Linear Block Codes

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Matrix Form

- We know that linear combination of linearly independent vectors can generates space or subspace.
- If there are k linearly independent vectors of vector space V_n defined over GF(2), these k vectors can be written into a matrix form with size of $k \times n$ below.

$$G = \begin{bmatrix} g_{00} & g_{01} & \cdots & g_{0,n-1} \\ g_{10} & g_{11} & \cdots & g_{1,n-1} \\ \vdots & \vdots & & \vdots \\ g_{k-1,0} & g_{k-1,1} & \cdots & g_{k-1,n-1} \end{bmatrix}$$

- These k linearly independent vectors can generate 2^k possible linear combinations, i.e., becomes a k-dimension vector subspace.
- This subspace is also called the *row space* of $\mathbf{G}$.

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- **Example 2.6**: In the following matrix $\mathbf{G}$, the third row is replaced by addition of the second and third rows, and the first and second rows are permuted, generating matrix $\mathbf{G}'$:

$$\mathbf{G} = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 \end{bmatrix} \qquad \mathbf{G}' = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

verify that both matrices generate the same three-dimension subspace.

- **Solution**:

$$\mathbf{G} = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 \end{bmatrix} \qquad \mathbf{G}' = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

The three-dimension subspace of vector space V_5 based on **G**:

$$\begin{aligned}
0 \bullet (10110) \oplus 0 \bullet (01001) \oplus 0 \bullet (11011) &= (00000) \\
0 \bullet (10110) \oplus 0 \bullet (01001) \oplus 1 \bullet (11011) &= (11011) \\
0 \bullet (10110) \oplus 1 \bullet (01001) \oplus 0 \bullet (11011) &= (01001) \\
0 \bullet (10110) \oplus 1 \bullet (01001) \oplus 1 \bullet (11011) &= (10010) \\
1 \bullet (10110) \oplus 0 \bullet (01001) \oplus 0 \bullet (11011) &= (10110) \\
1 \bullet (10110) \oplus 0 \bullet (01001) \oplus 1 \bullet (11011) &= (01101) \\
1 \bullet (10110) \oplus 1 \bullet (01001) \oplus 0 \bullet (11011) &= (11111) \\
1 \bullet (10110) \oplus 1 \bullet (01001) \oplus 1 \bullet (11011) &= (00100)
\end{aligned}$$

Dual subspace matrix

- For the vector space S, which is the row space of $k \times n$ matrix $\mathbf{G}$.

$$\mathbf{G} = \begin{bmatrix} \mathbf{g}_0 \\ \mathbf{g}_1 \\ \vdots \\ \mathbf{g}_{k-1} \end{bmatrix}$$

- If S_d is the dual space of S, the dimension of S_d is $n-k$. S_d is the row space of matrix $\mathbf{H}$ which is composed by $n-k$ linearly independent vectors $\mathbf{h}_0, \mathbf{h}_1, \ldots, \mathbf{h}_{n-k-1}$.

$$\mathbf{H} = \begin{bmatrix} \mathbf{h}_0 \\ \mathbf{h}_1 \\ \vdots \\ \mathbf{h}_{n-k-1} \end{bmatrix} = \begin{bmatrix} h_{00} & h_{01} & \ldots & h_{0,n-1} \\ h_{10} & h_{11} & \ldots & h_{1,n-1} \\ \vdots & \vdots & & \vdots \\ h_{n-k-1,0} & h_{n-k-1,1} & \ldots & h_{n-k-1,n-1} \end{bmatrix}$$

- **Property**: $\mathbf{g}_i \circ \mathbf{h}_j = 0$

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- **Example 2.7**: The vector subspace S is generated by matrix $\mathbf{G}$. Verify that the generated vector subspace S_d by matrix $\mathbf{H}$ is the dual vector space of S.

$$\mathbf{G} = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 \end{bmatrix} \mathbf{H} = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

- **Solution**:

$$\begin{aligned} S &= \{(00000),(11011),(10110),(01001),(10010),(11111),(01101),(00100)\} \\ S_d &= \{(00000),(01001),(10010),(11011)\} \end{aligned}$$

$$\mathbf{g}_i \circ \mathbf{h}_j = 0$$

$$\begin{array}{cc} \cdots & \cdots \\ (10110)\circ(01001) = 0 & (10110)\circ(10010) = 0 \\ (01001)\circ(01001) = 0 & (01001)\circ(10010) = 0 \\ \cdots & \cdots \\ (01101)\circ(01001) = 0 & (01101)\circ(10010) = 0 \end{array}$$

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Introduction of linear block codes

