence based on a given input sequence.

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Convolutional Encoder-1

- A convolutional encoder takes a k-tuple $\mathbf{m}_i$ of message elements as input, and generates the n-tuple $\mathbf{c}_i$ of coded elements as the output at a given time i.
- $\mathbf{c}_i$ depends not only on the input k-tuple $\mathbf{m}_i$ at time instant i but also on the previous k-tuples, $\mathbf{m}_j$ $(i - K \leq j < i)$.

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Convolutional Encoder-2

- This CC encoder takes two input elements and generates at the same instant three output elements. Code Rate $R_c = 2/3$.

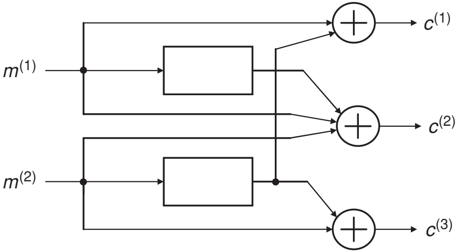

Figure: An example of a convolutional encoder

- Note: convolutional encoders of different structures can generate the same convolutional code.

Convolutional Encoder-3

Another example of CC encoder:

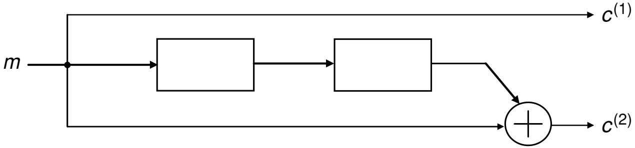

Figure: The structure of a systematic convolutional encoder of rate $R_c = 1/2$

- Note:the message elements appear explicitly in the output sequence together with the redundant elements, so it is a systematic code.

Convolutional Encoder-3

- In the given convolutional encoder below operating in GF(2), the input message k-tuple is simply one bit.
- At time instant i, the encoder takes one bit m_i and generates an output of two bits $c_i^{(1)}$ and $c_i^{(2)}$.
- The input sequence denoted by $\mathbf{m} = (m_0, m_1, m_2, \ldots)$.
- The output sequences denoted by $\mathbf{c}^{(1)} = (c_0^{(1)}, c_1^{(1)}, c_2^{(1)}, \ldots)$ and $\mathbf{c}^{(2)} = (c_0^{(2)}, c_1^{(2)}, c_2^{(2)}, \ldots)$

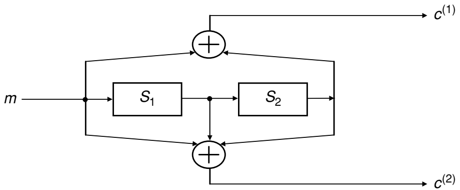

Figure: A CC encoder of rate $R_c = 1/2$

Convolutional Encoder-4

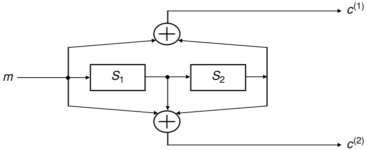

Figure: A CC encoder of rate $R_c = 1/2$

If message sequence $\mathbf{m} = (1, 0, 0, 0)$, then the states of the memory and the output sequences are

Table 6.1 Generator sequences for the FSSM of Figure 6.3

i	m	S_1	S_2	$c^{(1)}$	$c^{(2)}$
0	1	0	0	1	1
1	0	1	0	0	1
2	0	0	1	1	1
3	0	0	0	0	0

Convolutional Encoder-5

- The output sequences can be obtained as the convolution between the input sequence and the two impulse responses of the encoder.
- The impulse responses of the encoder $\mathbf{g}^{(1)}$ and $\mathbf{g}^{(2)}$ can be obtained by applying the unit impulse input sequence $\mathbf{m} = (1, 0, 0, \ldots)$ and observing the resulted outputs $c_i^{(1)}$ and $c_i^{(2)}$.

$$
\begin{aligned}
\mathbf{g}^{(1)} &= \left(g_0^{(1)}, g_1^{(1)}, g_2^{(1)}, \ldots, g_K^{(1)}\right) \\
\mathbf{g}^{(2)} &= \left(g_0^{(2)}, g_1^{(2)}, g_2^{(2)}, \ldots, g_K^{(2)}\right)
\end{aligned}
$$

- For this example, if the input is the unit impulse, then $\mathbf{c}^{(1)} = \mathbf{g}^{(1)}$, $\mathbf{c}^{(2)} = \mathbf{g}^{(2)}$,

$$
\begin{aligned}
\mathbf{g}^{(1)} &= (101) \\
\mathbf{g}^{(2)} &= (111)
\end{aligned}
$$

