Convolutional Codes

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10/03/2014

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Convolutional Codes in Systematic Form -1

- We know in a systematic code, message information can be seen directly extracted from the encoded information.
- In a CC, this means

$$\begin{aligned} c^{(i)} &= m^{(i)}, i = 1, 2, \ldots, k \\ g_i^{(j)} &= \begin{cases} 1 & j = i \\ 0 & j \neq i \end{cases} \end{aligned}$$

- The transfer function for a systematic convolutional code is of the form

$$\mathbf{G}(D) = \begin{bmatrix} 1 & 0 & \ldots & 0 & G_1^{k+1}(D) & G_1^{k+2}(D) & \ldots & G_1^n(D) \\ 0 & 1 & \ldots & 0 & G_2^{k+1}(D) & G_2^{k+2}(D) & \ldots & G_2^n(D) \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & 1 & G_k^{k+1}(D) & G_k^{k+2}(D) & \ldots & G_k^n(D) \end{bmatrix}$$

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Convolutional Codes in Systematic Form -2

- **Example 6.2**: Determine the transfer function of the systematic convolutional code as given in Figure 6.7, and then obtain the code sequence for the input sequence $\mathbf{m} = (1101)$.

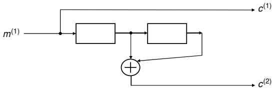

Figure: A systematic convolutional encoder

- The transfer function is

$$\mathbf{G}(D) = [\ 1 \quad D + D^2\ ]$$

- The message polynomial in D-domain: $m(D) = 1 + D + D^3$. Hence the output sequences:

$$\begin{aligned} C^{(1)}(D) &= m(D)G^{(1)}(D) = 1 + D + D^3 \\ C^{(2)}(D) &= m(D)G^{(2)}(D) = (1 + D + D^3)(D + D^2) = D + D^3 + D^4 + D^5 \end{aligned}$$

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Catastrophic Convolutional Encoders -1

- **Example**: Consider the $C_{conv}(2,1,2)$ encoder whose generator matrix is given by

$$\mathbf{G}(D) = [\ 1+D \quad 1+D^2\ ]$$

- In this case, we know that

$$HCF\{1+D, 1+D^2\} = 1+D$$

  where HCF stands for highest common factor. It is also referred to as GCD (Greatest common divisor).

- If the infinite input sequence is $\mathbf{m}(D) = \frac{1}{1+D} = 1 + D + D^2 + \ldots$, the output sequence for this encoder is

$$\begin{aligned} C^{(1)}(D) &= 1 \\ C^{(2)}(D) &= 1+D \end{aligned}$$

  which gives the output sequence with weight of 3, as $\mathbf{c} = (11, 01)$ followed by infinite sequences of 0s.

Catastrophic Convolutional Encoders -2

continue...

- If this output sequence is transmitted over a BSC and three non-zero bits are flipped due to channel noise, then the received sequence will be all zero.
- A maximum likely decoder will produce the all-zero codeword as its estimate, since this is a valid codeword and it agrees exactly with the received sequence.
- Thus the estimated input sequence $\mathbf{m}(D) = \mathbf{0}$. This implies that an infinite number of decoding error caused by a finite number of channel errors.
- This is undesirable and such an encoder is subject to catastrophic error propagation, referred to as a catastrophic encoder.

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Catastrophic Convolutional Encoders -3

- The ways to tell if an encoder is non-catastrophic:
  - No infinite Hamming Weight input sequence produces a finite output sequences. (Or all infinite Hamming weight input sequences produce infinite Hamming weight output sequences).
  - $HCF\{G^{(1)}(D), G^{(1)}(D), \ldots, G^{(n)}(D)\} = D^{l}, l \geq 0$.
  - Systematic linear convolutional code is inherently non-catastrophic.
  - etc.

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Catastrophic Convolutional Encoders-4

- **Example**: Consider the encoder $\tilde{\mathbf{G}}(D) = [1 + D + D^2 + D^3, 1 + D^3]$, tell if the encoder is catastrophic.
- **Solution**:

$$HCF\{1 + D + D^2 + D^3, 1 + D^3\} = 1 + D$$

As $1 + D$ is not in the form D^l, implies that the encoder is catastrophic.