- Message information is grouped into a k-bits block;
- There are 2^k possible messages;
- The generic denotation of a message: $\mathbf{m} = (m_0, m_1, \ldots, m_{k-1})$;
- The encoder encodes each k-bits source message into a n-bits codeword (or code vector): $\mathbf{c} = (c_0, c_1, \ldots, c_{n-1})$;
- The encoding procedure is a bijective assignment between 2^k vectors of the message vector space and 2^k out of the 2^n possible vectors of the encoded vector space.
- The k bits are information bits, the $n - k$ bits are redundancy. The coding rate $R = k/n$.
- **Definition 2.1**: A block code of length n and 2^k codewords are said to be a linear block code $C_b(n, k)$, if the 2^k codewords form a vector subspace, of dimension k, of the vector space V_n of all the vectors of length n with components in the field GF(2).
- **Property**: The sum of any two codewords is also a codeword.

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Generator matrix $\mathbf{G}$

- A linear block code $C_b(n, k)$ is a k-dimension vector subspace of the vector space V_n;
- This k-dimension vector subspace is generated by k linearly independent n-components vectors, $\mathbf{g}_0, \mathbf{g}_1, \dots, \mathbf{g}_{k-1}$;
- Each possible codeword $\mathbf{c}$ is a linear combination of the k vectors $\mathbf{g}_0, \mathbf{g}_1, \dots, \mathbf{g}_{k-1}$:

$$\mathbf{c} = m_0 \bullet \mathbf{g}_0 \oplus m_1 \bullet \mathbf{g}_1 \oplus \dots \oplus m_{k-1} \bullet \mathbf{g}_{k-1}$$

- Write the k linearly independent vectors $\mathbf{g}_0, \mathbf{g}_1, \dots, \mathbf{g}_{k-1}$ into matrix form:

$$\mathbf{G} = \begin{bmatrix} \mathbf{g}_0 \\ \mathbf{g}_1 \\ \vdots \\ \mathbf{g}_{k-1} \end{bmatrix} = \begin{bmatrix} g_{00} & g_{01} & \cdots & g_{0,n-1} \\ g_{10} & g_{11} & \cdots & g_{1,n-1} \\ \vdots & \vdots & & \vdots \\ g_{k-1,0} & g_{k-1,1} & \cdots & g_{k-1,n-1} \end{bmatrix}$$

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Generator matrix $\mathbf{G}$

- The matrix $\mathbf{G}$ is called the **generator matrix**.
- The matrix mechanism for generating any code word of a message vector $\mathbf{m} = (m_0, m_1, \ldots, m_{k-1})$:

$$
\begin{aligned}
\mathbf{c} = \mathbf{m} \circ \mathbf{G} \quad &= \quad (m_0, m_1, \ldots, m_{k-1}) \circ \begin{bmatrix} g_{00} & g_{01} & \cdots & g_{0,n-1} \\ g_{10} & g_{11} & \cdots & g_{1,n-1} \\ \vdots & \vdots & & \vdots \\ g_{k-1,0} & g_{k-1,1} & \cdots & g_{k-1,n-1} \end{bmatrix} \\
&= \quad (m_0, m_1, \ldots, m_{k-1}) \circ \begin{bmatrix} \mathbf{g}_0 \\ \mathbf{g}_1 \\ \vdots \\ \mathbf{g}_{k-1} \end{bmatrix} \\
&= \quad m_0 \bullet \mathbf{g}_0 \oplus m_1 \bullet \mathbf{g}_1 \oplus \ldots \oplus m_{k-1} \bullet \mathbf{g}_{k-1}
\end{aligned}
$$

- The k linearly independent rows of the generator matrix $\mathbf{G}$ generate the linear block code $C_b(n, k)$.

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Generator matrix $\mathbf{G}$

- **Example 2.8**: Consider the following generator matrix of size 4×7 and obtain the codeword corresponding the message vector $\mathbf{m} = (1001)$:

$$\mathbf{G} = \begin{bmatrix} \mathbf{g}_0 \\ \mathbf{g}_1 \\ \mathbf{g}_2 \\ \mathbf{g}_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

- **Solution**:

$$\begin{aligned} \mathbf{c} = \mathbf{m} \circ \mathbf{G} &= 1 \bullet \mathbf{g}_0 \oplus 0 \bullet \mathbf{g}_1 \oplus 0 \bullet \mathbf{g}_2 \oplus 1 \bullet \mathbf{g}_3 \\ &= (1101000) \oplus (1010001) = (0111001) \end{aligned}$$

- **Question**: if the message vector $\mathbf{m} = (1110)$, what is the corresponding codeword?