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Convolutional Encoder-6

- Impulse responses are known as the generator sequences of the CC.
- The encoded sequences can be expressed as

$$\begin{aligned}\mathbf{c}^{(1)} &= \mathbf{m} * \mathbf{g}^{(1)} \\ \mathbf{c}^{(2)} &= \mathbf{m} * \mathbf{g}^{(2)}\end{aligned}$$

where $*$ denotes discrete convolution modulo 2. For an integer $l \geq 0$

$$c_l^i = \sum_{i=0}^{K} m_{l-i} g_i^{(j)} = m_l g_0^{(j)} + m_{l-1} g_1^{(j)} + \ldots + m_{l-K} g_K^{(j)}$$

For the given example of Figure 6.3, there are

$$\begin{aligned}c_l^{(1)} &= \sum_{i=0}^{2} m_{l-i} g_i^{(1)} = m_l + m_{l-2} \\ c_l^{(2)} &= \sum_{i=0}^{2} m_{l-i} g_i^{(2)} = m_l + m_{l-1} + m_{l-2}\end{aligned}$$

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- **Example:** let message sequence $\mathbf{m} = (1, 0, 1, 1)$, then calculate the output sequences.
- **Solution**:

$$\begin{aligned} \mathbf{c}^{(1)} &= (1011) * (101) = (100111) \\ \mathbf{c}^{(2)} &= (1011) * (111) = (110001) \end{aligned}$$

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Description in the D-Transform Domain -1

- As we know, convolution in the time domain becomes multiplication in the spectral domain.
- This indicates a better convolutional codes based on expression given in D-transform domain or the delay domain.
- In D-transform domain, Convolution $*$ operation becomes multiplication.
- The sequences can use a polynomial form expressed in the variable D and the exponent of this variable determines the position of the element in the sequence.
- For example, the message sequence $\mathbf{m}^{(i)} = (m_0^{(i)}, m_1^{(i)}, m_2^{(i)}, \ldots)$ can be represented in polynomial form as

$$M^{(i)}(D) = m_0^{(i)} + m_1^{(i)} D + m_2^{(i)} D^2 + \ldots$$

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Description in the D-Transform Domain -2

- Similarly, impulse responses or the generate sequences can also adopt a polynomial form:

$$\begin{aligned} \mathbf{g}_i^{(j)} &= \left(g_{i0}^{(j)}, g_{i1}^{(j)}, g_{i2}^{(j)}, \ldots\right) \\ G_i^{(j)}(D) &= g_{i0}^{(j)} + g_{i1}^{(j)} D + g_{i2}^{(j)} D^2 + \ldots \end{aligned}$$

- $G_i^{(j)}(D)$ is used as a generic expression for a convolution encoder with multiple inputs and outputs. $G_i^{(j)}(D)$ shows the relation between input i and output j.
- This polynomial can be regarded as generator polynomial for each output sequence of encoder.
- In this input-to-output path, the number of delays or memory units D is called the length of the register which is equal to the degree of the generator polynomial for this path.

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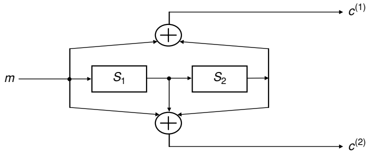
Description in the D-Transform Domain -3



- Use the example of encoder shown in Fig. 6.3, polynomial expression of the output sequences can be obtained as:

$$
\begin{aligned}
C^{(1)}(D) &= M(D)G^{(1)}(D) = M(D)(1+D^2) = c_0^{(1)} + c_1^{(1)}D + c_2^{(1)}D^2 + \dots \\
C^{(2)}(D) &= M(D)G^{(2)}(D) = M(D)(1+D+D^2) = c_0^{(2)} + c_1^{(2)}D + c_2^{(2)}D^2 + \dots
\end{aligned}
$$

- By multiplexing output polynomial $C^{(1)}(D)$ and $C^{(2)}(D)$, the code sequence in polynomial can be written as

$$C(D) = C^{(1)}(D^2) + DC^{(2)}(D^2)$$

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- **Example**: let message sequence $\mathbf{m} = (1, 0, 1, 1)$, determine the output sequence using D-transform domain.
- **Solution**:

$$\begin{aligned}
C^{(1)}(D) &= (1 + D^2 + D^3)(1 + D^2) = 1 + D^3 + D^4 + D^5 \\
C^{(2)}(D) &= (1 + D^2 + D^3)(1 + D + D^2) = 1 + D + D^5 \\
\mathbf{c}^{(1)} &= (100111) \\
\mathbf{c}^{(2)} &= (110001) \\
\mathbf{c} &= (11, 01, 00, 10, 10, 11)
\end{aligned}$$

- Note: each output sequence of the encoder has K bits more than the corresponding input sequence. Thus the code sequence becomes $2K$ bits longer for a $C_{conv}(2, 1, 2)$.
- Work on the Example 6.1 at home.
  - with $\mathbf{m} = (100011)$, to calculate the output sequences.