General Structure of Finite Impulse Response FSSMs

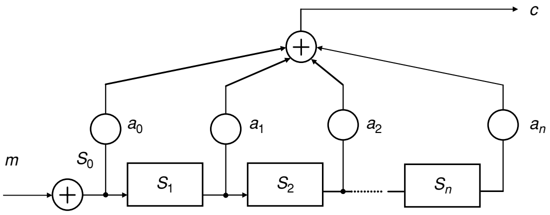

Figure: An FIR FSSM

- The coefficients of these structure are defined over GF(q), i.e., $a_i \in GF(q)$.
- The transfer function for this FIR FSSM shown in the figure is

$$G(D) = \frac{C(D)}{M(D)} = a_0 + a_1 D + a_2 D^2 + \ldots + a_n D^n$$

General Structure of Infinite Impulse Response FSSMs

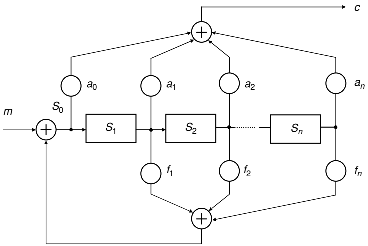

Figure: An IIR FSSM

- An infinite impulse response (IIR) structure contains feedback coefficients that connect the outputs of the registers to an adder, placed at the input.
- The transfer function for this IIR FSSM shown in the figure is

$$G(D) = \frac{C(D)}{M(D)} = \frac{a_0 + a_1 D + a_2 D^2 + \ldots + a_n D^n}{1 + f_1 D + f_2 D^2 + \ldots + f_n D^n}$$

Relation Between the Systematic and Non-Systematic Forms -1

- For a given non-systematic encoder, we can obtain its equivalent systematic form.
- The conversion method consists of converting the transfer function $G(D)$ of a non-systematic form into an expression of a systematic form by means of matrix operation.

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Relation Between the Systematic and Non-Systematic Forms -2

- **Example 6.4**: Determine the equivalent systematic version of the convolutional encoder generated by the transfer function

$$G(D) = G_{ns}(D) = [\ 1 + D^2 \quad 1 + D + D^2]$$

- **Solution**:

$$G_s(D)[\ 1 \quad \frac{1+D+D^2}{1+D^2}]$$

As the new transfer function consist of $\frac{1+D+D^2}{1+D^2}$, this procedure concerts the original FIR FSSM into IIR FSSM.

- Hence, we can see a non-systematic CC encoder with FIR transfer function has an equivalent systematic form with IIR transfer function.

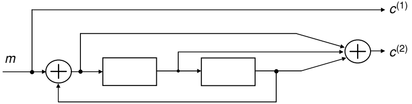

Figure: Equivalent systematic CC encoder of encoder Fig 6.3.

Modified State Diagram

- The state diagram can be modified to provide a complete description of the Hamming Weight of all non-zero codewords, i.e., a codeword weight enumerating function (WEF) for the code.
- The modified state diagram starts and ends in the all-zero State S_a.
- In the modified state diagram, the self-loop of S_a is omitted.
- In the modified state diagram branches emerging and arriving at the states are denoted by the term X^i, where i is the weight of the code sequence that corresponds to that branch.

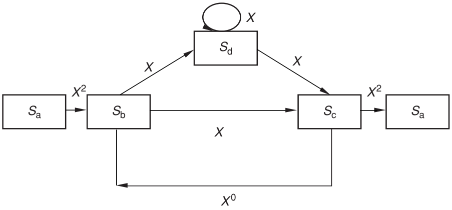

Figure: A modified state diagram of the state diagram Fig 6.5.

Minimum Free Distance

- We can determine the code WEF of a code by considering the modified state diagram of the encoder as a signal flow graph and applying Mason's rule to compute the generating function $T(X)$:

$$T(X) = \sum_i A_i X^i$$

where A_i is the number of sequence of weight i.

- The paths starting and arriving at the all-zero state S_a have a weight that can be calculated by adding the exponents i of the corresponding term of the form X^i.
- In the given figure, the path $S_aS_bS_cS_a$ has a total weight of 5, and path $S_aS_bS_dS_cS_a$ is of weight 6.
- Hence, the minimum distance of this code is 5 which is called minimum free distance $d_f = 5$.

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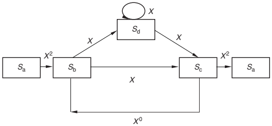
To find generating Function $T(X)$



- A simple way to find the generating function $T(X)$:
  - Assuming the input of the modified diagram is 1, so the output of the diagram is the generating function $T(X)$.
  - Using the name of the states as phantom variables to estimate $T(X)$.
  - As S_b has two inputs and S_c, S_d also has two inputs, there are relation:

$$
\begin{aligned}
S_b &= X^2 + S_c \\
S_c &= XS_b + XS_d = S_d = XS_b + XS_d \\
T(X) &= X^2 S_c
\end{aligned}
$$

Then we can obtain $S_c = \frac{X^3}{1-2X}$, hence $T(X) = \frac{X^5}{1-2X}$.

To find generating Function $T(X)$ -2

Continuing...

- $T(X) = X^5 + 2X^6 + 4X^7 + \ldots.$
- It means there is one path of weight 5, and two paths of weight 6, four path of weight 7, and so on.
- The minimum free distance of this code is $d_f = 5$.

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Viterbi Decoding Algorithm

Let's refer to the Power Point Slides...