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Codewords of a linear block code $C_b(7,4)$

Messages	Codewords
0 0 0 0	0 0 0 0 0 0 0
0 0 0 1	1 0 1 0 0 0 1
0 0 1 0	1 1 1 0 0 1 0
0 0 1 1	0 1 0 0 0 1 1
0 1 0 0	0 1 1 0 1 0 0
0 1 0 1	1 1 0 0 1 0 1
0 1 1 0	1 0 0 0 1 1 0
0 1 1 1	0 0 1 0 1 1 1
1 0 0 0	1 1 0 1 0 0 0
1 0 0 1	0 1 1 1 0 0 1
1 0 1 0	0 0 1 1 0 1 0
1 0 1 1	1 0 0 1 0 1 1
1 1 0 0	1 0 1 1 1 0 0
1 1 0 1	0 0 0 1 1 0 1
1 1 1 0	0 1 0 1 1 1 0
1 1 1 1	1 1 1 1 1 1 1

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Codewords of a linear block code $C_b(7,4)$

Messages	Codewords
0 0 0 0	0 0 0 ‖ 0 0 0 0
0 0 0 1	1 0 1 ‖ 0 0 0 1
0 0 1 0	1 1 1 ‖ 0 0 1 0
0 0 1 1	0 1 0 ‖ 0 0 1 1
0 1 0 0	0 1 1 ‖ 0 1 0 0
0 1 0 1	1 1 0 ‖ 0 1 0 1
0 1 1 0	1 0 0 ‖ 0 1 1 0
0 1 1 1	0 0 1 ‖ 0 1 1 1
1 0 0 0	1 1 0 ‖ 1 0 0 0
1 0 0 1	0 1 1 ‖ 1 0 0 1
1 0 1 0	0 0 1 ‖ 1 0 1 0
1 0 1 1	1 0 0 ‖ 1 0 1 1
1 1 0 0	1 0 1 ‖ 1 1 0 0
1 1 0 1	0 0 0 ‖ 1 1 0 1
1 1 1 0	0 1 0 ‖ 1 1 1 0
1 1 1 1	1 1 1 ‖ 1 1 1 1

Block codes in systematic form

- In the previous linear block code, the last four bits of each codeword are the same as the message bits;
- Namely, the message appears inside the codeword;
- The first three bits are the so-called parity check or redundancy bits.
- This particular form of the codeword is called **systematic form**.

$n-k$ parity check bits	k message bits

- Note: the parity check bits can also be placed at the end of the codeword.

Block codes in systematic form

- A systematic linear block code $C_b(n,k)$ is uniquely specified by a generator matrix of the form:

$$
\mathbf{G} = \begin{bmatrix} \mathbf{g}_0 \\ \mathbf{g}_1 \\ \vdots \\ \mathbf{g}_{k-1} \end{bmatrix} = \begin{bmatrix} p_{00} & p_{01} & \dots & p_{0,n-k-1} & 1 & 0 & 0 & \dots & 0 \\ p_{10} & p_{11} & \dots & p_{1,n-k-1} & 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots \\ p_{k-1,0} & p_{k-1,1} & \dots & p_{k-1,n-k-1} & 0 & 0 & 0 & \dots & 1 \end{bmatrix}
$$
$$
= \begin{bmatrix} P & I_k \end{bmatrix}
$$

- Submatrix P is of size $k \times (n-k)$;
- Submatrix I_k is of size $k \times k$.
- Generator matrix $\mathbf{G}$ is of size $k \times n$.

Block codes in systematic form

- Parity check equations:

$$
\begin{aligned}
c_j &= m_0 \bullet p_{0,j} + m_1 \bullet p_{1,j} + \ldots + m_{k-1} \bullet p_{k-1,j} & 0 \leq j < n-k \\
c_j &= c_{n-k+i} = m_i & 0 \leq i \leq k-1, n-k \leq j < n
\end{aligned}
$$

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Block codes in systematic form

- **Example 2.9**: List the parity check equations for the linear block code $C_b(7,4)$ blow:

$$\mathbf{c} = \mathbf{m} \circ \mathbf{G} = (m_0, m_1, m_2, m_3) \circ \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

- **Solution**:

$$\begin{aligned} c_0 &= m_0 \oplus m_2 \oplus m_3 \\ c_1 &= m_0 \oplus m_1 \oplus m_2 \\ c_2 &= m_1 \oplus m_2 \oplus m_3 \\ c_3 &= m_0 \\ c_4 &= m_1 \\ c_5 &= m_2 \\ c_6 &= m_3 \end{aligned}$$