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Description in the D-Transform Domain -4

- If $M^{(i)}(D)$ is the polynomial expression of the input sequence at input i and $C^{(j)}(D)$ is the polynomial expression of the output j generated by this input, then the polynomial $G_i^{(j)}(D) = C^{(j)}(D)/M^{(i)}(D)$. It is the transfer function that relates input i and output j.
- For a general encoder or FSSM structure which has k inputs and n outputs, there will be kn transfer functions that can be arranged in matrix form as

$$\mathbf{G}(D) = \begin{bmatrix} G_1^{(1)}(D) & G_1^{(2)}(D) & \dots & G_1^{(n)}(D) \\ G_2^{(1)}(D) & G_2^{(2)}(D) & \dots & G_2^{(n)}(D) \\ \vdots & \vdots & & \vdots \\ G_k^{(1)}(D) & G_k^{(2)}(D) & \dots & G_k^{(n)}(D) \end{bmatrix}$$

Description in the D-Transform Domain -5

- A CC $C_{conv}(n, k, K)$ produces the output sequences expressed in polynomial form as

$$\mathbf{C}(D) = \mathbf{M}(D)\mathbf{G}(D)$$

where $\mathbf{M}(D) = \left(M^{(1)}(D), M^{(2)}(D), \ldots, M^{(k)}(D)\right)$
and $\mathbf{C}(D) = \left(C^{(1)}(D), C^{(2)}(D), \ldots, C^{(n)}(D)\right)$

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Description in the *D*-Transform Domain -5

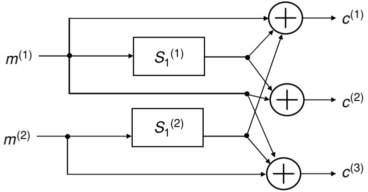

Figure: Encoder of $C_{conv}(3, 2, 1)$ of code rate $R_c = 2/3$.

- The input vector is $\mathbf{m} = \left(m_0^{(1)} m_0^{(2)}, m_1^{(1)} m_1^{(2)}, m_2^{(1)} m_2^{(2)}, \ldots\right)$.
- It can be written into separate input sequences: $\mathbf{m}^{(1)} = \left(m_0^{(1)}, m_1^{(1)}, m_2^{(1)}, \ldots\right)$ and $\mathbf{m}^{(2)} = \left(m_0^{(2)}, m_1^{(2)}, m_2^{(2)}, \ldots\right)$.
- Impulse responses or the generator sequence which relates input i and j: $\mathbf{g}_i^{(j)} = \left(g_{i,0}^{(j)}, g_{i,1}^{(j)}, \ldots, g_{i,K}^{(j)}\right)$

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Description in the D-Transform Domain -6

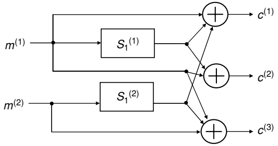

Figure: Encoder of $C_{conv}(3, 2, 1)$ of code rate $R_c = 2/3$.

- The generator sequences:

$$\begin{array}{lll} \mathbf{g}_1^{(1)} = (11) & \mathbf{g}_1^{(2)} = (11) & \mathbf{g}_1^{(3)} = (10) \\ \mathbf{g}_2^{(1)} = (01) & \mathbf{g}_2^{(2)} = (00) & \mathbf{g}_2^{(3)} = (11) \end{array}$$

- Expression for the generator polynomials in the D domain are

$$\begin{array}{ccc} G_1^{(1)}(D) = 1 + D & G_1^{(2)}(D) = 1 + D & G_1^{(3)}(D) = 1 \\ G_2^{(1)}(D) = D & G_2^{(2)}(D) = 0 & G_2^{(3)}(D) = 1 + D \end{array}$$

- If the message vector is $\mathbf{m}^{(1)} = (101)$ and $\mathbf{m}^{(2)} = (011)$, calculate the code sequences.

