A Probability Primer

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Outline

Sample Space and Event

- **Sample space**: a list of all possible outcomes. E.g.,
  - Months of all the students' birthday: [Jan, Feb, ..., Dec];
  - Weather: [sunny, cloudy, rain, ..., snow, foggy];
  - The "7-trinsskalaen" system: [12, 10, 7, 4, 02, 00, -3];
  - Tossing a coin: [Head, Tail];
  - ...
- **Event**: A set of possible outcomes. E.g.,
  - Today's weather
  - ...

Relative Frequency and Probability

- Suppose that we let $N(A)$ be the number of times we observe the event A occurring in n trials, while n represents the number of trials.
  - **Relative Frequency**: The relative frequency of event A is the proportion of times that event A occurs in n trials. It is calculated as

$$\text{Relative frequency of the event } A = \frac{N(A)}{n}$$

  - **Probability**: The probability of event A occurring is the relative frequency of event A occurring as the number of trials approaches infinity. It can be denoted by $p(A)$.

$$p(A) = \lim_{n \to \infty} \frac{N(A)}{n}$$

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Mutually Exclusive

- These two events are mutually exclusive if it is impossible for the same trial to result in both events.
- **Mutual exclusive**: Events A and B are mutually exclusive if and only if $p(A \text{ or } B) = p(A \cup B) = p(A) + p(B)$
- Example of two mutually exclusive events: One dice is rolled.
  - Event A is rolling a 1 or 2.
  - Event B is rolling a 4 or 5.
  - Ask the probability of event A or B?
- Solution:
  - The probability of event A is $1/3$ and the probability of event B is $1/3$.
  - The probability of A or B is $2/3$.

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Mutually Exclusive

- Example of two non-mutually exclusive events: One die is rolled.
  - Event A is rolling a 1 or 2.
  - Event B is rolling a 2 or 3.
  - Ask the probability of A or B?
- Solution:
  - The probability of event A is $1/3$ and the probability of event B is $1/3$.
  - The probability of A or B is NOT $2/3$.
  - A or B will only occur when a 1, 2, or 3 is rolled. Then means that the probability of event A or B occurring is $1/2$,

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Collectively Exhaustive

- Some events, when taken together, must occur.
- Collections of events, at least one of which must occur, are called collectively exhaustive.
- **Collectively Exhaustive**: The events A or B or C are collectively exhaustive if $p(A \cup B \cup C) = 1$.
- Example of collectively exhaustive events: One dice is rolled.
  - Event A is rolling a number less than 5.
  - Event B is rolling a number greater than 3.
  - Any roll of a dice will satisfy either A or B

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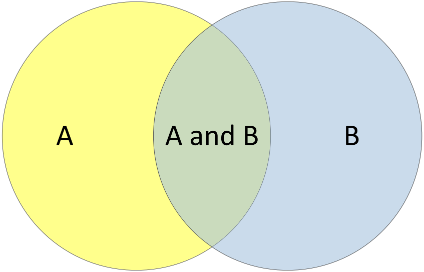
The Basic Addition Property of Probability



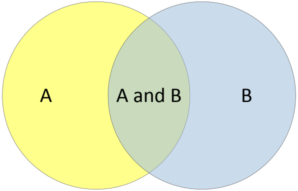
- If two events are non-mutually exclusive, there is

$$
\begin{aligned}
p(A \cup B) &= p(A) + p(B) - p(A \cap B) \\
p(A \cap B) &= p(A) + p(B) - p(A \cup B)
\end{aligned}
$$

- If two events are mutually exclusive, there is

$$p(A \cap B) = 0$$

Conditional Probability



- Without any condition, the probability of event B occurs is equal to the area of circle B relative to the total area of A and B.