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Parity check matrix $\mathbf{H}$

- We know $\mathbf{G} \Rightarrow S \Rightarrow S_d \Leftarrow \mathbf{H}$.

$$
\begin{aligned}
\mathbf{H} &= \begin{bmatrix}
1 & 0 & \dots & 0 & p_{00} & p_{1,0} & \dots & p_{k-1,0} \\
0 & 1 & \dots & 0 & p_{01} & p_{1,1} & \dots & p_{k-1,1} \\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\
0 & 0 & \dots & 1 & p_{0,n-k-1} & p_{1,n-k-1} & \dots & p_{k-1,n-k-1}
\end{bmatrix} \\
&= \begin{bmatrix} I_{n-k} & & P^T \end{bmatrix}
\end{aligned}
$$

- **Property**:

$$
\begin{aligned}
\mathbf{g}_i &= (p_{i0}, \dots, p_{ij}, \dots, p_{i,n-k-1}, \quad 0, \dots, \underbrace{1}_{i}, \dots, \underbrace{0}_{k-1}) \\
\mathbf{h}_j &= (0, \dots, \underbrace{1}_{j}, \dots, \underbrace{0}_{n-k-1}, \quad p_{0j}, \dots, p_{ij}, \dots, p_{k-1,j})
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{g}_i \circ \mathbf{h}_j &= p_{ij} \oplus p_{ij} = 0 \\
\mathbf{G} \circ \mathbf{H}^T &= \mathbf{0}
\end{aligned}
$$

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Parity check matrix $\mathbf{H}$

- As there is:

$$\begin{aligned} \mathbf{c} &= \mathbf{m} \circ \mathbf{G} \\ \mathbf{G} \circ \mathbf{H}^T &= \mathbf{0} \end{aligned}$$

- hence

$$\mathbf{c} \circ \mathbf{H}^T = \mathbf{m} \circ \mathbf{G} \circ \mathbf{H}^T = \mathbf{0}$$

- The codeword in systematic form is expressed as:

$$\mathbf{c} = (c_0, \ldots, c_j, \ldots, c_{n-k-1}, m_0, m_1, \ldots, m_{k-1})$$

$$\mathbf{h}_j = (0, \ldots, \underbrace{1}_{j}, \ldots, \underbrace{0}_{n-k-1}, p_{0j}, p_{1j}, \ldots, p_{k-1,j})$$

- Thus

$$\begin{aligned} \mathbf{c} \circ \mathbf{h}_j &= c_j \oplus p_{0j} \bullet m_0 \oplus p_{1j} \bullet m_1, \ldots, p_{k-1,j} \bullet m_{k-1} = 0 \\ c_j &= p_{0j} \bullet m_0 \oplus p_{1j} \bullet m_1, \ldots, p_{k-1,j} \bullet m_{k-1} \end{aligned}$$

- It means parity check matrix $\mathbf{H}$ also specifies completely a given block code.

Parity check matrix **H**

- **Example 2.10**: Determine the parity check matrix **H** for the linear block code $C_b(7,4)$ generated by the generator matrix:

$$\mathbf{G} = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} P & I_k \end{bmatrix}$$

- **Solution**:

$$\mathbf{H} = \begin{bmatrix} I_{n-k} & P^T \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}$$

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Syndrome Error Detection

- The components of the codeword $\mathbf{c} = (c_0, c_1, \ldots, c_{n-1})$ are taken from GF(2), i.e., $c_i \in \text{GF}(2)$.
- The received vector is denoted by $\mathbf{r} = (r_0, r_1, \ldots, r_{n-1})$, there is also $r_i \in \text{GF}(2)$.
- Error pattern is modeled by $\mathbf{e} = (e_0, e_1, \ldots, e_{n-1})$, $e_i \in \text{GF}(2)$.
- The error vector $\mathbf{e}$ has non-zero components in the positions when errors occur.
- What we are interested in is to detect error and correct the received vector.

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Syndrome Error Detection

There is

$$\mathbf{r} = \mathbf{c} \oplus \mathbf{e}$$

Hence

$$\mathbf{c} = \mathbf{r} \oplus \mathbf{e}$$

Since any codeword fits the condition:

$$\mathbf{c} \circ \mathbf{H}^T = \mathbf{0}$$

an error-detection mechanism can be implemented based on the above expression:

$$\begin{aligned} \mathbf{s} &= \mathbf{r} \circ \mathbf{H}^T \\ &= (\mathbf{c} \oplus \mathbf{e}) \circ \mathbf{H}^T \\ &= \mathbf{c} \circ \mathbf{H}^T \oplus \mathbf{e} \circ \mathbf{H}^T = \mathbf{e} \circ \mathbf{H}^T \end{aligned}$$