Description in the D-Transform Domain -7

- As the message vector is $\mathbf{m}^{(1)} = (101)$ and $\mathbf{m}^{(2)} = (011)$, the message polynomials are

$$M^{(1)}(D) = 1 + D^2 \quad M^{(2)}(D) = D + D^2$$

- The code polynomials are

$$\begin{aligned} C^{(1)}(D) &= M^{(1)}(D)G_1^{(1)}(D) + M^{(2)}(D)G_2^{(1)}(D) = 1 + D \\ C^{(2)}(D) &= M^{(1)}(D)G_1^{(2)}(D) + M^{(2)}(D)G_2^{(2)}(D) = 1 + D + D^2 + D^3 \\ C^{(3)}(D) &= M^{(1)}(D)G_1^{(3)}(D) + M^{(2)}(D)G_2^{(3)}(D) = 1 + D + D^2 + D^3 \end{aligned}$$

- Hence, the output sequence at each branch is

$$\mathbf{c}^{(1)} = (1100) \quad \mathbf{c}^{(2)} = (1111) \quad \mathbf{c}^{(3)} = (1111)$$

- The code sequence is $\mathbf{c} = (111, 111, 011, 011)$

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A short summary on convolutional code encoder

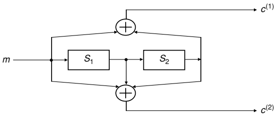
- The general structure of the encoder can be designed to have different memory level K_i in each of its branches.
- The memory order of the encoder is defined as the maximum register length, i.e.,

$$K = \max_{1 \leq i \leq k} (K_i)$$

  where i represent the input index.
- For a given $C_{conv}(n, k, K)$, if the input vector is a sequence of kL information bits, then the code sequence contains $n(L+K)$ bits.
- The amount $n_A = n(K+1)$ is the maximum number of output bits that one given input bit can influence, which is defined as constraint length of the code.
- Generally speaking the code rate of $C_{conv}(n, k, K)$ is k/n if $L \gg K$. However, for a given finite input sequence of length L, the code rate would be

$$R_c = \frac{kL}{n(L+K)}$$

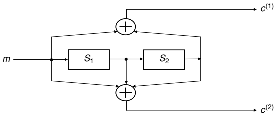
Representation of Connections



- When the input sequence is $\mathbf{m} = (100011)$, the corresponding output sequences can be given as follows:

Input m_i	State at t_i	State at t_{i+1}	$c^{(1)}$	$c^{(2)}$
–	0 0	0 0	–	–
1	0 0	1 0	1	1
0	1 0	0 1	0	1
0	0 1	0 0	1	1
0	0 0	0 0	0	0
1	0 0	1 0	1	1
1	1 0	1 1	1	0
0	1 1	0 1	1	0
0	0 1	0 0	1	1
0	0 0	0 0	0	0

State Diagram Representation -1



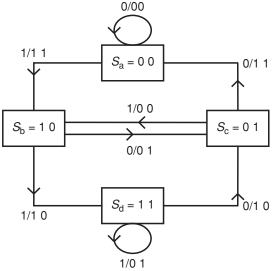
- For this encoder, we can also make a table which list all the possible transitions for constructing a state diagram of a convolutional encoder.
- Note: Table 6.5 there is NO relationship between the adjacent rows.

Table 6.5 Table of all the possible transitions for constructing a state diagram of a convolutional encoder

Input m_i	State at t_i	State at t_{i+1}	$c^{(1)}$	$c^{(2)}$
–	0 0	0 0	–	–
0	0 0	0 0	0	0
1	0 0	1 0	1	1
0	0 1	0 0	1	1
1	0 1	1 0	0	0
0	1 0	0 1	0	1
1	1 0	1 1	1	0
0	1 1	0 1	1	0
1	1 1	1 1	0	1

State Diagram Representation -2

- The state diagram is a pictorial representation of the evolution of the state sequences for the codes.
- The state of the FSSM that forms the encoder of a $1/n$ rate convolutional code is defined as the content of its K register stages.
- For this case there are four states, labelled $S_a = 00$, $S_b = 10$, $S_c = 01$, and $S_d = 11$.
- There are two possible transitions emerging from and arriving at any of these states, as there are only two possible input value, "1" or "0".



State Diagram Representation -3

- This state diagram has four states. However, each state only has two possible exits (i.e., transitions to the other states) only to the specific states.
- This kind of memory will be useful in determining if some transitions are not allowed in the decoded sequence to assist the decision for error correction.

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Trellis Representation -1

- Trellis diagram can clearly describes the start of the state sequence but also the repetitive structure of the state evolution.
- Trellis diagram is a state versus time instant representation.

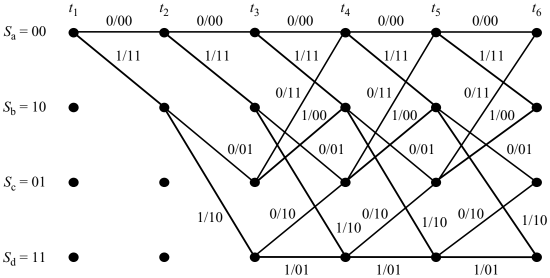

Figure: Trellis representation of the convolutional encoder of Figure 6.3.

- There are $2^K = 4$ possible states and the state structure becomes repetitive after time instant t_4.