$$p(B) = \frac{p(B)}{p(A \cup B)}$$

- What is the probability of event B, conditional on event A occurring?
- It would be the overlapping area of A and B to the area of circle A, that is

$$p(B|A) = \frac{p(A \cap B)}{p(A)}$$

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Conditional Probability

- **Example**: Suppose a dice is rolled. Event A is the roll takes a value less than 4. Event B is that the roll is an odd number. What is the probability of the roll being an odd number given that event A has occurred?
- **Solution**:
  - Event A will occur when a 1, 2, or 3 is rolled.
  - Event B will occur when a 1, 3, or 5 is rolled.
  - So of the three values that encompass event A, two of them are associated with event B.
  - $p(A) = 1/2$, $p(B) = 1/2$, $p(A \cap B) = 1/3$, so

$$p(B|A) = \frac{p(A \cap B)}{p(A)} = \frac{1/3}{1/2} = \frac{2}{3}$$

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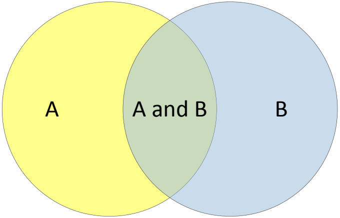
Joint Probability



- The probability of two events occurring at the same time, is the joint probability of these two events.
- The joint probability of event A and B is

$$p(A \cap B) = p(B|A)p(A) = p(A|B)p(B)$$

Independent Events

- Two events are independent if knowledge that one event has occurred does not cause you to adjust the probability of the other event occurring.
- **Definition**: Two events, A and B, are independent, if $p(A|B) = p(A)$.
- As there is $p(A|B) = \frac{p(A\cap B)}{p(B)}$, so

$$
\begin{aligned}
p(A|B) &= \frac{p(A \cap B)}{p(B)} = p(A) \\
&\Downarrow \\
p(A \cap B) &= p(A) \cdot p(B)
\end{aligned}
$$

- it means that the joint probability of two independent events is the product of the probabilities of each individual event.

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Independent Events

- **Example**: Suppose Tom rolls a dice and Jerry roll a dice.
  - Event A is Tom roll is a 3.
  - Event B is Jerry roll is a 3.
  - The probability of Tom's roll being a 3 if Jerry's roll is a 3 is $1/6$.
  - This is exactly the same as the probability of Tom's roll being a 3 and having no information about Jerry's roll.
  - So, the probability of both Tom and Jerry's rolls being a 3 can be calculated as:
$$p(A \cap B) = p(A) \cdot p(B) = 1/36$$

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Bayes' Theorem

- As we know there is:

$$\begin{aligned} p(A|B) &= \frac{p(A \cap B)}{p(B)} \\ p(A \cap B) &= p(A|B) \cdot p(B) \end{aligned}$$

- If we change A and B, there is

$$\begin{aligned} p(B|A) &= \frac{p(A \cap B)}{p(A)} \\ p(A \cap B) &= p(B|A) \cdot p(A) \end{aligned}$$

- Combining these two equations:

$$\begin{aligned} p(A \cap B) &= p(A|B) \cdot p(B) = p(B|A) \cdot p(A) \\ p(A|B) &= \frac{p(B|A) \cdot p(A)}{p(B)} \end{aligned}$$

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Simple Example of Bayes' Theorem

- **Example**: Suppose there is a school with 60% boys and 40% girls as its students.
  - The female students wear trousers or skirts in equal numbers;
  - The boys all wear trousers;
  - An observer sees a (random) student from a distance.
  - What the observer can see is that this student is wearing trousers.
  - What is the probability this student is a girl?