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Syndrome Error Detection

- $\mathbf{s}$ is called syndrome vector;
- If syndrome vector is all-zero vector, then the received vector is a valid codeword;
- When the syndrome vector contains at least one non-zero component, an error is detected in the received vector.
- Note: it is possible that the syndrome vector can be the all-zero vector even though the errors occurs in the received vector.
- If error patterns are equal to one of the codewords, i.e., $\mathbf{e} = \mathbf{c}$, $\mathbf{e}$ is not a all-zero component vector, there is

$$\mathbf{s} = \mathbf{e} \circ \mathbf{H}^T = \mathbf{0}$$

- **Q:** How many undetectable non-zero error pattern exist?
- **A:** $2^k - 1$.

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Syndrome Error Detection

- **Example 2.11:** For the linear block code $C_b(7,4)$, the parity check matrix is listed blow. Obtain the analytical expression of the syndrome vector bits.

$$\mathbf{H} = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}$$

Syndrome Error Detection

$$\mathbf{H} = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}$$

- **Solution**:
- Assuming $\mathbf{r} = (r_0, r_1, r_2, r_3, r_4, r_5, r_6)$, then

$$\mathbf{s} = (s_0, s_1, s_2) = \mathbf{r} \circ \mathbf{H}^T = (r_0, r_1, r_2, r_3, r_4, r_5, r_6) \circ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\begin{aligned} s_0 &= r_0 \oplus r_3 \oplus r_5 \oplus r_6 \\ s_1 &= r_1 \oplus r_3 \oplus r_4 \oplus r_5 \\ s_2 &= r_2 \oplus r_4 \oplus r_5 \oplus r_6 \end{aligned}$$

Syndrome Error Detection

- Syndrome vector actually is dependent on the error vector. Thus the bits of the syndrome vector can be expressed as:

$$
\begin{aligned}
s_0 &= e_0 \oplus e_{n-k} \bullet p_{00} \oplus e_{n-k+1} \bullet p_{10} \oplus \ldots \oplus e_{n-1} \bullet p_{k-1,0} \\
s_1 &= e_1 \oplus e_{n-k} \bullet p_{01} \oplus e_{n-k+1} \bullet p_{11} \oplus \ldots \oplus e_{n-1} \bullet p_{k-1,1} \\
&\vdots \\
s_{n-k-1} &= e_{n-k-1} \oplus e_{n-k} \bullet p_{0,n-k-1} \oplus e_{n-k+1} \bullet p_{1,n-k-1} \oplus \ldots \oplus e_{n-1} \bullet p_{k-1,n-k-1}
\end{aligned}
$$

Example 2.12

- **Example 2.12**: For the linear block code $C_b(7,4)$, the transmitted codeword $\mathbf{c}$ is affected by channel noise and received as the vector $\mathbf{r} = (0001010)$. The syndrome vector is $\mathbf{s} = (001)$, so the syndrome bits can be expressed by components in the error vector as below. To decode the transmitted codeword $\mathbf{c}$.

$$\mathbf{H} = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}$$

$$\begin{aligned} 0 &= e_0 \oplus e_3 \oplus e_5 \oplus e_6 \\ 0 &= e_1 \oplus e_3 \oplus e_4 \oplus e_5 \\ 1 &= e_2 \oplus e_4 \oplus e_5 \oplus e_6 \end{aligned}$$

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Example 2.12

e_0	e_1	e_2	e_3	e_4	e_5	e_6
0	0	1	0	0	0	0
1	1	1	1	0	0	0
1	0	0	0	0	0	1
0	1	0	1	0	0	1
0	0	0	1	0	1	0
1	1	0	0	0	1	0
0	1	1	0	0	1	1
1	0	1	1	0	1	1
0	1	0	0	1	0	0
1	0	0	1	1	0	0
1	1	1	0	1	0	1
0	0	1	1	1	0	1
1	0	1	0	1	1	0
0	1	1	1	1	1	1
1	1	0	1	1	1	1
0	0	0	0	1	1	1

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Example 2.12

- There are $2^4 = 16$ different error patterns satisfy the equations;
- The probability of i errors occur is higher than that of $i + 1$ errors occur;
- In the channel like BSC, the error pattern with the smallest number of non-zero components is considered as the true error pattern.
- Therefore, for the previous case, $\mathbf{e} = (0010000)$ is considered as the true error pattern, so

$$\mathbf{c} = \mathbf{r} \oplus \mathbf{e} = (0001010) \oplus (0010000) = (0011010)$$