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Simple Example of Bayes' Theorem

- **Solution**:
  - Assuming the event A is that the student observed is a girl, and the event B is that the student observed is wearing trousers.
  - What we need to calculate is $p(A|B)$.
  - $p(A)$ is the probability that the student is a girl regardless of any other information, so $p(A) = 0.4$.
  - $p(B|A)$ is the probability of the student wearing trousers given that the student is a girl $p(B|A) = 0.5$.
  - $p(B)$ is the probability of a (randomly selected) student wearing trousers regardless of any other information, so $p(B) = 0.6 + 0.4 \times 0.5 = 0.8$.
  - So the probability that the observed student who is wearing trousers is a girl can be calculated by Bayes' Theorem:

$$
\begin{aligned}
p(A|B) &= \frac{p(B|A) \cdot p(A)}{p(B)} \\
&= \frac{0.5 \times 0.4}{0.8} = 0.25
\end{aligned}
$$

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Discrete Random Variables

- A random variable X takes a value x_i from the alphabet $\mathbf{X}$ with probability $p(X = x_i)$.
- The vector of probabilities is $\mathbf{p}_X$.
- **Example**: A dice has a number at each side of the cube, so

$$
\begin{array}{rcllllllr}
\mathbf{X} & = & [& 1, & 2, & 3, & 4, & 5, & 6 \quad] \\
\mathbf{p}_X & = & [& 1/6, & 1/6, & 1/6, & 1/6, & 1/6, & 1/6 \quad]
\end{array}
$$

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The Expected Value of A Random Variable

- It is often helpful to have a measure of the central tendency of a random variable.
- The mean value of a random variable is useful.
- It is also called the expected value of the random variable
- **Definition**: The Expected Value of a (discrete) random variable is the sum of all the possible values that random variable can take on each weighted by its probability of occurring.

$$E(X) = \sum_{x_i \in \mathbf{X}} x_i p(x_i)$$

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The Expected Value of A Random Variable

- **Example**: A dice has a number at each side of the cube, so

$$\begin{array}{lcl} \mathbf{X} & = & [\ 1,\ 2,\ 3,\ 4,\ 5,\ 6\ ] \\ \mathbf{p}_X & = & [\ 1/6,\ 1/6,\ 1/6,\ 1/6,\ 1/6,\ 1/6\ ] \end{array}$$

Calculate the expected value of rolling a dice:

$$E_X = 1 \times 1/6 + 2 \times 1/6 + 3 \times 1/6 + 4 \times 1/6 + 5 \times 1/6 + 6 \times 1/6 = 3.5$$

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Joint Probability Distribution

- A joint probability distribution of two random variables identifies the probability of any pair of outcomes occurring together. e.g., $p(X = i, Y = j)$
- Assuming variable X is taken from sample space $\mathbf{X} = [1,\ 2,\ 3]$, with probability vector of $\mathbf{p}_X = [0.5,\ 0.3,\ 0.2]$.
- Assuming variable Y is taken from sample space $\mathbf{Y} = [1,\ 2,\ 3,\ 4]$, with probability vector of $\mathbf{p}_Y = [0.1,\ 0.2,\ 0.4,\ 0.3]$.
- We can construct a table:

	$Y = 1$	$Y = 2$	$Y = 3$	$Y = 4$
$X = 1$	0.05	0.1	0.2	0.15
$X = 2$	0.03	0.06	0.12	0.09
$X = 3$	0.02	0.04	0.08	0.06

- We can see

$$\sum_{i=1}^{|\mathbf{X}|} \sum_{j=1}^{|\mathbf{Y}|} p(x_i, y_j) = 1$$

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Marginal Distribution

- Marginal distribution is the probability that one random variable takes on any of its values regardless of the value of the other random variable.

	$Y = 1$	$Y = 2$	$Y = 3$	$Y = 4$	Marginal prob X
$X = 1$	0.05	0.1	0.2	0.15	0.5
$X = 2$	0.03	0.06	0.12	0.09	0.3
$X = 3$	0.02	0.04	0.08	0.06	0.2
Marginal prob Y	0.1	0.2	0.4	0.3	

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Summary

- Sample space and event
- Probability
- Mutual exclusive and Collectively exhaustive
- Conditional probability
- Joint probability
- The Bayes' Theorem
- Expected value
- Joint probability distribution
- Marginal distribution

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